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A Mathematical Origami Puzzle

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The recent axiomization of Origami has led to numerous breakthroughs in both mathematics and in understanding of the ancient art of paper folding. We propose a puzzle whose solution demonstrates the power of mathematical origami. This puzzle is accessible to the geometry student and could be used as supplemental geometry instruction as an extension of traditional compass and straight edge constructions. Detailed images and photos are provided to guide the audience through the puzzle's solution.

Introduction

Puzzle: *Can a piece of paper, with any four points marked on it, be folded in such a way that all four points end up as the four corners of a rectangle?* We begin our exploration by noticing that if we start with the four points as the corners of a rectangle, making the appropriate folds is easy.

Next, try adjusting one of the four points, as pictured in Figure 1, left. After exploring this case with a sheet of paper, it is not hard to see that you can fold the paper so that the points form the corners of a rectangle, as shown in Figure 1, right. For a more challenging case, draw three points with a fourth point inside the triangle formed by connecting the three points, as in Figure 2.

After exploring this case with a sheet of paper, it is not hard to see that you can fold the paper so that the points form the corners of a rectangle

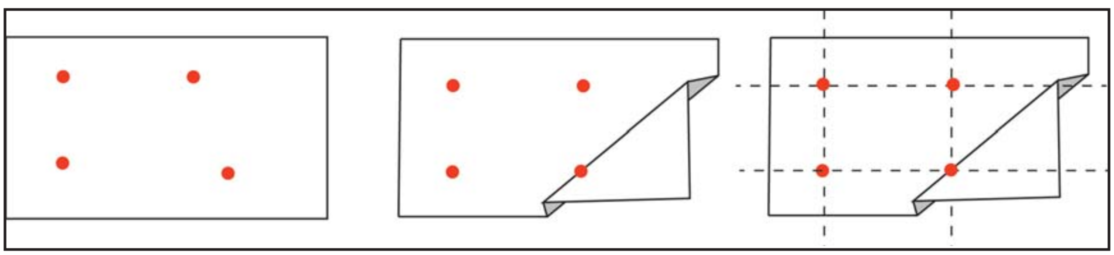


Fig 1 (Left) Initial setup; (Middle) Point moved; (Right) Rectangle

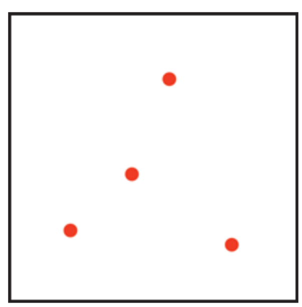


Fig 2 A more challenging rectangle folding task

Try folding for a few minutes and make a conjecture. Before we can start trying to prove any of these conjectures, we need a system of mathematically rigorous tools at our disposal.

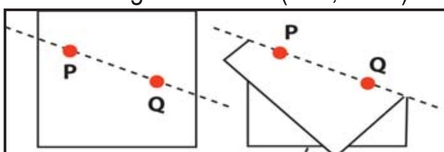
Origami in a Formal Mathematical Setting

To begin, we note that every definition from Euclidean geometry remains the same: a line extends infinitely, a rectangle is a quadrilateral with four right angles, etc. In addition to

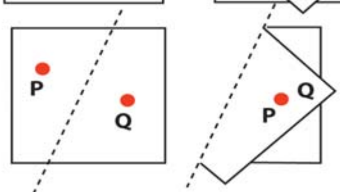
Euclidean geometry definitions, we must also understand the two main parts of origami: paper and folding. For our purposes, paper is *infinitely thin* and *transparent*. Being infinitely thin allows us to fold the paper as many times as we like; transparency allows us to always see a point or line drawn on the paper. An effective way to simulate these properties is to use transparency paper or lightweight paper and heavy markers. To make a fold in origami, we hold one side of the paper still and fold the other side over a straight line. In origami, the term *fold* refers both to the action of folding and to the line produced by that fold.

With clearly defined terminology, we can now discuss the axioms of origami. In 1989, Humiaka Huzita defined seven axioms that mathematically describe origami (Hull, 2008) listed in Table 1 with examples.

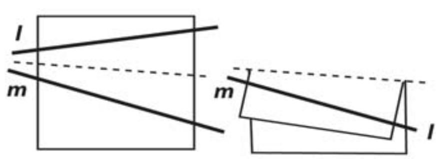
Table 1 Origami Axioms (Hull, 2008)



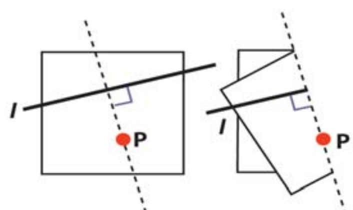
Axiom 1: Given two points P and Q there is a unique fold that passes through both of them.



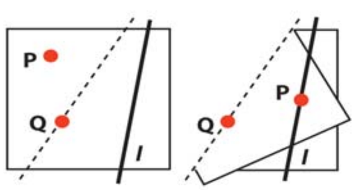
Axiom 2: Given two points P and Q, there is a unique fold that places P onto Q.



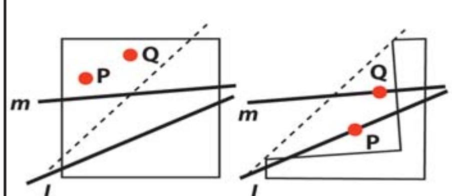
Axiom 3: Given two lines *l* and *m*, there is a fold that places *l* onto *m*.



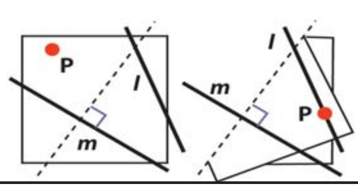
Axiom 4: Given a point P and a line *l*, there is a unique fold perpendicular to *l* that passes through point P.



Axiom 5: Given two points P, Q and a line *l*, there is a fold that places P onto *l* and passes through Q.



Axiom 6: Given two points P, Q and two lines *l*, *m*, there is a fold that places P onto *l* and Q onto *m*.



Axiom 7: Given one point P and two lines *l* and *m* there is a fold that places P onto Q and is perpendicular to *m*.

For our purposes, paper is infinitely thin and transparent. Being infinitely thin allows us to fold the paper as many times as we like; transparency allows us to always see a point or line drawn on the paper.

With a system of axioms in place, we have a framework for solving the original puzzle. Before solving it though, we can try solving a simpler problem as a warm-up: *Is it possible to fold any four points onto a line?* If this were possible, we could fold that line into a rectangle, as demonstrated in Figure 3.

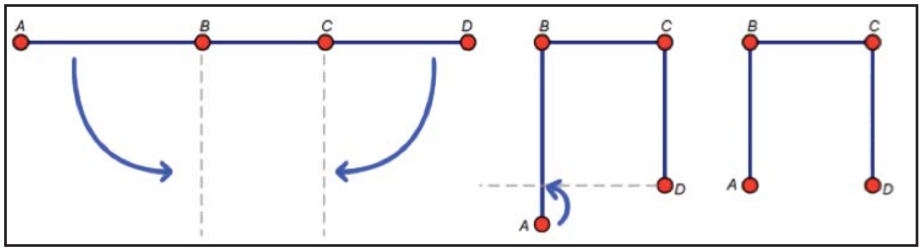


Fig 3 Folding a line into a rectangle

It turns out not only can we fold four points onto a line, but we can fold any collection of *n* points onto a line. We can demonstrate this using induction. Given any two points, by Axiom 1, there is a unique fold that passes through those two points. For reference we will label that fold *l*. Now assume that *n-1* points have been folded to line *l*. Now by Axiom 1, we can make a fold through P_1 and P_n and we can call it l_2. Finally by Axiom 3 we can fold l_2 onto *l*, and all *n* points have been folded to the line. We use the process in Figure 4 to fold 5 points onto a single line.

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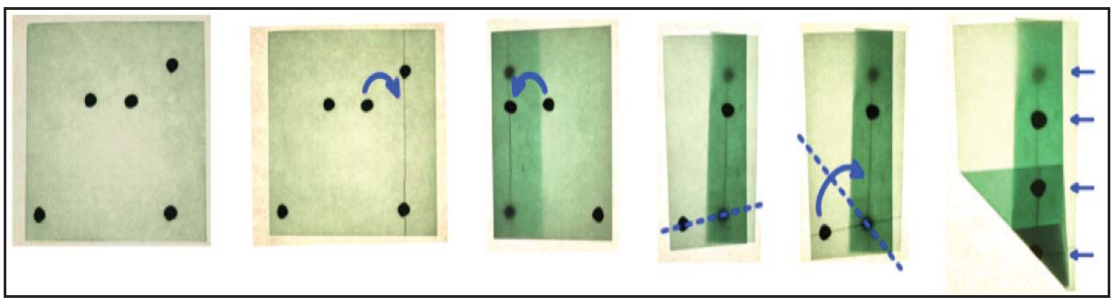


Photo 1. Folding 5 points to a line using our strategy

Notice that in Photo 1 the final line appears to contain only four points because two points end up in the same place. To answer our original problem, we want four distinct points on a line (recall Figure 3). However, because the method outlined above does not yield distinct points, we need a more specific proof to guarantee that all four points remain distinct. Below, we provide such an argument.

Lemma: It is possible to fold any four points onto a line such that those four points remain distinct.
Proof: Let A, B, C and D be distinct points on paper.
Step 1: Examine points A, B and C. If they are collinear go to Step 2. If they are not collinear, we need to fold them into distinct points on a line. To do this, we can fold lines AB, BC, and CA by Axiom 1 [Fig. 4a]. Then, by Axiom 4, we can fold a line through each vertex that is perpendicular to the opposite side of the triangle [Fig 4b]. One of these lines must intersect a segment at a point other than a vertex, call this point i [Fig. 4c]. Without loss of generality, we can assume that the line through i is the line that passes through point B and is perpendicular to AC. Now we can fold B to the intersection point i [Fig. 4d]. At this step, we now have three points as distinct points on a line.

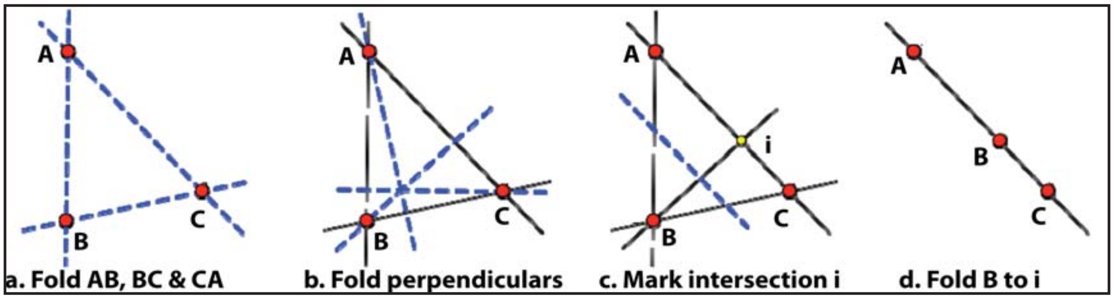


Fig 4 Steps to fold 3 distinct points on a line

Step 2: A, B, and C are now contained on a line, call it l_{ABC} .Now we can fold a line l through point D that is perpendicular to l_{ABC}. There are three possible cases for the intersection of line l.

Case 1: The intersection of lines l and l_{ABC} does not occur at points A, B or C, but instead at another point on l_{ABC}. Then we can fold D to that intersection point (indicated in yellow in Figure 5).

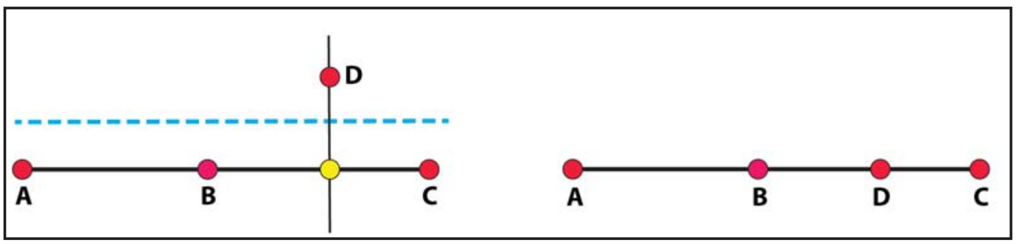


Fig 5 Four distinct points, Case 1

Case 2: The intersection of lines l and l_{ABC} is an outer point, as pictured in Figure 6.

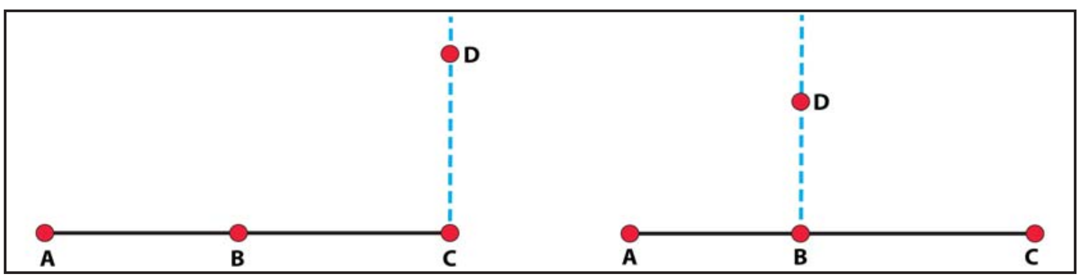


Fig 6 Two scenarios for case 2

In this situation, we can fold a line through D and the other outside point. From here, we can fold the lines perpendicular to this new line that run through each of the other points. Then we can fold each of those points to the intersection points as demonstrated by Figure 7.

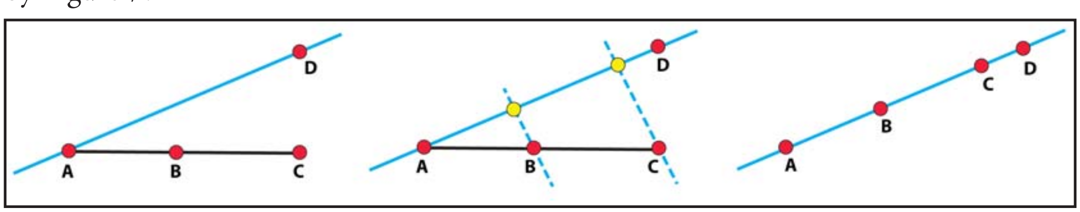


Fig 7 Folding process to solve case 2

With a system of axioms in place, we have a framework for solving the original puzzle. Before solving it though, we can try solving a simpler problem as a warm-up

Case 3: The intersection of line l and l_{ABC} is the inner point. In this case, we can fold the line connecting one of the outside points and D. Then we can either fold all points to the line by the process from Case 1 (as demonstrated in Figure 8), or the perpendicular line through A intersects D and we can proceed as in Case 2 (as shown in Figure 9).

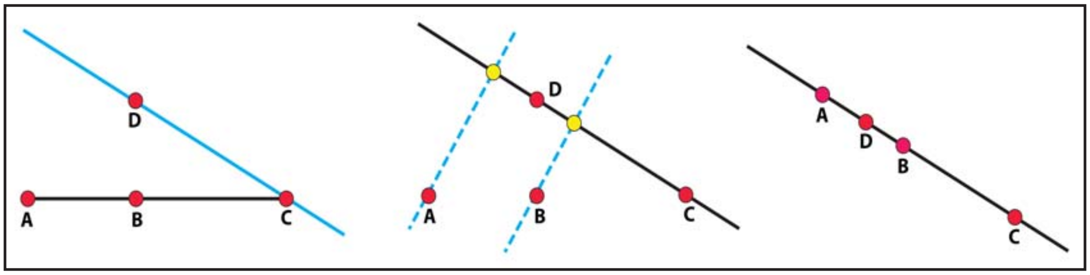


Fig 8 Folding sequence for Case 3 that results in the configuration in Case 1

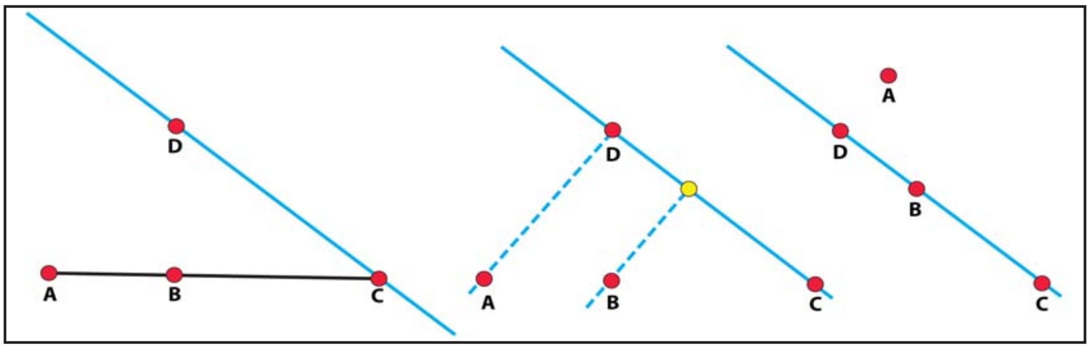
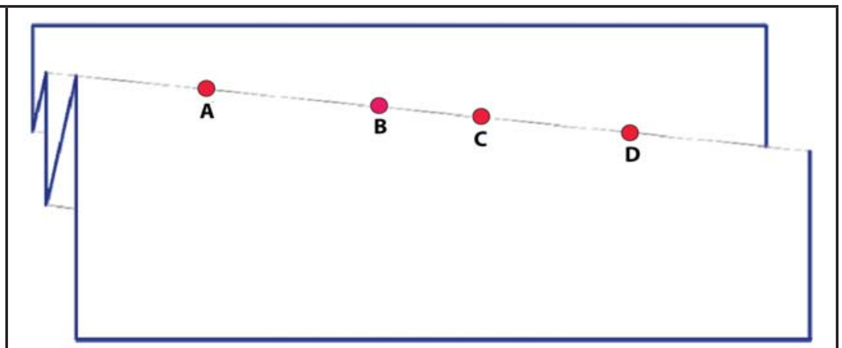
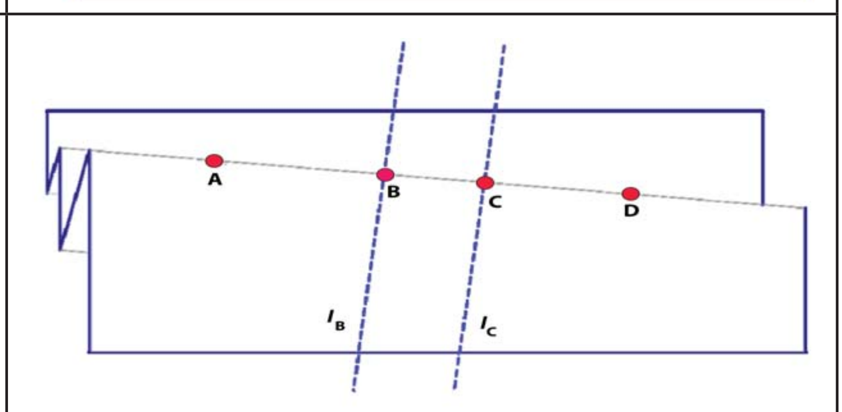
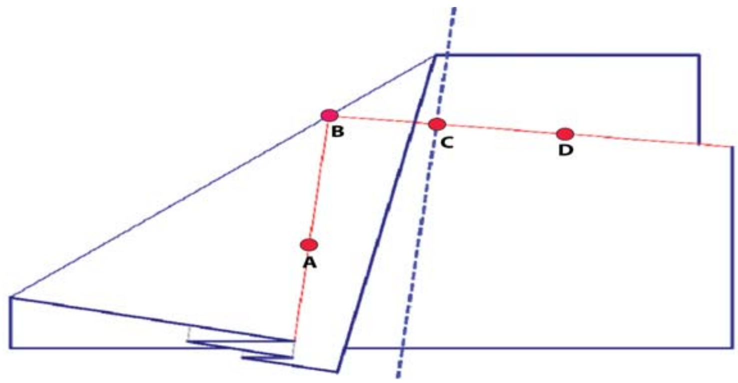
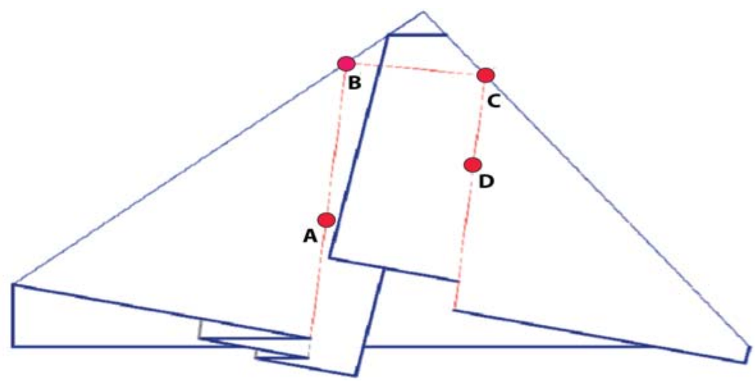
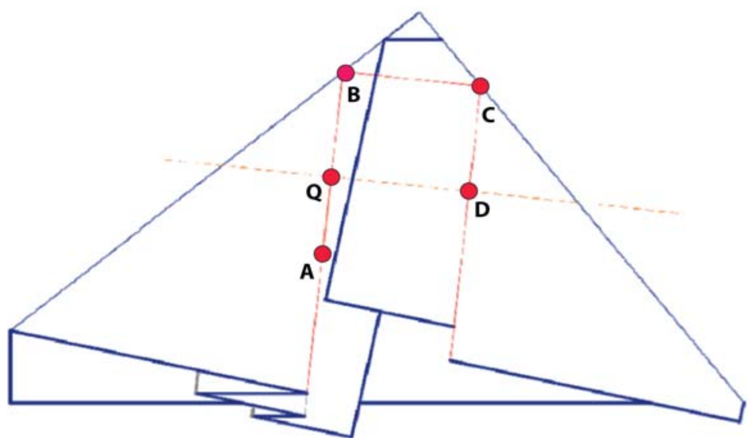
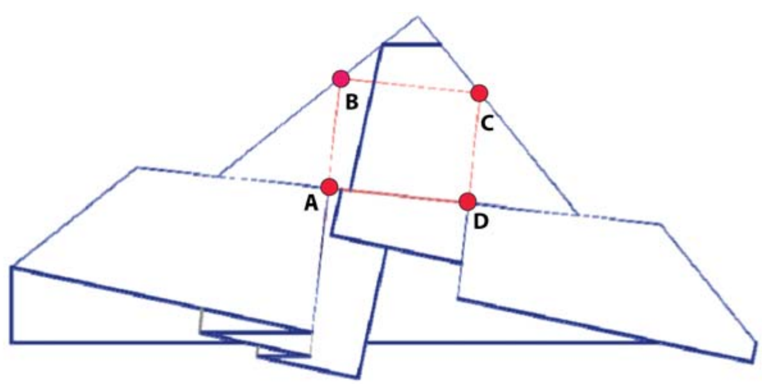


Fig 9 Folding sequence for Case 3 that results in the configuration examined in Case 2

Once we have this lemma, folding our four points onto a rectangle becomes a simple problem. *Puzzle Solution:* Any four distinct points can be folded to four distinct corners of a rectangle.

Once we have this lemma, folding our four points onto a rectangle becomes a simple problem.

Proof: By the lemma we can fold the four points onto the line AD. We will label the points in order on the line as A, B, C, D.	
Using Axiom 4, we can make a fold perpendicular to BC that passes through point B, called l_B. We can do the same thing through point C, calling that line l_C.	

Now by Axiom 5, there is a fold through point B that places A onto line l_B.	
Similarly there is a fold through point C that places D onto line l_C.	
Without loss of generality we can assume that A is further away from B than C is from D. Therefore, using Axiom 4 again, we can fold a perpendicular line to l_C that passes through D.	
We can now label the intersection of this fold with l_B as Q. Then by Axiom 2 we can fold A onto Q and we have successfully constructed a rectangle.	

This construction is just one of many amazing constructions based on the origami axioms. . Many of these axioms are based on compass and straight edge constructions from classical geometry.

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You better believe it!

"The associative architecture of the brain contributes to false memories, and to the ease with which politicians and companies manipulate our behavior and beliefs." Buonomano, D. (2011). *Brain bugs: How the brain's flaws shape our lives, 16.* W. W. Norton & Company, NY.

In Conclusion

This construction is just one of many amazing constructions based on the origami axioms. Many of these axioms are based on compass and straight edge constructions from classical geometry. Axiom 6, however, requires the ability to slide the paper as you make a fold. This results in a non-Euclidean geometry in which angles can be trisected and cubes doubled. For even more spectacular results see the work of Eric Demaine from MIT. He proved that anything that can be represented by a straight-line skeleton can be folded from a sheet of paper (2008). In addition, when folded from a two-color sheet of paper, the colors can be distributed in any possible way. This means that a lifelike giraffe could be folded with colored spots as in Photo 2.

Photo 2 John Montroll's Spotted Giraffe as folded by the first author

Works Cited

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