Fractions as Numbers and Extensions of the Number System: Developing Activities Based on Research

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Abstract: *Existing studies have shown fractions present significant challenges for K-8 students and pre-service teachers. They suggest that students may fail to see fractions as numbers and have difficulty moving beyond the part-whole realization, and thus they often think of fractions as objects disconnected from the number system. To address this issue, the current study proposes several activities that can be used in mathematics courses for pre-service elementary/secondary teachers or students to help them recognize fractions as numbers with the same visual representations as whole numbers (e.g., the number line) and on which the same kinds of operations can be used. Along with a description of the activities, the paper also provides observations about what happened when we gave the activities to a group of 28 prospective elementary teachers, highlighting evidence of their understanding of fractions as numbers.*

Keywords. *fraction sense, early grades, number, activities*

1 Research on Understanding Fractions as Numbers

The concept of fractions presents significant challenges for K-8 students as reported in numerous studies on students' understanding of fractions (e.g., Erlwanger, 1973; Mack, 1990; Post, Cramer, Lesh, Harel, & Behr, 1993). Studies suggest that students may fail to see fractions as numbers (e.g., Hannula, 2003; Post, et al., 1993), and have difficulty moving beyond the part-whole realization (e.g., Lamon, 2007; Mack, 1990). Students often think of fractions as objects disconnected from the number system, not as numbers (Erlwanger, 1973; Hannula, 2003; Mack, 1990; Post, et al., 1993). However, it has been reported that students who do not conceive of fractions as numbers have trouble appreciating how operations on fractions are similar to or different from operations on whole numbers. One such example is fraction addition, when students add numerators and denominators separately, ignoring the unit of addition (Erlwanger, 1973; Mack, 1990; Stafylidou & Vosniadou, 2004).

These difficulties may be a reflection of how fractions are taught in K-8 classrooms, and thus of what teachers understand about fractions. If teachers themselves do not have adequate knowledge of fractions as numbers within a coherent number system, or do not recognize the need to help students see fractions as numbers, it is unlikely that they will teach fractions in ways that move students beyond the part/whole conception. Later in this article, we explore prospective teachers' interactions around fractions as numbers.

2 Mathematical History of the Development of Fractions as Numbers

The history of mathematics reveals that the development of the idea that fractions are numbers, as well as the acceptance of this idea, was neither natural nor straightforward. Early in the development, ancient Greek mathematicians in the fifth century BCE thought in terms of relationships (ratios) between whole numbers, which can be considered as a precursor to the concept of fractions (Tropfke, 1980, p. 129), but not yet accepted as numbers. Centuries later in medieval Europe, a number would be associated to a relationship between two whole numbers, using the concept of value (e.g., the value of the ratio $8 : 4$ is 2; if we multiply this value by the second term, 2×4, we obtain the first term, 8). By the 13th century, the value (i.e., the relationship between two whole numbers) was described as a number also in cases where the value was not a whole number (Tropfke, 1980, p. 130).

In contrast to the Greeks, the Egyptians (1650 BCE) did have a symbol system for unit fractions to represent parts of a whole, but did not have a common arithmetic that worked on both whole numbers and fractions (Berlinghoff & Gouvea, 2004). When computing fractions from fractions (which corresponds to our multiplication of fractions), ancient Egyptians would not just assume commutativity but would, for instance, compute one-third of two-thirds on one hand and two-thirds of one-third on the other (Papyrus Rhind, Problem 61).

These historical observations show that early versions of fractional quantities were already in existence within several ancient cultures, but also that it took centuries for mathematicians to come to accept fractions as numbers that are operated on by the same arithmetic by which whole numbers are operated on. Considering that the development of fractions as numbers took mathematicians centuries, we cannot assume that conceiving of fractions as numbers will be trivial for today's students. This observation explains why there is a need for teacher's mathematical knowledge for teaching fractions as numbers, which supports the understanding of fractions as part of a coherent number system. This paper later provides several activities that can promote pre-service teachers' and elementary/middle school students' understanding of fractions as numbers, and thus as part of a number system.

3 Fractions as Numbers in the Common Core State Standards

The Common Core State Standards for Mathematics (2010) introduce fractions in Grade 3, first as parts of wholes and then as numbers on a number line. First, a fraction $\frac{1}{b}$ is understood as "the quantity formed by 1 part when a whole part is partitioned into b equal parts[and] a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$" (p. 24). Then, a fraction $\frac{1}{b}$ is visualized on a number line as "the end point of the part based at 0" after "defining the interval from 0 to 1 as the whole and partitioning it into b equal parts," and a fraction $\frac{a}{b}$ is marked off where the end point of a length of $\frac{1}{b}$ is located (p. 24). In Grade 4, the Standards emphasize two additional extensions from whole numbers to fractions, that can help students understand fractions as numbers. One is the idea of order, and the second is that new fractions can be built by "applying and extending previous understandings of operations with whole number" (p. 30). Fractions are connected to the concept of measurement and location on a number line (p. 31).

The way that the Common Core State Standards addresses fractions makes explicit the connection between fractions as parts of a whole and fractions as numbers. The connection can be visualized as the actions of partitioning on a unit interval and measuring on a number line, which highlights the sizes of the intervals that fractions represent and their locations on the number line. Students' difficulties with placing fractions on number lines have been reported in existing research (e.g., Post et al., 1993). However, number line diagrams have also been used to help students conceive of fractions as numbers (e.g., Vamvakoussi & Vosniadou, 2007).

These observations from existing research on the teaching and learning of fractions, the historical development of fractions as numbers, and the introduction to fractions as numbers in the Common Core State Standards provide motivation and a rationale for developing activities that promote fractions as numbers that can be used to better prepare pre-service elementary, middle school, and secondary mathematics teachers for the students and classes that they eventually will teach. Pre-service and in-service teachers' mathematical knowledge of teaching fractions as numbers would not only serve as strong content knowledge for themselves, but also better equip teachers to address students' difficulties with fractions, when those difficulties are rooted in a lack of understanding of fractions as numbers.

4 Activities to Extend the Number System to Fractions

The purpose of the activities introduced in this section is to make explicit that the same arithmetic that operates on whole numbers also operates on fractions. These activities can be used in mathematics courses for pre-service elementary/secondary teachers or in math classes for elementary/secondary students to help them recognize fractions as numbers with the same visual representations that are used for whole numbers (e.g., the number line) and on which the same kinds of operations can be used. The following principles, derived from research about learning mathematics and from experience, guided the development of the activities: a) Students learn best through active involvement with multiple concrete models; b) Physical aids are just one of a number of useful models that can support students in the acquisition of concepts: verbal, pictorial, symbolic and real-world representations are also important; c) Students should have opportunities to talk together and with their teacher about mathematical ideas; and d) Curriculum must focus on the development of conceptual knowledge prior to formal work with symbols and algorithms (Cramer, Behr, Post, & Lesh 1997).

Along with a description of the activities, we will also provide observations about what happened when we gave the activities to a group of 28 undergraduates who were registered in a mathematics content course for prospective elementary teachers. The content course was the second of three mathematics courses that prospective elementary teachers were required to take. The second course focused on rational number operations, ratios and proportions. The prospective teachers, grouped in pairs, received strips of adding machine paper, carefully measured and marked by one of the authors. Prospective teachers in each pair took turns folding and explained to each other what they were doing and why. Prospective teachers were then asked to share their methods and results with the whole group. One guiding principle was for prospective teachers to have opportunities to extend the number system by using the midpoint to find new fractions between given points and to also include fractions beyond the unit interval (i.e., mixed or improper fractions). Research has shown that some prospective teachers have problems with improper fractions (Toluk-Uçar, 2009).

4.1 Activity 1. Folding paper strips to find additional numbers on the number line

Each pair of prospective teachers received a paper strip (about 50 cm long) representing the interval $[0, 5]$. Only the positions corresponding to the numbers 0, 2 and 5 were marked on the strip (drawn to scale), and the corresponding numbers were written next to these points on the line. No marks for 1, 3, or 4 were provided. Prospective teachers were first asked to use folding or any other method to find the location for 1, 4 and 3 on the number line relative to 1, 2 and 5. All prospective teachers folded the interval $[0, 2]$ in half to find the point corresponding to 1. To find the point corresponding to 3, several methods were used. One pair folded the strip like an accordion to divide the interval $[2, 5]$ into three equal parts. Another pair used the mark for 2 to crease the strip and overlap 0 onto the strip to mark the position for 4, and then folded the interval $[2, 4]$ in half to find 3 (Figure 1). Next, prospective teachers were asked to extend the number line beyond whole numbers, by

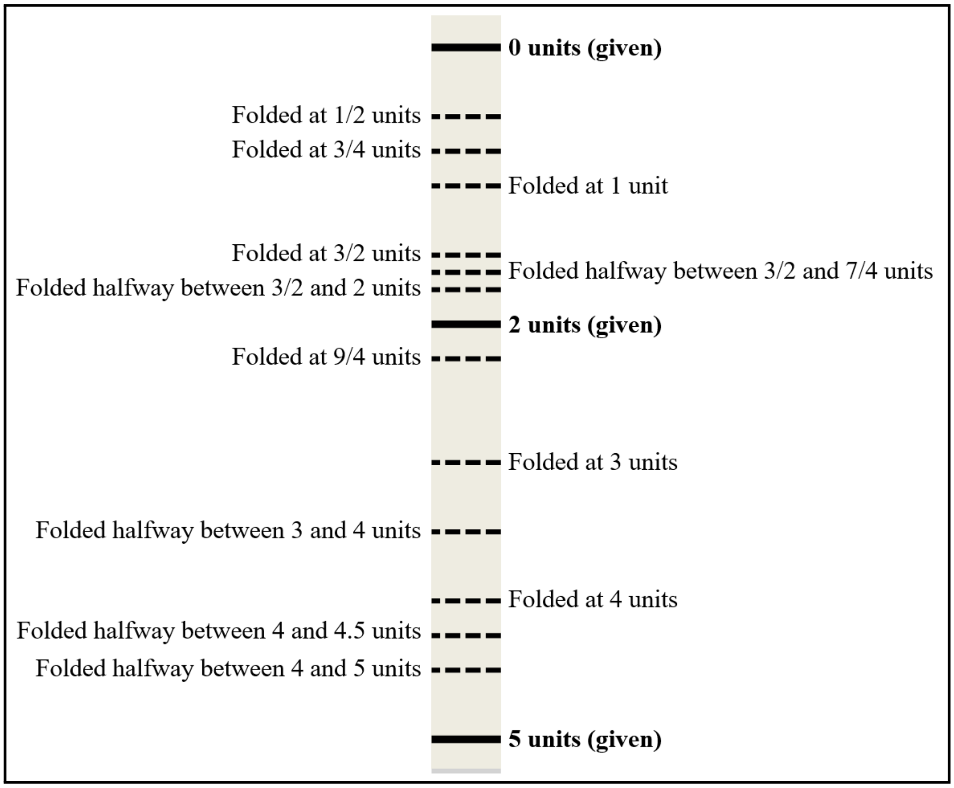

Fig. 1: *Author recreation of Activity 1 folded paper strip.*

finding the location for $\frac{1}{2}$ and $\frac{3}{2}$ on the number line. Prospective teachers folded the interval $[0, 1]$ to find $\frac{1}{2}$, and the interval $[1, 2]$ to find $\frac{3}{2}$, which many prospective teachers referred to as 1 and $\frac{1}{2}$. Thus prospective teachers extended to fractions a process they used previously to find whole numbers on the number line. After that prospective teachers were asked to fold the interval $[3, 4]$ in half and determine to what number did the new crease correspond. Prospective teachers identified the new number as 3 and $\frac{1}{2}$. They also identified the fraction obtained by folding the interval $[3/2, 2]$ in half, as 1 and $\frac{3}{4}$. Next, prospective teachers were asked to describe how they would locate $\frac{3}{4}$ and $\frac{7}{4}$ on the number line strip. They showed understanding that the process could be extended beyond halves because they did not have difficulty identifying numbers already marked on the number line for which the new numbers would be midpoints. Because we wanted to also stress that fractions could extend the number line system beyond the interval $[0, 1]$, we asked them to find a fraction between $\frac{3}{2}$ and $\frac{7}{4}$ on their paper strip, and to find two fractions in the interval $[4, 5]$. Finally we asked them to verbalize how they could find a fraction between any two fractions on their paper strip. Their verbal explanations gave evidence that they understood that the same process for locating the midpoint between two numbers on the number line could be used for any two fractions.

4.2 Activity 2. Zooming in on the number line

In order to develop prospective teachers' understanding that in mathematics we can represent fractions with as much precision as needed on the number line, the idea of zooming in using a computer program (SimCalc) was used. The basic idea behind zooming in is that we consider a given segment and magnify it. A difference between this mathematical zooming in and the zooming in in photography is that only the length of the segment is magnified; the width of the line and the size of the marks for the points remain the same size. The instructor displayed a number line with marks for whole numbers 0 to 5. One unmarked dot was placed slightly above the number line and prospective teachers were asked to estimate its location (Figure 2).

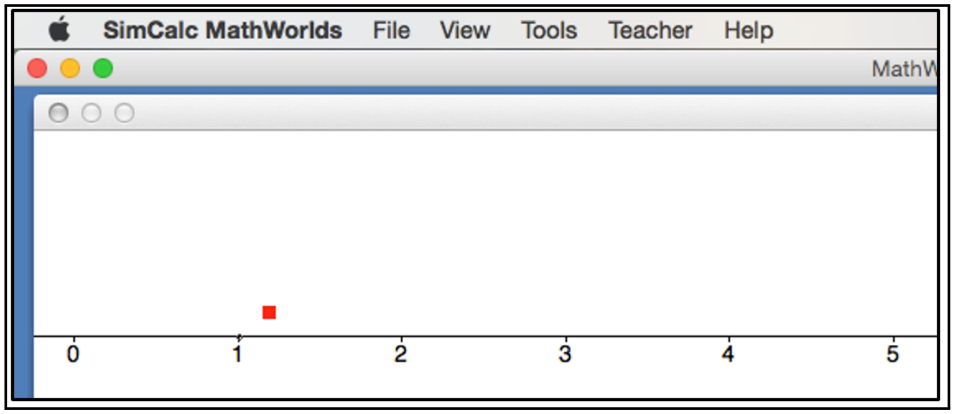

Fig. 2: *Location of point to be determined.*

Prospective teachers gave evidence that they understood that an important property of numbers also applies to fractions, namely that numbers can be located on the number line in a precise way. They did not have any difficulty identifying the interval in which the point was located and some prospective teachers were able to give more precision (e.g., "less than one and one half"). The instructor changed the scale on the number line to make the picture clear, and introduced additional marks without labels between the whole numbers (Figure 3), and repeated the process (Figure 4).

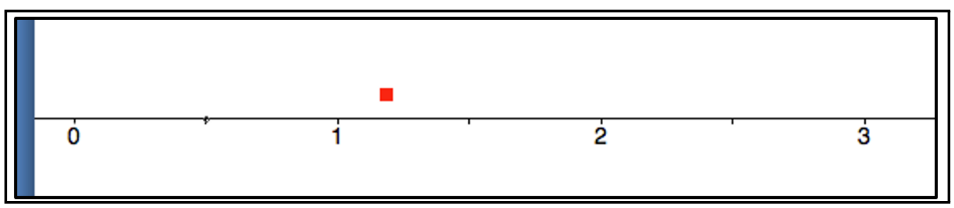

Fig. 3: *Zooming in and adding unlabeled marks.*

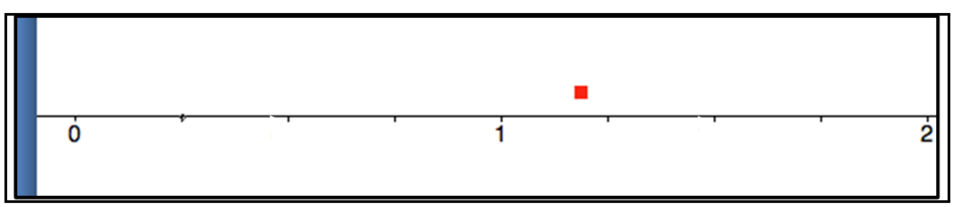

Fig. 4: *Zooming in again.*

Prospective teachers had little difficulty identifying the fractions corresponding to the new marks. The process was repeated several times, with the scale on the number line "stretched out" and new marks introduced at the midpoints without labels. Prospective teachers used the previous labeled marks as points of reference to correctly identify the new marks and eventually identify the location of the mark above the number line.

4.3 Activity 3. Using the average to find a fraction between two given fractions

The goal of this activity was for prospective teachers to make the connection between their method of finding the midpoint by folding and the arithmetical procedure for finding the average (i.e., adding the two numbers and dividing by 2), that is, for prospective teachers to realize that the midpoint between two numbers on the number line corresponds to their average. For example, $\frac{3}{4}$ is the average between $\frac{1}{2}$ and 1, because $\frac{1+\frac{1}{2}}{2} = \frac{\frac{3}{2}}{2} = \frac{3}{4}$. This method was used by the participants to find a new fraction between two given fractions, giving some evidence that they understood that fractions can be operated on in the same way as numbers. To illustrate, when prospective teachers were asked to find a fraction between $\frac{1}{2}$ and $\frac{3}{4}$, they computed the average. In other words, prospective teachers mentally found common denominators for the fractions, and added them. Then, to divide by 2, they multiplied by $\frac{1}{2}$, or simply multiplied the denominator of the sum by 2. Prospective teachers did not have difficulty finding fractions between a) $\frac{1}{4}$ and $\frac{1}{5}$, b) $\frac{3}{4}$ and 1, c) $\frac{3}{2}$ and $\frac{7}{4}$, d) $\frac{9}{40}$ and $\frac{1}{4}$. Besides finding the average, other methods to find the midpoint were mentioned by prospective teachers. For example, to find the midpoint between $\frac{1}{2}$ and $\frac{3}{4}$, a prospective teacher realized that the interval had a length of $\frac{1}{4}$, mentally divided the interval into two equal parts of length $\frac{1}{8}$, and added $\frac{1}{8}$ to the lower number to find the midpoint. Another prospective teacher mentioned the idea that the midpoint is equidistant from the two given fractions. Of course, these two ideas can be connected to the average. For instance, to express an unknown fraction x that is equidistant from $\frac{1}{2}$ and 1, we could use the idea that, on the number line, one of the conventions is that equal differences between numbers are represented by equal distances between the corresponding points. We can thus write the equation $1 - x = x - \frac{1}{2}$. Solving for x, we would see that x is indeed the average of $\frac{1}{2}$ and 1.

4.4 Activity 4. Using the idea of finding the midpoint to measure to within a given error bound

The goal for this activity was to provide a bridge between the concept of number density and measurement, and to emphasize that the process of finding the midpoint could be repeated as many times as required for a desired accuracy. Prospective teachers were asked to use the idea of midpoint to find the number corresponding to the location of a given point, to the nearest $\frac{1}{32}$ of the unit, on a long strip of paper (32 inches in length), where the entire strip of paper represented a unit interval (Figure 5).

By making the mystery location of the given point correspond to $\frac{11}{32}$ or to a number slightly larger or smaller than $\frac{11}{32}$, we were able to vividly demonstrate to prospective teachers the method of using successive midpoints to generate measurement accuracy because several folds were necessary. Specifically, prospective teachers first folded the unit interval in half and determined that the number was smaller than $\frac{1}{2}$. Then, they folded the interval $\left[0, \frac{1}{2}\right]$ in half and determined that the number was between $\frac{1}{4}$ and $\frac{1}{2}$. They repeated the process and found that the number was less than $\frac{3}{8}$, and continued halving until they found that $\frac{11}{32}$ was the first fraction obtained by halving that satisfied the given bound for error. Most pairs did not have difficulty identifying the fractions corresponding to the endpoints of the successively smaller intervals. However, in one case prospective teachers correctly identified the length of the interval they had just obtained $\left(\frac{1}{8}\right)$ but wrongly identified the endpoint as $\frac{1}{8}$ instead of $\frac{3}{8}$. This could also happen with younger students.

The process of measuring using the principle of the midpoint differs from the usual way of measuring in important ways. Usually when measuring, a new smaller unit is introduced when needed and the smaller unit is iterated as necessary. Often students fail to see this iterating process as an extension of the number system (Park, Gler, & McCrory , 2013).

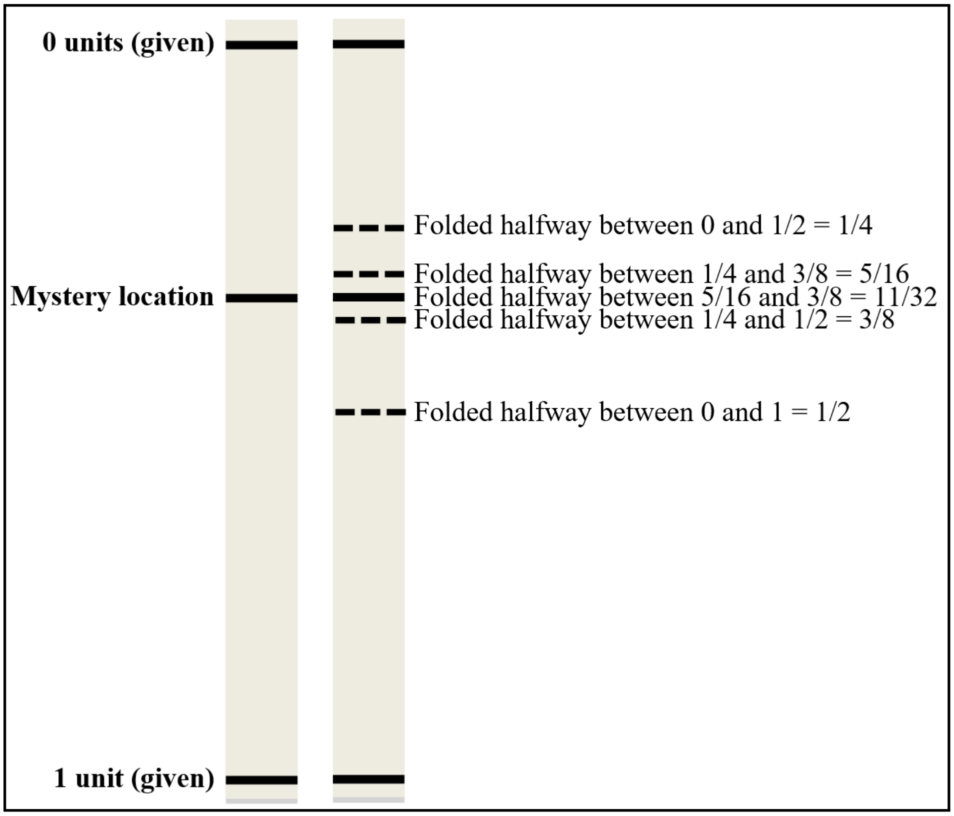

Fig. 5: *Author recreation of Activity 4 folded paper strip.*

5 Final Remarks

Addressing explicitly the mathematical ideas behind the process of extending the concept of number to fractions from whole numbers is important not only for students but also in courses for prospective teachers. Without strong content knowledge about fractions as numbers in a coherent number system and knowledge about students' difficulty understanding this aspect of fractions, the teacher's instruction may not explicitly address fractions as numbers, but only address fractions as part of a whole. To provide better opportunities for pre-service teachers and students to learn fractions as numbers, we presented four activities explicitly addressing their density as numbers using various visual realizations. In terms of developing the idea of density, it is important that students and prospective teachers realize that the process of finding the midpoint could be repeated indefinitely. Finding the midpoint between two fractions once is not enough to develop this concept. Interestingly, we can find a parallel in the history of approximation methods. In particular, the Babylonians (1600 BCE) had an approximation method for the square root of a number that was remarkably efficient (Flores, 2014). Using modern notation, if a is an approximation to $\sqrt{n}$, then $a + \frac{1}{2} \times \frac{1}{a}\left(n - a^2\right)$ is a better approximation. However, we have not found Babylonian texts that explicitly mention that the method could be repeated if more precision is needed or desired. It was not until Heron of Alexandria, in the first century CE (Thomas, 1957, p. 471), who used a method mathematically equivalent to the Babylonian method, where a better approximation is given by $\frac{a+\frac{n}{a}}{2}$, that we find an explicit mention that the method could be repeated if more precision is needed. We should not assume that, just because students (and prospective teachers) are able to find the midpoint between two fractions, that they, on their own, will come up with the idea that the process can be repeated indefinitely.

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