Using Algebra Fact Families to Solve Equations

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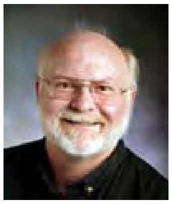

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Beginning algebra texts usually present two basic methods for solving basic single variable equations, such as

$$3x + 5 = 11. \qquad (1)$$

These two methods, often referred to as *Balancing Equations* (Method 1) and *Undoing Equations* (Method 2), are based on the idea that the problem-solver must take an action upon the equation itself. While these two approaches are very familiar and widely used, students sometimes have difficulty in making the transition from the concrete mechanics used in Method 1 to the abstract concepts underlying Method 2.

This article presents *Replacement Equations* (Method 3) as an alternative approach to help students make this transition more effectively (see Lay & Lay, 1990). The method of Replacement Equations is based on equivalent forms of equations, similar to the concept of "fact families" that children learn when learning arithmetic combinations. The replacement approach builds directly on basic arithmetic concepts with which the child may be familiar, and it provides a more seamless transition from arithmetic to pre-algebra and algebra. Perhaps more importantly, Method 3 does not require the student to act upon the equation so much as the student is required to recognize equivalent forms of the equation. We begin with a brief discussion of the two traditional approaches.

Traditional Approaches

Method 1: Balancing Equations

In this method the idea of a balance scale is utilized, so that the student solves an equation by "doing the same thing" to both sides of an equation to maintain balance. Using equation (1) above as an example, students can view the '=' sign as the fulcrum of a balance scale. It is reasonable that subtracting 5 from both sides of the equation will not alter the balance. Often, students are taught to write the process something like the following:

$$\begin{array}{rcl} 3x + 5 & = & 11 \\ \underline{\quad -5} & & \underline{-5} \\ 3x & = & 6 \end{array}$$

Using similar reasoning, students see that dividing both sides of the new equation by 3 will again preserve the balance. Thus, we have

$$\frac{3x}{3} = \frac{6}{3}$$

or $x = 2$. Because this approach is easy to picture using the concept of a balance scale, students are often comfortable using this method for equations like (1).

However, this approach often has significant limitations. The two-step process of adding/subtracting and multiplying/dividing is based, to a great extent, on the visual form of the equation. For example, if equation (1) were given as

$$5 + 3x = 11 \qquad (2)$$

it may not be clear to some students how to proceed. Based on the approach used for solving equation (1), one should subtract the second addend, or $3x$, from both sides. Any experienced mathematics teacher can testify to the incorrect results that may follow, including $5 = 8x$ or even $5x = 8$. Similarly, if equation (1) were originally given as

$$11 = 3x + 5 \qquad (3)$$

similar mistakes can occur, especially since the "left-to-right" sense of the equation is now reversed.

Method 2: "Undoing" Equations

In this method the student considers how a number, represented by x, is acted upon to produce a particular result expressed on the "other side" of the equation. Using the example

$$2(x + 3) – 1 = 13 \qquad (4a)$$

a student might ask the question, "What happened to me?" from the point of view of the variable. In short, this is what happened, in order, to x: someone added 3, multiplied that result by 2, and then subtracted 1 from that, to yield a result of 13. If these operations are viewed as, say, wrapping a package, it seems reasonable to unwrap it by undoing the events in reverse order. Hence, to undo the last operation of subtracting 1, we should first add 1. Then, to undo the effect of multiplying by 2, we will divide by 2. Finally, we would subtract 3 to undo the original action of adding 3. The process might take on the following schematic form.

Doing (Top to bottom)	Undoing (Bottom to top)
Step 1. Start with x	Step 5. You have $x = 4$
Step 2. Add 3 [$x + 3$]	Step 4. Subtract 3 [7 – 3]
Step 3. Multiply by 2 [$2(x + 3)$]	Step 3. Divide by 2 [14/2]
Step 4. Subtract 1 [$2(x + 3) – 1$]	Step 2. Add 1 [13 + 1]
Step 5. Obtain $2(x + 3) – 1 = 13$	Step1. Start with $2(x + 3) – 1 = 13$

Written more succinctly, we would probably see the following sequence of equivalent equations on a student's paper:

$$\begin{aligned} 2(x + 3) - 1 &= 13 \\ 2(x + 3) &= 14 \\ x + 3 &= 7 \end{aligned}$$

$$x = 4$$

Again, some of the concerns noted with Method 1 are relevant here. For example, if the original equation is in the equivalent form

$$1 = 13 - 2(x + 3) \qquad (4b)$$

the third step of the process, namely, "subtract the result from 13," is a little difficult to articulate. Perhaps more importantly, the "undoing" step for this is less clear.

Teachers of early algebra students know that students often use a hybrid approach of both methods. For example, in equation (4b), they would likely distribute the negative two and proceed from there. Although this will yield the correct answer, in doing so the student has changed the nature of the problem by not recognizing it as a simple difference of the form A = B – C. More importantly, the student's cues may be based more on the equation's appearance rather than its mathematical form. This practice can, over time, diminish the student's maturation as an effective user of algebra.

As a bridge between these mechanical and conceptual approaches, there is a third method of solving equations, that of Replacement Equations, that may help students to make the transition, not only to solve equations more skillfully, but to more fully understand the underlying concepts upon which equations are based.

Method 3: Replacement Equations

Fact Families

Young children are exposed to the notion of "fact families" when working with natural numbers and the four basic operations. There are two basic types of fact families: one for addition and subtraction, and the other for multiplication and division. An example of an addition/subtraction fact family (2 + 3 = 5) is given in Table 1. Please note that while there are eight combinations in the family, five of the combinations are connected by the symmetric property of equality (for example, 2 + 3 = 5 is equivalent to 5 = 2 + 3) and the commutative property of addition (for example, 2 + 3 = 5 is equivalent to 3 + 2 = 5). Thus, there are essentially three unique members of this family: 2 + 3 = 5, 5 – 3 = 2, and 5 – 2 = 3.

Table 1
Family of Facts for Addition and Subtraction

2 + 3 = 5	5 = 2 + 3
3 + 2 = 5	5 = 3 + 2
5 – 3 = 2	2 = 5 – 3
5 – 2 = 3	3 = 5 – 2

The fact families for addition/subtraction can be summarized as follows. Any addition/subtraction problem features one sum, **S**, and two addends, both labeled **A** for simplicity. If we want the sum, add the two addends. If we want an addend, subtract the other addend from the sum. Symbolically, then, the fact families for addition/subtraction can be summarized as

$$\mathbf{S = A + A} \text{ and } \mathbf{A = S - A.} \qquad (5)$$

As illustrated above, the equation A = S – A actually names two differences (for example, 5 – 3 = 2 and 5 – 2 = 3).

Similarly, there are eight combinations in a multiplication/division fact family, such as $2 \times 3 = 6$ (Table 2). A multiplication/division problem features one product, **P**, and two factors, again both labeled **F** for simplicity. If we want the product, multiply the two factors. If we want a factor, divide the product by the other factor. Thus,

$$\mathbf{P} = \mathbf{F} \times \mathbf{F} \text{ and } \mathbf{F} = \mathbf{P} \div \mathbf{F} \qquad (6)$$

As before, the eight combinations in Table 2 reduce to three essentially different statements: $2 \times 3 = 6$, $2 = 6 \div 3$, and $3 = 6 \div 2$.

Table 2
Family of Facts for Multiplication and Division

$2 \times 3 = 6$	$6 = 2 \times 3$
$3 \times 2 = 6$	$6 = 3 \times 2$
$6 \div 3 = 2$	$2 = 6 \div 3$
$6 \div 2 = 3$	$3 = 6 \div 2$

Applications to Algebra

The key to using the fact families is to identify which of the two forms an equation is currently in, namely, an addition/subtraction equation or a multiplication/division equation. In the case of a single operation equation, the choice is clear. For example, $2 + 3 = 5$ is in addition/subtraction form, and so other equivalent forms would be $5 - 2 = 3$ or $5 - 3 = 2$. Suppose now that one of the terms were missing, say, 2. Then our equation could appear as $x + 3 = 5$, $5 - x = 3$, or $5 - 3 = x$. All three of these equations are interchangeable since they are equivalent statements. Thus, if a student were given the perhaps troublesome equation $5 - x = 3$, she could "trade it in" for its equivalent form $x + 3 = 5$, which is perhaps more easily dealt with. Likewise, since $3 = 6 \div 2$ is in multiplication/division form, equivalent forms would be $2 = 6 \div 3$ or $2 \times 3 = 6$. Again, the division equation $3 = 6 \div x$ could be easily exchanged for the multiplication equation $3x = 6$.

Suppose that the equation has more than one operation, such as in our original equation,

$$3x + 5 = 11. \qquad (1)$$

There are two operations contained in the left-hand side: multiplication (3 times x) and addition ($3x$ plus 5). Under the order of operations, the last operation performed would be addition. Since the form of an expression is always *based on its last operation*, the equation $3x + 5 = 11$ is currently in the form of an addition/subtraction equation, with $3x$ and 5 as addends and 11 as the sum. Thus, when viewed as an Algebra Fact Family, equation (1) could be expressed as any one of eight equivalent equations (Table 3).

It is critical to note that, unlike the procedures described in Method 1, we did not "do" anything to the equation; we merely represented it as an Algebra Fact Family having two addends and a sum. Thus, once an equation is identified as being an addition/subtraction equation or a multiplication/division equation, it can easily be repeatedly "traded" for an equivalent form until the equation is solved. Let's return to equation (1) to illustrate.

$3x + 5 = 11$	Looking at the term $3x$, we see an addition equation with 11 as the sum and 3x and 5 as addends (**A** + **A** = **S**). So, trade for the equivalent form,
$3x = 11 - 5$	(**A** = **S** – **A**).

$3x = 6$	Computation. Now, looking at x, we see a multiplication equation with 6 as the product (**F** × **F** = **P**). So, trade for the equivalent form,
$x = 6 \div 3$	(**F** = **P** ÷ **F**).
$x = 2$	Computation.

Please note that using the familiar notion of Fact Families from arithmetic, we were able to replace old equations with new ones. The only skill needed is to appropriately recognize the three pieces of the equation. This may be a new way of thinking for some students and provides another method teachers can use to help build meaning into algebraic techniques. However, it is a skill that can be developed rapidly and, once developed, renders the solution of even a multi-operation equation quite easy.

Table 3
Family of Algebra Facts for Addition and Subtraction

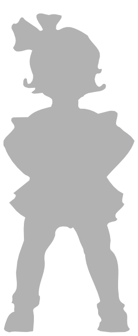
$3x + 5 = 11$	$11 = 3x + 5$
$5 + 3x = 11$	$11 = 5 + 3x$
$11 - 5 = 3x$	$3x = 11 – 5$
$11 - 3x = 5$	$5 = 11 - 3x$

Usefulness for Students

Every teacher of pre-algebra or algebra knows that rational expressions or equations involving fractions have been the bane of many students. However, if the equation is viewed as a member of an Algebraic Fact Family, that is, an equation comprised of three terms, two of which are linked by an operation, its solution is rendered much easier. For example, consider

$$\frac{20}{x-2} = 4. \qquad (7)$$

The following is a recent conversation with an early algebra student, Nayla, using the Algebra Fact Family approach.



Teacher:	What form do you think this equation is in?
Nayla:	Well, maybe a subtraction equation?
Teacher:	What would be the last operation to carry out?
Nayla:	Oh! Division, so this is a division equation with 4 and the "bottom," $x – 2$ as the factors, like 8 over 2 would be 4.
Teacher:	Good. Can we trade the equation in?
Nayla:	Sure. It's the same as 20 = 4 times $x – 2$.
Teacher:	Since we want to get a value for x, can we use a different trade?
Nayla:	Try 20 over 4 equals $x – 2$.
Teacher:	And then?
Nayla:	Since 20 over 4 is 5, we have $x – 2 = 5$, so x is just 2 + 5, or 7.

Please note that in using the Algebra Fact Families, Nayla is intuitively making the transition to Method 2 of using Inverse Operations to solve equations. Indeed, the whole idea of Inverse Operations is based on Fact Families; that is, viewing each equation as having three terms and one operation at any given step. Thus, Algebra Fact Families can help ease the transition into this more sophisticated notion of inverse operations.

Below are some more advanced problems our students have successfully used the Fact family approach on, often with a dialogue similar to Nayla's.

$$\frac{2}{3}x + (-\frac{1}{2}) = \frac{7}{2} \quad (8)$$

$$5x - 2 = 7 - 2x \quad (9)$$

Solve for x: $y - y_1 = m(x - x_1)$ (10)

Equation 8 is an addition equation with sum 7/2; thus, it can be traded for the subtraction equation (2/3) x = 7/2 – (-1/2). Simplifying, we have the division equation (2/3) x = 4, which can be traded for the division equation x = 4 divided by 2/3, or x = 6. In equation (9), by grouping ($5x$ – 2), we see the equation as being in the subtraction form A = B – C, where A and C are the addends and B is the sum.

Eighth grade student Juan summarized his solution for problem 10 using this hybrid approach:

"We can think of the equation as C = m A, which I can swap out for A = C/m, so that we have $x - x_1 = \frac{y - y_1}{m}$. Then, adding the x_1, we have $x = x_1 + \frac{y - y_1}{m}$."

Summary

This article has presented the use of Algebra Fact Families as an alternative way to understand and solve equations. The idea is based on the fact families of arithmetic that children learn at an early age, and is based on the developmental concept of "reversibility" described by child psychologist Jean Piaget (1970, 1985). As described above, the method of solving equations using inverse operations (Method 2) is based on the same idea underlying the Algebra Fact Families. However, the Algebra Fact Families make explicit the "undoing" process that many students have difficulty grasping due to its implicit nature. When applied to algebraic equations, the use of Algebra Fact Families gives the student the ability to rewrite an equation using an equivalent form. Using Algebra Fact Families gives the student a dependable tool in solving any equation, since an equation can always be viewed as a relationship between three terms and one operation. Perhaps most importantly, students can move away from the practice of balancing equations, in which the student must take action upon the equation, and move towards an understanding of the nature of the equation itself based on its form.

References

Lay, L., & Lay, S. (1990). *Principles of algebra* (4th ed.). New York: Meredith Publications.
Piaget, J. (1970). *Genetic epistemology*. New York: Columbia University Press.
Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development.* Chicago: University of Chicago Press.

Quote:

Theories have four stages of acceptance:

i) this is worthless nonsense;

ii) this is an interesting, but perverse, point of view;

iii) this is true, but quite unimportant;

iv) I always said so. – J. B. S. Haldane