Summing Squares and Cubes of Integers

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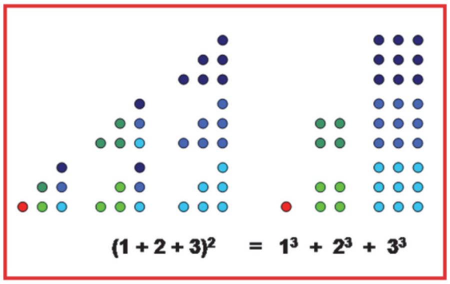
Abstract

Recreational mathematics can provide students with opportunities to explore mathematics in meaningful ways. Elementary number theory is one area of mathematics that lends itself readily to recreational mathematics. In this article, the author provides two examples from elementary number theory with results that students might find surprising, and which may be used to motivate them to study additional topics from number theory.



There are some mathematical topics that lend themselves to exploration and investigation by students much more readily than others. Some have practical applications that may motivate the students to further study. Cryptography, for example, is a topic that many students find interesting and worthy of additional study. Others may offer surprising results or may simply be entertaining to the students. The Fibonacci sequence and Pascal's triangle, for example, have both been used to motivate students to explore mathematics. Still others may simply align with a particular student's strengths, causing that student to want to investigate further. A topic that relies heavily on pattern recognition may play to one student's strengths while a different topic that requires computation may inspire another student.

Topics which are readily accessible to non-mathematicians while allowing them to engage in meaningful exploration fall under the umbrella of ***recreational mathematics.*** A commonly-accepted, albeit informal, definition of recreational mathematics is that it is mathematics that is carried out for entertainment or self-education rather than as a research or application-based professional activity. Recreational mathematics can be fun and has often inspired students to pursue further study of mathematical topics. While providing entertainment and amusement (or, indeed, *amazement*), it can also lay the groundwork for the study of more "serious" mathematics. Certainly, there have been occasions when recreational mathematics has led to unexpected, but meaningful, utility. For example, much of the foundation of actuarial science – the methods used to assess risk in insurance and other areas – utilizes the probability and statistical methods that grew out of the recreational mathematics subject of gambling in the 17^{th} century and following. The famous Königsberg Bridge problem, posed in a manner that could be considered as recreational mathematics in the 18^{th} century, led to the development of graph theory by Euler and others (Singmaster, 1993).

Recreational mathematics can be fun and has often inspired students to pursue further study of mathematical topics.

One area of mathematics that seems to lend itself readily to recreational mathematics is that of ***elementary number theory*** – the study of equations having integer or rational solutions. What follows is an example of two surprising results that students may find quite interesting.

Sums of Squares

Students who have studied the Pythagorean Theorem are likely familiar with the Pythagorean triple {3, 4, 5} and the fact that

(1) $3^2 + 4^2 = 5^2$

However, most are unaware that this identity can be extended to an entire family of identities by following a simple pattern involving the sum of the squares of consecutive integers: we will take the final term in each equation, and successively multiply the base number of that final term by a certain ratio (to be given below), and use that result as the base number of the first term in the next identity. We also add one additional term on both sides of the equation with each successive identity. The ratios that we use will be the sequence $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$. That is, we will use the fraction $\frac{2}{1}$ to obtain the second identity, then we will use the fraction $\frac{3}{2}$ to obtain the third identity, then $\frac{4}{3}$, then $\frac{5}{4}$, etc.

To begin, the final term in the identity $3^2 + 4^2 = 5^2$ is the term 5^2. We multiply the base number 5 by $\frac{2}{1}$ to obtain 10, and that will become the base number of the first term in the next identity. This identity will grow from having two terms on the left and one term on the right to having three terms on the left and two terms on the right:

(2) $10^2 + 11^2 + 12^2 = 13^2 + 14^2$

with both sides equaling 365. The final term in this identity is the term 14^2. Taking the base number of the final term, 14, and multiplying now by $\frac{3}{2}$ gives the initial base number for the next identity, 21. The new identity will grow to four terms on the left and three terms on the right:

(3) $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$

with both sides equaling 2030. Continuing in this manner, we see that $27 \cdot \frac{4}{3} = 36$ and the next identity is

(4) $36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$

and the one following that is

(5) $55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$

Apostol and Mnatsakanian (2011) point out that G.J. Doster showed that this family of identities is generalized by the formula

(6) $\sum_{k=0}^{n-1}(k+m)^2 + (m+n)^2 = \sum_{k=0}^{n-1}(k+m+n+1)^2$ where $m = n(2n + 1)$.

For example, if $n = 1$, then $m = 3$ and we get the identity

$$\sum_{k=0}^{0}(k+3)^2 + (3+1)^2 = \sum_{k=0}^{0}(k+3+1+1)^2$$

which is $3^2 + 4^2 = 5^2$. Similarly, if $n = 2$ then $m = 10$ and we obtain the identity

$$\sum_{k=0}^{1}(k+10)^2 + (10+2)^2 = \sum_{k=0}^{1}(k+10+2+1)^2$$

which is $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.

Unlike the first method that we used to obtain these identities, which required us multiplying the final term of one of the identities by one of the ratios $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}$, etc., in order to determine the first term of the next identity, Doster's formula allows us to obtain any of the equations in this family of identities without the requirement of knowing any of the earlier identities. For example, if $n = 8$ then $m = 136$ and we get the identity $136^2 + 137^2 + 138^2 + \cdots + 144^2 = 145^2 + 146^2 + \cdots + 152^2$, with the left hand side of this equation being the sum of nine terms while the right hand side of the equation is the sum of eight terms.

Sums of Cubes

Students who have taken integral calculus may also be familiar with the formulas for the sum of the first n integers and the sum of the first n cubes:

(7) $\quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ and

(8) $\quad 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

We see that if we square the formula for the sum of the first n integers then we obtain the formula for the sum of the first n cubes. Stated simply, *the sum of the first n cubes is the square of the sum of the first n integers.* The conventional way to prove these two formulas is by induction. However, a nice visual that exhibits this formula is provided by Barbeau (2014) as follows.

Suppose we wish to show that $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$ visually. We construct the following 4×4 array of numbers found in the corner of a standard multiplication table:

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

We will sum the numbers in this array in two different ways. First, we sum the rows going across as follows:

$(1 + 2 + 3 + 4) + (2 + 4 + 6 + 8) + (3 + 6 + 9 + 12) + (4 + 8 + 12 + 16) =$
$(1 + 2 + 3 + 4) + 2(1 + 2 + 3 + 4) + 3(1 + 2 + 3 + 4) + 4(1 + 2 + 3 + 4).$

Factoring out the common factor of $(1 + 2 + 3 + 4)$ gives the sum as

$(1 + 2 + 3 + 4)[1 + 2 + 3 + 4]$, which is $(1 + 2 + 3 + 4)^2$.

The second way that we will sum the numbers in the array is by summing along L-shaped gnomons, displayed as either shaded or unshaded in the table below:

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

The sum of all the entries in the array along the unshaded (or, respectively, shaded) gnomons is
$1 + (2 + 4 + 2) + (3 + 6 + 9 + 6 + 3) + (4 + 8 + 12 + 16 + 12 + 8 + 4).$

Rearranging the order of the terms gives

$$1 + [(2 + 2) + 4] + [(3 + 6) + (6 + 3) + 9] + [(4 + 12) + (8 + 8) + (12 + 4) + 16] =$$
$$1 + [4 + 4] + [9 + 9 + 9] + [16 + 16 + 16 + 16] = 1 + 2\cdot4 + 3\cdot9 + 4\cdot16 =$$
$$1 + 2\cdot2^2 + 3\cdot3^2 + 4\cdot4^2 = 1^3 + 2^3 + 3^3 + 4^3.$$

Thus, by adding all the numbers in this array in two different ways, we see that

$$(1 + 2 + 3 + 4)^2 = 1^3 + 2^3 + 3^3 + 4^3.$$

Notice the manner in which the terms were rearranged: the first entry in each row of a gnomon is matched with the first entry immediately above the corner entry of that gnomon. For example, in the fourth set of terms, the 4 that is the first entry in the bottom row gets matched with the 12 that is immediately above the corner entry of 16. Then, matching terms across the *row* of the gnomon with the entries proceeding upward in the *column* of the gnomon produces the desired matches (4 and 12; 8 and 8; 12 and 4; with the remaining corner entry of 16 left by itself).

This can be shown to work in the general case of adding the first n cubes as follows:

1	2	3	$\cdots$	$\cdots$	$n - 1$	n
2	4	6	$\cdots$	$\cdots$	$2(n - 1)$	$2n$
3	6	9	$\cdots$	$\cdots$	$3(n - 1)$	$3n$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$(n - 2)n$
n - 1	$2(n - 1)$	$3(n - 1)$	$\cdots$	$\cdots$	$(n - 1)^2$	$(n - 1)n$
n	$2n$	$3n$	$\cdots$	$(n - 2)n$	$(n - 1)n$	n^2

Adding row by row gives the sum of all the entries in the table as

$$[1 + 2 + 3 + \cdots + n] + [2 + 4 + 6 + \cdots + 2n] + [3 + 6 + 9 + \cdots + 3n] + \cdots +$$
$$[n + 2n + 3n + \cdots + n^2] = [1 + 2 + 3 + \cdots + n] + 2[1 + 2 + 3 + \cdots + n] +$$
$$3[1 + 2 + 3 + \cdots + n] + \cdots + n[1 + 2 + 3 + \cdots + n] =$$
$$[1 + 2 + 3 + \cdots + n]\,(1 + 2 + 3 + \cdots + n) = (1 + 2 + 3 + \cdots + n)^2.$$

Alternatively, adding along the L-shaped gnomons gives the sum of all the entries in the table as

$$1 + (2 + 4 + 2) + (3 + 6 + 9 + 6 + 3) + \cdots + [(n - 1) + 2(n - 1) + 3(n - 1) + \cdots + (n - 1)^2 + \cdots +$$
$$3(n - 1) + 2(n - 1) + (n - 1)] + [n + 2n + 3n + \cdots + (n - 1)n + n^2 + (n - 1)n + \cdots + 3n + 2n + n].$$

Rearranging the terms in the manner outlined above, we see that this sum is equivalent to

$$1 + [(2 + 2) + 4] + [(3 + 6) + (6 + 3) + 9] + \cdots$$
$$+ [(n + (n - 1)n) + (2n + (n - 2)n) + \cdots + ((n - 2)n + 2n) + ((n - 1)n + n) + n^2] =$$
$$1 + [4 + 4] + [9 + 9 + 9] + \cdots + [n^2 + n^2 + \cdots + n^2] =$$
$$1 + 2\cdot4 + 3\cdot9 + \cdots + n\cdot n^2 = 1 + 2\cdot2^2 + 3\cdot3^2 + \cdots + n\cdot n^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

Thus, by adding all the entries in the array in two different ways, we see that

$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$, that is, *the sum of the first n cubes is the square of the sum of the first n integers*.

Summary

Properties of sums of integers, squares of integers, cubes of integers, odd integers, even integers, etc., lend themselves to a great deal of exploration by the motivated student. Sometimes, merely providing them with the initial few identities allows them to identify the entire family of identities, offer their own proofs, and extend those identities. These activities can be fun, while inspiring them to explore mathematics in meaningful ways.

The particular examples discussed in this article can be presented in a manner that aligns nicely with the Standards for Mathematical Practice that mathematics educators are called upon to develop in their students as part of the Common Core State Standards, in particular the standard that students should "look for and express regularity in repeated reasoning." For example, if students were presented with the first two or three identities from the first example – (1), (2), and perhaps (3) – they could then be asked to find a pattern and determine a method that would yield later identities, and test their results. Similarly with the second example discussed in this article, students could be presented with the array and shaded and unshaded gnomons used to demonstrate that the sum of the first four cubes is equal to the square of the sum of the first four integers, then asked to explore the pattern for the first five integers, and generalize the results to the first *n* integers.

As a closing example, what would some of your students do if they were challenged to explore the following pattern?

$$1 = 1^3$$
$$3 + 5 = 2^3$$

References

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