Construction of sundials via vectors

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Abstrakt

A method based on oblique projection is presented for construction of sundials. The derived formulas are classical, but usage of vectors and projections renders a coherent presentation rather than a number of special cases. The presented work is aimed to be useful for those taking a beginning module on vector algebra.

1. Introduction

In [4] is pointed out that standard types of sundials can be described via projection from an equatorial sundial. We give details on using oblique projection and vectors for construction of sundials. It is a concrete example on the efficiency of vectors in describing objects in the physical three-dimensional space and can therefore be useful for those learning vector algebra. For the finer details on finding the time with the help of the Sun, such as adjustment with respect to the non-uniform orbital motion of the Earth around the Sun (equation of time) and longitudinal correction, see [1, 7, 8], where also history of sundials and further references can be found.

Projection is inherent when constructing horizontal or vertical sundials but the mathematical language of vector algebra is seldom exploited in their description; an account having vectors is [6] but is more oriented towards matrices and for one type of horizontal dial. Thus, the details carried out using vectors and potential use of the material in the undergraduate curriculum count as the main benefits of the present work.

For the outline of the presentation, details for a construction of a sundial, known as equatorial, are given in Section 2 with Figs. 1–2 to aid the explanation. In Section 3, oblique projection is used for the construction of a number of standard sundials. The circular face of an equatorial sundial is projected onto a plane, rendering an ellipse with its shape and location depending on the direction of the projection. Derivations of formulas involve vector algebra. The obtained formulas are classical (description of sundials can be found already in [3, Chapt. 27]), but using vectors gives a coherent framework rather than a set of separate cases for each dial. We are not after the most general results, but focus is on horizontal projection. In Section 4, a parameterization is constructed of the obtained ellipse for various kinds of oblique projection onto the horizontal plane. Two examples showing how the obtained formulas can be used to generate different types of sundials can be found in Section 5. In Section 6 are some remarks on possible generalizations. In the appendix is a program written in the computer software Matlab for generating the projected ellipse and hour points for horizontal sundials. The data and figure generated can be of help when building such sundials or when learning how they function.

2. Equatorial sundials

The angle ϕ in Fig. 1 is the latitude and d the declination of the Sun (angular distance from the celestial equator). The angle d varies with the inclination of the axis of the Earth, between approximately $-23.5°$ to $23.5°$ during one year. Since the distance from the Sun to the Earth is large, light rays from the Sun striking the Earth can be assumed to be parallel (illustrated by the lines L_1 and L_2). The style in Fig. 1 is oriented in the north–south direction pointing towards north with an angle ϕ to the horizontal plane (this plane is tangent to the sphere at the point B). The reader is encouraged to show that the angle BDC equals ϕ implying (use alternate angles) that the style is parallel to the axis of the Earth, and that the height of the Sun, ABE, is $90° - \phi + d$. The reader is also advised to think through the construction when the latitude satisfies $0° < \phi < 23.5°$. We work on the northern hemisphere and assume hereafter that the latitude is greater than $23.5°$.
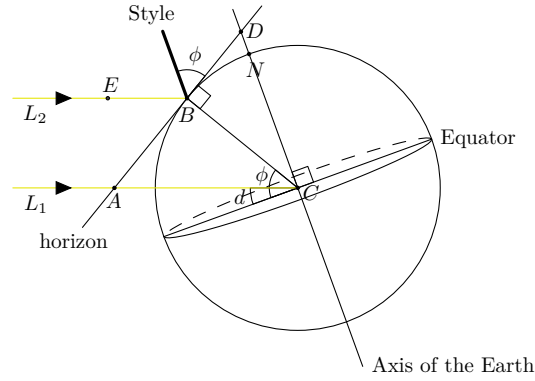
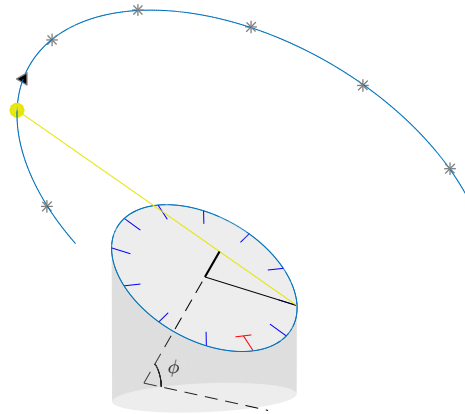Figur 1: Orientation of the style parallel with the axis of the Earth and pointing towards north

Figur 2: A sundial (equatorial)

If the style is oriented as in Fig. 1, a sundial can be constructed by drawing a circle in a plane with the style as normal, see Fig. 2. The plane is then parallel with the equatorial plane since the style is parallel with the axis of the Earth. The style can be thought of as being put at the centre of the Earth due to the large distance to the Sun. The apparent motion of the Sun during a day happens in a plane parallel with the plane containing the face of the sundial; this apparent motion is 15° per hour. In Fig. 2, every other hour has been marked uniformly along the boundary of the face. The style in Fig. 2 is oriented in the north–south direction facing north and the shadow of the style at 12 noon (local sun time) will fall on the segment marked with a T (for twelve). The shadow in Fig. 2 falls on the segment corresponding to 8 a.m. (sun time); the dial is viewed from north-east. An equatorial sundial has to have hour marks also on the back of the dial (the south facing side) since this side will be illuminated when the declination is negative. We describe how the circular face can be projected onto horizontal planes for simpler readings.

3.   Projection for constructing sundials

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be an orthonormal basis (positively oriented) in physical space with initial point O. Through O passes three mutually orthogonal coordinate axes with the basis vectors being in the positive direction of these axes. The notation $\boldsymbol{e}(x, y, z)^t$ (t: transpose) is used as a short-hand[†] for the left-hand

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[†] Hackman, P., Boken med Kossan på, ed. 15, Linköping University, 1999. (In Swedish)

side in the definition

$$(\boldsymbol{e}_1 \ \boldsymbol{e}_2 \ \boldsymbol{e}_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3,$$

where x, y and z are the coordinates in the given basis. The directed line segment between two points P and Q, with starting point at P, is written $\overrightarrow{PQ}$, and the same notation is used for the vector that this directed segment represents. The position vector $\overrightarrow{OP}$ is thought of as fixed at O.

For the description of the horizontal projection of the face of an equatorial sundial, we use Fig. 3 on page 7. We first describe how the equatorial face is aligned in Fig. 3, and then turn to the remaining parts and notation in that figure.

3.1 Vectorial description of the equatorial sundial

The basis vectors are chosen such that the position vector for the style (sometimes called gnomon) becomes

$$\boldsymbol{v} = \boldsymbol{e}(\cos\phi, 0, \sin\phi)^t, \tag{1}$$

where the angle ϕ corresponds to the latitude, that is $\boldsymbol{e}_1$ is oriented from south towards north, $\boldsymbol{e}_2$ points in the westerly direction and $\boldsymbol{e}_3$ points upwards from the horizontal plane; Fig. 3 is viewed from north-east. The style is placed with the starting point

$$M : (0, 0, z_M), \quad z_M > 0. \tag{2}$$

The two vectors

$$\boldsymbol{f}_1 = \boldsymbol{e}R(\sin\phi, 0, -\cos\phi)^t \qquad \text{and} \qquad \boldsymbol{f}_2 = \boldsymbol{e}R(0, 1, 0)^t \tag{3}$$

are orthogonal, and orthogonal to $\boldsymbol{v}$, where $R > 0$ is a real number.

The circle C described by the curve which the tip of the vector

$$\overrightarrow{OM} + \boldsymbol{f}_1 \cos t + \boldsymbol{f}_2 \sin t, \quad 0 \le t \le 2\pi, \tag{4}$$

traverses[†], has radius R and centre at the starting point M, and lies in a plane orthogonal to the style $\boldsymbol{v}$. The style is parallel to the axis of the Earth (see Fig. 1), therefore the plane containing C is parallel with the equatorial plane. The directed line segment between M and the point on C corresponding to 12 noon represents the vector $\boldsymbol{f}_1$, and $\boldsymbol{f}_2$ is represented by the directed line segment between M and the point on C corresponding to 6 a.m., see Fig. 3.

The style of the dial is adjusted lengthwise to fit the circle C in the sense that the shadow of the tip of the style shall fall on the boundary of the circular face as indicated in Fig. 3. The angle between the shadow and corresponding sun ray through the tip of the style is d since the plane through C is parallel to the equatorial plane by construction. The style itself is orthogonal to this plane, and the radius of C (also the length of the shadow) is R. Trigonometry in a right-angled triangle gives the length ℓ of the style to

$$\ell = R\tan d. \tag{5}$$

The tip P of the style is defined by the equality

$$\overrightarrow{OP} = \overrightarrow{OM} + \ell\boldsymbol{v},$$

which with the use of (1) and (2) can be written as

$$\overrightarrow{OP} = \boldsymbol{e} \begin{pmatrix} \ell\cos\phi \\ 0 \\ z_M + \ell\sin\phi \end{pmatrix}. \tag{6}$$

To construct other types of sundials than an equatorial, we use projection. The construction is general and can be used to construct a sundial on a flat wall for example; as mentioned in the introduction focus is on projection to the horizontal plane $z = 0$. We then turn to the description of the remaining parts in Fig. 3.

[†] The sum of a position vector and a free vector is according to common convention and construction a position vector, thus the tip of the vector in (4) is well-defined.

3.2   Projection of the equatorial dial

Let $\boldsymbol{w}$ be a vector of unit length chosen such that a circular cylinder can be constructed from the circle C and a generating line with direction $\boldsymbol{w}$. The intersection of the cylinder and the plane $z = 0$ is an ellipse (this elementary result is shown in [5, s. 7–8]; to see its validity note the shape of a shadow from a circular window onto a planar surface).

Projection in the direction of $\boldsymbol{w}$ onto the horizontal plane $z = 0$ is called oblique to indicate that $\boldsymbol{w}$ is not necessarily orthogonal to the projection plane. Projection of the points M and P renders M' and P', respectively. The projection of the directed line segment between M and P, representing the style, gives a directed line segment $\overrightarrow{M'P'}$ in the plane $z = 0$.

A triangular-shaped object with corners at the points M', P' and where the dashed line in Fig. 3 from M' intersects $P'P$ (the dashed lines between $M'M$ and $P'P$ are parallel with the style), is casting a triangular-shaped shadow with one corner on the ellipse giving the corresponding time mark in the plane $z = 0$ as the one on the circular face. A pole (or person) placed at P' with direction $\boldsymbol{w}$ casts a shadow along that time mark.

To have the shadow of the style of the circular face to exactly reach the circle C, as in Fig. 3, the style itself has to stretch or shrink during a year depending on the declination d as specified by (5). This implies that the position P is changing, which in turn means that the projected point P' is changing as well. Therefore, to tell the correct time with a pole at P', this pole has to move its position during a year. Examples will be given in Section 5.

We derive formulas for the construction, and concentrate on the case when the projection vector $\boldsymbol{w}$ is

$$\boldsymbol{w} = \boldsymbol{e}(\cos\alpha, 0, \sin\alpha)^t, \tag{7}$$

with $0 < \alpha \le \pi/2$. The projected points M' and P' are located on the x-axis due to the position (2) and orientation (1) of the style, and we find the coordinates of these points.

The (oblique) projection of M in the direction $-\boldsymbol{w}$ gives the point M' in the plane $z = 0$. The point M' can be found by determining a parameter c_M such that

$$\overrightarrow{OM} - c_M\boldsymbol{w}$$

is in the plane $z = 0$, meaning that the z-component is zero. The point M is defined in (2) and the direction $\boldsymbol{w}$ in (7), thus

$$\overrightarrow{OM} - c_M\boldsymbol{w} = \boldsymbol{e}\begin{pmatrix} -c_M\cos\alpha \\ 0 \\ z_M - c_M\sin\alpha \end{pmatrix}. \tag{8}$$

Since we project onto $z = 0$, we obtain the equation

$$z_M - c_M\sin\alpha = 0,$$

and the constant c_M is therefore

$$c_M = \frac{z_M}{\sin\alpha}.$$

Using this in (8), the projected point M' of the starting point M of the style becomes

$$M' : \left(-z_M\frac{\cos\alpha}{\sin\alpha}, 0, 0 \right). \tag{9}$$

Similarly, the projected point P' of the tip P of the style onto $z = 0$ along $-\boldsymbol{w}$ can be found. In Section 6 is a formula for oblique projection, however, we use the above strategy of determining a constant since calculations become straightforward and transparent when projecting onto $z = 0$.

We have to find a constant c_P such that

$$\overrightarrow{OP} - c_P\boldsymbol{w}$$

is in the plane $z = 0$. Using (6) and (7),

$$\overrightarrow{OP} - c_P \boldsymbol{w} = \boldsymbol{e} \begin{pmatrix} \ell \cos\phi - c_P \cos\alpha \\ 0 \\ z_M + \ell \sin\phi - c_P \sin\alpha \end{pmatrix}. \tag{10}$$

Projecting onto $z = 0$ means c_P shall be chosen as a solution to

$$z_M + \ell \sin\phi - c_P \sin\alpha = 0,$$

giving

$$c_P = \frac{z_M + \ell \sin\phi}{\sin\alpha}.$$

Using this value of c_P in (10), the projection P' of P is

$$P' : \left(\ell\cos\phi - \frac{z_M + \ell\sin\phi}{\sin\alpha} \cos\alpha, 0, 0 \right). \tag{11}$$

The projected point P' is the position on the x-axis (going from south to north), where a pole should be placed with direction $\boldsymbol{w}$, to cast a shadow on the corresponding hour point as the shadow of style of the equatorial sundial is indicating on C. The position of P' is changing during a year since, as mentioned above, the length of the style ℓ depends on the declination d in accordance with (5). Choices of the latitude and direction of the projection can simplify the derived expressions, for example, $\alpha = \pi/2$ or $\phi = \alpha$, as will be exploited in the examples in Section 5.

It is the changing position of the height of the Sun during a year which is compensated for with the movement of P' (the shadow from a pole fixed in the horizontal plane will in general move slightly each day for a given time). The change of the position of the Sun during a year, for a fixed time, is described by a curved known as an analemma (resembling an elongated figure eight). This change in position has to be corrected for when reading a sundial, along with longitudinal correction, to find the standard time (the time shown on clocks, phones and wristwatches).

4.   Projection of the circular boundary of the equatorial sundial

The projection C' of the circle C from (4) onto $z = 0$ is as mentioned in the previous section an ellipse. We give a parameterization of C'; notation is as in Fig. 3.

The point representing the tip of $\overrightarrow{OM} + \boldsymbol{f}_1$ is denoted P_1, and the tip of $\overrightarrow{OM} + \boldsymbol{f}_2$ is P_2. The axes of the ellipse are obtained by projection of $\overrightarrow{MP_1}$ respectively $\overrightarrow{MP_2}$ onto the plane $z = 0$ along the direction $-\boldsymbol{w}$. We start with the projection of $\overrightarrow{MP_1}$ (representing $\boldsymbol{f}_1$). The projection of the point M has already been found, see (9).

From (2) and (3), we can obtain an expression for P_1. To determine the projection of P_1 the constant c_{P_1} is chosen such that

$$\overrightarrow{OP_1} - c_{P_1} \boldsymbol{w} = \boldsymbol{e} \begin{pmatrix} R\sin\phi - c_{P_1}\cos\alpha \\ 0 \\ z_M - R\cos\phi - c_{P_1}\sin\alpha \end{pmatrix}$$
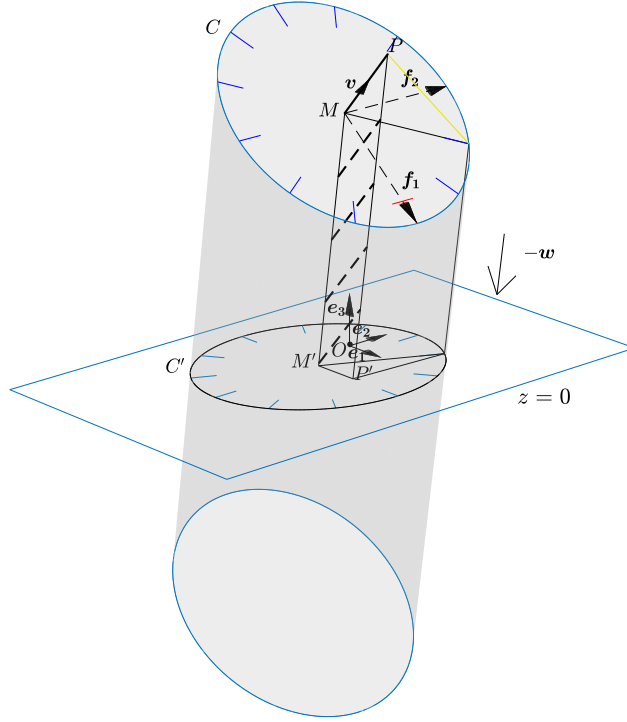
lie in $z = 0$. We get

$$c_{P_1} = \frac{z_M - R\cos\phi}{\sin\alpha}.$$

The projection of P_1 is therefore

$$P'_1 : \left(R\sin\phi - \frac{z_M - R\cos\phi}{\sin\alpha} \cos\alpha, 0, 0 \right). \tag{12}$$

Then $\boldsymbol{f}'_1 = \overrightarrow{M'P'_1}$ is a semi-axis to the ellipse. Using M' from (9) and P'_1 from (12),

$$\boldsymbol{f}'_1 = \boldsymbol{e} \left(R\sin\phi + \frac{R\cos\phi}{\sin\alpha} \cos\alpha, 0, 0 \right)^t. \tag{13}$$

Figur 3: Projection for construction of horizontal sundials

(M', P' and C' given in (9), (11) respectively (15))

The projection $\boldsymbol{f}_2'$ of $\overrightarrow{MP_2}$ (representing $\boldsymbol{f}_2$) is the other semi-axis, and from (3) and the type of projection (1), we find without calculations

$$\boldsymbol{f}_2' = (0, R, 0)^t. \tag{14}$$

The ellipse C', being the oblique projection along $-\boldsymbol{w}$ of the circle C onto $z = 0$, is then described by the curve which the tip of the vector

$$\overrightarrow{OM'} + \boldsymbol{f}_1' \cos t + \boldsymbol{f}_2' \sin t$$

traverses for $0 \le t \le 2\pi$. Using (9), (13) and (14), the position vector from O to a point on this ellipse can be described by

$$\boldsymbol{e}\left(-z_M \frac{\cos \alpha}{\sin \alpha} + \left(R \sin \phi + \frac{R \cos \phi}{\sin \alpha} \cos \alpha \right) \cos t, R \sin t, 0 \right)^t. \tag{15}$$

Small (positive) values in (15) for the angle α, defining the direction of the projection, render an ellipse, in the horizontal plane $z = 0$, being elongated in the x-direction, whilst $\alpha = \pi/2$ gives an ellipse elongated in the y-direction. Therefore, there is a value for α which gives a circle, we shall return to this value in Section 6. The shape of the ellipse is independent of the position (height) z_M of the style.

Using the value of the parameter t in (15) corresponding to the hour points on the circle C, we find where to place the corresponding hour points on the ellipse C'. It can sometimes be simpler to place those lines by giving an angle of a radius vector starting at M'. The choice $t = 0$ in (4) corresponds to the hour point for 12 noon, and when t increases the circle is traversed counter clockwise. The angle $\tilde{t}$ for the point on the ellipse corresponding to t on C, is measured from M' out from the axis $\boldsymbol{f}_1'$; using the coordinates in (15) and trigonometry for a right-angled triangle,

$$\tan \tilde{t} = \frac{R \sin t}{R \left(\sin \phi + \frac{\cos \phi \cos \alpha}{\sin \alpha} \right) \cos t} = \frac{\tan t}{\sin \phi + \frac{\cos \phi \cos \alpha}{\sin \alpha}}. \tag{16}$$

This expression is not defined for t being a multiple of a right angle. We return to this in the examples in the next section.

5. Examples of horizontal sundials

Ex. 1: The angle α in (7) is chosen to be 90°, meaning we have orthogonal projection onto the plane $z = 0$. The points constituting the ellipse C' can be found from (15),

$$C' : (R\sin\phi\cos t, R\sin t, 0).$$

Projection of the starting point M of the style is M', and using (9) it can be seen that M' coincides with the point O. The point where a pole should be placed, orthogonal to $z = 0$, to cast a shadow on the ellipse C', is by using (11) with ℓ from (5) (see further Fig. 4(a))

$$P' : (\ell\cos\phi, 0, 0)$$

The angle $\tilde{t}$ for a point on the ellipse is from (16) (same as given in [2])

$$\tan\tilde{t} = \frac{\tan t}{\sin\phi}.$$

This expression is not defined when t is a multiple of a right angle since $\tan t$ is not defined then. When t corresponds to 90°, the given parameterization (4) of the circle C gives the point on the equatorial sundial corresponding to 6 a.m. . This point is on the axis with direction $\boldsymbol{f}_2$, with projection $\boldsymbol{f}_2'$, and this latter vector is orthogonal to $\boldsymbol{f}_1'$. The angle $\tilde{t}$ measured from the axis corresponding to $\boldsymbol{f}_1'$ is therefore also 90°. A similar discussion can be done for other angles being a multiple of a right angle.

This type of horizontal sundial is known as analemmatic and studied in detail in [2] and [8, Chapt. 8.1–8.3]. Generating the position P' and the ellipse with hour points can be done with the Matlab program given in the Appendix.

Ex. 2: The angle α in (7) is taken to be equal to the latitude ϕ, thus we project along the direction (1) of the style. The points constituting the ellipse C' is obtained from (15),

$$C' : \left(-z_M\frac{\cos\phi}{\sin\phi} + R\frac{\cos t}{\sin\phi}, R\sin t, 0\right).$$

The projection M' of M (starting point of the style) coincides with the projection P' of P (the tip of the style). This means no change in position is needed in the horizontal plane, instead a pole can be fixated at M'. The pole is oriented south to north pointing towards the north with the angle $\alpha = \phi$ as for an equatorial sundial, but the shadow now falls on a horizontal plane with hour points along an ellipse, see further Fig. 4(b).

The angle $\tilde{t}$ on the ellipse is obtained from (16) (same as [8, Chapt. 5.3, Eqn. (1)])

$$\tan\tilde{t} = \sin\phi\tan t.$$

As in the previous example, the angle can be determined by a straightforward argument when t corresponds to a right-angle.

The ellipse and hour marks for this sundial can be generated using the Matlab program in the Appendix.

6. Some remarks

The type of projection can be changed. Placing a point at some height above the circle C of an equatorial sundial, one can define a projection of C onto $z = 0$ by taking straight lines from the point above C passing through the circle and finding their intersection with $z = 0$. This corresponds to the intersection of a cone and a plane, and such an intersection does not only give ellipses but conics, see further [4].
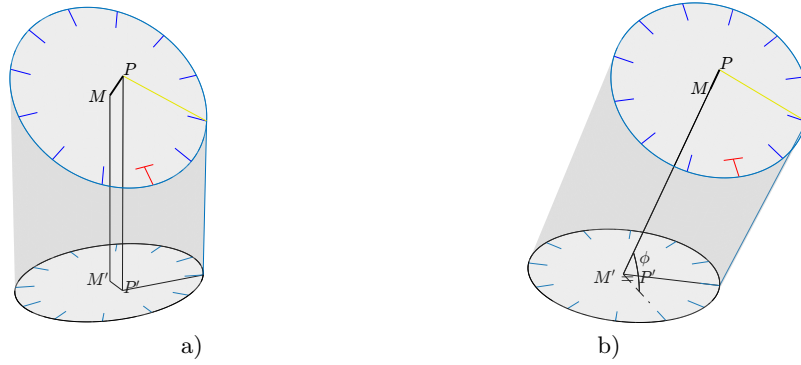
Figur 4: a) Ex. 1: Sundial via orthogonal projection   b) Ex. 2: Sundial via parallel projection

For oblique projection onto a plane passing through the point O there is the formula

$$\boldsymbol{u}_1 = \boldsymbol{u} - \frac{\boldsymbol{u} \cdot \boldsymbol{n}}{\boldsymbol{w} \cdot \boldsymbol{n}} \boldsymbol{w},$$

where the plane has unit normal $\boldsymbol{n}$, with $\boldsymbol{u}$ the vector projected onto the plane in the direction $\boldsymbol{w}$, and $\boldsymbol{u}_1$ the resulting vector (the symbol $\cdot$ is the standard scalar product). If the plane does not contain O then a translation is first needed. For the horizontal plane $z = 0$ with unit normal $\boldsymbol{n} = \boldsymbol{e}(0,0,1)^t$, the reader can check using the formula that the projection of the tip P of the style along $\boldsymbol{w}$ in (7) gives (11).

The term containing $\cos t$ in the x-coordinate in the representation (15) of the ellipse C' can be reformulated using the addition formula for cosine,

$$R \sin \phi \cos t + \frac{R \cos \phi \cos t}{\sin \alpha} \cos \alpha = R(\sin \phi \sin \alpha + \cos \phi \cos \alpha) \frac{\cos t}{\sin \alpha}$$
$$= R \cos(\phi - \alpha) \frac{\cos t}{\sin \alpha}.$$

For $\alpha = (90° + \phi)/2$, this expression reduces to $R \cos t$, meaning the intersection with the plane $z = 0$ is a circle. A pole placed at P' (position changes with the declination d) along south to north pointing north with angle α, will then cast a shadow on hour points lying along a circle, see further [8, Chapt. 8.5] (for names attached to this sundial like Lambert and Foster, see [9]). Data and figure for this sundial can be generated with the Matlab program in the Appendix.

Referenser

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Appendix

The program gives coordinates (positive in the direction of north and west, respectively) for generating hour points between 04:00 and 20:00 for horizontal sundials on the northern hemisphere with latitude greater than 23.5°. Coordinates for the point P' is given for the first day of each month. A figure of the ellipse C' with hour points is generated. When building a sundial, the height of a pole at P' can be calculated manually using the set radius R and trigonometry to make sure the height is such that the shadow of the pole reaches the ellipse.

```matlab
% Generating data in Matlab for horizontal sundials.

% Give the latitude phi, direction of projection alpha and radius R
% (Ex.1 is alpha=90, Ex. 2 is alpha=phi, and Ex. in remark is alpha=(90+phi)/2)
phi=59;
alpha=90;
R=1;

% Declination for the first day of each month: January,...,December
d=[-22.97 -16.95 -7.39 4.73 15.22 22.12 23.06 17.87 8.08 -3.39 -14.59 -21.88]';

% Angles are transformed to radians
phi=(phi/360)*2*pi;
alpha=(alpha/360)*2*pi;
d=(d/360)*2*pi;

% The ellipse is drawn using expression (15) with z_M=0;
% north-south direction (x) and west (y) is in the figure as on maps
t=(0:0.01:2*pi)';
x=(R*sin(phi)+(R*cos(phi)*cos(alpha)/sin(alpha)))*cos(t);
y=R*sin(t);
plot(y,x,'Color','black')
axis equal
hold on
axis off

% Hour points on the ellipse is from choices of t. Drawn for
% 04.00 to 20.00
t1=-(8*pi/12)+(pi/12)*(0:1:16)';
xt=(R*sin(phi)+(R*cos(phi)*cos(alpha)/sin(alpha)))*cos(t1);
yt=R*sin(t1);
plot(yt,xt,'*','Color','black')

% A line is drawn showing the variation of the position P'
ell=R*tan(d);
Pr=[0.*d,ell.*cos(phi)-((ell.*sin(phi))./sin(alpha)).*cos(alpha)];
line([0 0],[min(Pr(:,2)) max(Pr(:,2))])

% Text is printed to screen to identify hour points
fprintf('\t time \t x (north) \t y (west)\n')
fprintf('\t %2d \t %8.4f \t %8.4f \n',[(4:1:20)' xt -yt]')

% Text is printed to screen for the position of P' for each month
% (1=Jan,...,12=Dec)
fprintf('\t Month \t \t P''\n')
fprintf('\t %2d \t %8.4f\n',[(1:12)' Pr(:,2)]')

% Text for the figure, some hour points, P' and cardinal directions
text(yt(1)-0.1*R,xt(1),'4')
text(yt(9)-0.03*R,xt(9)+0.1*R,'12')
text(yt(17)+0.08*R,xt(17),'20')
plot(0,0,'x','Color','black','MarkerFaceColor','black','Markersize',[14])
text(0+0.05*R,max(Pr(:,2)),'P''')
text(yt(15)+0.10*R,xt(15),'E')
text(yt(3)-0.15*R,xt(3),'W')
text(-yt(9)-0.02*R,xt(9)-0.1*R,'N')
text(-yt(9)-0.03*R,-xt(9)-0.1*R,'S')
```