Shape Affects the Sound of a Drum: Modeling Area and Perimeter

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Abstract: *The sound of a drum is based on the wave equation from physics. The surface area of the drumhead and its fixed perimeter are key parameters. This paper uses area and perimeter to model and answer the question, Can a rectangular and a circular drum make the same sound?*

Keywords. *measurement, calculus, physics, modeling, multiple representations*

1 Introduction

The trained ear can hear the shape of a drum! The sound of a drum is based on the area and perimeter of the drumhead, and the tension of the drumhead's skin. Mark Kac (1966) opened a debate regarding whether it is always possible to hear the shape of a drum. In 1991, Carolyn Gordon and her colleagues "found two drums that have different shapes yet exactly the same sound when played" (Greenwald & Nestler, 2001, p. 12), but these were no ordinary drums. For practical purposes, the shape of a drumhead determines the sound of the drum. And in all cases, the sound depends on the area and perimeter of the drumhead. Greenwald and Nestler got us interested in the shapes and sounds of drums and how this topic might be an enticing way to get students to think deeply about the ideas of area and perimeter.

Understanding area and perimeter - and knowing which is which - is important for students in Grades 3-12 and beyond. For example,

- In Grade 3, students are expected to "recognize perimeter as an attribute of plane figures and distinguish between linear and area measures" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 22).
- In Grade 4, they "apply the area and perimeter formulas for rectangles in real world and mathematical problems" (p. 31).
- In Grade 7, students should "know the formulas for the area and circumference of a circle and use them to solve problems" (p. 50).

Greenwald and Nestler (2001) made a statement that especially intrigued us: They claimed "that a circle and a rectangle can never have both the same area and the same perimeter" (p. 14). Greenwald and Nestler suggested verifying this claim using computer algebra and plotting functions in three dimensions, but we have found a high school-friendly approach that we wish to share with teachers and teacher educators. We think that high school students could investigate Greenwald and

Nestler's claim in a variety of ways and with a variety of tools. Moreover, such an investigation would reinforce and deepen their understanding of the area and perimeter of rectangles and circles while developing their reasoning and problem-solving skills.

The purpose of this article is to show some possible solution paths. In doing so, we hope to stimulate the creation of instructional tasks that integrate the high school conceptual categories of algebra, functions, geometry, and modeling with mathematical processes (NCTM, 2000) and mathematical practices (NGA Center & CCSSO, 2010). Our investigation of this claim about a drum begins with quantitative reasoning in the geometric realm, but quickly moves to abstract reasoning in the world of algebra and functions, and finally returns to the original context of the shape and sound of a drum. Thus, it is a nice example of the Common Core's modeling cycle, shown in Fig 1.

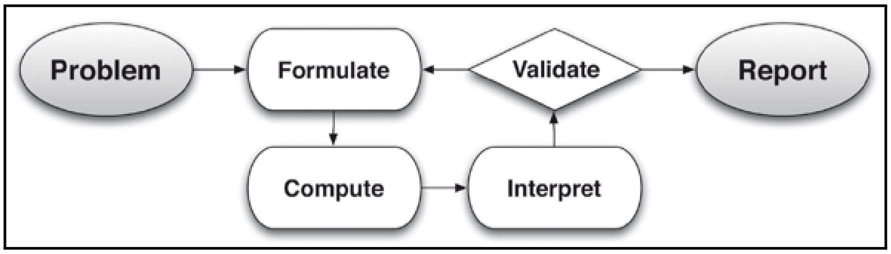

Fig. 1: *A schematic diagram of mathematical modeling (NGA Center & CCSS), 2010, p. 72).*

2 A Rectangle and a Circle Can't Have Both Equal Areas and Equal Perimeters

2.1 Formulating the Problem

We assume equal perimeters and equal areas and seek a contradiction, noting that the perimeter of a circle is its circumference. Let the side lengths of the rectangle be the positive numbers x and y, and let radius of the circle be the positive number r, all three measured using the same units. Then the perimeter of the rectangle = $2(x+y)$, and the circumference of the circle = $2\pi r$. The assumption that the rectangle and the circle have equal perimeters implies that

$$2\pi r = 2(x+y) \Leftrightarrow r = \frac{x+y}{\pi}. \tag{1}$$

If, in addition, the rectangle and circle have equal areas, then

$$\pi r^2 = xy \Leftrightarrow r = \sqrt{\frac{xy}{\pi}}. \tag{2}$$

Equating the r values in equations (1) and (2) yields

$$\frac{x+y}{\pi} = \sqrt{\frac{xy}{\pi}} \Leftrightarrow x+y = \frac{\pi\sqrt{xy}}{\sqrt{\pi}} \Leftrightarrow x+y = \sqrt{\pi xy}. \tag{3}$$

The question becomes, *Does equation (3) have any solutions for positive values of x and y?* If so, we have found another solution to the problem posed by Mark Kac in 1966. If not, we have found the contradiction we seek.

2.2 Reformulation and Computation

We now try to solve equation (3). We use a graphical approach and begin by letting $y = b$ for an arbitrary positive real number b. Thus, equation (3) becomes $x + b = \sqrt{\pi x b}$. We then define two functions: $f_1(x) = x + b$ and $f_2(x) = \sqrt{\pi b}\sqrt{x}$. (By substituting b for y, these are now functions of one variable instead of two, which makes the problem accessible to high school students.) The graphs of these functions are shown in Fig 2.

Try creating these graphs yourself for various positive values of b. In all cases, it appears that $f_1(x) > f_2(x)$ and that the essential relationship between the two graphs is unchanged as b varies. The positive number b acts as a dilation factor with (0, 0) serving as the center of dilation. Said another way, any two versions of Fig 2 that are obtained by using different values of b are geometrically similar to one another. (This is a nice connection between the world of function graphing and the world of plane geometry.) Although this visual evidence is convincing, it falls short of the contradiction needed for a formal proof.

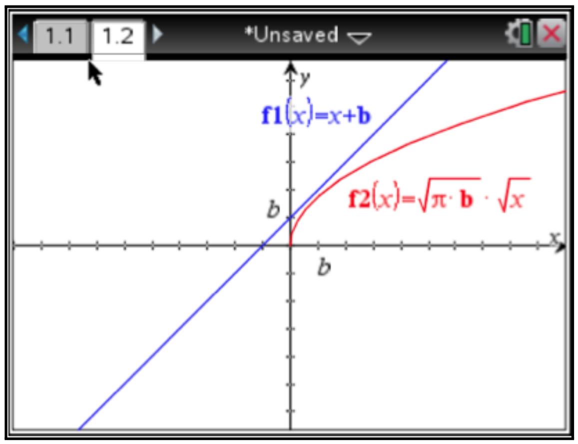

Fig. 2: *The graphs for f_1 and f_2 with a scale of b units on both axes.*

2.3 A Second Reformulation

We can take the graphical approach one step further by graphing $f_3(x) = f_1(x) - f_2(x)$,as shown in Fig 3. As may be expected at this point, regardless of the particular value of $b > 0$, it appears that the graph of $y = f_3(x)$ is strictly positive. Furthermore, Fig 4 illustrates that, for the case $b = 1$, the minimum feature of the graphing software indicates that the least value of f_3 is about 0.215, which occurs when $x = 0.785$. By dilation, the minimum point for the general case would be approximately $(0.785b, 0.215b)$. Once again, we have strong evidence of a contradiction of our original assumption that *a circle and a rectangle exist that have equal areas and equal perimeters*. But we still do not have a proof.

2.4 Modeling With Calculus

A proof that $f_3(x) = x + b - \sqrt{\pi b}\sqrt{x}$ is strictly positive throughout its domain requires some calculus. So, for completeness, we now confirm with certainty that the minimum value of f_3 is positive. We begin by setting the derivative of this function to zero:

$$f_3'(x) = \left[1 - \frac{\sqrt{\pi b}}{2\sqrt{x}} = 0\right]$$

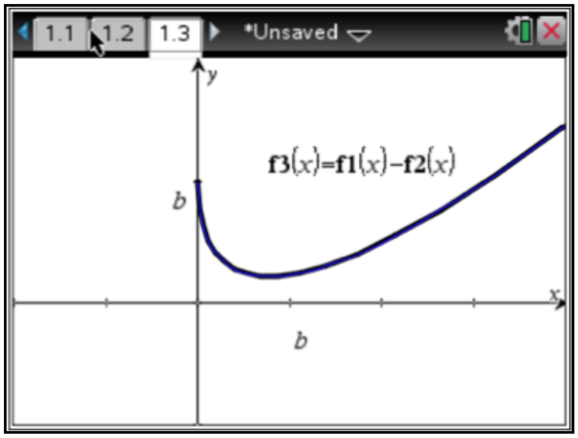

Fig. 3: *The graph of f_3 with a scale of b units on both axes.*

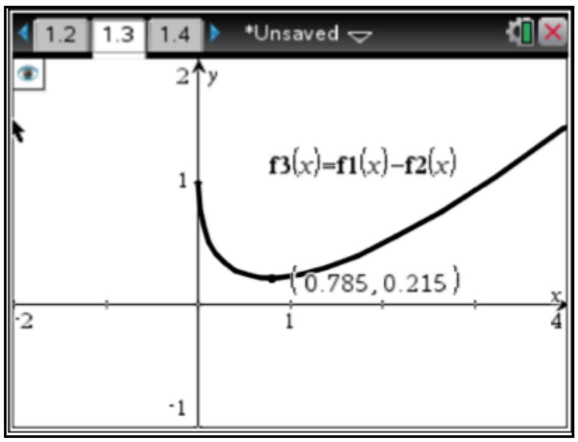

Fig. 4: *The graph of f_3 with its minimum point shown for the case $b = 1$.*

We then solve the bracketed equation:

$$\begin{aligned} \frac{\sqrt{\pi b}}{2\sqrt{x}} &= 1 \\ 2\sqrt{x} &= \sqrt{\pi b} \\ 4x &= \pi b \\ x &= \frac{\pi b}{4} \end{aligned}$$

Notice that the second derivative $f_3''(x) = \frac{\sqrt{\pi b}}{4\sqrt{x^3}}$ is undefined at 0, where a local maximum occurs, and is strictly positive on the rest of the domain of f_3, $(0, \infty)$, and hence the critical value of $x = \frac{\pi b}{4}$ will produce an absolute minimum value of the function:

$$f_3\left(\frac{\pi b}{4}\right) = \frac{\pi b}{4} + b - \sqrt{\pi b}\sqrt{\frac{\pi b}{4}} = \frac{\pi b}{4} + b - \frac{\pi b}{2} = b - \frac{\pi b}{4} = b\left(1 - \frac{\pi}{4}\right)$$

because $4 > \pi$, $\frac{\pi}{4} < 1$, and $1 - \frac{\pi}{4} > 0$. Because b is also positive, the minimum value of f_3 is a positive number and $f_3(x)$ is never zero. Thus $x + y = \sqrt{\pi xy}$ has no solution, and it is impossible for a circle and a rectangle to be both equal in area and in perimeter.

2.5 Reporting the Results and Looking Back

Thus we conclude that a rectangle and a circle can never have both the same area and the same perimeter. Returning to the original context, a rectangular drum will never sound like a round drum! This conclusion was reached using multiple representations; it was demonstrated by a precalculus graphical approach and formally proved using calculus techniques. Before moving on we note that

- Reducing the number of independent variables from two to one was key; this avoided the need for three-dimensional graphing and other techniques beyond the high school curriculum.
- The basic fact that $4 > \pi$ was crucial in the formal proof.
- As Fig 5 shows, if we check the calculus solution against the precalculus solution, we obtain consistent results.

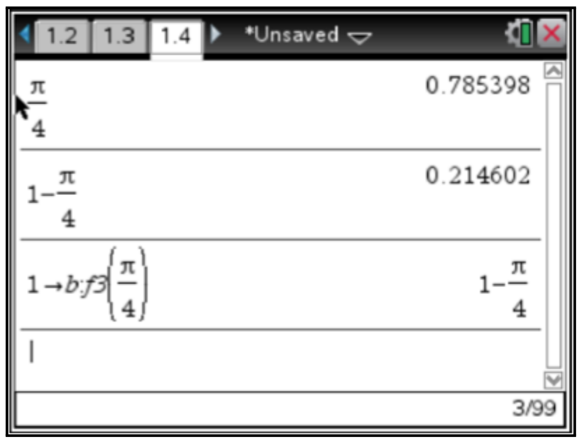

Fig. 5: *A comparison of approximate and exact results.*

3 Final Thoughts

The "shape of a drum" problem that we have solved should be presented to students in a way that encourages reasoning and sense making by requiring the "students to figure things out for themselves" (NCTM, 2009, p. 11). By putting the onus on the students in this manner, the drum problem has the potential to address the Standards for Mathematical Practice if students

- make sense of the problem,
- persevere in solving it,
- combine quantitative and abstract reasoning,
- construct logical arguments and use precise thinking,
- employ mathematical models and the modeling process, and
- "use appropriate tools strategically" (NGA Center & CCSSO, 2010, pp. 78).

In addition, we hope that our paper helps you give students an opportunity to connect geometry, algebra, and functions, to deepen their proficiency in reasoning and proof, and to solidify important ideas about area and perimeter. We encourage you to think of ways to provide some scaffolding for students while they work on this problem without spoiling their potential for invention and discovery. Moreover, we look forward to hearing how you have used the drum problem in your classroom.

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