VINBERG'S CHARACTERIZATION OF DYNKIN DIAGRAMS USING SUBADDITIVE FUNCTIONS WITH APPLICATION TO DTr-PERIODIC MODULES

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Let R be an Artin algebra. Given two indecomposable modules M,N, let $\mathrm{Irr}(M,N) = \mathrm{rad}(M,N)/\mathrm{rad}^2(M,N)$ be the bimodule of irreducible maps [5] and denote by a_{MN} the length of $\mathrm{Irr}(M,N)$ as an $\mathrm{End}(N)$-module, by a'_{MN} its length as an $\mathrm{End}(M)$-module. Note that in case M is not injective, then a_{MN} is equal to the multiplicity of N occuring in the middle term of the Auslander-Reiten sequence starting with M, whereas if N is not projective, then a'_{MN} is equal to the multiplicity of M occurring in the middle term of the Auslander-Reiten sequence ending with N. The Auslander-Reiten quiver $A(R)$ has as vertices the isomorphism classes of the indecomposable R-modules, and there is an arrow $[M] \to [N]$ provided $\mathrm{Irr}(M,N) \neq o$. We endow this arrow with the valuation (a_{MN}, a'_{MN}), and, in this way we obtain a valuated quiver. We denote the Auslander-Reiten translations by $A = \mathrm{DTr}$, $A^{-} = \mathrm{TrD}$. An indecomposable module is called stable provided $A^{n}M \neq o, A^{-n}M \neq o$ for all $n \in \mathbb{N}$. The full subquiver $A_s(R)$ of $A(R)$ consisting of the isomorphism classes of stable modules is called the stable Auslander-Reiten-quiver. Any component of the stable Auslander-Reiten quiver determines (uniquely) a Cartan matrix, and we call it its Cartan class. Also, a module M is called periodic provided $A^{p}M \approx M$ for some $p \in \mathbb{N}$.

Theorem. The Cartan class of a component of the stable Auslander-Reiten quiver of an Artin algebra containing periodic modules is either a Dynkin diagram or A_{∞}.

In the case of R being an algebra of finite representation type over an algebraically closed field, this is the famous result of Riedtmann [5], the extension to arbitrary Artin algebras of finite representation type being due to Todorov [9]. Todorov also has considered the general case of components of $A_s(R)$ containing periodic modules and reduced their Cartan classes to Dynkin diagrams or A_∞, A_∞^∞, B_∞, C_∞, D_∞. Thus, our only contribution is the elimination of the possibilities A_∞^∞, B_∞, C_∞, D_∞ (lemma 3). Note that the other cases actually do occur.

We will provide a rather elementary self-contained proof of the theorem using only the structure theorem for Riedtmann quivers and Auslander's theorem on the existence of indecomposable modules of arbitrarily large length in any infinite component of an Auslander-Reiten quiver. It was the technique of Todorov which motivated the present presentation: her sole use of length functions and inequalities seemed to ask for an axiomatic treatment using additive and subadditive functions (copying the additivity property of the ordinary length function on Auslander-Reiten sequences). This notion of an additive function was introduced by Bautista [2]. It was M. Auslander who pointed out during his visit to Bielefeld in June 1979 that the methods of Todorov should furnish an interesting combinatorial characterization of the Dynkin diagrams. In fact, such a characterization follows from the investigations of Vinberg in [10]: namely, the Dynkin diagrams are the only finite Cartan matrices with subadditive functions which are not additive. We will need an extension of this result to Cartan matrices which are not necessarily finite and provide a direct proof of the general result. In the same way, one also characterizes the Cartan matrices with additive functions; in the finite case, this result again is due to Vinberg [10], and also to Berman, Moody and Wonenburger [3]; it will be used in a forthcoming paper [4] to deal with binary polyhedral groups.

The authors are indebted to many participants of the Ottawa conference 1979, in particular P. Gabriel, M.I. Platzeck and I. Reiten, for stimulating discussions on this topic and helpful remarks concerning the final form of the manuscript.

1. A characterization of Dynkin diagrams

Let I be an index set. A Cartan matrix C on I is a function $C : I \times I \longrightarrow \mathbb{Z}$ satisfying the following properties

(1) $C_{ii} = 2$ for all $i \in I$.

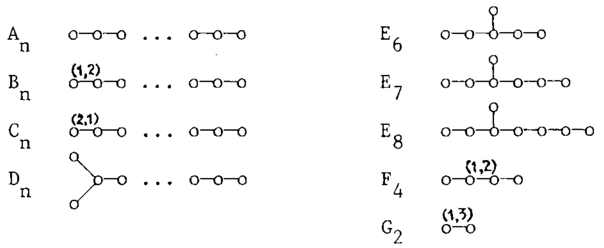
(2) $C_{ij} \leq 0$ for all $i \neq j$ in I.

(3) $C_{ij} = 0$ if and only if $C_{ji} = 0$.

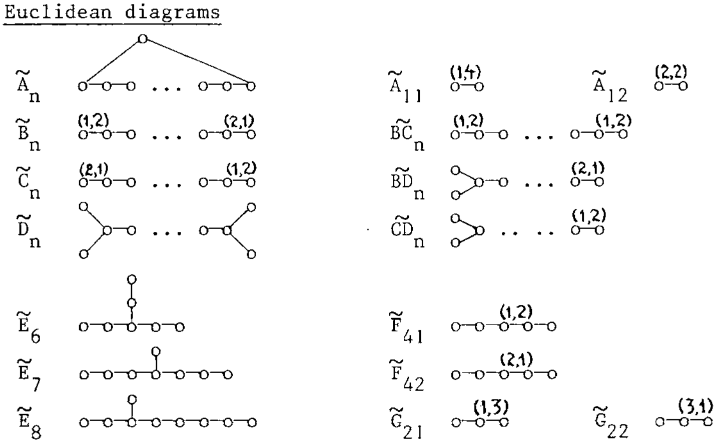
Note that we write C_{ij} instead of $C(i,j)$. The underlying graph of C has as vertices the elements of I, and edges $\{i,j\}$ for all pairs $i \neq j$ with $C_{ij} \neq 0$.

Of course, the easiest way to write down those Cartan matrices we will be interested in, is to start with the underlying graph and add to the edges pairs of numbers $\underset{i}{\circ}\overset{(|C_{ij}|,|C_{ji}|)}{\longrightarrow}\underset{j}{\circ}$ in case $C_{ij}C_{ji} \neq 1$, the "valuation".

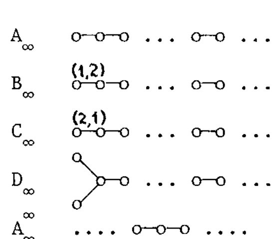
The Cartan matrix C will be called connected in case the underlying graph is connected. In particular, we are interested in the Dynkin diagrams



the Euclidean diagrams



and the following infinite diagrams

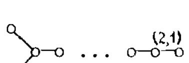


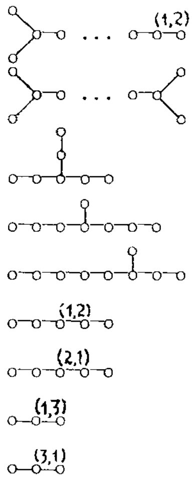
Let C be a Cartan matrix on I. By a <u>subadditive function for</u> C we will mean a function $d : I \longrightarrow \mathbb{N} = \{1,2,3,\ldots\}$ satisfying $\sum_{i\in I} d_i C_{ij} \geq o$ for all $j \in I$. Again, we write d_i instead of $d(i)$. Such a function is called <u>additive</u> provided we even have $\sum_{i\in I} d_i C_{ij} = o$ for all $j \in I$. (In case I is finite, an additive function is also called a null root [2]. Note that in case I is infinite, the existence of a subadditive function immediately implies that for fixed j, all but a finite number of C_{ij} are zero.)

<u>Lemma 1</u>. Let C be a Euclidean diagram. Then any subadditive function for C is additive.

<u>Proof</u>: Let C^t be the transpose of C, thus $C^t_{ij} = C_{ji}$ for all $i,j \in I$. With C also C^t is a Euclidean diagram. Now for every Euclidean diagram, there is an additive funtion h, see the table below. Given the Euclidean diagram C, let us denote by ∂ a fixed additive function for C^t, thus $\partial C^t = o$. Let d be a subadditive function for C. Then $(dC)\partial^t = d(C\partial^t) = o$. By assumption, the components of dC are $\geq o$, those of ∂ are $> o$. Therefore the equality $(dC)\partial^t = o$ implies that all components of dC are zero, which means that d is additive.

In the tables below, we have listed for every Euclidean diagram C an additive function h for C.
[Note that any other additive function for C is an integral multiple of this h. Namely, given a second additive function h' for C, we can form a non-trivial linear combination of h and h' which vanishes for some $i \in I$. However, it is well-known (and easy to see) that the Cartan matrices of Dynkin diagrams are regular. Thus the linear combination has to be the zero function, and therefore h' is a $\mathbb{Q}$-multiple of h. Since $h_i = 1$ for some $i \in I$, we see that h' even has to be an $\mathbb{N}$-multiple of h.]

Type	diagram	h
$\tilde{A}_{11}$		21
$\tilde{A}_{12}$		11
$\tilde{B}_n$		11...11
$\tilde{C}_n$		12...21
$\tilde{BC}_n$		22...21
$\tilde{BD}_n$		$^{1}_{1}$2...22



$\tilde{CD}_n$	$^1_1 2 \ldots 2 1$
$\tilde{D}_n$	$^1_1 2 \ldots 2 ^1_1$
$\tilde{E}_6$	1 2 12321
$\tilde{E}_7$	2 1234321
$\tilde{E}_8$	3 12345642
$\tilde{F}_{41}$	12321
$\tilde{F}_{42}$	12342
$\tilde{G}_{21}$	121
$\tilde{G}_{22}$	123

Given two Cartan matrices C on I and C' on I', then we call C' <u>smaller</u> than C provided $I' \subseteq I$ and $|C'_{ij}| \leq |C_{ij}|$ for all i,j in I'.

<u>Lemma 2</u>. Let C, C' be two different Cartan matrices, with C' smaller than C. Let d be a subadditive function for C. Then $d|I'$ is a subadditive function for C' which is not additive.

<u>Proof</u>: Let $j \in I'$, then

$$2d_j \geq \sum_{\substack{i \in I \\ i \neq j}} d_i |C_{ij}| \geq \sum_{\substack{i \in I' \\ i \neq j}} d_i |C_{ij}| \geq \sum_{\substack{i \in I' \\ i \neq j}} d_i |C'_{ij}|$$

shows that $d|C'$ is subadditive, again. If I' is a proper subset of I, choose $j \in I'$, $i \in I \setminus I'$ which are neighbors, then

$$\sum_{\substack{i \in I' \\ i \neq j}} d_i |C_{ij}| > \sum_{\substack{i \in I' \\ i \neq j}} d_i |C_{ij}|,$$

thus $d|C'$ is not additive. If $|C'_{ij}| < |C_{ij}|$ for some i,j in I', then

$$\sum_{\substack{i \in I' \\ i \neq j}} d_i |C_{ij}| > \sum_{\substack{i \in I' \\ i \neq j}} d_i |C'_{ij}|,$$

thus again, $d|C'$ is not additive.

Lemma 3. Every subadditive function for any one of A_∞^∞, B_∞, C_∞ or D_∞ is additive and bounded.

Proof: Consider first A_∞^∞. We may assume $I = \mathbb{Z}$, with edges $\{i,i+1\}$. Given $d : \mathbb{Z} \to \mathbb{N}$, there is some $i \in \mathbb{Z}$, where d takes its minimum. But the subadditivity means $2d_i \geq d_{i-1}+d_{i+1}$, which combined with $d_{i-1} \geq d_i$, $d_{i+1} \geq d_i$ gives $d_{i-1} = d_i = d_{i+1}$. By induction, we see that d is constant.

In writing down a subadditive function d, we will use the valued graph and attach to each vertex i the numbers d_i. In case B_∞,

$$d_0 \overset{(1,2)}{\text{———}} d_1 \text{———} d_2 \text{———} d_3 \ \cdots$$

we obtain from d a subaddtive function on A_∞^∞, namely

$$\cdots\ d_2 \text{———} d_1 \text{———} d_0 \text{———} d_1 \text{———} d_2 \ \cdots .$$

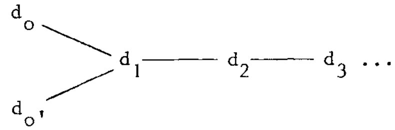
In case C_∞, we obtain from

$$d_0 \overset{(2,1)}{\text{———}} d_1 \text{———} d_2 \text{———} d_3 \ \cdots$$

a subadditive function on A_∞^∞, namely

$$\cdots\ d_2 \text{———} d_1 \text{———} 2d_0 \text{———} d_1 \text{———} d_2 \ \cdots .$$

In case D_∞, we obtain from



a subadditive function on A_∞^∞, namely

$$\cdots\ d_2 \text{———} d_1 \text{———} d_0+d_{0'} \text{———} d_1 \text{———} d_2 \ \cdots .$$

In all three cases, the obtained function on A_∞^∞ has to be constant, thus d is additive and bounded.

Theorem. Let C be a connected Cartan matrix and d a subadditive function for C.

(a) C is either a Dynkin diagram, a Euclidean diagram or one of A_∞^∞, A_∞, B_∞, C_∞, D_∞.

(b) If d is not additive, then C is a Dynkin diagram or A_∞.

(c) If d is unbounded, then C is A_∞.

Proof: If C is neither a Dynkin diagram nor one of A_∞, A_∞^∞, B_∞, C_∞, D_∞, then there exists a Euclidean diagram C' which is smaller than C (an easy verification).
Now if $C' \neq C$, then $d|C'$ cannot be additive, according to lemma 2. This is a contradiction, since $d|C'$ must be additive, according to lemma 1. This proves (a). If d is not additive, then Euclidean diagrams and A_∞^∞, B_∞, C_∞ and D_∞ cannot occur according to lemma 1 and lemma 3, this proves (b). If d is unbounded, then I has to be infinite, and only A_∞ remains according to lemma 3.

Remarks. For any Euclidean diagram, we have seen in the table of lemma 1 an additive function. Restricting these functions to proper subdiagrams, we obtain for all Dynkin diagrams subadditive functions which then cannot be additive. Thus, the Dynkin diagrams are the only Cartan matrices on a finite index set for which there exist subadditive functions which are not additive. This characterization of the Dynkin diagrams is due to Vinberg [10]. Also, there are the obvious additive functions on A_∞^∞, B_∞, C_∞, D_∞ (see lemma 3), and for A_∞, there are both additive functions, and subadditive functions which are not additive, for example

$$1 - 2 - 3 - 4 - 5 \ \dots$$
$$2 - 4 - 5 - 6 - 7 \ \dots$$

Finally, there are no additive functions for a Dynkin diagram C (since C is a regular matrix). Thus, the Euclidean diagrams are the only Cartan matrices on finite index sets with additive functions. This characterization of the Euclidean diagrams is due to Vinberg [10] and Berman-Moody-Wonenburger [3].

2. The application

For a quiver $\Gamma = (\Gamma_o, \Gamma_1)$ with Γ_o the set of vertices and Γ_1 the set of arrows, we always will assume that it does not have loops or double arrows. If x is a vertex, we denote by x^+ the set of endpoints of arrows with starting point x, and by x^- the set of starting points of arrows with endpoint x. In case the sets x^+ and x^- are finite for all x, we will call the quiver locally finite.

A Riedtmann quiver $\Delta = (\Delta_o, \Delta_1, \tau)$ is given by a quiver (Δ_o, Δ_1), together with an injective function $\tau : \Delta_o' \to \Delta_o$ defined on a subset Δ_o' of Δ_o satisfying $(\tau x)^+ = x^-$. Given an arrow $\alpha : y \to x$, there is a unique arrow $\tau x \to y$ and this arrow will be denoted by $\sigma\alpha$. A Riedtmann quiver is called stable provided τ is defined on all of Δ_o and is also surjective. Of course, any Riedtmann quiver has a unique maximal stable Riedtmann subquiver. (These concepts have been introduced in [5], there, a Riedtmann quiver is called "Darstellungs-köcher".) A vertex x of a Riedtmann quiver Δ will be called periodic provided $\tau^p(x) = x$ for some $p \in \mathbb{N}$. We will be interested in stable Riedtmann quivers containing periodic elements.

An important example of a Riedtmann quiver is the following: let Γ be an oriented tree (a quiver with underlying graph a tree), and define $\mathbb{Z}\Gamma$ as follows: its vertices are the elements of $\mathbb{Z} \times \Gamma_o$, and given an arrow $\alpha : x \to y$, there are arrows $(n,\alpha) : (n,x) \to (n,y)$ and $\sigma(n,\alpha) : (n+1,y) \to (n,x)$ for all $n \in \mathbb{Z}$. Finally, let $\tau(n,\alpha) = (n+1,\alpha)$. Note that in this way, we obtain a stable Riedtmann quiver.

Given a quiver (Γ_o, Γ_1), a function $a : \Gamma_1 \to \mathbb{N} \times \mathbb{N}$ will be called a valuation, and $\Gamma = (\Gamma_o, \Gamma_1, a)$ a valued quiver. The image of $\alpha : x \to y$ will be denoted by (a_α, a'_α), or also (a_{xy}, a'_{xy}). If Γ is a valued quiver, we can associate with it a Cartan-matrix $C = C(\Gamma)$ on the index set Γ_o as follows: for $x \in \Gamma_o$, let $C_{xx} = 2$, for $x \neq y$ in Γ_o, let $C_{xy} = -a_{xy} - a'_{yx}$, where $a_{xy} = o = a'_{xy}$ in case there is no arrow with starting point x and endpoint y. In case we deal with a valued oriented tree Γ, then (Γ_o, Γ_1) and C together determine the valuation.

A valued Riedtmann quiver $\Delta = (\Delta_o, \Delta_1, \tau, a)$ is given by a Riedtmann quiver $(\Delta_o, \Delta_1, \tau)$ and a valuation a for (Δ_o, Δ_1) such that $a_{\sigma\alpha} = a'_\alpha$,

$a'_{\sigma\alpha} = a_\alpha$ for all $\alpha : y \to x$ with $x \in \Delta'_o$. A typical example is again the following: let (Γ_o, Γ_1, a) be a valued oriented tree, and define on $\mathbb{Z}(\Gamma_o, \Gamma_1)$ a valuation by $a_{(n,\alpha)} = a_\alpha = a'_{\sigma(n,\alpha)}$ and $a'_{(n,\alpha)} = a'_\alpha = a_{\sigma(n,\alpha)}$. This valued Riedtmann quiver is denoted by $\mathbb{Z}(\Gamma_o, \Gamma_1, a)$.

Proposition. Let Γ, Γ' be valued oriented trees. Then $\mathbb{Z}\Gamma$ and $\mathbb{Z}\Gamma'$ are isomorphic if and only if the Cartan matrices $C(\Gamma)$ and $C(\Gamma')$ are isomorphic. Given any stable valued Riedtmann quiver Δ, there is a valued oriented tree Γ and a group G of automorphisms of $\mathbb{Z}\Gamma$ such that Δ is isomorphic to $\mathbb{Z}\Gamma/G$.

In case Δ is isomorphic to $\mathbb{Z}\Gamma/G$ for some valued oriented tree Γ, we call $C(\Gamma)$ the Cartan class of Δ; it is uniquely determined by Δ.

The proof of the proposition follows immediately from the corresponding result on Riedtmann quivers without valuations [5]. Namely, if $\Delta = (\Delta_o, \Delta_1, \tau, a)$ is a valued Riedtmann quiver, and $(\Delta_o, \Delta_1, \tau) = \mathbb{Z}(\Gamma_o, \Gamma_1)/G$ for some oriented tree (Γ_o, Γ_1), then using the projection from $\mathbb{Z}(\Gamma_o, \Gamma_1)$ onto $(\Delta_o, \Delta_1, \tau)$, the valuation a of Δ gives rise to a valuation on $\mathbb{Z}(\Gamma_o, \Gamma_1)$, also denoted by a, in such a way that $\mathbb{Z}(\Gamma_o, \Gamma_1)$ becomes a valued Riedtmann quiver. The canonical embedding of (Γ_o, Γ_1) into $\mathbb{Z}(\Gamma_o, \Gamma_1)$ given by $x \mapsto (o, x)$ endows (Γ_o, Γ_1) with a valuation, again denoted by a, and clearly $(\Delta_o, \Delta_1, \tau, a) = \mathbb{Z}(\Gamma_o, \Gamma_1, a)/G$. If x is a sink in (Γ_o, Γ_1), denote by $\sigma_x(\Gamma_o, \Gamma_1, a)$ the full valued subquiver of $\mathbb{Z}(\Gamma_o, \Gamma_1, a)$ with vertices (o, y) for $y \neq x$, and $(1, x)$. It is obvious that the Cartan matrices $C(\Gamma_o, \Gamma_1, a)$ and $C(\sigma_x(\Gamma_o, \Gamma_1, a))$ are isomorphic. This shows the unicity of the Cartan matrix associated to $\mathbb{Z}(\Gamma_o, \Gamma_1, a)$.

If $\Delta = (\Delta_o, \Delta_1, \tau, a)$ is a valued Riedtmann quiver, a subadditive function ℓ for Δ is, by definition, a function $\ell : \Delta_o \to \mathbb{N}$ satisfying

$$\ell(x) + \ell(\tau x) \geq \sum_{y \in x^-} \ell(y) a'_{yx} ,$$

for all $x \in \Delta'_o$. Such a function is called additive, provided we always have equality (for all $x \in \Delta'_o$).

Theorem. Let $\Delta = (\Delta_0, \Delta_1, \tau, a)$ be a stable valued Riedtmann quiver which is connected, and contains a periodic vertex. Assume there is a subadditive function ℓ for Δ.

(a) The Cartan class of Δ is either a Dynkin diagram, a Euclidean diagram, or one of A_∞, A_∞^∞, B_∞, C_∞, D_∞.

(b) If ℓ is not additive, then the Cartan class of Δ is a Dynkin diagram, or A_∞.

(c) If ℓ is unbounded, then the Cartan class of Δ is A_∞.

Proof: First note that the existence of a subadditive function implies that Δ is locally finite.

Let us show that any vertex of Δ has to be periodic. For, let x be periodic, say $\tau^p x = x$. Now,

$\tau^p(x^+) = (\tau^p(x))^+ = x^+$ shows that τ^p induces a permutation on the finite set x^+, and therefore τ^{pm} the identity on x^+, for some $m \in \mathbb{N}$. Thus any $y \in x^+$ also is periodic. Similarly, any $y \in x^-$ is periodic. But in this way, using in addition τ, we can reach any other vertex of Δ, since we assume that Δ is connected.

Let Δ be a quotient of $\mathbb{Z}\Gamma$, with Γ a valuated oriented tree with Cartan matrix C. We can assume that $\Gamma = \{0\} \times \Gamma$ is embedded into $\mathbb{Z}\Gamma$, and denote the corresponding map $\Gamma \longrightarrow \mathbb{Z}\Gamma \longrightarrow \Delta$ just by $u \mapsto \tilde{u}$. By definition of C, we have

$$C_{uv} = \begin{cases} 2 & u = v \\ -a_{\tilde{u}\tilde{v}} & u \longrightarrow v \\ & \text{in case} \\ -a'_{\tilde{v}\tilde{u}} & v \longrightarrow u \\ 0 & \text{otherwise} \end{cases}$$

Assume now there is given a subadditive function ℓ for Δ. We consider first the case where there exists a fixed number p with $\tau^p x = x$ for all vertices x of Δ. For example, this clearly is true in case Γ is finite. From ℓ we obviously obtain a τ-invariant subadditive function d for Δ, by

$$d(x) = \sum_{i=0}^{p-1} \ell(\tau^i x),$$

and d is additive if and only if ℓ is. Namely, $\tau^p x = x$ shows that $d(x) = d(\tau x)$, thus

$$2d(x) = d(x) + d(\tau x) = \sum_{i=0}^{p-1} \ell(\tau^i x) + \sum_{i=1}^{p} \ell(\tau^i x)$$

$$= \sum_{i=0}^{p-1} [\ell(\tau^i x) + \ell(\tau(\tau^i x))]$$

$$\geq \sum_{i=0}^{p-1} \sum_{y \in (\tau^i x)^-} \ell(y)\, a'_{y,\tau^i x}$$

$$= \sum_{i=0}^{p-1} \sum_{z \in x^-} \ell(\tau^i z)\, a'_{\tau^i z,\tau^i x}$$

$$= \sum_{z \in x^-} \sum_{i=0}^{p-1} \ell(\tau^i z)\, a'_{z,x}$$

$$= \sum_{z \in x^-} d(z) a'_{z,x}$$

where we have written $y \in (\tau^i x)^- = \tau^i(x^-)$ in the form $y = \tau^i z$, and used that $a'_{\tau^i z,\tau^i x} = a'_{z,x}$ for all x,z. Thus d is a τ-invariant subadditive funtion for Δ which is additve iff ℓ is additive. We consider now the composed map $\Gamma \longrightarrow \Delta \longrightarrow \mathbb{N}$, given by $u \mapsto d(\tilde{u})$.
Note that $\tilde{u}^-$ is the disjoint union of $\{\tilde{v} \mid v \in u^-\}$ and $\{\tau\tilde{v} \mid v \in u^+\}$, thus

$$2d(\tilde{u}) \geq \sum_{z \in \tilde{u}^-} d(z)\, a'_{z,\tilde{u}}$$

$$= \sum_{v \in u^-} d(\tilde{v})\, a'_{\tilde{v}\tilde{u}} + \sum_{v \in u^+} d(\tilde{v})\, a'_{\tau\tilde{v},\tilde{u}}$$

$$= \sum_{v \in u^-} d(\tilde{v})\, a'_{\tilde{v}\tilde{u}} + \sum_{v \in u^+} d(\tilde{v})\, a_{\tilde{u}\tilde{v}}$$

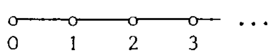
$$= -\sum_{v \in u^-} d(\tilde{v})\, c^t_{vu} - \sum_{v \in u^+} d(\tilde{v})\, c^t_{vu}$$

$$= -\sum_{v \neq u} d(\tilde{v})\, c^t_{vu} .$$

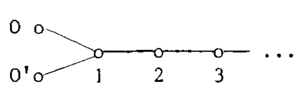
This shows that we obtain in this way a subadditive function for C^t, which is additive or unbounded iff ℓ is additive, or unbounded, respectively. Thus, the existence of a subadditive function ℓ on Δ implies that C has to be a Dynkin or Euclidean diagram or one of A_∞, A_∞^∞, B_∞, C_∞, D_∞. In case ℓ is additive, C must be Dynkin or A_∞, and in case ℓ is unbonded, C must be of the form A_∞.

Finally, consider the case where we only have for every vertex x of Δ a number $p(x)$ depending on x with $\tau^{p(x)}(x) = x$. In particular, Γ is infinite. Choosing a finite subdiagram Γ' of Γ, and Δ' the stable Riedtmann quiver generated by Γ', we see that Γ' has to be a Dynkin diagram or a Euclidean diagram. As a consequence, Γ only can be one of A_∞, A_∞^∞, B_∞, C_∞, or D_∞.

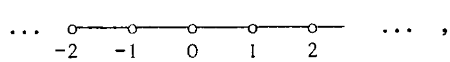
We claim that for Γ of type A_∞, A_∞^∞, B_∞, C_∞, or D_∞, any automorphism group G of $\mathbb{Z}\Gamma$ containing an element g with $g(n,x) = (n+p,x)$ for some $(n,x) \in \mathbb{Z}\Gamma$ and some $p \geq 1$, must contain a translation (an automorphism of the form $(m,y) \mapsto (m+q,y)$ for all $(m,y) \in \mathbb{Z}\Gamma$). Namely, in the cases A_∞, B_∞, C_∞, we use the following numbering



of the vertices of Γ. Any automorphism of $\mathbb{Z}\Gamma$ maps a subset of the form $\mathbb{Z}\times\{x\}$ into itself (this is clear for $x = 0$, since $\mathbb{Z}\times\{0\} = \{(n,x) \mid |(n,x)^+| = 1\}$, and follows by induction for the remaining x). If now $g(n,x) = (n+p,x)$ for some (n,x), then also all neighbours (m,y) of (n,x) will satisfy $g(m,y) = (m+p,y)$. Similarly, for D_∞, use the numbering



Then the subsets $\mathbb{Z}\times\{0,0'\}$, and $\mathbb{Z}\times\{x\}$ with $x \geq 1$ are mapped into themselves by any automorphism. If $g(n,x) = (n+p,x)$ for some (n,x), then also $g(m,y) = (m+p,y)$ for all neighbours with $y \geq 1$. If $(m,0)$ is a neighbour of $(n,1)$, and $g(n,1) = (n+p,1)$, then we only can conclude that $g^2(m,0) = (m+2p,0)$, however this then implies that g^2 is a translation. Finally consider the case A



where we may assume that $g(n,0) = (n+p,0)$, for some n,p. If $(m,1)$ is a neighbour of $(n,0)$, then either $g(m,1) = (m+p,1)$, and then g is a translation, or else $g(m,1) = (m+p,-1)$, and then at least g^2 is a translation.

As a consequence, we see that in all cases there is a fixed number q with $\tau^q(z) = z$ for all vertices z in Δ, thus we are in the previous case, and the theorem is proved.

An immediate consequence of this result is the theorem stated in the introduction: Note that the Auslander-Reiten quiver is always locally finite. Consider a component C of $A_s(R)$ containing a periodic module, and let ℓ be the ordinary length function, it clearly is subadditive. Note that ℓ is additive on C if and only if C is even a component of the complete Auslander-Reiten quiver $A(R)$. We may assume that R is connected. Now, if ℓ is not additive on C, then the Cartan class of C can only be a Dynkin diagram or A_∞, by part (b). If, on the other hand, ℓ is additive, then R cannot be of finite representation type, since there exists a component of the Auslander-Reiten quiver without projective modules, namely C. But then ℓ cannot be bounded on C, by a theorem of Auslander [1], see also [7]. Thus, we can apply (c) and see that the Cartan class of C is A_∞.

As a first application, we obtain Riedtmann's theorem [5], and its generalisation to arbitrary Artin algebras due to Todorov [9]:

<u>Corollary 1</u>. Let R be an Artin algebra of finite representation type. Let C be a connected component of $A_s(R)$. Then the Cartan class of C is a Dynkin diagram.

<u>Proof:</u> We only have to exclude the case A_∞. But this case is impossible since for any automorphism group G, $\mathbb{Z}A_\infty/G$ has infinitely many points.

As a second application, we can describe completely those components of the Auslander-Reiten quiver which contain a periodic module but no projective ones.

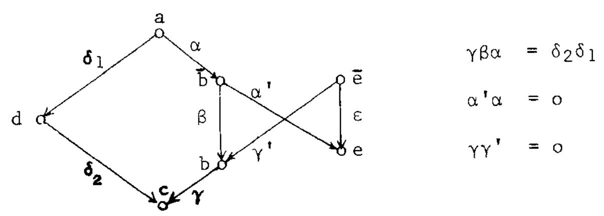
<u>Corollary 2</u>. Let R be an Artin algebra and C a connected component of $A(R)$ which contains only periodic modules. Then C is a quasi-serial component (in the sense of [6]).

Proof: Since we deal with a component of $A(R)$, the ordinary length function is additive. Thus, the Cartan class is A_∞. But this then implies that C is quasi-serial.

3. Example

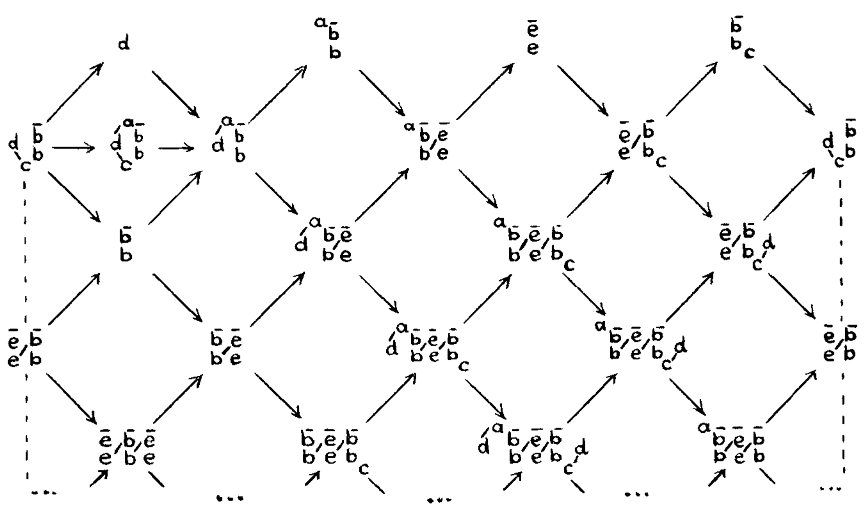
We have seen that a component of the Auslander-Reiten quiver with only periodic modules is quasi-serial. Let us exhibit the example of a component with stable part of Cartan class A_∞ containing periodic modules which is not quasi-serial.

Consider the Artin algebra R defined by the following quiver with relations



$$\gamma\beta\alpha = \delta_2\delta_1$$
$$\alpha'\alpha = 0$$
$$\gamma\gamma' = 0$$

and its component C containing the simple module corresponding to the vertex d. Then C has the following form (We denote any module by its composition factors in a suggestive way, the dotted lines have to be identified in order to form a cylinder):



Further examples can be built by using suitable regular enlargements and regular co-enlargements of tame quivers, see [8].

Remark. Note that the example above gives an algebra with infinitely many indecomposables which are both preprojective and preinjective in the sense of Auslander and Smalø. Namely, in C all modules containing the composition factor corresponding to the vertex a are preprojective, those containing the composition factor corresponding to the vertex c are preinjective.

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