Trigonometric Functions through Right Triangle Similarities

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Abstract: *This article presents an introduction to the trigonometric functions tangent, cosecant, secant, and cotangent. Students understand these functions as quotients of the sine and cosine functions only. However, applying right triangle geometry to a triangle constructed within the unit circle develops the remaining trigonometric functions as ratios of side lengths, fostering stronger student understanding.*

Keywords. *Trigonometry, unit circle, mathematical connections, dynamic mathematics software*

1 Introduction

One of the classes that I have taught over the past few years is a technology course for future secondary mathematics teachers. Part of the curriculum is the use of *Geometer's Sketchpad* (GSP). Among the numerous "Aha" moments that inevitably occur deals with the trigonometric functions. Students typically understand tangent, cosecant, secant, and cotangent functions as quotients of the sine and cosine functions. For example, most initially describe the tangent function as "sine over cosine." When students construct right triangles from the unit circle and discover that the trigonometric ratios can be demonstrated as line segments in the same manner as sine and cosine, they are often quite surprised. The trigonometric functions begin to make more sense conceptually when students are able to construct physical representations (i.e., segments) for each function.

The National Council of Teachers of Mathematics (NCTM), in its *Principles and Standards for School Mathematics* (NCTM, 2000), states that students should "use trigonometric relationships to determine lengths and angle measures" (p. 308). Authors of the recently published *Common Core State Standards Initiative* (CCSS) recommend that students "understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles" (High School Geometry: Similarity, Right Triangles, and Trigonometry Standard 6).

In addition to addressing the aforementioned NCTM and CCSS standards, the activity described herein also engages students in reasoning, sense-making, and proof. I use *GSP5* when I teach this lesson, but it could also be taught without the technology or with other dynamic tools such as *GeoGebra* or *TI-Nspire*. Using technology allows for students to investigate the domains and ranges of all six trigonometric functions easily and helps foster conjecturing and hypothesis-testing.

2 Preparing for the Activity

To prepare students for this lesson, the following topics should be reviewed:

1. **Mean Proportional**. Recall that the mean proportional of a right triangle is formed by constructing the altitude to the triangle's hypotenuse. This construction forms 3 similar right triangles (to be developed within the lesson);

2. **Geometric Mean**. Recall that a number is a geometric mean between two numbers if it satisfies the equation $m = \sqrt{ab}$. For example, 6 is the geometric mean of 3 and 12 because $\frac{3}{6} = \frac{6}{12} \rightarrow 6 = \sqrt{3 \times 12}$;

3. **Sine and Cosine Functions**. Recall that $sin\left(\theta\right) = \frac{\texttt{opposite side length}}{\texttt{hypotenuse length}}$ and $cos\left(\theta\right) = \frac{\texttt{adjacent side length}}{\texttt{hypotenuse length}}$;

4. **The Pythagorean Identity for Sine and Cosine**. Recall that $sin^2\left(\theta\right) + cos^2\left(\theta\right) = 1$.

3 The Activity

Begin by considering Fig 1, where $\overline{BD}$ is the altitude to the hypotenuse of right triangle $\triangle ABC$ (i.e., the mean proportional of $\triangle ABC$).

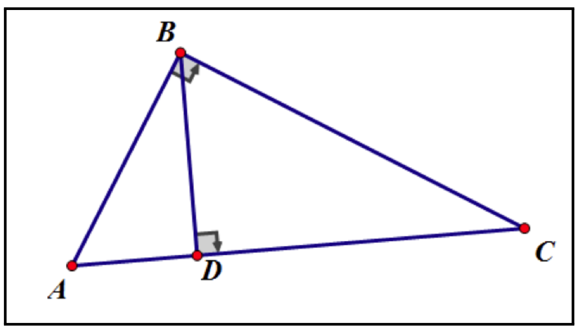

Fig. 1: *Right triangle $\triangle ABC$ with mean proportional $\overline{BD}$.*

The altitude, $\overline{BD}$, creates three similar triangles, $\triangle ABC \sim \triangle ADB \sim \triangle BDC$. Similarity may be confirmed through the Angle-Angle (AA) postulate for triangles. Look at the following ratios from these particular similar triangles.

$$\triangle ABC \sim \triangle ADB \rightarrow \frac{AC}{AB} = \frac{AB}{AD}$$
$$\triangle ABC \sim \triangle BDC \rightarrow \frac{AC}{BC} = \frac{BC}{CD}$$
$$\triangle ADB \sim \triangle BDC \rightarrow \frac{AD}{BD} = \frac{BD}{CD}$$

Note that AB, BC, and BD are geometric means within their respective ratios. Compare the first two ratios, $\frac{AC}{AB} = \frac{AB}{AD}$ and $\frac{AC}{BC} = \frac{BC}{CD}$, and the segments of the triangles in Figure A. These ratios come from the original right triangle and one of the triangles created by the altitude $\overline{BD}$. Observe that $\overline{AB}$ and $\overline{BC}$ are the legs of the right triangle $\triangle ABC$, the hypotenuse is $\overline{AC}$, and that $\overline{AD}$ and $\overline{CD}$ are the two segments of the hypotenuse. Our first theorem is that each leg of the right triangle is the geometric mean between the hypotenuse and that segment of the hypotenuse adjacent to the leg.

Now examine the third ratio, $\frac{AD}{BD} = \frac{BD}{CD}$. This ratio comes from the two right triangles created by the altitude $\overline{BD}$. The segment $\overline{BD}$ is the altitude to the hypotenuse, while $\overline{AD}$ and $\overline{CD}$ are two parts of the hypotenuse. Our second theorem is that the altitude to the hypotenuse is the geometric mean between the two segments of the hypotenuse.

As a result of the ratios of similar triangles, we have two facts with which to work. The proofs of these theorems are given above with the arguments about ratios from the similar triangles. These facts can now be applied to a special triangle constructed within the unit circle. The directions for the sketch in GSP5 are given at the end of this article. The instructions can be used as a basis for a student discovery lesson or as a teacher demonstration lesson.

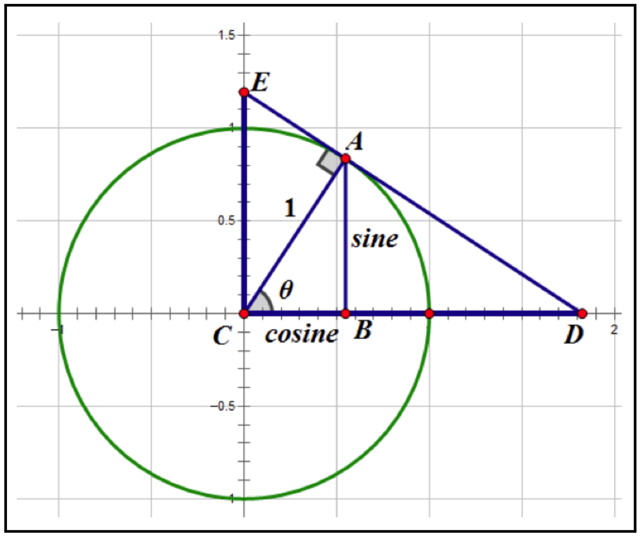

Fig. 2: *Similar right triangles constructed within the unit circle.*

3.1 Identifying Tangent

Using the fact that $\triangle ACD \sim \triangle BCA$, the proportion $\frac{AC}{BC} = \frac{AD}{AB}$ is valid. Substituting from the figure,$\frac{AC}{BC} = \frac{AD}{AB} \rightarrow \frac{AD}{1} = \frac{sin(\theta)}{cos(\theta)} = tan(\theta)$. The ratio of the length of $\overline{AD}$ over 1 is equal to the value of the tangent of the angle θ.

3.2 Identifying Secant

In right triangle $\triangle ACD$, each leg of the right triangle is the geometric mean between the hypotenuse and that segment of the hypotenuse adjacent to the leg. Therefore, $\frac{CD}{AC} = \frac{AC}{CB} \rightarrow \frac{CD}{1} = \frac{1}{cos(\theta)} \rightarrow CD = \frac{1}{cos(\theta)} = sec(\theta)$.

3.3 Identifying Cotangent

In right triangle $\triangle ECD$, the altitude $\overline{AC}$ is the geometric mean between the two segments of the hypotenuse. This implies $\frac{EA}{AC} = \frac{AC}{AD} \rightarrow \frac{EA}{1} = \frac{1}{tan(\theta)} \rightarrow EA = \frac{1}{tan(\theta)} = cot(\theta)$.

3.4 Identifying Cosecant

Before we develop cosecant, we need to manipulate the Pythagorean Identity,

$$sin^2(\theta) + cos^2(\theta) = 1 \tag{1}$$

Dividing each term of (1) by $sin^2(\theta)$,

$$\frac{sin^2(\theta)}{sin^2(\theta)}+\frac{cos^2(\theta)}{sin^2(\theta)}=\frac{1}{sin^2(\theta)} \tag{2}$$

Simplifying,

$$1+cot^2(\theta)=csc^2(\theta) \tag{3}$$

Now consider $\triangle CDE \sim \triangle ACE$. This implies $\frac{DE}{CE}=\frac{CE}{EA}\rightarrow\frac{tan(\theta)+cot(\theta)}{CE}=\frac{CE}{cot(\theta)}\rightarrow CE^2=cot(\theta)\left(tan(\theta)+cot(\theta)\right)=1+cot^2(\theta)=csc^2(\theta)\rightarrow CE=csc(\theta)$.

4 Conclusions

As the extension of the activity, an action button can be created such that the point A travels about the circle. The triangles automatically adjust. Students can start to visually understand the domain and range of each of the six trigonometric functions and points where certain trigonometric functions are not defined.

References

Common Core State Standards Initiative (2010). *Common core standards for mathematics*. (Also available at http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf)

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

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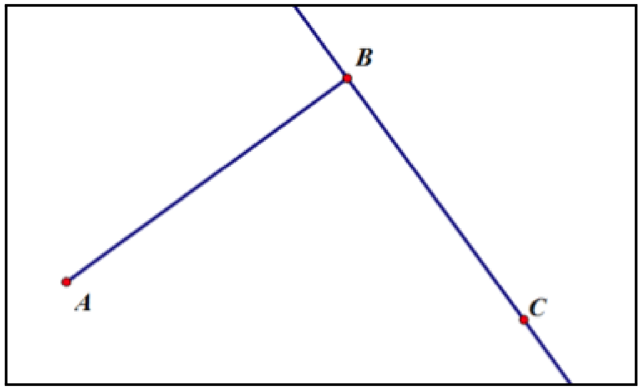
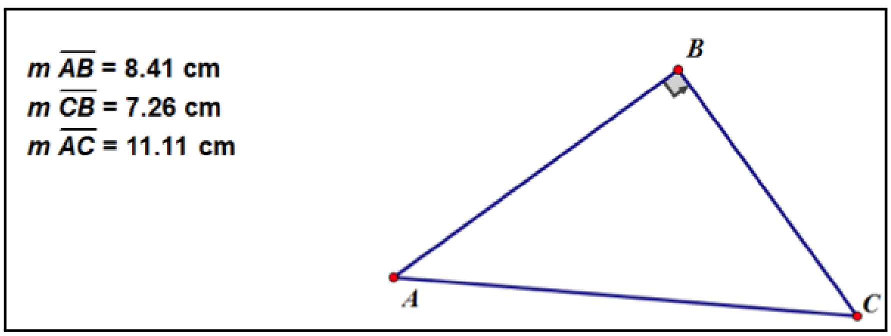
Dr. Moyer has 15 years of teaching experience at the secondary level. His interests lie in using technology to improve instruction and student achievement. Dr. Moyer regularly uses graphing calculators, Geometer's Sketchpad, and Fathom as part of his teaching methods. He is particularly interested in improving student achievement in geometry.

Student Activity Sheet for *Geometer's Sketchpad*

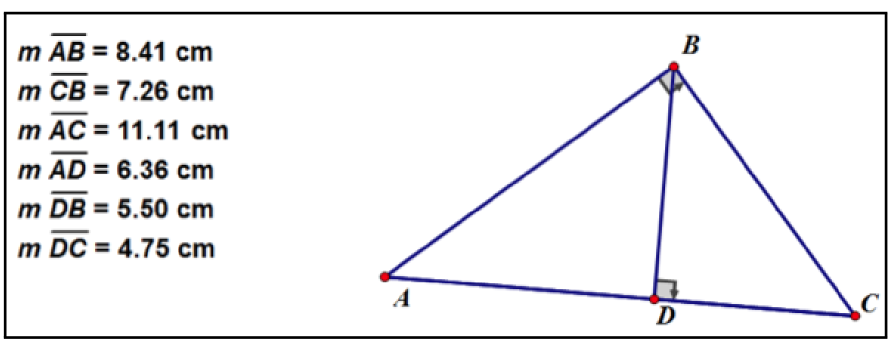
1. Let's review some geometry that we will need for this activity. In GSP, construct an arbitrary segment $\overline{AB}$. Select B and $\overline{AB}$, then choose "Perpendicular" from the GSP "Construct" menu. Next, construct an arbitrary point, C, along the perpendicular. These steps are highlighted below.



2. Hide the perpendicular (select it then choose "Hide Perpendicular Line" from the GSP "Display" menu). Construct the right triangle $\triangle ABC$ with right angle $\angle B$. Use the marker tool and draw an angle marker from $\overline{AB}$ to $\overline{BC}$. Measure the lengths of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ as shown below.



3. To construct the altitude $\overline{BD}$, select B and segment $\overline{AC}$ and choose "Perpendicular" from the GSP "Construct" menu. Label the intersection of the perpendicular and $\overline{AC}$ as D. Hide the perpendicular. Construct and measure the segments $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$. Mark the perpendicular at D as illustrated below.

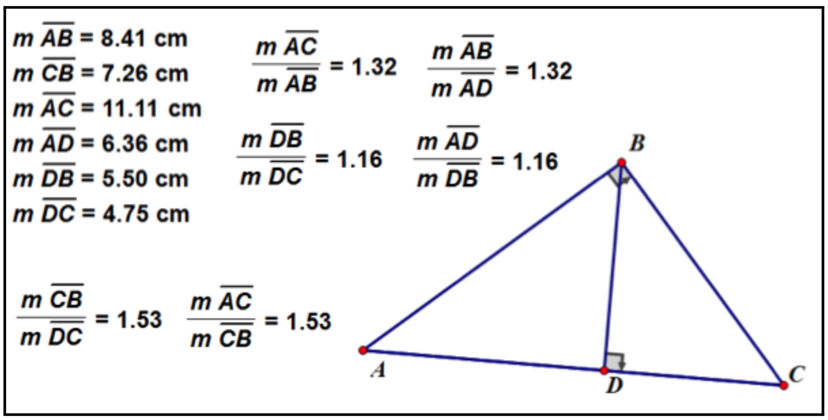


4. Measure $\angle BAC$. Calculate the measures of $\angle BCD$, $\angle ABD$, and $\angle CBD$. What can now be claimed about all three triangles? Write the corresponding ratios.

5. Construct and record the following ratios. Select the segments in the order shown, then choose "Ratio" from the GSP "Measure" menu. To select $\overline{AC}$, you may need to select $\overline{AD}$ twice. Make sure that the whole segment is highlighted when selecting

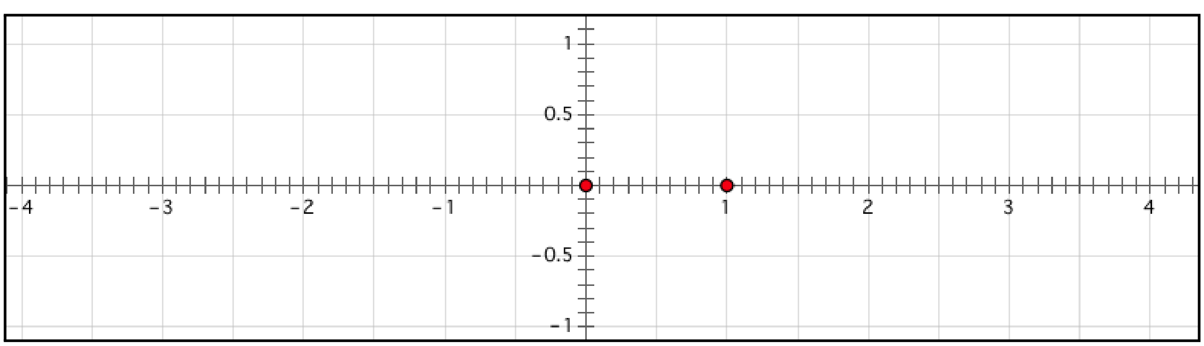
$$\frac{AC}{AB} = \_\_\_\_ \quad \frac{AB}{AD} = \_\_\_\_ \quad \frac{AC}{BC} = \_\_\_\_ \quad \frac{BC}{CD} = \_\_\_\_ \quad \frac{AD}{BD} = \_\_\_\_ \quad \frac{BD}{CD} = \_\_\_\_$$

When you have completed this task, your sketch should look similar to the following (specific values will vary).

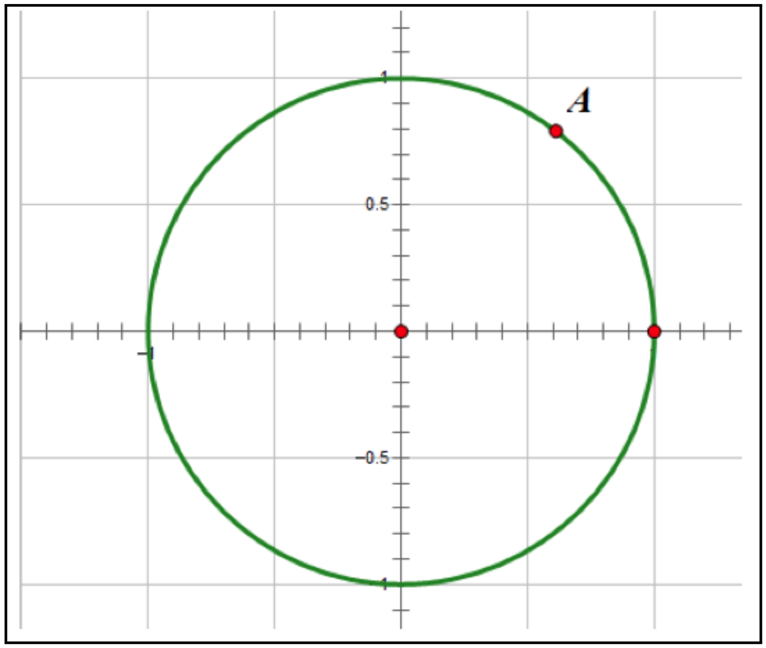


6. Drag any of the points A, B, or C. Make an observation about the displayed ratios.

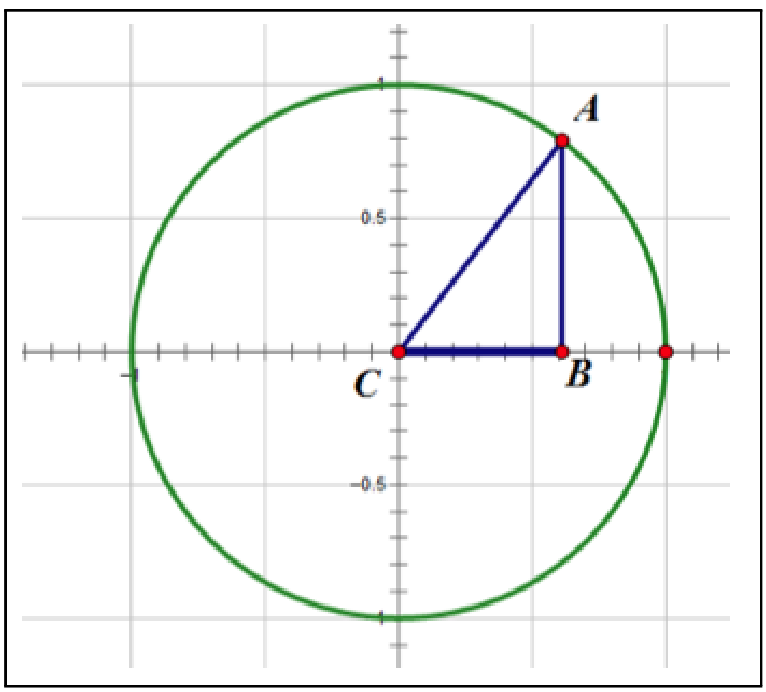
7. Now for the trigonometry. In a new sketch, create a coordinate system by choosing "Graph > Grid Form > Square." Drag the unit point $(1, 0)$ to the right until the x-axis goes from -4 to 4 as suggested below.



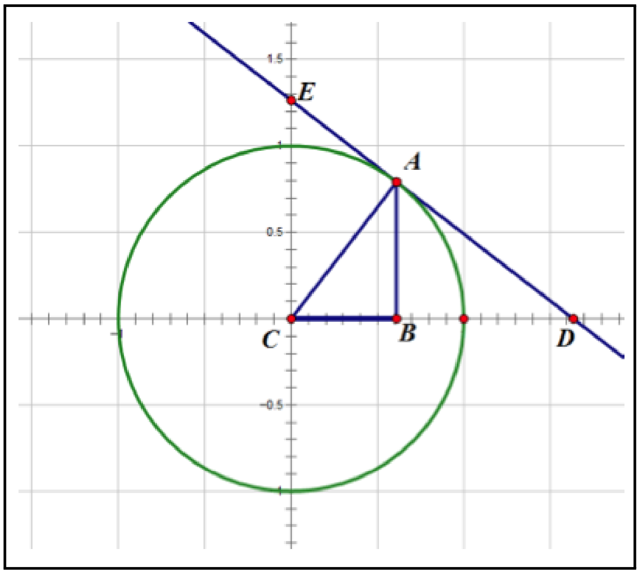
8. Construct the unit circle by selecting the compass tool, clicking on the origin and attaching the circle to the unit point. Create any point A on the circle in the first quadrant as shown below.



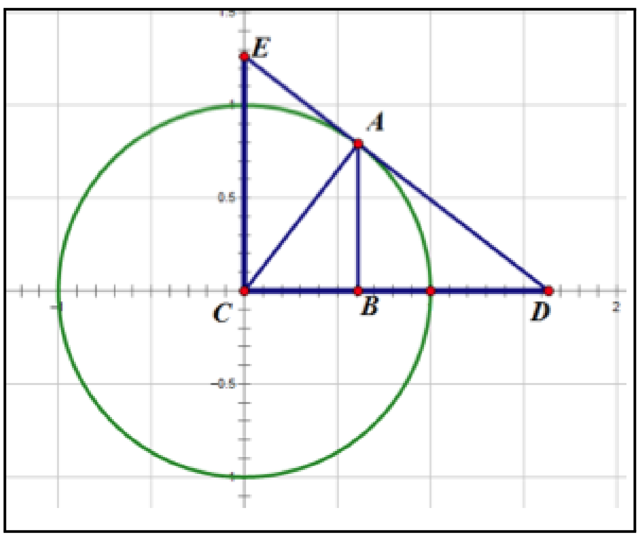
9. Construct a perpendicular from A to the x-axis. Label the intersection as point B. Then hide the perpendicular. Label the origin as point C. To do this, choose the text tool and click on the origin. Your sketch should look similar to the following.



10. Construct the tangent line to the circle at point A. Recalling that a tangent is perpendicular to a radius of the circle at a point of tangency, select the point A and the radius $\overline{AC}$, then select "Perpendicular Line" from the GSP "Construct" menu. Label the intersection of this tangent and each of the $x-$ and $y-axes$ as points D and E, respectively, as suggested below.

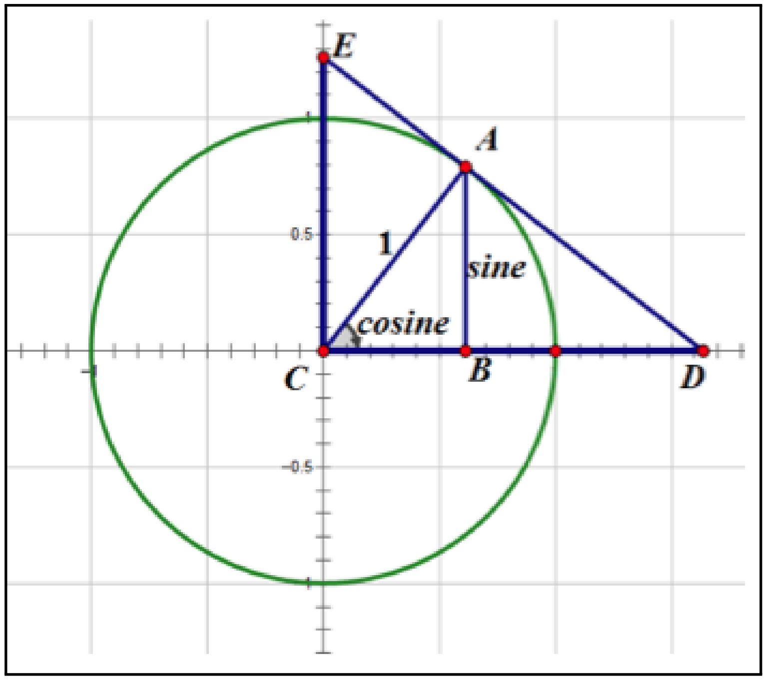


11. Hide the perpendicular and construct the following segments: $\overline{CD}$, $\overline{DA}$, $\overline{AE}$, and $\overline{EC}$ as shown below.



Name five right triangles in your sketch. What is true about all five of your triangles?

12. Create the angle marker for $\angle ACB$. Select the GSP marker tool, take the pen and "draw" an arc from $\overline{AC}$ to $\overline{CB}$. To label segments in GSP, choose the text tool and double-click on the particular segment. Label $\overline{AC}$ as 1. Previously, you defined the length of $\overline{AB}$ as the value of the sine function for the marked angle; likewise, the length of $\overline{BC}$ is the value of the cosine function. Label those two segments as such as shown below.



13. Recall from the review that the altitude constructed to the hypotenuse of a right triangle creates three similar triangles, and that two theorems follow as a direct result. With that mind, find the correct proportions within the five triangles to find a representation for the length of $\overline{AD}$ and segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

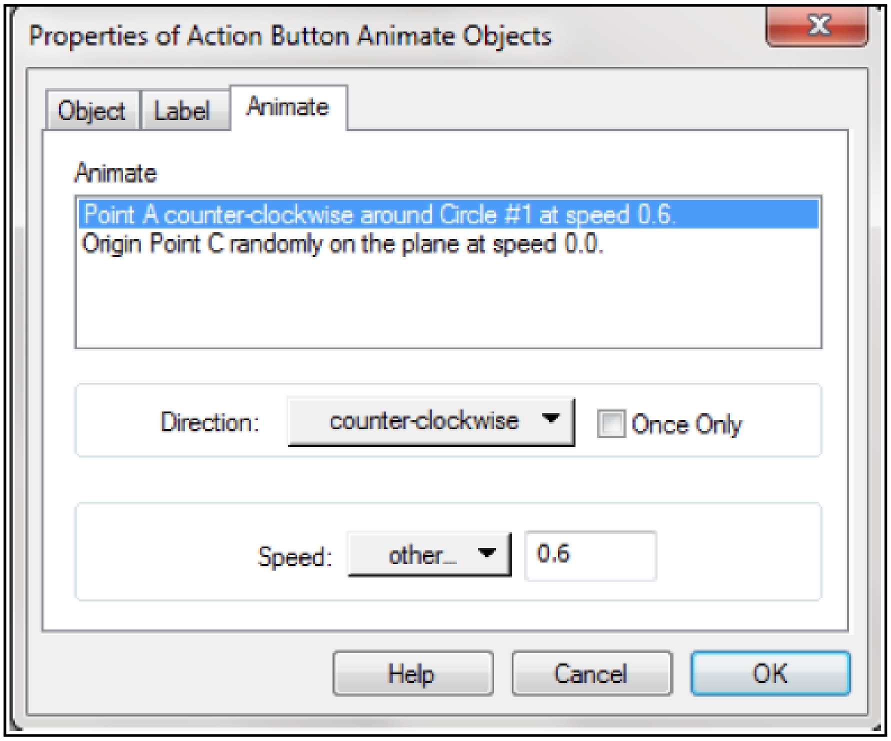
14. Again, use the geometric information and find a proportion that expresses the relationship between the lengths of $\overline{AE}$ and $\overline{AC}$ and $\overline{AD}$.

15. Once more, find a proportion that expresses the relationship between the length of $\overline{CE}$ and $\overline{AE}$ and $\overline{DE}$. (Hint: To finish the simplification, you will need to manipulate $sin^2(\theta)+cos^2(\theta) = 1$.)

16. Last time, find a proportion that expresses the relationship between the length of $\overline{CD}$ and $\overline{AC}$ and $\overline{BC}$.

17. Label the segments $\overline{AE}$, $\overline{CD}$, and $\overline{CE}$ accordingly.

18. As an extension, animate point A about the circle. To do this in GSP, select A and the circle, then "Edit > Action Buttons > Animate." Edit the dialogue box as shown.



To change speeds, pull down the speed menu, choose other, and enter the desired number. When complete, an action button will appear.

19. As point A travels about the circle, make an observation about the domain and range of each of the six trigonometric functions.