INVARIANT OPTIMAL CONTROL ON THE THREE-DIMENSIONAL SEMI-EUCLIDEAN GROUP: CONTROL AFFINE AND QUADRATIC HAMILTON-POISSON SYSTEMS

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Abstract

In this thesis we consider invariant control systems and Hamilton-Poisson systems on the three-dimensional semi-Euclidean group $\mathsf{SE}(1,1)$. We first classify the left-invariant control affine systems (under detached feedback equivalence). We provide a complete list of normal forms, as well as classifying conditions. As a corollary to this classification, we derive controllability criteria for control affine systems on $\mathsf{SE}(1,1)$. Secondly, we consider quadratic Hamilton-Poisson systems on the (minus) Lie-Poisson space $\mathfrak{se}(1,1)^*_-$. These systems are classified up to an affine isomorphism. Six normal forms are identified for the homogeneous case, whereas sixteen representatives (including several infinite families) are obtained for the inhomogeneous systems. Thereafter we consider the stability and integration of the normal forms obtained. For all homogeneous systems, and a subclass of inhomogeneous systems, we perform a complete stability analysis and derive explicit expressions for all integral curves. (The extremal controls of a large class of optimal control problems on $\mathsf{SE}(1,1)$ are linearly related to these integral curves.) Lastly, we discuss the Riemannian and sub-Riemannian problems. The (left-invariant) Riemannian and sub-Riemannian structures on $\mathsf{SE}(1,1)$ are classified, up to isometric group automorphisms and scaling. Explicit expressions for the geodesics of the normalised structures are found.

Key words and phrases. semi-Euclidean group, (detached) feedback equivalence, left-invariant control affine system, Hamilton-Poisson system, Lyapunov stability, the energy-Casimir method, elliptic function, sub-Riemannian structure, optimal control.

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Introduction

Invariant geometric control theory is the study of invariant control systems evolving on Lie groups. In the language of differential geometry, a (left-)invariant control system on a Lie group G consists of a family of left-invariant vector fields $\Xi = (\Xi_u)$ on G, smoothly parametrised by controls. A trajectory of such a system is an integral curve of the (nonautonomous) vector field $\Xi_{u(t)}$, where $u(\cdot)$ is a “admissible control.” The first major consideration of control theory is the *controllability* of a system. That is, given any initial state of the system, does there exist an admissible control transforming the system into any given end state? Using the geometric tools of Lie theory, a number of power results have been developed to answer the question of controllability for invariant control systems (see, *e.g.*, [42]).

Assuming the control system under consideration is controllable, a natural question to ask is whether there exists an admissible control that transforms the system to some end state in an optimal manner. More formally, given a (left-invariant) control system, some (practical) cost function and specified boundary conditions, can one determine a control and trajectory that minimises the cost function, subject to the control system and boundary data? Invariant optimal control theory is concerned with the study of such problems, as well as the development of tools for solving them.

Recent efforts have been devoted to the study of invariant optimal control problems with affine dynamics (*i.e.*, where the underlying control system Ξ is affine), particularly on low-dimensional Lie groups. Numerous important problems can be modeled in this fashion. Such problems include the ball-plate problem [27], Euler’s elastic problem [41, 28], motion of a free rigid body [11] and the sub-Riemannian length-minimisation problem [36, 21, 42]. More general approaches (as opposed to the treatment of a specific problem) have also been considered. For instance, the classification, under (detached) feedback equivalence, of control systems on three-dimensional Lie groups [18, 13, 16, 19], the classification of control systems on $\mathsf{SE}(2)$ under state space equivalence [1] and the investigation of (detached feedback equivalence) class representatives on $\mathsf{SE}(2)$ [2, 4]. The latter approach to control systems and optimal control problems on Lie groups has been facilitated by the development of theoretical tools in [14, 20, 15, 17].

The authors of [15, 17] note that the problem of determining the extremal controls for a large class of (invariant) optimal control problems reduces to the study of the integral curves of a quadratic Hamilton-Poisson system on the dual of the Lie algebra. (Here the Poisson structure is the Lie-Poisson bracket.) A natural approach is again to classify the systems and investigate the ensuant normal forms. In this vein, quadratic Hamilton-Poisson systems have recently been considered on $\mathfrak{se}(2)^*_-$ (the dual of the Euclidean Lie algebra with the minus Lie-Poisson bracket) [3] and the orthogonal Lie-Poisson space $\mathfrak{so}(3)^*_-$ [5].

Quadratic Hamilton-Poisson systems on Lie-Poisson spaces may be considered independently of control theory. Indeed, these systems appear naturally in a variety of fields of mathematical physics. The study of such systems has received increasing attention in recent years. For instance, (spectral and Lyapunov) stability as well as (numerical and analytical) integration for systems on $\mathfrak{se}(1,1)^*_-$, $\mathfrak{se}(2)^*_-$ and $\mathfrak{so}(3)^*_-$ were treated in [9], [12] and [22], respectively. The equivalence of Hamilton-Poisson systems was considered in [22, 43, 44].

In this thesis we consider invariant optimal control on the (three-dimensional) semi-Euclidean group $\mathsf{SE}(1,1)$. We outline the topics covered. Chapter 1 is concerned with $\mathsf{SE}(1,1)$ itself. We show that $\mathsf{SE}(1,1)$ is a connected and simply connected (matrix) Lie group (in fact, the group of motions of the Minkowski plane) and determine its Lie algebra $\mathfrak{se}(1,1)$. We investigate various algebraic properties of the group, particularly as those properties pertain to control theory. We also determine the group of Lie algebra automorphisms (which shall be used for several classifications in later chapters) as well as the adjoint and coadjoint orbits (in particular, the coadjoint orbits reflect the Lie-Poisson structure on the dual space $\mathfrak{se}(1,1)^*$).

In chapter 2 we treat the left-invariant control affine systems on $\mathsf{SE}(1,1)$. The equivalence of such systems under detached feedback equivalence is shown to reduce to the equivalence of affine subspaces of $\mathfrak{se}(1,1)$ under Lie algebra automorphisms. Accordingly, we classify the affine subspaces of the semi-Euclidean Lie algebra. Class representatives, as well as classification conditions, are provided. We then reinterpret these results as a classification of control systems. As a corollary, we determine the controllable systems, thereby establishing controllability criteria for systems on $\mathsf{SE}(1,1)$.

Chapter 3 is devoted to the equivalence of (quadratic) Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{se}(1,1)^*_-$. We consider equivalence of Hamilton-Poisson systems up to affine isomorphisms, and prove several useful results for classification. Using this "affine equivalence," the homogeneous quadratic Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$ are classified. This classification is then used to obtain a classification of the inhomogeneous systems. In both cases, normal forms are identified for each equivalence class.

Chapter 4 investigates (some of) the Hamilton-Poisson representatives obtained in chapter 3. (This constitutes the main part of this thesis.) In particular, we consider all homogeneous systems and a subclass of the inhomogeneous systems. We perform a complete (Lyapunov) stability analysis of all systems under consideration. We also consider integration of the associated equations of motion for these systems. Explicit expressions for all integral curves are found. These are typically in terms of elementary functions. However, for two of the systems considered, Jacobi elliptic functions are required. A consequence of this integration is that we have obtained, up to an affine isomorphism, the extremal controls for a class of optimal control problems on $\mathsf{SE}(1,1)$.

Finally, in chapter 5 we consider some optimal control problems on $\mathsf{SE}(1,1)$. In particular, we treat the Riemannian and sub-Riemannian length-minimisation problem. (That is, we determine the Riemannian and sub-Riemannian geodesics on $\mathsf{SE}(1,1)$.) We begin this investigation by introducing a natural equivalence relation between left-invariant Riemannian and sub-Riemannian structures, *viz.* equivalence up to isometric group automorphisms and scaling. The Riemannian and sub-Riemannian structures on $\mathsf{SE}(1,1)$ are then classified under this equivalence relation. (For the sub-Riemannian case, we identify a single representative structure; a single-parameter family of structures is obtained in the Riemannian case.) We then consider the Riemannian and sub-Riemannian problem associated to the class representatives and find explicit expressions for all geodesics.

Appendix A covers the necessary prerequisites for an understanding of the results obtained in this thesis. We have also used MATHEMATICA 8 throughout to assist with calculations; the code we have written may be found in appendix B.

Original Contributions

To the best of our knowledge, the following contributions in this thesis are original:

Chapter 3. A characterisation of affine equivalence of homogeneous quadratic Hamilton-Poisson (QHP) systems in terms of linear isomorphisms (proposition 3.1.6). A necessary condition for affine equivalence of inhomogeneous QHP systems (proposition 3.1.7). A complete classification of homogeneous QHP systems on $\mathfrak{se}(1,1)^*_-$ under affine equivalence (theorem 3.2.1, corollary 3.2.2). A result permitting the normalisation of the homogeneous part of an inhomogeneous QHP system on $\mathfrak{se}(1,1)^*_-$ (proposition 3.3.1). Calculation of the linear Poisson symmetries for the normalised homogeneous QHP systems on $\mathfrak{se}(1,1)^*_-$ (proposition 3.3.2). A complete classification of inhomogeneous QHP systems on $\mathfrak{se}(1,1)^*_-$ under affine equivalence (theorems 3.3.4, 3.3.6, 3.3.8, 3.3.10, 3.3.12 and 3.3.14 and the accompanying lemmas).

Chapter 4. A sufficient condition to be an integral curve of a Hamilton-Poisson system on $\mathfrak{se}(1,1)^*_-$ (proposition 4.1.2). A complete stability analysis of the normalised quadratic Hamilton-Poisson systems H_1 through H_5, as well as all inhomogeneous systems associated to H_0, H_1, H_2 and H_3 (propositions 4.2.1, 4.2.2, 4.2.3, 4.2.5, 4.2.9, 4.3.1, 4.3.2, 4.4.1, 4.4.2, 4.4.3, 4.5.1, 4.5.2, 4.5.3, 4.6.1 and 4.6.13). Calculation of the integral curves of the aforementioned Hamilton-Poisson systems (for H_3, proposition 4.2.4; for H_4, propositions 4.2.7 and 4.2.8; for H_5, propositions 4.2.10, 4.2.11 and 4.2.12; for $H^{(0)}_{2,\alpha}$, proposition 4.3.3; for $H^{(1)}_{3,\alpha}$, proposition 4.4.4; for $H^{(2)}_{2,\delta}$, proposition 4.5.4; for $H^{(3)}_1$, propositions 4.6.4, 4.6.5, 4.6.6, 4.6.7, 4.6.8, 4.6.9, 4.6.10, 4.6.11 and 4.6.12; for $H^{(3)}_2$, propositions 4.6.15, 4.6.16, 4.6.17, 4.6.18, 4.6.19, 4.6.20, 4.6.21, 4.6.22 and 4.6.23). Linear Poisson symmetries of the Hamilton-Poisson systems H_4 and $H^{(3)}_2$ reversing the sign of the Casimir function (propositions 4.2.6 and 4.6.14).

Chapter 5. A complete classification of left-invariant sub-Riemannian structures on $\mathsf{SE}(1,1)$ under isometric group automorphisms ($\mathfrak{L}$-isometries) (theorem 5.3.1). Calculation of the (reduced) extremal curves corresponding to (unit-speed) Riemannian geodesics on $\mathsf{SE}(1,1)$ (proposition 5.2.3). Calculation of (explicit) expressions for the Riemannian geodesics associated to all (left-invariant) Riemannian structures on $\mathsf{SE}(1,1)$ (propositions 5.2.4, 5.2.5 and 5.2.6).

Notation

We briefly outline the notational conventions we shall employ. Lie groups are denoted using uppercase characters in a sans serif typeface (*e.g.*, G). Lie algebras are denoted using lowercase letters in a Fraktur typeface (*e.g.*, $\mathfrak{g}$). We shall also employ the following notation:

- $\mathbf{1}$ identity element of a Lie group.
- $\rtimes$ semidirect product of Lie groups (normal subgroup on the left).
- $C^\infty(\mathsf{M})$ the set of (smooth) real-valued functions on a smooth manifold M.
- $\mathsf{Vec}(\mathsf{M})$ the set of (smooth) vector fields on a smooth manifold M.
- $\mathsf{GL}(\mathsf{V})$ group of invertible linear transformations of a vector space V.

$\mathfrak{gl}(\mathsf{V})$ Lie algebra of $\mathsf{GL}(\mathsf{V})$.

$\langle S \rangle$ linear span of a subset $S \subseteq \mathfrak{g}$ or of elements $B_1, \ldots, B_\ell \in \mathfrak{g}$.

$\mathsf{Lie}(S)$ Lie algebra generated by $S \subseteq \mathfrak{g}$, *i.e.*, the smallest Lie subalgebra containing S.

$\langle \cdot, \cdot \rangle$ natural pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, $(p, X) \mapsto p(X)$ between $\mathfrak{g}^*$ and $\mathfrak{g}$.

$\mathbf{d}F$ linearisation of $F \in \mathsf{C}^\infty(\mathsf{M})$; the linearisation at x is denoted $\mathbf{d}F(x)$.

$T\phi$ tangent map (differential) of a smooth map ϕ between manifolds; the tangent map at x is denoted $T_x\phi$.

$\mathbf{D}X$ linearisation of $X \in \mathsf{Vec}(\mathsf{M})$; the linearisation at x is denoted $\mathbf{D}X(x)$.

$X[F]$ directional derivative of $F \in \mathsf{C}^\infty(\mathsf{M})$ in the direction of $X \in \mathsf{Vec}(\mathsf{M})$.

Chapter 1

The Semi-Euclidean Group $\mathsf{SE}(1,1)$

This chapter introduces the three-dimensional semi-Euclidean group $\mathsf{SE}(1,1)$ and investigates some of its properties. We begin by showing that $\mathsf{SE}(1,1)$ is a Lie group and that it decomposes as the semi-direct product $\mathbb{R}^2 \rtimes \mathsf{SO}(1,1)_0$. We also discuss the relationship of $\mathsf{SE}(1,1)$ to the Minkowski plane $\mathbb{R}^{1,1}$. Specifically, we show that $\mathsf{SE}(1,1)$ is the group of (orientation-preserving) motions of $\mathbb{R}^{1,1}$.

Next we consider the topological properties of $\mathsf{SE}(1,1)$, showing that it is connected, simply connected and non-compact. We then calculate the Lie algebra $\mathfrak{se}(1,1)$ and consider some algebraic properties. We show that the centres of $\mathsf{SE}(1,1)$ and $\mathfrak{se}(1,1)$ are trivial. Consequently, we have that $\mathsf{SE}(1,1)$ is the only (connected) Lie group with Lie algebra $\mathfrak{se}(1,1)$, up to Lie group isomorphisms. We also calculate the adjoint representations Ad_g, $g \in \mathsf{SE}(1,1)$ and ad_X, $X \in \mathfrak{se}(1,1)$, and prove that $\mathsf{SE}(1,1)$ is unimodular, completely solvable and not nilpotent. Lastly, we calculate the exponential map $\exp : \mathfrak{se}(1,1) \to \mathsf{SE}(1,1)$ (which turns out to be a diffeomorphism) and find the group of Lie algebra automorphisms $\mathsf{Aut}(\mathfrak{se}(1,1))$. (The latter group is used in several places in this thesis for classification; *e.g.*, for the classification of control affine systems in chapter 2.)

Lastly, the adjoint and coadjoint orbits of $\mathsf{SE}(1,1)$ are determined. The coadjoint orbits are the symplectic leaves of the minus Lie-Poisson structure on $\mathfrak{se}(1,1)^*$ (see section A.1.4 and section A.3.3). Thus, calculating the coadjoint orbits gives insight into the structure of the Lie-Poisson structure on $\mathfrak{se}(1,1)^*$. (We consider Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$ in chapter 3 and chapter 4.) In addition, having calculated the adjoint and coadjoint orbits, we are able to prove that there does not exist an invariant and nondegenerate bilinear form on $\mathfrak{se}(1,1)$.

1.1 The Lie Group $\mathsf{SE}(1,1)$

The **semi-Euclidean group** is defined as

$$\mathsf{SE}(1,1) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}.$$

(This definition is made in retrospect, after selecting the signature $(-1,1)$ for the Lorentzian inner product, and determining the affine transformations that preserve this structure. See section 1.1.1.)

1.1.1 PROPOSITION. $\mathsf{SE}(1,1)$ *is a matrix Lie group.*

PROOF. We show that $\mathsf{SE}(1,1)$ is a closed subgroup of $\mathsf{GL}(3,\mathbb{R})$. For brevity, let

$$m(x,y,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix}.$$

Then $m(x,y,\theta)^{-1} = m(w\sinh\theta - v\cosh\theta, v\sinh\theta - w\cosh\theta, -\theta)$ and $m(x,y,\theta)m(v,w,\vartheta) = m(x + v\cosh\theta + w\sinh\theta, y + w\cosh\theta + v\sinh\theta, \theta + \vartheta)$. That is, $\mathsf{SE}(1,1)$ is an abstract subgroup of $\mathsf{GL}(3,\mathbb{R})$. It remains to show that $\mathsf{SE}(1,1)$ is closed in $\mathsf{GL}(3,\mathbb{R})$. Let $(g_n)_{n\in\mathbb{N}}$, $g_n = m(x_n, y_n, \theta_n)$ be a sequence in $\mathsf{SE}(1,1)$ such that $\lim_{n\to\infty} g_n = g \in \mathsf{GL}(3,\mathbb{R})$. (If $g \notin \mathsf{GL}(3,\mathbb{R})$, then there is nothing to prove.) Suppose $x_n \to x$, $y_n \to y$ and $\theta_n \to \theta$ as $n \to \infty$. We have $x, y, \theta \in \mathbb{R}$, since $\mathbb{R}$ is closed. Consequently, $g = m(x,y,\theta) \in \mathsf{SE}(1,1)$. That is, every sequence in $\mathsf{SE}(1,1)$ that converges in $\mathsf{GL}(3,\mathbb{R})$ converges in $\mathsf{SE}(1,1)$. Therefore $\mathsf{SE}(1,1)$ is a matrix Lie group. ■

The next three results concern two distinguished subgroups of the semi-Euclidean group and show that $\mathsf{SE}(1,1)$ decomposes as a semi-direct product of those subgroups. (Section A.1.1 discusses the semi-direct product of Lie groups.)

1.1.2 LEMMA. *The* pseudo-orthogonal group $\mathsf{SO}(1,1)_0 = \left\{ \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$ *is diffeomorphic to* $\mathbb{R}$.

PROOF. Define the map $\phi : \mathbb{R} \to \mathsf{SO}(1,1)_0$, $\theta \mapsto \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$. Since $\cosh(\cdot)$ is injective, we have

$$\phi(\theta) = \phi(\vartheta) \iff \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} = \begin{bmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{bmatrix} \iff \theta = \vartheta.$$

Hence, ϕ is well-defined and injective. If $g = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \in \mathsf{SO}(1,1)_0$, then $\theta \in \mathbb{R}$ and $\phi(\theta) = g$. Therefore ϕ is bijective. Moreover, $T_x\phi$ clearly has full rank for every $x \in \mathbb{R}$. It follows from the inverse function theorem (see, *e.g.*, [33]) that $\mathsf{SO}(1,1)_0$ is diffeomorphic to $\mathbb{R}$. ■

1.1.3 PROPOSITION. *The subsets of* $\mathsf{SE}(1,1)$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} : x, y \in \mathbb{R} \right\} \quad \textit{and} \quad \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

are closed Lie subgroups of $\mathsf{SE}(1,1)$*, isomorphic to* (*the Abelian group*) $\mathbb{R}^2$ *and* (*the pseudo-orthogonal group*) $\mathsf{SO}(1,1)_0$*, respectively.*

PROOF. Let

$$\mathsf{G} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} : \theta \in \mathbb{R} \right\}, \qquad \mathsf{H} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} : \theta \in \mathbb{R} \right\}.$$

It is clear that $\mathsf{G} \cong \mathbb{R}^2$ and $\mathsf{H} \cong \mathsf{SO}(1,1)_0$, and that G and H are abstract subgroups of $\mathsf{SE}(1,1)$. As $\mathbb{R}^2$ is closed, it follows that G is a closed subgroup of $\mathsf{SE}(1,1)$. Therefore, by Cartan's theorem (theorem A.1.1), we have that G is a closed Lie subgroup of $\mathsf{SE}(1,1)$. By lemma 1.1.2, we have that $\mathsf{G} = \mathsf{SO}(1,1)_0$ is diffeomorphic to $\mathbb{R}$, and so, again by Cartan's theorem, H is a closed Lie subgroup of $\mathsf{SE}(1,1)$. ∎

1.1.4 Proposition. $\mathsf{SE}(1,1)$ *decomposes as the semi-direct product* $\mathbb{R}^2 \rtimes \mathsf{SO}(1,1)_0$.

Proof. We have that $\mathbb{R}^2$ is a normal subgroup of $\mathsf{SE}(1,1)$. Indeed,

$$\begin{bmatrix} 1 & 0 & 0 \\ v & \cosh\vartheta & \sinh\vartheta \\ w & \sinh\vartheta & \cosh\vartheta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ v & \cosh\vartheta & \sinh\vartheta \\ w & \sinh\vartheta & \cosh\vartheta \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ x\cosh\vartheta + y\sinh\vartheta & 1 & 0 \\ x\sinh\vartheta + y\cosh\vartheta & 0 & 1 \end{bmatrix} \in \mathbb{R}^2,$$

and so $\mathbb{R}^2$ is normal in $\mathsf{SE}(1,1)$. It is clear that $\mathsf{SO}(1,1)_0 \cap \mathbb{R}^2 = \{\mathbf{1}\}$. Lastly, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\vartheta & \sinh\vartheta \\ y & \sinh\vartheta & \cosh\vartheta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix},$$

and so $\mathsf{SE}(1,1) = \mathbb{R}^2\,\mathsf{SO}(1,1)_0$. (As $\mathbb{R}^2$ and $\mathsf{SO}(1,1)_0$ are both subgroups of $\mathsf{SE}(1,1)$, we have $\mathbb{R}^2\,\mathsf{SO}(1,1)_0 \subseteq \mathsf{SE}(1,1)$.) Therefore $\mathsf{SE}(1,1)$ is the semi-direct product $\mathbb{R}^2 \rtimes \mathsf{SO}(1,1)_0$. ∎

1.1.1 Group of motions of the Minkowski plane

In this section we show that $\mathsf{SE}(1,1)$ is the group of orientation-preserving motions of the Minkowski plane. (In fact, $\mathsf{SE}(1,1)$ is the group of hyperbolic rotations and translations.) We first recall some concepts from Lorentzian (*i.e.*, Minkowski) geometry. The following exposition draws from [40, 23].

The **Minkowski plane** $\mathbb{R}^{1,1}$ is the pair $(\mathbb{R}^2, \odot)$, where $\odot$ denotes the **Lorentzian inner product**:

$$\mathbf{x} \odot \mathbf{y} = -x_1 y_1 + x_2 y_2, \qquad \mathbf{x} = (x_1, x_2),\ \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2.$$

The **Lorentzian norm** of $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ is defined as

$$\|\mathbf{x}\|_{\mathsf{Lor}} = \sqrt{\mathbf{x} \odot \mathbf{x}} = \sqrt{-x_1^2 + x_2^2}\,.$$

(Strictly speaking, the Lorentzian norm is not a norm, as $\|\cdot\|_{\mathsf{Lor}} \not\geq 0$. However, given the obvious analogy between Minkowski and Euclidean space, similar terminology is typically employed.) The norm $\|\mathbf{x}\|_{\mathsf{Lor}}$ can be positive, zero, or positive imaginary. A vector $\mathbf{x}$ is called **spacelike** if $\|\mathbf{x}\|_{\mathsf{Lor}} > 0$, **lightlike** if $\|\mathbf{x}\|_{\mathsf{Lor}} = 0$ and **timelike** if $\|\mathbf{x}\|_{\mathsf{Lor}}$ is imaginary. A timelike or lightlike vector $\mathbf{x} = (x_1, x_2)$ is called **positive** (resp. **negative**) if $x_1 > 0$ (resp. $x_1 < 0$).

A **Lorentz transformation** is a bijective map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves the Lorentzian norm, *i.e.*, $\|\phi(\mathbf{x})\|_{\mathsf{Lor}} = \|\mathbf{x}\|_{\mathsf{Lor}}$ for every $\mathbf{x} \in \mathbb{R}^2$. Every Lorentz transformation is linear (cf.

[40]). We shall call ϕ a **Lorentz isometry** if it is bijective and $\|\phi(\mathbf{x})-\phi(\mathbf{y})\|_{\mathsf{Lor}} = \|\mathbf{x}-\mathbf{y}\|_{\mathsf{Lor}}$ for every $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$. (As $\|\cdot\|_{\mathsf{Lor}}$ is not a proper norm, Lorentz isometries are not distance-preserving maps in the usual sense. The terminology is again inspired by the analogy with Euclidean space.) A Lorentz transformation is a Lorentz isometry. The converse, however, does not hold in general. Furthermore, it turns out that all Lorentz isometries are affine maps.

1.1.5 PROPOSITION. *Every Lorentz transformation is a Lorentz isometry. Every Lorentz isometry fixing the origin is a Lorentz transformation.*

PROOF. Let ϕ be a Lorentz transformation. Since ϕ is linear, for every $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$ we have $\|\phi(\mathbf{x})-\phi(\mathbf{y})\|_{\mathsf{Lor}} = \|\phi(\mathbf{x}-\mathbf{y})\|_{\mathsf{Lor}} = \|\mathbf{x}-\mathbf{y}\|_{\mathsf{Lor}}$. Hence ϕ is a Lorentz isometry. Let ψ be a Lorentz isometry such that $\psi(\mathbf{0}) = \mathbf{0}$. We have, for every $\mathbf{x} \in \mathbb{R}^2$, $\|\psi(\mathbf{x})\|_{\mathsf{Lor}} = \|\psi(\mathbf{x}) - \psi(\mathbf{0})\|_{\mathsf{Lor}} = \|\mathbf{x}-\mathbf{0}\|_{\mathsf{Lor}} = \|\mathbf{x}\|_{\mathsf{Lor}}$. Thus ψ is a Lorentz transformation. ■

1.1.6 PROPOSITION. *Every Lorentz isometry is affine.*

PROOF. Let ϕ be a Lorentz isometry such that $\phi(\mathbf{0}) = \mathbf{b}$. Define $\psi(\mathbf{x}) = \phi(\mathbf{x}) - \mathbf{b}$. Then we have $\psi(\mathbf{0}) = \phi(\mathbf{0}) - \mathbf{b} = \mathbf{0}$. Moreover, $\|\psi(\mathbf{x}) - \psi(\mathbf{y})\|_{\mathsf{Lor}} = \|\phi(\mathbf{x}) - \mathbf{b} - \phi(\mathbf{y}) + \mathbf{b}\|_{\mathsf{Lor}} = \|\phi(\mathbf{x}) - \phi(\mathbf{y})\|_{\mathsf{Lor}} = \|\mathbf{x}-\mathbf{y}\|_{\mathsf{Lor}}$. That is, ψ is a Lorentz isometry fixing the origin, and so by proposition 1.1.5 it is linear. Therefore $\psi(\mathbf{x}) = A\mathbf{x}$ for some $A \in \mathsf{GL}(2,\mathbb{R})$. It follows that $\phi(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, *i.e.*, ϕ is affine. ■

A Lorentz isometry $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ is said to be an **orientation-preserving motion of $\mathbb{R}^{1,1}$** if $\det A = 1$ and $\mathbf{x} \mapsto A\mathbf{x}$ transforms positive timelike vectors into positive timelike vectors.

1.1.7 THEOREM. *The semi-Euclidean group* $\mathsf{SE}(1,1)$ *is exactly the group of orientation-preserving motions of the Minkowski plane.*

PROOF. Let $\phi : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ be an orientation-preserving motion of $\mathbb{R}^{1,1}$. Since $\det A = 1$ and $\mathbf{x} \mapsto A\mathbf{x}$ preserves positive timelike vectors, it follows that $A \in \mathsf{SO}(1,1)_0$. (Indeed, $\mathsf{SO}(1,1)_0$ is typically *defined* to be the group of such maps; cf. [40].) Identifying elements $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ with the column vector $\begin{bmatrix}1 & \mathbf{x}\end{bmatrix}^\top = \begin{bmatrix}1 & x_1 & x_2\end{bmatrix}^\top$, we can write ϕ in matrix form as

$$\phi = \begin{bmatrix}1 & 0\\ \mathbf{b} & A\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ b_1 & \cosh\theta & \sinh\theta\\ b_2 & \sinh\theta & \cosh\theta\end{bmatrix}.$$

Thus $\phi \in \mathsf{SE}(1,1)$. Conversely, every element of $\mathsf{SE}(1,1)$ is of this form. Therefore $\mathsf{SE}(1,1)$ is the group of orientation-preserving motions of $\mathbb{R}^{1,1}$. ■

1.1.2 Topological properties of $\mathsf{SE}(1,1)$

We show that $\mathsf{SE}(1,1)$ is diffeomorphic to $\mathbb{R}^3$. It follows that $\mathsf{SE}(1,1)$ inherits the topological properties of $\mathbb{R}^3$.

1.1.8 PROPOSITION. $\mathsf{SE}(1,1)$ *is diffeomorphic to* $\mathbb{R}^3$.

PROOF. By lemma 1.1.2, $\mathsf{SO}(1,1)_0$ is diffeomorphic to $\mathbb{R}$. Since $\mathsf{SE}(1,1) = \mathbb{R}^2 \rtimes \mathsf{SO}(1,1)_0$, we have that $\mathsf{SE}(1,1)$ is diffeomorphic to $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. ■

1.1.9 COROLLARY. $\mathsf{SE}(1,1)$ *is connected, simply connected and non-compact.*

PROOF. By proposition 1.1.8, $\mathsf{SE}(1,1)$ is diffeomorphic to $\mathbb{R}^3$. Since the topological properties of connectedness, simply connectedness and compactness are preserved by diffeomorphisms, the result follows from the corresponding properties of $\mathbb{R}^3$. ■

1.1.3 Algebraic properties of $\mathsf{SE}(1,1)$

1.1.10 THEOREM. *The Lie algebra of* $\mathsf{SE}(1,1)$ *is*

$$\mathfrak{se}(1,1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} : x, y, \theta \in \mathbb{R} \right\}.$$

PROOF. For brevity, let

$$m(x,y,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} \qquad \text{and} \qquad M(x,y,\theta) = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix}.$$

Let $\mathfrak{g} = \{M(x,y,\theta) : x,y,\theta \in \mathbb{R}\}$. Consider a smooth curve $g(\cdot) : (-\varepsilon, \varepsilon) \to \mathsf{SE}(1,1)$ defined by $g(t) = m(x(t), y(t), \theta(t))$, where $x(0) = y(0) = \theta(0) = 0$. We have $g(0) = \mathbf{1}$, and so $\dot{g}(0) \in T_{\mathbf{1}}\mathsf{SE}(1,1) = \mathfrak{se}(1,1)$. However, $\dot{g}(0) = M(\dot{x}(0), \dot{y}(0), \dot{\theta}(0)) \in \mathfrak{g}$. Thus $\mathfrak{se}(1,1) \subseteq \mathfrak{g}$. For the converse, let $X = M(x,y,\theta) \in \mathfrak{g}$. Then $h(\cdot) : t \mapsto m(xt, yt, \theta t)$ is a smooth curve in $\mathsf{SE}(1,1)$ such that $h(0) = \mathbf{1}$ and $\dot{h}(0) = X$. That is, $X \in \mathfrak{se}(1,1)$, and so $\mathfrak{g} \subseteq \mathfrak{se}(1,1)$. Hence $\mathsf{se}(1,1) = \mathfrak{g}$. ■

Define the **standard (ordered) basis** of $\mathfrak{se}(1,1)$ to be $(E_i)_{i=1}^3$, where

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(E_1 and E_2 are the infinitesimal generators of translations and E_3 the infinitesimal generator of hyperbolic rotations.) The commutator relations (in terms of the matrix commutator $[X,Y] = XY - YX$) of the standard basis elements are given in table 1.1.

We denote by $(E_i^*)_{i=1}^3$ the dual basis for $\mathfrak{se}(1,1)^*$. That is, each element E_i^*, $i = 1,2,3$ is defined by $\langle E_i^*, E_j \rangle = \delta_{ij}$, $i,j = 1,2,3$. We shall write elements of $\mathfrak{se}(1,1)$ in coordinates as column vectors. On the other hand, elements of the dual space $\mathfrak{se}(1,1)^*$ will be written as row vectors.

1.1.11 PROPOSITION. *The centres* $\mathsf{Z}(\mathsf{SE}(1,1))$ *and* $\mathsf{Z}(\mathfrak{se}(1,1))$ *are trivial.*

PROOF. We first show that $\mathsf{Z}(\mathsf{SE}(1,1))$ is trivial. Let $g \in \mathsf{Z}(\mathsf{SE}(1,1))$. Then $ghg^{-1}h^{-1} = \mathbf{1}$ for every $h \in \mathsf{SE}(1,1)$. In particular,

$$\begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \mathbf{1}.$$

$[\cdot,\cdot]$	E_1	E_2	E_3
E_1	0	0	$-E_2$
E_2	0	0	$-E_1$
E_3	E_2	E_1	0

Table 1.1: Commutator relations for the standard basis $(E_i)_{i=1}^3$ of $\mathfrak{se}(1,1)$.

That is,

$$\begin{bmatrix} 1 & 0 & 0 \\ \cosh\theta - 1 & 1 & 0 \\ \sinh\theta & 0 & 1 \end{bmatrix} = \mathbf{1}.$$

This implies that $\theta = 0$. Next, we must have

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\vartheta & \sinh\vartheta \\ 0 & \sinh\vartheta & \cosh\vartheta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\vartheta & \sinh\vartheta \\ 0 & \sinh\vartheta & \cosh\vartheta \end{bmatrix}^{-1} = \mathbf{1}.$$

That is,

$$\begin{bmatrix} 1 & 0 & 0 \\ (1-\cosh\vartheta)x - y\sinh\vartheta & 1 & 0 \\ (1-\cosh\vartheta)y - x\sinh\vartheta & 0 & 1 \end{bmatrix} = \mathbf{1}.$$

The only solution to this equation, for every value of $\vartheta \in \mathbb{R}$, is $x = y = 0$. Thus $g = \mathbf{1}$, and so $\mathsf{Z}(\mathsf{SE}(1,1)) = \{\mathbf{1}\}$.

By proposition A.1.5 we have that $\mathsf{Z}(\mathfrak{se}(1,1))$ is the Lie algebra of $\mathsf{Z}(\mathsf{SE}(1,1))$. Since the latter subgroup is trivial, it follows that $\mathsf{Z}(\mathfrak{se}(1,1))$ is trivial, *i.e.*, $\mathsf{Z}(\mathfrak{se}(1,1)) = \{0\}$. ■

1.1.12 THEOREM. *The only connected Lie group (up to isomorphism) with Lie algebra* $\mathfrak{se}(1,1)$ *is* $\mathsf{SE}(1,1)$.

PROOF. Let G be a connected Lie group with Lie algebra $\mathfrak{se}(1,1)$. By theorem A.1.12, we have that G is isomorphic (as a Lie group) to $\mathsf{SE}(1,1)/\mathsf{N}$, where N is a discrete normal subgroup of $\mathsf{SE}(1,1)$. (As $\mathsf{SE}(1,1)$ is simply connected, the universal cover $\widetilde{\mathsf{SE}}(1,1)$ is $\mathsf{SE}(1,1)$ itself.) By proposition A.1.11, N is a subgroup of the centre $\mathsf{Z}(\mathsf{SE}(1,1)) = \{\mathbf{1}\}$, and so N is trivial. Thus G is isomorphic to $\mathsf{SE}(1,1)$. ■

1.1.13 PROPOSITION. *In terms of the standard basis* $(E_i)_{i=1}^3$, *the adjoint representations of* $\mathsf{SE}(1,1)$ *and* $\mathfrak{se}(1,1)$ *are*

$$\mathrm{Ad}_g = \begin{bmatrix} \cosh\theta & \sinh\theta & -y \\ \sinh\theta & \cosh\theta & -x \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{ad}_X = \begin{bmatrix} 0 & \theta & -y \\ \theta & 0 & -x \\ 0 & 0 & 0 \end{bmatrix},$$

respectively. Here

$$g = \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} \in \mathsf{SE}(1,1) \quad \text{and} \quad X = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & \theta \\ y & \theta & 0 \end{bmatrix} \in \mathfrak{se}(1,1).$$

PROOF. In terms of the ordered basis $(E_i)_{i=1}^3$, the i^{th} column of the matrix Ad_g is the image of E_i under Ad_g, written in coordinates. We have

$$\text{Ad}_g E_1 = gE_1g^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ \cosh\theta & 0 & 0 \\ \sinh\theta & 0 & 0 \end{bmatrix} = \cosh\theta E_1 + \sinh\theta E_2$$

$$\text{Ad}_g E_2 = gE_2g^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ \sinh\theta & 0 & 0 \\ \cosh\theta & 0 & 0 \end{bmatrix} = \sinh\theta E_1 + \cosh\theta E_2$$

$$\text{Ad}_g E_3 = gE_3g^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ -y & 0 & 1 \\ -x & 1 & 0 \end{bmatrix} = -yE_1 - xE_2 + E_3.$$

Therefore the matrix Ad_g takes the required form. Next, let

$$g(\cdot) : (-\varepsilon, \varepsilon) \to \mathsf{SE}(1,1), \qquad t \mapsto \begin{bmatrix} 1 & 0 & 0 \\ v(t) & \cosh\vartheta(t) & \sinh\vartheta(t) \\ w(t) & \sinh\vartheta(t) & \cosh\vartheta(t) \end{bmatrix}$$

be a curve in $\mathsf{SE}(1,1)$ such that $g(0) = \mathbf{1}$ and $\dot{g}(0) = X$. Since ad is the linearisation of Ad, we have

$$\text{ad}_X = \left.\frac{d}{dt}\right|_{t=0} \text{Ad}_{g(t)} = \begin{bmatrix} 0 & \dot{\vartheta}(0) & -\dot{w}(0) \\ \dot{\vartheta}(0) & 0 & -\dot{v}(0) \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \theta & -y \\ \theta & 0 & -x \\ 0 & 0 & 0 \end{bmatrix}.$$ ∎

1.1.14 PROPOSITION. *The semi-Euclidean group* $\mathsf{SE}(1,1)$ *and its Lie algebra* $\mathfrak{se}(1,1)$ *are*

(*i*) *not nilpotent*;

(*ii*) *completely solvable*;

(*iii*) *exponential*;

(*iv*) *solvable*;

(*v*) *not simple*; *and,*

(*vi*) *not semisimple.*

Furthermore, $\mathsf{SE}(1,1)$ *is unimodular.*

PROOF. It suffices to prove the first six properties for $\mathfrak{se}(1,1)$, since $\mathsf{SE}(1,1)$ is connected and simply connected (by corollary 1.1.9) and thus shares the same properties. (See section A.1.5.)

The spectrum of ad_X, $X = xE_1 + yE_2 + \theta E_3 \in \mathfrak{se}(1,1)$ is $(0, -\theta, \theta)$. Hence the spectrum of ad_{E_3} is nonzero, and so $\mathfrak{se}(1,1)$ is not nilpotent. On the other hand, since the eigenvalues of ad_X are real for every $X \in \mathfrak{se}(1,1)$, we have that $\mathfrak{se}(1,1)$ is completely solvable. This implies (proposition A.1.15) that $\mathfrak{se}(1,1)$ is exponential and solvable. By proposition 1.1.3 and proposition 1.1.4, $\mathbb{R}^2$ is a normal closed Lie subgroup of $\mathsf{SE}(1,1)$. Consequently, $T_{\mathbf{1}}\mathbb{R}^2 \cong \langle E_1, E_2 \rangle$ is an ideal of $\mathfrak{se}(1,1)$. Thus $\mathfrak{se}(1,1)$ is not simple. Since $\mathfrak{se}(1,1)$ is solvable, by proposition A.1.17 it cannot be semisimple.

Finally, from proposition 1.1.13, we have $\text{tr}(\text{ad}_X) = 0$ for every $X \in \mathfrak{se}(1,1)$. Therefore $\mathsf{SE}(1,1)$ is unimodular (see proposition A.1.16). ∎

1.1.15 PROPOSITION. *The exponential map* $\exp : \mathfrak{se}(1,1) \to \mathsf{SE}(1,1)$ *is given by*

$$\exp X = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\theta}\left[x \sinh\theta + y(\cosh\theta - 1)\right] & \cosh\theta & \sinh\theta \\ \frac{1}{\theta}\left[y \sinh\theta + x(\cosh\theta - 1)\right] & \sinh\theta & \cosh\theta \end{bmatrix}$$

where $X = xE_1 + yE_2 + \theta E_3 \in \mathfrak{se}(1,1)$. *(If* $\theta = 0$*, then* $\exp X$ *may be obtained by taking the limit* $\theta \to 0$.*)*

PROOF. The result follows by using the series expansion of the matrix exponential, *viz.*

$$\exp X = \mathbf{1} + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots.$$

(We used MATHEMATICA to compute this series expansion. See section B.1 for the code.) If $\theta \neq 0$, we get the expression given in the statement of the proposition. If $\theta = 0$, then

$$\exp X = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix}.$$

This is exactly the limit $\theta \to 0$ of the general expression for $\exp X$. ■

1.1.16 PROPOSITION. *The exponential map* $\exp : \mathfrak{se}(1,1) \to \mathsf{SE}(1,1)$ *is a diffeomorphism.*

PROOF. From proposition 1.1.14, we have that $\mathfrak{se}(1,1)$ is exponential, *i.e.*, the exponential map is a diffeomorphism. (See section A.1.5.) ■

1.1.17 PROPOSITION. *In terms of the standard basis* $(E_i)_{i=1}^3$*, the automorphism group of* $\mathfrak{se}(1,1)$ *is*

$$\mathsf{Aut}(\mathfrak{se}(1,1)) = \left\{ \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} : v, w, x, y \in \mathbb{R},\ \varsigma \in \{-1, 1\},\ x^2 \neq y^2 \right\}.$$

PROOF. Let $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ and write ψ in terms of $(E_i)_{i=1}^3$ as

$$\psi = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Since ψ is an automorphism, we have $\psi \cdot [E_i, E_j] = [\psi \cdot E_i, \psi \cdot E_j]$ for every $i, j = 1, 2, 3$. In particular, we have

$$\psi \cdot [E_2, E_3] = [\psi \cdot E_2, \psi \cdot E_3] \qquad \iff \qquad \begin{bmatrix} -a_1 - b_3c_2 + b_2c_3 \\ -b_1 - a_3c_2 + a_2c_3 \\ -c_1 \end{bmatrix} = 0.$$

This implies that $c_1 = 0$, $a_1 = b_2c_3 - b_3c_2$ and $b_1 = a_2c_3 - a_3c_2$. Next, using $\psi \cdot [E_1, E_3] = [\psi \cdot E_1, \psi \cdot E_3]$, we get

$$\begin{bmatrix} a_2(c_3^2 - 1) - a_3c_2c_3 \\ b_2(c_3^2 - 1) - b_3c_2c_3 \\ -c_2 \end{bmatrix} = 0.$$

Hence $c_2 = 0$ and $a_2(c_3^2 - 1) = b_2(c_3^2 - 1) = 0$. Since $\det \psi = c_3^2(b_2^2 - a_2^2)$, we cannot have both a_2 and b_2 zero, and so $c_3 = \pm 1$. That is,

$$\psi = \begin{bmatrix} \pm b_2 & a_2 & a_3 \\ \pm a_2 & b_2 & b_3 \\ 0 & 0 & \pm 1 \end{bmatrix}.$$

Setting $v = a_3$, $w = b_3$, $x = \pm b_2$, $y = a_2$ and $\varsigma = \pm 1$ yields

$$\psi = \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}, \tag{1.1.1}$$

where $\det \psi = x^2 - y^2 \neq 0$. Conversely, every linear isomorphism of this form preserves the Lie bracket. (See the supporting Mathematica code in section B.1.1.) Hence, every element of $\mathsf{Aut}(\mathfrak{se}(1,1))$ is of the form (1.1.1), as claimed. ∎

1.1.18 Proposition. *The elements $E_1 + E_2$ and $E_1 - E_2$ are eigenvectors of every automorphism of the form*

$$\begin{bmatrix} x & y & v \\ y & x & w \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.1.2}$$

Furthermore, $\langle E_1, E_2 \rangle$ is an invariant subspace and $\langle E_1 + E_2 \rangle \cup \langle E_1 - E_2 \rangle$ is an invariant subset of every automorphism.

Proof. Let

$$\psi = \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \in \mathsf{Aut}(\mathfrak{se}(1,1)).$$

We have $\psi \cdot (E_1 + E_2) = (x+y)(E_1 + \varsigma E_2)$ and $\psi \cdot (E_1 - E_2) = (x-y)(E_1 - \varsigma E_2)$. Consequently, $\langle E_1 + E_2 \rangle \cup \langle E_1 - E_2 \rangle$ is an invariant subset and $\langle E_1, E_2 \rangle$ is an invariant subspace of ψ, and hence of every automorphism. Furthermore, if ψ is of the form (1.1.2), then $\varsigma = 1$, and so $E_1 + E_2$, $E_1 - E_2$ are eigenvectors of ψ. ∎

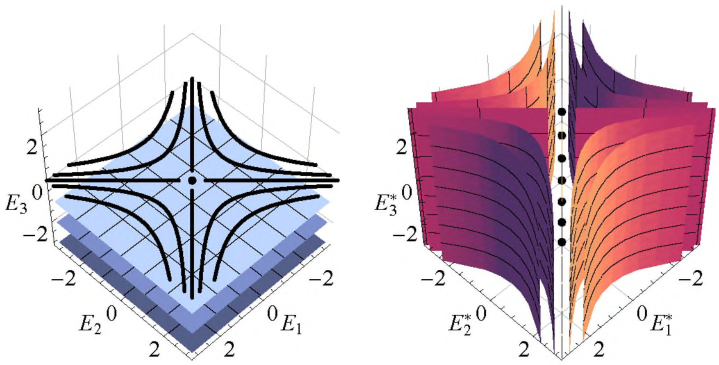
1.2 Adjoint and Coadjoint Orbits

We (briefly) recall the necessary concepts of the adjoint and coadjoint representations. (For further details, see section A.1.4.) Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. The **adjoint representation of** G is the map $\mathrm{Ad} : \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$, where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{Ad}_g X = gXg^{-1}$. The **coadjoint representation of** G is the map $\mathrm{Ad}^* : \mathsf{G} \to \mathsf{GL}(\mathfrak{g}^*)$, $g \mapsto \mathrm{Ad}^*_{g^{-1}}$ where $\mathrm{Ad}^*_{g^{-1}}$ is defined by the equation

$$\langle \mathrm{Ad}^*_{g^{-1}} p, X \rangle = \langle p, \mathrm{Ad}_{g^{-1}} X \rangle, \qquad p \in \mathfrak{g}, \ X \in \mathfrak{g}.$$

(That is, $\mathrm{Ad}^*_{g^{-1}}$ is the dual of $\mathrm{Ad}_{g^{-1}}$.) The **adjoint orbit** through an element $X \in \mathfrak{g}$ is the set $\mathfrak{Orb}(X) = \{\mathrm{Ad}_g X : g \in \mathsf{G}\}$. Similarly, the **coadjoint orbit** through $p \in \mathfrak{g}^*$ is defined as $\mathfrak{orb}(p) = \{\mathrm{Ad}^*_{g^{-1}} p : g \in \mathsf{G}\}$.

The following two results determine the adjoint and coadjoint orbits of $\mathsf{SE}(1,1)$. Section B.1.2 and section B.1.3 list the supporting Mathematica code. The adjoint and coadjoint orbits are graphed in figure 1.1.



(a) Adjoint orbits (b) Coadjoint orbits

Figure 1.1: Adjoint and coadjoint orbits of $\mathsf{SE}(1,1)$

1.2.1 Proposition. *We have the following adjoint orbits of* $\mathsf{SE}(1,1)$ *through* $X = xE_1 + yE_2 + \theta E_3 \in \mathfrak{se}(1,1)$:

(*i*) *If* $X = 0$, *then the orbit of* X *is trivial,* i.e., $\mathfrak{Orb}(X) = \{0\}$.

(*ii*) *If* $\theta = 0$, $x^2 + y^2 \neq 0$ *and* $x^2 \neq y^2$, *then the orbit of* X *is the hyperbola* $\mathfrak{Orb}(X) = \{(x\cosh\vartheta + y\sinh\vartheta)E_1 + (x\sinh\vartheta + y\cosh\vartheta)E_2 : \vartheta \in \mathbb{R}\}$.

(*iii*) *If* $\theta = 0$, $x^2 + y^2 \neq 0$ *and*

(*a*) $x - y = 0$, *then the orbit of* X *is the ray* $\mathfrak{Orb}(X) = \{t\,\mathrm{sgn}(x)(E_1 + E_2) : t > 0\}$.

(*b*) $x + y = 0$, *then the orbit of* X *is the ray* $\mathfrak{Orb}(X) = \{t\,\mathrm{sgn}(x)(E_1 - E_2) : t > 0\}$.

(*iv*) *If* $\theta \neq 0$, *then the orbit of* X *is the* (*affine*) *plane* $\mathfrak{Orb}(X) = \theta E_3 + \langle E_1, E_2\rangle$.

Proof. If $X = 0$, then $\mathrm{Ad}_g X = 0$ for every $g \in \mathsf{SE}(1,1)$, and so $\mathfrak{Orb}(X) = \{0\}$. Suppose $\theta = 0$ and $x^2 + y^2 \neq 0$. Let

$$g = \begin{bmatrix} 1 & 0 & 0 \\ v & \cosh\vartheta & \sinh\vartheta \\ w & \sinh\vartheta & \cosh\vartheta \end{bmatrix} \in \mathsf{SE}(1,1).$$

If $x^2 \neq y^2$, then

$$\mathrm{Ad}_g X = \begin{bmatrix} x\cosh\vartheta + y\sinh\vartheta \\ x\sinh\vartheta + y\cosh\vartheta \\ 0 \end{bmatrix}.$$

Thus the orbit of X is the hyperbola $\mathfrak{Orb}(X) = \{(x\cosh\vartheta + y\sinh\vartheta)E_1 + (x\sinh\vartheta + y\cosh\vartheta)E_2 : \vartheta \in \mathbb{R}\}$. If $x - y = 0$, then

$$\mathrm{Ad}_g X = \begin{bmatrix} x(\cosh\vartheta + \sinh\vartheta) \\ x(\sinh\vartheta + \cosh\vartheta) \\ 0 \end{bmatrix}.$$

The expression $\cosh\vartheta + \sinh\vartheta$, $\vartheta \in \mathbb{R}$ takes any strictly positive real value. Consequently, the orbit of X is the ray $\mathfrak{Orb}(X) = \{\operatorname{sgn}(x)t(E_1 + E_2) : t > 0\}$. If $x + y = 0$, then

$$\operatorname{Ad}_g X = \begin{bmatrix} x(\cosh\vartheta - \sinh\vartheta) \\ x(\sinh\vartheta - \cosh\vartheta) \\ 0 \end{bmatrix},$$

and since $\cosh\vartheta - \sinh\vartheta$, $\vartheta \in \mathbb{R}$ again takes any strictly positive real value, we have $\mathfrak{Orb}(X) = \{\operatorname{sgn}(x)t(E_1 - E_2) : t > 0\}$.

Suppose $\theta \neq 0$. Then

$$\operatorname{Ad}_g X = \begin{bmatrix} x\cosh\vartheta + y\sinh\vartheta - w\theta \\ x\sinh\vartheta + y\cosh\vartheta - v\theta \\ \theta \end{bmatrix}.$$

Since $\theta \neq 0$, the expressions $x\cosh\vartheta + y\sinh\vartheta - w\theta$ and $x\sinh\vartheta + y\cosh\vartheta - v\theta$ (for $v, w, \vartheta \in \mathbb{R}$) take any value in $\mathbb{R}$. Hence, the orbit through X is the (affine) plane $\mathfrak{Orb}(X) = \theta E_3 + \langle E_1, E_2\rangle$. Since the preceding calculations account for any possible value of X, the classification of adjoint orbits is complete. ■

1.2.2 PROPOSITION. *We have the following coadjoint orbits of* $\mathsf{SE}(1,1)$ *through* $p = xE_1^* + yE_2^* + \theta E_3^* \in \mathfrak{se}(1,1)^*$:

(*i*) *If* $x = y = 0$, *then the orbit of* p *is the point* $\mathfrak{orb}(p) = \{\theta E_3^*\}$.

(*ii*) *If* $x^2 + y^2 \neq 0$ *and* $x^2 \neq y^2$, *then the orbit of* p *is the hyperbolic cylinder* $\mathfrak{orb}(p) = \{(x\cosh\vartheta + y\sinh\vartheta)E_1^* + (x\sinh\vartheta + y\cosh\vartheta)E_2^* + tE_3^* : \vartheta, t \in \mathbb{R}\}$.

(*iii*) *If* $x^2 + y^2 \neq 0$ *and*

(*a*) $x - y = 0$, *then the orbit of* p *is the plane* $\mathfrak{orb}(p) = \{s\operatorname{sgn}(x)(E_1^* + E_2^*) + tE_3^* : s > 0,\ t \in \mathbb{R}\}$.

(*b*) $x + y = 0$, *then the orbit of* p *is the plane* $\mathfrak{orb}(p) = \{s\operatorname{sgn}(x)(E_1^* - E_2^*) + tE_3^* : s > 0,\ t \in \mathbb{R}\}$.

PROOF. From proposition 1.1.13, we have

$$\operatorname{Ad}_{g^{-1}} = \begin{bmatrix} \cosh\vartheta & \sinh\vartheta & -w \\ \sinh\vartheta & \cosh\vartheta & -v \\ 0 & 0 & 1 \end{bmatrix}, \qquad g^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ v & \cosh\vartheta & \sinh\vartheta \\ w & \sinh\vartheta & \cosh\vartheta \end{bmatrix} \in \mathsf{SE}(1,1).$$

By definition $\langle \operatorname{Ad}_{g^{-1}}^* p, X\rangle = \langle p, \operatorname{Ad}_{g^{-1}} X\rangle$ for any $p \in \mathfrak{se}(1,1)^*$ and $X \in \mathfrak{se}(1,1)^*$. Thus we have (in matrix form) $\operatorname{Ad}_{g^{-1}}^* p = p \cdot \operatorname{Ad}_{g^{-1}}$.

If $x = y = 0$, then $\operatorname{Ad}_{g^{-1}}^* p = \theta E_3^*$ for every $g \in \mathsf{SE}(1,1)$, and so $\mathfrak{orb}(p) = \{\theta E_3^*\}$. Suppose $x^2 + y^2 \neq 0$. If $x^2 \neq y^2$, then

$$\operatorname{Ad}_{g^{-1}}^* p = \begin{bmatrix} x\cosh\vartheta + y\sinh\vartheta \\ x\sinh\vartheta + y\cosh\vartheta \\ \theta - wx - vy \end{bmatrix}^\top.$$

Since $\theta - wx - vy$, $v, w \in \mathbb{R}$ takes any value in $\mathbb{R}$, it follows that the orbit of p is the hyperbolic cylinder $\mathfrak{orb}(p) = \{(x\cosh\vartheta + y\sinh\vartheta)E_1^* + (x\sinh\vartheta + y\cosh\vartheta)E_2^* + tE_3^* : \vartheta, t \in \mathbb{R}\}$. If $x - y = 0$, then

$$\mathrm{Ad}^*_{g^{-1}}\, p = \begin{bmatrix} x(\cosh\vartheta + \sinh\vartheta) \\ x(\cosh\vartheta + \sinh\vartheta) \\ \theta - wx - vx \end{bmatrix}^\top .$$

The expression $\theta - wx - vy$ (for $v, w \in \mathbb{R}$) takes any real value, and $\cosh\vartheta + \sinh\vartheta$, $\vartheta \in \mathbb{R}$ takes any value strictly greater than zero. Consequently, the orbit through p is the plane $\mathfrak{orb}(p) = \{s\,\mathrm{sgn}(x)(E_1^* + E_2^*) + tE_3^* : s > 0,\ t \in \mathbb{R}\}$. Similarly, if $x + y = 0$, then

$$\mathrm{Ad}^*_{g^{-1}}\, p = \begin{bmatrix} x(\cosh\vartheta - \sinh\vartheta) \\ x(\sinh\vartheta - \cosh\vartheta) \\ \theta - wx + vx \end{bmatrix}^\top .$$

Here $\theta - wx + vx$, $v, w \in \mathbb{R}$ (resp. $\cosh\vartheta - \sinh\vartheta$, $\vartheta \in \mathbb{R}$) takes any value in $\mathbb{R}$ (resp. any strictly positive value in $\mathbb{R}$), and so $\mathfrak{orb}(p) = \{s\,\mathrm{sgn}(x)(E_1^* - E_2^*) + tE_3^* : s > 0,\ t \in \mathbb{R}\}$. These cases account for every possible value of p, and so the classification of coadjoint orbits is complete. ∎

Recall that a **nondegenerate invariant bilinear form** on the Lie algebra $\mathfrak{g}$ of a connected Lie group G is a bilinear form $\mathcal{B}(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that $\mathcal{B}(X, \cdot) = 0$ implies $X = 0$ and $\mathcal{B}([X,Y],Z) + \mathcal{B}(X,[Y,Z]) = 0$ for every $X, Y, Z \in \mathfrak{g}$. The existence of such a form implies that the adjoint and coadjoint orbits of G are linearly related (proposition A.1.14). Having calculated the adjoint and coadjoint orbits of $\mathsf{SE}(1,1)$, we are able to show that $\mathfrak{se}(1,1)$ does not admit any such bilinear form.

1.2.3 PROPOSITION. *There does not exist a nondegenerate bilinear form on* $\mathfrak{se}(1,1)$.

PROOF. Suppose otherwise. Then by proposition A.1.14 there exists a linear isomorphism $\psi : \mathfrak{se}(1,1) \to \mathfrak{se}(1,1)^*$ such that $\psi \cdot \mathfrak{Orb}(X) = \mathfrak{orb}(\psi \cdot X)$ for every $X \in \mathfrak{se}(1,1)$. Let $X = xE_1 + yE_2 \neq 0$, where $x^2 \neq y^2$. By proposition 1.2.1, $\mathfrak{Orb}(X)$ is the hyperbola $\{(x\cosh\vartheta + y\sinh\vartheta)E_1 + (x\sinh\vartheta + y\cosh\vartheta)E_2 : \vartheta \in \mathbb{R}\}$, and so $\dim(\mathfrak{Orb}(X)) = 1$. Since all coadjoint orbits are even dimensional, we thus have $\dim(\psi\cdot\mathfrak{Orb}(X)) \neq \dim(\mathfrak{orb}(\psi\cdot X))$. This contradicts the assumption that ψ is a linear isomorphism. Therefore no such ψ exists, and so $\mathfrak{se}(1,1)$ does not admit a nondegenerate invariant bilinear form. ∎

Chapter 2

Classification of Full-Rank Control Systems

In this chapter we study the left-invariant control affine systems on $\mathsf{SE}(1,1)$. We do so by employing a natural equivalence relation (detached feedback equivalence) between control systems, and classifying the systems under this equivalence relation. The problem of equivalence of control systems on a (simply connected) Lie group under detached feedback equivalence is shown to reduce to the problem of classifying affine subspaces of the associated Lie algebra. Having obtained a classification of control systems, we are able to determine some controllability criteria for systems on $\mathsf{SE}(1,1)$.

Detached feedback equivalence was introduced in [20] as a natural restriction of feedback equivalence. (Feedback equivalence is the weakest equivalence relation that establishes a one-to-one correspondence between trajectories of equivalent systems; cf. [14].) The full-rank left-invariant control affine systems on all three-dimensional Lie groups have recently been classified under detached feedback equivalence (see [18, 13, 16, 19]).

2.1 Preliminaries

We briefly recall some concepts from invariant control theory, as detailed in section A.2. An ℓ-input left-invariant **control affine system** $\Sigma = (\mathsf{G}, \Xi)$ on a (real, finite-dimensional) connected matrix Lie group G is a control system of the form

$$\dot{g} = g\,\Xi(\mathbf{1}, u) = g(A + u_1B_1 + \ldots + u_\ell B_\ell), \qquad g \in \mathsf{G},\ u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell. \tag{2.1.1}$$

(Here $A, B_1, \ldots, B_\ell \in \mathfrak{g}$ and $B_1, \ldots, B_\ell$ are linearly independent.) A **trajectory-control** pair of Σ is a pair $(g(\cdot), u(\cdot))$, where $u(\cdot)$ is an admissible control and $g(\cdot)$ is a trajectory. The **trace** of Σ is the ℓ-dimensional affine subspace $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$. Σ is said to have **full rank** if $\mathsf{Lie}(\Gamma) = \mathfrak{g}$, *i.e.*, Γ generates the entire Lie algebra. (The full-rank condition is necessary for controllability of Σ.) Henceforth, we shall always assume that the systems under consideration have full rank. For convenience, we write (2.1.1) as $\Sigma : A + u_1B_1 + \ldots + u_\ell B_\ell$. Σ is called **homogeneous** if $A \in \Gamma^0$, and **inhomogeneous**, otherwise.

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ be two left-invariant control affine systems, with traces Γ and Γ', respectively. Σ and Σ' are said to be **detached feedback equivalent** (or ***DF*-equivalent**) if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$.

2.1.1 Proposition. *DF-equivalence is an equivalence relation.*

Proof. Let $\Sigma = (\mathsf{G}, \Xi)$, $\Sigma' = (\mathsf{G}, \Xi')$ and $\Sigma'' = (\mathsf{G}, \Xi'')$ be left-invariant control affine systems. Let $\phi = \mathrm{id}_{\mathsf{G}}$ and $\varphi = \mathrm{id}_{\mathbb{R}^\ell}$. Then $T_g\phi = \mathrm{id}_{T_g\mathsf{G}}$ for every $g \in \mathsf{G}$. Hence $T_g\phi \cdot \Xi(g,u) = \Xi(g,u) = \Xi(\phi(g), \varphi(u))$. That is, Σ is DF-equivalent to itself, and so DF-equivalence has the reflexive property.

Next, suppose Σ is DF-equivalent to Σ'. Then there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. Consequently, $(T_g\phi)^{-1} \cdot \Xi'(\phi(g), \varphi(u)) = \Xi(g,u)$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. As $(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}$, we have $T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g), \varphi(u)) = \Xi(\phi^{-1}(\phi(g)), \varphi^{-1}(\varphi(u)))$ for every $\phi(g) \in \mathsf{G}$ and $\varphi(u) \in \mathbb{R}^\ell$. That is, Σ' is DF-equivalent to Σ, and so DF-equivalence is symmetric.

Finally, suppose Σ is DF-equivalent to Σ' and Σ' is DF-equivalent to Σ''. That is, there exist diffeomorphisms $\phi_1, \phi_2 : \mathsf{G} \to \mathsf{G}$ and $\varphi_1, \varphi_2 : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_g\phi_1 \cdot \Xi(g,u) = \Xi'(\phi_1(g), \varphi_1(u)) \qquad \text{and} \qquad T_g\phi_2 \cdot \Xi'(g,u) = \Xi''(\phi_2(g), \varphi_2(u))$$

for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. Let $\phi = \phi_2 \circ \phi_1$ and $\varphi = \varphi_2 \circ \varphi_1$. We have, for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$,

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \Xi(g,u) \\ &= T_{\phi_1(g)}\phi_2 \cdot \Xi'(\phi_1(g), \varphi_1(u)) \\ &= \Xi''(\phi_2(\phi_1(g)), \varphi_2(\varphi_1(u))) = \Xi''(\phi(g), \varphi(u)). \end{aligned}$$

Thus Σ is DF-equivalent to Σ'', and so DF-equivalence is transitive. ■

The following two results demonstrate that detached feedback equivalence is natural, in the sense that it preserves the trajectory-control pairs and the controllability of equivalent systems.

2.1.2 Proposition. *If Σ is DF-equivalent to Σ', then the trajectory-control pairs of Σ and Σ' are in a one-to-one correspondence.*

Proof. Let $(g(\cdot), u(\cdot))$ be a trajectory-control pair of Σ. Since Σ and Σ' are DF-equivalent, there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. We will show that $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is the unique trajectory-control pair for Σ' corresponding to $(g(\cdot), u(\cdot))$. Indeed, for almost every t we have

$$\begin{aligned} \frac{d}{dt}\phi(g(t)) &= T_{g(t)}\phi \cdot \dot{g}(t) \\ &= T_{g(t)}\phi \cdot \Xi(g(t), u(t)) = \Xi'(\phi(g(t)), \varphi(u(t))). \end{aligned}$$

That is, $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is a trajectory-control pair of Σ'. Suppose $\phi(g_1(\cdot)) = \phi(g_2(\cdot))$ and $\varphi(u_1(\cdot)) = \varphi(u_2(\cdot))$, where $(g_1(\cdot), u_1(\cdot))$ and $(g_2(\cdot), u_2(\cdot))$ are trajectory-control pairs of Σ. Applying ϕ^{-1} and φ^{-1}, we have $g_1(\cdot) = g_2(\cdot)$ and $u_1(\cdot) = u_2(\cdot)$. Hence, trajectory-control pairs are mapped injectively from Σ to Σ'. Next, let $(g'(\cdot), u'(\cdot))$ be a trajectory-control pair of Σ'. Then $(\phi^{-1}(g'(\cdot)), \phi^{-1}(u'(\cdot)))$ is the trajectory-control pair of Σ that is mapped to $(g'(\cdot), u'(\cdot))$ by $\phi \times \varphi$. Thus, trajectory-control pairs are mapped surjectively. Therefore, the trajectory-control pairs of Σ and Σ' are in a one-to-one correspondence. ■

2.1.3 Proposition. *Suppose Σ and Σ' are DF-equivalent. Σ is controllable if and only if Σ' is controllable.*

Proof. Let $\mathcal{A}$ and $\mathcal{A}'$ denote the attainable sets (from identity) of Σ and Σ', respectively. Suppose Σ is controllable, *i.e.*, $\mathcal{A} = \mathsf{G}$ (see proposition A.2.2). Since Σ is DF-equivalent to Σ' (say, with respect to the diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$), there is a one-to-one correspondence between the trajectories of the two systems. In particular, $g(\cdot)$ is a trajectory of Σ if and only if $\phi(g(\cdot))$ is a trajectory of Σ'. Accordingly,

$$\begin{aligned}\mathcal{A}' &= \big\{h(T) : h(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma',\ h(0) = \mathbf{1}\big\}\\ &= \{\phi(g(T)) : g(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma,\ g(0) = \mathbf{1}\}\\ &= \phi(\mathcal{A}) = \mathsf{G}.\end{aligned}$$

Therefore Σ' is controllable. Swapping rôles of Σ and Σ', the same argument above shows that if Σ' is controllable, then Σ is controllable. ■

As a final step, we show that for systems evolving on a simply connected Lie group, detached feedback equivalence may be characterised at the level of the Lie algebra. For brevity, introduce the following notation:

$$\Xi_{u_1} = \Xi(\cdot, u_1), \qquad \Xi_{u_1\cdots u_k} = [\Xi_{u_1}, \Xi_{u_2\cdots u_k}], \qquad k > 1.$$

(Here $u_1, \ldots, u_k \in \mathbb{R}^\ell$.) We begin with a technical lemma.

2.1.4 Lemma. *The Lie algebra $\mathfrak{g}$ is given by $\mathfrak{g} = \mathsf{span}\{\Xi_{u_1\cdots u_k}(\mathbf{1}) : u_1, \ldots, u_k \in \mathbb{R}^\ell,\ k \in \mathbb{N}\}$.*

Proof. We have $\Gamma = \{\Xi_u(\mathbf{1}) : u \in \mathbb{R}^\ell\}$. Consequently, using the characterisation of $\mathsf{Lie}(\Gamma)$ in section A.1.1, we have

$$\begin{aligned}\mathsf{Lie}(\Gamma) &= \mathsf{span}\{A_1, [A_1, A_2], [A_1, [A_2, A_3]], \ldots, [A_1, [A_2, \ldots, [A_{k-1}, A_k] \cdots]] : A_i \in \Gamma,\ k \in \mathbb{N}\}\\ &= \mathsf{span}\{\Xi_{u_1}(\mathbf{1}), \ldots, [\Xi_{u_1}(\mathbf{1}), [\Xi_{u_2}(\mathbf{1}), \ldots, [\Xi_{u_{k-1}}(\mathbf{1}), \Xi_{u_k}(\mathbf{1})] \cdots]] : u_i \in \mathbb{R}^\ell,\ k \in \mathbb{N}\}\\ &= \mathsf{span}\{\Xi_{u_1\cdots u_k}(\mathbf{1}) : u_i \in \mathbb{R}^\ell,\ k \in \mathbb{N}\}.\end{aligned}$$

Since Σ has full rank (by assumption), we have $\mathfrak{g} = \mathsf{Lie}(\Gamma)$, and the proof is complete. ■

2.1.5 Theorem. (cf. [20]) *Suppose G is simply connected. Σ and Σ' are DF-equivalent if and only if there exists a Lie algebra automorphism $\psi : \mathsf{G} \to \mathsf{G}$ such that $\psi \cdot \Gamma = \Gamma'$.*

Proof. Suppose Σ and Σ' are DF-equivalent. Then there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\phi_*\Xi_u = \Xi'_{\varphi(u)}$, *i.e.*, $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. We may assume that $\phi(\mathbf{1}) = \mathbf{1}$. Indeed, suppose $\phi(\mathbf{1}) = a$. Then $(L_{a^{-1}} \circ \phi)(\mathbf{1}) = \mathbf{1}$ and

$$\begin{aligned}T_g(L_{a^{-1}} \circ \phi) \cdot \Xi(g, u) &= T_{\phi(g)}L_{a^{-1}} \cdot T_g\phi \cdot \Xi(g, u)\\ &= T_{\phi(g)}L_{a^{-1}} \cdot \Xi'(\phi(g), \varphi(u))\\ &= \Xi'(a^{-1}\phi(g), \varphi(u)) = \Xi'((L_{a^{-1}} \circ \phi)(g), \varphi(u)).\end{aligned}$$

That is, Σ and Σ' are also DF-equivalent with respect to $L_{a^{-1}} \circ \phi$ and φ. Hence, by replacing ϕ with $L_{a^{-1}} \circ \phi$, we can alway arrange for ϕ to preserve identity.

We first show that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. We have $\Gamma = \{\Xi_u(\mathbf{1}) : u \in \mathbb{R}^\ell\}$. Since $T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}) = \Xi'_{\varphi(u)}(\phi(\mathbf{1})) = \Xi'_{\varphi(u)}(\mathbf{1}) \in \Gamma'$, it follows that $T_{\mathbf{1}}\phi \cdot \Gamma \subseteq \Gamma'$. Moreover, as ϕ is a diffeomorphism, $T_{\mathbf{1}}\phi$ is a linear isomorphism, and so $\dim(T_{\mathbf{1}}\phi \cdot \Gamma) = \dim(\Gamma')$. Thus $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.

It remains to show that $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism. Since the pushforward by ϕ preserves the Lie bracket of vector fields, we have $\phi_*[\Xi_{u_1\cdots u_k}, \Xi_{v_1\cdots v_m}] = [\phi_*\Xi_{u_1\cdots u_k}, \phi_*\Xi_{v_1\cdots v_m}]$ for every $u_1, \ldots, u_k, v_1, \ldots, v_m \in \mathbb{R}^\ell$. Furthermore, as the Lie bracket of two left-invariant vector fields is left-invariant (see section A.2), we have that $\Xi_{u_1\cdots u_k}$, $\Xi_{v_1\cdots v_m}$ and $[\Xi_{u_1\cdots u_k}, \Xi_{v_1\cdots v_m}]$ are left-invariant, for every $u_1, \ldots, u_k, v_1, \ldots, v_m \in \mathbb{R}^\ell$ and $k, m \in \mathbb{N}$. Consequently,

$$\begin{aligned}(\phi_*[\Xi_{u_1\cdots u_k}, \Xi_{v_1\cdots v_m}])(\phi(\mathbf{1})) &= T_{\mathbf{1}}\phi \cdot [\Xi_{u_1\cdots u_k}, \Xi_{v_1\cdots v_m}](\mathbf{1}) \\ &= T_{\mathbf{1}}\phi \cdot [\Xi_{u_1\cdots u_k}(\mathbf{1}), \Xi_{v_1\cdots v_m}(\mathbf{1})].\end{aligned}$$

In a similar fashion,

$$\begin{aligned}[\phi_*\Xi_{u_1\cdots u_k}, \phi_*\Xi_{v_1\cdots v_m}](\phi(\mathbf{1})) &= [(\phi_*\Xi_{u_1\cdots u_k})(\phi(\mathbf{1})), (\phi_*\Xi_{v_1\cdots v_m})(\phi(\mathbf{1}))] \\ &= [T_{\mathbf{1}}\phi \cdot \Xi_{u_1\cdots u_k}(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_{v_1\cdots v_m}(\mathbf{1})].\end{aligned}$$

That is, we have $T_{\mathbf{1}}\phi \cdot [\Xi_{u_1\cdots u_k}(\mathbf{1}), \Xi_{v_1\cdots v_m}(\mathbf{1})] = [T_{\mathbf{1}}\phi \cdot \Xi_{u_1\cdots u_k}(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_{v_1\cdots v_m}(\mathbf{1})]$ for every $u_1, \ldots, u_k, v_1, \ldots, v_m \in \mathbb{R}^\ell$. Let $A, B \in \mathfrak{g}$. By lemma 2.1.4, we can write $A = \sum_i \alpha_i \Xi_{u_1\cdots u_{k_i}}(\mathbf{1})$ and $B = \sum_j \beta_j \Xi_{v_1\cdots v_{m_j}}(\mathbf{1})$ for some $\alpha_i, \beta_j \in \mathbb{R}$. Consequently,

$$\begin{aligned}T_{\mathbf{1}}\phi \cdot [A, B] &= T_{\mathbf{1}}\phi \cdot \left[\textstyle\sum_i \alpha_i \Xi_{u_1\cdots u_{k_i}}(\mathbf{1}), \sum_j \beta_j \Xi_{v_1\cdots v_{m_j}}(\mathbf{1})\right] \\ &= \textstyle\sum_{i,j} \alpha_i\beta_j T_{\mathbf{1}}\phi \cdot [\Xi_{u_1\cdots u_{k_i}}(\mathbf{1}), \Xi_{v_1\cdots v_{m_j}}(\mathbf{1})] \\ &= \textstyle\sum_{i,j} \alpha_i\beta_j [T_{\mathbf{1}}\phi \cdot \Xi_{u_1\cdots u_{k_i}}(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_{v_1\cdots v_{m_j}}(\mathbf{1})] \\ &= \left[T_{\mathbf{1}}\phi \cdot \left(\textstyle\sum_i \alpha_i \Xi_{u_1\cdots u_{k_i}}(\mathbf{1})\right), T_{\mathbf{1}}\phi \cdot \left(\sum_j \beta_j \Xi_{v_1\cdots v_{m_j}}(\mathbf{1})\right)\right] = [T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot B].\end{aligned}$$

That is, $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.

Conversely, suppose there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$. As G is simply connected, there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi = \psi$ (see theorem A.1.6). Since $\Xi'(\mathbf{1}, \cdot)$ is injective, the corestriction $\Xi'(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \Gamma' = \operatorname{im} \Xi'(\mathbf{1}, \cdot)$ has an inverse, say $\xi' : \Gamma' \to \mathbb{R}^\ell$. Let $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the diffeomorphism defined by $\varphi(u) = \xi'(T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u))$. Then $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = (\xi')^{-1}(\varphi(u)) = \Xi'(\mathbf{1}, \varphi(u))$ for every $u \in \mathbb{R}^\ell$. Lastly, we have $\phi = L_{\phi(g)} \circ \phi \circ L_{g^{-1}}$ for every $g \in \mathsf{G}$, and so

$$\begin{aligned}T_g\phi \cdot \Xi(g, u) &= T_g(L_{\phi(g)} \circ \phi \circ L_{g^{-1}}) \cdot \Xi(g, u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot T_g L_{g^{-1}} \cdot \Xi(g, u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot \Xi'(\mathbf{1}, \varphi(u)) = \Xi'(\phi(g), \varphi(u)).\end{aligned}$$

That is, Σ is DF-equivalent to Σ'. ■

Accordingly, the classification of systems (on a simply connected Lie group) under detached feedback equivalence is reduced to the classification of affine subspaces Γ of $\mathfrak{g}$ under Lie algebra automorphisms. Since $\mathsf{SE}(1,1)$ is simply connected (proposition 1.1.9), we shall follow this approach to the classification.

2.1.1 $\mathfrak{L}$-equivalence of affine subspaces

Let $\mathfrak{g}$ be an n-dimensional (real) Lie algebra and let $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ be an ℓ-dimensional affine subspace of $\mathfrak{g}$, where $B_1, \ldots, B_\ell$ are linearly independent. (We shall employ similar terminology and notation for affine subspaces as we do for control affine systems.) Γ is said to have **full rank** if $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. Γ is called **homogeneous** if $A \in \Gamma^0$, and **inhomogeneous**, otherwise. We shall also refer to Γ as an $(\ell, 0)$-affine subspace if it is homogeneous, and as an $(\ell, 1)$-affine subspace if it is inhomogeneous.

Let Γ and Γ' be affine subspaces of $\mathfrak{g}$. We say that Γ is **$\mathfrak{L}$-equivalent** to Γ' if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$.

2.1.6 Proposition. *$\mathfrak{L}$-equivalence is an equivalence relation.*

Proof. Let Γ, Γ' and Γ'' be affine subspaces of $\mathfrak{g}$. We have $\mathrm{id} \cdot \Gamma = \Gamma$, *i.e.*, Γ is $\mathfrak{L}$-equivalent to itself. Thus $\mathfrak{L}$-equivalence is reflexive. Next, suppose Γ is $\mathfrak{L}$-equivalent to Γ'. Then there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \Gamma'$. Consequently, $\psi^{-1} \cdot \Gamma' = \Gamma$. Since ψ^{-1} is an automorphism, it follows that Γ' is $\mathfrak{L}$-equivalent to Γ. Thus $\mathfrak{L}$-equivalence is symmetric. Finally, suppose Γ is $\mathfrak{L}$-equivalent to Γ' and Γ' is $\mathfrak{L}$-equivalent to Γ''. That is, there exist automorphisms ψ_1, ψ_2 such that $\psi_1 \cdot \Gamma = \Gamma'$ and $\psi_2 \cdot \Gamma' = \Gamma''$. Then $\psi_2 \cdot \psi_1 \cdot \Gamma = \psi_2 \cdot \Gamma' = \Gamma''$, *i.e.*, Γ and Γ'' are $\mathfrak{L}$-equivalent. Hence $\mathfrak{L}$-equivalence is transitive. ■

Two $\mathfrak{L}$-equivalent affine subspaces must have the same dimension and homogeneity. (This is because automorphisms preserve both dimension and homogeneity of affine subspaces.) Furthermore, the full-rank condition is invariant under $\mathfrak{L}$-equivalence. Indeed, we have the following result.

2.1.7 Proposition. *Let $\psi : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism. Γ has full rank if and only if $\psi \cdot \Gamma$ has full rank.*

Proof. Let $\psi \in \mathsf{Aut}(\mathfrak{g})$. Using the characterisation of $\mathsf{Lie}(\Gamma)$ given in section A.1.1, we have

$$\begin{aligned}
\mathsf{Lie}(\psi \cdot \Gamma) &= \mathsf{span}\,\{B_1, [B_1, B_2], \ldots, [B_1, [B_2, \ldots, [B_{k-1}, B_k] \cdots]] : B_i \in \psi \cdot \Gamma,\ k \in \mathbb{N}\} \\
&= \mathsf{span}\,\{\psi \cdot A_1, \ldots, [\psi \cdot A_1, [\psi \cdot A_2, \ldots, [\psi \cdot A_{k-1}, \psi \cdot A_k] \cdots]] : A_i \in \Gamma,\ k \in \mathbb{N}\} \\
&= \mathsf{span}\,\{\psi \cdot A_1, \psi \cdot [A_1, A_2], \ldots, \psi \cdot [A_1, [A_2, \ldots, [A_{k-1}, A_k] \cdots]] : A_i \in \Gamma,\ k \in \mathbb{N}\} \\
&= \psi \cdot \mathsf{Lie}(\Gamma).
\end{aligned}$$

Therefore, if $\mathsf{Lie}(\Gamma) = \mathfrak{g}$, then $\mathsf{Lie}(\psi \cdot \Gamma) = \mathfrak{g}$, and so $\psi \cdot \Gamma$ has full rank. Conversely, if $\psi \cdot \Gamma$ has full rank, for $\psi \in \mathsf{Aut}(\mathfrak{g})$, then $\Gamma = \psi^{-1} \cdot (\psi \cdot \Gamma)$ has full rank. ■

The following characterisations of the full-rank condition are used throughout the classification.

2.1.8 Proposition. *Two affine subspaces $\Gamma = A + \Gamma^0$ and $\Gamma' = A' + \Gamma'^0$ are $\mathfrak{L}$-equivalent if and only if there exists an automorphism ψ such that $\psi \cdot A \in \Gamma'$ and $\psi \cdot \Gamma^0 = \Gamma'^0$.*

Proof. Suppose Γ and Γ' are $\mathfrak{L}$-equivalent. Thus there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \Gamma'$. Then $\psi \cdot A \in \psi \cdot \Gamma = \Gamma'$, *i.e.*, $\psi \cdot A = A' + B'$ for some $B' \in \Gamma'^0$. Consequently, from $\psi \cdot \Gamma = \Gamma'$, we have $\psi \cdot A + \psi \cdot \Gamma^0 = A' + \Gamma'^0$, *i.e.*, $A' + B' + \psi \cdot \Gamma^0 = A' + \Gamma'^0$. This implies that $\psi \cdot \Gamma^0 = \Gamma'^0$ (since $B' \in \Gamma'^0$).

Conversely, suppose $\psi \cdot \Gamma^0 = \Gamma'^0$ and $\psi \cdot A \in \Gamma'$. Then $\psi \cdot A = A' + B'$ for some $B' \in \Gamma'^0$. Consequently, $\psi \cdot \Gamma = \psi \cdot A + \psi \cdot \Gamma^0 = A' + B' + \Gamma'^0 = A' + \Gamma'^0 = \Gamma'$. Thus Γ and Γ' are $\mathfrak{L}$-equivalent. ■

2.1.9 Proposition. *Let $\mathfrak{g}$ be a three-dimensional Lie algebra.*

(*i*) *A $(1,1)$-affine subspace $\Gamma = A + \langle B\rangle$ of $\mathfrak{g}$ has full rank if and only if A, B and $[A,B]$ are linearly independent.*

(*ii*) *A $(2,0)$-affine subspace $\Gamma = \langle B_1, B_2\rangle$ of $\mathfrak{g}$ has full rank if and only if B_1, B_2 and $[B_1,B_2]$ are linearly independent.*

(*iii*) *Any $(2,1)$-affine subspace of $\mathfrak{g}$ has full rank.*

Proof.

(*i*) We have that $\{A,B\}$ is linearly independent. Suppose $[A,B] \in \langle A,B\rangle$. Then $\mathsf{Lie}(\Gamma) = \mathsf{Lie}(\{A,B\}) = \langle A,B\rangle \neq \mathfrak{g}$, *i.e.*, Γ does not have full rank. Conversely, if $\{A,B,[A,B]\}$ is linearly independent, then we must have $\dim(\mathsf{Lie}(\Gamma)) = \dim(\mathsf{Lie}(\{A,B,[A,B]\})) = \dim(\mathfrak{g})$, since $\mathfrak{g}$ is three-dimensional. That is, $\mathsf{Lie}(\Gamma) = \mathfrak{g}$.

(*ii*) The set $\{B_1,B_2\}$ is linearly independent. If $[B_1,B_2] \in \langle B_1,B_2\rangle$, then we have $\mathsf{Lie}(\Gamma) = \mathsf{Lie}(\{B_1,B_2\}) = \langle B_1,B_2\rangle \neq \mathfrak{g}$, *i.e.*, Γ does not have full rank. For the converse, suppose $\{B_1,B_2,[B_1,B_2]\}$ is linearly independent. Then $\dim(\mathsf{Lie}(\Gamma)) = \dim(\mathfrak{g})$, and so $\mathsf{Lie}(\Gamma) = \mathfrak{g}$.

(*iii*) Let $\Gamma = A + \langle B_1,B_2\rangle$ have full rank, where $\{A,B_1,B_2\}$ is linearly independent. Since $\mathfrak{g}$ is three-dimensional, we have $\dim(\mathsf{Lie}(\Gamma)) = \dim(\langle\Gamma\rangle) = \dim(\mathfrak{g})$, *i.e.*, $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. ■

The next result enables one to gain a "pre-classification" of homogeneous subspaces based on the classification of the inhomogeneous affine subspaces of dimension one less (thereby reducing the computations that need to be performed).

2.1.10 Proposition. *Let Γ be a full-rank $(\ell+1,0)$-affine subspace of a Lie algebra $\mathfrak{g}$. Suppose $\{\Gamma_i : i \in I\}$ is an exhaustive collection of $\mathfrak{L}$-equivalence class representatives for $(\ell,1)$-affine subspaces of $\mathfrak{g}$. Then Γ is $\mathfrak{L}$-equivalent to at least one element of $\{\langle\Gamma_i\rangle : i \in I\}$.*

Proof. Let $\Gamma = \langle A, B_1, \ldots, B_\ell\rangle$. Then $A + \langle B_1,\ldots,B_\ell\rangle$ is a full-rank $(\ell,1)$-affine subspace, and we have $\Gamma = \langle A + \langle B_1,\ldots,B_\ell\rangle\rangle$. Furthermore, $A + \langle B_1,\ldots,B_\ell\rangle$ is $\mathfrak{L}$-equivalent to Γ_i for some $i \in I$. Hence, there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi\cdot(A + \langle B_1,\ldots,B_\ell\rangle) = \Gamma_i$. Accordingly,

$$\psi\cdot\Gamma = \psi\cdot\langle A + \langle B_1,\ldots,B_\ell\rangle\rangle = \langle\psi\cdot(A + \langle B_1,\ldots,B_\ell\rangle)\rangle = \langle\Gamma_i\rangle.$$

That is, Γ is $\mathfrak{L}$-equivalent to $\langle\Gamma_i\rangle$ for some $i \in I$. ■

2.2 Classification Under Detached Feedback Equivalence

In this section we classify all full-rank affine subspaces of the semi-Euclidean Lie algebra $\mathfrak{se}(1,1)$ under $\mathfrak{L}$-equivalence. We then reinterpret these results as a classification, under detached feedback equivalence, of the full-rank control affine systems on $\mathsf{SE}(1,1)$. In addition, we provide complete classifying conditions for the class representatives. Finally, we obtain controllability criteria in terms of the detached feedback equivalence representatives. Table 2.1 provides a summary of the results obtained. Section B.2 lists supporting Mathematica code that verifies the calculations.

We outline the approach followed in the classification. We first distinguish between the dimension of the affine subspaces (since affine subspaces of different dimensions cannot be $\mathfrak{L}$-equivalent). This separation of cases constitutes sections 2.2.1 and 2.2.2. For the latter section, we further distinguish between the homogeneous (*i.e.*, $(2,0)$) and the inhomogeneous (*i.e.*, $(2,1)$) affine subspaces. Note that there are no $(0,0)$- or $(1,0)$-affine subspaces of $\mathfrak{se}(1,1)$ that have full rank. Furthermore, there is only one $(3,0)$-affine subspace, namely $\mathfrak{se}(1,1)$ itself. (Accordingly, we are only concerned with the $(1,1)$-, $(2,0)$- and $(2,1)$-affine subspaces.) Lastly, the classifying conditions separate the various cases within each proof, and consist of various conditions on the linear part Γ^0 of the affine subspace.

We recall the automorphism group of $\mathfrak{se}(1,1)$. In terms of the standard basis $(E_i)_{i=1}^3$, any automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ is of the form

$$\psi = \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}.$$

(See proposition 1.1.17.) Here $\varsigma \in \{-1, 1\}$ and $x^2 \neq y^2$. (If we refer to an arbitrary automorphism, we shall assume it is of this form.) Lastly, let $(E_i^*)_{i=1}^3$ denote the dual of the standard basis. We shall consider the basis elements E_i^*, $i = 1, 2, 3$ as projections onto the i^{th} axis. (Indeed, for $X = x_1E_1 + x_2E_2 + x_3E_3 \in \mathfrak{se}(1,1)$, we have $E_i^*(X) = x_i$, $i = 1, 2, 3$.)

2.2.1 One-dimensional affine subspaces

2.2.1 PROPOSITION. *Any $(1,1)$-affine subspace $\Gamma = A + \Gamma^0$ is $\mathfrak{L}$-equivalent to exactly one of the following affine subspaces:*

$$\Gamma_1^{(1,1)} = E_1 + \langle E_3 \rangle \qquad E_3^*(\Gamma^0) \neq \{0\}$$

$$\Gamma_{2,\alpha}^{(1,1)} = \alpha E_3 + \langle E_1 \rangle \qquad E_3^*(\Gamma^0) = \{0\}.$$

Here $\alpha > 0$ parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

PROOF. Suppose that $E_3^*(\Gamma) \neq \{0\}$. Then Γ is of the form $\Gamma = a_1E_1 + a_2E_2 + \langle b_1E_1 + b_2E_2 + E_3 \rangle$ and

$$\psi_1 = \begin{bmatrix} 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_1 \cdot \Gamma = a_1E_1 + a_2E_2 + \langle E_3 \rangle$. We have $[a_1E_1 + a_2E_2, E_3] = a_1[E_1, E_3] + a_2[E_2, E_3] = -a_1E_2 - a_2E_2$. By proposition 2.1.9, since Γ has full rank, the set $\{a_1E_1 + a_2E_2, E_3, -a_2E_1 - a_1E_2\}$ is linearly independent. That is,

$$\begin{vmatrix} a_1 & 0 & -a_2 \\ a_2 & 0 & -a_1 \\ 0 & 1 & 0 \end{vmatrix} \neq 0 \qquad \iff \qquad a_1^2 - a_2^2 \neq 0.$$

Consequently, we have an automorphism

$$\psi_2 = \begin{bmatrix} \frac{a_1}{a_1^2 - a_2^2} & -\frac{a_2}{a_1^2 - a_2^2} & 0 \\ -\frac{a_2}{a_1^2 - a_2^2} & \frac{a_1}{a_1^2 - a_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (\det \psi_2 = a_1^2 - a_2^2 \neq 0)$$

such that $\psi_2 \cdot \Gamma = E_1 + \langle E_3 \rangle$. Thus Γ is $\mathfrak{L}$-equivalent to $\Gamma_1^{(1,1)}$.

Suppose $E_3^*(\Gamma^0) = \{0\}$. Then we have $\Gamma = a_1E_1 + a_2E_2 + a_3E_3 + \langle b_1E_1 + b_2E_2 \rangle$, where $a_3 \neq 0$. (If $a_3 = 0$, then Γ does not have full rank.) Thus

$$\psi_3 = \begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_3 \cdot \Gamma = a_3E_3 + \langle b_1E_1 + b_2E_2 \rangle$. We have $[a_3E_3, b_1E_1 + b_2E_2] = a_3b_1[E_3, E_1] + a_3b_2[E_2, E_3] = a_3b_1E_2 - a_3b_2E_1$. Accordingly, the set $\{a_3E_3, b_1E_1 + b_2E_2, a_3b_1E_2 - a_3b_2E_1\}$ is linearly independent (again by proposition 2.1.9). Equivalently,

$$\begin{vmatrix} 0 & b_1 & -a_3b_2 \\ 0 & b_2 & -a_3b_1 \\ a_3 & 0 & 0 \end{vmatrix} \neq 0 \qquad \iff \qquad a_3^2(b_1^2 - b_2^2) \neq 0.$$

Hence $b_1^2 \neq b_2^2$, and so

$$\psi_4 = \begin{bmatrix} \frac{b_1}{b_1^2-b_2^2} & -\frac{b_2}{b_1^2-b_2^2} & 0 \\ -\frac{\operatorname{sgn}(a_3)b_2}{b_1^2-b_2^2} & \frac{\operatorname{sgn}(a_3)b_1}{b_1^2-b_2^2} & 0 \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix} \qquad (\det \psi = b_1^2 - b_2^2 \neq 0)$$

is an automorphism such that $\psi_4 \cdot \Gamma = \operatorname{sgn}(a_3)a_3E_3 + \langle E_1 \rangle = |a_3|E_3 + \langle E_1 \rangle$. Therefore Γ is $\mathfrak{L}$-equivalent to $\Gamma_{2,\alpha}^{(1,1)}$, where $\alpha = |a_3| > 0$.

We have that $\langle E_1, E_2 \rangle$ is an invariant subspace of every automorphism of $\mathfrak{se}(1,1)$ (see proposition 1.1.18). Accordingly, $\psi \cdot \langle E_1 \rangle \subseteq \langle E_1, E_2 \rangle$ for every automorphism ψ. Hence $\Gamma_1^{(1,1)}$ cannot be $\mathfrak{L}$-equivalent to $\Gamma_{2,\alpha}^{(1,1)}$.

Suppose $\Gamma_{2,\alpha}^{(1,1)}$ is $\mathfrak{L}$-equivalent to $\Gamma_{2,\alpha'}^{(1,1)}$, for some $\alpha, \alpha' > 0$. That is, there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that $\psi \cdot \Gamma_{2,\alpha}^{(1,1)} = \Gamma_{2,\alpha'}^{(1,1)}$. Then from proposition 2.1.8 we have $\psi \cdot (\alpha E_3) \in \Gamma_{2,\alpha'}^{(1,1)} = \alpha' E_3 + \langle E_1 \rangle$, *i.e.*,

$$\alpha v E_1 + \alpha w E_2 + \varsigma \alpha E_3 \in \alpha' E_3 + \langle E_1 \rangle.$$

This implies $\alpha = \varsigma \alpha'$. As $\alpha, \alpha' > 0$ and $\varsigma \in \{-1, 1\}$, it follows that $\alpha = \alpha'$. ∎

2.2.2 Two-dimensional affine subspaces

We begin with the homogeneous case. The classification of $(2,0)$-affine subspaces follows easily from the results for the one-dimensional case.

2.2.2 Proposition. *Any $(2,0)$-affine subspace is $\mathfrak{L}$-equivalent to $\Gamma^{(2,0)} = \langle E_1, E_3 \rangle$.*

Proof. By proposition 2.1.10, we have that any $(2,0)$-affine subspace Γ is $\mathfrak{L}$-equivalent to $\langle \Gamma_1^{(1,1)} \rangle = \langle E_1, E_3 \rangle$ or $\langle \Gamma_{3,\alpha}^{(1,1)} \rangle = \langle \alpha E_3, E_1 \rangle = \langle E_1, E_3 \rangle$. ∎

2.2.3 Proposition. *Any* $(2,1)$*-affine subspace* $\Gamma = A + \Gamma^0$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following affine subspaces*:

$$\begin{aligned} \Gamma_1^{(2,1)} &= E_2 + \langle E_1, E_3\rangle && E_3^*(\Gamma^0) \neq \{0\},\ E_1 + E_2, E_1 - E_2 \notin \Gamma^0 \\ \Gamma_2^{(2,1)} &= E_1 + \langle E_1 + E_2, E_3\rangle && E_3^*(\Gamma^0) \neq \{0\},\ E_1 \pm E_2 \in \Gamma^0 \\ \Gamma_{3,\alpha}^{(2,1)} &= \alpha E_3 + \langle E_1, E_2\rangle && E_3^*(\Gamma^0) = \{0\}. \end{aligned}$$

Here $\alpha > 0$ *parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.*

Proof. Suppose that $E_3^*(\Gamma^0) \neq \{0\}$, $E_1 + E_2 \notin \Gamma^0$ and $E_1 - E_2 \notin \Gamma^0$. We have $\Gamma = a_1E_1 + a_2E_2 + \langle b_1E_1 + b_2E_2, c_1E_1 + c_2E_2 + E_3\rangle$ with $b_1^2 \neq b_2^2$. Then

$$\psi_1 = \begin{bmatrix} 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_1 \cdot \Gamma = a_1E_1 + a_2E_2 + \langle b_1E_1 + b_2E_2, E_3\rangle$. Next, we have an automorphism

$$\psi_2 = \begin{bmatrix} \frac{b_1}{b_1^2-b_2^2} & -\frac{b_2}{b_1^2-b_2^2} & 0 \\ -\frac{b_2}{b_1^2-b_2^2} & \frac{b_1}{b_1^2-b_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (\det \psi_2 = b_1^2 - b_2^2 \neq 0)$$

such that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_1'E_1 + a_2'E_2 + \langle E_1, E_3\rangle$ for some $a_1', a_2' \in \mathbb{R}$. Since $\psi_2 \cdot \psi_1 \cdot \Gamma$ is inhomogeneous, it follows that $a_2' \neq 0$. Consequently, we have an automorphism $\psi_3 = \operatorname{diag}\left(\frac{1}{a_2'}, \frac{1}{a_2'}, 1\right)$ such that $\psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = \frac{a_1'}{a_2'}E_1 + E_2 + \langle \frac{1}{a_2}E_1, E_3\rangle = E_2 + \langle E_1, E_3\rangle$. Thus Γ is $\mathfrak{L}$-equivalent to $\Gamma_1^{(2,1)}$.

Suppose $E_3^*(\Gamma^0) \neq \{0\}$ and $E_1 \pm E_2 \in \Gamma^0$. Then $\Gamma = a_1E_1 + a_2E_2 + \langle E_1 \pm E_2, b_1E_1 + b_2E_2 + E_3\rangle$, and we have an automorphism

$$\psi_4 = \begin{bmatrix} 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $\psi_4 \cdot \Gamma = a_1E_1 + a_2E_2 + \langle E_1 \pm E_2, E_3\rangle$. Since $\psi_4 \cdot \Gamma$ is inhomogeneous, $a_1E_1 + a_2E_2$ is not a scalar multiple of $E_1 \pm E_2$. That is, $\begin{vmatrix} a_1 & 1 \\ a_2 & \pm 1 \end{vmatrix} \neq 0$, or, equivalently, $a_1 \mp a_2 \neq 0$. Therefore

$$\psi_5 = \begin{bmatrix} \frac{a_1}{a_1^2-a_2^2} & -\frac{a_2}{a_1^2-a_2^2} & 0 \\ \mp\frac{a_2}{a_1^2-a_2^2} & \pm\frac{a_1}{a_1^2-a_2^2} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

is an automorphism such that $\psi_5 \cdot \psi_4 \cdot \Gamma = E_1 + \left\langle \frac{a_1 \pm a_2}{a_1^2 - a_2^2}(E_1 + E_2), \pm E_3 \right\rangle = E_1 + \langle E_1 + E_2, E_3\rangle$. Thus Γ is $\mathfrak{L}$-equivalent to $\Gamma_2^{(2,1)}$.

Suppose $E_3^*(\Gamma^0) = \{0\}$. Then $\Gamma = a_3E_3 + \langle E_1, E_2\rangle$ and $\psi_6 = \operatorname{diag}(1, \operatorname{sgn}(a_3), \operatorname{sgn}(a_3))$ is an automorphism such that $\psi_6 \cdot \Gamma = \operatorname{sgn}(a_3)a_3E_3 + \langle E_1, \operatorname{sgn}(a_3)E_2\rangle = |a_3|E_3 + \langle E_1, E_2\rangle$. Hence Γ is $\mathfrak{L}$-equivalent to $\Gamma_{3,\alpha}^{(2,1)}$, where $\alpha = |a_3| > 0$.

From proposition 1.1.18, $\langle E_1, E_2\rangle$ is an invariant subspace and $\langle E_1 + E_2\rangle \cup \langle E_1 - E_2\rangle$ is an invariant subset of any automorphism of $\mathfrak{se}(1,1)$. Hence, no two of $\Gamma^{(2,1)}_1$, $\Gamma^{(2,1)}_2$ and $\Gamma^{(2,1)}_{3,\alpha}$ can be $\mathfrak{L}$-equivalent.

Suppose that $\Gamma^{(2,1)}_{3,\alpha}$ is $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{3,\alpha'}$, for some $\alpha, \alpha' > 0$. Thus, there exists an automorphism ψ such that $\psi \cdot \Gamma^{(2,1)}_{3,\alpha} = \Gamma^{(2,1)}_{3,\alpha'}$. Hence, by proposition 2.1.8 we have $\psi \cdot (\alpha E_3) \in \Gamma^{(2,1)}_{3,\alpha'} = \alpha' E_3 + \langle E_1, E_2\rangle$, *i.e.*,

$$\alpha v E_1 + \alpha w E_2 + \varsigma \alpha E_3 \in \alpha' E_3 + \langle E_1, E_2\rangle.$$

This implies that $\alpha = \varsigma\alpha'$. As $\alpha, \alpha' > 0$ and $\varsigma \in \{-1, 1\}$, it follows that $\alpha = \alpha'$. ■

2.2.3 Classification of full-rank control systems

Having classified the full-rank affine subspaces of $\mathfrak{se}(1,1)$ in sections 2.2.1 and 2.2.2, we can now reinterpret those results as a classification, under detached feedback equivalence, of the full-rank left-invariant control affine systems on $\mathsf{SE}(1,1)$. As a corollary, we determine the controllable systems.

2.2.4 Theorem. *Let Σ be a full-rank left-invariant control affine system on $\mathsf{SE}(1,1)$ with trace $\Gamma = A + \Gamma^0$.*

(*i*) *If Σ is an inhomogeneous single-input control affine system, then it is DF-equivalent to exactly one of the following systems*:

$$\Sigma^{(1,1)}_1 : E_1 + uE_3 \qquad E_3^*(\Gamma^0) \neq \{0\}$$
$$\Sigma^{(1,1)}_{2,\alpha} : \alpha E_3 + uE_1 \qquad E_3^*(\Gamma^0) = \{0\}.$$

Here $\alpha > 0$ parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

(*ii*) *If Σ is a homogeneous two-input control affine system, then it is DF-equivalent to the system $\Sigma^{(2,0)} : u_1E_1 + u_2E_3$.*

(*iii*) *If Σ is an inhomogeneous two-input control affine system, then it is DF-equivalent to exactly one of the following systems*:

$$\Sigma^{(2,1)}_1 : E_2 + u_1E_1 + u_2E_3 \qquad E_3^*(\Gamma^0) \neq \{0\}, E_1 + E_2, E_1 - E_2 \notin \Gamma^0$$
$$\Sigma^{(2,1)}_2 : E_1 + u_1(E_1 + E_2) + u_2E_3 \qquad E_3^*(\Gamma^0) \neq \{0\}, E_1 \pm E_2 \in \Gamma^0$$
$$\Sigma^{(2,1)}_{3,\alpha} : \alpha E_3 + u_1E_1 + u_2E_2 \qquad E_3^*(\Gamma^0) = \{0\}.$$

Here $\alpha > 0$ parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

(*iv*) *If Σ is a homogeneous three-input control affine system, then it is DF-equivalent to the system $\Sigma^{(3,0)} : u_1E_1 + u_2E_2 + u_3E_3$.*

Type	Conditions	Representative ($\alpha > 0$)
$(1,1)$	$E_3^*(\Gamma^0) \neq \{0\}$	$\Sigma_1^{(1,1)} : E_1 + uE_3$
	$E_3^*(\Gamma^0) = \{0\}$	$\Sigma_{2,\alpha}^{(1,1)} : \alpha E_3 + uE_1$
$(2,0)$		$\Sigma^{(2,0)} : u_1E_1 + u_2E_3$
$(2,1)$	$E_3^*(\Gamma^0) \neq \{0\}$, $E_1 + E_2, E_1 - E_2 \notin \Gamma^0$	$\Sigma_1^{(2,1)} : E_2 + u_1E_1 + u_2E_3$
	$E_3^*(\Gamma^0) \neq \{0\}$, $E_1 \pm E_2 \in \Gamma^0$	$\Sigma_2^{(2,1)} : E_1 + u_1(E_1 + E_2) + u_2E_3$
	$E_3^*(\Gamma^0) = \{0\}$	$\Sigma_{3,\alpha}^{(2,1)} : \alpha E_3 + u_1E_1 + u_2E_2$
$(3,0)$		$\Sigma^{(3,0)} : u_1E_1 + u_2E_2 + u_3E_3$

Table 2.1: Classification of full-rank left-invariant control affine systems on $\mathsf{SE}(1,1)$

PROOF. We illustrate by proving item (*i*). (The proof of the other items is similar.) Let Σ be a single-input inhomogeneous control affine system, with trace $\Gamma = A + \Gamma^0$. Since Γ is a $(1,1)$-affine subspace of $\mathfrak{se}(1,1)$, by proposition 2.2.1 it is $\mathfrak{L}$-equivalent to exactly one of the subspaces $\Gamma_1^{(1,1)} = E_1 + \langle E_3 \rangle$ or $\Gamma_{2,\alpha}^{(1,1)} = \alpha E_3 + \langle E_1 \rangle$. That is, there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that $\psi \cdot \Gamma = \Gamma_1^{(1,1)}$ if $E_3^*(\Gamma^0) \neq \{0\}$ and $\psi \cdot \Gamma = \Gamma_{2,\alpha}^{(1,1)}$ if $E_3^*(\Gamma^0) = \{0\}$. By theorem 2.1.5, we thus have that

$$\Sigma \text{ is } DF\text{-equivalent to } \begin{cases} \Sigma_1^{(1,1)} & \text{if } E_3^*(\Gamma^0) \neq \{0\} \\ \Sigma_{2,\alpha}^{(1,1)} & \text{if } E_3^*(\Gamma^0) = \{0\}. \end{cases}$$

Inspection of the $\mathfrak{L}$-equivalence classifications of two-input systems in proposition 2.2.2 and proposition 2.2.3 yields the results of items (*ii*) and (*iii*). Lastly, it is clear that any $(3,0)$-affine subspace is $\mathfrak{L}$-equivalent to the subspace $\langle E_1, E_2, E_3 \rangle$. It follows that any three-input homogeneous control affine system is DF-equivalent to the system $\Sigma^{(3,0)}$. ■

2.2.5 COROLLARY. *Any controllable left-invariant control affine system on* $\mathsf{SE}(1,1)$ *is* DF*-equivalent to exactly one of the following systems*:

$$\Sigma^{(2,0)} : u_1E_1 + u_2E_3 \qquad \Sigma_1^{(2,1)} : E_2 + u_1E_1 + u_2E_3 \qquad \Sigma^{(3,0)} : u_1E_1 + u_2E_2 + u_3E_3.$$

PROOF. Let Σ be a left-invariant control affine system on $\mathsf{SE}(1,1)$, with trace $\Gamma = A + \Gamma^0$. By proposition A.2.3, Σ is controllable if and only if $\mathsf{Lie}(\Gamma^0) = \mathfrak{se}(1,1)$. Furthermore, controllability is preserved under detached feedback equivalence (proposition 2.1.3). Consequently, no single-input system on $\mathsf{SE}(1,1)$ is controllable. Since $[E_3, E_1] = E_2$, we have $\mathsf{Lie}(\langle E_1, E_3 \rangle) = \mathfrak{se}(1,1)$. Thus $\Sigma^{(2,0)}$ is controllable, and so any controllable homogeneous two-input system on $\mathsf{SE}(1,1)$ is DF-equivalent to $\Sigma^{(2,0)}$. Next, we have $[E_1, E_2] = 0$ and $[E_1 + E_2, E_3] = -E_2 - E_1$. Thus $\mathsf{Lie}(\langle E_1, E_2 \rangle) \neq \mathfrak{se}(1,1)$ and $\mathsf{Lie}(\langle E_1 + E_2, E_3 \rangle) \neq \mathfrak{se}(1,1)$. Hence, neither $\Sigma_2^{(2,1)}$ nor $\Sigma_{3,\alpha}^{(2,1)}$ is controllable. It follows that any controllable inhomogeneous two-input system is DF-equivalent to $\Sigma_1^{(2,1)}$. Finally, any three-input system on $\mathsf{SE}(1,1)$ is controllable, and is DF-equivalent to $\Sigma^{(3,0)}$. ■

Chapter 3

Classification of Quadratic Hamilton-Poisson Systems

In chapter 2, we classified (under detached feedback equivalence) all full-rank left-invariant control affine systems on $\mathsf{SE}(1,1)$. A natural next step is to consider the associated (left-invariant) optimal control problems for some given cost and boundary conditions (see section A.3). In particular in this thesis, we consider optimal control problems with fixed time and quadratic cost:

$$\dot{g} = g(A + u_1 B_1 + \ldots + u_\ell B_\ell), \quad g(\cdot) : [0,T] \to \mathsf{SE}(1,1),\ u(\cdot) : [0,T] \to \mathbb{R}^\ell \tag{3.0.1}$$

$$g(0) = \mathbf{1},\ g(T) = g_1, \quad g_1 \in \mathsf{SE}(1,1),\ T > 0 \text{ fixed} \tag{3.0.2}$$

$$\mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t))\, dt \to \min. \tag{3.0.3}$$

Here the cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$, $u \mapsto u^\top Q u$ is a positive definite quadratic form on $\mathbb{R}^\ell$. Using Pontryagin's Maximum Principle (section A.3.4), the optimal control problem (3.0.1)-(3.0.2)-(3.0.3) is lifted to a family of invariant Hamiltonian functions on the cotangent bundle $T^*\mathsf{SE}(1,1)$. This is then reduced to a single Hamiltonian function $H \in \mathsf{C}^\infty(\mathfrak{se}(1,1)^*)$ on the minus Lie-Poisson space $\mathfrak{se}(1,1)^*_-$. The extremal controls for (3.0.1)-(3.0.2)-(3.0.3) are linearly related to the integral curves of the Hamilton-Poisson system $(\mathfrak{se}(1,1)^*_-, H)$. (See, in particular, theorem A.3.8.) Furthermore, by theorem A.3.8, H is of the form $H = H_{A,\mathcal{Q}}$, where

$$H_{A,\mathcal{Q}}(p) = \langle p, A \rangle + \mathcal{Q}(p). \tag{3.0.4}$$

(Here $A \in \mathfrak{se}(1,1)$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{se}(1,1)^*$.) Thus the problem of determining the extremal controls for an optimal control problem (3.0.1)-(3.0.2)-(3.0.3) is reduced to the problem of finding the integral curves of the Hamilton-Poisson system (3.0.4).

In this chapter we shall consider all (quadratic) Hamilton-Poisson systems of the form $(\mathfrak{se}(1,1)^*_-, H_{A,\mathcal{Q}})$, where $\mathcal{Q}$ is positive semidefinite. We classify such systems up to affine isomorphisms, beginning with the homogeneous systems (*i.e.*, those for which $A = 0$). Based on the classification of homogeneous systems, we then arrive at a classification of the inhomogeneous systems (where $A \neq 0$). We obtain normalised class representatives for all equivalence classes (see table 3.1). Chapter 4 is concerned with the stability analysis and integration of (some of) the normal forms obtained.

3.1 Preliminaries

We recall some notational conventions and some concepts relating to Lie-Poisson spaces. Let $\mathfrak{g}$ be an (real) n-dimensional Lie algebra with dual space $\mathfrak{g}^*$. Denote the standard (ordered) basis of $\mathfrak{g}$ by $(E_i)_{i=1}^n$ and the dual basis by $(E_i^*)_{i=1}^n$. In terms of these bases, we will write elements of the Lie algebra as column vectors and elements of the dual space as row vectors. (Consequently, the pairing $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is given by matrix multiplication: $\langle p, X\rangle = pX$.)

The **(minus) Lie-Poisson bracket** $\{\cdot,\cdot\}$ on $\mathfrak{g}^*$ is given by

$$\begin{aligned}\{F,G\}(p) &= -\left\langle \mathrm{ad}^*_{\mathbf{d}F(p)}\, p, \mathbf{d}G(p)\right\rangle \\ &= -\left\langle p, [\mathbf{d}F(p), \mathbf{d}G(p)]\right\rangle, \qquad F,G \in \mathsf{C}^\infty(\mathfrak{g}^*).\end{aligned}$$

(Here $\mathrm{ad}^*_{\mathbf{d}F(p)}$ is the dual of the adjoint map $\mathrm{ad}_{\mathbf{d}F(p)} = [\mathbf{d}F(p), \cdot]$ and $[\cdot,\cdot]$ denotes the Lie bracket on $\mathfrak{g}$. As $\mathbf{d}F(p)$ and $\mathbf{d}G(p)$ are linear functions on $\mathfrak{g}^*$, they are elements of $\mathfrak{g}^{**} \cong \mathfrak{g}$.) The Lie-Poisson space $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. A **linear Poisson automorphism** is a linear isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $\{F,G\}\circ\Psi = \{F\circ\Psi, G\circ\Psi\}$ for every $F,G \in \mathsf{C}^\infty(\mathfrak{g}^*)$. (Linear Poisson automorphisms are exactly the dual maps of Lie algebra automorphisms; see proposition A.3.3.)

The **Hamiltonian vector field** $\vec{H}$ associated to a Hamiltonian function $H \in \mathsf{C}^\infty(\mathfrak{g}^*)$ is defined by $\vec{H}[F] = \{F, H\}$. In coordinates, $\vec{H}(p) = \mathrm{ad}^*_{\mathbf{d}H(p)}\, p$. A **Casimir function** is a function $C \in \mathsf{C}^\infty(\mathfrak{g}^*)$ such that $\vec{C} = 0$, *i.e.*, Casimir functions Poisson-commute with every element of $\mathsf{C}^\infty(\mathfrak{g}^*)$.

A **quadratic Hamilton-Poisson system** is a pair $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$, where $\mathfrak{g}^*_-$ is a Lie-Poisson space and $H_{A,\mathcal{Q}} \in \mathsf{C}^\infty(\mathfrak{g}^*)$ is a Hamiltonian function of the form $H_{A,\mathcal{Q}}(p) = L_A(p) + \mathcal{Q}(p)$. Here $L_A(p) = \langle p, A\rangle$, $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{g}^*$. In coordinates, $H_{A,\mathcal{Q}}$ takes the form $H_{A,\mathcal{Q}}(p) = pA + \frac{1}{2}pQp^\top$. (Here $Q \in \mathbb{R}^{n\times n}$ is the symmetric positive semidefinite matrix associated to $\mathcal{Q}$.) If no ambiguity results, we shall identify a Hamilton-Poisson system with its Hamiltonian function.

The system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is called **homogeneous** if $A = 0$, and **inhomogeneous**, otherwise. We abbreviate a homogeneous Hamilton-Poisson system $H_{0,\mathcal{Q}}$ by $H_\mathcal{Q}$. Furthermore, notice that to every inhomogeneous system $H_{A,\mathcal{Q}}$ is associated a homogeneous system $H_\mathcal{Q}$. (This fact shall prove useful in relating the classification of inhomogeneous systems to that of the homogeneous systems.)

It is useful to consider the Hamiltonian vector field corresponding to a quadratic Hamiltonian function $H_{A,\mathcal{Q}}$. Indeed, let $H_{A,\mathcal{Q}}(p) = L_A(p) + H_\mathcal{Q}(p) = \langle p, A\rangle + \mathcal{Q}(p)$. We have $\vec{H}_\mathcal{Q}(p) = \mathrm{ad}^*_{\mathbf{d}H_\mathcal{Q}(p)}\, p$ and $\vec{L}_A(p) = \mathrm{ad}^*_{\mathbf{d}L_A(p)}\, p = \mathrm{ad}^*_A\, p$. Therefore, since $X \mapsto \mathrm{ad}^*_X\, p$ is a linear map on $\mathfrak{g}$ for every $p \in \mathfrak{g}^*$, we get

$$\begin{aligned}\vec{H}_{A,\mathcal{Q}}(p) &= \mathrm{ad}^*_{\mathbf{d}(L_A + H_\mathcal{Q})(p)}\, p \\ &= \mathrm{ad}^*_{A + \mathbf{d}H_\mathcal{Q}(p)}\, p \\ &= \mathrm{ad}^*_A\, p + \mathrm{ad}^*_{\mathbf{d}H_\mathcal{Q}(p)}\, p = \vec{L}_A(p) + \vec{H}_\mathcal{Q}(p).\end{aligned}$$

That is, $\vec{H}_{A,\mathcal{Q}}$ decomposes as the sum $\vec{L}_A + \vec{H}_\mathcal{Q}$.

Two quadratic Hamilton-Poisson systems $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ on a (minus) Lie-Poisson space $\mathfrak{g}^*_-$ are said to be **affinely equivalent** (or **A-equivalent**) if there exists an affine isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$.

3.1.1 PROPOSITION. *Affine equivalence is an equivalence relation.*

PROOF. Let $H_{A,\mathcal{Q}}$, $H_{B,\mathcal{R}}$ and $H_{C,\mathcal{S}}$ be quadratic Hamilton-Poisson systems on a (minus) Lie-Poisson space $\mathfrak{g}^*_-$. We have $\mathrm{id}_{\mathfrak{g}^*} \cdot H_{A,\mathcal{Q}} = H_{A,\mathcal{Q}} \circ \mathrm{id}_{\mathfrak{g}^*}$, and so $H_{A,\mathcal{Q}}$ is A-equivalent to itself. Thus A-equivalence is reflexive. Next, suppose that $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{B,\mathcal{R}}$. Thus there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$. Then $\Psi_0^{-1} \cdot \vec{H}_{B,\mathcal{R}} = \vec{H}_{A,\mathcal{Q}} \circ \Psi^{-1}$, and so $H_{B,\mathcal{R}}$ is equivalent to $H_{A,\mathcal{Q}}$. Hence the symmetric property is satisfied. Lastly, suppose that $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{B,\mathcal{R}}$ and $H_{B,\mathcal{R}}$ is A-equivalent to $H_{C,\mathcal{S}}$. Thus, there exist affine isomorphisms $\Psi : p \mapsto \Psi_0(p) + q$ and $\Psi' : p \mapsto \Psi'_0(p) + q'$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$ and $\Psi'_0 \cdot \vec{H}_{B,\mathcal{R}} = \vec{H}_{C,\mathcal{S}} \circ \Psi'$. Then $\Psi' \cdot \Psi : p \mapsto (\Psi'_0 \cdot \Psi_0)(p) + \Psi'_0(q) + q'$ is an affine isomorphism such that $(\Psi'_0 \cdot \Psi_0) \cdot \vec{H}_{A,\mathcal{Q}} = \Psi'_0 \cdot \vec{H}_{B,\mathcal{R}} \circ \Psi = \vec{H}_{C,\mathcal{S}} \circ (\Psi' \circ \Psi)$, and so $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{C,\mathcal{S}}$. Therefore A-equivalence satisfies the transitivity property. ■

The next proposition demonstrates that A-equivalence is a natural equivalence relation (in the sense that affine equivalence preserves the appropriate properties of equivalent systems, specifically the integral curves and equilibrium points).

3.1.2 PROPOSITION. *If $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{B,\mathcal{R}}$, then the integral curves and equilibrium points of $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are in a one-to-one correspondence.*

PROOF. Let $p(\cdot)$ be an integral curve of $\vec{H}_{A,\mathcal{Q}}$. That is, $\dot{p}(t) = \vec{H}_{A,\mathcal{Q}}(p(t))$. Since $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are A-equivalent, there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$. We will show that $\Psi(p(\cdot))$ is the unique integral curve of $\vec{H}_{B,\mathcal{R}}$ corresponding to $p(\cdot)$. Indeed,

$$\begin{aligned}\frac{d}{dt}\Psi(p(t)) &= T_{p(t)}\Psi \cdot \dot{p}(t) \\ &= \Psi_0 \cdot \vec{H}_{A,\mathcal{Q}}(p(t)) = \vec{H}_{B,\mathcal{R}}(\Psi(p(t))).\end{aligned}$$

That is, $\Psi(p(\cdot))$ is an integral curve of $\vec{H}_{B,\mathcal{R}}$. Suppose $\Psi(p_1(\cdot)) = \Psi(p_2(\cdot))$, where $p_1(\cdot)$ and $p_2(\cdot)$ are integral curves of $\vec{H}_{A,\mathcal{Q}}$. Then $p_1(\cdot) = \Psi^{-1}(\Psi(p_1(\cdot))) = \Psi^{-1}(\Psi(p_2(\cdot))) = p_2(\cdot)$, and so integral curves are mapped injectively from $H_{A,\mathcal{Q}}$ to $H_{B,\mathcal{R}}$. Next, let $p'(\cdot)$ be an integral curve of $\vec{H}_{B,\mathcal{R}}$. We have

$$\begin{aligned}\frac{d}{dt}\Psi^{-1}(p'(t)) &= T_{p'(t)}\Psi^{-1} \cdot \dot{p}'(t) \\ &= \Psi_0^{-1} \cdot \vec{H}_{B,\mathcal{R}}(p'(t)) = \vec{H}_{A,\mathcal{Q}}(\Psi^{-1}(p'(t))),\end{aligned}$$

and so $\Psi^{-1}(p'(\cdot))$ is an integral curve of $\vec{H}_{A,\mathcal{Q}}$. Hence the integral curves are mapped surjectively. Therefore the integral curves of $\vec{H}_{A,\mathcal{Q}}$ and $\vec{H}_{B,\mathcal{R}}$ are in a one-to-one correspondence.

Let p_e be an equilibrium state of $\vec{H}_{A,\mathcal{Q}}$, *i.e.*, $\vec{H}_{A,\mathcal{Q}}(p_e) = 0$. We have $\vec{H}_{B,\mathcal{R}}(\Psi(p_e)) = \Psi_0 \cdot \vec{H}_{A,\mathcal{Q}}(p_e) = 0$. That is, $\Psi(p_e)$ is an equilibrium point of $\vec{H}_{B,\mathcal{R}}$. If p_e and q_e are equilibria of $\vec{H}_{A,\mathcal{Q}}$ such that $\Psi(p_e) = \Psi(q_e)$, then $p_e = \Psi^{-1}(\Psi(p_e)) = \Psi^{-1}(\Psi(q_e)) = q_e$. Lastly, let p'_e be an equilibrium point of $\vec{H}_{B,\mathcal{R}}$. Then $\vec{H}_{A,\mathcal{Q}}(\Psi^{-1}(p'_e)) = \Psi_0^{-1} \cdot \vec{H}_{B,\mathcal{R}}(p'_e) = 0$, *i.e.*, $\Psi^{-1}(p'_e)$ is an equilibrium point of $\vec{H}_{A,\mathcal{Q}}$. Therefore the equilibria of $\vec{H}_{A,\mathcal{Q}}$ and $\vec{H}_{B,\mathcal{R}}$ are in a one-to-one correspondence. ■

From the definition of affine equivalence, it is straightforward to arrive at the following three sufficient conditions. For later reference, we shall denote these conditions by ($\mathfrak{E}1$), ($\mathfrak{E}2$) and ($\mathfrak{E}3$).

3.1.3 Proposition. ([15, 17]) *Let $H_{A,\mathcal{Q}}$ be a quadratic Hamilton-Poisson system on $\mathfrak{g}^*_-$. Then $H_{A,\mathcal{Q}}$ is A-equivalent to*

($\mathfrak{E}1$) *$H_{A,\mathcal{Q}} \circ \Psi$, for any linear Poisson automorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$;*

($\mathfrak{E}2$) *$H_{A,\mathcal{Q}} + C$, for any Casimir function $C : \mathfrak{g}^* \to \mathbb{R}$;*

($\mathfrak{E}3$) *$H_{A,r\mathcal{Q}}$, for any $r \neq 0$.*

Proof.

($\mathfrak{E}1$) Let $F \in C^\infty(\mathfrak{g}^*)$ and let $G = H_{A,\mathcal{Q}} \circ \Psi$. Then

$$\begin{aligned}(\vec{H}_{A,\mathcal{Q}} \circ \Psi)[F] &= \vec{H}_{A,\mathcal{Q}}[F] \circ \Psi = \{F, H_{A,\mathcal{Q}}\} \circ \Psi \\ &= \{F \circ \Psi, G\} = \vec{G}[F \circ \Psi] = (\Psi \cdot \vec{G})[F].\end{aligned}$$

Since F is arbitrary, it follows that $\Psi \cdot \vec{G} = \vec{H}_{A,\mathcal{Q}} \circ \Psi$, *i.e.*, $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{A,\mathcal{Q}} \circ \Psi$.

($\mathfrak{E}2$) Let $G = H_{A,\mathcal{Q}} + C$. Then for every $F \in C^\infty(\mathfrak{g}^*)$ we have

$$\begin{aligned}\vec{G}[F] &= \{F, H_{A,\mathcal{Q}} + C\} = \{F, H_{A,\mathcal{Q}}\} + \{F, C\} \\ &= \{F, H_{A,\mathcal{Q}}\} = \vec{H}_{A,\mathcal{Q}}[F].\end{aligned}$$

That is, $\vec{G} = \vec{H}_{A,\mathcal{Q}}$, and so $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{A,\mathcal{Q}} + C$.

($\mathfrak{E}3$) Let Ψ be the linear isomorphism $\Psi : p \mapsto \frac{1}{r}p$. Then

$$\begin{aligned}(\Psi \cdot \vec{H}_{A,\mathcal{Q}})(p) - (\vec{H}_{A,r\mathcal{Q}} \circ \Psi)(p) &= \tfrac{1}{r}\vec{H}_{A,\mathcal{Q}}(p) - \vec{H}_{A,r\mathcal{Q}}(\tfrac{1}{r}p) \\ &= \tfrac{1}{r}\vec{L}_A(p) + \tfrac{1}{r}\vec{H}_{\mathcal{Q}}(p) - \vec{L}_A(\tfrac{1}{r}p) - \vec{H}_{r\mathcal{Q}}(\tfrac{1}{r}p).\end{aligned}$$

Since $\vec{L}_A$ is linear, we have that $\vec{L}_A(\frac{1}{r}p) = \frac{1}{r}\vec{L}_A(p)$. Moreover, as $\mathbf{d}H_{r\mathcal{Q}}(p) = r\mathbf{d}H_{\mathcal{Q}}(p) = \mathbf{d}H_{\mathcal{Q}}(rp)$ and $\operatorname{ad}^*_{r\mathbf{d}H_{\mathcal{Q}}(p)} p = r \operatorname{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)} p$, we get

$$\begin{aligned}\tfrac{1}{r}\vec{H}_{\mathcal{Q}}(p) - \vec{H}_{r\mathcal{Q}}(\tfrac{1}{r}p) &= \tfrac{1}{r} \operatorname{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)} p - \operatorname{ad}^*_{\mathbf{d}H_{r\mathcal{Q}}(\frac{1}{r}p)} (\tfrac{1}{r}p) \\ &= \tfrac{1}{r} \operatorname{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)} p - \tfrac{1}{r} \operatorname{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(p)} p = 0.\end{aligned}$$

That is, $\Psi \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{A,r\mathcal{Q}} \circ \Psi$, and so $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{A,r\mathcal{Q}}$. ∎

The classification of inhomogeneous quadratic Hamilton-Poisson systems shall draw upon those linear Poisson automorphisms that leave a homogeneous system invariant (in a certain sense). This motivates the following terminology. Let $H_{\mathcal{Q}}$ be a homogeneous quadratic Hamilton-Poisson system on $\mathfrak{g}^*_-$. By a **linear Poisson symmetry** we mean a linear Poisson automorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $H_{\mathcal{Q}} \circ \Psi = H_{r\mathcal{Q}} + C$, for some $r > 0$ and some Casimir function C. (In other words, a linear Poisson symmetry of $H_{\mathcal{Q}}$ is a linear Poisson automorphism that leaves the Hamiltonian function H_Q invariant up to dilations or the addition of a Casimir function.)

The next proposition characterises A-equivalence for homogeneous Hamilton-Poisson systems in terms of linear isomorphisms. We shall require a couple of technical lemmas for the proof.

3.1.4 Lemma. *Let H_Q be a homogeneous quadratic Hamilton-Poisson system on $\mathfrak{g}^*_-$ and let $\Psi_0 : \mathfrak{g}^* \to \mathfrak{g}^*$ be a linear isomorphism. The maps $p \mapsto \mathbf{d}H_Q(p)$ and $p \mapsto \mathrm{ad}^*_{\mathbf{d}H_Q(\Psi_0(p))}\, q$, $q \in \mathfrak{g}^*$ are linear.*

Proof. Let $p_1, p_2 \in \mathfrak{g}^*$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Write H_Q in coordinates as $H_Q(p) = \frac{1}{2} p Q p^\top$, where Q is a symmetric positive semidefinite matrix. Then

$$\mathbf{d}H_Q(p) = \tfrac{1}{2}(pQ + Qp^\top) = \tfrac{1}{2} p (Q + Q^\top) = pQ.$$

(As Q is symmetric, we have $Q = Q^\top$.) This is clearly a linear map. Next, since the map $X \mapsto \mathrm{ad}^*_X\, q$ is linear, we have

$$\begin{aligned}
\mathrm{ad}^*_{\mathbf{d}H_Q(\Psi_0(\lambda_1 p_1 + \lambda_2 p_2))}\, q &= \mathrm{ad}^*_{\mathbf{d}H_Q(\lambda_1 \Psi_0(p_1) + \lambda_2 \Psi_0(p_2))}\, q \\
&= \mathrm{ad}^*_{\lambda_1 \mathbf{d}H_Q(\Psi_0(p_1)) + \lambda_2 \mathbf{d}H_Q(\Psi_0(p_2))}\, q \\
&= \lambda_1\, \mathrm{ad}^*_{\mathbf{d}H_Q(\Psi_0(p_1))}\, q + \lambda_2\, \mathrm{ad}^*_{\mathbf{d}H_Q(\Psi_0(p_2))}\, q.
\end{aligned}$$

Thus $p \mapsto \mathrm{ad}^*_{\mathbf{d}H_Q(\Psi_0(p))}\, q$ is linear. ∎

3.1.5 Lemma. *Let H_Q be a homogeneous quadratic Hamilton-Poisson system on $\mathfrak{g}^*_-$ and let $\Psi_0 : \mathfrak{g}^* \to \mathfrak{g}^*$ be a linear isomorphism. Then both $T_0(\Psi_0 \cdot \vec{H}_Q)$ and $T_0(\vec{H}_Q \circ \Psi_0)$ are the zero map.*

Proof. We have $T_0(\Psi_0 \cdot \vec{H}_Q) = T_{\vec{H}_Q(0)} \Psi_0 \cdot \mathbf{D}\vec{H}_Q(0)$ and $T_0(\vec{H}_Q \circ \Psi_0) = \mathbf{D}\vec{H}_Q(\Psi_0(0)) \cdot T_0\Psi_0 = \mathbf{D}\vec{H}_Q(0) \cdot T_0\Psi_0$. We show that $\mathbf{D}\vec{H}_Q(0) = 0$, from which the result follows. For brevity, let $H = H_Q$. The linearised vector field $\mathbf{D}\vec{H}(0)$ is given in coordinates as $\mathbf{D}\vec{H}(0) = \left[\frac{\partial \vec{H}_i}{\partial p_j}(0)\right]_{i,j=1}^n$, where $\vec{H}_i$ is the i^{th} component of $\vec{H}$, given by

$$\vec{H}_i(p) = -\langle p, [E_i, \mathbf{d}H(p)]\rangle, \qquad i = 1, \ldots, n.$$

(See section A.3.2.) Write H in coordinates as $H(p) = \frac{1}{2} p Q p^\top$, where $Q = [q_{ij}]_{i,j=1}^n$ is symmetric and positive semidefinite. Expressed in terms of a basis $(E_i)_{i=1}^n$ of $\mathfrak{g}$, we have

$$\mathbf{d}H(p) = pQ = \sum_{k=1}^n p_k q_{k1} E_1 + \cdots + \sum_{k=1}^n p_k q_{kn} E_n.$$

Let the Lie bracket be specified by $[E_i, E_j] = \sum_{\ell=1}^n c_{ij}^\ell E_\ell$. Then

$$\begin{aligned}
[E_i, \mathbf{d}H(p)] &= \sum_{k=1}^n p_k q_{k1} [E_i, E_1] + \cdots + \sum_{k=1}^n p_k q_{kn} [E_i, E_n] \\
&= \sum_{k,\ell=1}^n p_k q_{k1} c_{i1}^\ell E_\ell + \cdots + \sum_{k,\ell=1}^n p_k q_{kn} c_{in}^\ell E_\ell \\
&= \sum_{k,\ell=1}^n p_k q_{k\ell} c_{i\ell}^1 E_1 + \cdots + \sum_{k,\ell=1}^n p_k q_{k\ell} c_{i\ell}^n E_n, \qquad i = 1, \ldots, n.
\end{aligned}$$

Thus we have the following coordinate expression for $\vec{H}_i$:

$$\begin{aligned}\vec{H}_i(p) &= -\langle p, [E_i, \mathbf{d}H(p)] \\ &= -\sum_{k,\ell=1}^{n} p_1 p_k q_{k\ell} c_{i\ell}^1 - \cdots - \sum_{k,\ell=1}^{n} p_n p_k q_{k\ell} c_{i\ell}^n \\ &= -\sum_{k,\ell,m=1}^{n} p_m p_k q_{k\ell} c_{i\ell}^m, \qquad i = 1, \ldots, n.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial \vec{H}_i}{\partial p_j}(p) &= -\sum_{k,\ell,m=1}^{n} \frac{\partial}{\partial p_j}(p_m p_k) q_{k\ell} c_{i\ell}^m \\ &= -\sum_{k,\ell,m=1}^{n} (\delta_{jm} p_k + p_m \delta_{jk}) q_{k\ell} c_{i\ell}^m, \qquad i, j = 1, \ldots, n.\end{aligned}$$

It is clear that $\frac{\partial \vec{H}_i}{\partial p_j}(0) = 0$ for each $i, j = 1, \ldots, n$. Consequently $\mathbf{D}\vec{H}(0) = 0$, and so $T_0(\Psi_0 \cdot \vec{H}_\mathcal{Q})$ and $T_0(\vec{H}_\mathcal{Q} \circ \Psi_0)$ are both the zero map. ■

3.1.6 Proposition. *Let $H_\mathcal{Q}$ and $H_\mathcal{R}$ be homogeneous quadratic Hamilton-Poisson systems on $\mathfrak{g}^*_-$. $H_\mathcal{Q}$ is A-equivalent to $H_\mathcal{R}$ if and only if there exists a linear isomorphism $\Psi_0 : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $\Psi_0 \cdot \vec{H}_\mathcal{Q} = \vec{H}_\mathcal{R} \circ \Psi_0$.*

Proof. Suppose there exists a linear isomorphism $\Psi_0 : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $\Psi_0 \cdot \vec{H}_\mathcal{Q} = \vec{H}_\mathcal{R} \circ \Psi_0$. Since every linear isomorphism is affine, it follows that $\vec{H}_\mathcal{Q}$ and $H_\mathcal{R}$ are A-equivalent.

Conversely, suppose there exists an affine isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_\mathcal{Q} = \vec{H}_\mathcal{R} \circ \Psi$. From the bilinearity of the Lie bracket, we have that ad^*_X is a linear map on $\mathfrak{g}^*$, for every $X \in \mathfrak{g}$. Similarly, $X \mapsto \mathrm{ad}^*_X p$ is a linear map on $\mathfrak{g}$ for every $p \in \mathfrak{g}^*$. Lastly, by lemma 3.1.4, the differential $\mathbf{d}H_\mathcal{Q}(p)$ is linear in p. Consequently,

$$\begin{aligned}\vec{H}_\mathcal{R}(\Psi_0(p) + q) &= \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(p)+q)}(\Psi_0(p) + q) \\ &= \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(p)+q)} \Psi_0(p) + \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(p)+q)}(q) \\ &= \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(p))} \Psi_0(p) + \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(p))} q + \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(q)} \Psi_0(p) + \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(q)} q \\ &= (\vec{H}_\mathcal{R} \circ \Psi_0)(p) + \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(p))} q + \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(q)} \Psi_0(p) + \vec{H}_\mathcal{R}(q) \\ &= (\vec{H}_\mathcal{R} \circ \Psi_0)(p) + F(p) + G(p) + \vec{H}_\mathcal{R}(q).\end{aligned}$$

(Here $F(p) = \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(p))} q$ and $G(p) = \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(q)} \Psi_0(p)$.) Using this, we can expand terms in $(\Psi_0 \cdot \vec{H}_\mathcal{Q})(p) - (\vec{H}_\mathcal{R} \circ \Psi)(p) = 0$, to get

$$(\Psi_0 \cdot \vec{H}_\mathcal{Q})(p) - (\vec{H}_\mathcal{R} \circ \Psi_0)(p) - F(p) - G(p) - \vec{H}_\mathcal{R}(q) = 0. \tag{3.1.1}$$

Take $p = 0$ in (3.1.1). We have $\vec{H}_\mathcal{Q}(0) = \mathrm{ad}^*_{\mathbf{d}H_\mathcal{Q}(0)} 0 = 0$, $\vec{H}_\mathcal{R}(0) = 0$, $F(0) = \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(\Psi_0(0))} q = 0$ and $G(0) = \mathrm{ad}^*_{\mathbf{d}H_\mathcal{R}(0)} \Psi_0(0) = 0$. Thus $\vec{H}_\mathcal{R}(q) = 0$. Interpret both sides of (3.1.1) as maps from $\mathfrak{g}^*$ to $\mathfrak{g}^*$, and linearise both sides at the origin:

$$T_0(\Psi_0 \cdot \vec{H}_\mathcal{Q}) - T_0(\vec{H}_\mathcal{R} \circ \Psi_0) - T_0 F - T_0 G = 0. \tag{3.1.2}$$

From lemma 3.1.5 we have that $T_0(\Psi_0 \cdot \vec{H}_{\mathcal{Q}}) = T_0(\vec{H}_{\mathcal{R}} \circ \Psi_0) = 0$. By lemma 3.1.4, F is a linear map. Similarly, by the linearity of $\mathrm{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(q)}$ and Ψ_0 we have that G is linear. For a linear map L, we make the identification $T_0L \longleftrightarrow L$. The last two terms on the left-hand side of (3.1.2) are then $T_0F = F$ and $T_0G = G$, from which it follows that $F + G = 0$. From (3.1.1) we then get that $\Psi_0 \cdot \vec{H}_{\mathcal{Q}} - \vec{H}_{\mathcal{R}} \circ \Psi_0 = 0$. ■

Lastly, we show that if two inhomogeneous Hamilton-Poisson systems are A-equivalent, then their associated homogeneous systems must also be equivalent. In particular, we shall make use of the contrapositive form of this result, *viz.* if the associated homogeneous systems of two inhomogeneous Hamilton-Poisson systems are not equivalent, then the inhomogeneous systems cannot be equivalent.

3.1.7 PROPOSITION. *Let $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ be A-equivalent inhomogeneous quadratic Hamilton-Poisson systems on $\mathfrak{g}^*_-$ of the form $H_{A,\mathcal{Q}} = L_A + H_{\mathcal{Q}}$ and $H_{B,\mathcal{R}} = L_B + H_{\mathcal{R}}$. Then $H_{\mathcal{Q}}$ is A-equivalent to $H_{\mathcal{R}}$.*

PROOF. Suppose there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$. We have

$$\begin{aligned}\vec{H}_{B,\mathcal{R}}(\Psi_0(p) + q) &= \vec{L}_B(\Psi_0(p) + q) + \vec{H}_{\mathcal{R}}(\Psi_0(p) + q) \\ &= (\vec{L}_B \circ \Psi_0)(p) + \vec{L}_B(q) + (\vec{H}_{\mathcal{R}} \circ \Psi_0)(p) + \vec{H}_{\mathcal{R}}(q) + F(p) + G(p)\end{aligned}$$

where $F(p) = \mathrm{ad}^*_{\mathbf{d}H_{\mathcal{R}}(\Psi_0(p))}\, q$ and $G(p) = \mathrm{ad}^*_{\mathbf{d}H_{\mathcal{R}}(q)}\, \Psi_0(p)$. (See the proof of proposition 3.1.6.) Using this we can expand terms in $(\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}})(p) - (\vec{H}_{B,\mathcal{R}} \circ \Psi)(p) = 0$, to get

$$\begin{aligned}(\Psi_0 \cdot \vec{L}_A)(p) + (\Psi_0 \cdot \vec{H}_{\mathcal{Q}})(p) - (\vec{L}_B \circ \Psi_0)(p) - \vec{L}_B(q) - (\vec{H}_{\mathcal{R}} \circ \Psi_0)(p) \\ - \vec{H}_{\mathcal{R}}(q) - F(p) - G(p) = 0.\end{aligned} \tag{3.1.3}$$

Setting $p = 0$ yields $\vec{L}_B(q) + \vec{H}_{\mathcal{R}}(q) = 0$. We may interpret both sides of 3.1.3 as maps from $\mathfrak{g}^*$ to $\mathfrak{g}^*$, and so may linearise both sides at the origin:

$$T_0(\Psi_0 \cdot \vec{L}_A) + T_0(\Psi_0 \cdot \vec{H}_{\mathcal{Q}}) - T_0(\vec{L}_B \circ \Psi_0) - T_0(\vec{H}_{\mathcal{R}} \circ \Psi_0) - T_0F - T_0G = 0.$$

By lemma 3.1.5 we have $T_0(\Psi_0 \cdot \vec{L}_A) = T_0(\Psi_0 \cdot \vec{H}_{\mathcal{Q}}) = 0$. Furthermore, F and G are linear maps. (This follows from lemma 3.1.4 and the linearity of $\mathrm{ad}^*_{\mathbf{d}H_{\mathcal{Q}}(q)}$ and Ψ_0.) We make the identification $T_0F \longleftrightarrow F$ and $T_0G \longleftrightarrow G$. Then (43) becomes $(\Psi_0 \cdot \vec{L}_A)(p) - (\vec{L}_B \circ \Psi_0)(p) - F(p) - G(p) = 0$, and so (3.1.3) reduces to $(\Psi_0 \cdot \vec{H}_{\mathcal{Q}})(p) - (\vec{H}_{\mathcal{Q}} \circ \Psi_0)(p) = 0$. That is, $H_{\mathcal{Q}}$ is A-equivalent to $H_{\mathcal{R}}$. ■

3.1.1 The (minus) Lie-Poisson structure on $\mathfrak{se}(1,1)^*$

Let $H \in C^\infty(\mathfrak{se}(1,1)^*)$ be a Hamiltonian function. Recall that the equations of motion for H are given componentwise by $\dot{p}_i = -\langle p, [E_i, \mathbf{d}H(p)]\rangle$ for $i = 1, 2, 3$. (See section A.3.2.) Explicitly, we have the following equations of motion for H:

$$\begin{cases} \dot{p}_1 = \dfrac{\partial H}{\partial p_3} p_2 \\ \dot{p}_2 = \dfrac{\partial H}{\partial p_3} p_1 \\ \dot{p}_3 = -\dfrac{\partial H}{\partial p_2} p_1 - \dfrac{\partial H}{\partial p_1} p_2. \end{cases} \tag{3.1.4}$$

3.1.8 Proposition. *In terms of the dual basis $(E_i^*)_{i=1}^3$, the group of linear Poisson automorphisms of* $\mathfrak{se}(1,1)^*_-$ *is*

$$\left\{ p \mapsto p \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} : v, w, x, y \in \mathbb{R},\ \varsigma \in \{-1, 1\},\ x^2 \neq y^2 \right\}.$$

Proof. By proposition 1.1.17, every Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ is of the form

$$\psi = \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}.$$

Since linear Poisson automorphisms of a (minus) Lie-Poisson space $\mathfrak{g}^*_-$ are exactly the dual maps of the Lie algebra automorphisms of $\mathfrak{g}$ (see proposition A.3.3), and using the convention that elements of $\mathfrak{g}^*$ are written as row vectors, it follows that every linear Poisson automorphism of $\mathfrak{se}(1,1)^*_-$ takes the form

$$\Psi : p \mapsto p \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}. \qquad \blacksquare$$

3.1.9 Proposition. *The function $C(p) = p_1^2 - p_2^2$ is the only functionally independent Casimir function on* $\mathfrak{se}(1,1)^*_-$*.*

Proof. Let $F = f(C)$, where $f : \mathbb{R} \to \mathbb{R}$ is arbitrary. For brevity, write F_{p_i} and C_{p_i} for the partial derivatives $\partial F / \partial p_i$ and $\partial C / \partial p_i$, respectively. Then $F_{p_3} = \dot{f}(C) C_{p_3} = 0$ and

$$-p_1 F_{p_2} - p_2 F_{p_1} = -p_1 \dot{f}(C) C_{p_2} - p_2 \dot{f}(C) C_{p_1} = 2\dot{f}(C)(p_1 p_2 - p_2 p_1) = 0.$$

Consequently, $\vec{F} = 0$, and so F is a Casimir function. By the Cauchy-Kowalevski theorem for the existence and uniqueness of solutions to partial differential equations, it follows that F is the only solution to this partial differential equation, *i.e.*, the only Casimir function. Therefore $C(p) = p_1^2 - p_2^2$ is the only functionally independent Casimir function on $\mathfrak{se}(1,1)^*_-$. $\blacksquare$

Henceforth,, whenever we mention a Casimir function C on $\mathfrak{se}(1,1)^*_-$, we shall be referring to the specific Casimir function $C(p) = p_1^2 - p_2^2$.

3.2 Homogeneous Systems

We now proceed to classify, under affine equivalence, the homogeneous quadratic Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$. By proposition 3.1.6, it suffices to consider equivalence of these systems under *linear* isomorphisms. We shall make use of the sufficient conditions ($\mathfrak{E}1$), ($\mathfrak{E}2$) and ($\mathfrak{E}3$) of proposition 3.1.3 to obtain a list of potential representatives. In the general case, one would then employ linear isomorphisms in order to further simplify the potential representatives. However, for homogeneous systems on $\mathfrak{se}(1,1)^*_-$ it turns out that the use of ($\mathfrak{E}1$), ($\mathfrak{E}2$) and ($\mathfrak{E}3$) is sufficient to arrive at a complete classification (a fact we capture in a corollary to the following theorem). To complete the classification, we verify that none of the potential representatives are equivalent to each other. Table 3.1 provides a summary of the results obtained. Section B.3.1 lists the supporting Mathematica code.

3.2.1 THEOREM. *Any homogeneous quadratic Hamilton-Poisson system H_Q on $\mathfrak{se}(1,1)^*_-$ is A-equivalent to exactly one of the following systems:*

$$H_0(p) = 0 \qquad H_1(p) = \tfrac{1}{2}p_1^2 \qquad H_2(p) = \tfrac{1}{2}(p_1 + p_2)^2$$
$$H_3(p) = \tfrac{1}{2}p_3^2 \qquad H_4(p) = \tfrac{1}{2}(p_1^2 + p_3^2) \qquad H_5(p) = \tfrac{1}{2}\big[(p_1 + p_2)^2 + p_3^2\big].$$

PROOF. Let $H_Q(p) = \frac{1}{2}pQp^\top$ be an arbitrary homogeneous quadratic Hamilton-Poisson system, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix} \qquad (a_1, a_2, a_3 \geq 0)$$

is positive semidefinite (PSD). Consider the case $a_3 = 0$. The 2×2 principal minors of Q are $a_1a_2 - b_1^2$, $-b_2^2$ and $-b_3^3$. Since Q is PSD, the principal minors must be nonnegative, and so $b_2 = b_3 = 0$. Furthermore, every PSD matrix $Q = [q_{ij}]$ satisfies the inequality $|q_{ij}| \leq \sqrt{q_{ii}q_{jj}} \leq \frac{1}{2}(q_{ii} + q_{jj})$. In particular, $|b_1| \leq \frac{1}{2}(a_1 + a_2)$, which implies that $4b_1^2 \leq (a_1 + a_2)^2$. This motivates the following three (sub)cases: $a_1 + a_2 = 0$, $4b_1^2 \neq (a_1 + a_2)^2 > 0$ or $4b_1^2 = (a_1 + a_2)^2 > 0$. If $a_1 + a_2 = 0$, then $a_1 = a_2 = b_1 = 0$ (as $a_1, a_2 \geq 0$), hence $H_Q = H_0$.

Suppose $4b_1^2 \neq (a_1 + a_2)^2 > 0$. If $b_1 \neq 0$, then

$$\Psi_1 : p \mapsto p\psi_1, \qquad \psi_1 = \begin{bmatrix} x & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism (for $x^2 \neq 1$) such that

$$\psi_1 Q \psi_1^\top = \begin{bmatrix} a_2 + 2b_1x + a_1x^2 & b_1 + (a_1 + a_2)x + b_1x^2 & 0 \\ b_1 + (a_1 + a_2)x + b_1x^2 & a_1 + 2b_1x + a_2x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The discriminant of the quadratic $b_1 + (a_1 + a_2)x + b_1x^2$ is $\Delta = (a_1 + a_2)^2 - 4b_1^2$. Since $4b_1^2 < (a_1 + a_2)^2$, we have $\Delta > 0$, and so the equation $b_1 + (a_1 + a_2)x + b_1x^2 = 0$ has two distinct real solutions for x, *viz.*, $x = -\frac{a_1 + a_2 \pm \sqrt{(a_1+a_2)^2 - 4b_1^2}}{2b_1}$. In particular, there is a solution for x with $x^2 \neq 1$. Consequently, we get $Q' = \psi_1 Q \psi_1^\top = \operatorname{diag}(a_1', a_2', 0)$, for some $a_1', a_2' \geq 0$. (If $b_1 = 0$, then we have $Q = \operatorname{diag}(a_1, a_2, 0)$, and so $Q' = Q$ with $a_1' = a_1$ and $a_2' = a_2$.) If $a_1' = a_2' = 0$, then $H_Q \circ \Psi_1 = H_0$. Otherwise, $\Psi_2 : p \mapsto p\psi_2$, $\psi_2 = \operatorname{diag}\big(\frac{1}{\sqrt{a_1' + a_2'}}, \frac{1}{\sqrt{a_1' + a_2'}}, 1\big)$ is an automorphism such that $\psi_2 Q' \psi_2^\top = \operatorname{diag}\big(\frac{a_1'}{a_1' + a_2'}, \frac{a_2'}{a_1' + a_2'}, 0\big)$. Therefore $H_Q \circ (\Psi_1 \circ \Psi_2) + \frac{a_2'}{2(a_1' + a_2')}C = H_1$, and so H_Q is A-equivalent to H_1.

Suppose $4b_1^2 = (a_1 + a_2)^2 > 0$, *i.e.*, $b_1 = \frac{1}{2}\sigma(a_1 + a_2)$ for some $\sigma \in \{-1, 1\}$. Then $\Psi_3 : p \mapsto p\psi_3$, $\psi_3 = \operatorname{diag}\big(\sqrt{\frac{2}{a_1 + a_2}}, \sigma\sqrt{\frac{2}{a_1 + a_2}}, \sigma\big)$ is an automorphism such that

$$\psi_3 Q \psi_3^\top = \begin{bmatrix} \frac{2a_1}{a_1 + a_2} & 1 & 0 \\ 1 & \frac{2a_2}{a_1 + a_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $H_Q \circ \Psi_3 - \frac{a_1 - a_2}{2(a_1 + a_2)}C = H_2$, and so H_Q is A-equivalent to H_2.

On the other hand, consider the case $a_3 > 0$. We have an automorphism

$$\Psi_4 : p \mapsto p\psi_4, \qquad \psi_4 = \begin{bmatrix} 1 & 0 & -\frac{b_2}{a_3} \\ 0 & 1 & -\frac{b_3}{a_3} \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$Q' = \tfrac{1}{a_3}\psi_4 Q \psi_4^\top = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since Q is PSD, we have that Q' is also PSD. Moreover, the lower right component of Q' is invariant under linear Poisson automorphisms:

$$\psi Q' \psi^\top = \begin{bmatrix} a_1'x^2 + 2b_1xy + a_2y^2 + v^2 & \varsigma b_1x^2 + \varsigma(a_1+a_2)xy + \varsigma b_1y^2 + vw & \varsigma v \\ \varsigma b_1x^2 + \varsigma(a_1+a_2)xy + \varsigma b_1y^2 + vw & a_2x^2 + 2b_1xy + a_1y^2 + w^2 & \varsigma w \\ \varsigma v & \varsigma w & 1 \end{bmatrix}.$$

Accordingly, we can use the same argument as for the case $a_3 = 0$ in order to show that H_Q is A-equivalent to H_3, H_4 or H_5.

It remains to be verified that no two representatives are equivalent. As a typical case, we show that H_1 is not A-equivalent to H_2. Suppose there exists a linear isomorphism $\Psi : p \mapsto p\psi$, $\psi = [\psi_{ij}]$ such that $(\Psi \cdot \vec{H}_1)(p) = (\vec{H}_2 \circ \Psi)(p)$. Then

$$\begin{cases} \psi_{31}p_1p_2 = 0 \\ \psi_{32}p_1p_2 = 0 \\ \psi_{33}p_1p_2 = [(\psi_{11}+\psi_{12})p_1 + (\psi_{21}+\psi_{22})p_2 + (\psi_{31}+\psi_{32})p_3]^2 \end{cases}$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing coefficients, it follows that $\psi_{31} = \psi_{32} = 0$, $\psi_{11} = -\psi_{12}$ and $\psi_{22} = -\psi_{21}$. Consequently $\det\psi = 0$, a contradiction. Hence no such Ψ exists, and so H_1 is not equivalent to H_2. Verifying that none of the other representatives are A-equivalent follows a similar argument. (See section B.3.1 for the Mathematica code that performs these verifications.) ■

3.2.2 Corollary. *Let H_Q be a homogeneous quadratic Hamilton-Poisson system on $\mathfrak{se}(1,1)^*_-$. There exists a linear Poisson automorphism Ψ and real numbers $r > 0$, $k \in \mathbb{R}$ such that $H_{rQ} \circ \Psi + kC = H_i$ for exactly one $i \in \{0, \ldots, 5\}$.*

Proof. This follows from the proof of theorem 3.2.1. ■

3.3 Inhomogeneous Systems

Having classified the homogeneous quadratic Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$, we now move on to the inhomogeneous systems. (A summary of the results obtained is given in table 3.1.) This classification will make use of the results for the homogeneous case (in a sense that will be made apparent below). The next result ensures we can always bring the homogeneous part of an inhomogeneous system to one of the six normal forms found in theorem 3.2.1.

3.3.1 Proposition. *Let $(\mathfrak{se}(1,1)^*_-, H_{A,Q})$ be an inhomogeneous quadratic Hamilton-Poisson system. Then $H_{A,Q}$ is A-equivalent to the system $L_B + H_i$ for some $B \in \mathfrak{se}(1,1)$ and exactly one $i \in \{0, \ldots, 5\}$.*

PROOF. We have $H_{A,\mathcal{Q}} = L_A + H_{\mathcal{Q}}$. Since $H_{\mathcal{Q}}$ is a homogeneous quadratic Hamilton-Poisson system, by corollary 3.2.2 there exists a linear Poisson automorphism $\Psi : p \mapsto p\psi$, $r > 0$, $k \in \mathbb{R}$ and exactly one $i \in \{0, \dots, 5\}$ such that $H_{r\mathcal{Q}} \circ \Psi + kC = H_i$. (That is, $H_{\mathcal{Q}}$ is A-equivalent to H_i using the sufficient conditions ($\mathfrak{E}$1), ($\mathfrak{E}$2) and ($\mathfrak{E}$3).) Consequently,

$$H_{A,r\mathcal{Q}} \circ \Psi + kC = L_A \circ \Psi + H_{r\mathcal{Q}} \circ \Psi + kC = L_{\psi \cdot A} + H_i = L_B + H_i,$$

where $B = \psi \cdot A$. That is, $H_{A,\mathcal{Q}}$ is A-equivalent to $L_B + H_i$. ■

Using the previous result, we shall assume, without loss of generality, that any inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{se}(1,1)^*_-$ is of the form $L_A + H_i$, for some $A \in \mathfrak{se}(1,1)$ and exactly one $i \in \{0, \dots, 5\}$. Moreover, proposition 3.1.7 implies that an inhomogeneous system of the form $L_A + H_i$ cannot be A-equivalent to a system of the form $L_B + H_j$ for $i \neq j$. Consequently, we classify the inhomogeneous systems of each different form independently of each other.

We outline the approach followed for the classification of inhomogeneous systems. We desire the homogeneous part of each inhomogeneous normal form to be as simple as possible, *i.e.*, to be one of the normal forms $H_0, \dots, H_5$. Thus, for each family $\{L_A + H_i : A \in \mathfrak{se}(1,1)\}$ of inhomogeneous systems, we shall first find the linear Poisson symmetries of the homogeneous system H_i. Linear Poisson symmetries leave H_i invariant up to dilations or addition of a Casimir. Neither dilations nor the addition of the quadratic Casimir function $C(p) = p_1^2 - p_2^2$ affect the linear part of the inhomogeneous system. Accordingly, we can use those linear Poisson symmetries to normalise the linear part L_A of the inhomogeneous representatives, while leaving the homogeneous part invariant. (The problem of normalising L_A is reduced to normalising elements of the Lie algebra $\mathfrak{se}(1,1)$ under dual maps of the linear Poisson symmetries.) We shall then employ general affine isomorphisms in order to further simplify the representatives. To complete the classification, we verify that none of the representatives obtained are equivalent. The calculations for these verifications can become quite lengthy and tedious. As such, we do not include full details, but rather illustrate the approach for some typical cases. The remaining cases are covered by the accompanying MATHEMATICA code. Sections B.3.3 and B.3.4 list the MATHEMATICA code.

The next result determines the linear Poisson symmetries of the homogeneous normal forms $H_0, \dots, H_5$. We denote a symmetry by $\Psi^{(i)}$ if it is a symmetry of H_i. The supporting MATHEMATICA code may be found in section B.3.2.

3.3.2 PROPOSITION. *The linear Poisson symmetries of H_i, for each $i = 0, \dots, 5$, are the linear Poisson automorphisms of the form $\Psi^{(i)} : p \mapsto p\psi^{(i)}$, where for each H_i, $\psi^{(i)}$ is of the form given below*:

$$H_0 \;:\; \psi^{(0)} = \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \qquad H_3 \;:\; \psi^{(3)} = \begin{bmatrix} x & y & 0 \\ \varsigma y & \varsigma x & 0 \\ 0 & 0 & \varsigma \end{bmatrix}$$

$$H_1 \;:\; \psi^{(1)} = \begin{bmatrix} x & 0 & v \\ 0 & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}, \begin{bmatrix} 0 & y & v \\ \varsigma y & 0 & w \\ 0 & 0 & \varsigma \end{bmatrix} \qquad H_4 \;:\; \psi^{(4)} = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm\varsigma & 0 \\ 0 & 0 & \varsigma \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 & 0 \\ \pm\varsigma & 0 & 0 \\ 0 & 0 & \varsigma \end{bmatrix}$$

$$H_2 \;:\; \psi^{(2)} = \begin{bmatrix} x & y & v \\ y & x & w \\ 0 & 0 & 1 \end{bmatrix} \qquad H_5 \;:\; \psi^{(5)} = \begin{bmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \; \textit{with} \; (x+y)^2 = 1$$

Proof. Clearly H_0 is invariant under any linear Poisson automorphisms of $\mathfrak{se}(1,1)^*_-$, and hence any automorphism is a linear Poisson symmetry of H_0. Suppose $\Psi : p \mapsto p\psi$ is an arbitrary automorphism. For H_1, we have

$$(H_1 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} x^2 & \varsigma xy & 0 \\ \varsigma xy & y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top.$$

We must have either $y = 0$ or $x = 0$ if H_1 is to be preserved. For $y = 0$ we get

$$(H_1 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top = x^2 H_1(p).$$

This is clearly a dilation of H_1 by $x^2 > 0$, so Ψ is a linear Poisson symmetry of the form $p \mapsto p\psi^{(1)}$. For $x = 0$, we have

$$(H_1 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} 0 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top.$$

Then $H_1 \circ \Psi + \frac{y^2}{2}C = y^2 H_1$, a dilation of H_1 by $y^2 > 0$. Therefore Ψ is again a linear Poisson symmetry of the form $p \mapsto p\psi^{(1)}$. For H_2,

$$(H_2 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} (x+y)^2 & \varsigma(x+y)^2 & 0 \\ \varsigma(x+y)^2 & (x+y)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} p^\top.$$

We see that $\varsigma = 1$, whence $H_2 \circ \Psi = (x+y)^2 H_2$, a dilation of H_2 by $(x+y)^2 > 0$. Therefore Ψ is a linear Poisson symmetry of the form $p \mapsto p\psi^{(2)}$. For H_3,

$$(H_3 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} v^2 & vw & \varsigma v \\ vw & w^2 & \varsigma w \\ \varsigma v & \varsigma w & 1 \end{bmatrix} p^\top.$$

Hence $v = w = 0$ for H_3 to be preserved, in which case Ψ is a linear Poisson symmetry of the form $p \mapsto p\psi^{(3)}$. For H_4, we have

$$(H_4 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} v^2 + x^2 & \varsigma xy + vw & \varsigma v \\ \varsigma xy + vw & w^2 + y^2 & \varsigma w \\ \varsigma v & \varsigma w & 1 \end{bmatrix} p^\top.$$

This implies that $v = w = 0$ and either $y = 0$ or $x = 0$. If $x = 0$ then

$$(H_4 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} 0 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} p^\top.$$

We see that $y = \pm 1$, whence $H_4 \circ \Psi + \frac{1}{2}C = H_4$. Similarly, if $y = 0$ then $x = \pm 1$. Therefore Ψ is a linear Poisson symmetry of the form $p \mapsto p\psi^{(4)}$. Finally, for H_5,

$$(H_5 \circ \Psi)(p) = \tfrac{1}{2}p \begin{bmatrix} v^2 + (x+y)^2 & \varsigma(x+y)^2 + vw & \varsigma v \\ \varsigma(x+y)^2 + vw & w^2 + (x+y)^2 & \varsigma w \\ \varsigma v & \varsigma w & 1 \end{bmatrix} p^\top.$$

Then $v = w = 0$ and $\varsigma = (x+y)^2 = 1$, and so $H_5 \circ \Psi = H_5$. Therefore Ψ is a linear Poisson symmetry of the form $p \mapsto p\psi^{(5)}$.

Conversely, if Ψ is an automorphism of the form $p \mapsto p\psi^{(i)}$, $i \in \{0, \dots, 5\}$, then it preserves H_i up to dilations and the addition of a Casimir function, and so is a linear Poisson symmetry. ∎

The next six subsections perform the classification of inhomogeneous quadratic Hamilton-Poisson systems of the form $L_A + H_i$, for each $i = 0, \dots, 5$. We shall annotate an inhomogeneous system representative with a superscript (i) if it is associated to the homogeneous normal form H_i.

3.3.1 Inhomogeneous systems associated to H_0

3.3.3 Lemma. *Let $A = \sum_{i=1}^3 a_i E_i \in \mathfrak{se}(1,1)$ (with $A \neq 0$). There exists an automorphism of the form $\psi^{(0)}$ such that $\psi^{(0)} \cdot A \in \{E_1, E_1 + E_2, \alpha E_3 : \alpha > 0\}$.*

Proof. Suppose $a_3 = 0$. If $a_1^2 \neq a_2^2$, then

$$\psi_1^{(0)} = \begin{bmatrix} \frac{a_1}{a_1^2 - a_2^2} & -\frac{a_2}{a_1^2 - a_2^2} & 0 \\ -\frac{a_2}{a_1^2 - a_2^2} & \frac{a_1}{a_1^2 - a_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_1^{(0)} \cdot A = E_1$. If $a_1 = a \neq 0$ and $a_2 = \pm a$, then $\psi_2^{(0)} = \operatorname{diag}\left(\frac{1}{a}, \pm\frac{1}{a}, \pm 1\right)$ yields $\psi_2^{(0)} \cdot A = E_1 + E_2$.

Suppose $a_3 \neq 0$. Then

$$\psi_3^{(0)} = \begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & \operatorname{sgn}(a_3) & -\operatorname{sgn}(a_3)\frac{a_2}{a_3} \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix}$$

is an automorphism such that $\psi_3^{(0)} \cdot A = \alpha E_3$, where $\alpha = |a_3| > 0$. ∎

3.3.4 Theorem. *Any inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{se}(1,1)^*_-$ of the form $H_{A,\mathcal{Q}} = L_A + H_0$ is A-equivalent to exactly one of the following systems:*

$$H_1^{(0)}(p) = p_1 \qquad\qquad H_{2,\alpha}^{(0)}(p) = \alpha p_3.$$

Here $\alpha > 0$ parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

Proof. From lemma 3.3.3 there exists a linear Poisson symmetry $\Psi^{(0)} : p \mapsto p\psi^{(0)}$ of H_0 such that $L_A \circ \Psi^{(0)} = L_{\psi^{(0)} \cdot A}$ is equal to one of L_{E_1}, $L_{E_1+E_2}$, or $L_{\alpha E_3}$, for some $\alpha > 0$. Since $H_{A,\mathcal{Q}} \circ \Psi^{(0)} = L_A \circ \Psi^{(0)} + H_0$, it follows that $H_{A,\mathcal{Q}}$ is A-equivalent to one of the systems

$$G_1(p) = p_1, \qquad G_2(p) = p_1 + p_2, \qquad G_{3,\alpha}(p) = \alpha p_3.$$

We show that G_1 is A-equivalent to G_2. Indeed, we have a linear isomorphism

$$\Psi : p \mapsto p \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$(\Psi \cdot \vec{G}_1)(p) = \begin{bmatrix} 0 \\ 0 \\ -p_2 \end{bmatrix}^\top \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -p_2 \end{bmatrix}^\top = (\vec{G}_2 \circ \Psi)(p).$$

That is, $\Psi \cdot \vec{G}_1 = \vec{G}_2 \circ \Psi$, and so G_1 and G_2 are A-equivalent.

We show that G_1 is not A-equivalent to $G_{3,\alpha}$. Indeed, suppose otherwise. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p[\psi_{ij}]$ such that $\Psi_0 \cdot \vec{G}_1 = \vec{G}_{3,\alpha} \circ \Psi$. That is,

$$\begin{bmatrix} -\psi_{31}p_2 \\ -\psi_{32}p_2 \\ -\psi_{33}p_2 \end{bmatrix}^\top = \begin{bmatrix} \alpha(\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3 + q_2) \\ \alpha(\psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3 + q_1) \\ 0 \end{bmatrix}^\top,$$

for all $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$. Comparing coefficients of p_3, we have $\psi_{31} = \psi_{32} = 0$. Furthermore, from the last equation $\psi_{33} = 0$. Thus $\det[\psi_{ij}] = 0$, a contradiction. Therefore no such Ψ exists, and so G_1 is not A-equivalent to $G_{3,\alpha}$.

Lastly, we show that $G_{3,\alpha}$ is A-equivalent to $G_{3,\alpha'}$ only if $\alpha = \alpha'$. Suppose there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p[\psi_{ij}]$ such that $\Psi_0 \cdot \vec{G}_{3,\alpha} = \vec{G}_{3,\alpha'} \circ \Psi$. Then

$$\begin{bmatrix} \alpha(\psi_{21}p_1 + \psi_{11}p_2) \\ \alpha(\psi_{22}p_1 + \psi_{12}p_2) \\ \alpha(\psi_{23}p_1 + \psi_{13}p_2) \end{bmatrix}^\top = \begin{bmatrix} \alpha'(\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3 + q_2) \\ \alpha'(\psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3 + q_1) \\ 0 \end{bmatrix}^\top.$$

From the last equation, we see that $\psi_{23} = \psi_{13} = 0$. Comparing coefficients of p_1 in the first equation, and p_2 in the second, we see that $\alpha\psi_{21} = \alpha'\psi_{12}$ and $\alpha\psi_{12} = \alpha'\psi_{21}$. If $\psi_{21} \neq 0$, then $\frac{\alpha}{\alpha'} = \frac{\psi_{12}}{\psi_{21}} = \frac{\alpha'}{\alpha}$, *i.e.*, $\alpha^2 = (\alpha')^2$. As $\alpha, \alpha' > 0$, it follows that $\alpha = \alpha'$. If $\psi_{21} = 0$, then $\psi_{12} = 0$ and $\det[\psi_{ij}] = \psi_{11}\psi_{22}\psi_{33}$. Consequently, $\psi_{11}, \psi_{22} \neq 0$. Comparing coefficients of p_2 in the first equation and p_1 in the second, we have $\alpha\psi_{11} = \alpha'\psi_{22}$ and $\alpha\psi_{22} = \alpha'\psi_{11}$. As before, this implies that $\alpha = \alpha'$.

Therefore $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of $H_1^{(0)} = G_1$ or $H_{3,\alpha}^{(0)} = G_{3,\alpha}$, where $\alpha > 0$.■

3.3.2 Inhomogeneous systems associated to H_1

3.3.5 Lemma. *Let $A = \sum_{i=1}^3 a_i E_i \in \mathfrak{se}(1,1)$ (with $A \neq 0$). There exists an automorphism of the form $\psi^{(1)}$ such that $\psi^{(1)} \cdot A \in \{E_1 + \beta E_2, \alpha E_3 : \alpha > 0,\ \beta \geq 0\}$.*

Proof. Suppose $a_3 = 0$. If $a_2 = 0$, then $a_1 \neq 0$ (since $A \neq 0$), so $\psi_1^{(1)} = \operatorname{diag}\left(\frac{1}{a_1}, \frac{1}{a_1}, 1\right)$ is an automorphism such that $\psi_1^{(1)} \cdot A = E_1 = E_1 + \beta E_2$, where $\beta = 0$. If $a_2 \neq 0$ and $a_1 = 0$, then the automorphism

$$\psi_2^{(1)} = \begin{bmatrix} 0 & \frac{1}{a_2} & 0 \\ \frac{1}{a_2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives $\psi_2^{(1)} \cdot A = E_1 = E_1 + \beta E_2$, where $\beta = 0$. If $a_2 \neq 0$ and $a_1 \neq 0$, then

$$\psi_3^{(1)} = \begin{bmatrix} 0 & \frac{1}{a_2} & 0 \\ \operatorname{sgn}\left(\frac{a_1}{a_2}\right)\frac{1}{a_2} & 0 & 0 \\ 0 & 0 & \operatorname{sgn}\left(\frac{a_1}{a_2}\right) \end{bmatrix}$$

is an automorphism such that $\psi_2^{(1)} \cdot A = E_1 + \beta E_2$, where $\beta = \left|\frac{a_1}{a_2}\right| > 0$.

Suppose $a_3 \neq 0$. Then

$$\psi_4^{(1)} = \begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & \operatorname{sgn}(a_3) & -\operatorname{sgn}(a_3)\frac{a_2}{a_3} \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix}$$

is an automorphism such that $\psi_4^{(1)} \cdot A = \alpha E_3$, where $\alpha = |a_3| > 0$. ■

3.3.6 Theorem. *Any inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{se}(1,1)^*_-$ of the form $H_{A,\mathcal{Q}} = L_A + H_1$ is A-equivalent to exactly one of the following systems:*

$$H_1^{(1)}(p) = p_1 + \tfrac{1}{2}p_1^2 \qquad H_2^{(1)}(p) = p_1 + p_2 + \tfrac{1}{2}p_1^2 \qquad H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \tfrac{1}{2}p_1^2.$$

Here $\alpha > 0$ parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

Proof. From lemma 3.3.5 there exists a linear Poisson symmetry $\Psi^{(1)} : p \mapsto p\psi^{(1)}$ of H_1 such that $L_A \circ \Psi^{(1)} = L_{\psi^{(1)}\cdot A}$ is equal to one of $L_{E_1+\beta E_2}$ or $L_{\alpha E_3}$, for some $\beta \geq 0$ or $\alpha > 0$. Since $H_{A,\mathcal{Q}} \circ \Psi^{(1)} = L_A \circ \Psi^{(1)} + H_1$, it follows that $H_{A,\mathcal{Q}}$ is A-equivalent to one of the systems

$$G_{1,\beta}(p) = p_1 + \beta p_2 + \tfrac{1}{2}p_1^2, \qquad G_{2,\alpha}(p) = \alpha p_3 + \tfrac{1}{2}p_1^2.$$

We show that $G_{1,\beta}$ is A-equivalent to $G_{1,1}$ when $\beta > 0$. Indeed, if $\beta > 0$, then $\Psi : p \mapsto p \operatorname{diag}\left(1, \frac{1}{\beta}, \frac{1}{\beta}\right)$ is a linear isomorphism such that

$$(\Psi \cdot \vec{G}_{1,\beta})(p) = \begin{bmatrix} 0 \\ 0 \\ -\beta p_1 - p_2 - p_1 p_2 \end{bmatrix}^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ -p_1 - \frac{1}{\beta}p_2 - \frac{1}{\beta}p_2 p_2 \end{bmatrix}^\top = (\vec{G}_{1,1} \circ \Psi)(p).$$

Therefore $G_{1,\beta}$, $\beta > 0$ is A-equivalent to $G_{1,1}$.

We show that $G_{1,0}$ is not A-equivalent to $G_{1,1}$. Indeed, suppose otherwise. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p[\psi_{ij}]$ such that $\Psi_0 \cdot \vec{G}_{1,0} = \vec{G}_{1,1} \circ \Psi$. That is,

$$\begin{bmatrix} -\psi_{31}(1+p_1)p_2 \\ -\psi_{32}(1+p_1)p_2 \\ -\psi_{33}(1+p_1)p_2 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ -\psi_{11}p_1 - \psi_{21}p_2 - \psi_{31}p_3 - q_1 - (\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3 + q_2) \\ \times(\psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3 + q_1 + 1) \end{bmatrix}^\top$$

for every $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$. From the first two equations, we see that $\psi_{31} = \psi_{32} = 0$. Comparing coefficients of p_1^2 and p_2^2, we have $\psi_{11}\psi_{12} = 0$ and $\psi_{21}\psi_{22} = 0$. Suppose $\psi_{11} = 0$; then $\psi_{22} = 0$, otherwise $\det[\psi_{ij}] = \psi_{33}(\psi_{11}\psi_{22} - \psi_{21}\psi_{12}) = 0$. We are left with the equality

$$-\psi_{33}(1+p_1)p_2 = -\psi_{21}p_1 - q_2 - (\psi_{12}p_2 + q_2)(\psi_{21}p_1 + q_1 + 1).$$

Comparing coefficients of p_1, we have $(1+q_1)\psi_{12}=0$, which implies that $1+q_1=0$. This leaves the equality $(\psi_{21}\psi_{12}-\psi_{33})p_1p_2+(\psi_{21}-\psi_{33}+q_2\psi_{21})p_2-1=0$, which is a contradiction for $p_1=p_2=0$. A similar contradiction is reached if $\psi_{12}=0$. Hence, no such Ψ exists, *i.e.*, $G_{1,0}$ is not A-equivalent to $G_{1,1}$.

A similar argument shows that $G_{1,0}$ and $G_{1,1}$ are not A-equivalent to $G_{2,\alpha}$. (See section B.3.4 for the Mathematica code that performs this verification.)

Lastly, we show that $G_{2,\alpha}$ is A-equivalent to $G_{2,\alpha'}$ only if $\alpha=\alpha'$. Suppose there exists an affine isomorphism $\Psi : p\mapsto \Psi_0(p)+q$, $\Psi_0(p)=p[\psi_{ij}]$ such that $\Psi_0\cdot\vec{G}_{2,\alpha}=\vec{G}_{2,\alpha'}\circ\Psi$. Then

$$\begin{bmatrix}\alpha\psi_{21}p_1-\psi_{31}p_1p_2+\alpha\psi_{11}p_2\\ \alpha\psi_{22}p_1-\psi_{32}p_1p_2+\alpha\psi_{12}p_2\\ \alpha\psi_{23}p_1-\psi_{33}p_1p_2+\alpha\psi_{13}p_2\end{bmatrix}^\top$$
$$=\begin{bmatrix}\alpha'(\psi_{12}p_1+\psi_{22}p_2+\psi_{32}p_3+q_2)\\ \alpha'(\psi_{11}p_1+\psi_{21}p_2+\psi_{31}p_3+q_1)\\ -(\psi_{11}p_1+\psi_{21}p_2+\psi_{31}p_3+q_1)(\psi_{12}p_1+\psi_{22}p_2+\psi_{32}p_3+q_2)\end{bmatrix}^\top.$$

We see that $\psi_{31}=\psi_{32}=0$. Comparing coefficients of p_1 and p_2 in the first two equations, we have

$$\begin{cases}\alpha\psi_{21}=\alpha'\psi_{12}\\ \alpha\psi_{12}=\alpha'\psi_{21}\end{cases}\quad\text{and}\quad\begin{cases}\alpha\psi_{11}=\alpha'\psi_{22}\\ \alpha\psi_{22}=\alpha'\psi_{11}\end{cases}$$

If $\psi_{21}\neq 0$, then $\frac{\alpha}{\alpha'}=\frac{\psi_{12}}{\psi_{21}}=\frac{\alpha'}{\alpha}$. Since $\alpha,\alpha'>0$, this implies that $\alpha=\alpha'$. If $\psi_{21}=0$, then $\psi_{12}=0$, since $\alpha,\alpha'>0$. Then $\det[\psi_{ij}]=\psi_{11}\psi_{22}\psi_{33}\neq 0$. Consequently, $\psi_{11}\neq 0$, and so $\frac{\alpha}{\alpha'}=\frac{\psi_{22}}{\psi_{11}}=\frac{\alpha'}{\alpha}$, whence $\alpha=\alpha'$.

Therefore $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of $H_1^{(1)}=G_{1,0}$, $H_1^{(1)}=G_{1,1}$ or $H_{3,\alpha}^{(1)}=G_{2,\alpha}$, where $\alpha>0$. ∎

3.3.3 Inhomogeneous systems associated to H_2

3.3.7 Lemma. *Let* $A=\sum_{i=1}^3 a_iE_i\in\mathfrak{se}(1,1)$ *(with* $A\neq 0$*). There exists an automorphism of the form* $\psi^{(2)}$ *such that* $\psi^{(2)}\cdot A\in\{E_1, E_1+\sigma E_2, \delta E_3 : \delta\neq 0,\ \sigma=\pm1\}$.

Proof. Suppose $a_3=0$. If $a_1^2\neq a_2^2$, then

$$\psi_1^{(2)}=\begin{bmatrix}\frac{a_1}{a_1^2-a_2^2} & -\frac{a_2}{a_1^2-a_2^2} & 0\\ -\frac{a_2}{a_1^2-a_2^2} & \frac{a_1}{a_1^2-a_2^2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

is an automorphism such that $\psi_1^{(2)}\cdot A=E_1$. If $a_1=a\neq 0$ and $a_2=\pm a$, then the automorphism $\psi_2^{(2)}=\operatorname{diag}\left(\frac{1}{a},\frac{1}{a},1\right)$ yields $\psi_2^{(2)}\cdot A=E_1+\sigma E_2$, where $\sigma=\pm1$.

Suppose $a_3\neq 0$. The automorphism

$$\psi_3^{(2)}=\begin{bmatrix}1 & 0 & -\frac{a_1}{a_3}\\ 0 & 1 & -\frac{a_2}{a_3}\\ 0 & 0 & 1\end{bmatrix}$$

gives $\psi_3^{(2)}\cdot A=\delta E_3$, where $\delta=a_3\neq 0$. ∎

3.3.8 Theorem. *Any inhomogeneous quadratic Hamilton-Poisson system on* $\mathfrak{se}(1,1)^*_-$ *of the form* $H_{A,\mathcal{Q}} = L_A + H_2$ *is A-equivalent to exactly one of the following systems:*

$$\begin{aligned} H_1^{(2)}(p) &= p_1 + \tfrac{1}{2}(p_1+p_2)^2 \\ H_2^{(2)}(p) &= p_1 + p_2 + \tfrac{1}{2}(p_1+p_2)^2 \\ H_{3,\delta}^{(2)}(p) &= \delta p_3 + \tfrac{1}{2}(p_1+p_2)^2. \end{aligned}$$

Here $\delta \neq 0$ *parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.*

Proof. From lemma 3.3.7 there exists a linear Poisson symmetry $\Psi^{(2)} : p \mapsto p\psi^{(2)}$ of H_2 such that $L_A \circ \Psi^{(2)} = L_{\psi^{(2)}\cdot A}$ is equal to one of L_{E_1}, $L_{E_1+\sigma E_2}$ or $L_{\delta E_3}$, for some $\sigma \in \{-1, 1\}$ or $\delta \neq 0$. Since $H_{A,\mathcal{Q}} \circ \Psi^{(2)} = L_A \circ \Psi^{(2)} + H_2$, it follows that $H_{A,\mathcal{Q}}$ is A-equivalent to one of the systems

$$G_1(p) = p_1 + \tfrac{1}{2}(p_1+p_2)^2, \quad G_{2,\sigma}(p) = p_1 + \sigma p_2 + \tfrac{1}{2}(p_1+p_2)^2, \quad G_{3,\delta}(p) = \delta p_3 + \tfrac{1}{2}(p_1+p_2)^2.$$

We show that G_1 is A-equivalent to $G_{2,-1}$. Indeed,

$$\Psi : p \mapsto p \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that

$$(\Psi \cdot \vec{G}_1)(p) = \begin{bmatrix} 0 \\ 0 \\ -p_2 - (p_1+p_2)^2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -p_2 - (p_1+p_2)^2 \end{bmatrix}^\top = (\vec{G}_{2,-1} \circ \Psi)(p).$$

That is, $\Psi \cdot \vec{G}_1 = \vec{G}_{1,-1} \circ \Psi$, and so G_1 is A-equivalent to $G_{2,-1}$.

We show that G_1 is not A-equivalent to $G_{2,1}$. Indeed, suppose otherwise. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p[\psi_{ij}]$ such that $\Psi_0 \cdot \vec{G}_1 = \vec{G}_{2,1} \circ \Psi$. That is,

$$\begin{bmatrix} \psi_{31}[p_2 + (p_1+p_2)^2] \\ \psi_{32}[p_2 + (p_1+p_2)^2] \\ \psi_{33}[p_2 + (p_1+p_2)^2] \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ [1 + (\psi_{11}+\psi_{12})p_1 + (\psi_{21}+\psi_{22})p_2 + (\psi_{31}+\psi_{32})p_3 + q_1 + q_2] \\ \times [(\psi_{11}+\psi_{12})p_1 + (\psi_{21}+\psi_{22})p_2 + (\psi_{31}+\psi_{32})p_3 + q_1 + q_2] \end{bmatrix}^\top$$

for every $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$. We see that $\psi_{31} = \psi_{32} = 0$. Comparing coefficients of p_1^2 and p_2^2 on both sides, we get $\psi_{33} = (\psi_{11}+\psi_{12})^2$ and $\psi_{33} = (\psi_{22}+\psi_{21})^2$. If $\psi_{11}+\psi_{12} = \psi_{22}+\psi_{21}$, then we are left with the equality

$$q_1+q_2+(q_1+q_2)^2+(1+2q_1+2q_2)(\psi_{21}+\psi_{22})p_1+(\psi_{21}+\psi_{22}-\psi_{33}+2(\psi_{21}+\psi_{22})(q_1+q_2))p_2 = 0.$$

Consider the coefficient of p_1. Since $\psi_{21}+\psi_{22} \neq 0$ (otherwise $\det[\psi_{ij}] = 0$) we have $1 + 2q_1 + 2q_2 = 0$. This leaves the equality $-\frac{1}{4} - \psi_{33}p_2 = 0$, which is a contradiction for $p_2 = 0$. A similar contradiction is reached if $\psi_{11}+\psi_{12} = -(\psi_{22}+\psi_{21})$. Hence no such Ψ exists, *i.e.*, G_1 is not A-equivalent to $G_{2,1}$.

A similar argument shows that neither G_1 nor $G_{2,1}$ are A-equivalent to $G_{3,\delta}$. (See section B.3.4 for the MATHEMATICA code that performs these verifications.)

Lastly, we show that $G_{3,\delta}$ is A-equivalent to $G_{3,\delta'}$ only if $\delta = \delta'$. Suppose there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p[\psi_{ij}]$ such that $\Psi_0 \cdot \vec{G}_{3,\delta} = \vec{G}_{3,\delta'} \circ \Psi$. Then

$$\begin{bmatrix} \delta\psi_{21}p_1 + \delta\psi_{11}p_2 - \psi_{31}(p_1+p_2)^2 \\ \delta\psi_{22}p_1 + \delta\psi_{12}p_2 - \psi_{32}(p_1+p_2)^2 \\ \delta\psi_{23}p_1 + \delta\psi_{13}p_2 - \psi_{33}(p_1+p_2)^2 \end{bmatrix}^\top$$
$$= \begin{bmatrix} \delta'(\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3 + q_2) \\ \delta'(\psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3 + q_1) \\ -[(\psi_{11} + \psi_{12}p_1 + (\psi_{21} + \psi_{22})p_2 + (\psi_{12} + \psi_{32})p_3 + q_1 + q_2]^2 \end{bmatrix}^\top .$$

We see that $\psi_{31} = \psi_{32} = 0$. Comparing coefficients of p_1 and p_2 in the first two equations, we have

$$\begin{cases} \delta\psi_{21} = \delta'\psi_{12} \\ \delta\psi_{12} = \delta'\psi_{21} \end{cases} \quad \text{and} \quad \begin{cases} \delta\psi_{11} = \delta'\psi_{22} \\ \delta\psi_{22} = \delta'\psi_{11} \end{cases}$$

If $\psi_{21} \neq 0$, then $\frac{\delta}{\delta'} = \frac{\psi_{12}}{\psi_{21}} = \frac{\delta'}{\delta}$. (If $\psi_{21} = 0$, then $\psi_{12} = 0$ and $\psi_{11}, \psi_{22} \neq 0$. A similar argument to that below then shows that $\delta = \delta'$.) This implies that $\psi_{12} = \pm\psi_{21}$ and $\delta = \pm\delta'$. If $\delta = \delta'$, there is nothing to prove. If $\delta = -\delta'$, then we are left with

$$\begin{bmatrix} -\delta' q_2 \\ -\delta' q_1 \\ -\delta'\psi_{23}p_1 - \delta'\psi_{13}p_2 - \psi_{33}(p_1+p_2)^2 + ((\psi_{12} - \psi_{22})(p_1 - p_2) + q_1 + q_2)^2 \end{bmatrix}^\top = 0.$$

Thus $q_1 = q_2 = 0$ and $\psi_{13} = \psi_{23} = 0$ (from the coefficients of p_1 and p_2, respectively). We are left with the equality $(\psi_{12} - \psi_{22})^2(p_1 - p_2)^2 - \psi_{33}(p_1 + p_2)^2 = 0$. Setting $p_1 = 1$, $p_2 = -1$, this implies that $\psi_{12} = \psi_{22}$, a contradiction since $\det[\psi_{ij}] = \psi_{33}(\psi_{12}^2 - \psi_{22}^2)$. Thus we must have $\delta = \delta'$.

Therefore $H_{A,Q}$ is A-equivalent to exactly one of $H_1^{(2)} = G_1$, $H_2^{(2)} = G_{2,1}$ or $H_{3,\delta}^{(2)} = G_{3,\delta}$, where $\delta \neq 0$. ■

3.3.4 Inhomogeneous systems associated to H_3

3.3.9 LEMMA. *Let $A = \sum_{i=1}^3 a_i E_i \in \mathfrak{se}(1,1)$ (with $A \neq 0$). There exists an automorphism of the form $\psi^{(3)}$ such that $\psi^{(3)} \cdot A \in \{E_1 + \beta E_3, E_1 + E_2 + \gamma E_3, \alpha E_3 : \alpha > 0,\ \beta \geq 0,\ \gamma \in \mathbb{R}\}$.*

PROOF. Suppose $a_3 = 0$. If $a_1^2 \neq a_2^2$, then

$$\psi_1^{(3)} = \begin{bmatrix} \frac{a_1}{a_1^2 - a_2^2} & -\frac{a_2}{a_1^2 - a_2^2} & 0 \\ -\frac{a_2}{a_1^2 - a_2^2} & \frac{a_1}{a_1^2 - a_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_1^{(3)} \cdot A = E_1 = E_1 + \beta E_3$, where $\beta = 0$. If $a_1 = a \neq 0$ and $a_2 = \pm a$, then $\psi_2^{(3)} = \operatorname{diag}\left(\frac{1}{a}, \pm\frac{1}{a}, \pm 1\right)$ gives $\psi_2^{(3)} \cdot A = E_1 + E_2 = E_1 + E_2 + \gamma E_3$, where $\gamma = 0$.

Suppose $a_3 \neq 0$. If $a_1^2 \neq a_2^2$, then

$$\psi_3^{(3)} = \begin{bmatrix} \frac{a_1}{a_1^2-a_2^2} & -\frac{a_2}{a_1^2-a_2^2} & 0 \\ -\operatorname{sgn}(a_3)\frac{a_2}{a_1^2-a_2^2} & \operatorname{sgn}(a_3)\frac{a_1}{a_1^2-a_2^2} & 0 \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix}$$

yields $\psi_3^{(3)} \cdot A = E_1 + \beta E_3$, where $\beta = |a_3| > 0$. If $a_1 = a \neq 0$ and $a_2 = \pm a$, then $\psi_4^{(3)} = \operatorname{diag}\left(\frac{1}{a}, \pm\frac{1}{a}, \pm 1\right)$ is an automorphism such that $\psi_4^{(3)} \cdot A = E_1 + E_2 + \gamma E_3$, where $\gamma = \pm a_3 \neq 0$. Otherwise $a_1 = a_2 = 0$, in which case we have an automorphism $\psi_5^{(3)} = \operatorname{diag}(1, \operatorname{sgn}(a_3), \operatorname{sgn}(a_3))$ such that $\psi_5^{(3)} \cdot A = \alpha E_3$, where $\alpha = |a_3| > 0$. ■

3.3.10 THEOREM. *Any inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{se}(1,1)^*_-$ of the form $H_{A,\mathcal{Q}} = L_A + H_3$ is A-equivalent to exactly one of the following systems*:

$$H_1^{(3)}(p) = p_1 + \tfrac{1}{2}p_3^2 \qquad H_2^{(3)}(p) = p_1 + p_2 + \tfrac{1}{2}p_3^2 \qquad H_3^{(3)}(p) = \tfrac{1}{2}p_3^2.$$

PROOF. From lemma 3.3.9 there exists a linear Poisson symmetry $\Psi^{(3)} : p \mapsto p\psi^{(3)}$ of H_3 such that $L_A \circ \Psi^{(3)} = L_{\psi^{(3)}\cdot A}$ is equal to one of $L_{E_1+\beta E_3}$, $L_{E_1+E_2}$, $L_{E_1+E_2+\gamma E_3}$ or $L_{\alpha E_3}$, for some $\alpha > 0$ or $\gamma \in \mathbb{R}$. Since $H_{A,\mathcal{Q}} \circ \Psi^{(3)} = L_A \circ \Psi^{(3)} + H_3$, it follows that $H_{A,\mathcal{Q}}$ is A-equivalent to one of the systems

$$G_{1,\beta}(p) = p_1 + \beta p_3 + \tfrac{1}{2}p_3^2, \qquad G_{2,\gamma}(p) = p_1 + p_2 + \gamma p_3 + \tfrac{1}{2}p_3^2, \qquad G_{3,\alpha}(p) = \alpha p_3 + \tfrac{1}{2}p_3^2.$$

We show that $G_{1,\beta}$ is A-equivalent to $G_{1,0}$. Indeed, $\Psi : p \mapsto p + \beta E_3^*$ is an affine isomorphism such that

$$\vec{G}_{1,\beta}(p) = \begin{bmatrix} p_2(\beta + p_3) \\ p_1(\beta + p_3) \\ -p_2 \end{bmatrix}^\top = (\vec{G}_{1,0} \circ \Psi)(p).$$

That is, $\vec{G}_{1,\beta} = \vec{G}_{1,0} \circ \Psi$, and so $G_{1,\beta}$ is A-equivalent to $G_{1,0}$. Similarly, the affine isomorphism $p \mapsto p + \gamma E_3^*$ shows that $G_{2,\gamma}$ is A-equivalent to $G_{2,0}$, and the affine isomorphism $p \mapsto p + \alpha E_3^*$ shows that $G_{3,0}$ is A-equivalent to $G_{3,0}$.

We show that $G_{1,0}$ is not A-equivalent to $G_{2,0}$. Indeed, suppose otherwise. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p[\psi_{ij}]$ such that $\Psi_0 \cdot \vec{G}_{1,0} = \vec{G}_{2,0} \circ \Psi$. That is,

$$\begin{aligned} &\begin{bmatrix} -\psi_{31}p_2 + \psi_{21}p_1p_3 + \psi_{11}p_2p_3 \\ -\psi_{32}p_2 + \psi_{22}p_1p_3 + \psi_{12}p_2p_3 \\ -\psi_{33}p_2 + \psi_{23}p_1p_3 + \psi_{13}p_2p_3 \end{bmatrix}^\top \\ &\quad = \begin{bmatrix} (\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3 + q_2)(\psi_{13}p_1 + \psi_{23}p_2 + \psi_{33}p_3 + q_3) \\ (\psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3 + q_1)(\psi_{13}p_1 + \psi_{23}p_2 + \psi_{33}p_3 + q_3) \\ -(\psi_{11} + \psi_{12})p_1 - (\psi_{21} + \psi_{22})p_2 - (\psi_{31} + \psi_{32})p_3 - (q_1 + q_2) \end{bmatrix}^\top \end{aligned}$$

for every $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$. From the coefficients of p_3^2 in the first two equations, we see that $\psi_{31}\psi_{33} = \psi_{32}\psi_{33} = 0$. We have $\psi_{33} \neq 0$, whence $\psi_{31} = \psi_{32} = 0$. (Indeed, suppose $\psi_{33} = 0$. Considering the coefficients of p_1, p_2 and p_3 in the last equation, we have $\psi_{31} = -\psi_{32}$, $\psi_{21} = -\psi_{22}$ and $\psi_{11} = -\psi_{12}$. This implies that $\det[\psi_{ij}] = 0$, a contradiction.) The coefficient of p_1^2 is $-\psi_{12}\psi_{13}$ in the first equation, and $-\psi_{11}\psi_{13}$ in the second. Since

$\det[\psi_{ij}] = \psi_{33}(\psi_{11}\psi_{22} - \psi_{21}\psi_{12}) \neq 0$, we have $\psi_{13} = 0$. Similarly, considering the coefficients of p_2^2, we have $\psi_{23} = 0$. This leave us with the equality

$$\begin{bmatrix} (\psi_{21} - \psi_{12}\psi_{33})p_1p_3 + (\psi_{11} - \psi_{22}\psi_{33})p_2p_3 - q_2\psi_{33}p_3 - q_3\psi_{12}p_1 - q_3\psi_{22}p_2 - q_2q_3 \\ (\psi_{22} - \psi_{11}\psi_{33})p_1p_3 + (\psi_{12} - \psi_{21}\psi_{33})p_2p_3 - q_1\psi_{33}p_3 - q_3\psi_{11}p_1 - q_3\psi_{21}p_2 - q_1q_3 \\ (\psi_{11} + \psi_{12})p_1 + (\psi_{21} + \psi_{22} - \psi_{33})p_2 + q_1 + q_2 \end{bmatrix}^\top = 0.$$

The coefficients of p_1p_3 imply that $\psi_{21} = \psi_{12}\psi_{33}$ and $\psi_{22} = \psi_{11}\psi_{33}$, whence $\det[\psi_{ij}] = (\psi_{11} - \psi_{12})(\psi_{11} + \psi_{12})\psi_{33}^2$. But, considering the coefficient of p_1 in the last equation, we have $\psi_{11} = -\psi_{12}$. Hence $\det[\psi_{ij}] = 0$, a contradiction. Thus no such Ψ exists, *i.e.*, G_1 is not A-equivalent to $G_{2,0}$.

A similar argument shows that $G_{1,0}$ is not A-equivalent to $G_{3,0}$, and $G_{2,0}$ is not A-equivalent to $G_{3,0}$. (See section B.3.4 for the Mathematica code that performs these verifications.)

Therefore $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of $H_1^{(3)} = G_{1,0}$, $H_2^{(3)} = G_{2,0}$ or $H_3^{(3)} = G_{3,0}$. ■

3.3.5 Inhomogeneous systems associated to H_4

3.3.11 Lemma. *Let* $A = \sum_{i=1}^3 a_iE_i \in \mathfrak{se}(1,1)$ *(with* $A \neq 0$*). There exists an automorphism of the form* $\psi^{(4)}$ *such that* $\psi^{(4)} \cdot A \in \{\beta E_1 + \alpha E_2, \gamma E_1 + \beta E_2 + \alpha E_3 : \alpha > 0,\ \beta \geq 0,\ \gamma \in \mathbb{R}\}$.

Proof. Suppose $a_3 = 0$. If $a_2 \neq 0$ and $a_1 = 0$, then $\psi_1^{(4)} = \mathrm{diag}(1, \mathrm{sgn}(a_2), \mathrm{sgn}(a_2))$ is an automorphism such that $\psi_1^{(4)} \cdot A = \alpha E_2 = \beta E_1 + \alpha E_2$, where $\beta = 0$ and $\alpha = |a_2| > 0$. If $a_2 \neq 0$ and $a_1 \neq 0$, then the automorphism $\psi_2^{(4)} = \mathrm{diag}(\mathrm{sgn}(a_1), \mathrm{sgn}(a_1a_2)\,\mathrm{sgn}(a_1), \mathrm{sgn}(a_1a_2))$ yields $\psi_2^{(4)} \cdot A = \beta E_1 + \alpha E_2$, where $\beta = |a_1| > 0$ and $\alpha = |a_2| > 0$. If $a_2 = 0$, then $a_1 \neq 0$ (since $A \neq 0$), and the automorphism

$$\psi_3^{(4)} = \begin{bmatrix} 0 & \mathrm{sgn}(a_1) & 0 \\ \mathrm{sgn}(a_1) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives $\psi_3^{(4)} \cdot A = \alpha E_2 = \beta E_1 + \alpha E_2$, where $\beta = 0$ and $\alpha = |a_1| > 0$.

Suppose $a_3 \neq 0$. If $a_2 = 0$ and $a_1 = 0$, then $\psi_4^{(4)} = \mathrm{diag}(1, \mathrm{sgn}(a_3), \mathrm{sgn}(a_3))$ gives $\psi_4^{(4)} \cdot A = \alpha E_3 = \gamma E_1 + \beta E_2 + \gamma E_3$, where $\gamma = \beta = 0$ and $\alpha = |a_3| > 0$. If $a_2 = 0$ and $a_1 \neq 0$, then

$$\psi_5^{(4)} = \begin{bmatrix} 0 & \mathrm{sgn}(a_1a_3) & 0 \\ \mathrm{sgn}(a_3)\,\mathrm{sgn}(a_1a_3) & 0 & 0 \\ 0 & 0 & \mathrm{sgn}(a_3) \end{bmatrix}$$

is an automorphism such that $\psi_5^{(4)} \cdot A = \beta E_2 + \alpha E_3 = \gamma E_1 + \beta E_2 + \alpha E_3$, where $\gamma = 0$, $\beta = |a_1| > 0$ and $\alpha = |a_3| > 0$. If $a_2 \neq 0$, then $\psi_4^{(4)} = \mathrm{diag}(\mathrm{sgn}(a_2a_3), \mathrm{sgn}(a_3)\,\mathrm{sgn}(a_2a_3), \mathrm{sgn}(a_3))$ yields $\psi_4^{(4)} \cdot A = \gamma E_1 + \beta E_2 + \alpha E_3$, where $\gamma = \mathrm{sgn}(a_2a_3)a_1 \in \mathbb{R}$, $\beta = |a_2| > 0$ and $\alpha = |a_3| > 0$. ■

3.3.12 Theorem. *Any inhomogeneous quadratic Hamilton-Poisson system on* $\mathfrak{se}(1,1)^*_-$ *of the form* $H_{A,\mathcal{Q}} = L_A + H_4$ *is A-equivalent to exactly one of the following systems*:

$$H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \tfrac{1}{2}(p_1^2 + p_3^2) \qquad H_{2,\alpha_1,\alpha_2}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \tfrac{1}{2}(p_1^2 + p_3^2).$$

Here $\alpha > 0$ and $\alpha_1 \geq \alpha_2 > 0$ parametrise a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.

Proof. From lemma 3.3.11 there exists a linear Poisson symmetry $\Psi^{(4)} : p \mapsto p\psi^{(4)}$ of H_4 such that $L_A \circ \Psi^{(4)} = L_{\psi^{(4)} \cdot A}$ is equal to one of $L_{\beta E_1 + \alpha E_2}$ or $L_{\gamma E_1 + \beta E_2 + \alpha E_3}$, for some $\alpha > 0$, $\beta \geq 0$ or $\gamma \in \mathbb{R}$. Since $H_{A,\mathcal{Q}} \circ \Psi^{(4)} = L_A \circ \Psi^{(4)} + H_4$, it follows that $H_{A,\mathcal{Q}}$ is A-equivalent to one of the systems

$$G_{1,\alpha,\beta}(p) = \beta p_1 + \alpha p_2 + \tfrac{1}{2}(p_1^2 + p_3^2), \qquad G_{2,\alpha,\beta,\gamma}(p) = \gamma p_1 + \beta p_2 + \alpha p_3 + \tfrac{1}{2}(p_1^2 + p_3^2).$$

We have that $G_{2,\alpha,\beta,\gamma}$ is A-equivalent to $G_{2,0,\beta,\gamma}$. Indeed, $\Psi : p \mapsto p + \alpha E_3^*$ is an affine isomorphism such that

$$\vec{G}_{2,\alpha,\beta,\gamma}(p) = \begin{bmatrix} p_2(\alpha + p_3) \\ p_1(\alpha + p_3) \\ -\gamma p_3 - p_1(\beta + p_2) \end{bmatrix}^\top = (\vec{G}_{2,0,\beta,\gamma} \circ \Psi)(p).$$

That is, $\vec{G}_{2,\alpha,\beta,\gamma} = \vec{G}_{2,0,\beta,\gamma} \circ \Psi$, and so $G_{2,\alpha,\beta,\gamma}$ is A-equivalent to $G_{2,0,\beta,\gamma}$. Next, $\Psi : p \mapsto \operatorname{diag}(-1, 1, -1)$ is a linear isomorphism such that

$$\begin{aligned}(\Psi \cdot \vec{G}_{2,0,\beta,\gamma})(p) &= \begin{bmatrix} p_2 p_3 \\ p_1 p_3 \\ -\gamma p_2 - p_1(\beta + p_2) \end{bmatrix}^\top \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -p_2 p_3 \\ p_1 p_3 \\ \gamma p_2 + p_1(\beta + p_2) \end{bmatrix}^\top = (\vec{G}_{2,0,\beta,-\gamma} \circ \Psi)(p).\end{aligned}$$

Thus $G_{2,0,\beta,\gamma}$ is A-equivalent to $G_{2,0,\beta,-\gamma}$. Consequently, we may assume that $\gamma \geq 0$, *i.e.*, we have a potential family of representatives $G_{2,0,\beta_1,\beta_2}(p) = \beta_1 p_1 + \beta_2 p_2 + \frac{1}{2}(p_1^2 + p_3^2)$, with $\beta_1, \beta_2 \geq 0$ and β_1, β_2 not both zero. If $\beta_2 > 0$, then $G_{2,0,\beta_1,\beta_2} = G_{1,\alpha,\beta}$, where $\alpha = \beta_2 > 0$ and $\beta = \beta_1 \geq 0$. If $\beta_1 > 0$, then

$$\Psi : p \mapsto p \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3.1}$$

is a linear isomorphism such that

$$\begin{aligned}(\Psi \cdot \vec{G}_{2,0,\beta_1,\beta_2})(p) &= \begin{bmatrix} p_2 p_3 \\ p_1 p_3 \\ -\beta_1 p_2 - p_1(\beta_2 + p_2) \end{bmatrix}^\top \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} p_1 p_3 \\ p_2 p_3 \\ -\alpha p_2 - p_1(\beta + p_2) \end{bmatrix}^\top = (\vec{G}_{1,\alpha,\beta} \circ \Psi)(p),\end{aligned}$$

where $\alpha = \beta_1 > 0$ and $\beta = \beta_2 \geq 0$. Hence $G_{2,0,\beta_1,\beta_2}$, $\beta_1 > 0$ is A-equivalent to $G_{1,\alpha,\beta}$.

Introduce a new family of potential representatives $G_{3,\alpha}(p) = \alpha p_1 + \frac{1}{2}(p_1^2 + p_3^2)$. The linear isomorphism (3.3.1) then yields

$$(\Psi \cdot \vec{G}_{1,\alpha,0})(p) = \begin{bmatrix} p_2 p_3 \\ p_1 p_3 \\ -p_1(\alpha + p_2) \end{bmatrix}^\top \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} p_1 p_3 \\ p_2 p_3 \\ -p_1(\alpha + p_2) \end{bmatrix}^\top = (\vec{G}_{3,\alpha} \circ \Psi)(p).$$

That is, $G_{1,\alpha,0}$ is A-equivalent to $G_{3,\alpha}$. To summarize the results thus far, we have two potential families of representatives, *viz.*

$$G_{1,\alpha_1,\alpha_2}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \tfrac{1}{2}(p_1^2 + p_3^2), \qquad G_{3,\alpha}(p) = \alpha p_1 + \tfrac{1}{2}(p_1^2 + p_3^2).$$

(Here $\alpha, \alpha_1, \alpha_2 > 0$.)

Once again using (3.3.1), we have

$$\begin{aligned}(\Psi \cdot \vec{G}_{1,\alpha_1,\alpha_2})(p) &= \begin{bmatrix} p_2p_3 \\ p_1p_3 \\ -\alpha_1 p_2 - p_1(\alpha_2 + p_2)\end{bmatrix}^\top \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix} \\ &= \begin{bmatrix} p_1p_3 \\ p_2p_3 \\ -\alpha_1 p_2 - p_1(\alpha_2 + p_2)\end{bmatrix}^\top = (\vec{G}_{1,\alpha_2,\alpha_1} \circ \Psi)(p).\end{aligned}$$

That is, G_{1,α_1,α_2} is A-equivalent to G_{1,α_2,α_1}. Consequently, we may assume that $\alpha_1 \geq \alpha_2$.

We show that G_{1,α_1,α_2} is not A-equivalent to $G_{3,\alpha}$ for any $\alpha > 0$, $\alpha_1 \geq \alpha_2 > 0$. Indeed, suppose otherwise. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p[\psi_{ij}]$ such that $\Psi_0 \cdot \vec{G}_{1,\alpha_1,\alpha_2} = \vec{G}_{3,\alpha} \circ \Psi$. That is,

$$\begin{aligned}&\begin{bmatrix} \psi_{21}p_1p_3 + \psi_{11}p_2p_3 - \psi_{31}p_1p_2 - \alpha_1\psi_{31}p_2 - \alpha_2\psi_{31}p_1 \\ \psi_{22}p_1p_3 + \psi_{12}p_2p_3 - \psi_{32}p_1p_2 - \alpha_1\psi_{32}p_2 - \alpha_2\psi_{32}p_1 \\ \psi_{23}p_1p_3 + \psi_{13}p_2p_3 - \psi_{33}p_1p_2 - \alpha_1\psi_{33}p_2 - \alpha_2\psi_{33}p_1 \end{bmatrix}^\top \\ &\quad = \begin{bmatrix} (\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3 + q_2)(\psi_{13}p_1 + \psi_{23}p_2 + \psi_{33}p_3 + q_3) \\ (\psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3 + q_1)(\psi_{13}p_1 + \psi_{23}p_2 + \psi_{33}p_3 + q_3) \\ -(\alpha + \psi_{11}p_1 + \psi_{21}p_2 + \psi_{31}p_3 + q_1)(\psi_{12}p_1 + \psi_{22}p_2 + \psi_{32}p_3 + q_2)\end{bmatrix}^\top\end{aligned}$$

for every $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{R}$. Comparing coefficients of p_1^2, p_2^2 and p_3^2 in the first and second equations, we have

$$\begin{cases} \psi_{12}\psi_{13} = 0 \\ \psi_{22}\psi_{23} = 0 \\ \psi_{32}\psi_{33} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \psi_{11}\psi_{13} = 0 \\ \psi_{21}\psi_{23} = 0 \\ \psi_{31}\psi_{33} = 0 \end{cases}$$

Suppose $\psi_{13} \neq 0$. Then $\psi_{12} = \psi_{11} = 0$. If $\psi_{23} \neq 0$, then $\psi_{22} = \psi_{21} = 0$, whence $\det[\psi_{ij}] = 0$, a contradiction. Hence $\psi_{23} = 0$. Similarly, if $\psi_{33} \neq 0$, then $\psi_{32} = \psi_{31} = 0$, which again implies $\det[\psi_{ij}] = 0$. Thus $\psi_{33} = 0$. The first two equations are now

$$\begin{cases} (\psi_{31} + \psi_{22}\psi_{13})p_1p_2 - (\psi_{21} - \psi_{32}\psi_{13})p_1p_3 + (\psi_{13}q_2 + \psi_{31}\alpha_2)p_1 + (\psi_{22}q_3 + \psi_{31}\alpha_1)p_2 \\ \qquad + \psi_{32}q_3p_3 + q_2q_3 = 0 \\ (\psi_{32} + \psi_{21}\psi_{13})p_1p_2 - (\psi_{22} - \psi_{31}\psi_{13})p_1p_3 + (\psi_{13}q_1 + \psi_{32}\alpha_2)p_1 + (\psi_{21}q_3 + \psi_{32}\alpha_1)p_2 \\ \qquad + \psi_{31}q_3p_3 + q_1q_3 = 0 \end{cases}$$

From the coefficients of p_3, we have $q_3 = 0$ (if $\psi_{31} = \psi_{32} = 0$, then $\det[\psi_{ij}] = 0$). The coefficients of p_2 then imply that $\psi_{31} = \psi_{32} = 0$ (since $\alpha_1 > 0$), a contradiction.

Suppose that $\psi_{13} = 0$ and $\psi_{23} \neq 0$. Then $\psi_{22} = \psi_{21} = 0$ and from the coefficients of p_3^2, we have $\psi_{33} = 0$. (If $\psi_{31} = \psi_{32} = 0$, then $\det[\psi_{ij}] = 0$, a contradiction.) The first two equations are now

$$\begin{cases} (\psi_{31} + \psi_{12}\psi_{23})p_1p_2 - (\psi_{11} - \psi_{32}\psi_{23})p_2p_3 + (\psi_{23}q_2 + \psi_{31}\alpha_1)p_2 + (\psi_{12}q_3 + \psi_{31}\alpha_2)p_1 \\ \qquad + \psi_{32}q_3p_3 + q_2q_3 = 0 \\ (\psi_{32} + \psi_{11}\psi_{23})p_1p_2 - (\psi_{12} - \psi_{31}\psi_{23})p_2p_3 + (\psi_{23}q_1 + \psi_{32}\alpha_1)p_2 + (\psi_{11}q_3 + \psi_{32}\alpha_2)p_1 \\ \qquad + \psi_{31}q_3p_3 + q_1q_3 = 0. \end{cases}$$

Comparing the coefficients of p_3, we see that $q_3 = 0$ (otherwise $\psi_{31} = \psi_{32} = 0$, whence $\det[\psi_{ij}] = 0$). The coefficients of p_1 we then get $\psi_{31} = \psi_{32} = 0$ (since $\alpha_2 > 0$), a contradiction.

Suppose that $\psi_{13} = 0$ and $\psi_{23} = 0$. Then $\psi_{33} \neq 0$, else $\det[\psi_{ij}] = 0$. From the coefficients of p_3^2, this implies that $\psi_{32} = \psi_{31} = 0$. The coefficients of p_1^2 and p_2^2 in the third equation gives

$$\begin{cases} \psi_{11}\psi_{12} = 0 \\ \psi_{21}\psi_{22} = 0. \end{cases}$$

Moreover, we have $\det[\psi_{ij}] = \psi_{33}(\psi_{11}\psi_{22} - \psi_{21}\psi_{12}) \neq 0$. Thus we must have either $\psi_{11} = \psi_{22} = 0$ or $\psi_{21} = \psi_{12} = 0$. If the former case holds, then we have the equations

$$\begin{cases} (\psi_{21} - \psi_{12}\psi_{33})p_1p_3 - q_2\psi_{33}p_3 - q_3\psi_{12}p_1 - q_2q_3 = 0 \\ (\psi_{12} - \psi_{21}\psi_{33})p_2p_3 - q_1\psi_{33}p_3 - q_3\psi_{21}p_2 - q_1q_3 = 0. \end{cases}$$

As $\psi_{33}, \psi_{21}, \psi_{12} \neq 0$, we have $q_1 = q_2 = q_3 = 0$. The third equation is now

$$(\psi_{21}\psi_{12} - \psi_{33})p_1p_2 + (\alpha\psi_{12} - \psi_{33}\alpha_2)p_1 - \psi_{33}\alpha_1p_2 = 0.$$

This implies that $\alpha_1\psi_{33} = 0$, a contradiction, since $\alpha_1 > 0$ and $\psi_{33} \neq 0$. (The situation $\psi_{21} = \psi_{12} = 0$ leads to a similar contradiction.) Therefore, in all cases, we have a contradiction. Thus G_{1,α_1,α_2} cannot be A-equivalent to $G_{3,\alpha}$.

A similar argument shows that G_{1,α_1,α_2} is A-equivalent to $G_{1,\alpha_1',\alpha_2'}$ only if $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$ and $G_{3,\alpha}$ is A-equivalent to $G_{3,\alpha'}$ only if $\alpha = \alpha'$. (See section B.3.4 for the Mathematica code that performs these verifications.)

Therefore $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of $H^{(4)}_{1,\alpha} = G_{3,\alpha}$ or $H^{(4)}_{2,\alpha_1,\alpha_2} = G_{2,\alpha_1,\alpha_2}$, where $\alpha > 0$ and $\alpha_1 \geq \alpha_2 > 0$. ■

3.3.6 Inhomogeneous systems associated to H_5

3.3.13 Lemma. *Let $A = \sum_{i=1}^3 a_iE_i \in \mathfrak{se}(1,1)$ (with $A \neq 0$). There exists an automorphism of the form $\psi^{(5)}$ such that $\psi^{(5)} \cdot A \in \{\beta E_1 + \gamma E_3, \delta E_1 + \alpha E_2 + \gamma E_3 : \alpha > 0,\ \beta \geq 0,\ \gamma \in \mathbb{R},\ \delta \neq 0\}$.*

Proof. Suppose $a_1^2 \neq a_2^2$. Then we have an automorphism

$$\psi_1^{(5)} = \begin{bmatrix} \operatorname{sgn}(a_1 + a_2)\frac{a_1}{a_1 - a_2} & -\operatorname{sgn}(a_1 + a_2)\frac{a_2}{a_1 - a_2} & 0 \\ -\operatorname{sgn}(a_1 + a_2)\frac{a_2}{a_1 - a_2} & \operatorname{sgn}(a_1 + a_2)\frac{a_1}{a_1 - a_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $\psi_1^{(5)} \cdot A = \beta E_1 + \gamma E_3$, where $\beta = |a_1 + a_2| > 0$ and $\gamma = a_3 \in \mathbb{R}$.

Suppose $a_1 = a \neq 0$ and $a_2 = \pm a$. Then $\psi_2^{(5)} = \operatorname{diag}(\pm\operatorname{sgn}(a), \pm\operatorname{sgn}(a), 1)$ is an automorphism such that $\psi_2^{(5)} \cdot A = \delta E_1 + \alpha E_2 + \gamma E_3$, where $\delta = \pm|a| \neq 0$, $\alpha = |a| > 0$ and $\gamma = a_3 \in \mathbb{R}$.

Lastly, suppose $a_1 = a_2 = 0$. Then $A = \gamma E_3 = \beta E_1 + \gamma E_3$, where $\beta = 0$ and $\gamma = a_3 \in \mathbb{R}$. ■

3.3.14 Theorem. *Any inhomogeneous Hamilton-Poisson system on* $\mathfrak{se}(1,1)^*_-$ *of the form* $H_{A,\mathcal{Q}} = L_A + H_5$ *is A-equivalent to exactly one of the following systems*:

$$\begin{aligned}
H^{(5)}_{1,\alpha}(p) &= \alpha p_1 + \tfrac{1}{2}\big[(p_1+p_2)^2 + p_3^2\big] \\
H^{(5)}_{2}(p) &= p_1 - p_2 + \tfrac{1}{2}\big[(p_1+p_2)^2 + p_3^2\big] \\
H^{(5)}_{3,\alpha}(p) &= \alpha(p_1+p_2) + \tfrac{1}{2}\big[(p_1+p_2)^2 + p_3^2\big].
\end{aligned}$$

Here $\alpha > 0$ *parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.*

Proof. From lemma 3.3.13 there exists a linear Poisson symmetry $\Psi^{(5)} : p \mapsto p\psi^{(5)}$ of H_5 such that $L_A \circ \Psi^{(5)} = L_{\psi^{(5)}\cdot A}$ is equal to one of $L_{\beta E_1 + \gamma E_3}$, $L_{\delta E_1 + \alpha E_2 + \gamma E_3}$, for some $\alpha > 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$ or $\delta \neq 0$. Since $H_{A,\mathcal{Q}} \circ \Psi^{(5)} = L_A \circ \Psi^{(5)} + H_5$, it follows that $H_{A,\mathcal{Q}}$ is A-equivalent to one of the systems

$$\begin{aligned}
G_{1,\beta,\gamma}(p) &= \beta p_1 + \gamma p_3 + \tfrac{1}{2}\big[(p_1+p_2)^2 + p_3^2\big], \\
G_{2,\alpha,\gamma,\delta}(p) &= \delta p_1 + \alpha p_2 + \gamma p_3 + \tfrac{1}{2}\big[(p_1+p_2)^2 + p_3^2\big].
\end{aligned}$$

We have that $G_{1,\beta,\gamma}$ is A-equivalent to $G_{1,\beta,0}$. Indeed, $\Psi : p \mapsto p + \gamma E_3^*$ is an affine isomorphism such that

$$\vec{G}_{1,\beta,\gamma}(p) = \begin{bmatrix} p_2(\gamma + p_3) \\ p_1(\gamma + p_3) \\ -\beta p_2 - (p_1+p_2)^2 \end{bmatrix}^\top = (\vec{G}_{1,\beta,0} \circ \Psi)(p).$$

Thus $G_{1,\beta,\gamma}$ is A-equivalent to $G_{1,\beta,0}$. Consequently, we may assume that $\beta = \alpha > 0$, since the case $\beta = 0$ reduces to one of the homogeneous systems.

In a similar manner, the affine isomorphism $\Psi : p \mapsto p + \gamma E_3^*$ may be used to show that $G_{2,\alpha,\gamma,\delta}$ is A-equivalent to $G_{2,\alpha,0,\delta}$.

Consider the family $G_{2,\alpha,\delta,0}$. Suppose $\delta^2 \neq \alpha^2$. Then

$$\Psi : p \mapsto p \begin{bmatrix} \frac{\delta}{|\delta+\alpha|} & \frac{\alpha}{|\delta+\alpha|} & 0 \\ \frac{\alpha}{|\delta+\alpha|} & \frac{\delta}{|\delta+\alpha|} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that

$$\begin{aligned}
(\Psi \cdot \vec{G}_{2,\alpha,\delta,0})(p) &= \begin{bmatrix} p_2 p_3 \\ p_1 p_3 \\ -p_1^2 - p_2(\delta + p_2) - p_1(\alpha + 2p_2) \end{bmatrix}^\top \begin{bmatrix} \frac{\delta}{|\delta+\alpha|} & \frac{\alpha}{|\delta+\alpha|} & 0 \\ \frac{\alpha}{|\delta+\alpha|} & \frac{\delta}{|\delta+\alpha|} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{|\delta+\alpha|}(\alpha p_1 + \delta p_2) p_3 \\ \frac{1}{|\delta+\alpha|}(\delta p_1 + \alpha p_2) p_3 \\ -p_1^2 - p_2(\delta + p_2) - p_1(\alpha + 2p_2) \end{bmatrix}^\top = (\vec{G}_{1,|\delta+\alpha|} \circ \Psi)(p).
\end{aligned}$$

That is, $G_{2,\alpha,\delta,0}$ ($\delta^2 \neq \alpha^2$) is A-equivalent to $G_{1,|\delta+\alpha|}$.

Suppose $\delta^2 = \alpha^2$. Introduce two new families of potential representatives:

$$G_3(p) = p_1 - p_2 + \tfrac{1}{2}\big[(p_1+p_2)^2 + p_3^2\big], \qquad G_{4,\alpha}(p) = \alpha(p_1+p_2) + \tfrac{1}{2}\big[(p_1+p_2)^2 + p_3^2\big].$$

(Here $\alpha > 0$.) If $\alpha = -\delta > 0$, then

$$\Psi : p \mapsto p \begin{bmatrix} \frac{1+\delta}{2} & \frac{1-\delta}{2} & 0 \\ \frac{1-\delta}{2} & \frac{1+\delta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism such that

$$\begin{aligned} (\Psi \cdot \vec{G}_{2,\alpha,\delta,0})(p) &= \begin{bmatrix} p_2 p_3 \\ p_1 p_3 \\ -p_1^2 + p_1(\delta - 2p_2) - p_2(\delta + p_2) \end{bmatrix}^\top \begin{bmatrix} \frac{1+\delta}{2} & \frac{1-\delta}{2} & 0 \\ \frac{1-\delta}{2} & \frac{1+\delta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\left[-(\delta-1)p_1 + (\delta+1)p_2\right] p_3 \\ \frac{1}{2}\left[(\delta-1)p_1 - (\delta-1)p_2\right] p_3 \\ -p_1^2 + p_1(\delta - 2p_2) - p_2(\delta + p_2) \end{bmatrix}^\top = (\vec{G}_3 \circ \Psi)(p). \end{aligned}$$

Hence $G_{2,\alpha,\delta,0}$ ($\alpha = -\delta > 0$) is A-equivalent to G_3. If $\alpha = \delta > 0$, then $G_{2,\alpha,\delta,0} = G_{4,\alpha}$.

Thus we have the potential representatives $G_{1,\alpha}$, G_3 and $G_{4,\alpha}$. A straightforward calculation confirms that none of these representatives are equivalent and that $G_{1,\alpha}$ and $G_{4,\alpha}$ are unique representatives for unique values of the parameter $\alpha > 0$. However, the calculations are extremely lengthy and tedious, and we shall not present them here. Nonetheless, section B.3.4 lists the Mathematica code that performs these verifications.

Therefore $H_{A,\mathcal{Q}}$ is A-equivalent to exactly one of $H^{(5)}_{1,\alpha} = G_{1,\alpha}$, $H^{(5)}_{2} = G_3$ or $H^{(5)}_{3,\alpha} = G_{4,\alpha}$, where $\alpha > 0$. ■

Homogeneous Systems	Inhomogeneous Systems
$H_0(p) = 0$	$H_1^{(0)}(p) = p_1$
	$H_{2,\alpha}^{(0)}(p) = \alpha p_3$
$H_1(p) = \frac{1}{2}p_1^2$	$H_1^{(1)}(p) = p_1 + \frac{1}{2}p_1^2$
	$H_2^{(1)}(p) = p_1 + p_2 + \frac{1}{2}p_1^2$
	$H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \frac{1}{2}p_1^2$
$H_2(p) = \frac{1}{2}(p_1 + p_2)^2$	$H_1^{(2)}(p) = p_1 + \frac{1}{2}(p_1 + p_2)^2$
	$H_2^{(2)}(p) = p_1 + p_2 + \frac{1}{2}(p_1 + p_2)^2$
	$H_{3,\delta}^{(2)}(p) = \delta p_3 + \frac{1}{2}(p_1 + p_2)^2$
$H_3(p) = \frac{1}{2}p_3^2$	$H_1^{(3)}(p) = p_1 + \frac{1}{2}p_3^2$
	$H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2}p_3^2$
	$H_3^{(3)}(p) = \frac{1}{2}p_3^2$
$H_4(p) = \frac{1}{2}(p_1^2 + p_3^2)$	$H_{1,\alpha}^{(4)}(p) = \alpha p_1 + \frac{1}{2}(p_1^2 + p_3^2)$
	$H_{2,\alpha_1,\alpha_2}^{(4)}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2}(p_1^2 + p_3^2)$
$H_5(p) = \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$	$H_{1,\alpha}^{(5)}(p) = \alpha p_1 + \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$
	$H_2^{(5)}(p) = p_1 - p_2 + \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$
	$H_{3,\alpha}^{(5)}(p) = \alpha(p_1 + p_2) + \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$
$\alpha > 0$, $\alpha_1 \geq \alpha_2 > 0$, $\delta \neq 0$	

Table 3.1: Classification of quadratic Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$

Chapter 4

Stability and Integration of Hamilton-Poisson Systems

In chapter 3, we classified a class of quadratic Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$ under affine equivalence. Table 3.1 lists class representatives for both the homogeneous systems (of the form $H_Q(p) = \frac{1}{2}pQp^\top$) and the inhomogeneous systems (of the form $H_{A,Q}(p) = pA + \frac{1}{2}pQp^\top$). The purpose of this chapter is to investigate a number of the normal forms we have obtained. Specifically, we shall consider the (nontrivial) homogeneous representatives $H_1, \ldots, H_5$ as well as the first eleven inhomogeneous representatives (*i.e.*, those associated to the homogeneous systems H_0, H_1, H_2 and H_3).

For each system, we begin by investigating the (Lyapunov) stability nature of the equilibrium states. Stability is proved by means of the (extended) energy-Casimir method. Instability usually requires a direct approach, employing the definition of Lyapunov instability. However, a number of cases may be shown to be unstable by showing spectral instability. (See section A.5 for details on the methods we employ.)

Following the stability analysis, we turn to the problem of finding the integral curves of the system. The systems whose integral curves are lines are easily integrated, so we simply state the general form of each integral curve. Most of the remaining systems may be integrated in terms of elementary functions. However, the systems H_4 and $H_1^{(3)}$ require the use of Jacobi elliptic functions. For the more involved systems (specifically, H_4, H_5, $H_1^{(3)}$ and $H_2^{(3)}$), the integration is typically subdivided into several subsections, depending on the various different configurations of the system.

The general approach to integration for the systems requiring Jacobi elliptic functions (*i.e.*, H_4 and $H_1^{(3)}$) is as follows. For each different configuration of the system, we use the two constants of motion (the Hamiltonian function and the Casimir function $C(p) = p_1^2 - p_2^2$) to transform the equations of motion into a single (separable) differential equation, which (possibly after some reduction; see section A.6.2) may be integrated in terms of Jacobi elliptic functions. This yields an expression for a single component of the candidate integral curve. The constants of motion are then used again to find expressions for the remaining two components. (For degenenerate configurations of the system, we can typically avoid explicit integration by taking a suitable limit of the expressions obtained for nondegenerate cases.) The intent is to find a prospective expression for a single integral curve. (As such, we neglect any constants of integration.) Having found such an expression, we verify that it is an integral curve, by showing that the equations of motion are satisfied. Lastly, we complete the analysis by proving a result concerning *all* integral curves of the system for that particular configuration.

For the other systems, a more *ad hoc* approach is employed. For H_5 and $H_2^{(3)}$, we are able to transform the Hamiltonian equation into a (separable) differential equation in one variable, which is easily solved. The remaining two components of the integral curves are then obtained by means of the equations of motion and the Casimir function. For $H_{2,\alpha}^{(0)}$, $H_{3,\alpha}^{(1)}$ and $H_{3,\delta}^{(2)}$, the first two equations of motion are linear, which allows us to use the matrix exponential for integration.

We also graph typical configurations of each system (*sans* those with lines for integral curves). More precisely, we graph the level sets $H^{-1}(h_0)$ and $C^{-1}(c_0)$ and their intersection. (Here H is the Hamiltonian function, $C(p) = p_1^2 - p_2^2$ is the Casimir function and h_0, c_0 are typical values for H and C along an integral curve, respectively.) The stable equilibrium points (illustrated in blue) and unstable equilibrium points (illustrated in red) are also plotted in each case.

4.1 Preliminaries

The stability analysis requires the use of a suitable norm on $\mathfrak{se}(1,1)^*$. Since this space is finite-dimensional, all norms are equivalent. For simplicity, we use the Euclidean norm $\|p\| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. (Here $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^* \in \mathfrak{se}(1,1)^*$.)

Suppose $p(\cdot)$ is an integral curve of a Hamiltonian vector field $\vec{H}$ on $\mathfrak{se}(1,1)^*_-$ such that $C(p(t)) > 0$ for every t. We have $p_1(t)^2 \leq p_1(t)^2 - p_2(t)^2 = C(p(t))$, and so either $p_1(t) \leq -\sqrt{C(p(t))}$ or $p_1(t) \geq \sqrt{C(p(t))}$. The following proposition (the proof of which is immediate) asserts that the value of $p_1(\cdot)$ at $t = 0$ is sufficient to determine which case holds. (We employ this result implicitly throughout the integration.)

4.1.1 Proposition. *Suppose* $c_0 = C(p(0)) \geq 0$. *Then*

(*i*) $p_1(0) \leq -\sqrt{c_0}$ *if and only if* $p_1(t) \leq -\sqrt{c_0}$ *for all* t.

(*ii*) $p_1(0) \geq \sqrt{c_0}$ *if and only if* $p_1(t) \geq \sqrt{c_0}$ *for all* t.

Lastly, we prove a useful sufficient condition for a curve to be an integral curve of a Hamilton-Poisson system.

4.1.2 Proposition. *Let* H *be a Hamilton-Poisson system on* $\mathfrak{se}(1,1)^*_-$ *and let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an absolutely continuous curve such that* $\dot{p}_1 = p_2 \frac{\partial H}{\partial p_3}$, $C(p(t)) =$ *constant and* $H(p(t)) =$ *constant. Then* $p(\cdot)$ *is an integral curve of* $\vec{H}$.

Proof. We show that $p(\cdot)$ satisfies the equations of motion of H (equation (3.1.4)). For brevity, denote the partial derivative $\partial H / \partial p_i$ by H_{p_i}. By assumption, the first equation of motion holds true. Differentiating both sides of $C(p(t)) = p_1(t)^2 - p_2(t)^2 =$ constant, we get $2p_1\dot{p}_1 - 2p_2\dot{p}_2 = 0$, and so $\dot{p}_2 = \frac{p_1\dot{p}_2}{p_2} = \frac{p_1}{p_2} p_2 H_{p_3} = p_1 H_{p_3}$. Thus the second equation of motion is satisfied. Lastly, differentiate both sides of $H(p(t)) =$ constant, to get $\dot{p}_1 H_{p_1} + \dot{p}_2 H_{p_2} + \dot{p}_3 H_{p_3} = 0$. Solving for $\dot{p}_3$, the result is

$$\dot{p}_3 = \frac{-\dot{p}_1 H_{p_1} - \dot{p}_2 H_{p_2}}{H_{p_3}} = \frac{-p_2 H_{p_3} H_{p_1} - p_1 H_{p_3} H_{p_2}}{H_{p_3}} = -p_1 H_{p_2} - p_2 H_{p_1}.$$

Therefore $\dot{p}(t) = \vec{H}(p(t))$, *i.e.*, $p(\cdot)$ is an integral curve of $\vec{H}$. ∎

4.2 Homogeneous Systems

We begin our analysis of the affine equivalence representatives of table 3.1 by considering the homogeneous systems H_1 through H_5. The systems H_1 and H_2 have lines for integral curves, so we treat them here (the integration for these systems is immediate; as such, the stability analysis is the main effort). The remaining homogeneous systems H_3, H_4 and H_5 are treated in sections 4.2.1, 4.2.2 and 4.2.3, respectively. Of these three systems, H_3 and H_5 are integrated in terms of elementary functions, whereas the integral curves of H_4 are expressed in terms of Jacobi elliptic functions. (See section A.6 for further details on the Jacobi elliptic functions.)

Section B.4.1 and section B.4.2 list the supporting Mathematica code for H_1 and H_2, respectively. The equations of motion of the system $H_1(p) = \frac{1}{2}p_1^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -p_1 p_2. \end{cases}$$

These are immediately solved, to give the integral curve $p(t) = (c_1, c_2, c_3 - c_1 c_2 t)$, with $c_1, c_2, c_3 \in \mathbb{R}$. The equilibrium states of $\vec{H}_1$ are $\mathsf{e}_1^{\eta,\mu} = (\eta, 0, \mu)$ and $\mathsf{e}_2^{\nu,\mu} = (0, \nu, \mu)$, where $\eta, \mu \in \mathbb{R}$ and $\nu \neq 0$.

4.2.1 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *and* $\mathsf{e}_2^{\nu,\mu}$ *are unstable.*

Proof. Consider the states $\mathsf{e}_1^{\eta,\mu}$, $\eta \neq 0$. Fix a bounded open neighbourhood U of $\mathsf{e}_1^{\eta,\mu}$. Consider the integral curve $p(t) = (\eta, \delta, \mu - \eta\delta t)$, where $\delta > 0$. Since $\|p(0) - \mathsf{e}_1^{\eta,\mu}\| = \delta$, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Furthermore, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + \eta^2 + \lim_{t\to\infty}(\mu - \eta\delta t)^2 = \infty$. Hence there exists $t_1 > 0$ such that $p(t_1) \notin U$. It follows that the states $\mathsf{e}_1^{\eta,\mu}$, $\eta \neq 0$ are unstable.

Consider the states $\mathsf{e}_1^{0,\mu}$. Let U be a bounded open neighbourhood of $\mathsf{e}_1^{0,\mu}$. The curve $p(t) = (\frac{1}{2}\delta, \frac{\sqrt{3}}{2}\delta, \mu - \frac{\sqrt{3}}{4}\delta^2 t)$ is an integral curve of $\vec{H}_1$ for any $\delta > 0$. Furthermore, $\|p(0) - \mathsf{e}_1^{0,\mu}\| = \delta$. Accordingly, for any neighbourhood $V \subseteq U$ containing $\mathsf{e}_1^{0,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + \lim_{t\to\infty}(\mu - \frac{\sqrt{3}}{4}\delta^2 t)^2 = \infty$, and so there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the states $\mathsf{e}_1^{0,\mu}$ are unstable.

Consider the states $\mathsf{e}_2^{\nu,\mu}$. Fix a bounded open neighbourhood U of $\mathsf{e}_2^{\nu,\mu}$. We have that $p(t) = (\delta, \nu, \mu - \delta\nu t)$ is an integral curve of $\vec{H}_1$ for any $\delta > 0$. Thus, as $\|p(0) - \mathsf{e}_2^{\nu,\mu}\| = \delta$, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_2^{\nu,\mu}$ there exists $\delta > 0$ such that $p(0) \in V$. But $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + \nu^2 + \lim_{t\to\infty}(\mu - \delta\nu t)^2 = \infty$, and so there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the states $\mathsf{e}_2^{\nu,\mu}$ are unstable. ■

The equations of motion of the system $H_2(p) = \frac{1}{2}(p_1 + p_2)^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -(p_1 + p_2)^2. \end{cases}$$

The integral curves are of the form $p(t) = (c_1, c_2, c_3 - (c_1 + c_2)^2 t)$, for $c_1, c_2, c_3 \in \mathbb{R}$. The equilibrium states of $\vec{H}_2$ are $\mathsf{e}_1^{\eta,\mu} = (\eta, -\eta, \mu)$, where $\eta, \mu \in \mathbb{R}$.

4.2.2 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *are unstable.*

Proof. Fix a bounded open neighbourhood U of $\mathsf{e}_1^{\eta,\mu}$. We have that $p(t) = (\delta + \eta, \delta - \eta, \mu - 4\delta^2 t)$ is an integral curve of $\vec{H}_2$ (for any $\delta > 0$) such that $\|p(0) - \mathsf{e}_1^{\eta,\mu}\| = \sqrt{2}\,\delta$. Accordingly, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Furthermore, $\lim_{t\to\infty} \|p(t)\|^2 = (\delta - \eta)^2 + (\delta + \eta)^2 + \lim_{t\to\infty}(\mu - 4\delta^2 t)^2 = \infty$. Hence, there exists $t_1 > 0$ such that $p(t_1) \notin U$. It follows that the states $\mathsf{e}_1^{\eta,\mu}$ are unstable. ■

4.2.1 The system H_3

The equations of motion of the system $H_3(p) = \frac{1}{2}p_3^2$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = 0. \end{cases}$$

The equilibrium states of $\vec{H}_3$ are $\mathsf{e}_1^{\eta,\mu} = (\eta, \mu, 0)$ and $\mathsf{e}_2^{\mu} = (0, 0, \nu)$, where $\eta, \mu, \nu \in \mathbb{R}$ and $\nu \neq 0$. See section B.4.5 for accompanying Mathematica code.

4.2.3 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *and* e_2^{ν} *are unstable.*

Proof. Consider the states $\mathsf{e}_1^{\eta,\mu}$, $\eta \neq -\mu$. Fix a bounded open neighbourhood U of $\mathsf{e}_1^{\eta,\mu}$. We have that $p(t) = \big(\eta\cosh(\delta t) + \mu\sinh(\delta t), \eta\sinh(\delta t) + \mu\cosh(\delta t), \delta\big)$ is an integral curve of $\vec{H}_3$ for any $\delta > 0$. Indeed,

$$\begin{aligned} \dot{p}_1 &= \delta\eta\sinh(\delta t) + \delta\mu\cosh(\delta t) = p_2 p_3 \\ \dot{p}_2 &= \delta\mu\sinh(\delta t) + \delta\eta\cosh(\delta t) = p_1 p_3 \\ \dot{p}_3 &= 0. \end{aligned}$$

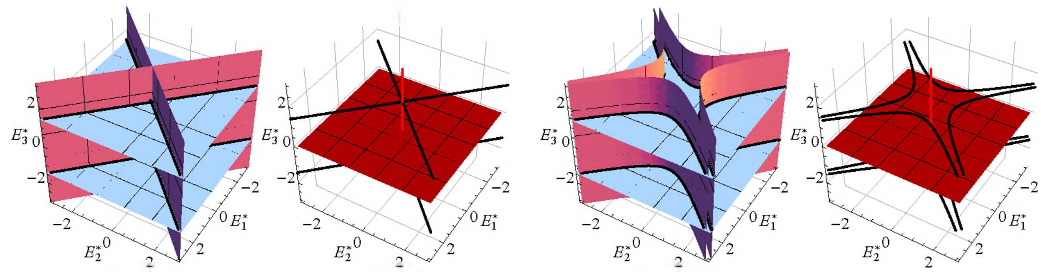
Moreover, we have $\|p(0) - \mathsf{e}_1^{\eta,\mu}\| = \delta$. Accordingly, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Furthermore,

$$\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + \lim_{t\to\infty} \big[(\eta^2 + \mu^2)\cosh(2\delta t) + 2\eta\mu\sinh(2\delta t)\big] = \infty.$$

Hence, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Thus the states $\mathsf{e}_1^{\eta,\mu}$, $\eta \neq -\mu$ are unstable.

Consider the states $\mathsf{e}_1^{\eta,-\eta}$, $\eta \neq 0$. Let U be a bounded open neighbourhood of $\mathsf{e}_1^{\eta,-\eta}$. The curve $p(t) = (\eta e^{\delta t}, -\eta e^{\delta t}, -\delta)$ is an integral curve of $\vec{H}_3$ for any $\delta > 0$. Indeed, $\dot{p}_1 = \delta\eta e^{\delta t} = p_2 p_3$, $\dot{p}_2 = -\delta\eta e^{\delta t} = p_1 p_3$ and $\dot{p}_3 = 0$. Since $\|p(0) - \mathsf{e}_1^{\eta,-\eta}\| = \delta$, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,-\eta}$, there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + 2\eta^2 \lim_{t\to\infty} e^{2\delta t} = \infty$. Consequently, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Thus the states $\mathsf{e}_1^{\eta,-\eta}$, $\eta \neq 0$ are unstable.

Consider the state $\mathsf{e}_1^{0,0}$. Fix a bounded open neighbourhood U of $\mathsf{e}_1^{0,0}$. We have that $p(t) = (\delta e^{\delta t}, \delta e^{\delta t}, \delta)$ is an integral curve of $\vec{H}_3$ for any $\delta > 0$. Indeed, $\dot{p}_1 = \delta^2 e^{\delta t} = p_2 p_3$, $\dot{p}_2 = \delta^2 e^{\delta t} = p_1 p_3$ and $\dot{p}_3 = 0$. Moreover, $\|p(0) - \mathsf{e}_1^{0,0}\| = \sqrt{3}\,\delta$. Therefore, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{0,0}$, there exists $\delta > 0$ such that $p(0) \in V$. Since $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 \lim_{t\to\infty}(1 + 2e^{2\delta t}) = \infty$, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Thus the state $\mathsf{e}_1^{0,0}$ is unstable.



(a) $c_0 = 0$ (b) $c_0 \neq 0$

Figure 4.1: Typical configurations of H_3

Consider the states e_2^ν. The linearisation of the vector field $\vec{H}_3$ is

$$\mathbf{D}\vec{H}_3(p) = \begin{bmatrix} 0 & p_3 & p_2 \\ p_3 & 0 & p_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The linearisation at e_2^ν has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\nu$. Since $\nu \neq 0$, $\mathbf{D}\vec{H}_3(\mathsf{e}_2^\nu)$ has a positive real eigenvalue. Hence the states e_2^ν are (spectrally) unstable. ■

Lastly, we determine the integral curves of $\vec{H}_3$. Typical configurations of H_3 are plotted in figure 4.1. (In the figure, we have $c_0 = C(p(0))$, where $p(\cdot)$ is an integral curve of $\vec{H}_3$.)

4.2.4 Proposition. *If $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ is an integral curve of $\vec{H}_3$ and $p_3(0) = \Omega$, then*

$$\begin{cases} p_1(t) = p_1(0)\cosh(\Omega t) + p_2(0)\sinh(\Omega t) \\ p_2(t) = p_1(0)\sinh(\Omega t) + p_2(0)\cosh(\Omega t) \\ p_3(t) = \Omega. \end{cases}$$

Proof. Since $\dot{p}_3 = 0$, we have $p_3(t) = \Omega$ for some $\Omega \in \mathbb{R}$. Let $P(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$. The differential equation $\dot{P}(t) = \begin{bmatrix} 0 & \Omega \\ \Omega & 0 \end{bmatrix} P(t)$ has the solution

$$P(t) = P(0)\exp\left(\begin{bmatrix} 0 & \Omega \\ \Omega & 0 \end{bmatrix} t\right) = P(0)\begin{bmatrix} \cosh(\Omega t) & \sinh(\Omega t) \\ \sinh(\Omega t) & \cosh(\Omega t) \end{bmatrix}.$$

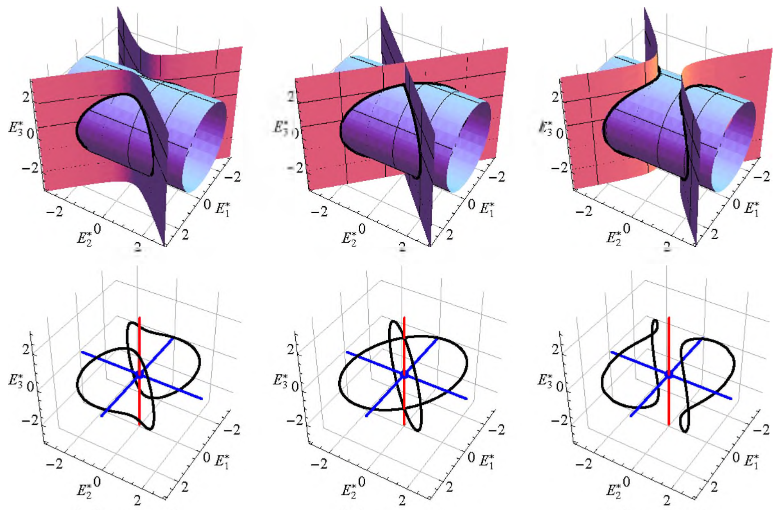
Therefore $p_1(t) = p_1(0)\cosh(\Omega t) + p_2(0)\sinh(\Omega t)$ and $p_2(t) = p_1(0)\sinh(\Omega t) + p_2(0)\cosh(\Omega t)$. ■

4.2.2 The system H_4

The equations of motion of the system $H_4(p) = \frac{1}{2}(p_1^2 + p_3^2)$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_1 p_2. \end{cases}$$

The equilibrium states of $\vec{H}_4$ are $\mathsf{e}_1^\mu = (\mu, 0, 0)$, $\mathsf{e}_2^\nu = (0, \nu, 0)$ and $\mathsf{e}_3^\nu = (0, 0, \nu)$, where $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$. Section B.4.4 lists the supporting Mathematica code.



(a) $c_0 > 0$ (b) $c_0 = 0$ (c) $c_0 < 0$

Figure 4.2: Typical configurations of H_4

4.2.5 PROPOSITION. *The equilibrium states* e_1^μ *and* e_2^ν *are stable, whereas the states* e_3^ν *are unstable.*

PROOF. As $\mathbf{d}(2H_4 - \frac{1}{2}C)(\mathsf{e}_1^0) = 0$ and $\mathbf{d}^2(2H_4 - \frac{1}{2}C)(\mathsf{e}_1^0) = \operatorname{diag}(1,1,2)$, the state e_1^0 is stable.

Consider the states e_1^μ, $\mu \neq 0$. Let $H_\lambda = \lambda_0 H_4 + \lambda_1 C$, where $\lambda_0 = 1$ and $\lambda_1 = -\frac{1}{2}$. We have

$$\mathbf{d}H_\lambda(p) = \begin{bmatrix} 0 \\ p_2 \\ p_3 \end{bmatrix}, \qquad \mathbf{d}^2 H_\lambda(p) = \operatorname{diag}(0,1,1).$$

Thus $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$. Moreover, since $W = \ker \mathbf{d}H_4(\mathsf{e}_1^\mu) \cap \ker \mathbf{d}C(\mathsf{e}_1^\mu) = \mathsf{span}\{E_2^*, E_3^*\}$, the restriction $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu)|_{W\times W} = \operatorname{diag}(1,1)$ is positive definite. Therefore the states e_1^μ, $\mu \neq 0$ are stable.

Consider the states e_2^ν. Define the energy function $H_\lambda = \lambda_0 H_4 + \lambda_1 C$, where $\lambda_0 = 1$ and $\lambda_1 = 0$. Then

$$\mathbf{d}H_\lambda(p) = \begin{bmatrix} p_1 \\ 0 \\ p_3 \end{bmatrix}, \qquad \mathbf{d}^2 H_\lambda(p) = \operatorname{diag}(1,0,1).$$

Furthermore, $W = \ker \mathbf{d}H_4(\mathsf{e}_1^\mu) \cap \ker \mathbf{d}C(\mathsf{e}_1^\mu) = \mathsf{span}\{E_1^*, E_3^*\}$. Accordingly, $\mathbf{d}H_4(\mathsf{e}_2^\nu) = 0$ and the restriction of $\mathbf{d}^2 H_4(\mathsf{e}_2^\nu)$ to $W \times W$ is positive definite. Hence the states e_2^ν are stable.

Consider the states e_3^ν. The linearisation of the vector field $\vec{H}_4$ is

$$\mathbf{D}\vec{H}_4(p) = \begin{bmatrix} 0 & p_3 & p_2 \\ p_3 & 0 & p_1 \\ -p_2 & -p_1 & 0 \end{bmatrix}.$$

The linearisation at e_3^ν has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\nu$. Since $\nu \neq 0$, $\mathbf{D}\vec{H}_4(\mathsf{e}_3^\nu)$ has a positive real eigenvalue. Hence the states e_3^ν are (spectrally) unstable. ■

We now proceed to find the integral curves of $\vec{H}_4$. Let $\bar{p}(\cdot)$ be an integral curve of $\vec{H}_4$ and let $c_0 = C(\bar{p}(0))$ and $h_0 = H_4(\bar{p}(0))$. Typical configurations of H_4 are graphed in figure 4.2.

We consider two cases depending on the sign of c_0. In fact, by the following result, we may assume $c_0 > 0$.

4.2.6 PROPOSITION. *The map* $\Psi : (p_1, p_2, p_3) \mapsto (p_2, p_1, p_3)$ *is a linear Poisson symmetry of* H_4 *such that* $C \circ \Psi = -C$.

PROOF. Ψ is clearly a linear Poisson automorphism. Furthermore, $(H_4 \circ \Psi)(p) = \frac{1}{2}(p_2^2 + p_3^2) = \frac{1}{2}(p_1^2 + p_3^2) - \frac{1}{2}(p_1^2 - p_2^2) = (H_4 - \frac{1}{2}C)(p)$. Therefore Ψ is a linear Poisson symmetry of H_4. Finally, we have $(C \circ \Psi)(p) = p_2^2 - p_1^2 = -C(p)$. ■

Accordingly, in order to find the integral curves of $\vec{H}_4$ when $c_0 < 0$, we find the integral curves when $c_0 > 0$ and apply the linear Poisson symmetry Ψ. In particular, if $\bar{q}(\cdot)$ is an integral curve of $\vec{H}_4$ such that $C(\bar{q}(0)) < 0$, then $\bar{p}(\cdot) = \Psi(\bar{q}(\cdot))$ is an integral curve of $\vec{H}_4$ such that $C(\bar{p}(0)) > 0$. Thus we assume, without loss of generality, that $c_0 > 0$.

Lastly, notice that if $h_0 = 0$, then $\bar{p}(\cdot)$ is constant. (Indeed, if $\bar{p}_1(t)^2 + \bar{p}_3(t)^2 = 2h_0 = 0$, then $\bar{p}_1(t) = \bar{p}_3(t) = 0$. Thus $-\bar{p}_2(t)^2 = c_0$, whence $c_0 \leq 0$ and $\bar{p}_2(t) = \pm\sqrt{-c_0}$. But $\bar{p}(t) = (0, 0, \pm\sqrt{-c_0})$ is an equilibrium point of $\vec{H}_4$.) Hence we always assume $h_0 > 0$.

4.2.2.1 Case *I*: $c_0 \neq 0$

By proposition 4.2.6, we may assume $c_0 > 0$. Notice that $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2 \leq \bar{p}_1(t)^2 + \bar{p}_3(t)^2 = 2h_0$. That is, $c_0 \leq 2h_0$. If $c_0 = 2h_0$, then $\bar{p}(\cdot)$ is constant. (Indeed, we have $\bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0 = 2h_0 = \bar{p}_1(t)^2 + \bar{p}_3(t)^2$, whence $-\bar{p}_2(t)^2 = \bar{p}_3(t)^2$. This implies that $\bar{p}(t) = (\pm\sqrt{c_0}, 0, 0)$, which is an equilibrium point of $\vec{H}_4$.) Assume $c_0 < 2h_0$. From the first equation of motion $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$, we get

$$\dot{\bar{p}}_1^2 = \bar{p}_2^2\bar{p}_3^2 = (2h_0 - \bar{p}_1^2)(\bar{p}_1^2 - c_0).$$

Take the square root of both sides and separate variables. We get

$$\frac{d\bar{p}_1}{\sqrt{(2h_0 - \bar{p}_1^2)(\bar{p}_1^2 - c_0)}} = \sigma_1\, dt$$

for some $\sigma_1 \in \{-1, 1\}$. Since $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2 \leq \bar{p}_1(t)^2 \leq \bar{p}_1(t)^2 + \bar{p}_3(t)^2 = 2h_0$, there exists $\sigma_2 \in \{-1, 1\}$ such that $\sqrt{c_0} \leq \sigma_2\bar{p}_1(t) \leq \sqrt{2h_0}$. Let $a = \sqrt{2h_0}$ and $b = \sqrt{c_0}$. Integrating both sides, we have

$$\int_{\sigma_2\bar{p}_1(t)}^{a} \frac{d\bar{p}_1}{\sqrt{(2h_0 - \bar{p}_1^2)(\bar{p}_1^2 - c_0)}} = \sigma_1 t. \tag{4.2.1}$$

Use the integral formula (A.6.8) to integrate the left-hand side of (4.2.1). The result is

$$\frac{1}{\sqrt{2h_0}}\,\mathrm{dn}^{-1}\left(\frac{\sigma_2}{\sqrt{2h_0}}\bar{p}_1(t), \sqrt{\frac{2h_0 - c_0}{2h_0}}\right) = \sigma_1 t \quad \Rightarrow \quad \bar{p}_1(t) = \sigma_2\Omega\,\mathrm{dn}(\Omega t, k).$$

Here $\Omega = \sqrt{2h_0}$ and $k = \sqrt{\frac{2h_0 - c_0}{2h_0}}$. As $0 < c_0 < 2h_0$, we have $0 < k < 1$. Moreover, since $\mathrm{dn}(\cdot, k)$ is even, the σ_1 can be eliminated. We now use the constant of motion $2h_0 = \bar{p}_1(t)^2 + \bar{p}_3(t)^2$ to find an expression for $\bar{p}_3(\cdot)$:

$$\bar{p}_3(t) = \sigma_2\sqrt{2h_0 - \bar{p}_1(t)^2} = \sigma_3\Omega\sqrt{1 - \mathrm{dn}^2(\Omega t, k)} = \sigma_3\Omega k\,\mathrm{sn}(\Omega t, k)$$

where $\sigma_3 \in \{-1,1\}$. (We have used the square relation (A.6.4).) In fact, since $\operatorname{sn}\big(\Omega t + \frac{2K}{\Omega}, k\big) = -\operatorname{sn}(\Omega t, k)$, we assume $\sigma_3 = 1$. Finally, using the derivative formula (A.6.1), integrate the equation

$$\dot{\bar{p}}_2(t) = \bar{p}_1\bar{p}_3 = \sigma_2\Omega\sqrt{2h_0 - c_0}\,\operatorname{dn}(\Omega t, k)\operatorname{sn}(\Omega t, k)$$

to get $\bar{p}_2(t) = -\sigma_2\sqrt{2h_0 - c_0}\,\operatorname{cn}(\Omega t, k)$. Rephrasing the constants in terms of Ω and k, we have the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = \sigma_2\Omega\operatorname{dn}(\Omega t, k) \\ \bar{p}_2(t) = -\sigma_2\Omega k\operatorname{cn}(\Omega t, k) \\ \bar{p}_3(t) = \Omega k\operatorname{sn}(\Omega t, k). \end{cases}$$

We verify that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_4$. Indeed,

$$\begin{aligned} \dot{\bar{p}}_1(t) &= -\sigma_2\Omega^2k^2\operatorname{cn}(\Omega t, k)\operatorname{sn}(\Omega t, k) = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= \sigma_2\Omega^2k\operatorname{dn}(\Omega t, k)\operatorname{sn}(\Omega t, k) = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= \Omega^2k\operatorname{cn}(\Omega t, k)\operatorname{dn}(\Omega t, k) = -\bar{p}_1(t)\bar{p}_2(t). \end{aligned}$$

That is, $\dot{\bar{p}}(t) = \vec{H}_3(\bar{p}(t))$. Furthermore, $\bar{p}(\cdot)$ is clearly defined over $\mathbb{R}$. Since $\operatorname{cn}(\Omega t, k)$ and $\operatorname{sn}(\Omega t, k)$ have period $\frac{4K}{\Omega}$, $\bar{p}_2(\cdot)$ and $\bar{p}_3(\cdot)$ have period $\frac{4K}{\Omega}$. Similarly, since $\operatorname{dn}(\Omega t, k)$ has period $\frac{2K}{\Omega}$, $\bar{p}_1(\cdot)$ has period $\frac{2K}{\Omega}$. Finally, as $\operatorname{sn}(\cdot, k)$ is odd and $\operatorname{dn}(\cdot, k)$, $\operatorname{cn}(\cdot, k)$ are even, we have that $\bar{p}_1(\cdot)$ and $\bar{p}_2(\cdot)$ are even and $\bar{p}_3(\cdot)$ is odd.

We now make an explicit statement regarding all integral curves of $\vec{H}_4$ for this case.

4.2.7 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_4$ such that $H_4(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$ and $c_0 < 2h_0$. There exist $t_0 \in \big[-\frac{2K}{\Omega}, \frac{2K}{\Omega}\big]$ and $\sigma \in \{-1,1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases} \bar{p}_1(t) = \sigma\Omega\operatorname{dn}(\Omega t, k) \\ \bar{p}_2(t) = -\sigma\Omega k\operatorname{cn}(\Omega t, k) \\ \bar{p}_3(t) = \Omega k\operatorname{sn}(\Omega t, k). \end{cases}$$

Here $4K$ is the period of $\operatorname{sn}(\cdot, k)$, $\Omega = \sqrt{2h_0}$ and $k = \sqrt{\frac{2h_0 - c_0}{2h_0}}$.

Proof. Let $\sigma = \operatorname{sgn}(p_1(0)) \in \{-1,1\}$. (If $p_1(0) = 0$, then $c_0 = -p_2(0)^2 \leq 0$, a contradiction.) Since $p_1(t)^2 \geq p_1(t)^2 - p_2(t)^2 = c_0$, we have $p_3(t)^2 = 2h_0 - p_1(t)^2 \leq 2h_0 - c_0 = \Omega^2k^2$. That is, $-\Omega k \leq p_3(t) \leq \Omega k$. Similarly, $-\Omega k \leq \bar{p}_3(t) \leq \Omega k$. Moreover, $\bar{p}_3\big(-\frac{K}{\Omega}\big) = -\Omega k$ and $\bar{p}_3\big(\frac{K}{\Omega}\big) = \Omega k$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_1 \in \big[-\frac{K}{\Omega}, \frac{K}{\Omega}\big]$ such that $\bar{p}_3(t_1) = p_3(0)$. Then

$$\bar{p}_1(t_1)^2 = 2h_0 - \bar{p}_3(t_1)^2 = 2h_0 - p_3(0)^2 = p_1(0)^2.$$

Since $\operatorname{sgn}(\bar{p}_1(t_1)) = \sigma = \operatorname{sgn}(p_1(0))$, we have $\bar{p}_1(t_1) = p_1(0)$. Lastly,

$$\bar{p}_2(t_1)^2 = \bar{p}_1(t_1)^2 - c_0 = p_1(0)^2 - c_0 = p_2(0)^2,$$

and so $\bar{p}_2(t_1) = \pm p_2(0)$. As $\bar{p}_1(\cdot)$ is even with period $\frac{2K}{\Omega}$, $\bar{p}_2(\cdot)$ is even with period $\frac{4K}{\Omega}$ and $\bar{p}_3(\cdot)$ is odd with period $\frac{4K}{\Omega}$, we have $\bar{p}_1\big(\frac{2K}{\Omega} - t_1\big) = \bar{p}_1(t_1)$, $\bar{p}_2\big(\frac{2K}{\Omega} - t_1\big) = -\bar{p}_2(t_1)$ and

$\bar{p}_3\left(\frac{2K}{\Omega} - t_1\right) = \bar{p}_3(t_1)$. Thus, there exists $t_0 \in \left[-\frac{2K}{\Omega}, \frac{2K}{\Omega}\right]$ (*i.e.*, $t_0 = t_1$ or $t_0 = -t_1$) such that $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_4$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ∎

4.2.2.2 Case II: $c_0 = 0$

If $\bar{p}_1(t) = 0$ for some t, then $\bar{p}(\cdot)$ is constant. Indeed, suppose $\bar{p}_1(t) = 0$ for some t. Then $\bar{p}_2(t) = 0$ (as $c_0 = 0$), whence $\bar{p}_3(t)^2 = 2h_0$. That is, $\bar{p}(t) = (0, 0, \pm\sqrt{2h_0})$, which is an equilibrium point of $\vec{H}_4$. Therefore we assume $\bar{p}_1(t) \neq 0$ for every t. Take the limit $c_0 \to 0$ of the integral curves obtained in case I, and allow for changes of sign. Since $k \to 1$ as $c_0 \to 0$, we get the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = \sigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{p}_2(t) = -\sigma\varsigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{p}_3(t) = \varsigma\Omega\tanh(\Omega t). \end{cases}$$

(Here $\sigma, \varsigma \in \{-1, 1\}$ and $\Omega = \sqrt{2h_0}$.) We verify that this is an integral curve of $\vec{H}_4$. Indeed,

$$\begin{aligned} \dot{\bar{p}}_1(t) &= -\sigma\Omega^2\,\mathrm{sech}(\Omega t)\tanh(\Omega t) = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= \sigma\varsigma\Omega^2\,\mathrm{sech}(\Omega t)\tanh(\Omega t) = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= \varsigma\Omega^2\,\mathrm{sech}^2(\Omega t) = -\bar{p}_1(t)\bar{p}_2(t). \end{aligned}$$

Thus $\dot{\bar{p}}(t) = \vec{H}_3(\bar{p}(t))$. Furthermore, $\bar{p}(\cdot)$ is clearly defined over $\mathbb{R}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_4$ for this case.

4.2.8 PROPOSITION. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_4$ such that $H_4(p(0)) = h_0 > 0$, $C(p(0)) = 0$ and $p_1(0) \neq 0$. There exist $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases} \bar{p}_1(t) = \sigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{p}_2(t) = -\sigma\varsigma\Omega\,\mathrm{sech}(\Omega t) \\ \bar{p}_3(t) = \varsigma\Omega\tanh(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{2h_0}$.

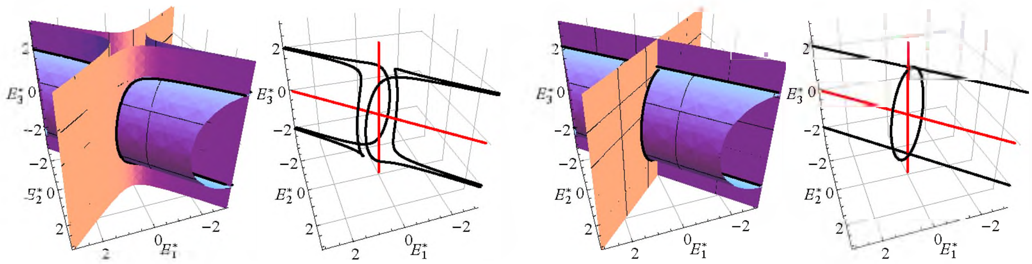
PROOF. Let $\sigma = \mathrm{sgn}(p_1(0)) \in \{-1, 1\}$ and $\varsigma = -\sigma\,\mathrm{sgn}(p_2(0)) \in \{-1, 1\}$. (As $p_1(0) \neq 0$, we have $p_2(0) \neq 0$.) Since $0 < p_1(t)^2$, we have $p_3(t)^2 = 2h_0 - p_1(t)^2 < 2h_0 = \Omega^2$. That is, $-\Omega < p_3(t) < \Omega$. Similarly, $-\Omega < \bar{p}_3(t) < \Omega$. Moreover, $\lim_{t\to\infty} \bar{p}_3(t) = \varsigma\Omega$ and $\lim_{t\to-\infty} \bar{p}_3(t) = -\varsigma\Omega$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0)^2 = 2h_0 - \bar{p}_3(t_0)^2 = 2h_0 - p_3(0)^2 = p_1(0)^2.$$

Since $\mathrm{sgn}(\bar{p}_1(t_0)) = \sigma = \mathrm{sgn}(p_1(0))$, we have $\bar{p}_1(t_0) = p_1(0)$. Finally,

$$\bar{p}_2(t_0)^2 = \bar{p}_1(t_0)^2 - c_0 = p_1(0)^2 - c_0 = p_2(0)^2.$$

But $\mathrm{sgn}(\bar{p}_2(t_0)) = -\sigma\varsigma = \mathrm{sgn}(p_2(0))$, and so $\bar{p}_2(t_0) = p_2(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_4$ passing through the same point at $t = 0$, they solve the same Cauchy problem, and hence are identical. ∎



(a) $c_0 \neq 0$ (b) $c_0 = 0$

Figure 4.3: Typical configurations of H_5

4.2.3 The system H_5

The equations of motion of the system $H_5(p) = \frac{1}{2}\left[(p_1 + p_2)^2 + p_3^2\right]$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -(p_1 + p_2)^2. \end{cases}$$

The equilibrium states of $\vec{H}_5$ are $\mathsf{e}_1^\mu = (\mu, -\mu, 0)$ and $\mathsf{e}_2^\nu = (0, 0, \nu)$, where $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$. The accompanying MATHEMATICA code for H_5 may be found in section B.4.5.

4.2.9 PROPOSITION. *The equilibrium states e_1^μ and e_2^ν are unstable.*

PROOF. Consider the states e_1^μ, $\mu \neq 0$. Fix a bounded open neighbourhood U of e_1^μ. The curve $p(t) = (\mu e^{\delta t}, -\mu e^{\delta t}, -\delta)$ is an integral curve of $\vec{H}_4$ for any $\delta > 0$. Indeed, $\dot{p}_1 = \delta\mu e^{\delta t} = p_2 p_3$, $\dot{p}_2 = -\delta\mu e^{\delta t} = p_1 p_3$ and $\dot{p}_3 = 0$. Furthermore, $\|p(0) - \mathsf{e}_1^\mu\| = \delta$. Accordingly, for any open neighbourhood $V \subseteq U$ of e_1^μ there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + 2\mu^2 \lim_{t\to\infty} e^{2\delta t} = \infty$. Thus, there exists $t_1 > 0$ such that $p(t_1) \notin V$. Therefore the states e_1^μ, $\mu \neq 0$ are unstable.

Consider the state e_1^0. Let U be any bounded open neighbourhood of e_1^0. We have that $p(t) = (\delta e^{\delta t}, -\delta e^{\delta t}, -\delta)$ with $\delta > 0$ is an integral curve of $\vec{H}_3$. Indeed, $\dot{p}_1 = \delta^2 e^{\delta t} = p_2 p_3$, $\dot{p}_2 = \delta^2 e^{\delta t} = p_1 p_3$ and $\dot{p}_3 = 0$. Moreover, $\|p(0) - \mathsf{e}_1^0\| = \sqrt{3}\,\delta$. Therefore, for any open neighbourhood $V \subseteq U$ of e_1^0, there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 \lim_{t\to\infty}(1 + 2e^{2\delta t}) = \infty$, and so there exists $t_1 > 0$ such that $p(t_1) \notin U$. Thus the state e_1^0 is unstable.

Consider the states e_2^ν. The linearisation of the vector field $\vec{H}_5$ is

$$\mathbf{D}\vec{H}_5(p) = \begin{bmatrix} 0 & p_3 & p_2 \\ p_3 & 0 & p_1 \\ -2(p_1 + p_2) & -2(p_1 + p_2) & 0 \end{bmatrix}.$$

The linearisation at e_2^ν has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\nu$. Since $\nu \neq 0$, $\mathbf{D}\vec{H}_5(\mathsf{e}_2^\nu)$ has a positive real eigenvalue. Hence the states e_2^ν are (spectrally) unstable. ∎

We now proceed to find the integral curves of $\vec{H}_5$. Let $\bar{p}(\cdot)$ be an integral curve of $\vec{H}_5$ and let $c_0 = C(\bar{p}(0))$ and $h_0 = H_5(\bar{p}(0))$. We consider two cases depending on the sign of c_0. The corresponding level sets $H_5^{-1}(h_0)$, $C^{-1}(c_0)$ and their intersection are plotted in figure 4.3.

If $h_0 = 0$, then $\bar{p}(\cdot)$ is constant. (Indeed, if $(\bar{p}_1(t) + \bar{p}_2(t))^2 + \bar{p}_3(t)^2 = 2h_0 = 0$, then $\bar{p}_1(t) + \bar{p}_2(t) = 0$ and $\bar{p}_3(t) = 0$. From the equations of motion we then have $\dot{\bar{p}}_1 = \dot{\bar{p}}_2 = 0$, whence $\bar{p}(t) = (\mu, -\mu, 0)$ for some $\mu \in \mathbb{R}$. Then $\bar{p}(t)$ is an equilibrium point of $\vec{H}_5$.) Thus we always assume $h_0 > 0$.

4.2.3.1 Case *I*: $c_0 \neq 0$

Substitute $\dot{\bar{p}}_3 = -(\bar{p}_1 + \bar{p}_2)^2$ into the equation $2h_0 = (\bar{p}_1(t) + \bar{p}_2(t))^2 + \bar{p}_3(t)^2$. The result is a separable differential equation in $\bar{p}_3$, *viz.*

$$\frac{d\bar{p}_3}{\bar{p}_3^2 - 2h_0} = dt. \tag{4.2.2}$$

We have $\bar{p}_3(t)^2 < (\bar{p}_1(t) + \bar{p}_2(t))^2 + \bar{p}_3(t)^2 = 2h_0$, *i.e.*, $\bar{p}_3(t)^2 < 2h_0$. We use the integral formula (A.6.13) to integrate the left-hand side of (4.2.2). The result is

$$-\frac{1}{\sqrt{2h_0}} \tanh^{-1}\left(\frac{\bar{p}_3(t)}{\sqrt{2h_0}}\right) = t.$$

Let $\Omega = \sqrt{2h_0}$ and solve for $\bar{p}_3(t)$, to get $\bar{p}_3(t) = -\Omega\tanh(\Omega t)$. Differentiating this expression with respect to t yields

$$\bar{p}_1(t) + \bar{p}_2(t) = \sigma\sqrt{-\dot{\bar{p}}_3(t)} = \sigma\Omega\,\mathrm{sech}(\Omega t)$$

for some $\sigma \in \{-1, 1\}$. Since $(\bar{p}_1(t) + \bar{p}_2(t))(\bar{p}_1(t) - \bar{p}_2(t)) = c_0$, we have

$$\bar{p}_1(t) - \bar{p}_2(t) = \sigma\frac{c_0}{\Omega\,\mathrm{sech}(\Omega t)} = \frac{\sigma c_0}{\Omega}\cosh(\Omega t).$$

Thus

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} = \begin{bmatrix} \sigma\Omega\,\mathrm{sech}(\Omega t) \\ \frac{\sigma c_0}{\Omega}\cosh(\Omega t) \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, we get the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = \dfrac{\sigma}{2\Omega}\left[\Omega^2\,\mathrm{sech}(\Omega t) + c_0\cosh(\Omega t)\right] \\ \bar{p}_2(t) = \dfrac{\sigma}{2\Omega}\left[\Omega^2\,\mathrm{sech}(\Omega t) - c_0\cosh(\Omega t)\right] \\ \bar{p}_3(t) = -\Omega\tanh(\Omega t). \end{cases}$$

We verify that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_5$. Indeed,

$$\begin{aligned} \dot{\bar{p}}_1(t) - \bar{p}_2(t)\bar{p}_3(t) &= \frac{\sigma}{2}\left[c_0\sinh(\Omega t) - \Omega^2\,\mathrm{sech}(\Omega t)\tanh(\Omega t)\right] \\ &\quad -\frac{\sigma}{2}\left[c_0\cosh(\Omega t) - \Omega^2\,\mathrm{sech}(\Omega t)\right]\tanh(\Omega t) = 0 \\ \dot{\bar{p}}_2(t) - \bar{p}_1(t)\bar{p}_3(t) &= -\frac{\sigma}{2}\left[c_0\sinh(\Omega t) + \Omega^2\,\mathrm{sech}(\Omega t)\tanh(\Omega t)\right] \\ &\quad +\frac{\sigma}{2}\left[c_0\cosh(\Omega t) + \Omega^2\,\mathrm{sech}(\Omega t)\right]\tanh(\Omega t) = 0 \\ \dot{\bar{p}}_3(t) + (\bar{p}_1(t) + \bar{p}_2(t))^2 &= (\sigma^2 - 1)\Omega^2\,\mathrm{sech}^2(\Omega t) = 0. \end{aligned}$$

That is, $\dot{\bar{p}}(t) = \vec{H}_5(\bar{p}(t))$. Furthermore, $\bar{p}(\cdot)$ is clearly defined over $\mathbb{R}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_5$ for this case.

4.2.10 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_5$ such that $H_5(p(0)) = h_0 > 0$ and $C(p(0)) = c_0 \neq 0$. There exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases} \bar{p}_1(t) = \dfrac{\sigma}{2\Omega}\left[\Omega^2 \operatorname{sech}(\Omega t) + c_0 \cosh(\Omega t)\right] \\ \bar{p}_2(t) = \dfrac{\sigma}{2\Omega}\left[\Omega^2 \operatorname{sech}(\Omega t) - c_0 \cosh(\Omega t)\right] \\ \bar{p}_3(t) = -\Omega \tanh(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{2h_0}$.

Proof. Let $\sigma = \operatorname{sgn}(p_1(0) + p_2(0)) \in \{-1, 1\}$. (If $p_1(0) + p_2(0) = 0$, then $c_0 = 0$, a contradiction.) We have $p_3(t)^2 < (p_1(t) + p_2(t))^2 + p_3(t)^2 = 2h_0 = \Omega^2$, and so $-\Omega < p_3(t) < \Omega$. Similarly, $-\Omega < \bar{p}_3(t) < \Omega$. Moreover, $\lim_{t\to\infty} \bar{p}_3(t) = -\Omega$ and $\lim_{t\to-\infty} \bar{p}_3(t) = \Omega$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\left(\bar{p}_1(t_0) + \bar{p}_2(t_0)\right)^2 = 2h_0 - \bar{p}_3(t_0)^2 = 2h_0 - p_3(0)^2 = \left(p_1(0) + p_2(0)\right)^2.$$

But $\operatorname{sgn}(\bar{p}_1(t_0) + \bar{p}_2(t_0)) = \sigma = \operatorname{sgn}(p_1(0) + p_2(0))$, and so $\bar{p}_1(t_0) + \bar{p}_2(t_0) = p_1(0) + p_2(0)$. Further,

$$\left(\bar{p}_1(t_0) + \bar{p}_2(t_0)\right)\left(\bar{p}_1(t_0) - \bar{p}_2(t_0)\right) = c_0 = \left(p_1(0) + p_2(0)\right)\left(p_1(0) - p_2(0)\right),$$

which implies that $\bar{p}_1(t_0) - \bar{p}_2(t_0) = p_1(0) - p_2(0)$. Thus, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{p}_1(t_0) \\ \bar{p}_2(t_0) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, we have $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_5$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ■

4.2.3.2 Case II: $c_0 = 0$

If $\bar{p}_1(t) = 0$ for some t, then $\bar{p}(\cdot)$ is constant. Indeed, suppose $\bar{p}_1(t) = 0$ for some t. Then $\bar{p}_2(t) = 0$ (as $c_0 = 0$), whence $\bar{p}_3(t)^2 = 2h_0$. Thus $\bar{p}(t) = (0, 0, \pm\sqrt{2h_0})$, which is an equilibrium point of $\vec{H}_5$. Therefore we assume $\bar{p}_1(t) \neq 0$ for all t.

Since $\bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0 = 0$, we have $\bar{p}_1(t) - \bar{p}_2(t) = 0$ or $\bar{p}_1(t) + \bar{p}_2(t) = 0$. Suppose $\boldsymbol{\bar{p}_1(t) - \bar{p}_2(t) = 0}$. By taking the limit $c_0 \to 0$ of the integral curves in proposition 4.2.10, we arrive at the following result.

4.2.11 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_5$ such that $H_5(p(0)) = h_0 > 0$, $C(p(0)) = 0$, $p_1(0) - p_2(0) = 0$ and $p_1(0) \neq 0$. There exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$*

such that $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ *is defined by*

$$\begin{cases} \bar{p}_1(t) = \dfrac{\sigma\Omega}{2}\operatorname{sech}(\Omega t) \\ \bar{p}_2(t) = \dfrac{\sigma\Omega}{2}\operatorname{sech}(\Omega t) \\ \bar{p}_3(t) = -\Omega\tanh(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{2h_0}$.

PROOF. Let $\sigma = \operatorname{sgn}(p_1(0)) \in \{-1, 1\}$. From the constant of motion $2h_0 = p_1(t)^2 + p_3(t)^2$, we have $p_3(t)^2 = 2h_0 - p_1(t)^2 < 2h_0 = \Omega^2$. That is, $-\Omega < p_3(t) < \Omega$. Similarly, $-\Omega < \bar{p}_3(t) < \Omega$. Moreover, $\lim_{t\to-\infty} \bar{p}_3(t) = \Omega$ and $\lim_{t\to\infty} \bar{p}_3(t) = -\Omega$. Since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0)^2 = 2h_0 - \bar{p}_3(t_0)^2 = 2h_0 - p_3(0)^2 = p_1(0)^2.$$

Since $\operatorname{sgn}(\bar{p}_1(t_0)) = \sigma = \operatorname{sgn}(p_1(0))$, we have $\bar{p}_1(t_0) = p_1(0)$. Lastly, $\bar{p}_2(t_0) = \bar{p}_1(t_0) = p_1(0) = p_2(0)$. Therefore, as $t \mapsto \bar{p}(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_5$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ∎

Suppose $\boldsymbol{\bar{p}_1(t) + \bar{p}_2(t) = 0}$. Let $\Omega = \sqrt{2h_0}$. Then $\bar{p}_3(t)^2 = \Omega^2$, *i.e.*, $\bar{p}_3(t) = \sigma_1\Omega$, for some $\sigma_1 \in \{-1, 1\}$. The first equation of motion becomes $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3 = -\sigma_1\Omega\bar{p}_1$. This is immediately solved, to get $\bar{p}_1(t) = \sigma_2 e^{-\sigma_1\Omega t}$ for some $\sigma_2 \in \{-1, 1\}$. Then we have $\bar{p}_2(t) = -\sigma_2 e^{-\sigma_1\Omega t}$, and hence the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = \sigma_2 e^{-\sigma_1\Omega t} \\ \bar{p}_2(t) = -\sigma_2 e^{-\sigma_1\Omega t} \\ \bar{p}_3(t) = \sigma_1\Omega. \end{cases}$$

It is straightforward to verify that $\dot{\bar{p}}(t) = \vec{H}_5(\bar{p}(t))$. Indeed, $\dot{\bar{p}}_1(t) = -\sigma_1\sigma_2\Omega e^{-\sigma_1\Omega t} = \bar{p}_2(t)\bar{p}_3(t)$, $\dot{\bar{p}}_2(t) = \sigma_1\sigma_2\Omega e^{-\sigma_1\Omega t} = \bar{p}_1(t)\bar{p}_3(t)$ and $\dot{\bar{p}}_3 = 0 = -(\bar{p}_1(t) + \bar{p}_2(t))^2$. Moreover, $\bar{p}(\cdot)$ is clearly defined over $\mathbb{R}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_5$ for this case.

4.2.12 PROPOSITION. *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_5$ *such that* $H_4(p(0)) = h_0 > 0$, $C(p(0)) = 0$, $p_1(0) + p_2(0) = 0$ *and* $p_1(0) \neq 0$. *There exist* $t_0 \in \mathbb{R}$ *and* $\sigma, \varsigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ *is defined by*

$$\begin{cases} \bar{p}_1(t) = \varsigma e^{\sigma\Omega t} \\ \bar{p}_2(t) = -\varsigma e^{\sigma\Omega t} \\ \bar{p}_3(t) = -\sigma\Omega. \end{cases}$$

Here $\Omega = \sqrt{2h_0}$.

PROOF. Let $\sigma = -\operatorname{sgn}(p_3(0)) \in \{-1, 1\}$ and $\varsigma = -\operatorname{sgn}(p_2(0))$. (If $p_3(0) = 0$, then $h_0 = 0$, a contradiction. If $p_2(0) = 0$, then $p_1(0) = 0$, contradicting the assumption.) We have

$$\lim_{t\to-\infty} \bar{p}_2(t) = \begin{cases} -\varsigma\infty & \text{if } \sigma = -1 \\ 0 & \text{if } \sigma = +1, \end{cases} \qquad \lim_{t\to\infty} \bar{p}_2(t) = \begin{cases} \varsigma\infty & \text{if } \sigma = +1 \\ 0 & \text{if } \sigma = -1. \end{cases}$$

Therefore, since $\bar{p}_2(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_2(t_0) = p_2(0)$. Then $\bar{p}_1(t_0) = -\bar{p}_2(t_0) = -p_2(0) = p_1(0)$. Finally, since $p_3(\cdot)$ and $\bar{p}_3(\cdot)$ are both constant and $\operatorname{sgn}(\bar{p}_3(t_0)) = \sigma = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_3(t_0) = p_3(0)$. Therefore, as $t \mapsto \bar{p}(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_5$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ■

4.3 Inhomogeneous Systems Associated to H_0

There are two inhomogeneous systems associated to the (trivial) Hamilton-Poisson system $H_0(p) = 0$, *viz.* $H_1^{(0)}(p) = p_1$ and $H_{2,\alpha}^{(0)}(p) = \alpha p_3$. The first system has lines for integral curves, and so we shall treat it here. The integral curves of $H_{2,\alpha}^{(0)}$ are hyperbolae; this system is considered in section 4.3.1. Section B.4.6 and section B.4.7 list the supporting Mathematica code for the stability analysis of $H_1^{(0)}$ and $H_{2,\alpha}^{(0)}$, respectively.

The equations of motion of the system $H_1^{(0)}(p) = p_1$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -p_2. \end{cases}$$

The integral curves are $p(t) = (c_1, c_2, c_3 - c_2 t)$, for $c_1, c_2, c_3 \in \mathbb{R}$. The equilibrium states of $\vec{H}_1^{(0)}$ are $\mathsf{e}^{\eta,\mu} = (\eta, 0, \mu)$, where $\eta, \mu \in \mathbb{R}$.

4.3.1 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *are unstable.*

Proof. Fix a bounded open neighbourhood U of $\mathsf{e}_1^{\eta,\mu}$. Consider the integral curve $p(t) = (\eta, \delta, \mu - \delta t)$, for $\delta > 0$. We have $\|p(0) - \mathsf{e}_1^{\eta,\mu}\| = \delta$. Accordingly, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Furthermore, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + \eta^2 + \lim_{t\to\infty}(\mu - \delta t)^2 = \infty$, and so there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the states $\mathsf{e}_1^{\eta,\mu}$ are unstable. ■

4.3.1 The system $H_{2,\alpha}^{(0)}$

The equations of motion of the system $H_{2,\alpha}^{(0)}(p) = \alpha p_3$ $(\alpha > 0)$ are

$$\begin{cases} \dot{p}_1 = \alpha p_2 \\ \dot{p}_2 = \alpha p_1 \\ \dot{p}_3 = 0. \end{cases}$$

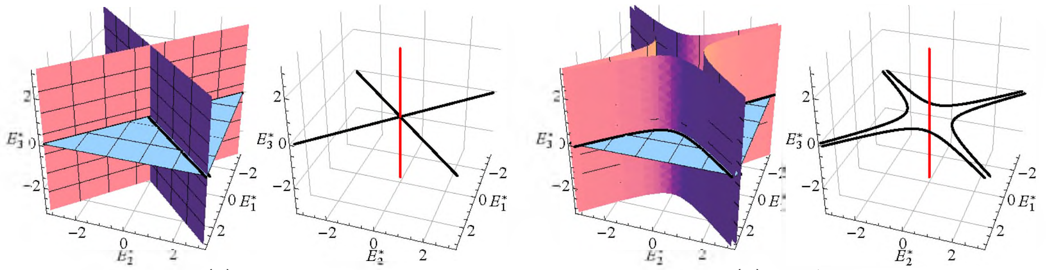
The equilibrium states of $\vec{H}_{2,\alpha}^{(0)}$ are $\mathsf{e}_1^{\mu} = (0, 0, \mu)$, where $\mu \in \mathbb{R}$.

4.3.2 Proposition. *The equilibrium states* e_1^{μ} *are unstable.*

Proof. The linearisation of the vector field $\vec{H}_{2,\alpha}^{(0)}$ is

$$\mathbf{D}\vec{H}_{3,\alpha}^{(1)}(p) = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The linearisation at e_1^{μ} has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \alpha$. Since $\alpha > 0$, $\mathbf{D}\vec{H}_{2,\alpha}^{(0)}(\mathsf{e}_1^{\mu})$ has a positive real eigenvalue. Hence the states e_1^{μ} are (spectrally) unstable. ■



(a) $c_0 = 0$ (b) $c_0 \neq 0$

Figure 4.4: Typical configurations of $H^{(0)}_{2,\alpha}$

Lastly, we determine the integral curves. The typical configurations of the system are graphed in figure 4.4. (In the figure we have $c_0 = C(p(0))$, where $p(\cdot)$ is an integral curve of $\vec{H}^{(0)}_{2,\alpha}$.)

4.3.3 PROPOSITION. *If $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ is an integral curve of $\vec{H}^{(0)}_{2,\alpha}$, then*

$$\begin{cases} p_1(t) = p_1(0)\cosh(\alpha t) + p_2(0)\sinh(\alpha t) \\ p_2(t) = p_1(0)\sinh(\alpha t) + p_2(0)\cosh(\alpha t) \\ p_3(t) = p_3(0). \end{cases}$$

PROOF. Let $P(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$. The differential equation $\dot{P}(t) = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} P(t)$ has the solution

$$P(t) = P(0)\exp\left(\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} t\right) = P(0)\begin{bmatrix} \cosh(\alpha t) & \sinh(\alpha t) \\ \sinh(\alpha t) & \cosh(\alpha t) \end{bmatrix}.$$

Therefore $p_1(t) = p_1(0)\cosh(\alpha t) + p_2(0)\sinh(\alpha t)$ and $p_2(t) = p_1(0)\sinh(\alpha t) + p_2(0)\cosh(\alpha t)$. Finally, as $\dot{p}_3 = 0$, we have $p_3(t) = p_3(0)$. ■

4.4 Inhomogeneous Systems Associated to H_1

Associated to the system $H_1(p) = \frac{1}{2}p_1^2$ are the three systems $H^{(1)}_1(p) = p_1 + \frac{1}{2}p_1^2$, $H^{(1)}_2(p) = p_1 + p_2 + \frac{1}{2}p_1^2$ and $H^{(1)}_{3,\alpha}(p) = \alpha p_3 + \frac{1}{2}p_1^2$. The first two of these systems have simple integral curves (in fact, lines), so we shall consider them here. The system $H^{(1)}_{3,\alpha}$ is slightly more involved, and is treated in section 4.4.1. Sections B.4.8, B.4.9 and B.4.10 list the MATHEMATICA code supporting the stability analysis of $H^{(1)}_1$, $H^{(1)}_2$ and $H^{(1)}_{3,\alpha}$, respectively.

The equations of motion of the system $H^{(1)}_1(p) = p_1 + \frac{1}{2}p_1^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -p_2(1 + p_1). \end{cases}$$

The integral curves are $p(t) = (c_1, c_2, c_3 - c_2(1 + c_1)t)$, for $c_1, c_2, c_3 \in \mathbb{R}$. The equilibrium states of $\vec{H}^{(1)}_1$ are $\mathsf{e}_1^{\eta,\mu} = (\eta, 0, \mu)$ and $\mathsf{e}_2^{\nu,\mu} = (-1, \nu, \mu)$, where $\eta, \nu, \mu \in \mathbb{R}$, $\nu \neq 0$.

4.4.1 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *and* $\mathsf{e}_2^{\nu,\mu}$ *are unstable.*

Proof. Consider the states $\mathsf{e}_1^{\eta,\mu}$, $\eta \neq -1$. Let U be a bounded open neighbourhood of $\mathsf{e}_1^{\eta,\mu}$. The integral curve $p(t) = (\eta, \delta, \mu - \delta(1+\eta)t)$ (for $\delta > 0$) satisfies $\|p(0) - \mathsf{e}_1^{\eta,\mu}\| = \delta$. Therefore, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + \eta^2 + \lim_{t\to\infty}(\mu - \delta(1+\eta)t)^2 = \infty$, and so there exists $t_1 > 0$ such that $p(t_1) \notin U$. Hence the states $\mathsf{e}_1^{\eta,\mu}$ are unstable.

Consider the states $\mathsf{e}_1^{-1,\mu}$. Let U be a bounded open neighbourhood of $\mathsf{e}_1^{-1,\mu}$. The curve $p(t) = (\delta - 1, \delta, \mu - \delta^2 t)$ is an integral curve of $\vec{H}_1^{(1)}$ for any $\delta > 0$, such that $\|p(0) - \mathsf{e}_1^{-1,\mu}\| = \sqrt{2}\,\delta$. Hence, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{-1,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Since $\lim_{t\to\infty} \|p(t)\|^2 = (\delta - 1)^2 + \delta^2 + \lim_{t\to\infty}(\mu - \delta^2 t)^2 = \infty$, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the states $\mathsf{e}_1^{-1,\mu}$ are unstable.

Consider the states $\mathsf{e}_2^{\nu,\mu}$. Fix a bounded open neighbourhood U of $\mathsf{e}_2^{\nu,\mu}$. We have that $p(t) = (\delta - 1, \nu, \mu - \nu\delta t)$ is an integral curve of $\vec{H}_1^{(1)}$ for any $\delta > 0$. Furthermore, $\|p(0) - \mathsf{e}_2^{\nu,\mu}\| = \delta$. Accordingly, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_2^{\nu,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. As $\lim_{t\to\infty} \|p(t)\|^2 = (\delta - 1)^2 + \nu^2 + \lim_{t\to\infty}(\mu - \nu\delta t)^2 = \infty$, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the states $\mathsf{e}_2^{\nu,\mu}$ are unstable. ∎

The equations of motion of the system $H_2^{(1)}(p) = p_1 + p_2 + \frac{1}{2}p_1^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -(p_1 + p_2 + p_1 p_2). \end{cases}$$

We solve these to get the integral curve $p(t) = (c_1, c_2, c_3 - (c_1 + c_2 + c_1 c_2)t)$, for some $c_1, c_2, c_3 \in \mathbb{R}$. The equilibrium states of $\vec{H}_2^{(1)}$ are $\mathsf{e}_1^{\eta,\mu} = \left(\eta, -\frac{\eta}{1+\eta}, \mu\right)$, where $\eta, \mu \in \mathbb{R}$, $\eta \neq -1$.

4.4.2 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *are unstable.*

Proof. Fix a bounded open neighbourhood U of $\mathsf{e}_1^{\eta,\mu}$. We have that $p(t) = \left(\eta, \delta - \frac{\eta}{1+\eta}, \mu - \delta(1+\eta)t\right)$ is an integral of $\vec{H}_2^{(1)}$ for any $\delta > 0$. Since $\|p(0) - \mathsf{e}_1^{\eta,\mu}\| = \delta$, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Furthermore, $\lim_{t\to\infty} \|p(t)\|^2 = \eta^2 + (\delta - \frac{\eta}{1+\eta})^2 + \lim_{t\to\infty}(\mu - \delta(1+\eta)t)^2 = \infty$. Thus there exists $t_1 > 0$ such that $p(t_1) \notin U$. It follows that the states $\mathsf{e}_1^{\eta,\mu}$ are unstable. ∎

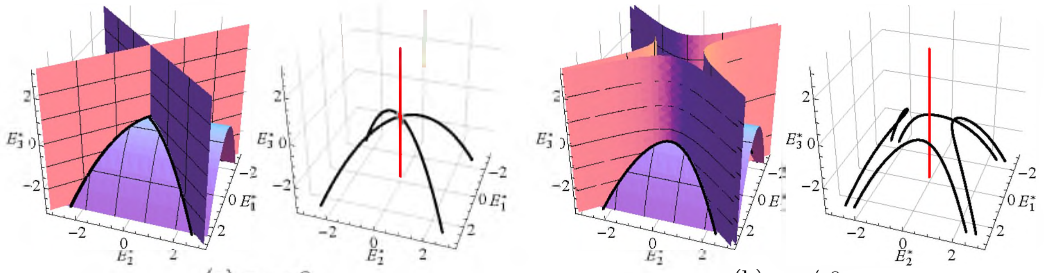
4.4.1 The system $H_{3,\alpha}^{(1)}$

The equations of motion of the system $H_{3,\alpha}^{(1)}(p) = \alpha p_3 + \frac{1}{2}p_1^2$ ($\alpha > 0$) are

$$\begin{cases} \dot{p}_1 = \alpha p_2 \\ \dot{p}_2 = \alpha p_1 \\ \dot{p}_3 = -p_1 p_2. \end{cases}$$

The equilibrium states of $\vec{H}_{3,\alpha}^{(1)}$ are $\mathsf{e}_1^{\mu} = (0, 0, \mu)$, where $\mu \in \mathbb{R}$.

4.4.3 Proposition. *The equilibrium states* e_1^{μ} *are unstable.*



(a) $c_0 = 0$ (b) $c_0 \neq 0$

Figure 4.5: Typical configurations of $H^{(1)}_{3,\alpha}$

Proof. The linearisation of the vector field $\vec{H}^{(1)}_{3,\alpha}$ is

$$\mathbf{D}\vec{H}^{(1)}_{3,\alpha}(p) = \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ -p_2 & -p_1 & 0 \end{bmatrix}.$$

The linearisation at e_1^{μ} has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\alpha$. Since $\alpha > 0$, $\mathbf{D}\vec{H}^{(1)}_{3,\alpha}(\mathsf{e}_1^{\mu})$ has a positive real eigenvalue. Hence the states e_1^{μ} are (spectrally) unstable. ■

Lastly, we derive the integral curves of $\vec{H}^{(1)}_{3,\alpha}$. Typical configurations of the system $H^{(1)}_{3,\alpha}$ are graphed in figure 4.5 (corresponding to $c_0 = 0$ and $c_0 \neq 0$, where $c_0 = C(p(0))$ and $p(\cdot)$ is an integral curve of $\vec{H}^{(1)}_{3,\alpha}$).

4.4.4 Proposition. *If $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ is an integral curve of $\vec{H}^{(1)}_{3,\alpha}$, then*

$$\begin{cases} p_1(t) = p_1(0)\cosh(\alpha t) + p_2(0)\sinh(\alpha t) \\ p_2(t) = p_1(0)\sinh(\alpha t) + p_2(0)\cosh(\alpha t) \\ p_3(t) = \frac{1}{2\alpha}\left(p_1(0)^2 - p_1(t)^2\right) + p_3(0). \end{cases}$$

Proof. Let $P(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$. The differential equation $\dot{P}(t) = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} P(t)$ has the solution

$$P(t) = P(0)\exp\left(\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} t\right) = P(0)\begin{bmatrix} \cosh(\alpha t) & \sinh(\alpha t) \\ \sinh(\alpha t) & \cosh(\alpha t) \end{bmatrix}.$$

Therefore $p_1(t) = p_1(0)\cosh(\alpha t) + p_2(0)\sinh(\alpha t)$ and $p_2(t) = p_1(0)\sinh(\alpha t) + p_2(0)\cosh(\alpha t)$. Let $h_0 = H^{(1)}_{3,\alpha}(p(0))$. Then $h_0 = \alpha p_3(t) + \frac{1}{2}p_1(t)^2$, and so $p_3(t) = \frac{1}{\alpha}\left(h_0 - \frac{1}{2}p_1(t)^2\right)$. Since $p_3(0) = \frac{1}{\alpha}\left(h_0 - \frac{1}{2}p_1(0)^2\right)$, we have $h_0 = \frac{1}{2}(p_1(0)^2 + 2\alpha p_3(0))$. Substituting this into the expression for $p_3(\cdot)$ gives the result. ■

4.5 Inhomogeneous Systems Associated to H_2

We consider those inhomogeneous representatives associated to $H_2(p) = \frac{1}{2}(p_1 + p_2)^2$, namely the systems $H^{(2)}_1(p) = p_1 + \frac{1}{2}(p_1 + p_2)^2$, $H^{(2)}_2(p) = p_1 + p_2 + \frac{1}{2}(p_1 + p_2)^2$ and $H^{(2)}_{3,\delta}(p) =$

$\delta p_3+\frac{1}{2}(p_1+p_2)^2$. The systems $H_1^{(2)}$ and $H_2^{(2)}$ are considered here, as they have lines for integral curves. The system $H_{3,\delta}^{(2)}$ is considered in section 4.5.1. The supporting Mathematica code for $H_1^{(2)}$, $H_2^{(2)}$ and $H_{3,\delta}^{(2)}$ is listed in sections B.4.11, B.4.12 and B.4.13, respectively.

The equations of motion of the system $H_1^{(2)}(p) = p_1 + \frac{1}{2}(p_1+p_2)^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -p_2 - (p_1+p_2)^2. \end{cases}$$

The integral curves are $p(t) = (c_1, c_2, c_3 - c_2 t - (c_1+c_2)^2 t)$, for $c_1, c_2, c_3 \in \mathbb{R}$. The equilibrium states of $\vec{H}_1^{(2)}$ are $\mathsf{e}_1^{\eta,\mu} = (\eta, \frac{1}{2}(-1-2\eta-\sqrt{4\eta+1}), \mu)$ and $\mathsf{e}_2^{\eta,\mu} = (\eta, \frac{1}{2}(-1-2\eta+\sqrt{4\eta+1}), \mu)$, where $\eta, \mu \in \mathbb{R}$, $\eta \geq -\frac{1}{4}$.

4.5.1 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *and* $\mathsf{e}_2^{\eta,\mu}$ *are unstable.*

Proof. Let $\mathsf{e}^{\eta,\mu} = (\eta, \epsilon, \mu)$, where $\epsilon = -\frac{1}{2}(1+2\eta+\sigma\sqrt{4\eta+1})$ and $\sigma \in \{-1, 1\}$. (We shall consider both $\mathsf{e}_1^{\eta,\mu}$ and $\mathsf{e}_2^{\eta,\mu}$ in the same argument.) Let U be a bounded open neighbourhood of $\mathsf{e}^{\eta,\mu}$. The curve $p(t) = (\eta, \epsilon - \sigma\delta, \mu - (\epsilon - \sigma\delta)t - (\eta + \epsilon - \sigma\delta)^2 t)$ is an integral curve of $\vec{H}_1^{(2)}$ for any $\delta > 0$, with $\|p(0) - \mathsf{e}^{\eta,\mu}\| = \delta$. Accordingly, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Furthermore,

$$\lim_{t\to\infty} \|p(t)\|^2 = (\epsilon - \sigma\delta)^2 + \eta^2 + \lim_{t\to\infty} \left[(\epsilon - \sigma\delta + (\eta + \epsilon - \sigma\delta)^2)t + \mu\right]^2.$$

Substituting for ϵ, we have $\epsilon - \sigma\delta + (\eta + \epsilon - \sigma\delta)^2 = \delta(\delta + \sqrt{1+4\eta}) > 0$. Consequently, $\lim_{t\to\infty} \|p(t)\|^2 = \infty$. Thus, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Hence the states $\mathsf{e}_1^{\eta,\mu}$ and $\mathsf{e}_2^{\eta,\mu}$ are unstable. ■

The equations of motion of the system $H_2^{(2)}(p) = p_1 + p_2 + \frac{1}{2}(p_1+p_2)^2$ are

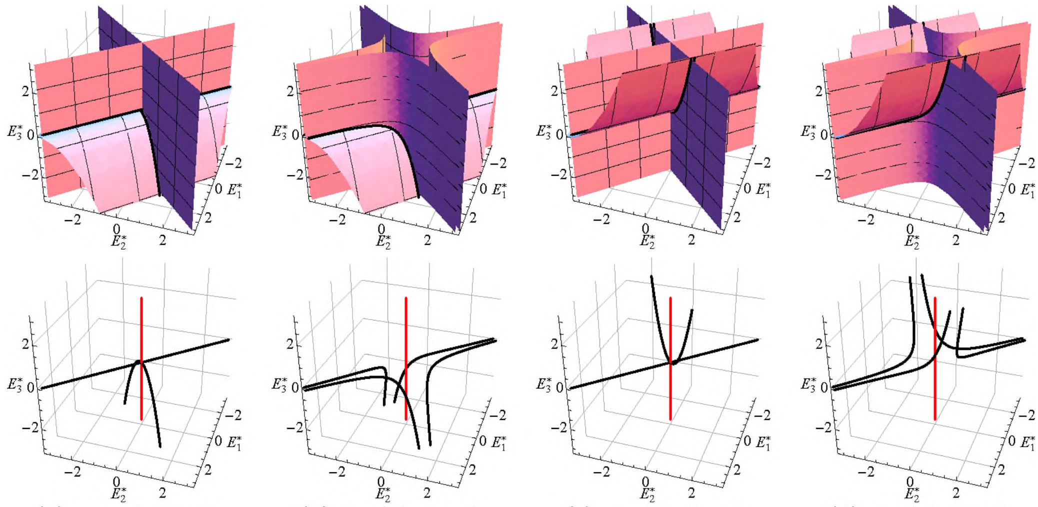
$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -(p_1+p_2)(1+p_1+p_2). \end{cases}$$

The integral curves are $p(t) = (c_1, c_2, c_3 - (c_1+c_2)(1+c_1+c_2)t)$, for $c_1, c_2, c_3 \in \mathbb{R}$. The equilibrium states of $\vec{H}_2^{(2)}$ are $\mathsf{e}_1^{\eta,\mu} = (\eta, -1-\eta, \mu)$ and $\mathsf{e}_2^{\eta,\mu} = (\eta, -\eta, \mu)$, where $\eta, \mu \in \mathbb{R}$.

4.5.2 Proposition. *The equilibrium states* $\mathsf{e}_1^{\eta,\mu}$ *and* $\mathsf{e}_2^{\eta,\mu}$ *are unstable.*

Proof. Consider the states $\mathsf{e}_1^{\eta,\mu}$. Fix a bounded open neighbourhood U of $\mathsf{e}_2^{\eta,\mu}$. We have that $p(t) = (\eta, -\delta-1-\eta, \mu - \delta(\delta+1)t)$ is an integral curve of $\vec{H}_2^{(2)}$ for any $\delta > 0$. Since $\|p(0) - \mathsf{e}_1^{\eta,\mu}\| = \delta$, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_1^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. Furthermore, as $\lim_{t\to\infty} \|p(t)\|^2 = \eta^2 + (\delta+\eta+1)^2 + \lim_{t\to\infty}(\mu - \delta(\delta+1)t)^2 = \infty$, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the states $\mathsf{e}_1^{\eta,\mu}$ are unstable.

Consider the states $\mathsf{e}_2^{\eta,\mu}$. Let U be a bounded open neighbourhood of $\mathsf{e}_2^{\eta,\mu}$. The integral curve $p(t) = (\eta, \delta-\eta, \mu - \delta(\delta+1)t)$, where $\delta > 0$, satisfies $\|p(0) - \mathsf{e}_2^{\eta,\mu}\| = \delta$. Thus, for any open neighbourhood $V \subseteq U$ of $\mathsf{e}_2^{\eta,\mu}$, there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \eta^2 + (\delta-\eta)^2 + \lim_{t\to\infty}(\mu - \delta(\delta+1)t)^2 = \infty$, and so there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the states $\mathsf{e}_2^{\eta,\mu}$ are unstable. ■



(a) $c_0 = 0$, $\delta > 0$ (b) $c_0 \neq 0$, $\delta > 0$ (c) $c_0 = 0$, $\delta < 0$ (d) $c_0 \neq 0$, $\delta < 0$

Figure 4.6: Typical configurations of $H^{(2)}_{3,\delta}$

4.5.1 The system $H^{(2)}_{3,\delta}$

The equations of motion of the system $H^{(2)}_{3,\delta}(p) = \delta p_3 + \frac{1}{2}(p_1 + p_2)^2$ ($\delta \neq 0$) are

$$\begin{cases} \dot{p}_1 = \delta p_2 \\ \dot{p}_2 = \delta p_1 \\ \dot{p}_3 = -(p_1 + p_2)^2. \end{cases}$$

The equilibrium states of $\vec{H}^{(2)}_{3,\delta}$ are $\mathsf{e}^{\mu}_1 = (0, 0, \mu)$, where $\mu \in \mathbb{R}$.

4.5.3 PROPOSITION. *The equilibrium states e^{μ}_1 are unstable.*

PROOF. The linearisation of the vector field $\vec{H}^{(2)}_{3,\delta}$ is

$$\mathbf{D}\vec{H}^{(2)}_{3,\delta}(p) = \begin{bmatrix} 0 & \delta & 0 \\ \delta & 0 & 0 \\ -2(p_1 + p_2) & -2(p_1 + p_2) & 0 \end{bmatrix}.$$

The linearisation at e^{μ}_1 has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\delta$. As $\delta \neq 0$, it follows that $\mathbf{D}\vec{H}^{(2)}_{3,\delta}(\mathsf{e}^{\mu}_1)$ has a positive real eigenvalue. Hence the states e^{μ}_1 are (spectrally) unstable. ■

The last step is to determine the integral curves of $\vec{H}^{(2)}_{3,\delta}$. There are several different configurations of the system (graphed in figure 4.6), depending on the initial conditions of the integral curve. (In figure 4.6, we have $c_0 = C(p(0))$, where $p(\cdot)$ is an integral curve of $\vec{H}^{(2)}_{3,\delta}$.)

4.5.4 PROPOSITION. *If $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ is an integral curve of $\vec{H}^{(2)}_{3,\delta}$, then*

$$\begin{cases} p_1(t) = p_1(0)\cosh(\delta t) + p_2(0)\sinh(\delta t) \\ p_2(t) = p_1(0)\sinh(\delta t) + p_2(0)\cosh(\delta t) \\ p_3(t) = \frac{1}{2\delta}\left[(p_1(0) + p_2(0))^2 - (p_1(t) + p_2(t))^2\right] + p_3(0). \end{cases}$$

Proof. Let $P(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$. The differential equation $\dot{P}(t) = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix} P(t)$ has the solution

$$P(t) = P(0) \exp\left(\begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix} t \right) = P(0) \begin{bmatrix} \cosh(\delta t) & \sinh(\delta t) \\ \sinh(\delta t) & \cosh(\delta t) \end{bmatrix}.$$

Therefore $p_1(t) = p_1(0)\cosh(\delta t) + p_2(0)\sinh(\delta t)$ and $p_2(t) = p_1(0)\sinh(\delta t) + p_2(0)\cosh(\delta t)$. Let $h_0 = H^{(2)}_{3,\delta}(p(t)) = \delta p_3(t) + \frac{1}{2}(p_1(t) + p_2(t))^2$. Then $p_3(t) = \frac{1}{\delta}\left[h_0 - \frac{1}{2}(p_1(t) + p_2(t))^2\right]$. Since $p_3(0) = \frac{1}{\delta}\left[h_0 - \frac{1}{2}(p_1(0) + p_2(0))^2\right]$, we have $h_0 = \frac{1}{2}\left[(p_1(0) + p_2(0))^2 + 2\delta p_3(0)\right]$. Substituting this into the expression for $p_3(\cdot)$ gives the result. ■

4.6 Inhomogeneous Systems Associated to H_3

The inhomogeneous systems associated to $H_3(p) = \frac{1}{2}p_3^2$ are $H^{(3)}_1(p) = p_1 + \frac{1}{2}p_3^2$, $H^{(3)}_2(p) = p_1 + p_2 + \frac{1}{2}p_3^2$ and $H^{(3)}_3(p) = \frac{1}{2}p_3^2$. The latter system is exactly the homogeneous system H_3, which we have dealt with in section 4.2.1; we shall not duplicate that effort here. The system $H^{(3)}_1$ (considered in the following section) will turn out to be the most involved of the systems we investigate, and requires the use of Jacobi elliptic functions for integration. (See section A.6 for more on the Jacobi elliptic functions, and, in particular, the reduction of integrals to the standard form.) Furthermore, it turns out that the vector field $\vec{H}^{(3)}_1$ is not complete (*i.e.*, the domain of every integral curve cannot be extended to $\mathbb{R}$). The integral curves of $H^{(3)}_2$ may be integrated in terms of elementary functions; this system is treated in section 4.6.2.

Supporting Mathematica code for the systems $H^{(3)}_1$ and $H^{(3)}_2$ may be found in section B.4.14 and section B.4.15, respectively. (This code verifies the stability calculations, including finding the equilibrium points of each vector field. Furthermore, code supporting the reduction to standard form for several cases of $H^{(3)}_1$ is also provided in section B.4.14).

4.6.1 The system $H^{(3)}_1$

The equations of motion of the system $H^{(3)}_1(p) = p_1 + \frac{1}{2}p_3^2$ are

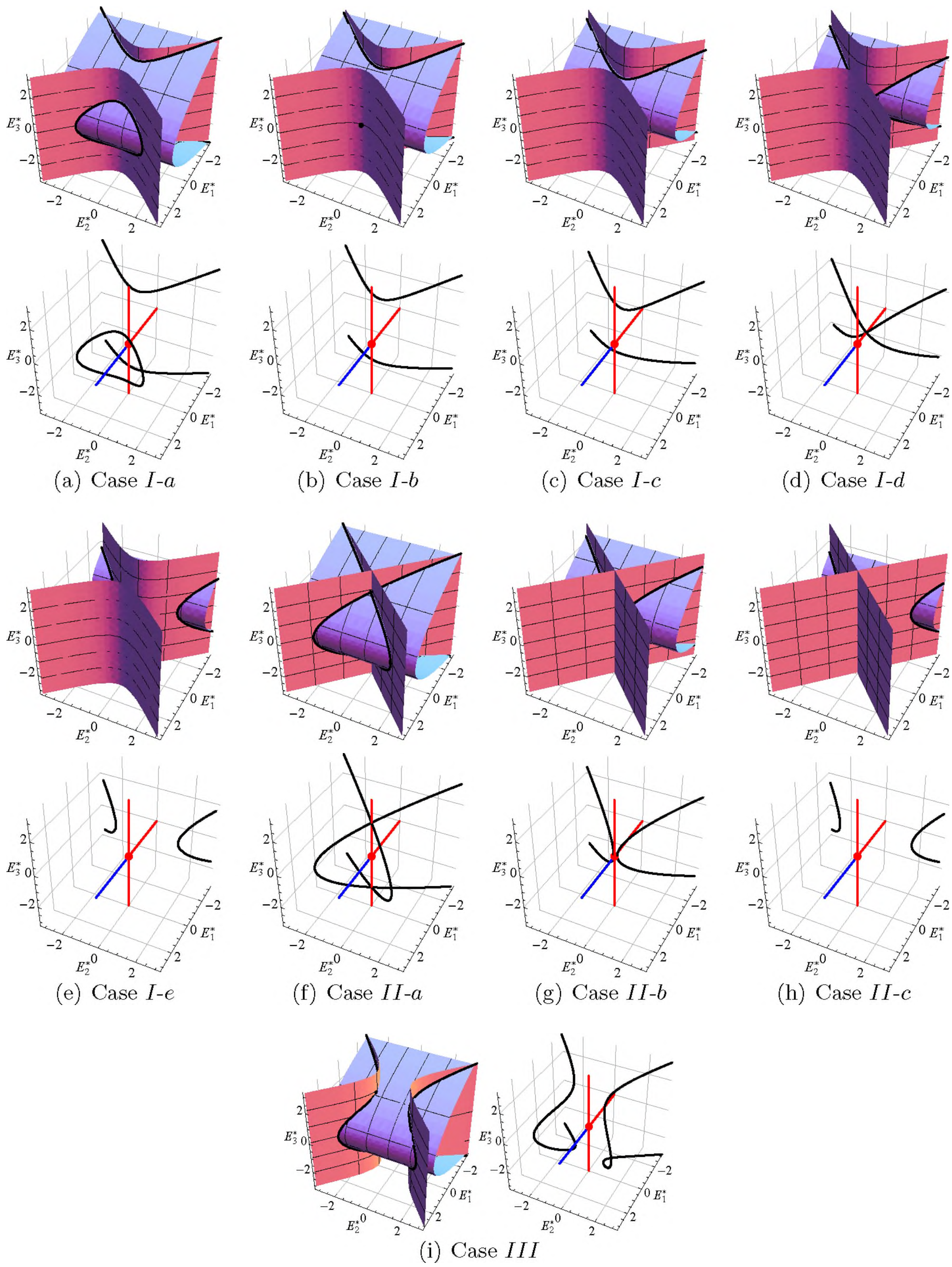
$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -p_2. \end{cases}$$

The equilibrium states of $\vec{H}^{(3)}_1$ are $\mathsf{e}^\mu_1 = (\mu, 0, 0)$ and $\mathsf{e}^\nu_2 = (0, 0, \nu)$, where $\nu, \mu \in \mathbb{R}$, $\nu \neq 0$.

4.6.1 Proposition. *The equilibrium states e^μ_1 are stable if $\mu > 0$, and unstable, otherwise. The equilibrium states e^ν_2 are unstable.*

Proof. Consider the states e^μ_1, $\mu > 0$. Let $H_\lambda = \lambda_0 H^{(3)}_1 + \lambda_1 C$, where $\lambda_0 = \mu$ and $\lambda_1 = -\frac{1}{2}$. We have

$$\mathbf{d}H_\lambda(p) = \begin{bmatrix} \mu - p_1 \\ p_2 \\ \mu p_3 \end{bmatrix}, \qquad \mathbf{d}^2 H_\lambda(p) = \operatorname{diag}(-1, 1, \mu).$$



(a) Case I-a (b) Case I-b (c) Case I-c (d) Case I-d

(e) Case I-e (f) Case II-a (g) Case II-b (h) Case II-c

(i) Case III

Figure 4.7: Typical configurations of $H_1^{(3)}$

Thus $\mathbf{d}H_\lambda(\mathsf{e}_1^\mu) = 0$. Moreover, $W = \ker \mathbf{d}H_1^{(3)}(\mathsf{e}_1^\mu) \cap \ker \mathbf{d}C(\mathsf{e}_1^\mu) = \mathsf{span}\{E_2^*, E_3^*\}$. Consequently, as $\mu > 0$, the restriction $\mathbf{d}^2 H_\lambda(\mathsf{e}_1^\mu)|_{W\times W} = \operatorname{diag}(1,\mu)$ is positive definite. Hence the states e_1^μ, $\mu > 0$ are stable.

Consider the state e_1^0. Let U be an open bounded neighbourhood of e_1^0. We have that $p(\cdot) : (-\infty, 0) \to \mathfrak{se}(1,1)^*$, $t \mapsto (-\frac{2}{t^2}, \frac{2}{t^2}, \frac{2}{t})$ is an integral curve of $\vec{H}_1^{(3)}$. Indeed, $\dot p_1 = \frac{4}{t^3} = p_2 p_3$, $\dot p_2 = -\frac{4}{t^3} = p_2 p_3$ and $\dot p_3 = -\frac{2}{t^2} = -p_2$. We have $\lim_{t\to-\infty} \|p(t) - \mathsf{e}_1^0\|^2 = \lim_{t\to-\infty} \frac{4(t^2+2)}{t^4} = 0$. Accordingly, for every neighbourhood $V \subseteq U$ of e_1^0, there exists $t_1 < 0$ such that $p(t_1) \in V$. Furthermore, $\lim_{t\to 0} \|p(t)\| = \infty$, and so there exists $t_2 < 0$ such that $p(t_2) \notin U$. Therefore the state e_1^0 is unstable.

Consider the states e_1^μ, $\mu < 0$. Fix a bounded open neighbourhood U of e_1^μ. The curve $p(\cdot)$ defined componentwise as

$$\begin{cases} p_1(t) = \mu\left[1 + 2\operatorname{csch}^2(\sqrt{-\mu}\, t)\right] \\ p_2(t) = -2\mu \coth(\sqrt{-\mu}\, t)\operatorname{csch}(\sqrt{-\mu}\, t) \\ p_3(t) = 2\sqrt{-\mu}\, \operatorname{csch}(\sqrt{-\mu}\, t). \end{cases}$$

is an integral curve of $\vec{H}_1^{(3)}$. Indeed,

$$\begin{aligned} \dot p_1 &= -4\mu\sqrt{-\mu}\, \coth(\sqrt{-\mu}\, t)\operatorname{csch}^2(\sqrt{-\mu}\, t) = p_2 p_3 \\ \dot p_2 &= 2\mu\sqrt{-\mu}\, \operatorname{csch}(\sqrt{-\mu}\, t)\left[\coth^2(\sqrt{-\mu}\, t) + \operatorname{csch}^2(\sqrt{-\mu}\, t)\right] = p_1 p_3 \\ \dot p_3 &= 2\mu \coth(\sqrt{-\mu}\, t)\operatorname{csch}(\sqrt{-\mu}\, t) = -p_2. \end{aligned}$$

Furthermore,

$$\lim_{t\to-\infty} \|p(t) - \mathsf{e}_1^\mu\|^2 = \lim_{t\to-\infty} 4\mu \operatorname{csch}^2(\sqrt{-\mu}\, t)\left[\mu \operatorname{csch}^2(\sqrt{-\mu}\, t) + \mu\coth^2(\sqrt{-\mu}\, t) - 1\right] = 0.$$

Thus, for every neighbourhood $V \subseteq U$ of e_1^μ, there exists $t_1 < 0$ such that $p(t_1) \in V$. However, since

$$\lim_{t\to 0} \|p(t) - \mathsf{e}_1^\mu\|^2 = \lim_{t\to 0} 4\mu \operatorname{csch}^2(\sqrt{-\mu}\, t)\left[\mu \operatorname{csch}^2(\sqrt{-\mu}\, t) + \mu\coth^2(\sqrt{-\mu}\, t) - 1\right] = \infty$$

there exists $t_2 < 0$ such that $p(t_2) \notin U$. Hence the states e_1^μ, $\mu < 0$ are unstable.

Consider the states e_2^ν. The linearisation of the vector field $\vec{H}_1^{(3)}$ is

$$\mathbf{D}\vec{H}_1^{(3)}(p) = \begin{bmatrix} 0 & p_3 & p_2 \\ p_3 & 0 & p_1 \\ -1 & -1 & 0 \end{bmatrix}.$$

The linearisation at e_2^ν has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\nu$. Since $\nu \neq 0$, $\mathbf{D}\vec{H}_1^{(3)}(\mathsf{e}_2^\nu)$ has a positive real eigenvalue. Hence the states e_2^ν are (spectrally) unstable. ∎

We now proceed to find the integral curves of $\vec{H}_1^{(3)}$. Let $\bar p(\cdot)$ be an integral curve of $\vec{H}_1^{(3)}$ and let $c_0 = C(\bar p(0))$ and $h_0 = H_1^{(3)}(\bar p(0))$. We consider three main cases depending on the sign of c_0. For the cases $c_0 > 0$ and $c_0 = 0$ we have several further subcases. (Table 4.1 lists the qualitative breakdown of cases; figure 4.7 plots typical configurations corresponding to these cases.) Consider the case $c_0 > 0$. The level sets $\{p : H_1^{(3)}(p) = h_0\}$ and $\{p : C(p) = c_0\}$

are tangent when the gradients of $H_1^{(3)}$ and C are parallel, *i.e.*,

$$\nabla H_1^{(3)}(p) = \lambda \nabla C(p) \qquad \iff \qquad \begin{bmatrix} 1 \\ 0 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2\lambda p_1 \\ -2\lambda p_2 \\ 0 \end{bmatrix}.$$

(Here $\lambda \in \mathbb{R}$.) If $\lambda = 0$, then comparing the first components we have $1 = 0$, a contradiction. Thus $\lambda \neq 0$ and so $p_2 = p_3 = 0$. Therefore $h_0^2 = p_1^2 = c_0$, *i.e.*, $h_0 = \sqrt{c_0}$ or $h_0 = -\sqrt{c_0}$. This motivates the (sub)cases $h_0 > \sqrt{c_0}$, $h_0 = \sqrt{c_0}$, $-\sqrt{c_0} < h_0 < \sqrt{c_0}$, $h_0 < -\sqrt{c_0}$ and $h_0 = -\sqrt{c_0}$. Setting $c_0 = 0$ yields the (sub)cases $h_0 < 0$, $h_0 = 0$ and $h_0 > 0$.

Conditions		Designation
$c_0 > 0$	$h_0 > \sqrt{c_0}$	Case *I-a*
	$h_0 = \sqrt{c_0}$	Case *I-b*
	$-\sqrt{c_0} < h_0 < \sqrt{c_0}$	Case *I-c*
	$h_0 = -\sqrt{c_0}$	Case *I-d*
	$h_0 < -\sqrt{c_0}$	Case *I-e*
$c_0 = 0$	$h_0 > 0$	Case *II-a*
	$h_0 = 0$	Case *II-b*
	$h_0 < 0$	Case *II-c*
$c_0 < 0$		Case *III*

Table 4.1: Qualitative breakdown of cases for $H_1^{(3)}$

4.6.1.1 Case *I*: $c_0 > 0$

4.6.1.1.1 Case *I-a*: $h_0 > \sqrt{c_0}$. Using the equation of motion $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ and the constants of motion $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)$ and $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, we get

$$\dot{\bar{p}}_1^2 = (\bar{p}_2\bar{p}_3)^2 = (\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1). \tag{4.6.1}$$

We shall reduce this equation to standard form before integrating. (Section A.6.2 discusses the reduction to standard form.) Let $X_1 = \bar{p}_1^2 - c_0$ and $X_2 = 2h_0 - 2\bar{p}_1$. Then $X_1 - \lambda X_2$ is a perfect square for $\lambda_1 = -(\delta + h_0)$ and $\lambda_2 = \delta - h_0$, where $\delta = \sqrt{h_0^2 - c_0}$. (As $h_0 > \sqrt{c_0}$, we have $\delta, \lambda_1, \lambda_2 \in \mathbb{R}$.) Accordingly, $X_1 - \lambda_1 X_2 = (\bar{p}_1 + \lambda_1)^2$ and $X_1 - \lambda_2 X_2 = (\bar{p}_1 + \lambda_2)^2$. Thus we have

$$X_1 X_2 = \left[A_1(\bar{p}_1 + \lambda_1)^2 + B_1(\bar{p}_1 + \lambda_2)^2\right]\left[A_2(\bar{p}_1 + \lambda_1)^2 + B_2(\bar{p}_1 + \lambda_2)^2\right]$$

where A_1, A_2, B_1, B_2 are given by

$$A_1 = \frac{1}{2}\left(1 - \frac{h_0}{\delta}\right), \qquad B_1 = \frac{1}{2}\left(1 + \frac{h_0}{\delta}\right), \qquad A_2 = \frac{1}{2\delta}, \qquad B_2 = -\frac{1}{2\delta}.$$

Take the square root of both sides in (4.6.1). After separating variables, we get the following equation:

$$\frac{d\bar{p}_1}{\sqrt{(\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1)}} = \frac{d\bar{p}_1}{\sqrt{X_1 X_2}} = \sigma_1 dt$$

for some $\sigma_1 \in \{-1, 1\}$. By the preceding calculations, this is

$$\frac{d\bar{p}_1}{\sqrt{[A_1(\bar{p}_1 + \lambda_1)^2 + B_1(\bar{p}_1 + \lambda_2)^2]\,[A_2(\bar{p}_1 + \lambda_1)^2 + B_2(\bar{p}_1 + \lambda_2)^2]}} = \sigma_1 dt. \tag{4.6.2}$$

We have $(\delta + h_0)(\delta - h_0) = \delta^2 - h_0^2 = -c_0 < 0$. As $\delta, h_0 > 0$, it follows that $\delta + h_0 > 0$, whence $\delta - h_0 < 0$. Thus $A_1 A_2 = \frac{\delta - h_0}{4\delta^2} < 0$. Make the change of variables $u = \frac{\bar{p}_1 + \lambda_1}{\bar{p}_1 + \lambda_2}$. Equation (4.6.2) then becomes

$$\frac{du}{\sqrt{-\left(u^2 + \frac{B_1}{A_1}\right)\left(u^2 - 1\right)}} = \sigma_1(\lambda_2 - \lambda_1)\sqrt{-A_1 A_2}\, t = \sigma_1 \sqrt{h_0 - \delta}\, dt. \tag{4.6.3}$$

Here $\frac{B_1}{A_1} = \frac{\delta + h_0}{\delta - h_0} < 0$. Since $\bar{p}_1(t)^2 \geq \bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$, we have $\bar{p}_1(t) \leq -\sqrt{c_0}$ or $\bar{p}_1(t) \geq \sqrt{c_0}$. (Each situation will give rise to different integral curves.)

Consider the case $\boldsymbol{\bar{p}_1(t) \leq -\sqrt{c_0}}$. Let $a = \sqrt{-\frac{B_1}{A_1}}$, $b = 1$ and $x = \frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2}$. Integrating both sides of (4.6.3), we have

$$\int_b^x \frac{du}{\sqrt{(a^2 - u^2)(u^2 - b^2)}} = \sigma_1 \sqrt{h_0 - \delta}\, dt. \tag{4.6.4}$$

As $h_0 + \delta > h_0 - \delta > 0$, we have $a = \sqrt{\frac{h_0 + \delta}{h_0 - \delta}} > 1 = b$. Apply the integral formula (A.6.7) to the left-hand side of (4.6.4). We have, for $b \leq x \leq a$,

$$\begin{aligned} &\frac{1}{a}\,\mathrm{nd}^{-1}\left(\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2}, \frac{\sqrt{a^2 - 1}}{a}\right) = \sigma_1 \sqrt{h_0 - \delta}\, t \\ \Rightarrow \quad &\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2} = \mathrm{nd}\left(a\sqrt{h_0 - \delta}\, t, \frac{\sqrt{a^2 - 1}}{a}\right). \end{aligned}$$

(As $\mathrm{nd}(\cdot, k)$ is even, we can eliminate the σ_1.) Let $\Omega = a\sqrt{h_0 - \delta} = \sqrt{\delta + h_0}$ and $k = \frac{\sqrt{a^2-1}}{a} = \sqrt{\frac{2\delta}{\delta + h_0}}$, $k' = \sqrt{\frac{h_0 - \delta}{h_0 + \delta}}$. (Since $\delta < h_0$ we have $2\delta = \delta + \delta < \delta + h_0$, whence $k < 1$; thus $0 < k < 1$.) Substituting for λ_1, λ_2 and solving for $\bar{p}_1(t)$, we get

$$\bar{p}_1(t) = \frac{(\delta - h_0)\,\mathrm{nd}(\Omega t, k) + (\delta + h_0)}{1 - \mathrm{nd}(\Omega t, k)} = \frac{(\delta + h_0)\,\mathrm{dn}(\Omega t, k) + (\delta - h_0)}{\mathrm{dn}(\Omega t, k) - 1}.$$

(We have rewritten the expression in terms of the basic Jacobi elliptic function $\mathrm{dn}(\cdot, k)$.)

Using the Casimir equation $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, we find

$$\begin{aligned}
\bar{p}_2(t)^2 &= \left(\frac{(\delta + h_0)\,\mathrm{dn}(\Omega t, k) + (\delta - h_0)}{\mathrm{dn}(\Omega t, k) - 1}\right)^2 - c_0 \\
&= \frac{(\delta + h_0)^2\,\mathrm{dn}^2(\Omega t, k) - 2c_0\,\mathrm{dn}(\Omega t, k) + (\delta - h_0)^2}{(\mathrm{dn}(\Omega t, k) - 1)^2} - c_0 \\
&= 2\delta\frac{(\delta + h_0)\,\mathrm{dn}^2(\Omega t, k) + (\delta - h_0)}{(\mathrm{dn}(\Omega t, k) - 1)^2} \\
&= 2\delta(\delta + h_0)\frac{\mathrm{dn}^2(\Omega t, k) - (k')^2}{(\mathrm{dn}(\Omega t, k) - 1)} \\
&= 2\delta(\delta + h_0)k^2\frac{\mathrm{cn}^2(\Omega t, k)}{(\mathrm{dn}(\Omega t, k) - 1)^2} \\
&= 4\delta^2\frac{\mathrm{cn}^2(\Omega t, k)}{(\mathrm{dn}(\Omega t, k) - 1)^2}.
\end{aligned}$$

(We have used the square relation (A.6.6) in the penultimate step.) Taking the square root of both sides yields

$$\bar{p}_2(t) = \sigma_2 2\delta\,\frac{\mathrm{cn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) - 1}$$

for some $\sigma_2 \in \{-1, 1\}$. Lastly, we use the equation $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$ to find an expression for $\bar{p}_3(\cdot)$:

$$\begin{aligned}
\bar{p}_3(t)^2 &= 2h_0 - 2\bar{p}_1(t) \\
&= -2\delta\frac{\mathrm{dn}(\Omega t, k) + 1}{\mathrm{dn}(\Omega t, k) - 1} \cdot \frac{\mathrm{dn}(\Omega t, k) - 1}{\mathrm{dn}(\Omega t, k) - 1} \\
&= -2\delta\frac{\mathrm{dn}^2(\Omega t, k) - 1}{(\mathrm{dn}(\Omega t, k) - 1)^2} \\
&= 2\delta k^2\frac{\mathrm{sn}^2(\Omega t, k)}{(\mathrm{dn}(\Omega t, k) - 1)^2}.
\end{aligned}$$

(We have used the square relation (A.6.4) in the last step.) Take the square root of both sides. The result is

$$\bar{p}_3(t) = \sigma_3\sqrt{2\delta}\,k\frac{\mathrm{sn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) - 1}$$

for some $\sigma_3 \in \{-1, 1\}$. Therefore we have the following (prospective) integral curve:

$$\begin{cases}
\bar{p}_1(t) = \dfrac{(\delta + h_0)\,\mathrm{dn}(\Omega t, k) + (\delta - h_0)}{\mathrm{dn}(\Omega t, k) - 1} \\
\bar{p}_2(t) = \sigma_2 2\delta\dfrac{\mathrm{cn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) - 1} \\
\bar{p}_3(t) = \sigma_3\sqrt{2\delta}\,k\dfrac{\mathrm{sn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) - 1}.
\end{cases}$$

We show that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$ if and only if $\sigma_2 = \sigma_3$. By proposition 4.1.2, it suffices to show that $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2 = \sigma_3$ (since we know $\bar{p}(\cdot)$ satisfies the equations for the constants of motion). Indeed,

$$\begin{aligned}\dot{\bar{p}}_1(t) - \bar{p}_2(t)\bar{p}_3(t) &= \frac{2\delta k^2\Omega^2 \operatorname{cn}(\Omega t, k)\operatorname{sn}(\Omega t, k)}{(\operatorname{dn}(\Omega t, k) - 1)^2} - \frac{\sigma_2\sigma_3 2\sqrt{2}\,\delta\sqrt{\delta}\,k \operatorname{cn}(\Omega t, k)\operatorname{sn}(\Omega t, k)}{(\operatorname{dn}(\Omega t, k) - 1)^2}\\ &= \frac{2\delta k(k\Omega - \sigma_2\sigma_3\sqrt{2\delta}\,)\operatorname{cn}(\Omega t, k)\operatorname{sn}(\Omega t, k)}{(\operatorname{dn}(\Omega t, k) - 1)^2}.\end{aligned}$$

Now $k\Omega - \sigma_2\sigma_3\sqrt{2\delta} = \sqrt{2\delta}\,(1 - \sigma_2\sigma_3)$, and so $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2\sigma_3 = 1$, *i.e.*, $\sigma_2 = \sigma_3$. Thus $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$.

Finally, since $\operatorname{dn}(\Omega t, k) = 1$ at the points $\frac{2nK}{\Omega}$, $n \in \mathbb{Z}$ (where $4K$ denotes the period of $\operatorname{sn}(\cdot, k)$) it follows that $\bar{p}(\cdot)$ is only defined on the open intervals $\left(\frac{2nK}{\Omega}, \frac{2(n+1)K}{\Omega}\right)$, $n \in \mathbb{Z}$. (This implies that the vector field $\vec{H}_1^{(3)}$ is not complete.)

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.2 Proposition. *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_1^{(3)}$ *such that* $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$, $h_0 > \sqrt{c_0}$ *and* $p_1(0) \leq -\sqrt{c_0}$.

(*i*) *There exist* $t_0 \in \left(0, \frac{2K}{\Omega}\right)$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : \left(0, \frac{2K}{\Omega}\right) \to \mathfrak{se}(1,1)^*$ *is defined by*

$$\begin{cases}\bar{p}_1(t) = \dfrac{(\delta + h_0)\operatorname{dn}(\Omega t, k) + (\delta - h_0)}{\operatorname{dn}(\Omega t, k) - 1}\\ \bar{p}_2(t) = \sigma 2\delta\,\dfrac{\operatorname{cn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) - 1}\\ \bar{p}_3(t) = \sigma k^2\Omega\,\dfrac{\operatorname{sn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) - 1}.\end{cases}$$

Here $4K$ *is the period of* $\operatorname{sn}(\cdot, k)$, $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{h_0 + \delta}$ *and* $k = \sqrt{\frac{2\delta}{h_0 + \delta}}$.

(*ii*) $t \mapsto \bar{p}(t + t_0)$ *is the unique maximal integral curve starting at* $\bar{p}(t_0)$.

Proof. (*i*) Let $\sigma = -\operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_3(0) = 0$, then $h_0 = p_1(0) \leq -\sqrt{c_0} < 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_2|_{(0,K/\Omega)}(t)) = \sigma$ and $\operatorname{sgn}(\bar{p}_2|_{(K/\Omega, 2K/\Omega)}(t)) = -\sigma$. Moreover, $\lim_{t\to 0}\bar{p}_2(t) = -\sigma\infty$ and $\lim_{t\to 2K/\Omega}\bar{p}_2(t) = \sigma\infty$. Therefore, since $\bar{p}_2(\cdot)$ is continuous, there exists $t_0 \in \left(0, \frac{2K}{\Omega}\right)$ such that $\bar{p}_2(t_0) = p_2(0)$. Then

$$\bar{p}_1(t_0)^2 = c_0 + \bar{p}_2(t_0)^2 = c_0 + p_2(0)^2 = p_1(0)^2.$$

But $\bar{p}_1(t_0) \leq -\sqrt{c_0}$ and $p_1(0) \leq -\sqrt{c_0}$, so $\operatorname{sgn}(\bar{p}_1(t_0)) = \operatorname{sgn}(p_1(0))$; it follows that $\bar{p}_1(t_0) = p_1(0)$. Finally, we have

$$\bar{p}_3(t_0)^2 = 2(h_0 - \bar{p}_1(t_0)) = 2(h_0 - p_1(0)) = p_3(0)^2.$$

Since $\operatorname{sgn}(\bar{p}_3(t_0)) = -\sigma = \operatorname{sgn}(p_3(0))$, we then have $\bar{p}_3(t_0) = p_3(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t + t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}(s + t_0)$, and so $s + t_0 \in \left(0, \frac{2K}{\Omega}\right)$. Consequently, $s \in \left(-t_0, \frac{2K}{\Omega} - t_0\right)$, *i.e.*,

$(-\varepsilon, \varepsilon) \subseteq \left(-t_0, \frac{2K}{\Omega} - t_0\right)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t + t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t + t_0)$. ■

Consider the case $\boldsymbol{\bar{p}_1(t) \geq \sqrt{c_0}}$. Let $a = -\sqrt{-\frac{B_1}{A_1}}$, $b = -1$ and $x = \frac{\bar{p}_1(t)+\lambda_1}{\bar{p}_1(t)+\lambda_2}$. Integrating both sides of (4.6.3), we get

$$\int_x^a \frac{du}{\sqrt{(a^2 - u^2)(u^2 - b^2)}} = \sigma_1 \sqrt{h_0 - \delta}\, dt.$$

From the case $\bar{p}_1(t) \leq -\sqrt{c_0}$ we have that $-b \leq -a$. Furthermore,

$$\begin{aligned}\int_{-x}^{-a} \frac{du}{\sqrt{(a^2 - u^2)(u^2 - b^2)}} &= -\int_x^a \frac{du}{\sqrt{(a^2 - u^2)(u^2 - b^2)}} \\ &= -\sigma_1 \sqrt{h_0 - \delta}\, t.\end{aligned}$$

Hence, for $-b \leq -x \leq -a$, we can apply the formula (A.6.8) for the integral on the left-hand side. This yields

$$\begin{aligned}&\frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{1}{a}\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2}, -\frac{\sqrt{a^2-1}}{a}\right) = -\sigma_1 \sqrt{h_0 - \delta}\, t \\ \Rightarrow \quad &\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2} = a\,\mathrm{dn}\left(-a\sqrt{h_0 - \delta}\, t, -\frac{\sqrt{a^2-1}}{a}\right).\end{aligned}$$

(Using the evenness of $\mathrm{dn}(\cdot, k)$, we can eliminate the σ_1.) Let $\Omega = -a\sqrt{h_0 - \delta} = \sqrt{h_0 + \delta}$ and $k = -\frac{\sqrt{a^2-1}}{a} = \sqrt{\frac{2\delta}{h_0+\delta}}$, $k' = \sqrt{\frac{h_0-\delta}{h_0+\delta}}$. Solving for $\bar{p}_1(t)$, we get

$$\bar{p}_1(t) = \frac{k'\sqrt{c_0}\,\mathrm{dn}(\Omega t, k) + k'(h_0 + \delta)}{\mathrm{dn}(\Omega t, k) + k'} = \sqrt{c_0}\,\frac{k'\,\mathrm{dn}(\Omega t, k) + 1}{\mathrm{dn}(\Omega t, k) + k'}.$$

Using the Casimir equation $\bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$, we have

$$\begin{aligned}\bar{p}_2(t)^2 &= \bar{p}_1(t)^2 - c_0 \\ &= \left[\sqrt{c_0}\,\frac{k'\,\mathrm{dn}(\Omega t, k) + 1}{\mathrm{dn}(\Omega t, k) + k'}\right]^2 - c_0 \\ &= \frac{c_0 k^2 (1 - \mathrm{dn}^2(\Omega t, k))}{(\mathrm{dn}(\Omega t, k) + k')^2} \\ &= \frac{c_0 k^4\,\mathrm{sn}^2(\Omega t, k)}{(\mathrm{dn}(\Omega t, k) + k')^2}.\end{aligned}$$

(We have used the square relation (A.6.4) in the last step.) Therefore, taking the square root of both sides, we get

$$\bar{p}_2(t) = \sigma_2 k^2 \sqrt{c_0}\,\frac{\mathrm{sn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) + k'}$$

for some $\sigma_2 \in \{-1, 1\}$. In fact, since $\mathrm{sn}\left(\Omega t + \frac{2K}{\Omega}, k\right) = -\,\mathrm{sn}(\Omega t, k)$, we assume $\sigma_2 = 1$. Lastly, differentiate the equation $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$ on both sides, to get $0 = \dot{\bar{p}}_1(t) + \bar{p}_3(t)\dot{\bar{p}}_3(t)$.

Hence,

$$\begin{aligned}\bar{p}_3(t) &= -\frac{\dot{\bar{p}}_1(t)}{\dot{\bar{p}}_3(t)} = \frac{\dot{\bar{p}}_1(t)}{\bar{p}_2(t)}\\ &= \left(\frac{\sqrt{c_0}\,k^4\Omega\,\mathrm{cn}(\Omega t,k)\,\mathrm{sn}(\Omega t,k)}{(\mathrm{dn}(\Omega t,k)+k')^2}\right)\left(\frac{k'+\mathrm{dn}(\Omega t,k)}{k^2\sqrt{c_0}\,\mathrm{sn}(\Omega t,k)}\right)\\ &= k\sqrt{2\delta}\,\frac{\mathrm{cn}(\Omega t,k)}{k'+\mathrm{dn}(\Omega t,k)}.\end{aligned}$$

Therefore, we have the following (prospective) integral curve:

$$\begin{cases}\bar{p}_1(t) = \sqrt{c_0}\,\dfrac{k'\,\mathrm{dn}(\Omega t,k)+1}{\mathrm{dn}(\Omega t,k)+k'}\\ \bar{p}_2(t) = k^2\sqrt{c_0}\,\dfrac{\mathrm{sn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k)+k'}\\ \bar{p}_3(t) = k\sqrt{2\delta}\,\dfrac{\mathrm{cn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k)+k'}.\end{cases}$$

We verify that $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$. By proposition 4.1.2, it suffices to show that $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$. (This is because we know $C(\bar{p}(t)) =$ constant, and $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ follows from the calculations we have done to get an expression for $\bar{p}_3(\cdot)$.) Indeed,

$$\begin{aligned}&h_0 - \bar{p}_1(t) - \tfrac{1}{2}\bar{p}_3(t)^2\\ &\quad= -\frac{\delta k^2\,\mathrm{cn}^2(\Omega t,k) + (\mathrm{dn}(\Omega t,k)+k')(\sqrt{c_0} - h_0k' + (k'\sqrt{c_0} - h_0)\,\mathrm{dn}(\Omega t,k))}{(\mathrm{dn}(\Omega t,k)+k')^2}\\ &\quad= -\frac{\delta(\mathrm{dn}^2(\Omega t,k) - (k')^2) + (\mathrm{dn}(\Omega t,k)+k')(\sqrt{c_0} - h_0k' + (k'\sqrt{c_0} - h_0)\,\mathrm{dn}(\Omega t,k))}{(\mathrm{dn}(\Omega t,k)+k')^2}\\ &\quad= \frac{(h_0 - k'\sqrt{c_0} - \delta)\,\mathrm{dn}(\Omega t,k) - \sqrt{c_0} + k'(h_0+\delta)}{\mathrm{dn}(\Omega t,k)+k'}.\end{aligned}$$

We have $-\sqrt{c_0} + k'(h_0+\delta) = h_0 - k'\sqrt{c_0} - \delta = 0$, and so $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$. Therefore $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$. Moreover, since $\mathrm{dn}(\Omega t,k) + k' > 0$ for every $t \in \mathbb{R}$ we have that $\bar{p}(\cdot)$ is defined over $\mathbb{R}$.

Lastly, since $\mathrm{dn}(\Omega t,k)$ and $\mathrm{cn}(\Omega t,k)$ are even and $\mathrm{sn}(\Omega t,k)$ is odd, it follows that $\bar{p}_1(\cdot)$, $\bar{p}_3(\cdot)$ are even and $\bar{p}_2(\cdot)$ is odd.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.3 Proposition. *Let $p(\cdot) : (-\varepsilon,\varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$, $h_0 > \sqrt{c_0}$ and $p_1(0) \geq \sqrt{c_0}$. There exists $t_0 \in \left[-\frac{2K}{\Omega}, \frac{2K}{\Omega}\right]$ such that $p(t) = \bar{p}(t+t_0)$ for every $t \in (-\varepsilon,\varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases}\bar{p}_1(t) = \sqrt{c_0}\,\dfrac{k'\,\mathrm{dn}(\Omega t,k)+1}{\mathrm{dn}(\Omega t,k)+k'}\\ \bar{p}_2(t) = k^2\sqrt{c_0}\,\dfrac{\mathrm{sn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k)+k'}\\ \bar{p}_3(t) = k\sqrt{2\delta}\,\dfrac{\mathrm{cn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k)+k'}.\end{cases}$$

Here $4K$ is the period of $\mathrm{sn}(\cdot,k)$, $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{h_0+\delta}$, $k = \sqrt{\frac{2\delta}{h_0+\delta}}$ and $k' = \sqrt{\frac{h_0-\delta}{h_0+\delta}}$.

PROOF. Let $\omega = \sqrt{2h_0 - 2\sqrt{c_0}}$. From the constant of motion $h_0 = p_1(t) + \frac{1}{2}p_3(t)^2$ we have $p_3(t)^2 = 2h_0 - 2p_1(t) \leq 2h_0 - 2\sqrt{c_0} = \omega^2$, *i.e.*, $-\omega \leq p_3(t) \leq \omega$. Similarly, $-\omega \leq \bar{p}_3(t) \leq \omega$. Moreover,

$$\bar{p}_3(0) = \frac{k\sqrt{2\delta}}{1+k'} = \frac{2\delta}{\sqrt{h_0-\delta} + \sqrt{h_0+\delta}} = \sqrt{2h_0 - 2\sqrt{c_0}} = \omega$$

and $\bar{p}_3\big(\frac{2K}{\Omega}\big) = -\frac{k\sqrt{2\delta}}{1+k'} = -\omega$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_1 \in \big[0, \frac{2K}{\Omega}\big]$ such that $\bar{p}_3(t_1) = p_3(0)$. Then $\bar{p}_1(t_1) = h_0 - \frac{1}{2}\bar{p}_3(t_1)^2 = h_0 - \frac{1}{2}p_3(0)^2 = p_1(0)$. Lastly,

$$\bar{p}_2^2(t_1)^2 = \bar{p}_1(t_1)^2 - c_0 = p_1(0)^2 - c_0 = p_2(0)^2,$$

and so $\bar{p}_2(t_1) = \pm p_2(0)$. Since $\bar{p}_1(\cdot)$ and $\bar{p}_3(\cdot)$ are even and $\bar{p}_2(\cdot)$ is odd, we have $\bar{p}_1(-t_1) = \bar{p}_1(t_1)$, $\bar{p}_2(-t_1) = -\bar{p}_2(t_1)$ and $\bar{p}_3(t_1) = \bar{p}_3(t_1)$. Hence, there exists $t_0 \in \big[-\frac{2K}{\Omega}, \frac{2K}{\Omega}\big]$ (*i.e.*, $t_0 = t_1$ or $t_0 = -t_1$) such that $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ■

4.6.1.1.2 Case ***I-b*****:** $h_0 = \sqrt{c_0}$. We have $\bar{p}_1(t)^2 \geq \bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$, and so either $\bar{p}_1(t) \leq -\sqrt{c_0}$ or $\bar{p}_1(t) \geq \sqrt{c_0}$. We claim that the latter situation leads to a constant integral curve. Indeed, suppose $\bar{p}_1(t) \geq \sqrt{c_0} = h_0$. Then $\bar{p}_3(t)^2 = 2h_0 - 2\bar{p}_1(t) \leq 0$, *i.e.*, $\bar{p}_3(t) = 0$. Then $\bar{p}_1(t) = h_0 - \frac{1}{2}\bar{p}_3(t)^2 = h_0$ and $\bar{p}_2(t) = \pm\sqrt{\bar{p}_1(t)^2 - c_0} = \pm\sqrt{h_0^2 - c_0} = 0$. Hence $\bar{p}(t) = (h_0, 0, 0)$, which is an equilibrium point of $\vec{H}_1^{(3)}$. Thus we suppose $\bar{p}_1(t) \leq -\sqrt{c_0}$.

Consider the integral curve of proposition 4.6.2. Denote by $q(\cdot)$ the curve obtained by limiting (from above) $h_0 \to \sqrt{c_0}$. Then

$$\begin{aligned}
q_1(t) &= \lim_{h_0 \to \sqrt{c_0}} \frac{(\delta + h_0)\,\mathrm{dn}(\Omega t, k) + (\delta - h_0)}{\mathrm{dn}(\Omega t, k) - 1} = h_0\left[1 - \csc^2\left(\sqrt{h_0}\,t\right)\right] \\
q_2(t) &= \lim_{h_0 \to \sqrt{c_0}} \sigma 2\delta\,\frac{\mathrm{cn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) - 1} = -\sigma 2h_0 \cot\left(\sqrt{h_0}\,t\right)\csc\left(\sqrt{h_0}\,t\right) \\
q_3(t) &= \lim_{h_0 \to \sqrt{c_0}} \sigma k^2\Omega\,\frac{\mathrm{sn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) - 1} = -\sigma 2\sqrt{h_0}\,\csc\left(\sqrt{h_0}\,t\right).
\end{aligned}$$

(If one takes the limit $h_0 \to \sqrt{c_0}$ of the integral curve in proposition 4.6.3, the result is the constant integral curve $\bar{p}(t) = (h_0, 0, 0)$.) Let $\bar{p}(t) = q\big(t - \frac{\pi}{2\sqrt{h_0}}\big)$. Thus we have the following (prospective) integral curve:

$$\begin{cases}
\bar{p}_1(t) = -h_0\left[1 + 2\tan^2\left(\sqrt{h_0}\,t\right)\right] \\
\bar{p}_2(t) = -\sigma 2h_0 \sec\left(\sqrt{h_0}\,t\right)\tan\left(\sqrt{h_0}\,t\right) \\
\bar{p}_3(t) = 2\sigma\sqrt{h_0}\,\sec\left(\sqrt{h_0}\,t\right).
\end{cases}$$

We have that $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$, *i.e.*, $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$. Indeed,

$$\begin{aligned}
\dot{\bar{p}}_1(t) &= -4h_0\sqrt{h_0}\,\tan\left(\sqrt{h_0}\,t\right)\sec^2\left(\sqrt{h_0}\,t\right) = \bar{p}_2(t)\bar{p}_3(t) \\
\dot{\bar{p}}_2(t) &= -\sigma 2h_0\sqrt{h_0}\,\sec\left(\sqrt{h_0}\,t\right)\left[\sec^2\left(\sqrt{h_0}\,t\right) + \tan^2\left(\sqrt{h_0}\,t\right)\right] = \bar{p}_1(t)\bar{p}_3(t) \\
\dot{\bar{p}}_3(t) &= \sigma 2h_0\,\mathrm{sech}\left(\sqrt{h_0}\,t\right)\tan\left(\sqrt{h_0}\,t\right) = -\bar{p}_2(t).
\end{aligned}$$

Lastly, since $\cos\left(\sqrt{h_0}\,t\right) = 0$ at the points $\frac{\pi}{\sqrt{h_0}}(2n - \frac{1}{2})$ and $\frac{\pi}{\sqrt{h_0}}(2n + \frac{1}{2})$, $n \in \mathbb{Z}$, it follows that $\bar{p}(\cdot)$ is only defined on the open intervals $\left(\frac{\pi}{\sqrt{h_0}}(2n - \frac{1}{2}), \frac{\pi}{\sqrt{h_0}}(2n + \frac{1}{2})\right)$, $n \in \mathbb{Z}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.4 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$, $h_0 = \sqrt{c_0}$ and $p_1(0) \leq -\sqrt{c_0}$.*

(*i*) *There exist $t_0 \in \left(-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}\right)$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \left(-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}\right) \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases} \bar{p}_1(t) = -\Omega^2\left[1 + 2\tan^2(\Omega t)\right] \\ \bar{p}_2(t) = -\sigma 2\Omega^2 \sec(\Omega t)\tan(\Omega t) \\ \bar{p}_3(t) = 2\sigma\Omega\sec(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{h_0}$.

(*ii*) *$t \mapsto \bar{p}(t + t_0)$ is the unique maximal integral curve starting at $\bar{p}(t_0)$.*

Proof. (*i*) Let $\sigma = \operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_3(0) = 0$, then $h_0 = p_1(0) \leq -\sqrt{c_0}$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_2|_{(-\pi/(2\Omega),0)}(t)) = \sigma$ and $\operatorname{sgn}(\bar{p}_2|_{(0,\pi/(2\Omega))}(t)) = -\sigma$. Moreover, $\lim_{t\to-\pi/(2\Omega)} \bar{p}_2(t) = \sigma\infty$ and $\lim_{t\to\pi/(2\Omega))} \bar{p}_2(t) = -\sigma\infty$. Therefore, since $\bar{p}_2(\cdot)$ is continuous, there exists $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ such that $\bar{p}_2(t_0) = p_2(0)$. Then

$$\bar{p}_1(t_0)^2 = \bar{p}_2(t_0)^2 + c_0 = p_2(0)^2 + c_0 = p_1(0)^2,$$

and since $\bar{p}_1(t_0), p_1(0) \leq -\sqrt{c_0}$ it follows that $\bar{p}_1(t_0) = p_1(0)$. Moreover,

$$\bar{p}_3(t_0)^2 = 2(h_0 - \bar{p}_1(t_0)) = 2(h_0 - p_1(0) = p_3(0)^2.$$

Since $\operatorname{sgn}(\bar{p}_3(t_0)) = \sigma = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_3(t_0) = p_3(0)$. Therefore, since $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t+t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}(s+t_0)$, and so $s+t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$. Consequently, $s \in (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} - t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} - t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t + t_0)$. ■

4.6.1.1.3 Case *I-c*: $-\sqrt{c_0} < h_0 < \sqrt{c_0}$. From $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ and the constants of motion $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$ and $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, we have

$$\begin{aligned} \dot{\bar{p}}_1^2 &= (\bar{p}_2\bar{p}_3)^2 = (\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1) \\ &= (\bar{p}_1 + \sqrt{c_0})(\bar{p}_1 - \sqrt{c_0})(2h_0 - 2\bar{p}_1). \end{aligned} \tag{4.6.5}$$

This must be reduced to standard form before we can solve it (see section A.6.2). Let $X_1 = \bar{p}_1 - \sqrt{c_0}$ and $X_2 = (\bar{p}_1 + \sqrt{c_0})(2h_0 - 2\bar{p}_1)$. Then $X_1 - \lambda X_2$ is a perfect square for $\lambda_1 = \frac{1}{2h_0+4\delta-6\sqrt{c_0}}$ and $\lambda_2 = \frac{1}{2h_0-4\delta-6\sqrt{c_0}}$, where $\delta = \sqrt{2(c_0 - \sqrt{c_0}\,h_0)}$. (As $-\sqrt{c_0} < h_0 < \sqrt{c_0}$ we have $c_0 - \sqrt{c_0}\,h_0 = \sqrt{c_0}\,(\sqrt{c_0} - h_0) > 0$; thus $\delta, \lambda_1, \lambda_2 \in \mathbb{R}$.) Accordingly,

$X_1 - \lambda_1 X_2 = (\bar{p}_1 - r_1)^2$ and $X_1 - \lambda_2 X_2 = (\bar{p}_1 - r_2)^2$, where $r_1 = \sqrt{c_0} - \delta$ and $r_2 = \sqrt{c_0} + \delta$. Hence

$$X_1 X_2 = \left[A_1(\bar{p}_1 - r_1)^2 + B_1(\bar{p}_1 - r_2)^2\right]\left[A_2(\bar{p}_1 - r_1)^2 + B_2(\bar{p}_1 - r_2)^2\right]$$

where

$$A_1 = \frac{1}{4\delta}, \qquad B_1 = -\frac{1}{4\delta}, \qquad A_2 = \frac{1}{4\delta\lambda_2}, \qquad B_2 = -\frac{1}{4\delta\lambda_1}.$$

Take the square root of both sides in (4.6.5). After separating variables, we have

$$\frac{d\bar{p}_1}{\sqrt{(\bar{p}_1 + \sqrt{c_0})(\bar{p}_1 - \sqrt{c_0})(2h_0 - 2\bar{p}_1)}} = \frac{d\bar{p}_1}{\sqrt{X_1 X_2}} = \sigma_1 dt$$

for some $\sigma_1 \in \{-1, 1\}$. By the preceding calculations, this is

$$\frac{d\bar{p}_1}{\sqrt{[A_1(\bar{p}_1 - r_1)^2 + B_1(\bar{p}_1 - r_2)^2]\,[A_2(\bar{p}_1 - r_1)^2 + B_2(\bar{p}_1 - r_2)^2]}} = \sigma_1 dt.$$

Since $h_0 - 2\delta - 3\sqrt{c_0} < \sqrt{c_0} - 2\delta - 3\sqrt{c_0} = -2(\sqrt{c_0} + \delta) < 0$, we have $A_1 A_2 = \frac{h_0 - 2\delta - 3\sqrt{c_0}}{8\delta^2} < 0$. Make the change of variables $u = \frac{\bar{p}_1 - r_1}{\bar{p}_1 - r_2}$. The result is

$$\frac{du}{\sqrt{-\left(u^2 - 1\right)\left(u^2 + \frac{B_2}{A_2}\right)}} = \sigma_1(r_1 - r_2)\sqrt{-A_1 A_2}\, t = -\sigma_1 \frac{1}{2}\sqrt{-\frac{1}{\lambda_2}}\, dt.$$

Since $\frac{B_2}{A_2} = -\frac{\lambda_2}{\lambda_1} < 0$, let $a = 1$, $b = \sqrt{\frac{\lambda_2}{\lambda_1}} < 1$ and $x = \frac{\bar{p}_1(t) - r_1}{\bar{p}_1(t) - r_2}$. Now integrate both sides:

$$\int_x^a \frac{du}{\sqrt{(a^2 - u^2)(u^2 - b^2)}} = -\sigma_1 \frac{1}{2}\sqrt{-\frac{1}{\lambda_2}}\, t.$$

We use the integral formula (A.6.8) for the left-hand side. Under the constraint $b \leq x \leq a$, the result is

$$\begin{aligned} \operatorname{dn}^{-1}\left(\frac{\bar{p}_1(t) - r_1}{\bar{p}_1(t) - r_2}, \sqrt{1 - b^2}\right) &= -\sigma_1 \frac{1}{2}\sqrt{-\frac{1}{\lambda_2}}\, t \\ \Rightarrow \qquad \frac{\bar{p}_1(t) - r_1}{\bar{p}_1(t) - r_2} &= \operatorname{dn}\left(\frac{1}{2}\sqrt{-\frac{1}{\lambda_2}}\, t, \sqrt{1 - b^2}\right). \end{aligned}$$

(As $\operatorname{dn}(\cdot, k)$ is even, we can eliminate the $-\sigma_1$.) Let $\Omega = \frac{1}{2}\sqrt{-\frac{1}{\lambda_2}} = \frac{1}{2}\sqrt{6\sqrt{c_0} - 2h_0 + 4\delta}$ and $k = \sqrt{1 - b^2} = 2\sqrt{\frac{\delta}{3\sqrt{c_0} - h_0 + 2\delta}}$, $k' = \sqrt{\frac{3\sqrt{c_0} - h_0 - 2\delta}{3\sqrt{c_0} - h_0 + 2\delta}}$. (As $b < 1$ we have $0 < k < 1$.) Solving for $\bar{p}_1(t)$, we get

$$\bar{p}_1(t) = \frac{r_2 \operatorname{dn}(\Omega t, k) - r_1}{\operatorname{dn}(\Omega t, k) - 1} = \frac{(\delta + \sqrt{c_0})\operatorname{dn}(\Omega t, k) + (\delta - \sqrt{c_0})}{\operatorname{dn}(\Omega t, k) - 1}.$$

Next, we use the Casimir equation $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$ to find $\bar{p}_2(t)$:

$$\begin{aligned} \bar{p}_2(t)^2 &= \bar{p}_1(t)^2 - c_0 \\ &= \left(\frac{(\delta + \sqrt{c_0})\operatorname{dn}(\Omega t, k) + (\delta - \sqrt{c_0})}{\operatorname{dn}(\Omega t, k) - 1}\right)^2 - c_0 \\ &= \frac{\delta(\delta + 2\sqrt{c_0})\operatorname{dn}^2(\Omega t, k) + 2\delta^2 \operatorname{dn}(\Omega t, k) + \delta(\delta - 2\sqrt{c_0})}{[\operatorname{dn}(\Omega t, k) - 1]^2}. \end{aligned}$$

We have $\delta(\delta + 2\sqrt{c_0}) + \delta(\delta - 2\sqrt{c_0}) = 2\delta^2$, and so

$$\begin{aligned}\bar{p}_2(t)^2 &= \frac{\left[\delta(\delta + 2\sqrt{c_0})\,\mathrm{dn}(\Omega t, k) + \delta(\delta - 2\sqrt{c_0})\right][\mathrm{dn}(\Omega t, k) + 1]}{[\mathrm{dn}(\Omega t, k) - 1]^2} \\ &= \delta(\delta + 2\sqrt{c_0})\frac{\left[\mathrm{dn}(\Omega t, k) + \frac{\delta - 2\sqrt{c_0}}{\delta + 2\sqrt{c_0}}\right][\mathrm{dn}(\Omega t, k) + 1]}{[\mathrm{dn}(\Omega t, k) - 1]^2}.\end{aligned}$$

We claim that $\frac{\delta - 2\sqrt{c_0}}{\delta + 2\sqrt{c_0}} = -k'$. Indeed,

$$\begin{aligned}&(\delta - 2\sqrt{c_0})^2(3\sqrt{c_0} - h_0 + 2\delta) \\ &\quad = 2\sqrt{c_0}\left(3\sqrt{c_0} - h_0 - 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right)\left(3\sqrt{c_0} - h_0 + 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right) \\ &\quad = 2\sqrt{c_0}\,(h_0 + \sqrt{c_0})^2.\end{aligned}$$

Similarly,

$$\begin{aligned}&(\delta + 2\sqrt{c_0})^2(3\sqrt{c_0} - h_0 - 2\delta) \\ &\quad = 2\sqrt{c_0}\left(3\sqrt{c_0} - h_0 + 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right)\left(3\sqrt{c_0} - h_0 - 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right) \\ &\quad = 2\sqrt{c_0}\,(h_0 + \sqrt{c_0})^2.\end{aligned}$$

Consequently, $\left(\frac{\delta - 2\sqrt{c_0}}{\delta + 2\sqrt{c_0}}\right)^2 = \frac{3\sqrt{c_0} - h_0 - 2\delta}{3\sqrt{c_0} - h_0 + 2\delta} = (k')^2$. As $(\delta - 2\sqrt{c_0})(\delta + 2\sqrt{c_0}) = -2(c_0 + \sqrt{c_0}\,h_0) < 0$ and $\delta + 2\sqrt{c_0} > 0$, it follows that $\delta - 2\sqrt{c_0} < 0$, whence $\frac{\delta - 2\sqrt{c_0}}{\delta + 2\sqrt{c_0}} = -k'$. Therefore

$$\begin{aligned}\bar{p}_2(t)^2 &= \delta(\delta + 2\sqrt{c_0})\frac{[\mathrm{dn}(\Omega t, k) - k'][\mathrm{dn}(\Omega t, k) + 1]}{[\mathrm{dn}(\Omega t, k) - 1]^2}\cdot\frac{\mathrm{dn}(\Omega t, k) + k'}{\mathrm{dn}(\Omega t, k) + k'} \\ &= \delta(\delta + 2\sqrt{c_0})\frac{\left[\mathrm{dn}^2(\Omega t, k) - (k')^2\right][\mathrm{dn}(\Omega t, k) + 1]}{[\mathrm{dn}(\Omega t, k) + k'][\mathrm{dn}(\Omega t, k) - 1]^2} \\ &= \delta(\delta + 2\sqrt{c_0})\frac{k^2\,\mathrm{cn}^2(\Omega t, k)\,[\mathrm{dn}(\Omega t, k) + 1]}{[\mathrm{dn}(\Omega t, k) + k'][\mathrm{dn}(\Omega t, k) - 1]^2}.\end{aligned}$$

(We have used the square relation (A.6.6) in the last step.) Taking the square root of both sides, we get

$$\bar{p}_2(t) = \sigma_2 k\sqrt{\delta(\delta + 2\sqrt{c_0})}\,\frac{\mathrm{cn}(\Omega t, k)\sqrt{\mathrm{dn}(\Omega t, k) + 1}}{\sqrt{\mathrm{dn}(\Omega t, k) + k'}\,[\mathrm{dn}(\Omega t, k) - 1]}$$

for some $\sigma_2 \in \{-1, 1\}$. Lastly, we use the equation $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$ to get an expression for $\bar{p}_3(\cdot)$:

$$\begin{aligned}\bar{p}_3(t)^2 &= 2h_0 - 2\bar{p}_1(t) \\ &= -\frac{2\left[(\delta + \sqrt{c_0} - h_0)\,\mathrm{dn}(\Omega t, k) + (\delta - \sqrt{c_0} + h_0)\right]}{\mathrm{dn}(\Omega t, k) - 1} \\ &= -2(\delta + \sqrt{c_0} - h_0)\frac{\left[\mathrm{dn}(\Omega t, k) + \frac{\delta - \sqrt{c_0} + h_0}{\delta + \sqrt{c_0} - h_0}\right]}{\mathrm{dn}(\Omega t, k) - 1}.\end{aligned}$$

We claim that $\frac{\delta-\sqrt{c_0}+h_0}{\delta+\sqrt{c_0}-h_0} = k'$. Indeed,

$$\begin{aligned}
&(\delta - \sqrt{c_0} + h_0)^2(3\sqrt{c_0} - h_0 + 2\delta) \\
&\quad = (\sqrt{c_0} - h_0)\left(3\sqrt{c_0} - h_0 - 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right)\left(3\sqrt{c_0} - h_0 + 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right) \\
&\quad = (\sqrt{c_0} - h_0)(\sqrt{c_0} + h_0)^2
\end{aligned}$$

and

$$\begin{aligned}
&(\delta + \sqrt{c_0} - h_0)^2(3\sqrt{c_0} - h_0 - 2\delta) \\
&\quad = (\sqrt{c_0} - h_0)\left(3\sqrt{c_0} - h_0 + 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right)\left(3\sqrt{c_0} - h_0 - 2\sqrt{2(c_0 - \sqrt{c_0}\,h_0)}\right) \\
&\quad = (\sqrt{c_0} - h_0)(\sqrt{c_0} + h_0)^2.
\end{aligned}$$

Thus $\left(\frac{\delta-\sqrt{c_0}+h_0}{\delta+\sqrt{c_0}-h_0}\right)^2 = (k')^2$. Since $(\delta-\sqrt{c_0}+h_0)(\delta+\sqrt{c_0}-h_0) = c_0 - h_0^2 > 0$ and $\delta+\sqrt{c_0}-h_0 > 0$, we have $\frac{\delta-\sqrt{c_0}+h_0}{\delta+\sqrt{c_0}-h_0} = k'$. Hence,

$$\begin{aligned}
\bar{p}_3(t)^2 &= -2(\delta + \sqrt{c_0} - h_0)\frac{\mathrm{dn}(\Omega t, k) + k'}{\mathrm{dn}(\Omega t, k) - 1} \cdot \frac{\mathrm{dn}(\Omega t, k) - 1}{\mathrm{dn}(\Omega t, k) - 1} \\
&= -2(\delta + \sqrt{c_0} - h_0)\frac{[\mathrm{dn}(\Omega t, k) + k']\,[\mathrm{dn}(\Omega t, k) - 1]}{[\mathrm{dn}(\Omega t, k) - 1]^2} \\
&= 2(\delta + \sqrt{c_0} - h_0)\frac{[\mathrm{dn}(\Omega t, k) + k']\,[1 - \mathrm{dn}(\Omega t, k)]}{[\mathrm{dn}(\Omega t, k) - 1]^2}.
\end{aligned}$$

Taking the square root of both sides yields

$$\bar{p}_3(t) = \sigma_3\sqrt{2(\delta + \sqrt{c_0} - h_0)}\,\frac{\sqrt{\mathrm{dn}(\Omega t, k) + k'}\,\sqrt{1 - \mathrm{dn}(\Omega t, k)}}{\mathrm{dn}(\Omega t, k) - 1}$$

for some $\sigma_3 \in \{-1, 1\}$. Thus we have the following (prospective) integral curve:

$$\begin{cases}
\bar{p}_1(t) = \dfrac{(\delta + \sqrt{c_0}\,)\,\mathrm{dn}(\Omega t, k) + (\delta - \sqrt{c_0}\,)}{\mathrm{dn}(\Omega t, k) - 1} \\
\bar{p}_2(t) = \sigma_2 k\sqrt{\delta(\delta + 2\sqrt{c_0}\,)}\,\dfrac{\mathrm{cn}(\Omega t, k)\sqrt{\mathrm{dn}(\Omega t, k) + 1}}{\sqrt{\mathrm{dn}(\Omega t, k) + k'}\,[\mathrm{dn}(\Omega t, k) - 1]} \\
\bar{p}_3(t) = \sigma_3\sqrt{2(\delta + \sqrt{c_0} - h_0)}\,\dfrac{\sqrt{\mathrm{dn}(\Omega t, k) + k'}\,\sqrt{1 - \mathrm{dn}(\Omega t, k)}}{\mathrm{dn}(\Omega t, k) - 1}.
\end{cases}$$

We will show that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$ if and only $\sigma_2 = \sigma_3$. By proposition 4.1.2, it suffices to show that $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2 = \sigma_3$ (since we know $\bar{p}(\cdot)$ satisfies the equations of the constants of motion). Indeed,

$$\begin{aligned}
\dot{\bar{p}}_1(t) - \bar{p}_2(t)\bar{p}_3(t) &= \frac{2\delta\Omega k^2\,\mathrm{cn}(\Omega t, k)\,\mathrm{sn}(\Omega t, k)}{[\mathrm{dn}(\Omega t, k) - 1]^2} \\
&\qquad - \frac{\sigma_2\sigma_3 k\sqrt{2\delta(\delta + 2\sqrt{c_0}\,)(\delta + \sqrt{c_0} - h_0)}\,\mathrm{cn}(\Omega t, k)\sqrt{1 - \mathrm{dn}^2(\Omega t, k)}}{[\mathrm{dn}(\Omega t, k) - 1]^2} \\
&= -\frac{k^2\left[\sigma_2\sigma_3\sqrt{\delta(\delta + 2\sqrt{c_0}\,)}\,\sqrt{2(\delta + \sqrt{c_0} - h_0)} - 2\delta\Omega\right]\mathrm{cn}(\Omega t, k)\,\mathrm{sn}(\Omega t, k)}{[\mathrm{dn}(\Omega t, k) - 1]^2}.
\end{aligned}$$

We have

$$\begin{aligned}
&\sigma_2\sigma_3\sqrt{2\delta(\delta+2\sqrt{c_0})(\delta+\sqrt{c_0}-h_0)}-2\delta\Omega\\
&\quad=\sigma_2\sigma_3\sqrt{4\sqrt{c_0}(\sqrt{c_0}-h_0)(3\sqrt{c_0}-h_0+2\delta)}-\delta\sqrt{6\sqrt{c_0}-2h_0+4\delta}\\
&\quad=\sigma_2\sigma_3\sqrt{4\sqrt{c_0}(\sqrt{c_0}-h_0)(3\sqrt{c_0}-h_0+2\delta)}-\sqrt{(2c_0-2\sqrt{c_0}\,h_0)(6\sqrt{c_0}-2h_0+4\delta)}\\
&\quad=(\sigma_2\sigma_3-1)\sqrt{4\sqrt{c_0}(\sqrt{c_0}-h_0)(3\sqrt{c_0}-h_0+2\delta)}\,,
\end{aligned}$$

and this is zero if and only if $\sigma_2\sigma_3 = 1$, *i.e.*, $\sigma_2 = \sigma_3$. In this case, we have $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$.

Finally, since $\mathrm{dn}(\Omega t, k) = 1$ at the points $\frac{2nK}{\Omega}$, $n \in \mathbb{Z}$ (where $4K$ denotes the period of $\mathrm{sn}(\cdot, k)$), it follows that $\bar{p}(\cdot)$ is only defined on the open intervals $\left(\frac{2nK}{\Omega}, \frac{2(n+1)K}{\Omega}\right)$, $n \in \mathbb{Z}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.5 PROPOSITION. *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_1^{(3)}$ *such that* $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ *and* $-\sqrt{c_0} < h_0 < \sqrt{c_0}$.

(*i*) *There exist* $t_0 \in \left(0, \frac{2K}{\Omega}\right)$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : \left(0, \frac{2K}{\Omega}\right) \to \mathfrak{se}(1,1)^*$ *is defined by*

$$\begin{cases}
\bar{p}_1(t) = \dfrac{(\delta+\sqrt{c_0})\,\mathrm{dn}(\Omega t,k)+(\delta-\sqrt{c_0})}{\mathrm{dn}(\Omega t,k)-1}\\[2ex]
\bar{p}_2(t) = \sigma k\sqrt{\delta(\delta+2\sqrt{c_0})}\,\dfrac{\mathrm{cn}(\Omega t,k)\sqrt{\mathrm{dn}(\Omega t,k)+1}}{\sqrt{\mathrm{dn}(\Omega t,k)+k'}\,[\mathrm{dn}(\Omega t,k)-1]}\\[2ex]
\bar{p}_3(t) = \sigma\sqrt{2(\delta+\sqrt{c_0}-h_0)}\,\dfrac{\sqrt{\mathrm{dn}(\Omega t,k)+k'}\sqrt{1-\mathrm{dn}(\Omega t,k)}}{\mathrm{dn}(\Omega t,k)-1}.
\end{cases}$$

Here $4K$ *is the period of* $\mathrm{sn}(\cdot, k)$ *and*

$$\delta = \sqrt{2(c_0-\sqrt{c_0}\,h_0)} \qquad \Omega = \tfrac{1}{2}\sqrt{6\sqrt{c_0}-2h_0+4\delta}$$

$$k = 2\sqrt{\tfrac{\delta}{3\sqrt{c_0}-h_0+2\delta}} \qquad k' = \sqrt{\tfrac{3\sqrt{c_0}-h_0-2\delta}{3\sqrt{c_0}-h_0+2\delta}}\,.$$

(*ii*) $t \mapsto \bar{p}(t + t_0)$ *is the unique maximal integral curve starting at* $\bar{p}(t_0)$.

PROOF. (*i*) Since $p_1(t)^2 \geq p_1(t)^2 - p_2(t)^2 = c_0$, we have $p_1(t) \leq -\sqrt{c_0}$ or $p_1(t) \geq \sqrt{c_0}$. If $p_1(t) \geq \sqrt{c_0}$, then $h_0 \geq h_0 - \frac{1}{2}p_3(t)^2 = p_1(t) \geq \sqrt{c_0}$, which is a contradiction. Thus $p_1(t) \leq -\sqrt{c_0}$.

Let $\sigma = \mathrm{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_3(0) = 0$, then $h_0 = p_1(0) \leq -\sqrt{c_0}$, a contradiction.) We have $\mathrm{sgn}(\bar{p}_2|_{(0,K/\Omega)}(t)) = \sigma$ and $\mathrm{sgn}(\bar{p}_2|_{(K/\Omega,2K/\Omega)}(t)) = -\sigma$. Moreover, $\lim_{t\to 0}\bar{p}_2(t) = \sigma\infty$ and $\lim_{t\to 2K/\Omega}\bar{p}_2(t) = -\sigma\infty$. Therefore, since $\bar{p}_2(\cdot)$ is continuous, there exists $t_0 \in \left(0, \frac{2K}{\Omega}\right)$ such that $\bar{p}_2(t_0) = p_2(0)$. Then

$$\bar{p}_1(t_0)^2 = c_0 + \bar{p}_2(t_0)^2 = c_0 + p_2(0)^2 = p_1(0)^2.$$

As $\bar{p}_1(t_0), p_1(0) \leq -\sqrt{c_0}$, we have $\bar{p}_1(t_0) = p_1(0)$. Lastly,

$$\bar{p}_3(t_0)^2 = 2(h_0 - \bar{p}_1(t_0)) = 2(h_0 - p_1(0)) = p_3(0)^2.$$

Since $\operatorname{sgn}(\bar{p}_3(t_0)) = \sigma = \operatorname{sgn}(p_3(0))$, it follows that $\bar{p}_3(t_0) = p_3(0)$. Therefore, as $t \mapsto \bar{p}(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t+t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}(s+t_0)$, and so $s + t_0 \in \left(0, \frac{2K}{\Omega}\right)$. Consequently, $s \in \left(-t_0, \frac{2K}{\Omega} - t_0\right)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq \left(-t_0, \frac{2K}{\Omega} - t_0\right)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t+t_0)$. ■

4.6.1.1.4 Case *I-d*: $h_0 = -\sqrt{c_0}$. We have $\bar{p}_1(t) = h_0 - \frac{1}{2}\bar{p}_3(t)^2 \leq h_0 = -\sqrt{c_0}$. If $\bar{p}_1(t) = -\sqrt{c_0} = h_0$ for some t, then $\bar{p}(\cdot)$ is constant. (Indeed, we have $\bar{p}(t) = (-\sqrt{c_0}, 0, 0)$, which is an equilibrium point of $\vec{H}_1^{(3)}$.) Assume $\bar{p}_1(t) < -\sqrt{c_0}$.

Consider the integral curve of proposition 4.6.5. Let $\bar{p}(\cdot)$ be the curve obtained by limiting (from above) $h_0 \to -\sqrt{c_0}$. Then

$$\begin{aligned}
\bar{p}_1(t) &= \lim_{h_0 \to -\sqrt{c_0}} \frac{(\delta + \sqrt{c_0})\operatorname{dn}(\Omega t, k) + (\delta - \sqrt{c_0})}{\operatorname{dn}(\Omega t, k) - 1} \\
&= h_0 \left[1 + 2\operatorname{csch}^2\left(\sqrt{-h_0}\, t\right)\right] \\
\bar{p}_2(t) &= \lim_{h_0 \to -\sqrt{c_0}} \sigma 2k \sqrt{\delta(\delta + 2\sqrt{c_0})}\, \frac{\operatorname{cn}(\Omega t, k)\sqrt{\operatorname{dn}(\Omega t, k) + 1}}{\sqrt{\operatorname{dn}(\Omega t, k) + k'}\,[\operatorname{dn}(\Omega t, k) - 1]} \\
&= -\sigma 2 h_0 \coth\left(\sqrt{-h_0}\, t\right)\operatorname{csch}\left(\sqrt{-h_0}\, t\right) \\
\bar{p}_3(t) &= \lim_{h_0 \to -\sqrt{c_0}} \sigma \sqrt{2(\delta + \sqrt{c_0} - h_0)}\, \frac{\sqrt{\operatorname{dn}(\Omega t, k) + k'}\sqrt{1 - \operatorname{dn}(\Omega t, k)}}{\operatorname{dn}(\Omega t, k) - 1} \\
&= \sigma 2\sqrt{-h_0}\operatorname{csch}\left(\sqrt{-h_0}\, t\right).
\end{aligned}$$

Now

$$\begin{aligned}
\dot{\bar{p}}_1(t) &= -4h_0\sqrt{-h_0}\coth\left(\sqrt{-h_0}\, t\right)\csc^2\left(\sqrt{-h_0}\, t\right) \\
&= \bar{p}_2(t)\bar{p}_3(t) \\
\dot{\bar{p}}_2(t) &= \sigma 2h_0\sqrt{-h_0}\coth^2\left(\sqrt{-h_0}\, t\right)\operatorname{csch}\left(\sqrt{-h_0}\, t\right) + \sigma 2 h_0 \sqrt{-h_0}\operatorname{csch}^2\left(\sqrt{-h_0}\, t\right) \\
&= \bar{p}_1(t)\bar{p}_3(t) \\
\dot{\bar{p}}_3(t) &= \sigma 2h_0 \coth\left(\sqrt{-h_0}\, t\right)\operatorname{csch}\left(\sqrt{-h_0}\, t\right) \\
&= -\bar{p}_2(t).
\end{aligned}$$

That is, $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$, and so $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$. Finally, notice that $\sinh(\Omega t) = 0$ when $t = 0$. Thus $\bar{p}(\cdot)$ is only defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.6 Proposition. *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_1^{(3)}$ *such that* $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ *and* $h_0 = -\sqrt{c_0}$.

(*i*) *There exist* $t_0 \in \mathbb{R}\backslash\{0\}$ *and* $\sigma, \varsigma \in \{-1,1\}$ *such that* $p(t) = \bar{p}_\varsigma(t+t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$, $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$ *and*

$$\begin{cases} \bar{p}_1(t) = -\Omega^2\left[1 + 2\operatorname{csch}^2(\Omega t)\right] \\ \bar{p}_2(t) = \sigma 2\Omega^2 \coth(\Omega t)\operatorname{csch}(\Omega t) \\ \bar{p}_3(t) = \sigma 2\Omega \operatorname{csch}(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{-h_0}$.

(*ii*) $t \mapsto \bar{p}_\pm(t+t_0)$ *is the unique maximal integral curve starting at* $\bar{p}_\pm(t_0)$.

PROOF. (*i*) Let $\sigma = \operatorname{sgn}(p_2(0)) \in \{-1,1\}$ and $\varsigma = \sigma \operatorname{sgn}(p_3(0)) \in \{-1,1\}$. (If $p_2(0) = 0$ or $p_3(0) = 0$, then $p(0) = (-\sqrt{c_0}, 0, 0)$, which is an equilibrium point of $\vec{H}_1^{(3)}$.) We have $\operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t)) = \sigma$. Moreover,

$$\lim_{t\to 0}\bar{p}_{+,2}(t) = \sigma\infty, \ \lim_{t\to\infty}\bar{p}_{+,2}(t) = 0 \qquad \text{and} \qquad \lim_{t\to-\infty}\bar{p}_{-,2}(t) = 0, \ \lim_{t\to 0}\bar{p}_{-,2}(t) = \sigma\infty.$$

Suppose $\operatorname{sgn}(p_3(0)) = 1$, so we have $\varsigma = \sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = -1 \\ (0,\infty) & \text{if } \sigma = +1 \end{cases}$$

such that $\bar{p}_{\sigma,2}(t_0) = p_2(0)$. Then

$$\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 + c_0 = p_2(0)^2 + c_0 = p_1(0)^2.$$

We have $p_1(0), \bar{p}_{\sigma,1}(t_0) \leq -\sqrt{c_0} < 0$, and so $\bar{p}_{\sigma,1}(t_0) = p_1(0)$. Moreover,

$$\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\sigma,3}(t_0)) = 1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

Suppose $\operatorname{sgn}(p_3(0)) = -1$, so $\varsigma = -\sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = +1 \\ (0,\infty) & \text{if } \sigma = -1 \end{cases}$$

such that $\bar{p}_{-\sigma,2}(t_0) = p_2(0)$. From $\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 = p_2(0)^2 = p_1(0)^2$ we again get $\bar{p}_{-\sigma,1}(t_0) = p_1(0)$. Similarly, as $\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2$ and $\operatorname{sgn}(\bar{p}_{-\sigma,3}(t_0)) = -1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{-\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ both solve the same Cauchy problem, and so are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_\pm(t+t_0)$. Suppose $p(0) = \bar{p}_-(t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}_-(s+t_0)$, and so $s + t_0 \in (-\infty, 0)$. Consequently, $s \in (-\infty, -t_0)$, *i.e.*, $(-\varepsilon,\varepsilon) \subseteq (-\infty, -t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}_-(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}_-(t+t_0)$.

Suppose $p(0) = \bar{p}_+(t_0)$. Similar to before, if $s \in (-\varepsilon,\varepsilon)$, then $s + t_0 \in (0,\infty)$, *i.e.*, $s \in (-t_0, \infty)$. Thus $(-\varepsilon,\varepsilon) \subseteq (-t_0,\infty)$, and so the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_+(t+t_0)$. The uniqueness of $t \mapsto \bar{p}_+(t+t_0)$ follows from its maximality. ■

4.6.1.1.5 Case *I-e*: $h_0 < -\sqrt{c_0}$. From the equation of motion $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$, the Hamiltonian equation $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)$ and the Casimir equation $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, we get

$$\dot{\bar{p}}_1^2 = (\bar{p}_2\bar{p}_3)^2 = (\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1). \tag{4.6.6}$$

Before integrating, we reduce this equation to standard form. (This process is detailed in section A.6.2.) Let $X_1 = \bar{p}_1^2 - c_0$ and $X_2 = 2h_0 - 2\bar{p}_1$. Then $X_1 - \lambda X_2$ is a perfect square for $\lambda_1 = -(\delta + h_0)$ and $\lambda_2 = \delta - h_0$, where $\delta = \sqrt{h_0^2 - c_0}$. (Since $h_0 < -\sqrt{c_0}$, we have $h_0^2 - c_0 > 0$, and so $\delta, \lambda_1, \lambda_2 \in \mathbb{R}$.) Accordingly, $X_1 - \lambda_1 X_2 = (\bar{p}_1 + \lambda_1)^2$ and $X_1 - \lambda_2 X_2 = (\bar{p}_1 + \lambda_2)^2$. Thus

$$X_1X_2 = \left[A_1(\bar{p}_1 + \lambda_1)^2 + B_1(\bar{p}_1 + \lambda_2)^2\right]\left[A_2(\bar{p}_1 + \lambda_1)^2 + B_2(\bar{p}_1 + \lambda_2)^2\right]$$

where A_1, A_2, B_1, B_2 are given by

$$A_1 = \frac{1}{2}\left(1 - \frac{h_0}{\delta}\right), \qquad B_1 = \frac{1}{2}\left(1 + \frac{h_0}{\delta}\right), \qquad A_2 = \frac{1}{2\delta}, \qquad B_2 = -\frac{1}{2\delta}.$$

Take the square root of both sides in (4.6.6) and separate variables. We get

$$\frac{d\bar{p}_1}{\sqrt{(\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1)}} = \frac{d\bar{p}_1}{\sqrt{X_1X_2}} = \sigma_1 dt$$

for some $\sigma_1 \in \{-1, 1\}$. By the preceding calculations, this is

$$\frac{d\bar{p}_1}{\sqrt{\left[A_1(\bar{p}_1 + \lambda_1)^2 + B_1(\bar{p}_1 + \lambda_2)^2\right]\left[A_2(\bar{p}_1 + \lambda_1)^2 + B_2(\bar{p}_1 + \lambda_2)^2\right]}} = \sigma_1 dt. \tag{4.6.7}$$

We have $A_1A_2 = \frac{\delta - h_0}{4\delta^2}$. Since $(\delta + h_0)(\delta - h_0) = \delta^2 - h_0^2 = -c_0 < 0$ and $\delta - h_0 > 0$, we have $\delta + h_0 < 0$. Consequently, $A_1A_2 > 0$. Make the change of variables $u = \frac{\bar{p}_1 + \lambda_1}{\bar{p}_1 + \lambda_2}$, to get

$$\frac{du}{\sqrt{\left(u^2 + \frac{B_1}{A_1}\right)(u^2 - 1)}} = \sigma_1(\lambda_2 - \lambda_1)\sqrt{A_1A_2}\, dt = \sigma_1\sqrt{\delta - h_0}\, dt.$$

Here $\frac{B_1}{A_1} = \frac{\delta + h_0}{\delta - h_0} < 0$. Let $a = -1$, $b = -\sqrt{\frac{h_0 + \delta}{h_0 - \delta}}$, $x = \frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2}$ and integrate both sides:

$$\begin{aligned}\int_{-a}^{-x} \frac{du}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} &= -\int_a^x \frac{du}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} \\ &= \sigma_1\sqrt{\delta - h_0}\, t.\end{aligned}$$

We have $h_0 - \delta < h_0 + \delta$ and $h_0 < -\sqrt{c_0} < 0$, and so $\frac{h_0 + \delta}{h_0 - \delta} < 1$. Thus $-b < -a$. Hence, under the constraint $-b < -a \leq -x$, we can use the integral formula (A.6.9) to integrate the left-hand side:

$$\begin{aligned} & \frac{1}{a}\operatorname{dc}^{-1}\left(\frac{1}{a}\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2}, \frac{b}{a}\right) = \sigma_1\sqrt{\delta - h_0}\, t \\ \Rightarrow\quad & \frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2} = a\operatorname{dc}\left(\sqrt{\delta - h_0}\, t, \frac{b}{a}\right).\end{aligned}$$

(The function $\mathrm{dc}(\cdot,k)$ is even, so we can eliminate the $a\sigma_1 = -\sigma_1$.) Let $\Omega = \sqrt{\delta - h_0}$ and $k = \frac{b}{a} = \sqrt{\frac{h_0+\delta}{h_0-\delta}}$, $k' = \sqrt{\frac{2\delta}{\delta - h_0}}$. (Since $\delta - h_0 > 0$ and $\delta + h_0 < 0$, we have $0 < k < 1$.) Now solve for $\bar{p}_1(t)$:

$$\bar{p}_1(t) = \frac{(h_0-\delta)\,\mathrm{dc}(\Omega,k) + (h_0+\delta)}{1+\mathrm{dc}(\Omega,k)} = \frac{(h_0+\delta)\,\mathrm{cn}(\Omega t,k) + (h_0-\delta)\,\mathrm{dn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)}.$$

Use the constant of motion $\bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$ to find an expression for $\bar{p}_2(\cdot)$:

$$\begin{aligned}\bar{p}_2(t)^2 &= \left(\frac{(h_0+\delta)\,\mathrm{cn}(\Omega t,k) + (h_0-\delta)\,\mathrm{dn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)}\right)^2 - c_0 \\ &= \frac{((\delta-h_0)^2 - c_0)\,\mathrm{dn}^2(\Omega t,k) - 2(\delta^2+c_0-h_0^2)\,\mathrm{dn}(\Omega t,k)\,\mathrm{cn}(\Omega t,k)}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2} \\ &\quad + \frac{((\delta+h_0)^2 - c_0)\,\mathrm{cn}^2(\Omega t,k)}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2}.\end{aligned}$$

We have $(\delta-h_0)^2 - c_0 + (\delta+h_0)^2 - c_0 = -2(\delta^2 + c_0 - h_0^2)$. Consequently,

$$\begin{aligned}\bar{p}_2(t)^2 &= \frac{((\delta-h_0)+\sqrt{c_0})\,\mathrm{dn}(\Omega t,k) + ((\delta+h_0)-\sqrt{c_0})\,\mathrm{cn}(\Omega t,k)]}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2} \\ &\quad\times \frac{((\delta-h_0)-\sqrt{c_0})\,\mathrm{dn}(\Omega t,k) + ((\delta+h_0)+\sqrt{c_0})\,\mathrm{cn}(\Omega t,k)}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2} \\ &= ((\delta-h_0)+\sqrt{c_0})\frac{\mathrm{dn}(\Omega t,k) + \frac{(\delta+h_0)-\sqrt{c_0}}{(\delta-h_0)+\sqrt{c_0}}\,\mathrm{cn}(\Omega t,k)]}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2} \\ &\quad\times ((\delta-h_0)-\sqrt{c_0})\frac{\mathrm{dn}(\Omega t,k) + \frac{(\delta+h_0)+\sqrt{c_0}}{(\delta-h_0)-\sqrt{c_0}}\,\mathrm{cn}(\Omega t,k)}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2}.\end{aligned}$$

We have

$$\left[\frac{(\delta+h_0)-\sqrt{c_0}}{(\delta-h_0)+\sqrt{c_0}}\right]^2 = \frac{h_0+\delta}{h_0-\delta} = k^2 \qquad \text{and} \qquad \left[\frac{(\delta+h_0)+\sqrt{c_0}}{(\delta-h_0)-\sqrt{c_0}}\right]^2 = \frac{h_0+\delta}{h_0-\delta} = k^2.$$

Since $\delta - h_0 > 0$ and $\delta + h_0 < 0$, it follows that $\frac{(\delta+h_0)-\sqrt{c_0}}{(\delta-h_0)+\sqrt{c_0}} = -k$ and $\frac{(\delta+h_0)+\sqrt{c_0}}{(\delta-h_0)-\sqrt{c_0}} = k$. Consequently,

$$\begin{aligned}\bar{p}_2(t)^2 &= ((\delta-h_0)+\sqrt{c_0})((\delta-h_0)-\sqrt{c_0})\frac{[\mathrm{dn}(\Omega t,k) - k\,\mathrm{cn}(\Omega t,k)]\,[\mathrm{dn}(\Omega t,k) + k\,\mathrm{cn}(\Omega t,k)]}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2} \\ &= ((\delta-h_0)^2 - c_0)\frac{\mathrm{dn}^2(\Omega t,k) - k^2\,\mathrm{cn}(\Omega t,k)}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2} \\ &= ((\delta-h_0)^2 - c_0)\frac{(k')^2}{[\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)]^2}.\end{aligned}$$

(We have used the square relation (A.6.6) in the last step.) Taking the square root of both sides and using the fact that $((\delta-h_0)^2 - c_0)(k')^2 = 4(h_0^2 - c_0^2) = 4\delta^2$, we get

$$\bar{p}_2(t) = \sigma_2\frac{2\delta}{\mathrm{dn}(\Omega t,k)+\mathrm{cn}(\Omega t,k)}$$

for some $\sigma_2 \in \{-1, 1\}$. Next, using $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3^2(t)$ yields

$$\begin{aligned}\bar{p}_3(t)^2 &= 2h_0 - 2\bar{p}_1(t)\\ &= 2\delta\frac{\mathrm{dn}(\Omega t,k) - \mathrm{cn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)}\cdot\frac{\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)}\\ &= 2\delta\frac{\mathrm{dn}^2(\Omega t,k) - \mathrm{cn}^2(\Omega t,k)}{[\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)]^2}\\ &= 2\delta\frac{(k')^2\,\mathrm{sn}^2(\Omega t,k)}{[\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)]^2}.\end{aligned}$$

(We have used (A.6.4) and (A.6.5) in the final step.) Therefore

$$\bar{p}_3(t) = \sigma_3\frac{\sqrt{2\delta}\,k'\,\mathrm{sn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)}$$

for some $\sigma_3 \in \{-1, 1\}$. Thus we have the following (prospective) integral curve:

$$\begin{cases}\bar{p}_1(t) = \dfrac{(h_0+\delta)\,\mathrm{cn}(\Omega t,k) + (h_0-\delta)\,\mathrm{dn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)}\\ \bar{p}_2(t) = \sigma_2\dfrac{2\delta}{\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)}\\ \bar{p}_3(t) = \sigma_3\dfrac{\sqrt{2\delta}\,k'\,\mathrm{sn}(\Omega t,k)}{\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)}.\end{cases}$$

We show that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$ if and only if $\sigma_2 = -\sigma_3$. By proposition 4.1.2, it suffices to show that $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2 = -\sigma_3$ (since we know that $\bar{p}(\cdot)$ satisfies the equations of the constants of motion). We have

$$\begin{aligned}\dot{\bar{p}}_1(t) - \bar{p}_2(t)\bar{p}_3(t) &= \frac{2\delta\Omega\left[k^2\,\mathrm{cn}^2(\Omega t,k) - \mathrm{dn}^2(\Omega t,k)\right]\mathrm{sn}(\Omega t,k)}{[\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)]^2} - \frac{\sigma_2\sigma_3 2\delta k'\sqrt{2\delta}\,\mathrm{sn}(\Omega t,k)}{[\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)]^2}\\ &= -\frac{2\delta k'(\sigma_2\sigma_3\sqrt{2\delta} + \Omega k')\,\mathrm{sn}(\Omega t,k)}{[\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)]^2}.\end{aligned}$$

As $k'\Omega = \sqrt{2\delta}$, we have that $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2\sigma_3 = -1$, *i.e.*, $\sigma_2 = -\sigma_3$. In this case $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$.

Finally, since $\mathrm{dn}(\Omega t,k) + \mathrm{cn}(\Omega t,k)$ is zero at the points $\frac{2(2n+1)K}{\Omega}$, $n \in \mathbb{Z}$ (where $4K$ denotes the period of $\mathrm{sn}(\cdot,k)$), it follows that $\bar{p}(\cdot)$ is only defined on the open intervals $\left(-\frac{2(2n+1)K}{\Omega}, \frac{2(2n+1)K}{\Omega}\right)$, $n \in \mathbb{Z}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.7 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and $h_0 < -\sqrt{c_0}$.*

(i) There exist $t_0 \in \left(-\frac{2K}{\Omega}, \frac{2K}{\Omega}\right)$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every

$t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \left(-\frac{2K}{\Omega}, \frac{2K}{\Omega}\right) \to \mathfrak{se}(1,1)^*$ is defined by

$$\begin{cases} \bar{p}_1(t) = \dfrac{(h_0 + \delta)\,\mathrm{cn}(\Omega t, k) + (h_0 - \delta)\,\mathrm{dn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) + \mathrm{cn}(\Omega t, k)} \\ \bar{p}_2(t) = \sigma \dfrac{2\delta}{\mathrm{dn}(\Omega t, k) + \mathrm{cn}(\Omega t, k)} \\ \bar{p}_3(t) = -\sigma \dfrac{\sqrt{2\delta}\, k' \,\mathrm{sn}(\Omega t, k)}{\mathrm{dn}(\Omega t, k) + \mathrm{cn}(\Omega t, k)}. \end{cases}$$

Here $4K$ is the period of $\mathrm{sn}(\cdot, k)$, $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{\delta - h_0}$, $k = \sqrt{\frac{h_0+\delta}{h_0-\delta}}$ and $k' = \sqrt{\frac{2\delta}{\delta - h_0}}$.

(ii) $t \mapsto \bar{p}(t + t_0)$ is the unique maximal integral curve starting at $\bar{p}(t_0)$.

Proof. *(i)* Let $\sigma = \mathrm{sgn}(p_2(0)) \in \{-1, 1\}$. (If $p_2(0) = 0$, then $p_1(0)^2 = c_0$. However, $p_1(0) = h_0 - \frac{1}{2}p_3(0)^2 \leq h_0 < -\sqrt{c_0}$, which is a contradiction.) We have $\mathrm{sgn}(\bar{p}_3|_{(-2K/\Omega,0)}(t)) = \sigma$ and $\mathrm{sgn}(\bar{p}_3|_{(0,2K/\Omega)}(t)) = -\sigma$. Moreover, $\lim_{t\to -2K/\Omega} \bar{p}_3(t) = \sigma\infty$ and $\lim_{t \to 2K/\Omega} \bar{p}_3(t) = -\sigma\infty$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \left(-\frac{2K}{\Omega}, \frac{2K}{\Omega}\right)$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}p_3(0)^2 = p_1(0).$$

Lastly,

$$\bar{p}_2(t_0)^2 = \bar{p}_1(t_0)^2 - c_0 = p_1(0)^2 - c_0 = p_2(0)^2.$$

As $\mathrm{sgn}(\bar{p}_2(t_0)) = \sigma = \mathrm{sgn}(p_2(0))$, we have $\bar{p}_2(t_0) = p_2(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(ii) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t+t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By *(i)*, $p(s) = \bar{p}(s + t_0)$, and so $s + t_0 \in \left(-\frac{2K}{\Omega} \frac{2K}{\Omega}\right)$. Consequently, $s \in \left(-\frac{2K}{\Omega} - t_0, \frac{2K}{\Omega} - t_0\right)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq \left(-\frac{2K}{\Omega} - t_0, \frac{2K}{\Omega} - t_0\right)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t + t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t + t_0)$. ■

4.6.1.2 Case II: $c_0 = 0$

4.6.1.2.1 Case II-a: $h_0 > 0$. As $\bar{p}_1(t)^2 \geq 0$, we have $\bar{p}_1(t) \leq 0$ or $\bar{p}_1(t) \geq 0$. If $\bar{p}_1(t) = 0$ for some t, then $\bar{p}(\cdot)$ is constant. (Indeed, $p(t) = (0, 0, \pm\sqrt{2h_0})$, which is an equilibrium point of $\vec{H}_1^{(3)}$.) Thus we assume either $\bar{p}_1(t) < 0$ or $\bar{p}_1(t) > 0$ for every t.

Consider the case $\boldsymbol{\bar{p}_1(t) < 0}$. Let $\bar{p}(\cdot)$ be the curve obtained by taking the limit $c_0 \to 0$ of the integral curve in proposition 4.6.2. Then

$$\begin{cases} \bar{p}_1(t) = -2\Omega^2 \,\mathrm{csch}^2(\Omega t) \\ \bar{p}_2(t) = \sigma 2\Omega^2 \,\mathrm{csch}^2(\Omega t) \\ \bar{p}_3(t) = \sigma 2\Omega \coth(\Omega t). \end{cases}$$

(Here $\Omega = \sqrt{\frac{h_0}{2}}$.) We have

$$\begin{aligned}
\dot{\bar{p}}_1(t) &= 4\Omega^3 \operatorname{csch}^2(\Omega t)\coth(\Omega t) = \bar{p}_2(t)\bar{p}_3(t)\\
\dot{\bar{p}}_2(t) &= -4\Omega^3 \operatorname{csch}^2(\Omega t)\coth(\Omega t) = \bar{p}_2(t)\bar{p}_3(t)\\
\dot{p}_3(t) &= -\sigma 2\Omega^2 \operatorname{csch}^2(\Omega t) = -\bar{p}_2(t).
\end{aligned}$$

That is, $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$. Finally, notice that $\sinh(\Omega t) = 0$ when $t = 0$. Thus $\bar{p}(\cdot)$ is only defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.8 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0 > 0$, $C(p(0)) = 0$ and $p_1(0) < 0$.*

(*i*) *There exist $t_0 \in \mathbb{R}\backslash\{0\}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{p}_\varsigma(t+t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$, $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$ and*

$$\begin{cases} \bar{p}_1(t) = -2\Omega^2 \operatorname{csch}^2(\Omega t)\\ \bar{p}_2(t) = \sigma 2\Omega^2 \operatorname{csch}^2(\Omega t)\\ \bar{p}_3(t) = \sigma 2\Omega \coth(\Omega t).\end{cases}$$

Here $\Omega = \sqrt{\frac{h_0}{2}}$.

(*ii*) *$t \mapsto \bar{p}_\pm(t + t_0)$ is the unique maximal integral curve starting at $\bar{p}_\pm(t_0)$.*

Proof. (*i*) Let $\sigma = \operatorname{sgn}(p_2(0)) \in \{-1, 1\}$ and $\varsigma = \sigma \operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_2(0) = 0$ or $p_3(0) = 0$, then $p_1(0) = 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t)) = \sigma$. Moreover,

$$\lim_{t\to 0} \bar{p}_{+,2}(t) = \sigma\infty, \ \lim_{t\to\infty} \bar{p}_{+,2}(t) = 0 \qquad \text{and} \qquad \lim_{t\to-\infty} \bar{p}_{-,2}(t) = 0, \ \lim_{t\to 0} \bar{p}_{-,2}(t) = \sigma\infty.$$

Suppose $\operatorname{sgn}(p_3(0)) = 1$, so we have $\varsigma = \sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = -1\\ (0, \infty) & \text{if } \sigma = +1\end{cases}$$

such that $\bar{p}_{\sigma,2}(t_0) = p_2(0)$. Then

$$\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 + c_0 = p_2(0)^2 + c_0 = p_1(0)^2.$$

We have $p_1(0), \bar{p}_{\sigma,1}(t_0) < 0$, and so $\bar{p}_{\sigma,1}(t_0) = p_1(0)$. Moreover,

$$\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\sigma,3}(t_0)) = 1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

Suppose $\operatorname{sgn}(p_3(0)) = -1$, so $\varsigma = -\sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = +1 \\ (0, \infty) & \text{if } \sigma = -1 \end{cases}$$

such that $\bar{p}_{-\sigma,2}(t_0) = p_2(0)$. From $\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 = p_2(0)^2 = p_1(0)^2$ we again get $\bar{p}_{-\sigma,1}(t_0) = p_1(0)$. Similarly, as $\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2$ and $\operatorname{sgn}(\bar{p}_{-\sigma,3}(t_0)) = -1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{-\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore $t \mapsto \bar{p}_\varsigma(t + t_0)$ and $t \mapsto p(t)$ both solve the same Cauchy problem, and so are identical.

(ii) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_\pm(t + t_0)$. Suppose $p(0) = \bar{p}_-(t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By *(i)*, $p(s) = \bar{p}_-(s + t_0)$, and so $s + t_0 \in (-\infty, 0)$. Consequently, $s \in (-\infty, -t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-\infty, -t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}_-(t + t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}_-(t + t_0)$.

Suppose $p(0) = \bar{p}_+(t_0)$. Similar to before, if $s \in (-\varepsilon, \varepsilon)$, then $s + t_0 \in (0, \infty)$, *i.e.*, $s \in (-t_0, \infty)$. Thus $(-\varepsilon, \varepsilon) \subseteq (-t_0, \infty)$, and so the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_+(t + t_0)$. The uniqueness of $t \mapsto \bar{p}_+(t + t_0)$ follows from its maximality. ■

Consider the case $\boldsymbol{\bar{p}_1(t) > 0}$. Limiting $c_0 \to 0$ in the integral curve of proposition 4.6.3 yields a constant integral curve. Thus we shall integrate the equations of motion.

We have $\bar{p}_1(t)^2 = \bar{p}_2(t)^2$, and so $\bar{p}_1(t) = \sigma_1 \bar{p}_2(t)$ for some $\sigma_1 \in \{-1, 1\}$. Using $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$, we get the differential equation

$$\dot{\bar{p}}_1 = \sigma_1 \bar{p}_1 \bar{p}_3 = \sigma_1 \sqrt{2}\, \bar{p}_1 \sqrt{h_0 - \bar{p}_1}\,.$$

Separating variables yields

$$\frac{d\bar{p}_1}{\bar{p}_1 \sqrt{h_0 - \bar{p}_1}} = \sigma_1 \sqrt{2}\, dt.$$

Make the change of variables $u = \sqrt{h_0 - \bar{p}_1}$. Then $d\bar{p}_1 = -2u\, du$ and

$$\frac{2\, du}{u^2 - h_0} = \sigma_1 \sqrt{2}\, dt.$$

As $\bar{p}_1(t) > 0$, we have $h_0 > h_0 - \bar{p}_1(t) = u^2$. The integral formula (A.6.13) gives

$$-\frac{2}{\sqrt{h_0}} \tanh^{-1}\left(\frac{u}{\sqrt{h_0}}\right) = \sigma_1 \sqrt{2}\, t.$$

Substituting for u and solving for $\bar{p}_1(\cdot)$, we get $\bar{p}_1(t) = h_0 \operatorname{sech}^2(\Omega t)$, where $\Omega = \sqrt{\frac{h_0}{2}}$. Then $\bar{p}_2(t) = \sigma_1 \bar{p}_1(t) = \sigma_1 h_0 \operatorname{sech}^2(\Omega t)$. Using the constant of motion $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$, we have

$$\bar{p}_3(t)^2 = 2(h_0 - \bar{p}_1(t)) = 2h_0 \tanh^2(\Omega t).$$

Thus $\bar{p}_3(t) = \sigma_2 2\Omega \tanh(\Omega t)$ for some $\sigma_2 \in \{-1, 1\}$. Hence we have the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = 2\Omega^2 \operatorname{sech}^2(\Omega t) \\ \bar{p}_2(t) = -\sigma_1 2\Omega^2 \operatorname{sech}^2(\Omega t) \\ \bar{p}_3(t) = \sigma_2 2\Omega \tanh(\Omega t)\,. \end{cases}$$

We verify that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$ if and only if $\sigma_1 = \sigma_2$. Since we know $\bar{p}(\cdot)$ satisfies the equations of the constants of motion, it suffices to show that $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_1 = \sigma_2$. (See proposition 4.1.2.) We have

$$\begin{aligned}\dot{\bar{p}}_1(t) - \bar{p}_2(t)\bar{p}_3(t) &= -4\Omega^3 \operatorname{sech}^2(\Omega t)\tanh(\Omega t) + \sigma_1\sigma_2 4\Omega^2 \operatorname{sech}^2(\Omega t)\tanh(\Omega t)\\ &= (\sigma_1\sigma_2 - 1)4\Omega^3 \operatorname{sech}^2(\Omega t)\tanh(\Omega t).\end{aligned}$$

Thus $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_1\sigma_2 = 1$, *i.e.*, $\sigma_1 = \sigma_2$. In this case, we have $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.9 PROPOSITION. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0 > 0$, $C(p(0)) = 0$ and $p_1(0) > 0$. There exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases}\bar{p}_1(t) = 2\Omega^2 \operatorname{sech}^2(\Omega t)\\ \bar{p}_2(t) = -\sigma 2\Omega^2 \operatorname{sech}^2(\Omega t)\\ \bar{p}_3(t) = \sigma 2\Omega \tanh(\Omega t)\,.\end{cases}$$

Here $\Omega = \sqrt{\frac{h_0}{2}}$.

PROOF. Let $\sigma = -\operatorname{sgn}(p_2(0)) \in \{-1, 1\}$. (If $p_2(0) = 0$, then $p_1(0) = 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_3|_{(0,\infty)}(t)) = -\operatorname{sgn}(\bar{p}_3|_{(-\infty,0)}(t)) = \sigma$. Moreover, $\lim_{t\to\infty} \bar{p}_3(t) = \sigma\infty$ and $\lim_{t\to-\infty} \bar{p}_3(t) = -\sigma\infty$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}p_3(0)^2 = p_1(0).$$

Furthermore, $\bar{p}_2(t_0)^2 = \bar{p}_1(t_0)^2 = p_1(0)^2 = p_2(0)^2$. Since $\operatorname{sgn}(\bar{p}_2(t_0)) = -\sigma = \operatorname{sgn}(p_2(0))$, we have $\bar{p}_2(t_0) = p_2(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ■

4.6.1.2.2 Case *II-b*: $h_0 = 0$. We have $\bar{p}_1(t) = h_0 - \frac{1}{2}\bar{p}_3(t)^2 \leq h_0 = 0$. If $\bar{p}_1(t) = 0$ for some t, then $\bar{p}(\cdot)$ is constant. (Indeed, we have $\bar{p}(t) = (0, 0, 0)$, which is an equilibrium point of $\vec{H}_1^{(3)}$.) Assume $\bar{p}_1(t) < 0$.

Let $\bar{p}(\cdot)$ be the limit $h_0 \to 0$ of the integral curve in proposition 4.6.8. Then

$$\begin{cases}\bar{p}_1(t) = -\dfrac{2}{t^2}\\ \bar{p}_2(t) = \sigma\dfrac{2}{t^2}\\ \bar{p}_3(t) = \sigma\dfrac{2}{t}.\end{cases}$$

We have $\dot{\bar{p}}_1(t) = \frac{4}{t^3} = \bar{p}_2(t)\bar{p}_3(t)$, $\dot{\bar{p}}_2(t) = -\sigma\frac{4}{t^3} = \bar{p}_1(t)\bar{p}_3(t)$ and $\dot{\bar{p}}_3(t) = -\sigma\frac{2}{t^2} = -\bar{p}_2(t)$. That is, $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$. Finally, notice that $\bar{p}(\cdot)$ is only defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.10 PROPOSITION. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = C(p(0)) = 0$ and $p_1(0) < 0$.*

(*i*) *There exist $t_0 \in \mathbb{R}\backslash\{0\}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{p}_\varsigma(t+t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$, $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$ and*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{2}{t^2} \\ \bar{p}_2(t) = \sigma\dfrac{2}{t^2} \\ \bar{p}_3(t) = \sigma\dfrac{2}{t}. \end{cases}$$

(*ii*) *$t \mapsto \bar{p}_\pm(t + t_0)$ is the unique maximal integral curve starting at $\bar{p}_\pm(t_0)$.*

PROOF. (*i*) Let $\sigma = \operatorname{sgn}(p_2(0)) \in \{-1, 1\}$ and $\varsigma = \sigma \operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_2(0) = 0$ or $p_3(0) = 0$, then $p_1(0) = 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t)) = \sigma$. Moreover,

$$\lim_{t\to 0} \bar{p}_{+,2}(t) = \sigma\infty, \; \lim_{t\to\infty} \bar{p}_{+,2}(t) = 0 \qquad \text{and} \qquad \lim_{t\to-\infty} \bar{p}_{-,2}(t) = 0, \; \lim_{t\to 0} \bar{p}_{-,2}(t) = \sigma\infty.$$

Suppose $\operatorname{sgn}(p_3(0)) = 1$, so we have $\varsigma = \sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = -1 \\ (0, \infty) & \text{if } \sigma = +1 \end{cases}$$

such that $\bar{p}_{\sigma,2}(t_0) = p_2(0)$. Then $\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 = p_2(0)^2 = p_1(0)^2$. We have $p_1(0) < 0$ and $\bar{p}_{\sigma,1}(t_0) < 0$, hence $\bar{p}_{\sigma,1}(t_0) = p_1(0)$. Moreover,

$$\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\sigma,3}(t_0)) = 1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

Suppose $\operatorname{sgn}(p_3(0)) = -1$, so $\varsigma = -\sigma$. Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous and $\operatorname{sgn}(p_2(0)) = \operatorname{sgn}(\bar{p}_{-,2}(t)) = \operatorname{sgn}(\bar{p}_{+,2}(t))$, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \sigma = +1 \\ (0, \infty) & \text{if } \sigma = -1 \end{cases}$$

such that $\bar{p}_{-\sigma,2}(t_0) = p_2(0)$. From $\bar{p}_{\sigma,1}(t_0)^2 = \bar{p}_{\sigma,2}(t_0)^2 = p_2(0)^2 = p_1(0)^2$ we again get $\bar{p}_{-\sigma,1}(t_0) = p_1(0)$. Similarly, as $\bar{p}_{\sigma,3}(t_0)^2 = 2h_0 - 2\bar{p}_{\sigma,1}(t_0) = 2h_0 - 2p_1(0) = p_3(0)^2$ and $\operatorname{sgn}(\bar{p}_{-\sigma,3}(t_0)) = -1 = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{-\sigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore $t \mapsto \bar{p}_\varsigma(t + t_0)$ and $t \mapsto p(t)$ both solve the same Cauchy problem, and so are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_\pm(t + t_0)$. Suppose $p(0) = \bar{p}_-(t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}_-(s + t_0)$, and so $s + t_0 \in (-\infty, 0)$. Consequently, $s \in (-\infty, -t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-\infty, -t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}_-(t + t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}_-(t + t_0)$.

Suppose $p(0) = \bar{p}_+(t_0)$. Similar to before, if $s \in (-\varepsilon, \varepsilon)$, then $s + t_0 \in (0, \infty)$, *i.e.*, $s \in (-t_0, \infty)$. Thus $(-\varepsilon, \varepsilon) \subseteq (-t_0, \infty)$, and so the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_+(t + t_0)$. The uniqueness of $t \mapsto \bar{p}_+(t + t_0)$ follows from its maximality. ∎

4.6.1.2.3 Case *II-c*: $h_0 < 0$. Consider the integral curve of proposition 4.6.7. Let $\bar{p}(\cdot)$ be the curve obtained by taking the limit $c_0 \to 0$ of this integral curve. Then

$$\begin{cases} \bar{p}_1(t) = -2\Omega^2 \sec^2(\Omega t) \\ \bar{p}_2(t) = -\sigma 2\Omega^2 \sec^2(\Omega t) \\ \bar{p}_3(t) = \sigma 2\Omega \tan(\Omega t). \end{cases}$$

(Here $\Omega = \sqrt{-\frac{h_0}{2}}$.) We have

$$\begin{aligned} \dot{\bar{p}}_1(t) &= -4\Omega^3 \sec^2(\Omega t)\tan(\Omega t) = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= -\sigma 4\Omega^3 \sec^2(\Omega t)\tan(\Omega t) = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= \sigma 2\Omega^2 \sec^2(\Omega t) = -\bar{p}_2(t). \end{aligned}$$

That is, $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$. Finally, since $\cos(\Omega t) = 0$ at the points $\frac{\pi}{\Omega}(2n - \frac{1}{2})$ and $\frac{\pi}{\Omega}(2n + \frac{1}{2})$, $n \in \mathbb{Z}$, it follows that $\bar{p}(\cdot)$ is only defined on the open intervals $\left(\frac{\pi}{\Omega}(2n - \frac{1}{2}), \frac{\pi}{\Omega}(2n + \frac{1}{2})\right)$, $n \in \mathbb{Z}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.11 PROPOSITION. *Let* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ *be an integral curve of* $\vec{H}_1^{(3)}$ *such that* $H_1^{(3)}(p(0)) = h_0 < 0$ *and* $C(p(0)) = 0$.

(*i*) *There exist* $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for every* $t \in (-\varepsilon, \varepsilon)$, *where* $\bar{p}(\cdot) : (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}) \to \mathfrak{se}(1,1)^*$ *is defined by*

$$\begin{cases} \bar{p}_1(t) = -2\Omega^2 \sec^2(\Omega t) \\ \bar{p}_2(t) = -\sigma 2\Omega^2 \sec^2(\Omega t) \\ \bar{p}_3(t) = \sigma 2\Omega \tan(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{-\frac{h_0}{2}}$.

(*ii*) $t \mapsto \bar{p}(t + t_0)$ *is the unique maximal integral curve starting at* $\bar{p}(t_0)$.

PROOF. (*i*) Let $\sigma = -\operatorname{sgn}(p_2(0)) \in \{-1, 1\}$. (If $p_2(0) = 0$, then $p_1(0) = 0$, and so $p_3(0)^2 = 2h_0 < 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_3|_{(-\pi/(2\Omega),0)}(t)) = -\sigma$ and $\operatorname{sgn}(\bar{p}_3|_{(0,\pi/(2\Omega))}(t)) = \sigma$. Furthermore, $\lim_{t\to-\pi/(2\Omega)} \bar{p}_3(t) = -\sigma\infty$ and $\lim_{t\to\pi/(2\Omega)} \bar{p}_3(t) = \sigma\infty$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}\bar{p}_3(0)^2 = p_1(0).$$

Lastly, we have $\bar{p}_2(t_0)^2 = \bar{p}_1(t_0)^2 = p_1(0)^2 = p_2(0)^2$. As $\operatorname{sgn}(\bar{p}_2(t_0)) = -\sigma = \operatorname{sgn}(p_2(0))$, it follows that $\bar{p}_2(t_0) = p_2(0)$. Therefore, since $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t+t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}(s+t_0)$, and so $s+t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$. Consequently, $s \in (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} + t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} + t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t + t_0)$. ∎

4.6.1.3 Case III: $c_0 < 0$

From the equation of motion $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ and the constants of motion $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)$ and $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$, we get

$$\dot{\bar{p}}_1^2 = (\bar{p}_2\bar{p}_3)^2 = (\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1). \tag{4.6.8}$$

This equation must be reduced to standard form before we can integrate (see section A.6.2 for a discussion of this process). To that end, let $X_1 = \bar{p}_1^2 - c_0$ and $X_2 = 2h_0 - 2\bar{p}_1$. Then $X_1 - \lambda X_2$ is a perfect square for $\lambda_1 = -(\delta + h_0)$ and $\lambda_2 = \delta - h_0$, where $\delta = \sqrt{h_0^2 - c_0}$. (As $c_0 < 0 \leq h_0^2$, we have $\delta, \lambda_1, \lambda_2 \in \mathbb{R}$.) Accordingly, $X_1 - \lambda_1 X_2 = (\bar{p}_1 + \lambda_1)^2$ and $X_1 - \lambda_2 X_2 = (\bar{p}_1 + \lambda_2)^2$. Thus we have

$$X_1X_2 = \left[A_1(\bar{p}_1 + \lambda_1)^2 + B_1(\bar{p}_1 + \lambda_2)^2\right]\left[A_2(\bar{p}_1 + \lambda_1)^2 + B_2(\bar{p}_1 + \lambda_2)^2\right]$$

where A_1, A_2, B_1 and B_2 are given by

$$A_1 = \frac{1}{2}\left(1 - \frac{h_0}{\delta}\right), \qquad B_1 = \frac{1}{2}\left(1 + \frac{h_0}{\delta}\right), \qquad A_2 = \frac{1}{2\delta}, \qquad B_2 = -\frac{1}{2\delta}.$$

Taking the square root of both sides in (4.6.8) and separating variables, we have

$$\frac{d\bar{p}_1}{\sqrt{(\bar{p}_1^2 - c_0)(2h_0 - 2\bar{p}_1)}} = \frac{d\bar{p}_1}{\sqrt{X_1X_2}} = \sigma_1 dt$$

for some $\sigma_1 \in \{-1, 1\}$. By the calculations above, this is

$$\frac{d\bar{p}_1}{\sqrt{[A_1(\bar{p}_1 + \lambda_1)^2 + B_1(\bar{p}_1 + \lambda_2)^2]\,[A_2(\bar{p}_1 + \lambda_1)^2 + B_2(\bar{p}_1 + \lambda_2)^2]}} = \sigma_1 dt. \tag{4.6.9}$$

We have $(\delta + h_0)(\delta - h_0) = -c_0 > 0$. We claim that $\delta > h_0$. Suppose otherwise, *i.e.*, $\delta < h_0$. (We cannot have $h_0 = \delta$ as $c_0 < 0$.) Then $\delta + h_0 < 0$, and hence $2\delta = (\delta + h_0) + (\delta - h_0) < 0$, a contradiction. Thus $\delta > h_0$, and so $A_1A_2 = \frac{\delta - h_0}{2\delta^2} > 0$. Make the change of variables $u = \frac{\bar{p}_1 + \lambda_1}{\bar{p}_1 + \lambda_2}$ in (4.6.9). The result is

$$\frac{du}{\sqrt{\left(u^2 + \frac{B_1}{A_1}\right)\left(u^2 - 1\right)}} = \sigma_1(\lambda_2 - \lambda_1)\sqrt{A_1A_2}\,dt = \sigma_1\sqrt{\delta - h_0}\,dt$$

where $\frac{B_1}{A_1} = \frac{\delta + h_0}{\delta - h_0} > 0$. Let $a = -1$ and $b = \sqrt{\frac{B_1}{A_1}} = \sqrt{\frac{\delta + h_0}{\delta - h_0}}$. Then we have

$$\frac{du}{\sqrt{(u^2 - a^2)(u^2 + b^2)}} = \sigma_1\sqrt{\delta - h_0}\,dt.$$

Let $x = -\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2}$ and integrate both sides. We get

$$-\int_{-a}^{-x} \frac{du}{\sqrt{(u^2 - a^2)(u^2 + b^2)}} = \int_{a}^{x} \frac{du}{\sqrt{(u^2 - a^2)(u^2 + b^2)}} = \sigma_1\sqrt{\delta - h_0}\,t.$$

Using the integral formula (A.6.10) to integrate the left-hand side, we have (for $-x \geq -a$),

$$\begin{aligned} &\frac{1}{\sqrt{a^2 + b^2}}\,\mathrm{nc}^{-1}\left(-\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2}, \frac{b}{\sqrt{a^2 + b^2}}\right) = -\sigma_1\sqrt{\delta - h_0}\,t \\ \Rightarrow\quad &\frac{\bar{p}_1(t) + \lambda_1}{\bar{p}_1(t) + \lambda_2} = -\,\mathrm{nc}\left(\sqrt{a^2 + b^2}\,\sqrt{\delta - h_0}\,t, \frac{b}{\sqrt{a^2 + b^2}}\right). \end{aligned}$$

(As $\mathrm{nc}(\cdot, k)$ is even, we can eliminate the $-\sigma_1$.) Let $\Omega = \sqrt{a^2+b^2}\sqrt{\delta - h_0} = \sqrt{2\delta}$ and $k = \frac{b}{\sqrt{a^2+b^2}} = \sqrt{\frac{\delta+h_0}{2\delta}}$, $k' = \sqrt{\frac{\delta-h_0}{2\delta}}$. (As $\delta > h_0$ we have $0 < k < 1$.) Solving for $\bar{p}_1(t)$, we get

$$\bar{p}_1(t) = \frac{(h_0 - \delta)\,\mathrm{nc}(\Omega t, k) + (h_0 + \delta)}{\mathrm{nc}(\Omega t, k) + 1} = \frac{(h_0 + \delta)\,\mathrm{cn}(\Omega t, k) + (h_0 - \delta)}{\mathrm{cn}(\Omega t, k) + 1}.$$

From the Casimir equation $\bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$, we have

$$\begin{aligned}
\bar{p}_2(t)^2 &= \bar{p}_1(t)^2 - c_0 \\
&= \left[\frac{(h_0 + \delta)\,\mathrm{cn}(\Omega t, k) + (h_0 - \delta)}{\mathrm{cn}(\Omega t, k) + 1}\right]^2 - c_0 \\
&= 2\delta \frac{(\delta + h_0)\,\mathrm{cn}^2(\Omega t, k) + (\delta - h_0)}{[\mathrm{cn}(\Omega t, k) + 1]^2} \\
&= (2\delta)^2 \frac{k^2\,\mathrm{cn}^2(\Omega t, k) + (k')^2}{[\mathrm{cn}(\Omega t, k) + 1]^2} \\
&= (2\delta)^2 \frac{\mathrm{dn}^2(\Omega t, k)}{[\mathrm{cn}(\Omega t, k) + 1]^2}.
\end{aligned}$$

(We have used the square relation (A.6.6) in the last step.) Taking the square root of both sides yields

$$\bar{p}_2(t) = \sigma_2 \Omega^2 \frac{\mathrm{dn}(\Omega t, k)}{\mathrm{cn}(\Omega t, k) + 1}$$

for some $\sigma_2 \in \{-1, 1\}$. Lastly we employ the constant of motion $h_0 = \bar{p}_1(t) + \frac{1}{2}\bar{p}_3(t)^2$ to find an expression for $\bar{p}_3(\cdot)$:

$$\begin{aligned}
\bar{p}_3(t)^2 &= 2h_0 - 2\bar{p}_1(t) \\
&= -2\delta \frac{\mathrm{cn}(\Omega t, k) - 1}{\mathrm{cn}(\Omega t, k) + 1} \cdot \frac{\mathrm{cn}(\Omega t, k) + 1}{\mathrm{cn}(\Omega t, k) + 1} \\
&= -2\delta \frac{\mathrm{cn}^2(\Omega t, k) - 1}{[\mathrm{cn}(\Omega t, k) + 1]^2} \\
&= 2\delta \frac{\mathrm{sn}^2(\Omega t, k)}{[\mathrm{cn}(\Omega t, k) + 1]^2}.
\end{aligned}$$

(We have used (A.6.5) for the final step.) Hence, taking the square root of both sides,

$$\bar{p}_3(t) = \sigma_3 \frac{\Omega\,\mathrm{sn}(\Omega t, k)}{\mathrm{cn}(\Omega t, k) + 1}$$

for some $\sigma_3 \in \{-1, 1\}$. Therefore we have the following (prospective) integral curve:

$$\begin{cases}
\bar{p}_1(t) = \dfrac{(h_0 + \delta)\,\mathrm{cn}(\Omega t, k) + (h_0 - \delta)}{\mathrm{cn}(\Omega t, k) + 1} \\[2ex]
\bar{p}_2(t) = \sigma_2 \Omega^2 \dfrac{\mathrm{dn}(\Omega t, k)}{\mathrm{cn}(\Omega t, k) + 1} \\[2ex]
\bar{p}_3(t) = \sigma_3 \dfrac{\Omega\,\mathrm{sn}(\Omega t, k)}{\mathrm{cn}(\Omega t, k) + 1}.
\end{cases}$$

We will show that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_1^{(3)}$ if and only if $\sigma_2 = -\sigma_3$. By proposition 4.1.2, it suffices to show that $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2 = -\sigma_3$ (since we know $\bar{p}(\cdot)$ satisfies the constants of motion). Indeed,

$$\begin{aligned}\dot{\bar{p}}_1(t) - \bar{p}_2(t)\bar{p}_3(t) &= -\frac{2\delta\Omega \operatorname{dn}(\Omega t, k)\operatorname{sn}(\Omega t, k)}{[\operatorname{cn}(\Omega t, k)+1]^2} - \frac{\sigma_2\sigma_3\Omega^3 \operatorname{dn}(\Omega t, k)\operatorname{sn}(\Omega t, k)}{[\operatorname{cn}(\Omega t, k)+1]^2}\\ &= -\frac{\Omega(\sigma_2\sigma_3\Omega^2 + 2\delta)\operatorname{dn}(\Omega t, k)\operatorname{sn}(\Omega t, k)}{[\operatorname{cn}(\Omega t, k)+1]^2}.\end{aligned}$$

Since $\Omega^2 = 2\delta$, we have $\dot{\bar{p}}_1 = \bar{p}_2\bar{p}_3$ if and only if $\sigma_2\sigma_3 = -1$, *i.e.*, $\sigma_2 = -\sigma_3$. In this case, $\dot{\bar{p}}(t) = \vec{H}_1^{(3)}(\bar{p}(t))$.

Finally, since $\operatorname{cn}(\Omega t, k) + 1 = 0$ at the points $\frac{4nK}{\Omega}$, $n \in \mathbb{Z}$ (where $4K$ denotes the period of $\operatorname{sn}(\cdot, k)$), it follows that $\bar{p}(\cdot)$ is only defined on the open intervals $\left(\frac{4nK}{\Omega}, \frac{4(n+1)K}{\Omega}\right)$, $n \in \mathbb{Z}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_1^{(3)}$ for this case.

4.6.12 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_1^{(3)}$ such that $H_1^{(3)}(p(0)) = h_0$ and $C(p(0)) = c_0 < 0$.*

(*i*) *There exist $t_0 \in \left(0, \frac{4K}{\Omega}\right)$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t+t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : (0, \frac{4K}{\Omega}) \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases}\bar{p}_1(t) = \dfrac{(h_0+\delta)\operatorname{cn}(\Omega t, k) + (h_0 - \delta)}{\operatorname{cn}(\Omega t, k)+1}\\ \bar{p}_2(t) = \sigma\Omega^2\dfrac{\operatorname{dn}(\Omega t, k)}{\operatorname{cn}(\Omega t, k)+1}\\ \bar{p}_3(t) = \sigma\dfrac{\Omega\operatorname{sn}(\Omega t, k)}{\operatorname{cn}(\Omega t, k)+1}.\end{cases}$$

Here $4K$ is the period of $\operatorname{sn}(\cdot, k)$, $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{2\delta}$ and $k = \sqrt{\frac{\delta + h_0}{2\delta}}$.

(*ii*) *$t \mapsto \bar{p}(t+t_0)$ is the unique maximal integral curve starting at $\bar{p}(t_0)$.*

Proof. (*i*) Let $\sigma = \operatorname{sgn}(p_2(0)) \in \{-1, 1\}$. (If $p_2(0) = 0$, then $p_1(0)^2 = c_0 < 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_3|_{(0,2K/\Omega)}(t)) = -\sigma$ and $\operatorname{sgn}(\bar{p}_3|_{(2K/\Omega,4K/\Omega)}(t)) = \sigma$. Moreover, $\lim_{t\to 0}\bar{p}_3(t) = -\sigma\infty$ and $\lim_{t\to 4K/\Omega}\bar{p}_3(t) = \sigma\infty$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \left(0, \frac{4K}{\Omega}\right)$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}p_3(0)^2 = p_1(0).$$

Lastly, we have

$$\bar{p}_2(t_0)^2 = \bar{p}_1(t_0)^2 - c_0 = p_1(0)^2 - c_0 = p_2(0)^2.$$

Since $\operatorname{sgn}(\bar{p}_2(t_0)) = \sigma = \operatorname{sgn}(p_2(0))$, it follows that $\bar{p}_2(t_0) = p_2(0)$. Therefore, as $t \mapsto \bar{p}(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_1^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t+t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}(s+t_0)$, and so $s + t_0 \in (0, \frac{4K}{\Omega})$. Consequently, $s \in (-t_0, \frac{4K}{\Omega} - t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-t_0, \frac{4K}{\Omega} - t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t+t_0)$. ∎

4.6.2 The system $H_2^{(3)}$

The equations of motion of the system $H_2^{(3)}(p) = p_1 + p_2 + \frac{1}{2}p_3^2$ are

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_1 p_3 \\ \dot{p}_3 = -(p_1 + p_2). \end{cases}$$

The equilibrium states of $\vec{H}_2^{(3)}$ are $\mathsf{e}_1^\mu = (\mu, -\mu, 0)$ and $\mathsf{e}_2^\nu = (0, 0, \nu)$, where $\nu, \mu \in \mathbb{R}$, $\nu \neq 0$.

4.6.13 Proposition. *The equilibrium states* e_1^μ *and* e_2^ν *are unstable.*

Proof. Consider the states e_1^μ, $\mu \neq 0$. Fix a bounded open neighbourhood U of e_1^μ. The curve $p(t) = (\mu e^{\delta t}, -\mu e^{\delta t}, -\delta)$ is an integral curve of $\vec{H}_2^{(3)}$ for any $\delta > 0$. Indeed, $\dot{p}_1 = \delta\mu e^{\delta t} = p_2 p_3$, $\dot{p}_2 = -\delta\mu e^{\delta t} = p_1 p_3$ and $\dot{p}_3 = 0 = -(p_1 + p_2)$. Furthermore, we have $\|p(0) - \mathsf{e}_1^\mu\| = \delta$. Accordingly, for any open neighbourhood $V \subseteq U$ of e_1^μ, there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + 2\mu^2 \lim_{t\to\infty}(e^{\delta t} - 1) = \infty$. Hence, there exists $t_1 > 0$ such that $p(t_1) \notin U$. Thus the states e_1^μ, $\mu \neq 0$ are unstable.

Consider the state e_1^0. Let U be a bounded open neighbourhood of e_1^0 and let $p(t) = (\delta e^{\delta t}, -\delta e^{\delta t}, -\delta)$. We have $\dot{p}_1 = \delta^2 e^{\delta t} = p_2 p_3$, $\dot{p}_2 = -\delta^2 e^{\delta t} = p_1 p_3$ and $\dot{p}_3 = 0 = -(p_1 + p_2)$, and so $p(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$ for any $\delta > 0$. Furthermore, $\|p(0) - \mathsf{e}_1^0\| = \sqrt{3}\,\delta$. Hence, for every open neighbourhood $V \subseteq U$ of e_1^0, there exists $\delta > 0$ such that $p(0) \in V$. However, $\lim_{t\to\infty} \|p(t)\|^2 = \delta^2 + 2\delta^2 \lim_{t\to\infty} e^{2\delta t} = \infty$, and so there exists $t_1 > 0$ such that $p(t_1) \notin U$. Therefore the state e_1^0 is unstable.

Consider the states e_2^ν. The linearisation of the vector field $\vec{H}_2^{(3)}$ is

$$\mathbf{D}\vec{H}_2^{(3)}(p) = \begin{bmatrix} 0 & p_3 & p_2 \\ p_3 & 0 & p_1 \\ 0 & -1 & 0 \end{bmatrix}.$$

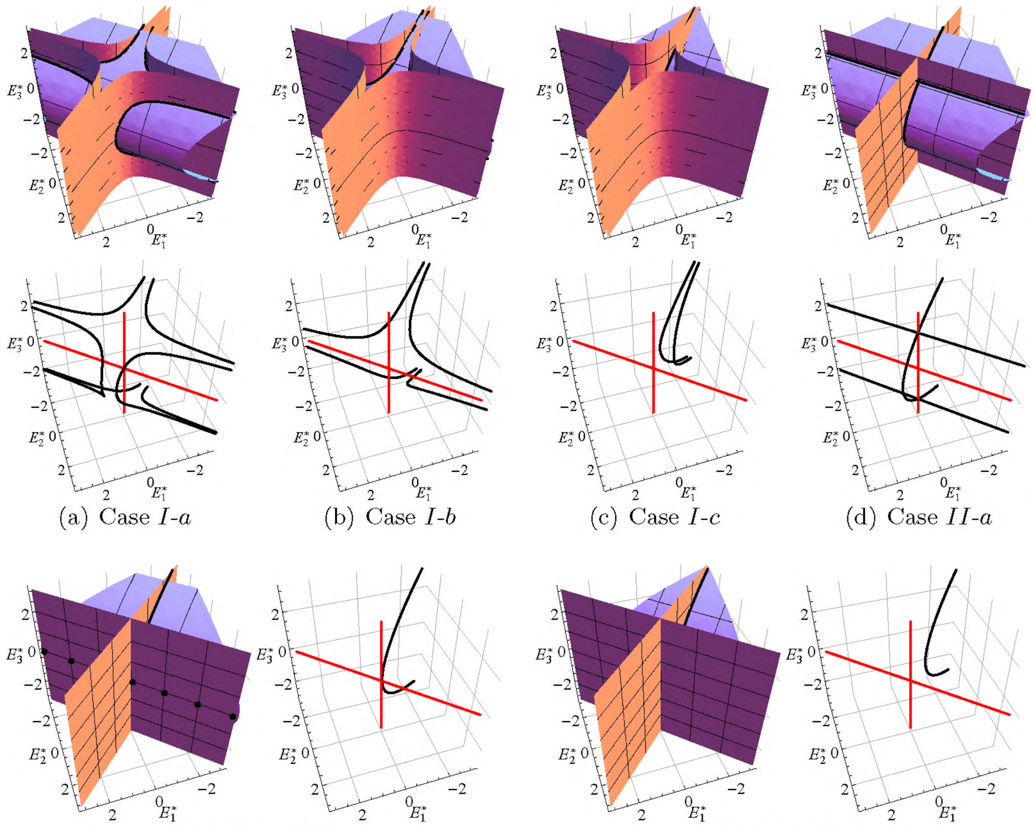
The linearisation at e_2^ν has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\nu$. Since $\nu \neq 0$, $\mathbf{D}\vec{H}_2^{(3)}$ has a positive real eigenvalue. Hence the states e_2^ν are (spectrally) unstable. ∎

We shall now find the integral curves of $\vec{H}_2^{(3)}$. Let $\bar{p}(\cdot)$ be an integral curve of $\vec{H}_2^{(3)}$ and let $c_0 = C(\bar{p}(0))$ and $h_0 = H_2^{(3)}(\bar{p}(0))$. We consider two main cases depending on whether c_0 is zero or not. In fact, the following proposition implies that we only need to consider the cases $c_0 > 0$ and $c_0 = 0$.

4.6.14 Proposition. *The map* $\Psi : (p_1, p_2, p_3) \mapsto (p_2, p_1, p_3)$ *is a linear Poisson symmetry of* $H_2^{(3)}$ *such that* $C \circ \Psi = -C$.

Proof. Ψ is clearly a linear Poisson automorphism. Furthermore, $(H_2^{(3)} \circ \Psi)(p) = p_2 + p_1 + \frac{1}{2}p_3^2 = H_2^{(3)}(p)$. Therefore Ψ is a linear Poisson symmetry of $H_2^{(3)}$. Finally, we have $(C \circ \Psi)(p) = p_2^2 - p_1^2 = -C(p)$. ∎

Accordingly, in order to find the integral curves of $\vec{H}_2^{(3)}$ when $c_0 < 0$, we find the integral curves when $c_0 > 0$ and apply the linear Poisson symmetry Ψ. In particular, if $\bar{q}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$ such that $C(\bar{q}(0)) < 0$, then $\bar{p}(\cdot) = \Psi(\bar{q}(\cdot))$ is an integral curve of $\vec{H}_2^{(3)}$ such that $C(\bar{p}(0)) > 0$.



(a) Case I-a (b) Case I-b (c) Case I-c (d) Case II-a

(e) Case II-b (f) Case II-c

Figure 4.8: Typical configurations of $H_2^{(3)}$

For each case $c_0 > 0$ and $c_0 = 0$, we have several further subcases. (Table 4.2 lists the qualitative breakdown of cases; figure 4.8 plots typical configurations of the system corresponding to these cases. The configurations for $c_0 < 0$ are also plotted along with those for $c_0 > 0$.) The level sets $\{p : C(p) = c_0\}$ (for $c_0 = 0$ or $c_0 > 0$) and $\{p : H_2^{(3)}(p) = h_0\}$ will be tangent to each other exactly when $\{p : H_2^{(3)}(p) = h_0\}$ is tangent to the plane $\{p : p_1 + p_2 = 0\}$. This occurs when the gradients of $H_2^{(3)}$ and $P(p) = p_1 + p_2$ are parallel, *i.e.*,

$$\nabla H_2^{(3)}(p) = \lambda \nabla P(p) \qquad \Longleftrightarrow \qquad \begin{bmatrix} 1 \\ 1 \\ p_3 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ 0 \end{bmatrix}.$$

(Here $\lambda \in \mathbb{R}$.) We see that $\lambda = 1$ and $p_3 = 0$. Consequently, $h_0 = p_1 + p_2 = 0$. This motivates the (sub)cases $h_0 > 0$, $h_0 = 0$ and $h_0 < 0$.

4.6.2.1 Case I: $c_0 \neq 0$

By proposition 4.6.14, we may assume $c_0 > 0$.

4.6.2.1.1 Case I-a: $h_0 > 0$. Since $\bar{p}_1(t)^2 \geq \bar{p}_1(t)^2 - \bar{p}_2(t)^2 = c_0$, we have either $\bar{p}_1(t) \leq -\sqrt{c_0}$ or $\bar{p}_1(t) \geq \sqrt{c_0}$.

Conditions		Designation
$c_0 \neq 0$	$h_0 > 0$	Case I-a
	$h_0 = 0$	Case I-b
	$h_0 < 0$	Case I-c
$c_0 = 0$	$h_0 > 0$	Case II-a
	$h_0 = 0$	Case II-b
	$h_0 < 0$	Case II-c

Table 4.2: Qualitative breakdown of cases for $H_2^{(3)}$

Consider the case $\boldsymbol{\bar{p}_1(t) \leq -\sqrt{c_0}}$. From the equation of motion $\dot{\bar{p}}_3 = -(\bar{p}_1 + \bar{p}_2)$ and the constant of motion $h_0 = \bar{p}_1(t) + \bar{p}_2(t) + \frac{1}{2}\bar{p}_3(t)^2$, we get the differential equation $\dot{\bar{p}}_3 = \frac{1}{2}\bar{p}_1^2 - h_0$. That is,

$$\frac{d\bar{p}_3}{\bar{p}_3^2 - 2h_0} = \frac{1}{2}\, dt. \tag{4.6.10}$$

Now $(\bar{p}_1(t) - \bar{p}_2(t))(\bar{p}_1(t) + \bar{p}_2(t)) = c_0 > 0$. Therefore we have

$$[\bar{p}_1(t) - \bar{p}_2(t) > 0 \text{ and } \bar{p}_1(t) + \bar{p}_2(t) > 0] \qquad \text{or} \qquad [\bar{p}_1(t) - \bar{p}_2(t) < 0 \text{ and } \bar{p}_1(t) + \bar{p}_2(t) < 0].$$

In the former case, $\bar{p}_2(t) < \bar{p}_1(t) < 0$, whence $\bar{p}_1(t) + \bar{p}_2(t) < 0$, a contradiction. Therefore $\bar{p}_1(t) - \bar{p}_2(t) < 0$ and $\bar{p}_1(t) + \bar{p}_2(t) < 0$. Consequently,

$$\bar{p}_3(t)^2 = 2(h_0 - \bar{p}_1(t) - \bar{p}_2(t)) > 2(h_0 - 2\bar{p}_1(t)) > 2h_0.$$

We use the formula (A.6.12) to integrate (4.6.10). The result is

$$-\frac{1}{\sqrt{2h_0}} \coth^{-1}\left(\frac{\bar{p}_3(t)}{\sqrt{2h_0}}\right) = \frac{1}{2}t.$$

After rearranging, this gives $\bar{p}_3(t) = -2\Omega\coth(\Omega t)$, where $\Omega = \sqrt{\frac{h_0}{2}}$. Next, differentiate $\bar{p}_3(\cdot)$ to get $\bar{p}_1(t) + \bar{p}_2(t)$ and use the Casimir equation $c_0 = (\bar{p}_1(t) - \bar{p}_2(t))(\bar{p}_1 + \bar{p}_2(t))$ to solve for $\bar{p}_1(t) - \bar{p}_2(t)$. The result is the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} = \begin{bmatrix} -2\Omega^2 \operatorname{csch}^2(\Omega t) \\ -\frac{c_0}{2\Omega^2} \sinh^2(\Omega t) \end{bmatrix}.$$

As $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, we can solve for $\bar{p}_1(\cdot)$ and $\bar{p}_2(\cdot)$. Thus we have the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = -\dfrac{1}{4\Omega^2}\left[4\Omega^4 \operatorname{csch}^2(\Omega t) + c_0 \sinh^2(\Omega t)\right] \\ \bar{p}_2(t) = -\dfrac{1}{4\Omega^2}\left[4\Omega^4 \operatorname{csch}^2(\Omega t) - c_0 \sinh^2(\Omega t)\right] \\ \bar{p}_3(t) = -2\Omega \coth(\Omega t). \end{cases}$$

We confirm that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Indeed,

$$\begin{aligned}
\dot{\bar{p}}_1(t) &= -\frac{c_0}{4\Omega}\sinh(2\Omega t) + 2\Omega^3\operatorname{csch}^2(\Omega t)\coth(\Omega t) = \bar{p}_2(t)\bar{p}_3(t) \\
\dot{\bar{p}}_2(t) &= \frac{c_0}{4\Omega}\sinh(2\Omega t) + 2\Omega^3\operatorname{csch}^2(\Omega t)\coth(\Omega t) = \bar{p}_1(t)\bar{p}_3(t) \\
\dot{\bar{p}}_3(t) &= 2\Omega^2\operatorname{csch}^2(\Omega t) = -(\bar{p}_1(t) + \bar{p}_2(t)).
\end{aligned}$$

Thus $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$. Finally, notice that $\sinh(\Omega t) = 0$ when $t = 0$. Thus $\bar{p}(\cdot)$ is only defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.15 PROPOSITION. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$ and $p_1(0) \leq -\sqrt{c_0}$.*

(*i*) *There exist $t_0 \in \mathbb{R}\setminus\{0\}$ and $\varsigma \in \{-1, 1\}$ such that $p(t) = \bar{p}_\varsigma(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$, $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$ and*

$$\begin{cases}
\bar{p}_1(t) = -\frac{1}{4\Omega^2}\left[4\Omega^4\operatorname{csch}^2(\Omega t) + c_0\sinh^2(\Omega t)\right] \\
\bar{p}_2(t) = -\frac{1}{4\Omega^2}\left[4\Omega^4\operatorname{csch}^2(\Omega t) - c_0\sinh^2(\Omega t)\right] \\
\bar{p}_3(t) = -2\Omega\coth(\Omega t).
\end{cases}$$

Here $\Omega = \sqrt{\frac{h_0}{2}}$.

(*ii*) *$t \mapsto \bar{p}_\pm(t + t_0)$ is the unique maximal integral curve starting at $\bar{p}_\pm(t_0)$.*

PROOF. (*i*) Let $\varsigma = -\operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_3(0) = 0$, then $p_1(0) + p_2(0) = h_0 > 0$ and $p_1(0) - p_2(0) > 0$, as $c_0 > 0$. Then $p_1(0) = \frac{1}{2}\left[\big(p_1(0) + p_2(0)\big) + \big(p_1(0) - p_2(0)\big)\right] > 0$, a contradiction.) We have

$$\lim_{t\to 0}\bar{p}_{+,2}(t) = -\infty, \ \lim_{t\to\infty}\bar{p}_{+,2}(t) = \infty \qquad \text{and} \qquad \lim_{t\to 0}\bar{p}_{-,2}(t) = -\infty, \ \lim_{t\to-\infty}\bar{p}_{-,2}(t) = \infty.$$

Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous, there exists

$$t_0 \in \begin{cases} (-\infty, 0) & \text{if } \varsigma = -1 \\ (0, \infty) & \text{if } \varsigma = +1 \end{cases}$$

such that $\bar{p}_{\varsigma,2}(t_0) = p_2(0)$. Then

$$\bar{p}_{\varsigma,1}(t_0)^2 = \bar{p}_{\varsigma,2}(t_0)^2 + c_0 = p_2(0)^2 + c_0 = p_1(0)^2.$$

We have $\bar{p}_{\varsigma,1}(t_0), p_1(0) \leq -\sqrt{c_0} < 0$, and so $\bar{p}_{\varsigma,1}(t_0) = p_1(0)$. Finally,

$$\bar{p}_{\varsigma,3}(t_0)^2 = 2(h_0 - \bar{p}_{\varsigma,1}(t_0) - \bar{p}_{\varsigma,2}(t_0)) = 2(h_0 - p_1(0) - p_2(0)) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\varsigma,3}(t_0)) = -\varsigma = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\varsigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(ii) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_{\pm}(t+t_0)$. Suppose $p(0) = \bar{p}_{-}(t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By *(i)*, $p(s) = \bar{p}_{-}(s+t_0)$, and so $s + t_0 \in (-\infty, 0)$. Consequently, $s \in (-\infty, -t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-\infty, -t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}_{-}(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}_{-}(t+t_0)$.

Suppose $p(0) = \bar{p}_{+}(t_0)$. Similar to before, if $s \in (-\varepsilon, \varepsilon)$, then $s + t_0 \in (0, \infty)$, *i.e.*, $s \in (-t_0, \infty)$. Thus $(-\varepsilon, \varepsilon) \subseteq (-t_0, \infty)$, and so the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_{+}(t+t_0)$. The uniqueness of $t \mapsto \bar{p}_{+}(t+t_0)$ follows from its maximality. ■

Consider the case $\boldsymbol{\bar{p}_1(t) \geq \sqrt{c_0}}$. Again from $\dot{\bar{p}}_3 = -(\bar{p}_1 + \bar{p}_2)$ and $h_0 = \bar{p}_1(t) + \bar{p}_2(t) + \frac{1}{2}\bar{p}_3(t)^2$ we get the differential equation

$$\frac{d\bar{p}_3}{\bar{p}_3^2 - 2h_0} = \frac{1}{2}\, dt. \tag{4.6.11}$$

Since $c_0 > 0$, we have

$$[\bar{p}_1(t) - \bar{p}_2(t) > 0 \text{ and } \bar{p}_1(t) + \bar{p}_2(t) > 0] \qquad \text{or} \qquad [\bar{p}_1(t) - \bar{p}_2(t) < 0 \text{ and } \bar{p}_1(t) + \bar{p}_2(t) < 0].$$

In the second case $0 < \bar{p}_1(t) < \bar{p}_2(t)$, and so $\bar{p}_1(t) + \bar{p}_2(t) > 0$, a contradiction. Thus $\bar{p}_1(t) - \bar{p}_2(t) > 0$ and $\bar{p}_1(t) + \bar{p}_2(t) > 0$. As a result

$$\bar{p}_3(t)^2 = 2(h_0 - \bar{p}_1(t) - \bar{p}_2(t)) < 2(h_0 - 2\bar{p}_1(t)) < 2h_0.$$

Use the integral formula (A.6.13) to integrate (4.6.11). We get

$$-\frac{1}{\sqrt{2h_0}} \tanh^{-1}\left(\frac{\bar{p}_3(t)}{\sqrt{2h_0}}\right) = \frac{1}{2}t,$$

i.e., $\bar{p}_3(t) = -2\Omega \tanh(\Omega t)$, where $\Omega = \sqrt{\frac{h_0}{2}}$. By differentiating the expression for $\bar{p}_3(t)$ to get $\bar{p}_1(t) + \bar{p}_2(t)$ and using $c_0 = \bar{p}_1(t)^2 - \bar{p}_2(t)^2$ to solve for $\bar{p}_1(t) - \bar{p}_2(t)$, we get the following equation:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{p}_1(t) \\ \bar{p}_2(t) \end{bmatrix} = \begin{bmatrix} -2\Omega^2 \operatorname{csch}^2(\Omega t) \\ -\frac{c_0}{2\Omega^2} \sinh^2(\Omega t) \end{bmatrix}.$$

As $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, this equation can be solved for $\bar{p}_1(t)$ and $\bar{p}_2(t)$. Thus we have the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = \dfrac{1}{4\Omega^2}\left[4\Omega^4 \operatorname{sech}^2(\Omega t) + c_0 \cosh^2(\Omega t)\right] \\ \bar{p}_2(t) = \dfrac{1}{4\Omega^2}\left[4\Omega^4 \operatorname{sech}^2(\Omega t) - c_0 \cosh^2(\Omega t)\right] \\ \bar{p}_3(t) = -2\Omega \tanh(\Omega t). \end{cases}$$

We have

$$\begin{aligned} \dot{\bar{p}}_1(t) &= \frac{c_0}{4\Omega} \sinh(2\Omega t) - 2\Omega^3 \operatorname{sech}^2(\Omega t) \tanh(\Omega t) = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= -\frac{c_0}{4\Omega} \sinh(2\Omega t) - 2\Omega^3 \operatorname{sech}^2(\Omega t) \tanh(\Omega t) = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= -2\Omega^2 \operatorname{sech}^2(\Omega t) = -(\bar{p}_1(t) + \bar{p}_2(t)). \end{aligned}$$

Therefore $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$, *i.e.*, $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Furthermore, $\bar{p}(\cdot)$ is defined over $\mathbb{R}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.16 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$ and $p_1(0) \geq \sqrt{c_0}$. There exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases} \bar{p}_1(t) = \dfrac{1}{4\Omega^2}\left[4\Omega^4 \operatorname{sech}^2(\Omega t) + c_0 \cosh^2(\Omega t)\right] \\ \bar{p}_2(t) = \dfrac{1}{4\Omega^2}\left[4\Omega^4 \operatorname{sech}^2(\Omega t) - c_0 \cosh^2(\Omega t)\right] \\ \bar{p}_3(t) = -2\Omega \tanh(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{\frac{h_0}{2}}$.

Proof. We have $p_1(t) \geq \sqrt{c_0}$. Then from $p_1(t)^2 - p_2(t)^2 = c_0 > 0$, it follows that $p_2(t) < p_1(t)$. Thus $p_1(t) - p_2(t) > 0$, and so $p_1(t) + p_2(t) > 0$. Therefore $p_3(t)^2 = 2(h_0 - p_1(t) - p_2(t)) < 2h_0$, and so $-\sqrt{2h_0} < p_3(t) < \sqrt{2h_0}$. Moreover, $\lim_{t\to\infty} \bar{p}_3(t) = -\sqrt{2h_0}$ and $\lim_{t\to-\infty} \bar{p}_3(t) = \sqrt{2h_0}$. Since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) + \bar{p}_2(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}p_3(0)^2 = p_1(0) + p_2(0),$$

and from $\bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = c_0 = p_1(0)^2 - p_2(0)^2$, we get

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{p}_1(t_0) \\ \bar{p}_2(t_0) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, it follows that $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ■

4.6.2.1.2 Case *I-b*: $h_0 = 0$. Consider the integral curve of proposition 4.6.15. Taking the limit $h_0 \to 0$ yields

$$\begin{cases} \bar{p}_1(t) = -\dfrac{4 + c_0 t^4}{4t^2} \\ \bar{p}_2(t) = -\dfrac{4 - c_0 t^4}{4t^2} \\ \bar{p}_3(t) = -\dfrac{2}{t}. \end{cases}$$

We have $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$. Indeed, $\dot{\bar{p}}_1(t) = \frac{4 - c_0 t^4}{2t^3} = \bar{p}_2(t)\bar{p}_3(t)$, $\dot{\bar{p}}_2(t) = \frac{4 + c_0 t^4}{2t^3} = \bar{p}_1(t)\bar{p}_3(t)$ and $\dot{\bar{p}}_3(t) = \frac{2}{t^2} = -(\bar{p}_1(t) + \bar{p}_2(t))$. Thus $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Finally, notice that $\bar{p}(\cdot)$ is only defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.17 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = 0$ and $C(p(0)) = c_0 > 0$.*

(*i*) *There exist* $t_0 \in \mathbb{R}\setminus\{0\}$ *and* $\varsigma \in \{-1,1\}$ *such that* $p(t) = \bar{p}_\varsigma(t+t_0)$ *for every* $t \in (-\varepsilon,\varepsilon)$, *where* $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$, $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$ *and*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{4+c_0t^4}{4t^2} \\ \bar{p}_2(t) = -\dfrac{4-c_0t^4}{4t^2} \\ \bar{p}_3(t) = -\dfrac{2}{t}. \end{cases}$$

(*ii*) $t \mapsto \bar{p}_\pm(t+t_0)$ *is the unique maximal integral curve starting at* $\bar{p}_\pm(t_0)$.

PROOF. (*i*) Since $p_1(t)^2 \geq p_1(t)^2 - p_2(t)^2 = c_0$, we have $p_1(t) \leq -\sqrt{c_0}$ or $p_1(t) \geq \sqrt{c_0}$. If $p_1(t) \geq \sqrt{c_0}$, then $p_1(t)+p_2(t) > 0$, and so $h_0 = p_1(t)+p_2(t)+\frac{1}{2}p_3(t)^2 > 0$, a contradiction. Thus $p_1(t) \leq -\sqrt{c_0}$. Similarly, $\bar{p}_1(t) \leq -\sqrt{c_0}$.

Let $\varsigma = -\operatorname{sgn}(p_3(0)) \in \{-1,1\}$. (If $p_3(0) = 0$, then $p_1(0)+p_2(0) = h_0 = 0$, whence $c_0 = 0$, a contradiction.) We have

$$\lim_{t\to 0}\bar{p}_{+,2}(t) = -\infty,\ \lim_{t\to\infty}\bar{p}_{+,2}(t) = \infty \qquad \text{and} \qquad \lim_{t\to 0}\bar{p}_{-,2}(t) = -\infty,\ \lim_{t\to -\infty}\bar{p}_{-,2}(t) = \infty.$$

Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous, there exists

$$t_0 \in \begin{cases} (-\infty,0) & \text{if } \varsigma = -1 \\ (0,\infty) & \text{if } \varsigma = +1 \end{cases}$$

such that $\bar{p}_{\varsigma,2}(t_0) = p_2(0)$. Then

$$\bar{p}_{\varsigma,1}(t_0)^2 = \bar{p}_{\varsigma,2}(t_0)^2 + c_0 = p_2(0)^2 + c_0 = p_1(0)^2.$$

We have $\bar{p}_{\varsigma,1}(t_0), p_1(0) \leq -\sqrt{c_0} < 0$, and so $\bar{p}_{\varsigma,1}(t_0) = p_1(0)$. Finally,

$$\bar{p}_{\varsigma,3}(t_0)^2 = 2(h_0 - \bar{p}_{\varsigma,1}(t_0) - \bar{p}_{\varsigma,2}(t_0)) = 2(h_0 - p_1(0) - p_2(0)) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\varsigma,3}(t_0)) = -\varsigma = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\varsigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_\pm(t+t_0)$. Suppose $p(0) = \bar{p}_-(t_0)$. Let $s \in (-\varepsilon,\varepsilon)$. By (*i*), $p(s) = \bar{p}_-(s+t_0)$, and so $s+t_0 \in (-\infty,0)$. Consequently, $s \in (-\infty,-t_0)$, *i.e.*, $(-\varepsilon,\varepsilon) \subseteq (-\infty,-t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}_-(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}_-(t+t_0)$.

Suppose $p(0) = \bar{p}_+(t_0)$. Similar to before, if $s \in (-\varepsilon,\varepsilon)$, then $s+t_0 \in (0,\infty)$, *i.e.*, $s \in (-t_0,\infty)$. Thus $(-\varepsilon,\varepsilon) \subseteq (-t_0,\infty)$, and so the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_+(t+t_0)$. The uniqueness of $t \mapsto \bar{p}_+(t+t_0)$ follows from its maximality. ■

4.6.2.1.3 Case *I-c*: $h_0 < 0$. Using the equations $\dot{\bar{p}}_3 = -(\bar{p}_1+\bar{p}_2)$ and $h_0 = \bar{p}_1(t)+\bar{p}_2(t)+\frac{1}{2}\bar{p}_3(t)^2$, we get the differential equation $\dot{\bar{p}}_3 = \frac{1}{2}\bar{p}_3^2 - h_0$. Separating variables yields

$$\frac{d\bar{p}_3}{\bar{p}_3^2 + (\sqrt{-2h_0})^2} = \frac{1}{2}\,dt.$$

This may be integrated using the formula (A.6.11):

$$\frac{1}{\sqrt{-2h_0}}\tan^{-1}\left(\frac{\bar{p}_3(t)}{\sqrt{-2h_0}}\right) = \frac{1}{2}t.$$

Solving for $\bar{p}_3(t)$, we get $\bar{p}_3(t) = 2\Omega\tan(\Omega t)$, where $\Omega = \sqrt{-\frac{h_0}{2}}$. Then $\bar{p}_1(t) + \bar{p}_2(t) = -\dot{\bar{p}}_3 = 2\Omega^2\sec^2(\Omega t)$. Next, using the Casimir equation $c_0 = (p_1(t)+p_2(t))(p_1(t)-p_2(t))$, we get $\bar{p}_1(t) - \bar{p}_2(t) = \frac{c_0}{2\Omega^2}\cos^2(\Omega t)$. That is,

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}p_1(t)\\ p_2(t)\end{bmatrix} = \begin{bmatrix}2\Omega^2\sec^2(\Omega t)\\ \frac{c_0}{2\Omega^2}\cos^2(\Omega t)\end{bmatrix}.$$

As $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ is invertible, this equation can be solved for $\bar{p}_1(t)$ and $\bar{p}_2(t)$. Thus we have the following (prospective) integral curve:

$$\begin{cases}\bar{p}_1(t) = -\dfrac{1}{4\Omega^2}\left[4\Omega^4\sec^2(\Omega t) + c_0\cos^2(\Omega t)\right]\\ \bar{p}_2(t) = -\dfrac{1}{4\Omega^2}\left[4\Omega^4\sec^2(\Omega t) - c_0\cos^2(\Omega t)\right]\\ \bar{p}_3(t) = 2\Omega\tan(\Omega t).\end{cases}$$

We have

$$\begin{aligned}\dot{\bar{p}}_1(t) &= \frac{c_0}{4\Omega}\sin(2\Omega t) - 2\Omega^3\sec^2(\Omega t)\tan(\Omega t) = \bar{p}_2(t)\bar{p}_3(t)\\ \dot{\bar{p}}_2(t) &= -\frac{c_0}{4\Omega}\sin(2\Omega t) - 2\Omega^3\sec^2(\Omega t)\tan(\Omega t) = \bar{p}_1(t)\bar{p}_3(t)\\ \dot{\bar{p}}_3(t) &= 2\Omega^2\sec^2(\Omega t) = -(\bar{p}_1(t)+\bar{p}_2(t)).\end{aligned}$$

That is, $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$, and so $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Finally, since $\cos(\Omega t) = 0$ at the points $\frac{\pi}{\Omega}(2n-\frac{1}{2})$ and $\frac{\pi}{\Omega}(2n+\frac{1}{2})$, $n\in\mathbb{Z}$, it follows that $\bar{p}(\cdot)$ is only defined on the open intervals $\left(\frac{\pi}{\Omega}(2n-\frac{1}{2}), \frac{\pi}{\Omega}(2n+\frac{1}{2})\right)$, $n\in\mathbb{Z}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.18 Proposition. *Let $p(\cdot) : (-\varepsilon,\varepsilon)\to\mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = h_0 < 0$ and $C(p(0)) = c_0 > 0$.*

(i) There exists $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ such that $p(t) = \bar{p}(t+t_0)$ for every $t\in(-\varepsilon,\varepsilon)$, where $\bar{p}(\cdot) : (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}) \to \mathfrak{se}(1,1)^*$ is defined by

$$\begin{cases}\bar{p}_1(t) = -\dfrac{1}{4\Omega^2}\left[4\Omega^4\sec^2(\Omega t) + c_0\cos^2(\Omega t)\right]\\ \bar{p}_2(t) = -\dfrac{1}{4\Omega^2}\left[4\Omega^4\sec^2(\Omega t) - c_0\cos^2(\Omega t)\right]\\ \bar{p}_3(t) = 2\Omega\tan(\Omega t).\end{cases}$$

Here $\Omega = \sqrt{-\frac{h_0}{2}}$.

(ii) $t\mapsto\bar{p}(t+t_0)$ is the unique maximal integral curve starting at $\bar{p}(t_0)$.

PROOF. (*i*) We have $\operatorname{sgn}(\bar{p}_3|_{(-\pi/(2\Omega),0)}(t)) = -1$ and $\operatorname{sgn}(\bar{p}_3|_{(0,\pi/(2\Omega))}(t)) = 1$. Furthermore, $\lim_{t\to-\pi/(2\Omega)} \bar{p}_3(t) = -\infty$ and $\lim_{t\to\pi/(2\Omega)} \bar{p}_3(t) = \infty$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) + \bar{p}_2(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}\bar{p}_3(0)^2 = p_1(0) + p_2(0),$$

and from $\bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = c_0 = p_1(0)^2 - p_2(0)^2$, we get

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1(t_0) \\ p_2(t_0) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, it follows that $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t+t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By (*i*), $p(s) = \bar{p}(s+t_0)$, and so $s+t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$. Consequently, $s \in (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} + t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} + t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t + t_0)$. ∎

4.6.2.2 Case *II*: $c_0 = 0$

4.6.2.2.1 Case *II*-*a*: $h_0 > 0$. As $\bar{p}_1(t)^2 = \bar{p}_2^2(t)$, we have $\bar{p}_1(t)+\bar{p}_2(t) = 0$ or $\bar{p}_1(t)-\bar{p}_2(t) = 0$. Similarly, since $\bar{p}_1^2(t) \geq 0$, we have $\bar{p}_1(t) \geq 0$ or $\bar{p}_1(t) \leq 0$. If $\bar{p}_1(t) = 0$ for some t, then $\bar{p}(\cdot)$ is constant. (Indeed, we have $\bar{p}(t) = (0, 0, \pm\sqrt{2h_0})$, which is an equilibrium point of $\vec{H}_2^{(3)}$.) Thus either $\bar{p}_1(t) < 0$ or $\bar{p}_1(t) > 0$.

Consider the case $\boldsymbol{\bar{p}_1(t) - \bar{p}_2(t) = 0}$, $\boldsymbol{\bar{p}_1(t) < 0}$. Take the limit $c_0 \to \infty$ in the integral curve of proposition 4.6.15. The result is

$$\begin{cases} \bar{p}_1(t) = -\Omega^2 \operatorname{csch}^2(\Omega t) \\ \bar{p}_2(t) = -\Omega^2 \operatorname{csch}^2(\Omega t) \\ \bar{p}_3(t) = -2\Omega \coth(\Omega t). \end{cases}$$

(Here $\Omega = \sqrt{\frac{h_0}{2}}$.) We have

$$\begin{aligned} \dot{\bar{p}}_1(t) &= 2\Omega^3 \operatorname{csch}^2(\Omega t) \coth(\Omega t) = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= 2\Omega^3 \operatorname{csch}^2(\Omega t) \coth(\Omega t) = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= 2\Omega^2 \operatorname{csch}^2(\Omega t) = -(\bar{p}_1(t) + \bar{p}_2(t)). \end{aligned}$$

Thus $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Finally, as $\sinh(\Omega t) = 0$ when $t = 0$, we have that $\bar{p}(\cdot)$ is only defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.19 PROPOSITION. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = h_0 > 0$, $C(p(0)) = 0$, $p_1(0) - p_2(0) = 0$ and $p_1(0) < 0$.*

(*i*) *There exist* $t_0 \in \mathbb{R}\setminus\{0\}$ *and* $\varsigma \in \{-1,1\}$ *such that* $p(t) = \bar{p}_\varsigma(t+t_0)$ *for every* $t \in (-\varepsilon,\varepsilon)$, *where* $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$, $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$ *and*

$$\begin{cases} \bar{p}_1(t) = -\Omega^2 \operatorname{csch}^2(\Omega t) \\ \bar{p}_2(t) = -\Omega^2 \operatorname{csch}^2(\Omega t) \\ \bar{p}_3(t) = -2\Omega \coth(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{\frac{h_0}{2}}$.

(*ii*) $t \mapsto \bar{p}_\pm(t+t_0)$ *is the unique maximal integral curve starting at* $\bar{p}_\pm(t_0)$.

Proof. (*i*) Let $\varsigma = -\operatorname{sgn}(p_3(0)) \in \{-1,1\}$. (If $p_3(0) = 0$, then $h_0 = p_1(0)+p_2(0)+\frac{1}{2}p_3(0)^2 = 0$, a contradiction.) We have

$$\lim_{t\to 0} \bar{p}_{+,2}(t) = -\infty, \ \lim_{t\to\infty} \bar{p}_{+,2}(t) = 0 \qquad \text{and} \qquad \lim_{t\to 0} \bar{p}_{-,2}(t) = -\infty, \ \lim_{t\to-\infty} \bar{p}_{-,2}(t) = 0.$$

Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous, there exists

$$t_0 \in \begin{cases} (-\infty,0) & \text{if } \varsigma = -1 \\ (0,\infty) & \text{if } \varsigma = +1 \end{cases}$$

such that $\bar{p}_{\varsigma,2}(t_0) = p_2(0)$. (As $p_1(0) = p_2(0)$, we have $p_2(0) < 0$.) Then $\bar{p}_{1,\varsigma}(t_0) = \bar{p}_{2,\varsigma}(t_0) = p_2(0) = p_1(0)$. Finally,

$$\bar{p}_{\varsigma,3}(t_0)^2 = 2(h_0 - \bar{p}_{\varsigma,1}(t_0) - \bar{p}_{\varsigma,2}(t_0)) = 2(h_0 - p_1(0) - p_2(0)) = p_3(0)^2.$$

As $\operatorname{sgn}(\bar{p}_{\varsigma,3}(t_0)) = -\varsigma = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\varsigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_\pm(t+t_0)$. Suppose $p(0) = \bar{p}_-(t_0)$. Let $s \in (-\varepsilon,\varepsilon)$. By (*i*), $p(s) = \bar{p}_-(s+t_0)$, and so $s + t_0 \in (-\infty,0)$. Consequently, $s \in (-\infty,-t_0)$, *i.e.*, $(-\varepsilon,\varepsilon) \subseteq (-\infty,-t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}_-(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}_-(t+t_0)$.

Suppose $p(0) = \bar{p}_+(t_0)$. Similar to before, if $s \in (-\varepsilon,\varepsilon)$, then $s+t_0 \in (0,\infty)$, *i.e.*, $s \in (-t_0,\infty)$. Thus $(-\varepsilon,\varepsilon) \subseteq (-t_0,\infty)$, and so the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_+(t+t_0)$. The uniqueness of $t \mapsto \bar{p}_+(t+t_0)$ follows from its maximality. ■

Consider the case $\boldsymbol{\bar{p}_1(t) - \bar{p}_2(t) = 0}$, $\boldsymbol{\bar{p}_1(t) > 0}$. Take the limit $c_0 \to 0$ in the integral curve of proposition 4.6.16. We get

$$\begin{cases} \bar{p}_1(t) = \Omega^2 \operatorname{sech}^2(\Omega t) \\ \bar{p}_2(t) = \Omega^2 \operatorname{sech}^2(\Omega t) \\ \bar{p}_3(t) = -2\Omega \tanh(\Omega t). \end{cases}$$

(Here $\Omega = \sqrt{\frac{h_0}{2}}$.) We have $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$. Indeed,

$$\begin{aligned} \dot{\bar{p}}_1(t) &= -2\Omega^3 \operatorname{sech}^2(\Omega t)\tanh(\Omega t) = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= -2\Omega^3 \operatorname{sech}^2(\Omega t)\tanh(\Omega t) = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= -2\Omega^2 \operatorname{sech}^2(\Omega t) = -(\bar{p}_1(t) + \bar{p}_2(t)). \end{aligned}$$

Therefore $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Furthermore, $\bar{p}(\cdot)$ is defined over $\mathbb{R}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.20 PROPOSITION. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = h_0 > 0$, $C(p(0)) = 0$, $p_1(0) - p_2(0) = 0$ and $p_1(0) > 0$. There exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(1,1)^*$ is defined by*

$$\begin{cases} \bar{p}_1(t) = \Omega^2 \operatorname{sech}^2(\Omega t) \\ \bar{p}_2(t) = \Omega^2 \operatorname{sech}^2(\Omega t) \\ \bar{p}_3(t) = -2\Omega \tanh(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{\frac{h_0}{2}}$.

PROOF. Since $p_1(t) > 0$ and $p_2(t) = p_1(t)$, we have $p_1(t) + p_2(t) > 0$, and so $p_3(t)^2 = 2(h_0 - p_1(t) - p_2(t)) < 2h_0$. Thus $-\sqrt{2h_0} < p_3(t) < \sqrt{2h_0}$. Similarly, we have $-\sqrt{2h_0} < \bar{p}_3(t) < \sqrt{2h_0}$. Moreover, $\lim_{t\to\infty} \bar{p}_3(t) = -\sqrt{2h_0}$ and $\lim_{t\to-\infty} \bar{p}_3(t) = \sqrt{2h_0}$. Since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) + \bar{p}_2(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}p_3(0)^2 = p_1(0) + p_2(0),$$

and from $\bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = p_1(0)^2 - p_2(0)^2$, we get

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1(t_0) \\ p_2(t_0) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, it follows that $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ∎

Consider the case $\boldsymbol{\bar{p}_1(t) + \bar{p}_2(t) = 0}$. If $\bar{p}_1(t) = -\bar{p}_2(t) = 0$ for some t, then $\bar{p}(t) = (0, 0, \pm\sqrt{2h_0})$, which is an equilibrium point of $\vec{H}_2^{(3)}$ (and so $\bar{p}(\cdot)$ is constant). Suppose $\bar{p}_1(t)$ and $\bar{p}_2(t)$ are not both zero for the same t. The equation $2h_0 = \bar{p}_3(t)^2$ yields $\bar{p}_3(t) = \sigma_1\sqrt{2h_0}$, for some $\sigma_1 \in \{-1, 1\}$. Then $\dot{\bar{p}}_1 = -\sigma_1\sqrt{2h_0}\,\bar{p}_1$, which has the solution $\bar{p}_1(t) = \sigma_2 e^{-\sigma_1\sqrt{2h_0}\,t}$, for some $\sigma_2 \in \{-1, 1\}$. Thus we have the following (prospective) integral curve:

$$\begin{cases} \bar{p}_1(t) = \sigma_2 e^{-\sigma_1\sqrt{2h_0}\,t} \\ \bar{p}_2(t) = -\sigma_2 e^{-\sigma_1\sqrt{2h_0}\,t} \\ \bar{p}_3(t) = \sigma_1\sqrt{2h_0}. \end{cases}$$

We verify that $\bar{p}(\cdot)$ is an integral curve. Indeed,

$$\begin{aligned} \dot{\bar{p}}_1(t) &= -\sigma_1\sigma_2\sqrt{2h_0}\, e^{-\sigma_1\sqrt{2h_0}\,t} = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= \sigma_1\sigma_2\sqrt{2h_0}\, e^{-\sigma_1\sqrt{2h_0}\,t} = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= 0 = -(\bar{p}_1(t) + \bar{p}_2(t)), \end{aligned}$$

and so $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$. Furthermore, $\bar{p}(\cdot)$ is clearly defined on $\mathbb{R}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.21 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = h_0 > 0$, $C(p(0)) = 0$ and $p_1(0) + p_2(0) = 0$ (with $p_1(0)$ and $p_2(0)$ not both zero). There exist $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \varsigma e^{-\sigma\sqrt{2h_0}\,t} \\ \bar{p}_2(t) = -\varsigma e^{-\sigma\sqrt{2h_0}\,t} \\ \bar{p}_3(t) = \sigma\sqrt{2h_0}\,. \end{cases}$$

Proof. Let $\sigma = \operatorname{sgn}(p_1(0)) \in \{-1, 1\}$ and $\varsigma = \operatorname{sgn}(p_3(0)) \in \{-1, 1\}$. (If $p_1(0) = 0$, then $p_2(0) = 0$, a contradiction. Similarly, if $p_3(0) = 0$, then $p_1(0) + p_2(0) = h_0 > 0$, a contradiction.) We have $\operatorname{sgn}(\bar{p}_1(t)) = \sigma$. Furthermore,

$$\lim_{t\to-\infty} \bar{p}_1(t) = \begin{cases} 0 & \text{if } \varsigma = -1 \\ \sigma\infty & \text{if } \varsigma = +1 \end{cases} \qquad \text{and} \qquad \lim_{t\to\infty} \bar{p}_1(t) = \begin{cases} \sigma\infty & \text{if } \varsigma = -1 \\ 0 & \text{if } \varsigma = +1. \end{cases}$$

Since $\bar{p}_1(\cdot)$ is continuous, $p_1(0) \neq 0$ and $\operatorname{sgn}(p_1(0)) = \sigma$, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_1(t_0) = p_1(0)$. Then $\bar{p}_2(t_0) = \bar{p}_1(t_0) = -p_1(0) = p_2(0)$. Finally,

$$\bar{p}_3(t_0)^2 = 2(h_0 - \bar{p}_1(t_0) - \bar{p}_2(t_0)) = 2(h_0 - p_1(0) - p_2(0)) = p_3(0)^2.$$

But $\operatorname{sgn}(\bar{p}_3(t_0)) = \varsigma = \operatorname{sgn}(p_3(0))$, and so $\bar{p}_3(t_0) = p_3(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. ∎

4.6.2.2.2 Case *II-b*: $h_0 = 0$. If $\bar{p}_1(t) + \bar{p}_2(t) = 0$ for some t, then we have $\bar{p}(t) = (\bar{p}_1(0), -\bar{p}_1(0), \pm\sqrt{2h_0}\,) = (\bar{p}_1(0), -\bar{p}_1(0), 0)$, which is an equilibrium point of $\vec{H}_2^{(3)}$ (and so $\bar{p}(\cdot)$ is constant). Assume $\bar{p}_1(t) + \bar{p}_2(t) \neq 0$ and $\bar{p}_1(t) \neq 0$ for every t.

Furthermore, since $\bar{p}_1(t)^2 > \bar{p}_1(t)^2 - \bar{p}_2(t)^2 = 0$, we have $\bar{p}_1(t) > 0$ or $\bar{p}_1(t) < 0$. We claim that the latter situation is impossible. Indeed, suppose $\bar{p}_1(t) > 0$. Since $\bar{p}_1(t) - \bar{p}_2(t) = 0$, we have $\bar{p}_2(t) > 0$ and so $h_0 = \bar{p}_1(t) + \bar{p}_2(t) + \frac{1}{2}\bar{p}_3(t)^2 > 0$, a contradiction. Thus $\bar{p}_1(t) < 0$ for every t.

Taking the limit $c_0 \to 0$ in the integral curve of proposition 4.6.17 yields

$$\begin{cases} \bar{p}_1(t) = -\dfrac{1}{t^2} \\ \bar{p}_2(t) = -\dfrac{1}{t^2} \\ \bar{p}_3(t) = -\dfrac{2}{t}. \end{cases}$$

We have $\dot{\bar{p}}_1(t) = \frac{2}{t^3} = \bar{p}_2(t)\bar{p}_3(t)$, $\dot{\bar{p}}_2(t) = \frac{2}{t^3} = \bar{p}_1(t)\bar{p}_3(t)$ and $\dot{\bar{p}}_3(t) = \frac{2}{t^2} = -(\bar{p}_1(t) + \bar{p}_2(t))$. That is, $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$, and so $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Furthermore, $\bar{p}(\cdot)$ is only defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.22 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = 0$, $C(p(0)) = 0$, $p_1(0) + p_2(0) \neq 0$ and $p_1(0) < 0$.*

(*i*) *There exist* $t_0 \in \mathbb{R}\setminus\{0\}$ *and* $\varsigma \in \{-1,1\}$ *such that* $p(t) = \bar{p}_\varsigma(t+t_0)$ *for every* $t \in (-\varepsilon,\varepsilon)$, *where* $\bar{p}_-(\cdot) = \bar{p}|_{(-\infty,0)}(\cdot)$, $\bar{p}_+(\cdot) = \bar{p}|_{(0,\infty)}(\cdot)$ *and*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{1}{t^2} \\ \bar{p}_2(t) = -\dfrac{1}{t^2} \\ \bar{p}_3(t) = -\dfrac{2}{t}. \end{cases}$$

(*ii*) $t \mapsto \bar{p}_\pm(t+t_0)$ *is the unique maximal integral curve starting at* $\bar{p}_\pm(t_0)$.

PROOF. (*i*) Let $\varsigma = -\operatorname{sgn}(p_3(0)) \in \{-1,1\}$. (If $p_3(0) = 0$, then $p_1(0) + p_2(0) = h_0 = 0$, a contradiction.) We have

$$\lim_{t\to 0} \bar{p}_{+,2}(t) = -\infty,\ \lim_{t\to\infty} \bar{p}_{+,2}(t) = 0 \qquad \text{and} \qquad \lim_{t\to 0} \bar{p}_{-,2}(t) = -\infty,\ \lim_{t\to-\infty} \bar{p}_{-,2}(t) = 0.$$

Since $\bar{p}_{-,2}(\cdot)$ and $\bar{p}_{+,2}(\cdot)$ are continuous, there exists

$$t_0 \in \begin{cases} (-\infty,0) & \text{if } \varsigma = -1 \\ (0,\infty) & \text{if } \varsigma = +1 \end{cases}$$

such that $\bar{p}_{\varsigma,2}(t_0) = p_2(0)$. We then have $\bar{p}_{\varsigma,1}(t_0)^2 = \bar{p}_{\varsigma,2}(t_0)^2 = p_2(0)^2 = p_1(0)^2$. Since $\bar{p}_{\varsigma,1}(t_0), p_1(0) < 0$, it follows that $\bar{p}_{\varsigma,1}(t_0) = p_1(0)$. Finally, $\bar{p}_{\varsigma,3}(t_0)^2 = -2(\bar{p}_{\varsigma,1}(t_0)+\bar{p}_{\varsigma,2}(t_0)) = -2(p_1(0) + p_2(0)) = p_3(0)^2$. As $\operatorname{sgn}(\bar{p}_{\varsigma,3}(t_0)) = -\varsigma = \operatorname{sgn}(p_3(0))$, we have $\bar{p}_{\varsigma,3}(t_0) = p_3(0)$. That is, $\bar{p}_\varsigma(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}_\varsigma(t+t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(*ii*) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_\pm(t+t_0)$. Suppose $p(0) = \bar{p}_-(t_0)$. Let $s \in (-\varepsilon,\varepsilon)$. By (*i*), $p(s) = \bar{p}_-(s+t_0)$, and so $s + t_0 \in (-\infty,0)$. Consequently, $s \in (-\infty,-t_0)$, *i.e.*, $(-\varepsilon,\varepsilon) \subseteq (-\infty,-t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}_-(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}_-(t+t_0)$.

Suppose $p(0) = \bar{p}_+(t_0)$. Similar to before, if $s \in (-\varepsilon,\varepsilon)$, then $s + t_0 \in (0,\infty)$, *i.e.*, $s \in (-t_0,\infty)$. Thus $(-\varepsilon,\varepsilon) \subseteq (-t_0,\infty)$, and so the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}_+(t+t_0)$. The uniqueness of $t \mapsto \bar{p}_+(t+t_0)$ follows from its maximality. ■

4.6.2.2.3 Case *II-c*: $h_0 < 0$. Limiting $c_0 \to 0$ in the integral curve of proposition 4.6.18, we get the following curve:

$$\begin{cases} \bar{p}_1(t) = -\Omega^2 \sec^2(\Omega t) \\ \bar{p}_2(t) = -\Omega^2 \sec^2(\Omega t) \\ \bar{p}_3(t) = 2\Omega \tan(\Omega t). \end{cases}$$

(Here $\Omega = \sqrt{-\frac{h_0}{2}}$.) We verify that $\dot{\bar{p}}(t) = \vec{H}_2^{(3)}(\bar{p}(t))$. Indeed,

$$\begin{aligned} \dot{\bar{p}}_1(t) &= -2\Omega^3 \sec^2(\Omega t)\tan(\Omega t) = \bar{p}_2(t)\bar{p}_3(t) \\ \dot{\bar{p}}_2(t) &= -2\Omega^3 \sec^2(\Omega t)\tan(\Omega t) = \bar{p}_1(t)\bar{p}_3(t) \\ \dot{\bar{p}}_3(t) &= 2\Omega^2 \sec^2(\Omega t) = -(\bar{p}_1(t) + \bar{p}_2(t)). \end{aligned}$$

Thus $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_2^{(3)}$. Furthermore, $\bar{p}(\cdot)$ is clearly defined over $\mathbb{R}$.

We now make an explicit statement regarding all integral curves of $\vec{H}_2^{(3)}$ for this case.

4.6.23 Proposition. *Let $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}_2^{(3)}$ such that $H_2^{(3)}(p(0)) = h_0 < 0$ and $C(p(0)) = 0$.*

(i) There exists $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) : (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega}) \to \mathfrak{se}(1,1)^*$ is defined by

$$\begin{cases} \bar{p}_1(t) = -\Omega^2 \sec^2(\Omega t) \\ \bar{p}_2(t) = -\Omega^2 \sec^2(\Omega t) \\ \bar{p}_3(t) = 2\Omega \tan(\Omega t). \end{cases}$$

Here $\Omega = \sqrt{-\frac{h_0}{2}}$.

(ii) $t \mapsto \bar{p}(t + t_0)$ is the unique maximal integral curve starting at $\bar{p}(t_0)$.

Proof. *(i)* We have $\operatorname{sgn}(\bar{p}_3|_{(-\pi/(2\Omega),0)}(t)) = -1$ and $\operatorname{sgn}(\bar{p}_3|_{(0,\pi/(2\Omega))}(t)) = 1$. Furthermore, $\lim_{t\to-\pi/(2\Omega)} \bar{p}_3(t) = -\infty$ and $\lim_{t\to\pi/(2\Omega)} \bar{p}_3(t) = \infty$. Therefore, since $\bar{p}_3(\cdot)$ is continuous, there exists $t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$ such that $\bar{p}_3(t_0) = p_3(0)$. Then

$$\bar{p}_1(t_0) + \bar{p}_2(t_0) = h_0 - \tfrac{1}{2}\bar{p}_3(t_0)^2 = h_0 - \tfrac{1}{2}\bar{p}_3(0)^2 = p_1(0) + p_2(0),$$

and from $\bar{p}_1(t_0)^2 - \bar{p}_2(t_0)^2 = 0 = p_1(0)^2 - p_2(0)^2$, we get

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1(t_0) \\ p_2(t_0) \end{bmatrix} = \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is invertible, it follows that $\bar{p}(t_0) = p(0)$. Therefore, as $t \mapsto \bar{p}(t + t_0)$ and $t \mapsto p(t)$ are both integral curves of $\vec{H}_2^{(3)}$ passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical.

(ii) We show that the domain of $p(\cdot)$ is no larger than that of $t \mapsto \bar{p}(t+t_0)$. Let $s \in (-\varepsilon, \varepsilon)$. By *(i)*, $p(s) = \bar{p}(s+t_0)$, and so $s+t_0 \in (-\frac{\pi}{2\Omega}, \frac{\pi}{2\Omega})$. Consequently, $s \in (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} + t_0)$, *i.e.*, $(-\varepsilon, \varepsilon) \subseteq (-\frac{\pi}{2\Omega} - t_0, \frac{\pi}{2\Omega} + t_0)$. Therefore the domain of $p(\cdot)$ is contained in that of $t \mapsto \bar{p}(t+t_0)$. Uniqueness now follows from the maximality of $t \mapsto \bar{p}(t + t_0)$. ■

Chapter 5

Optimal Control on $\mathsf{SE}(1,1)$

In this chapter we investigate invariant Riemannian and sub-Riemannian structures on the semi-Euclidean group. As the first step, we introduce a natural equivalence relation between such structures, *viz.* equivalence up to isometric group automorphisms. The left-invariant Riemannian and sub-Riemannian structures on $\mathsf{SE}(1,1)$ are then classified. This classification is related to some extent with the classification of control systems (in fact, the classification of subspaces) in chapter 2, and we shall draw upon some of those results. Up to equivalence, we identify a single-parameter family of sub-Riemannian structures on $\mathsf{SE}(1,1)$. On the other hand, a two-parameter family of Riemannian structures is obtained. By scaling, these may be reduced to a single representative and a single-parameter family of representatives, respectively.

Following the classification we determine the minimising geodesics for each Riemannian and sub-Riemannian equivalence class representative. These are expressed in terms of Jacobi elliptic functions, and typically involve several parameters. We outline the approach followed during the integration. Geodesics are (local) length minimisers, and so the problem of finding the geodesics is written as an optimal control problem on $\mathsf{SE}(1,1)$. Next we identify a family of Hamiltonian functions on the cotangent bundle $T^*\mathsf{SE}(1,1)$. By Pontryagin's Maximum Principle (see section A.3.4), this family of Hamiltonians reduces to a single homogeneous Hamiltonian H evolving on the minus Lie-Poisson space $\mathfrak{se}(1,1)^*_-$, *i.e.* a (homogeneous) Hamilton-Poisson system $(\mathfrak{se}(1,1)^*_-, H)$. Such systems have been classified in chapter 3 (up to affine equivalence), and the integral curves were calculated in chapter 4. We use the results of chapter 4 to determine the extremal controls for the optimal control problem. (The extremal controls are linearly related to the integral curves of $\vec{H}$.) The final step is to integrate the equations for the optimal trajectories on the group. This work is divided into several different cases, depending on initial conditions. The resultant curves are the geodesics on $\mathsf{SE}(1,1)$.

5.1 Preliminaries

We briefly recall some concepts from Riemannian and sub-Riemannian geometry, as detailed in section A.4. A **sub-Riemannian structure** on a (real, finite-dimensional) connected matrix Lie group G is a pair $(\mathcal{D}, \mathbf{g})$, where $\mathcal{D}$ is a distribution on G and $\mathbf{g}$ is a sub-Riemannian metric on $\mathcal{D}$. If $\mathcal{D} = T\mathsf{G}$, then we speak of a **Riemannian structure g**. We shall restrict to structures that are **left-invariant** (*i.e.*, $\mathcal{D}_g = g\,\mathcal{D}_\mathbf{1}$ and $\mathbf{g}_g(gX, gY) = \mathbf{g}_\mathbf{1}(X,Y)$ for every $X, Y \in \mathcal{D}_\mathbf{1}$ and $g \in \mathsf{G}$) and **bracket-generating** (*i.e.*, $\mathsf{Lie}(\mathcal{D}_\mathbf{1}) = \mathfrak{g}$). **Horizontal curves** are absolutely continuous curves $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) \in \mathcal{D}_{g(t)}$ for a.e. $t \in [0, T]$. By theorem A.4.1 (the Chow-Rashevskii theorem), there exists a horizontal curve joining any two points so long as $\mathcal{D}$ is bracket-generating. A (minimising) **geodesic** is a horizontal curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\ell(g(\cdot)) = d(g(0), g(T))$. Geodesics are solutions to the following problem:

$$\dot{g}(t) \in \mathcal{D}_{g(t)}, \quad g(\cdot) : [0, T] \to \mathsf{G}$$
$$g(0) = g_0, \ g(T) = g_1, \quad g_0, g_1 \in \mathsf{G}, \ T > 0 \text{ fixed}$$
$$\ell(g(\cdot)) = \int_0^T \sqrt{\mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t))}\, dt \to \min.$$

Theorem A.4.2 guarantees the existence of geodesics between any two points sufficiently close to each other. The above problem may be rewritten as the optimal control problem

$$\dot{g} = g(u_1 E_1 + \ldots + u_\ell E_\ell), \quad g(\cdot) : [0, T] \to \mathsf{G}, \ u(\cdot) : [0, T] \to \mathbb{R}^\ell$$
$$g(0) = g_0, \ g(T) = g_1, \quad g_0, g_1 \in \mathsf{G}, \ T > 0 \text{ fixed}$$
$$\mathcal{J}(u(\cdot)) = \int_0^T u_1(t)^2 + \ldots + u_\ell(t)^2\, dt \to \min.$$

(The **normal** geodesics are projections of normal extremals, whereas **abnormal** geodesics are projections of abnormal extremals.) We shall determine the geodesics for every sub-Riemannian (and Riemannian) structure on $\mathsf{SE}(1, 1)$. To that end, we introduce an equivalence relation between sub-Riemannian structures. Let $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ be two left-invariant sub-Riemannian structures on a Lie group G. We say that $(\mathcal{D}, \mathbf{g})$ is **$\mathfrak{L}$-isometric** to $(\mathcal{D}', \mathbf{g}')$ if there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that

$$T_g\phi \cdot \mathcal{D}_g = \mathcal{D}'_{\phi(g)} \qquad \text{and} \qquad \mathbf{g}_g(X, Y) = \mathbf{g}'_{\phi(g)}(T_g\phi \cdot X, T_g\phi \cdot Y) \tag{5.1.1}$$

for every $X, Y \in \mathcal{D}_g$ and $g \in \mathsf{G}$. (Notice that, in the Riemannian case, the first condition is trivially satisfied, since $\mathcal{D} = \mathcal{D}' = T\mathsf{G}$.) An automorphism $\phi : \mathsf{G} \to \mathsf{G}$ that satisfies the conditions of (5.1.1) is called an **$\mathfrak{L}$-isometry**.

5.1.1 Proposition. *Equivalence under $\mathfrak{L}$-isometries is an equivalence relation.*

Proof. Let $(\mathcal{D}, \mathbf{g})$, $(\mathcal{D}', \mathbf{g}')$ and $(\mathcal{D}'', \mathbf{g}'')$ be left-invariant sub-Riemannian structures on a matrix Lie group G. We have $T_g \,\mathrm{id} \cdot \mathcal{D}_g = \mathcal{D}_g = \mathcal{D}_{\mathrm{id}(g)}$ and $\mathbf{g}_g(X, Y) = \mathbf{g}_{\mathrm{id}(g)}(T_g \,\mathrm{id} \cdot X, T_g \,\mathrm{id} \cdot Y)$ for every $g \in \mathsf{G}$. Hence $(\mathcal{D}, \mathbf{g})$ is $\mathfrak{L}$-isometric to itself, and so equivalence under $\mathfrak{L}$-isometries has the reflexive property.

Next, suppose $(\mathcal{D}, \mathbf{g})$ is $\mathfrak{L}$-isometric to $(\mathcal{D}', \mathbf{g}')$. Then there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_g\phi \cdot \mathcal{D}_g = \mathcal{D}'_{\phi(g)}$ and $\mathbf{g}_g(X, Y) = \mathbf{g}'_{\phi(g)}(T_g\phi \cdot X, T_g\phi \cdot Y)$ for every $X, Y \in \mathcal{D}_g$ and $g \in \mathsf{G}$. Consequently, $(T_g\phi)^{-1} \cdot \mathcal{D}'_{\phi(g)} = \mathcal{D}_g$ and $\mathbf{g}_g((T_g\phi)^{-1} \cdot U, (T_g\phi)^{-1} \cdot V) = \mathbf{g}'_{\phi(g)}(U, V)$ for every $U, V \in \mathcal{D}'_{\phi(g)}$ and $g \in \mathsf{G}$. Since $(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}$, we have $T_{\phi(g)}\phi^{-1} \cdot \mathcal{D}'_{\phi(g)} = \mathcal{D}_{\phi^{-1}(\phi(g))}$ and $\mathbf{g}'_{\phi(g)}(X, Y) = \mathbf{g}_{\phi^{-1}(\phi(g))}(T_{\phi(g)}\phi^{-1} \cdot X, T_{\phi(g)}\phi^{-1} \cdot Y)$ for every $U, V \in \mathcal{D}_{\phi(g)}$ and $\phi(g) \in \mathsf{G}$. That is, $(\mathcal{D}', \mathbf{g}')$ is $\mathfrak{L}$-equivalent to $(\mathcal{D}, \mathbf{g})$, and so equivalence under $\mathfrak{L}$-isometries has the symmetry property.

Finally, suppose that $(\mathcal{D}, \mathbf{g})$ is $\mathfrak{L}$-isometric to $(\mathcal{D}', \mathbf{g}')$ and $(\mathcal{D}', \mathbf{g}')$ is $\mathfrak{L}$-isometric to $(\mathcal{D}'', \mathbf{g}'')$. Then there exist $\phi_1, \phi_2 \in \mathsf{Aut}(\mathsf{G})$ such that

$$T_g\phi_1 \cdot \mathcal{D}_g = \mathcal{D}'_{\phi_1(g)} \qquad \text{and} \qquad T_g\phi_1 \cdot \mathbf{g}_g(X, Y) = \mathbf{g}'_{\phi_1(g)}(T_g\phi_1 \cdot X, T_g\phi_1 \cdot Y)$$

for every $g \in \mathsf{G}$, $X, Y \in \mathcal{D}_g$, and

$$T_g\phi_2 \cdot \mathcal{D}'_g = \mathcal{D}''_{\phi_2(g)} \qquad \text{and} \qquad T_g\phi_2 \cdot \mathbf{g}'_g(X', Y') = \mathbf{g}''_{\phi_2(g)}(T_g\phi_2 \cdot X', T_g\phi_2 \cdot Y')$$

for every $g \in \mathsf{G}$, $X', Y' \in \mathcal{D}'_g$. Let $\phi = \phi_2 \circ \phi_1$. Then ϕ is an automorphism such that, for every $g \in \mathsf{G}$,

$$T_g\phi \cdot \mathcal{D}_g = T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \mathcal{D}_g = T_{\phi_1(g)}\phi_2 \cdot \mathcal{D}'_{\phi_1(g)} = \mathcal{D}''_{\phi_2(\phi_1(g))} = \mathcal{D}''_{\phi(g)}.$$

Furthermore, for every $g \in \mathsf{G}$ and $X, Y \in \mathcal{D}_g$,

$$\begin{aligned} T_g\phi \cdot \mathbf{g}_g(X, Y) &= T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \mathbf{g}_g(X, Y) \\ &= T_{\phi_1(g)}\phi_2 \cdot \mathbf{g}'_{\phi_1(g)}(T_g\phi_1 \cdot X, T_g\phi_1 \cdot Y) \\ &= \mathbf{g}''_{\phi_2(\phi_1(g))}(T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot X, T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot Y) \\ &= \mathbf{g}''_{\phi(g)}(T_g\phi \cdot X, T_g\phi \cdot Y). \end{aligned}$$

Thus $(\mathcal{D}, \mathbf{g})$ is $\mathfrak{L}$-isometric to $(\mathcal{D}'', \mathbf{g}'')$, and so equivalence under $\mathfrak{L}$-isometries is transitive. ∎

The next result shows that equivalence under $\mathfrak{L}$-isometries is natural, in that it preserves salient properties of the sub-Riemannian structure. Let $d(\cdot, \cdot)$ and $\ell(\cdot)$ denote the Carnot-Carathéodory metric and length functional, respectively, of $(\mathcal{D}, \mathbf{g})$ (as defined in section A.4). Similarly, let $d'(\cdot, \cdot)$ and $\ell'(\cdot)$ denote the Carnot-Carathéodory metric and length functional of $(\mathcal{D}', \mathbf{g}')$.

5.1.2 Proposition. *Suppose $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are $\mathfrak{L}$-isometric with respect to an automorphism $\phi : \mathsf{G} \to \mathsf{G}$.*

(*i*) *$\mathcal{D}$ is bracket-generating if and only if $\mathcal{D}'$ is bracket-generating.*

(*ii*) *The horizontal curves of $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are in a one-to-one correspondence.*

(*iii*) *The Carnot-Carathéodory distance and the length of curves is preserved. That is, we have $d(\cdot, \cdot) = d'(\phi(\cdot), \phi(\cdot))$ and $\ell(g(\cdot)) = \ell'(\phi(g(\cdot)))$ for every horizontal curve $g(\cdot) : [0, T] \to \mathsf{G}$.*

(*iv*) *The geodesics of $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are in a one-to-one correspondence.*

Proof.

(*i*) We have that $\mathcal{D}$ and $\mathcal{D}'$ are bracket-generating if and only if the subspaces $\mathcal{D}_\mathbf{1}$ and $\mathcal{D}'_\mathbf{1}$ have full rank, *i.e.*, $\mathsf{Lie}(\mathcal{D}_\mathbf{1}) = \mathsf{Lie}(\mathcal{D}'_\mathbf{1}) = \mathfrak{g}$. Furthermore, as ϕ is a Lie group automorphism, $T_\mathbf{1}\phi$ is a Lie algebra automorphism (see theorem A.1.6). Since $T_\mathbf{1}\phi \cdot \mathcal{D}_\mathbf{1} = \mathcal{D}'_\mathbf{1}$ and automorphisms preserve full rank subspaces (proposition 2.1.7), it follows that $\mathcal{D}$ is bracket-generating if and only if $\mathcal{D}'$ is bracket-generating.

(*ii*) Let $g(\cdot) : [0, T] \to \mathsf{G}$ be a horizontal curve of $(\mathcal{D}, \mathbf{g})$. Then $\dot{g}(t) \in \mathcal{D}_{g(t)}$ for a.e. $t \in [0, T]$. Accordingly, for a.e. $t \in [0, T]$, we have

$$\frac{d}{dt}\phi(g(t)) = T_{g(t)}\phi \cdot \dot{g}(t) \in T_{g(t)}\phi \cdot \mathcal{D}_{g(t)} = \mathcal{D}'_{\phi(g(t))}.$$

That is, $(\phi \circ g)(\cdot)$ is a horizontal curve of $(\mathcal{D}', \mathbf{g}')$. Since $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are arbitrary, we have that $g(\cdot)$ is a horizontal curve of $(\mathcal{D}, \mathbf{g})$ if and only if $(\phi \circ g)(\cdot)$ is a horizontal curve of $(\mathcal{D}', \mathbf{g}')$. That is, the horizontal curves of $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are in a one-to-one correspondence.

(*iii*) Let $g(\cdot) : [0, T] \to \mathsf{G}$ be a horizontal curve. Then

$$\begin{aligned} \ell'((\phi \circ g)(\cdot)) &= \int_0^T \sqrt{\mathbf{g}'_{\phi(g(t))}\left(\frac{d}{dt}\phi(g(t)), \frac{d}{dt}\phi(g(t))\right)}\, dt \\ &= \int_0^T \sqrt{\mathbf{g}'_{\phi(g(t))}\left(T_{g(t)}\phi \cdot \dot{g}(t), T_{g(t)}\phi \cdot \dot{g}(t)\right)}\, dt \\ &= \int_0^T \sqrt{\mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t))}\, dt = \ell(g(\cdot)). \end{aligned}$$

Thus the length functional is preserved by ϕ. Next, let $a, b \in \mathsf{G}$ be arbitrary. Since $\mathcal{D}$ is bracket-generating (by assumption), the Chow-Rashevskii theorem (theorem A.4.1) ensures the existence of a horizontal curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = a$ and $g(T) = b$. By item (*ii*), $\phi \circ g$ is a horizontal curve such that $\phi(g(0)) = \phi(a)$ and $\phi(g(T)) = \phi(b)$. Accordingly,

$$\begin{aligned} &d'(\phi(a), \phi(b)) \\ &\quad = d'\left((\phi \circ g)(0), (\phi \circ g)(T)\right) \\ &\quad = \inf\left\{\ell'(h(\cdot)) : h : [0, S] \to \mathsf{G} \text{ is horizontal, } h(0) = (\phi \circ g)(0),\ h(S) = (\phi \circ g)(T)\right\} \\ &\quad = \inf\left\{\ell'((\phi \circ f)(\cdot)) : f : [0, S] \to \mathsf{G} \text{ is horizontal, } f(0) = g(0),\ f(S) = g(T)\right\} \\ &\quad = \inf\left\{\ell(f(\cdot)) : f : [0, S] \to \mathsf{G} \text{ is horizontal, } f(0) = g(0),\ f(S) = g(T)\right\} \\ &\quad = d(g(0), g(T)) = d(a, b). \end{aligned}$$

That is, the Carnot-Carathéodory distance is preserved by ϕ.

(*iv*) Let $g(\cdot) : [0, T] \to \mathsf{G}$ be a geodesic of $(\mathcal{D}, \mathbf{g})$, *i.e.*, $d(g(0), g(T)) = \ell(g(\cdot))$. Then, using item (*iii*),

$$d'((\phi \circ g)(0), (\phi \circ g)(T)) = d(g(0), g(T)) = \ell(g(\cdot)) = \ell'((\phi \circ g)(\cdot)).$$

Thus $(\phi \circ g)(\cdot)$ is a geodesic of $(\mathcal{D}', \mathbf{g}')$. Since $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are arbitrary, we have that $g(\cdot)$ is a geodesic of $(\mathcal{D}, \mathbf{g})$ if and only if $(\phi \circ g)(\cdot)$ is a geodesic of $(\mathcal{D}', \mathbf{g}')$. That is, the geodesics of $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are in a one-to-one correspondence. ■

Next, we show that for (left-invariant) sub-Riemannian structures on simply connected Lie groups, equivalence under $\mathfrak{L}$-isometries may be characterised at the level of the Lie algebra.

5.1.3 Theorem. *Let $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ be sub-Riemannian structures on a simply connected Lie group G. $(\mathcal{D}, \mathbf{g})$ is $\mathfrak{L}$-isometric to $(\mathcal{D}', \mathbf{g}')$ if and only if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \mathcal{D}_\mathbf{1} = \mathcal{D}'_\mathbf{1}$ and $\mathbf{g}_\mathbf{1}(X, Y) = \mathbf{g}'_\mathbf{1}(\psi \cdot X, \psi \cdot Y)$ for every $X, Y \in \mathcal{D}_\mathbf{1}$.*

Proof. Suppose that $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are $\mathfrak{L}$-isometric with respect to an automorphism $\phi : \mathsf{G} \to \mathsf{G}$, *i.e.*, $T_g\phi \cdot \mathcal{D}_g = \mathcal{D}'_{\phi(g)}$ and $\mathbf{g}_g(X, Y) = \mathbf{g}'_{\phi(g)}(T_g\phi \cdot X, T_g\phi \cdot Y)$ for every $X, Y \in \mathcal{D}_g$

and $g \in \mathsf{G}$. By theorem A.1.6 we have that $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism. Hence, there exists a Lie algebra automorphism $\psi = T_{\mathbf{1}}\phi$ such that $\psi \cdot \mathcal{D}_{\mathbf{1}} = \mathcal{D}'_{\mathbf{1}}$ and $\mathbf{g}_{\mathbf{1}}(X, Y) = \mathbf{g}'_{\mathbf{1}}(\psi \cdot X, \psi \cdot Y)$ for every $X, Y \in \mathcal{D}_{\mathbf{1}}$.

Conversely, since G is a simply connected Lie group, there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi = \psi$ (theorem A.1.6). We have $\phi = L_{\phi(g)} \circ \phi \circ L_{g^{-1}}$ for every $g \in \mathsf{G}$. Accordingly,

$$\begin{aligned}
T_g\phi \cdot \mathcal{D}_g &= T_g(L_{\phi(g)} \circ \phi \circ L_{g^{-1}}) \cdot \mathcal{D}_g \\
&= T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot T_g L_{g^{-1}} \cdot \mathcal{D}_g \\
&= T_{\mathbf{1}}L_{\phi(g)} \cdot \psi \cdot \mathcal{D}_{\mathbf{1}} \\
&= T_{\mathbf{1}}L_{\phi(g)} \cdot \mathcal{D}'_{\mathbf{1}} = \mathcal{D}'_{\phi(g)}.
\end{aligned}$$

Furthermore, for every $X, Y \in \mathcal{D}_g$ and $g \in \mathsf{G}$, we have

$$\begin{aligned}
\mathbf{g}_g(X, Y) &= \mathbf{g}_{\mathbf{1}}(T_g L_{g^{-1}} \cdot X, T_g L_{g^{-1}} \cdot Y) \\
&= \mathbf{g}'_{\mathbf{1}}(\psi \cdot T_g L_{g^{-1}} \cdot X, \psi \cdot T_g L_{g^{-1}} \cdot Y) \\
&= \mathbf{g}'_{\mathbf{1}}(T_{\mathbf{1}}\phi \cdot T_g L_{g^{-1}} \cdot X, T_{\mathbf{1}}\phi \cdot T_g L_{g^{-1}} \cdot Y) \\
&= \mathbf{g}'_{\phi(g)}(T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot T_g L_{g^{-1}} \cdot X, T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot T_g L_{g^{-1}} \cdot Y) \\
&= \mathbf{g}'_{\phi(g)}(T_g(L_{\phi(g)} \circ \phi \circ L_{g^{-1}}) \cdot X, T_g(L_{\phi(g)} \circ \phi \circ L_{g^{-1}}) \cdot Y) \\
&= \mathbf{g}'_{\phi(g)}(T_g\phi \cdot X, T_g\phi \cdot Y).
\end{aligned}$$

Therefore $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}', \mathbf{g}')$ are $\mathfrak{L}$-isometric. ∎

The isometric group automorphisms, *i.e.*, $\mathfrak{L}$-isometries, preserve both the Lie group structure and the sub-Riemannian structure. It turns out that a dilation of the metric, while not preserving the sub-Riemannian structure, does not affect it in an appreciable fashion. This allows us to consider scaled sub-Riemannian structures as essentially the same. Let $d(\cdot, \cdot)$ and $\ell(\cdot)$ denote the Carnot-Carathéodory metric and length functional, respectively, of $(\mathcal{D}, \mathbf{g})$. Similarly, let $d'(\cdot, \cdot)$ and $\ell'(\cdot)$ denote the Carnot-Carathéodory metric and length functional of the dilated structure $(\mathcal{D}, r^2\mathbf{g})$, where $r > 0$.

5.1.4 PROPOSITION. *The following statements hold true regarding* $(\mathcal{D}, \mathbf{g})$ *and* $(\mathcal{D}, r^2\mathbf{g})$, $r > 0$:

(*i*) *The horizontal curves of* $(\mathcal{D}, \mathbf{g})$ *and* $(\mathcal{D}, r^2\mathbf{g})$ *are in a one-to-one correspondence.*

(*ii*) $d'(\cdot, \cdot) = rd(\cdot, \cdot)$ *and* $\ell'(\cdot) = r\ell(\cdot)$.

(*iii*) *The geodesics of* $(\mathcal{D}, \mathbf{g})$ *and* $(\mathcal{D}, r^2\mathbf{g})$ *are in a one-to-one correspondence.*

PROOF.

(*i*) Distributions are invariant under dilations. Furthermore, the class of horizontal curves depends only on the distribution and not on the sub-Riemannian metric. It follows that the horizontal curves of $(\mathcal{D}, \mathbf{g})$ are exactly those of $(\mathcal{D}, r^2\mathbf{g})$, and *vice versa*.

(*ii*) Let $g(\cdot) : [0, T] \to \mathsf{G}$ be a horizontal curve. We have

$$\ell'(g(\cdot)) = \int_0^T \sqrt{r^2\mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t))}\, dt = r\ell(g(\cdot)),$$

i.e., $\ell'(\cdot) = r\ell(\cdot)$. Similarly, for every $a, b \in \mathsf{G}$, we have

$$\begin{aligned} d'(a,b) &= \inf\left\{\ell'(g(\cdot)) : g : [0,T] \to \mathsf{G} \text{ is horizontal, } g(0) = a,\ g(T) = b\right\} \\ &= \inf\left\{r\ell(g(\cdot)) : g : [0,T] \to \mathsf{G} \text{ is horizontal, } g(0) = a,\ g(T) = b\right\} \\ &= rd(a,b). \end{aligned}$$

Thus $d'(\cdot,\cdot) = rd(\cdot,\cdot)$.

(*iii*) Let $g(\cdot) : [0,T] \to \mathsf{G}$ be a geodesic of $(\mathcal{D}, \mathbf{g})$. Then $d(g(0), g(T)) = \ell(g(\cdot))$. Using item (*ii*), we have

$$d'(g(0), g(T)) = rd(g(0), g(T)) = r\ell(g(\cdot)) = \ell'(g(\cdot)).$$

That is, $g(\cdot)$ is a geodesic of $(\mathcal{D}, r^2\mathbf{g})$. Conversely, if $g(\cdot)$ is a geodesic of $(\mathcal{D}, r^2\mathbf{g})$, then it is a geodesic of the scaled structure $(\mathcal{D}, \frac{1}{r^2}(r^2\mathbf{g})) = (\mathcal{D}, \mathbf{g})$. Thus the geodesics of $(\mathcal{D}, \mathbf{g})$ and $(\mathcal{D}, r^2\mathbf{g})$ are in a one-to-one correspondence. ■

5.2 The Riemannian Problem

We begin by classifying the left-invariant Riemannian structures on $\mathsf{SE}(1,1)$. We show that, up to $\mathfrak{L}$-isometries, there is a two-parameter family of class representatives. This may be further reduced to a single-parameter family by employing a suitable dilation. The results of this classification (before scaling) coincide with those obtained by the authors of [25]. (However, note that $\mathsf{SE}(1,1)$ is denoted Sol in [25], and a different basis is employed for $\mathfrak{se}(1,1)$, *viz.* $(E_1 - E_2, E_1 + E_2, -E_3)$.) Following the classification, we shall calculate the Riemannian geodesics for each normalised Riemannian structure on $\mathsf{SE}(1,1)$.

We briefly recall the automorphism group of $\mathfrak{se}(1,1)$. Let $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$. Then ψ is of the form

$$\psi = \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}, \qquad \varsigma \in \{-1, 1\},\ x^2 \neq y^2.$$

(See proposition 1.1.17.) If we refer to an unspecified automorphism, we shall assume it is of this form.

5.2.1 Theorem. *Every left-invariant Riemannian structure on* $\mathsf{SE}(1,1)$ *is* $\mathfrak{L}$*-isometric to exactly one of the structures* $\mu\,\mathbf{g}^\lambda$*, where* $\mathbf{g}^\lambda$ *is specified in the basis* $(E_i)_{i=1}^3$ *by*

$$\mathbf{g}^\lambda_{\mathbf{1}}(X,Y) = X^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} Y, \qquad X, Y \in \mathfrak{se}(1,1).$$

Here $\mu > 0$ *and* $0 < \lambda \leq 1$ *parametrise a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.*

Proof. Let $\mathbf{g}$ be a left-invariant Riemannian structure on $\mathsf{SE}(1,1)$, given in coordinates by

$$\mathbf{g}_{\mathbf{1}} = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix} \qquad (a_1, a_2, a_3 > 0).$$

(For brevity, we identify $\mathbf{g_1}$ with its associated matrix, in terms of the basis $(E_i)_{i=1}^3$.) Define the automorphism

$$\psi_1 = \begin{bmatrix} 1 & 0 & \frac{b_1 b_3 - a_2 b_2}{a_1 a_2 - b_1^2} \\ 0 & 1 & \frac{b_1 b_2 - a_1 b_3}{a_1 a_2 - b_1^2} \\ 0 & 0 & 1 \end{bmatrix}.$$

(Since $\mathbf{g_1}$ is positive definite, the 2×2 principal minor $a_1 a_2 - b_1^2$ is strictly positive.) We have

$$\psi_1^\top \mathbf{g_1} \psi_1 = \begin{bmatrix} a_1 & b_1 & 0 \\ b_1 & a_2 & 0 \\ 0 & 0 & a_3' \end{bmatrix},$$

for some $a_3' > 0$. Suppose $b_1 \neq 0$, and let

$$\psi_2 = \begin{bmatrix} 1 & -\frac{a_1 + a_2 + \sqrt{(a_1+a_2)^2 - 4b_1^2}}{2b_1} & 0 \\ -\frac{a_1 + a_2 + \sqrt{(a_1+a_2)^2 - 4b_1^2}}{2b_1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\psi_1^\top \mathbf{g_1} \psi_1$ is positive definite, it follows that $a_1 + a_2 > 2b_1$, whence $(a_1 + a_2)^2 - 4b_1^2 > 0$. Consequently, ψ_2 is an automorphism such that $(\psi_1\psi_2)^\top \mathbf{g_1} (\psi_1\psi_2) = \mathrm{diag}(a_1', a_2', a_3')$ for some $a_1', a_2' > 0$. (If $b_1 = 0$, then $\psi_2 = \mathrm{diag}(1,1,1)$ suffices, with $a_1' = a_1$ and $a_2' = a_2$.) Then the automorphism $\psi_3 = \mathrm{diag}\left(\sqrt{\frac{a_3'}{a_1}}, \sqrt{\frac{a_3'}{a_1}}, 1\right)$ yields $(\psi_1\psi_2\psi_3)^\top \mathbf{g_1} (\psi_1\psi_2\psi_3) = a_3' \,\mathrm{diag}(1, a_2'', 1)$, where $a_2'' > 0$. For brevity, let $\mathbf{g_1'} = (\psi_1\psi_2\psi_3)^\top \mathbf{g_1} (\psi_1\psi_2\psi_3)$. If $a_2'' > 1$, then

$$\psi_4 = \begin{bmatrix} 0 & \frac{1}{\sqrt{a_2''}} & 0 \\ \frac{1}{\sqrt{a_2''}} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi_4^\top \mathbf{g_1'} \psi_4 = a_3' \,\mathrm{diag}(1, \frac{1}{a_2''}, 1)$. Thus either $\mathbf{g}' = \mu\, \mathbf{g}^\lambda$ where $\mu = a_3'$ and $\lambda = a_2'' \leq 1$ or $\psi_4^\top \mathbf{g_1'} \psi_4 = \mu\, \mathbf{g}^\lambda$ where $\mu = a_3'$ and $\lambda = \frac{1}{a_2''} \leq 1$. Therefore, by theorem 5.1.3, $\mathbf{g}$ is $\mathfrak{L}$-isometric to the structure $\mu\, \mathbf{g}^\lambda$, where $\mu > 0$ and $0 < \lambda \leq 1$.

We show that $\mu\, \mathbf{g_1^\lambda}$ is $\mathfrak{L}$-isometric to $\mu' \mathbf{g_1^{\lambda'}}$ only if $\mu = \mu'$ and $\lambda = \lambda'$. Suppose there exists an automorphism ψ such that $\psi^\top \mathbf{g_1^\lambda} \psi = \frac{\mu'}{\mu} \mathbf{g_1^{\lambda'}}$. In matrix form,

$$\begin{bmatrix} x^2 + \lambda y^2 & xy(1+\lambda) & vx + \varsigma\lambda wy \\ xy(1+\lambda) & y^2 + x^2\lambda & vy + \varsigma\lambda wx \\ vx + \varsigma\lambda wy & vy\varsigma\lambda + wx & 1 + v^2 + \lambda w^2 \end{bmatrix} = \frac{\mu'}{\mu} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As $\lambda > 0$, we have $xy = 0$. Suppose $x = 0$; then $v = w = 0$, as $y \neq 0$, and $\frac{\mu'}{\mu} = 1$. Hence $\mu = \mu'$ (since $\mu, \mu' > 0$), and we are left with the equality $\mathrm{diag}(\lambda y^2, y^2, 1) = \mathrm{diag}(1, \lambda', 1)$. This implies that $\lambda\lambda' = 1$, and so $\lambda = \lambda' = 1$. Suppose $y = 0$; then $v = w = 0$, $\mu = \mu'$, and we have $\mathrm{diag}(x^2, \lambda x^2, 1) = \mathrm{diag}(1, \lambda', 1)$. This implies that $\lambda = \lambda'$. ■

Considering the classification of theorem 5.2.1, and the fact that dilations of Riemannian structures do not have an appreciable effect (proposition 5.1.4), we shall study the single-parameter family of Riemannian structures $\mathbf{g}^\lambda$ on $\mathsf{SE}(1,1)$. In particular, we find explicit expressions for the Riemannian geodesics. The Riemannian problem for $\mathbf{g}^\lambda$ may be written as

$$\begin{gathered}\dot g(t) \in T_{g(t)}\mathsf{SE}(1,1), \quad g(\cdot) : [0,T] \to \mathsf{SE}(1,1) \\ g(0) = \mathbf{1},\ g(T) = g_T, \quad g_T \in \mathsf{SE}(1,1),\ T > 0 \text{ fixed} \\ \ell(g(\cdot)) = \int_0^T \sqrt{\mathbf{g}^\lambda_{g(t)}(\dot g(t), \dot g(t))}\, dt \to \min.\end{gathered}$$

(Due to left invariance, we restrict to Riemannian geodesics starting at the identity, since this can always be arranged.) In accordance with the discussion in section A.4, we may write the Riemannian problem as an optimal control problem:

$$(\mathsf{R}) \quad \begin{cases} \dot g = g\,\Xi(\mathbf{1}, u) = g(u_1 E_1 + u_2 E_2 + u_3 E_3), \quad g(\cdot) : [0,T] \to \mathsf{SE}(1,1),\ u(\cdot) : [0,T] \to \mathbb{R}^\ell \\ g(0) = \mathbf{1},\ g(T) = g_T, \quad g_T \in \mathsf{SE}(1,1),\ T > 0 \text{ fixed} \\ \mathcal{J}(u(\cdot)) = \int_0^T u_1(t)^2 + \lambda u_2(t)^2 + u_3(t)^2\, dt \to \min. \end{cases}$$

By theorem A.4.2, solutions to (R) are guaranteed to exist (at least, locally; *i.e.*, for g_T in a sufficiently small neighbourhood of identity). The family $(H^\nu_u)_{u\in\mathbb{R}^3}$ of control-dependent Hamiltonian functions is specified by

$$H^\nu_u(p) = u_1 p_1 + u_2 p_2 + u_3 p_3 + \nu(u_1^2 + \lambda u_2^2 + u_3^2).$$

Consider the abnormal geodesics. Set $\nu = 0$. The maximality condition (A.3.8) of PMP implies

$$\frac{\partial H^0_u}{\partial u_i} = 0 \qquad \iff \qquad p_i = 0, \qquad i = 1, 2, 3.$$

Consequently, $p(t) = 0$ for all t, which contradicts the regularity condition $(\nu, p(t)) \not\equiv 0$ of the Maximum Principle. Hence, there are no abnormal extremals.

5.2.2 REMARK. The absence of abnormal geodesics for the structure $\mathbf{g}^\lambda$ is not unique to this structure. Indeed, there are no abnormal geodesics on Riemannian manifolds. (This fact is indicative of the differences between Riemannian and sub-Riemannian geometry.) See, *e.g.*, [37]. □

Consider the normal geodesics. Theorem A.3.8 implies that the (normal) extremal controls are given by

$$u(t) = Q^{-1}\mathbf{B}^\top p(t)^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top p(t)^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix} p(t)^\top.$$

That is, $u_1(t) = p_1(t)$, $u_2(t) = \frac{1}{\lambda} p_2(t)$ and $u_3(t) = p_3(t)$. Here $p(\cdot)$ is an integral curve of the Hamilton-Poisson system $(\mathfrak{se}(1,1)^*_-, H^{(\mathsf{R})})$, where $H^{(\mathsf{R})}$ is the (reduced) Hamiltonian function

$$H^{(\mathsf{R})}(p) = \tfrac{1}{2}\, p\,\mathbf{B}\, Q^{-1}\mathbf{B}^\top p^\top = \tfrac{1}{2}\left(p_1^2 + \tfrac{1}{\lambda} p_2^2 + p_3^2\right).$$

Let $(g(t), p(t))$ be an extremal trajectory of (R). We have

$$\mathbf{g}^{\lambda}_{g(t)}(\dot g(t), \dot g(t)) = 1 \iff u_1(t)^2 + \lambda u_2(t)^2 + u_3(t)^2 = 1 \iff H^{(\mathsf{R})}(p(t)) = \tfrac{1}{2}.$$

Thus, we assume that $H^{(\mathsf{R})}(p(t)) = \frac{1}{2}$, and so the resultant geodesics will have unit speed. The geodesic equations take the form

$$\begin{cases} \dot p = \vec{H}^{(\mathsf{R})}(p) & \text{(vertical subsystem)} \\ \dot g = g\,\Xi(\mathbf{1}, u) & \text{(horizontal subsystem).} \end{cases}$$

The vertical subsystem is described by the (quadratic and homogeneous) Hamilton-Poisson system $(\mathfrak{se}(1,1)^*_-, H^{(\mathsf{R})})$. This system is A-equivalent to the system $(\mathfrak{se}(1,1)^*_-, H_4)$, which we have already investigated. (See theorem 3.2.1 and section 4.2.2.) Indeed,

$$\Psi : p \mapsto p\psi, \qquad \psi = \operatorname{diag}\left(-\sqrt{\tfrac{\lambda}{\lambda+1}}, -\sqrt{\tfrac{\lambda}{\lambda+1}}, 1\right) \tag{5.2.1}$$

is a linear isomorphism such that $\Psi \cdot \vec{H}_4 = \vec{H}^{(\mathsf{R})} \circ \Psi$. (If $p(\cdot)$ is an integral curve of $\vec{H}_4$, then $\Psi(p(\cdot))$ is an integral curve of $\vec{H}^{(\mathsf{R})}$.) The following proposition determines the form of all integral curves of $\vec{H}^{(\mathsf{R})}$.

5.2.3 Proposition. *Let $\bar p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}^{(\mathsf{R})}$. Suppose that $H^{(\mathsf{R})}(\bar p(0)) = \frac{1}{2}$ and let $c_0 = C(\bar p(0)) \leq 1$.*

(a) (i) If $c_0 = 1$, then $p(t) = \big(\pm 1, 0, 0\big)$.

(ii) If $0 < c_0 < 1$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar p(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar p_1(t) = -\sigma\sqrt{\tfrac{\lambda + c_0}{\lambda+1}}\,\operatorname{dn}(\Omega t, k) \\ \bar p_2(t) = \sigma k\sqrt{\tfrac{\lambda + c_0}{\lambda+1}}\,\operatorname{cn}(\Omega t, k) \\ \bar p_3(t) = k\Omega\operatorname{sn}(\Omega t, k). \end{cases}$$

Here $\Omega = \sqrt{\frac{\lambda + c_0}{\lambda}}$ and $k = \sqrt{\frac{\lambda(1 - c_0)}{\lambda + c_0}}$.

(b) (i) If $c_0 = 0$ and $p_1(0) = 0$, then $p(t) = (0, 0, \pm 1)$.

(ii) If $c_0 = 0$ and $p_1(0) \neq 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar p(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar p_1(t) = -\sigma\sqrt{\tfrac{\lambda}{\lambda+1}}\,\operatorname{sech} t \\ \bar p_2(t) = \sigma\varsigma\sqrt{\tfrac{\lambda}{\lambda+1}}\,\operatorname{sech} t \\ \bar p_3(t) = \varsigma \tanh t. \end{cases}$$

(c) If $c_0 < 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar p(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar p_1(t) = \sigma k\sqrt{\tfrac{\lambda(1 - c_0)}{\lambda+1}}\,\operatorname{cn}(\Omega t, k) \\ \bar p_2(t) = -\sigma\sqrt{\tfrac{\lambda(1 - c_0)}{\lambda+1}}\,\operatorname{dn}(\Omega t, k) \\ \bar p_3(t) = k\Omega\operatorname{sn}(\Omega t, k). \end{cases}$$

Here $\Omega = \sqrt{1-c_0}$ *and* $k = \sqrt{\frac{\lambda+c_0}{\lambda(1-c_0)}}$.

Proof. We have that $\Psi \cdot \vec{H}_4 = \vec{H}^{(\mathsf{R})} \circ \Psi$, where Ψ is given in equation (5.2.1). Let $q(\cdot) = \Psi^{-1}(p(\cdot))$ and $\bar{q}(\cdot) = \Psi^{-1}(\bar{p}(\cdot))$ be the integral curves of $\vec{H}_4$ corresponding to $\bar{p}(\cdot)$ and $p(\cdot)$, respectively. Let $d_0 = C(q(0))$ and $h_0 = H_4(q(0)) > 0$. We have

$$d_0 = q_1(0)^2 - q_2(0)^2 = \tfrac{\lambda+1}{\lambda}(p_1(0)^2 - p_2(0)^2) = \tfrac{\lambda+1}{\lambda}c_0,$$

and so $\operatorname{sgn}(d_0) = \operatorname{sgn}(c_0)$. Since $c_0 = p_1(t)^2 - p_2(t)^2 \leq p_1(t)^2 + \frac{1}{\lambda}p_2(t)^2 + p_3(t)^2 = 1$, we have $c_0 \leq 1$. If $c_0 = 1$, then $p_1(t)^2 - p_2(t)^2 = 1 = p_1(t)^2 + \frac{1}{\lambda}p_2(t)^2 + p_3(t)^2$, whence $-p_2(t)^2 = \frac{1}{\lambda}p_2(t)^2 + p_3(t)^2$. This implies that $p_2(t) = p_3(t) = 0$, and so $p_1(t) = \pm 1$. Hence $p(t) = (\pm 1, 0, 0)$.

Suppose $0 < c_0 < 1$. By proposition 4.2.7, there exists $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $q(t) = \bar{q}(t + t_0)$ for every t, where $\bar{q}(\cdot)$ is given in the statement of that proposition. Consequently, there exists $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \Psi(q(t)) = \Psi(\bar{q}(t + t_0)) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where $\bar{p}(\cdot) = \Psi(\bar{q}(\cdot))$ is given by

$$\begin{cases} \bar{p}_1(t) = -\sigma\sqrt{2h_0}\sqrt{\frac{\lambda}{\lambda+1}}\operatorname{dn}\left(\sqrt{2h_0}\,t, k\right) \\ \bar{p}_2(t) = \sigma k\sqrt{2h_0}\sqrt{\frac{\lambda}{\lambda+1}}\operatorname{cn}\left(\sqrt{2h_0}\,t, k\right) \\ \bar{p}_3(t) = k\sqrt{2h_0}\operatorname{sn}\left(\sqrt{2h_0}\,t, k\right). \end{cases}$$

Here $k = \sqrt{\frac{2h_0 - d_0}{2h_0}}$. We have $\frac{1}{2} = H^{(\mathsf{R})}(\bar{p}(0)) = h_0 - \frac{d_0}{2(\lambda+1)}$, and so $h_0 = \frac{\lambda+d_0+1}{2(\lambda+1)} = \frac{\lambda+c_0}{2\lambda}$. Substituting for h_0 and d_0 in the expression for $\bar{p}(\cdot)$ completes the result.

Suppose $c_0 = 0$. If $p_1(0) = 0$, then $p_2(0) = 0$ and $p_3(0) = \pm 1$. Since $\Psi^{-1}((0,0,\pm 1)) = (0,0,\pm 1)$ is an equilibrium point of $\vec{H}_4$ (see proposition 4.2.5), the point $(0,0,\pm 1)$ is an equilibrium point of $\vec{H}^{(\mathsf{R})}$. Hence $p(t) = (0,0,\pm 1)$ is constant. If $p_1(0) \neq 0$, then from proposition 4.2.8 there exists $\sigma, \varsigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $q(t) = \bar{q}(t + t_0)$ for every t. (The expression for $\bar{q}(\cdot)$ may be found in the statement of proposition 4.2.8.) Thus, there exists $\sigma, \varsigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \Psi(q(t)) = \Psi(\bar{q}(t + t_0)) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar{p}_1(t) = -\sigma\sqrt{2h_0}\sqrt{\frac{\lambda}{\lambda+1}}\operatorname{sech}\left(\sqrt{2h_0}\,t\right) \\ \bar{p}_2(t) = \sigma\varsigma\sqrt{2h_0}\sqrt{\frac{\lambda}{\lambda+1}}\operatorname{sech}\left(\sqrt{2h_0}\,t\right) \\ \bar{p}_3(t) = \varsigma\sqrt{2h_0}\tanh\left(\sqrt{2h_0}\,t\right). \end{cases}$$

We have $\frac{1}{2} = H^{(\mathsf{R})}(\bar{p}(0)) = h_0$. Substituting this value into the above expression for $\bar{p}(\cdot)$ yields the result.

Suppose $c_0 < 0$. By lemma 4.2.6 and proposition 4.2.7, there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $q(t) = \bar{q}(t + t_0)$ for every t, where

$$\begin{cases} \bar{q}_1(t) = \sigma k\sqrt{2h_0 - d_0}\operatorname{cn}\left(\sqrt{2h_0 - d_0}\,t, k\right) \\ \bar{q}_2(t) = -\sigma\sqrt{2h_0 - d_0}\operatorname{dn}\left(\sqrt{2h_0 - d_0}\,t, k\right) \\ \bar{q}_3(t) = k\sqrt{2h_0 - d_0}\operatorname{sn}\left(\sqrt{2h_0 - d_0}\,t, k\right). \end{cases}$$

Here $k = \sqrt{\frac{2h_0}{2h_0 - d_0}}$. Therefore, there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \Psi(q(t)) = \Psi(\bar{q}(t + t_0)) = \bar{p}(t + t_0)$, where $\bar{p}(\cdot)$ is given by

$$\begin{cases} \bar{p}_1(t) = \sigma k \sqrt{2h_0 - d_0}\,\sqrt{\frac{\lambda}{\lambda+1}}\,\mathrm{cn}\left(\sqrt{2h_0 - d_0}\,t, k\right) \\ \bar{p}_2(t) = -\sigma \sqrt{2h_0 - d_0}\,\sqrt{\frac{\lambda}{\lambda+1}}\,\mathrm{dn}\left(\sqrt{2h_0 - d_0}\,t, k\right) \\ \bar{p}_3(t) = k\sqrt{2h_0 - d_0}\,\mathrm{sn}\left(\sqrt{2h_0 - d_0}\,t, k\right). \end{cases}$$

From $\frac{1}{2} = H^{(\mathsf{R})}(\bar{p}(0)) = h_0 - \frac{d_0}{2(1+\lambda)}$, we get $h_0 = \frac{\lambda + d_0 + 1}{2(1+\lambda)} = \frac{\lambda + c_0}{2\lambda}$. Substituting for h_0 and d_0 in the expression for $\bar{p}(\cdot)$ completes the result. ■

The horizontal subsystem is given in coordinates by

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 0 \\ \dot{x} & \dot{\theta}\sinh\theta & \dot{\theta}\cosh\theta \\ \dot{y} & \dot{\theta}\cosh\theta & \dot{\theta}\sinh\theta \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} (u_1 E_1 + u_2 E_2 + u_3 E_3) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ p_1 & 0 & p_3 \\ \frac{1}{\lambda}p_2 & p_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ p_1\cosh\theta + \frac{1}{\lambda}p_2\sinh\theta & p_3\sinh\theta & p_3\cosh\theta \\ p_1\sinh\theta + \frac{1}{\lambda}p_2\cosh\theta & p_3\cosh\theta & p_3\sinh\theta \end{bmatrix}. \end{aligned}$$

Equating components yields the following equations:

$$\begin{cases} \dot{x} = p_1\cosh\theta + \frac{1}{\lambda}p_2\sinh\theta \\ \dot{y} = p_1\sinh\theta + \frac{1}{\lambda}p_2\cosh\theta \\ \dot{\theta} = p_3. \end{cases} \tag{5.2.2}$$

As $g(0) = \mathbf{1}$, we have the initial conditions $x(0) = y(0) = \theta(0) = 0$. For brevity, we shall make the identification

$$(x, y, \theta) \in \mathbb{R}^3 \qquad \longleftrightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix}.$$

(Hence, we refer to (5.2.2) as the horizontal subsystem.)

We shall now integrate (5.2.2). Let $(\bar{g}(\cdot), \bar{p}(\cdot))$ be an extremal trajectory for the optimal control problem (R), where $\bar{p}(\cdot)$ is given in proposition 5.2.3. We write the geodesic $\bar{g}(\cdot)$ as the triple $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$. From the geodesic equations, we have $\dot{\bar{x}}(0) = \bar{p}_1(0)$, $\lambda\dot{\bar{y}}(0) = \bar{p}_2(0)$ and $\dot{\bar{\theta}}(0) = \bar{p}_3(0)$. Consequently, we shall have a total of five different qualitative cases, corresponding to those of proposition 5.2.3. Table 5.1 enumerates the cases and their designations.

For the purposes of integration, we make the change of variables $\bar{v} = \bar{x} + \bar{y}$ and $\bar{w} = \bar{x} - \bar{y}$. The horizontal subsystem becomes

$$\begin{cases} \dot{\bar{v}} = (\bar{p}_1 + \frac{1}{\lambda}\bar{p}_2)e^{\bar{\theta}} \\ \dot{\bar{w}} = (\bar{p}_1 - \frac{1}{\lambda}\bar{p}_2)e^{-\bar{\theta}} \\ \dot{\bar{\theta}} = \bar{p}_3. \end{cases}$$

Conditions		Designation
$\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 > 0$	$\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 = 1$	Case *I*-*a*
	$\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 1$	Case *I*-*b*
$\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 = 0$	$\dot{x}(0) = 0$	Case *II*-*a*
	$\dot{x}(0) \neq 0$	Case *II*-*b*
$\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 0$		Case *III*

Table 5.1: Qualitative breakdown of cases for the optimal control problem (R)

Case I**:** $\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 > 0$

Case I-a: $\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 = 1$

We have $c_0 = \bar{p}_1(0)^2 - \bar{p}_2(0)^2 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 = 1$, and so by proposition 5.2.3 $\bar{p}(\cdot)$ is constant. Indeed, we have $\bar{p}(t) = (\sigma, 0, 0)$, for some $\sigma \in \{-1, 1\}$. The horizontal subsystem is thus

$$\begin{cases} \dot{\bar{x}} = \sigma \cosh \bar{\theta} \\ \dot{\bar{y}} = \sigma \sinh \bar{\theta} \\ \dot{\bar{\theta}} = 0. \end{cases}$$

These are immediately solved, to give the geodesic $\bar{g}(t) = (\sigma t, 0, 0)$. Evaluating $\dot{\bar{g}}(0)$, we have $\sigma = \operatorname{sgn}(\dot{x}(0))$.

Case I-b: $\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 1$

We have $0 < c_0 = \bar{p}_1(0)^2 - \bar{p}_2(0)^2 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 1$. Hence, from proposition 5.2.3, case (a), (ii), the horizontal subsystem (in coordinates $(\bar{v}, \bar{w}, \bar{\theta})$) reads

$$\begin{cases} \dot{\bar{v}} = -\sigma\sqrt{\frac{\lambda+c_0}{\lambda+1}}\left[\operatorname{dn}(\Omega t, k) - \frac{k}{\lambda}\operatorname{cn}(\Omega t, k)\right] e^{\bar{\theta}(t)} \\ \dot{\bar{w}} = -\sigma\sqrt{\frac{\lambda+c_0}{\lambda+1}}\left[\operatorname{dn}(\Omega t, k) + \frac{k}{\lambda}\operatorname{cn}(\Omega t, k)\right] e^{-\bar{\theta}(t)} \\ \dot{\bar{\theta}} = k\Omega\operatorname{sn}(\Omega t, k). \end{cases}$$

(Here $\sigma \in \{-1, 1\}$, $\Omega = \sqrt{\frac{\lambda+c_0}{\lambda}}$ and $k = \sqrt{\frac{\lambda(1-c_0)}{\lambda+c_0}}$.) Separate variables in the last equation, and use equation (A.6.15) to integrate the right-hand side. We get $\bar{\theta}(t) = k\Omega\int \operatorname{sn}(\Omega t, k)\, dt = \ln\left[\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)\right] + c_{\bar{\theta}}$ with $c_{\bar{\theta}} \in \mathbb{R}$. From $\bar{\theta}(0) = 0$, it follows that $c_{\bar{\theta}} = -\ln(1-k)$. Thus $\bar{\theta}(t) = \ln\left[\frac{\operatorname{dn}(\Omega t,k) - k\operatorname{cn}(\Omega t,k)}{1-k}\right]$. The first equation of motion now becomes

$$\begin{aligned} \dot{\bar{v}} &= -\sigma\sqrt{\frac{\lambda+c_0}{\lambda+1}}\left[\operatorname{dn}(\Omega t, k) - \frac{k}{\lambda}\operatorname{cn}(\Omega t, k)\right]\frac{\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)}{1-k} \\ &= -\frac{\sigma}{1-k}\sqrt{\frac{\lambda+c_0}{\lambda+1}}\left[\operatorname{dn}^2(\Omega t, k) - k(1+\tfrac{1}{\lambda})\operatorname{dn}(\Omega t, k)\operatorname{cn}(\Omega t, k) + \tfrac{k^2}{\lambda}\operatorname{cn}^2(\Omega t, k)\right] \\ &= -\frac{\sigma}{1-k}\sqrt{\frac{\lambda+c_0}{\lambda+1}}\left[(1+\tfrac{1}{\lambda})\operatorname{dn}^2(\Omega t, k) - k(1+\tfrac{1}{\lambda})\operatorname{dn}(\Omega t, k)\operatorname{cn}(\Omega t, k) - \tfrac{(k')^2}{\lambda}\right] \\ &= -\frac{\sigma}{1-k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\left[\operatorname{dn}^2(\Omega t, k) - k\operatorname{dn}(\Omega t, k)\operatorname{cn}(\Omega t, k) - \tfrac{(k')^2}{\lambda+1}\right]. \end{aligned}$$

(We have used the square relation (A.6.6) in the penultimate step.) Integrating both sides, we have

$$\begin{aligned}\bar{v}(t) = &-\frac{\sigma}{1-k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\int_0^t \operatorname{dn}^2(\Omega t,k)\,dt\\ &+\frac{\sigma k}{1-k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt\\ &+\frac{\sigma(1+k)}{\lambda}\sqrt{\frac{\lambda+c_0}{\lambda+1}}\,t.\end{aligned}$$

We can integrate the first expression using the integral formula (A.6.14):

$$-\frac{\sigma}{1-k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\int_0^t \operatorname{dn}^2(\Omega t,k)\,dt = -\frac{\sigma}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,E(\operatorname{am}(\Omega t,k),k).$$

Similarly, using the derivative formula (A.6.1), we have

$$\frac{\sigma k}{1-k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt = \frac{\sigma k}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,\operatorname{sn}(\Omega t,k).$$

Consequently,

$$\begin{aligned}\bar{v}(t) &= -\frac{\sigma}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,E(\operatorname{am}(\Omega t,k),k)+\frac{\sigma k}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,\operatorname{sn}(\Omega t,k)+\frac{\sigma(1+k)}{\lambda}\sqrt{\tfrac{\lambda+c_0}{\lambda+1}}\,t\\ &= -\frac{\sigma}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t,k),k)-k\operatorname{sn}(\Omega t,k)-\tfrac{(k')^2}{\lambda+1}\Omega t\right].\end{aligned}$$

The last equation of motion is

$$\begin{aligned}\dot{\bar{w}} &= -\sigma\sqrt{\tfrac{\lambda+c_0}{\lambda+1}}\left[\operatorname{dn}(\Omega t,k)+\tfrac{k}{\lambda}\operatorname{cn}(\Omega t,k)\right]\frac{1-k}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}\cdot\frac{\operatorname{dn}(\Omega t,k)+k\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)+k\operatorname{cn}(\Omega t,k)}\\ &= -\sigma\sqrt{\tfrac{\lambda+c_0}{\lambda+1}}\left[\operatorname{dn}(\Omega t,k)+\tfrac{k}{\lambda}\operatorname{cn}(\Omega t,k)\right]\frac{(1-k)\left[\operatorname{dn}(\Omega t,k)+k\operatorname{cn}(\Omega t,k)\right]}{\operatorname{dn}^2(\Omega t,k)-k^2\operatorname{cn}^2(\Omega t,k)}\\ &= -\frac{\sigma}{\lambda(1+k)}\sqrt{\tfrac{\lambda+c_0}{\lambda+1}}\left[\lambda^2\operatorname{dn}^2(\Omega t,k)+k(\lambda+1)\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)+k^2\operatorname{cn}^2(\Omega t,k)\right]\\ &= -\frac{\sigma}{1+k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\left[\operatorname{dn}^2(\Omega t,k)+k\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)-\tfrac{(k')^2}{\lambda+1}\right].\end{aligned}$$

(We have used the square relation (A.6.6) in the penultimate step.) Separating variables and integrate both sides. We get

$$\begin{aligned}\bar{w}(t) = &-\frac{\sigma}{1+k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\int_0^t \operatorname{dn}^2(\Omega t,k)\,dt\\ &-\frac{\sigma k}{1+k}\frac{\sqrt{(\lambda+c_0)(\lambda+1)}}{\lambda}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt\\ &+\frac{\sigma(1-k)}{\lambda}\sqrt{\tfrac{\lambda+c_0}{\lambda+1}}\,t\\ = &-\frac{\sigma}{1+k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,E(\operatorname{am}(\Omega t,k),k)-\frac{\sigma k}{1+k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,\operatorname{sn}(\Omega t,k)+\frac{\sigma(1-k)}{\lambda}\sqrt{\tfrac{\lambda+c_0}{\lambda+1}}\,t\\ = &-\frac{\sigma}{1+k}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t,k),k)+k\operatorname{sn}(\Omega t,k)-\tfrac{(k')^2}{\lambda+1}\Omega t\right].\end{aligned}$$

Lastly, using $\begin{bmatrix}\bar{x}(t)\\ \bar{y}(t)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}\bar{v}(t)\\ \bar{w}(t)\end{bmatrix}$, we can find $\bar{x}(\cdot)$ and $\bar{y}(\cdot)$. Therefore we have the following expression for $\bar{g}(\cdot)$:

$$\begin{cases} \bar{x}(t) = -\dfrac{\sigma}{1-k^2}\sqrt{\dfrac{\lambda+1}{\lambda}}\left[E(\mathrm{am}(\Omega t,k),k) - k^2\,\mathrm{sn}(\Omega t,k) - \frac{1-k^2}{1+\lambda}\Omega t\right] \\ \bar{y}(t) = -\dfrac{\sigma k}{1-k^2}\sqrt{\dfrac{\lambda+1}{\lambda}}\left[E(\mathrm{am}(\Omega t,k),k) - \mathrm{sn}(\Omega t,k) - \frac{1-k^2}{1+\lambda}\Omega t\right] \\ \bar{\theta}(t) = \ln\left[\dfrac{\mathrm{dn}(\Omega t,k) - k\,\mathrm{cn}(\Omega t,k)}{1-k}\right]. \end{cases}$$

We now make an explicit statement regarding all Riemannian geodesics for this case.

5.2.4 Proposition. *Let $g(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ be a unit-speed geodesic on $(\mathsf{SE}(1,1), \mathbf{g}^\lambda)$ such that $g(0) = \mathbf{1}$ and $0 < \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 1$. Then $g(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ for every t, where $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$ is given by*

$$\begin{cases} \bar{x}(t) = -\dfrac{\sigma}{1-k^2}\sqrt{\dfrac{\lambda+1}{\lambda}}\left[E(\mathrm{am}(\Omega t,k),k) - k^2\,\mathrm{sn}(\Omega t,k) - \frac{1-k^2}{1+\lambda}\Omega t\right] \\ \bar{y}(t) = -\dfrac{\sigma k}{1-k^2}\sqrt{\dfrac{\lambda+1}{\lambda}}\left[E(\mathrm{am}(\Omega t,k),k) - \mathrm{sn}(\Omega t,k) - \frac{1-k^2}{1+\lambda}\Omega t\right] \\ \bar{\theta}(t) = \ln\left[\dfrac{\mathrm{dn}(\Omega t,k) - k\,\mathrm{cn}(\Omega t,k)}{1-k}\right]. \end{cases}$$

Here $c_0 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2$, $\Omega = \sqrt{\frac{\lambda+c_0}{\lambda}}$, $k = \sqrt{\frac{\lambda(1-c_0)}{\lambda+c_0}}$, $\sigma = -\,\mathrm{sgn}(\dot{x}(0))$ and ρ_0 satisfies the equation $\mathrm{dn}(\Omega\rho_0,k) = |\dot{x}(0)|\sqrt{\frac{\lambda+1}{\lambda+c_0}}$.

Proof. The curve $(g(\cdot), p(\cdot))$ is an extremal trajectory for (R), for some integral curve $p(\cdot)$ of $\vec{H}^{(\mathsf{R})}$, and corresponding to the optimal control $u(\cdot) = (p_1(\cdot), \lambda p_2(\cdot), p_3(\cdot))$. Let $c_0 = C(p(0))$. We have $c_0 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2$, and so $0 < c_0 < 1$. By proposition 5.2.3 there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t+t_0)$ for every t, where $\bar{p}(\cdot)$ is given under item (a), (ii) in the statement of that proposition. Let $\rho_0 = t_0$ and $\bar{u}(\cdot) = (\bar{p}_1(\cdot), \lambda\bar{p}_2(\cdot), \bar{p}_3(\cdot))$. Since $(g(\cdot), p(\cdot))$ is an extremal trajectory, we have

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)) = g(t)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)).$$

Similarly, if $\tilde{g}(t) = \bar{g}^{-1}(\rho_0)\bar{g}(t+\rho_0)$, then

$$\begin{aligned} \dot{\tilde{g}}(t) &= \bar{g}(\rho_0)^{-1}\dot{\bar{g}}(t+\rho_0) \\ &= \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)) = \tilde{g}(t)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)). \end{aligned}$$

Furthermore, $\tilde{g}(0) = \bar{g}(\rho_0)^{-1}\bar{g}(\rho_0) = \mathbf{1} = g(0)$. Since $t \mapsto g(t)$ and $t \mapsto \tilde{g}(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ satisfy the same differential equation, with the same initial conditions, they both solve the same Cauchy problem, and hence are identical. Finally, we have the following horizontal subsystem for $g(\cdot)$:

$$\begin{cases} \dot{x}(t) = \bar{p}_1(t+\rho_0)\cosh\theta + \frac{1}{\lambda}\bar{p}_2(t+\rho_0)\sinh\theta \\ \dot{y}(t) = \bar{p}_1(t+\rho_0)\sinh\theta + \frac{1}{\lambda}\bar{p}_2(t+\rho_0)\cosh\theta \\ \dot{\theta}(t) = \bar{p}_3(t+\rho_0). \end{cases}$$

(Here $x(0) = y(0) = \theta(0) = 0$.) Consequently, $\dot{x}(0) = \bar{p}_1(\rho_0) = -\sigma\sqrt{\frac{\lambda+c_0}{\lambda+1}}\,\mathrm{dn}(\Omega\rho_0, k)$. It follows that $\sigma = -\operatorname{sgn}(\dot{x}(0))$ and $\mathrm{dn}(\Omega\rho_0, k) = -\sigma\dot{x}(0)\sqrt{\frac{\lambda+1}{\lambda+c_0}} = |\dot{x}(0)|\sqrt{\frac{\lambda+1}{\lambda+c_0}}$. ■

Case II**:** $\dot{\bar{x}}(0)^2 - \lambda^2\dot{\bar{y}}(0)^2 = 0$

Case II-a**:** $\dot{\bar{x}}(0) = 0$

We have $c_0 = \bar{p}_1(0)^2 - \bar{p}_2(0)^2 = \dot{\bar{x}}(0)^2 - \lambda^2\dot{\bar{y}}(0)^2 = 0$ and $\bar{p}_1(0) = \dot{\bar{x}}(0) = 0$. By proposition 5.2.3, $\bar{p}(\cdot)$ is constant, given by $\bar{p}(t) = (0, 0, \sigma)$ for some $\sigma \in \{-1, 1\}$. The horizontal subsystem is thus

$$\begin{cases} \dot{\bar{x}} = 0 \\ \dot{\bar{y}} = 0 \\ \dot{\bar{\theta}} = \sigma. \end{cases}$$

These are immediately solved, to give the geodesic $\bar{g}(t) = (0, 0, \sigma t)$. Evaluating $\dot{\bar{g}}(0)$, we have $\sigma = \operatorname{sgn}(\dot{\bar{\theta}}(0))$.

Case II-b**:** $\dot{\bar{x}}(0) \neq 0$

For this case it proves easier to work with the original coordinates $(\bar{x}, \bar{y}, \bar{\theta})$, rather than $(\bar{v}, \bar{w}, \bar{\theta})$. Let $c_0 = C(\bar{p}(0)) = \bar{p}_1(0)^2 - \bar{p}_2(0)^2$. Then $c_0 = \dot{\bar{x}}(0)^2 - \lambda^2\dot{\bar{y}}(0)^2 = 0$ and $\bar{p}_1(0) = \dot{\bar{x}}(0) \neq 0$. Hence, using the expression for $\bar{p}(\cdot)$ in proposition 5.2.3, case (b), (ii), the horizontal subsystem takes the form

$$\begin{cases} \dot{\bar{x}} = -\sigma\sqrt{\frac{\lambda}{\lambda+1}}\,\mathrm{sech}\,t(\cosh\bar{\theta} - \frac{\varsigma}{\lambda}\sinh\bar{\theta}) \\ \dot{\bar{y}} = -\sigma\sqrt{\frac{\lambda}{\lambda+1}}\,\mathrm{sech}\,t(\sinh\bar{\theta} - \frac{\varsigma}{\lambda}\cosh\bar{\theta}) \\ \dot{\bar{\theta}} = \varsigma\tanh t. \end{cases}$$

(Here $\sigma, \varsigma \in \{-1, 1\}$.) The last equation can be immediately integrated, to get $\bar{\theta}(t) = \varsigma\int\tanh t\,dt = \varsigma\ln(\cosh t)$. (From $\bar{\theta}(0) = 0$, it follows that the constant of integration is zero.) Next, using the fact that $\cosh(\cdot)$ is even and $\sinh(\cdot)$ is odd, we have

$$\begin{aligned} \dot{\bar{x}} &= -\sigma\sqrt{\tfrac{\lambda}{\lambda+1}}\,\mathrm{sech}\,t\left[\cosh(\ln(\cosh t)) - \tfrac{1}{\lambda}\sinh(\ln(\cosh t))\right] \\ &= -\tfrac{\sigma}{2}\sqrt{\tfrac{\lambda}{\lambda+1}}\,\mathrm{sech}\,t\left[(\cosh t + \mathrm{sech}\,t) - \tfrac{1}{\lambda}\sinh t\tanh t\right] \\ &= -\tfrac{\sigma}{2\lambda}\sqrt{\tfrac{\lambda}{\lambda+1}}\left[(1+\lambda)\,\mathrm{sech}^2\,t - (1-\lambda)\right]. \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} \bar{x}(t) &= -\tfrac{\sigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\int_0^t \mathrm{sech}^2\,t\,dt + \tfrac{\sigma(1-\lambda)}{2\sqrt{\lambda(1+\lambda)}}t \\ &= -\tfrac{\sigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\tanh t + \tfrac{\sigma(1-\lambda)}{2\sqrt{\lambda(1+\lambda)}}t \\ &= \tfrac{\sigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[\tfrac{1-\lambda}{1+\lambda}t - \tanh t\right]. \end{aligned}$$

Lastly, we have

$$\begin{aligned}\dot{y} &= -\sigma\varsigma\sqrt{\tfrac{\lambda}{\lambda+1}}\,\operatorname{sech} t\left[\sinh(\ln(\cosh t)) - \tfrac{1}{\lambda}\cosh(\ln(\cosh t))\right]\\ &= -\tfrac{\sigma\varsigma}{2}\sqrt{\tfrac{\lambda}{\lambda+1}}\,\operatorname{sech} t\left[\sinh t\tanh t - \tfrac{1}{\lambda}(\cosh t + \operatorname{sech} t)\right]\\ &= \tfrac{\sigma\varsigma}{2\lambda}\sqrt{\tfrac{\lambda}{\lambda+1}}\,\left[(1+\lambda)\operatorname{sech}^2 t + (1-\lambda)\right].\end{aligned}$$

Integrating both sides yields

$$\begin{aligned}\bar{y}(t) &= \tfrac{\sigma\varsigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\int_0^t \operatorname{sech}^2 t\,dt + \tfrac{\sigma\varsigma(1-\lambda)}{2\lambda}\sqrt{\tfrac{\lambda}{\lambda+1}}\,t\\ &= \tfrac{\sigma\varsigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\tanh t + \tfrac{\sigma\varsigma(1-\lambda)}{2\lambda}\sqrt{\tfrac{\lambda}{\lambda+1}}\,t\\ &= \tfrac{\sigma\varsigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[\tfrac{1-\lambda}{1+\lambda}t + \tanh t\right].\end{aligned}$$

Therefore we have the following expression for $\bar{g}(\cdot)$:

$$\begin{cases}\bar{x}(t) = \tfrac{\sigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[\tfrac{1-\lambda}{1+\lambda}t - \tanh t\right]\\ \bar{y}(t) = \tfrac{\sigma\varsigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[\tfrac{1-\lambda}{1+\lambda}t + \tanh t\right]\\ \bar{\theta}(t) = \varsigma\ln(\cosh t).\end{cases}$$

We now make an explicit statement regarding all Riemannian geodesics for this case.

5.2.5 Proposition. *Let $g(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ be a unit-speed geodesic on $(\mathsf{SE}(1,1), \mathbf{g}^\lambda)$ such that $g(0) = \mathbf{1}$, $\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 = 0$ and $\dot{x}(0) \neq 0$. Then $g(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ for every t, where $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$ is given by*

$$\begin{cases}\bar{x}(t) = \tfrac{\sigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[\tfrac{1-\lambda}{1+\lambda}t - \tanh t\right]\\ \bar{y}(t) = \tfrac{\sigma\varsigma}{2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[\tfrac{1-\lambda}{1+\lambda}t + \tanh t\right]\\ \bar{\theta}(t) = \varsigma\ln(\cosh t).\end{cases}$$

Here $\sigma = -\operatorname{sgn}(\dot{x}(0))$, $\varsigma = \sigma\operatorname{sgn}(\dot{y}(0))$ and ρ_0 satisfies the equation $\operatorname{sech}\rho_0 = |\dot{x}(0)|\sqrt{\tfrac{\lambda+1}{\lambda}}$.

Proof. The curve $(g(\cdot), p(\cdot))$ is an extremal trajectory for (R), for some integral curve $p(\cdot)$ of $\vec{H}^{(\mathsf{R})}$, and corresponding to the optimal control $u(\cdot) = (p_1(\cdot), \lambda p_2(\cdot), p_3(\cdot))$. Let $c_0 = C(p(0))$. We have $c_0 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 = 0$. By proposition 5.2.3 there exist $\sigma, \varsigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t+t_0)$ for every t, where $\bar{p}(\cdot)$ is given under item (b), (ii) in the statement of that proposition. Let $\rho_0 = t_0$ and $\bar{u}(\cdot) = (\bar{p}_1(\cdot), \lambda\bar{p}_2(\cdot), \bar{p}_3(\cdot))$. Since $(g(\cdot), p(\cdot))$ is an extremal trajectory, we have

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)) = g(t)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)).$$

Similarly, if $\tilde{g}(t) = \bar{g}^{-1}(\rho_0)\bar{g}(t+\rho_0)$, then

$$\begin{aligned}\dot{\tilde{g}}(t) &= \bar{g}(\rho_0)^{-1}\dot{\bar{g}}(t+\rho_0)\\ &= \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)) = \tilde{g}(t)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)).\end{aligned}$$

Furthermore, $\tilde{g}(0) = \bar{g}(\rho_0)^{-1}\bar{g}(\rho_0) = \mathbf{1} = g(0)$. Since $t \mapsto g(t)$ and $t \mapsto \tilde{g}(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ satisfy the same differential equation, with the same initial conditions, they both solve the same Cauchy problem, and hence are identical. Finally, we have the following horizontal subsystem for $g(\cdot)$:

$$\begin{cases} \dot{x}(t) = \bar{p}_1(t+\rho_0)\cosh\theta + \frac{1}{\lambda}\bar{p}_2(t+\rho_0)\sinh\theta \\ \dot{y}(t) = \bar{p}_1(t+\rho_0)\sinh\theta + \frac{1}{\lambda}\bar{p}_2(t+\rho_0)\cosh\theta \\ \dot{\theta}(t) = \bar{p}_3(t+\rho_0). \end{cases}$$

(Here $x(0) = y(0) = \theta(0) = 0$.) Consequently, $\dot{x}(0) = \bar{p}_1(\rho_0) = -\sigma\sqrt{\frac{\lambda+c_0}{\lambda+1}}\operatorname{sech}\rho_0$ and $\dot{y}(0) = \frac{1}{\lambda}\bar{p}_2(\rho_0) = \frac{\sigma\varsigma}{\lambda}\sqrt{\frac{\lambda}{\lambda+1}}\operatorname{sech}\rho_0$. It follows that $\sigma = -\operatorname{sgn}(\dot{x}(0))$, $\varsigma = \sigma\operatorname{sgn}(\dot{y}(0))$ and $\operatorname{sech}\rho_0 = -\sigma\dot{x}(0)\sqrt{\frac{\lambda+1}{\lambda}} = |\dot{x}(0)|\sqrt{\frac{\lambda+1}{\lambda}}$. ■

Case III: $\dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 0$

Let $c_0 = C(\bar{p}(0)) = \bar{p}_1(0)^2 - \bar{p}_2(0)^2$. We have $c_0 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 0$. Accordingly, the horizontal subsystem in coordinates $(\bar{v}, \bar{w}, \bar{\theta})$ takes the form

$$\begin{cases} \dot{\bar{v}} = \sigma\sqrt{\frac{\lambda(1-c_0)}{\lambda+1}}\left[k\operatorname{cn}(\Omega t,k) - \frac{1}{\lambda}\operatorname{dn}(\Omega t,k)\right]e^{\bar{\theta}(t)} \\ \dot{\bar{w}} = \sigma\sqrt{\frac{\lambda(1-c_0)}{\lambda+1}}\left[k\operatorname{cn}(\Omega t,k) + \frac{1}{\lambda}\operatorname{dn}(\Omega t,k)\right]e^{-\bar{\theta}(t)} \\ \dot{\bar{\theta}} = k\Omega\operatorname{sn}(\Omega t,k). \end{cases}$$

(Here $\sigma \in \{-1, 1\}$, $\Omega = \sqrt{1-c_0}$ and $k = \sqrt{\frac{\lambda+c_0}{\lambda(1-c_0)}}$.) The last equation is separable, and so we have $\bar{\theta}(t) = k\Omega\int\operatorname{sn}(\Omega t,k)\,dt = \ln\left[\operatorname{dn}(\Omega t,k) - k\operatorname{cn}(\Omega t,k)\right] + c_{\bar{\theta}}$ for some $c_{\bar{\theta}} \in \mathbb{R}$. (We have used (A.6.15) to integrate the right-hand side.) The initial condition $\bar{\theta}(0) = 0$ yields $c_{\bar{\theta}} = -\ln(1-k)$. Consequently $\bar{\theta}(t) = \ln\left[\frac{\operatorname{dn}(\Omega t,k) - k\operatorname{cn}(\Omega t,k)}{1-k}\right]$. Substituting this into the first equation of the horizontal subsystem, we have

$$\begin{aligned} \dot{\bar{v}} &= \sigma\sqrt{\frac{\lambda(1-c_0)}{\lambda+1}}\left[k\operatorname{cn}(\Omega t,k) - \frac{1}{\lambda}\operatorname{dn}(\Omega t,k)\right]\frac{\operatorname{dn}(\Omega t,k) - k\operatorname{cn}(\Omega t,k)}{1-k} \\ &= -\frac{\sigma}{\lambda(1-k)}\sqrt{\frac{\lambda(1-c_0)}{\lambda+1}}\left[\operatorname{dn}^2(\Omega t,k) - k(\lambda+1)\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k) + \lambda k^2\operatorname{cn}^2(\Omega t,k)\right] \\ &= -\frac{\sigma}{1-k}\sqrt{\frac{(\lambda+1)(1-c_0)}{\lambda}}\left[\operatorname{dn}^2(\Omega t,k) - k\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k) - \frac{\lambda(k')^2}{\lambda+1}\right]. \end{aligned}$$

(We have used square relation (A.6.6) in the last step.) Integrating both sides, we get

$$\begin{aligned} \bar{v}(t) = &-\frac{\sigma}{1-k}\sqrt{\frac{(\lambda+1)(1-c_0)}{\lambda}}\int_0^t \operatorname{dn}^2(\Omega t,k)\,dt \\ &+\frac{\sigma k}{1-k}\sqrt{\frac{(\lambda+1)(1-c_0)}{\lambda}}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt \\ &+\sigma(1+k)\Omega\sqrt{\frac{\lambda}{\lambda+1}}\,t. \end{aligned}$$

Using the integral formula (A.6.14), we have

$$-\frac{\sigma}{1-k}\sqrt{\frac{(\lambda+1)(1-c_0)}{\lambda}}\int_0^t \operatorname{dn}^2(\Omega t,k)\,dt = -\frac{\sigma}{1-k}\sqrt{\frac{\lambda+1}{\lambda}}\,E(\operatorname{am}(\Omega t,k),k).$$

Next, from the derivative formula (A.6.1), we get

$$\frac{\sigma k}{1-k}\sqrt{\tfrac{(\lambda+1)(1-c_0)}{\lambda}}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt = \frac{\sigma k}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\operatorname{sn}(\Omega t,k).$$

Substituting these expressions into that for $\bar{v}(\cdot)$, we get

$$\begin{aligned}\bar{v}(t) &= -\frac{\sigma}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,E(\operatorname{am}(\Omega t,k),k) + \frac{\sigma k}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\operatorname{sn}(\Omega t,k) + \sigma(1+k)\Omega\sqrt{\tfrac{\lambda}{\lambda+1}}\,t\\ &= -\frac{\sigma}{1-k}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t,k),k) - k\operatorname{sn}(\Omega t,k) - \tfrac{\lambda(k')^2}{\lambda+1}\Omega t\right].\end{aligned}$$

Lastly, we have the equation

$$\begin{aligned}\dot{\bar{w}} &= \sigma\sqrt{\tfrac{\lambda(1-c_0)}{\lambda+1}}\left[k\operatorname{cn}(\Omega t,k) + \tfrac{1}{\lambda}\operatorname{dn}(\Omega t,k)\right]\frac{1-k}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}\cdot\frac{\operatorname{dn}(\Omega t,k)+k\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)+k\operatorname{cn}(\Omega t,k)}\\ &= \sigma\sqrt{\tfrac{\lambda(1-c_0)}{\lambda+1}}\left[k\operatorname{cn}(\Omega t,k) + \tfrac{1}{\lambda}\operatorname{dn}(\Omega t,k)\right]\frac{(1-k)\left[\operatorname{dn}(\Omega t,k)+k\operatorname{cn}(\Omega t,k)\right]}{\operatorname{dn}^2(\Omega t,k)-k^2\operatorname{cn}^2(\Omega t,k)}\\ &= \frac{\sigma}{\lambda(1+k)}\sqrt{\tfrac{\lambda(1-c_0)}{\lambda+1}}\left[\operatorname{dn}^2(\Omega t,k) + k(\lambda+1)\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k) + \lambda k^2\operatorname{cn}^2(\Omega t,k)\right]\\ &= \frac{\sigma}{1+k}\sqrt{\tfrac{(\lambda+1)(1-c_0)}{\lambda}}\left[\operatorname{dn}^2(\Omega t,k) + k\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k) - \tfrac{\lambda(k')^2}{\lambda+1}\right].\end{aligned}$$

Thus, using the integral formula (A.6.14) and the derivative formula (A.6.1), we get

$$\begin{aligned}\bar{w}(t) &= \frac{\sigma}{1+k}\sqrt{\tfrac{(\lambda+1)(1-c_0)}{\lambda}}\int_0^t \operatorname{dn}^2(\Omega t,k)\,dt\\ &\quad + \frac{\sigma k}{1+k}\sqrt{\tfrac{(\lambda+1)(1-c_0)}{\lambda}}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt\\ &\quad - \sigma(1-k)\Omega\sqrt{\tfrac{\lambda}{\lambda+1}}\,t\\ &= \frac{\sigma}{1+k}\sqrt{\tfrac{\lambda+1}{\lambda}}\,E(\operatorname{am}(\Omega t,k),k) + \frac{\sigma k}{1+k}\sqrt{\tfrac{\lambda+1}{\lambda}}\operatorname{sn}(\Omega t,k) - \sigma(1-k)\Omega\sqrt{\tfrac{\lambda}{\lambda+1}}\,t\\ &= \frac{\sigma}{1+k}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t,k),k) + k\operatorname{sn}(\Omega t,k) - \tfrac{\lambda(k')^2}{\lambda+1}\Omega t\right].\end{aligned}$$

Finally, solving for $\bar{x}(\cdot)$ and $\bar{y}(\cdot)$ using the equation $\begin{bmatrix}\bar{x}(t)\\ \bar{y}(t)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}\bar{v}(t)\\ \bar{w}(t)\end{bmatrix}$, we have the following expression for $\bar{g}(\cdot)$:

$$\begin{cases}\bar{x}(t) = -\dfrac{\sigma k}{1-k^2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t,k),k) - \operatorname{sn}(\Omega t,k) - \tfrac{\lambda(1-k^2)}{\lambda+1}\Omega t\right]\\ \bar{y}(t) = -\dfrac{\sigma}{1-k^2}\sqrt{\tfrac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t,k),k) - k^2\operatorname{sn}(\Omega t,k) - \tfrac{\lambda(1-k^2)}{\lambda+1}\Omega t\right]\\ \bar{\theta}(t) = \ln\left[\dfrac{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}{1-k}\right].\end{cases}$$

We now make an explicit statement regarding all Riemannian geodesics for this case.

5.2.6 PROPOSITION. *Let $g(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ be a unit-speed geodesic on $(\mathsf{SE}(1,1), \mathbf{g}^\lambda)$ such that $g(0) = \mathbf{1}$ and $\dot{x}(0)^2 - \lambda^2 \dot{y}(0)^2 < 0$. Then $g(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t + \rho_0)$ for every t, where $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$ is given by*

$$\begin{cases} \bar{x}(t) = -\dfrac{\sigma k}{1-k^2}\sqrt{\frac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t, k), k) - \operatorname{sn}(\Omega t, k) - \frac{\lambda(1-k^2)}{\lambda+1}\Omega t\right] \\ \bar{y}(t) = -\dfrac{\sigma}{1-k^2}\sqrt{\frac{\lambda+1}{\lambda}}\left[E(\operatorname{am}(\Omega t, k), k) - k^2\operatorname{sn}(\Omega t, k) - \frac{\lambda(1-k^2)}{\lambda+1}\Omega t\right] \\ \bar{\theta}(t) = \ln\left[\dfrac{\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)}{1-k}\right]. \end{cases}$$

Here $c_0 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2$, $\Omega = \sqrt{1-c_0}$, $k = \sqrt{\frac{\lambda+c_0}{\lambda(1-c_0)}}$, $\sigma = -\operatorname{sgn}(\dot{y}(0))$ and ρ_0 satisfies the equation $\operatorname{dn}(\Omega\rho_0, k) = |\dot{y}(0)|\sqrt{\frac{\lambda+1}{\lambda(1-c_0)}}$.

PROOF. The curve $(g(\cdot), p(\cdot))$ is an extremal trajectory for (R), for some integral curve $p(\cdot)$ of $\vec{H}^{(\mathsf{R})}$, and corresponding to the optimal control $u(\cdot) = (p_1(\cdot), \lambda p_2(\cdot), p_3(\cdot))$. Let $c_0 = C(p(0))$. We have $c_0 = \dot{x}(0)^2 - \lambda^2\dot{y}(0)^2 < 0$. By proposition 5.2.3 there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every t, where $\bar{p}(\cdot)$ is given under item (c) in the statement of that proposition. Let $\rho_0 = t_0$ and $\bar{u}(\cdot) = (\bar{p}_1(\cdot), \lambda\bar{p}_2(\cdot), \bar{p}_3(\cdot))$. Since $(g(\cdot), p(\cdot))$ is an extremal trajectory, we have

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)) = g(t)\,\Xi(\mathbf{1}, \bar{u}(t + \rho_0)).$$

Similarly, if $\tilde{g}(t) = \bar{g}^{-1}(\rho_0)\bar{g}(t + \rho_0)$, then

$$\begin{aligned} \dot{\tilde{g}}(t) &= \bar{g}(\rho_0)^{-1}\dot{\bar{g}}(t + \rho_0) \\ &= \bar{g}(\rho_0)^{-1}\bar{g}(t + \rho_0)\,\Xi(\mathbf{1}, \bar{u}(t + \rho_0)) = \tilde{g}(t)\,\Xi(\mathbf{1}, \bar{u}(t + \rho_0)). \end{aligned}$$

Furthermore, $\tilde{g}(0) = \bar{g}(\rho_0)^{-1}\bar{g}(\rho_0) = \mathbf{1} = g(0)$. Since $t \mapsto g(t)$ and $t \mapsto \tilde{g}(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ satisfy the same differential equation, with the same initial conditions, they both solve the same Cauchy problem, and hence are identical. Finally, we have the following horizontal subsystem for $g(\cdot)$:

$$\begin{cases} \dot{x}(t) = \bar{p}_1(t + \rho_0)\cosh\theta + \frac{1}{\lambda}\bar{p}_2(t + \rho_0)\sinh\theta \\ \dot{y}(t) = \bar{p}_1(t + \rho_0)\sinh\theta + \frac{1}{\lambda}\bar{p}_2(t + \rho_0)\cosh\theta \\ \dot{\theta}(t) = \bar{p}_3(t + \rho_0). \end{cases}$$

(Here $x(0) = y(0) = \theta(0) = 0$.) Consequently, $\dot{y}(0) = \bar{p}_2(\rho_0) = -\sigma\sqrt{\frac{\lambda(1-c_0)}{\lambda+1}}\operatorname{dn}(\Omega\rho_0, k)$. It follows that $\sigma = -\operatorname{sgn}(\dot{y}(0))$ and $\operatorname{dn}(\Omega\rho_0, k) = -\sigma\dot{y}(0)\sqrt{\frac{\lambda+1}{\lambda(1-c_0)}} = |\dot{y}(0)|\sqrt{\frac{\lambda+1}{\lambda(1-c_0)}}$. ∎

5.3 The Sub-Riemannian Problem

In this section we classify the invariant sub-Riemannian structures on $\mathsf{SE}(1,1)$. We show that, up to $\mathfrak{L}$-isometries, there is one single-parameter left-invariant sub-Riemannian structure on $\mathsf{SE}(1,1)$. This family of representatives may be reduced to a single representative by scaling. The results of this classification (up to $\mathfrak{L}$-isometries and scaling) are identical to that obtained in [6]. However, the authors of [6] employ a weaker equivalence relation than that considered here. Indeed, the classification in [6] is up to isometric diffeomorphisms and scaling. We demonstrate that the class of isometries can be restricted to those that are also group automorphisms. (Note that the authors of [6] refer to $\mathsf{SE}(1,1)$ as the "special hyperbolic group," and denote it $\mathsf{SH}(2)$.) Following the classification of sub-Riemannian structures, we shall calculate the geodesics on $\mathsf{SE}(1,1)$. A similar calculation, with essentially identical results, appears in the paper [21].

5.3.1 THEOREM. *Every left-invariant sub-Riemannian structure on* $\mathsf{SE}(1,1)$ *is* $\mathfrak{L}$*-isometric to the structure* $(\mathcal{D}, \lambda\, \mathbf{g})$*, where* $\mathcal{D}$ *and* $\mathbf{g}$ *are specified in the basis* (E_1, E_3) *by*

$$\mathcal{D}_\mathbf{1} = \langle E_1, E_3\rangle, \qquad \mathbf{g}_\mathbf{1}(X,Y) = X^\top \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} Y, \qquad X, Y \in \mathcal{D}_\mathbf{1}.$$

Here $\lambda > 0$ *parametrises a family of class representatives, each different value corresponding to a distinct (non-equivalent) representative.*

PROOF. Let $(\mathcal{D}', \mathbf{g}')$ be a left-invariant sub-Riemannian structure on $\mathsf{SE}(1,1)$. By proposition 2.2.2 every bracket-generating subspace of dimension 2 (*i.e.*, every full-rank $(2,0)$-affine subspace) of $\mathfrak{se}(1,1)$ is $\mathfrak{L}$-equivalent to $\langle E_1, E_3\rangle$. Consequently, there exists $\psi_1 \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that $\psi_1 \cdot \mathcal{D}_\mathbf{1} = \mathcal{D}'_\mathbf{1}$. Every automorphism of $\mathfrak{se}(1,1)$ is of the form

$$\psi = \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix}, \qquad x^2 \neq y^2,\ \varsigma \in \{-1, 1\}.$$

(See proposition 1.1.17.) Hence $\psi \cdot (aE_1 + bE_3) = (bv + ax)E_1 + (bw + \varsigma ay)E_2 + (\varsigma b)E_3$. Accordingly, ψ preserves the subspace $\mathcal{D}_\mathbf{1}$ if and only if $y = w = 0$. Write elements $aE_1 + bE_3 \in \mathcal{D}_\mathbf{1}$ as column vectors $\begin{bmatrix} a \\ b\end{bmatrix}$. In the basis (E_1, E_3), the (restricted) Lie algebra automorphisms $\psi : \mathcal{D}_\mathbf{1} \to \mathcal{D}_\mathbf{1}$ take the form $\psi = \begin{bmatrix} x & v \\ 0 & \varsigma\end{bmatrix}$.

We have that $\mathbf{g}''_\mathbf{1}(X,Y) = \mathbf{g}'_\mathbf{1}(\psi_1 \cdot X, \psi_1 \cdot Y)$ is a (positive definite) inner product on $\mathcal{D}_\mathbf{1}$. Identify $\mathbf{g}''_\mathbf{1}$ with its associated matrix, in terms of the basis (E_1, E_3). Let $\mathbf{g}''_\mathbf{1} = \begin{bmatrix} a_1 & b \\ b & a_2\end{bmatrix}$. As $\mathbf{g}''_\mathbf{1}$ is positive definite, we have $a_1, a_2 > 0$ and $a_1a_2 - b^2 > 0$. Hence

$$\psi_2 = \begin{bmatrix} \frac{\sqrt{a_1a_2 - b^2}}{\alpha} & -\frac{b}{a_1} \\ 0 & 1\end{bmatrix}$$

is an automorphism such that $\psi_2^\top \mathbf{g}''_\mathbf{1} \psi_2 = \operatorname{diag}(\lambda, \lambda, \lambda)$, where $\lambda = \frac{a_1a_2 - b^2}{a_1} > 0$. Therefore, we have $\psi_2\psi_1 \cdot \mathcal{D}_\mathbf{1} = \mathcal{D}'_\mathbf{1}$ and $\mathbf{g}'_\mathbf{1}(\psi_2\psi_1 \cdot X, \psi_2\psi_1 \cdot Y) = \lambda\, \mathbf{g}_\mathbf{1}(X,Y)$ for every $X, Y \in \mathcal{D}_\mathbf{1}$. Consequently, by theorem 5.1.3, $(\mathcal{D}', \mathbf{g}')$ is $\mathfrak{L}$-isometric to $(\mathcal{D}, \lambda\, \mathbf{g})$. ∎

By proposition 5.1.4, the factor of λ in the sub-Riemannian structure $(\mathcal{D}, \lambda\, \mathbf{g})$ does not have a considerable effect, and may be normalised. Hence we consider the sub-Riemannian problem for the structure $(\mathcal{D}, \mathbf{g})$. That is, we are considering the problem

$$\begin{gathered} \dot g(t) \in \mathcal{D}_{g(t)}, \quad g(\cdot) : [0,T] \to \mathsf{SE}(1,1) \\ g(0) = \mathbf{1},\ g(T) = g_T, \quad g_T \in \mathsf{SE}(1,1),\ T > 0 \text{ fixed} \\ \ell(g(\cdot)) = \int_0^T \sqrt{\mathbf{g}_{g(t)}(\dot g(t), \dot g(t))}\, dt \to \min. \end{gathered}$$

(In view of the left-invariance of the problem, we restrict to geodesics starting at the identity.) From section A.4, this problem may be interpreted as the optimal control problem

$$(\mathsf{SR})\quad \begin{cases} \dot g = g\,\Xi(\mathbf{1},u) = g(u_1E_1 + u_2E_3), \quad g(\cdot):[0,T]\to \mathsf{SE}(1,1),\ u(\cdot):[0,T]\to\mathbb{R}^\ell \\ g(0)=\mathbf{1},\ g(T)=g_T, \quad g_T\in\mathsf{SE}(1,1),\ T>0 \text{ fixed} \\ \mathcal{J}(u(\cdot)) = \int_0^T u_1(t)^2 + u_2(t)^2\,dt \to \min. \end{cases}$$

Theorem A.4.2 guarantees solutions to (SR) (at least, locally; *i.e.*, for g_T in a sufficiently small neighbourhood of identity). The family $(H_u^\nu)_{u\in\mathbb{R}^2}$ of control-dependent Hamiltonian functions is specified by

$$H_u^\nu(p) = u_1p_1 + u_2p_3 + \nu(u_1^2+u_2^2).$$

We first consider the abnormal geodesics. Setting $\nu = 0$ and applying the maximality condition of PMP yields

$$\frac{\partial H_u^0}{\partial u_i} = 0 \qquad \iff \qquad p_i = 0, \qquad i = 1,3.$$

Consequently, $H_u^0(p) = 0$ for every p. From the condition (A.3.7) of PMP, we get $\dot\xi(t) = (\dot g(t), \dot p(t)) = 0$, *i.e.*, $g(\cdot)$ and $p(\cdot)$ are constant. (From the boundary conditions, we thus have $g(t) = \mathbf{1}$.) That is, the only abnormal geodesics are constant.

5.3.2 REMARK. The fact that there are no nontrivial (*i.e.*, nonconstant) abnormal geodesics for the structure $(\mathcal{D}, \mathbf{g})$ is a consequence of the *contact structure* implicit in this space. It can be shown that there are no abnormal geodesics for contact sub-Riemannian manifolds. See, *e.g.*, [6, 37]. □

Consider the normal geodesics. From theorem A.3.8 we have that the (normal) extremal controls are

$$u(t) = Q^{-1}\mathbf{B}^\top p(t)^\top = \begin{bmatrix}1&0\\0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}^\top p(t)^\top = \begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}p(t)^\top.$$

That is, $u_1(t) = p_1(t)$ and $u_2(t) = p_3(t)$. Here $p(\cdot)$ is an integral curve of the Hamilton-Poisson system $(\mathfrak{se}(1,1)^*_-, H^{(\mathsf{SR})})$, where

$$H^{(\mathsf{SR})}(p) = \tfrac{1}{2}\,p\,\mathbf{B}\,Q^{-1}\mathbf{B}^\top p^\top = \tfrac{1}{2}(p_1^2+p_3^2).$$

This is exactly the Hamilton-Poisson system H_4 obtained in theorem 3.2.1 and studied in section 4.2.2. Let $(g(t), p(t))$ be an extremal trajectory of (SR). We have

$$\mathbf{g}_{g(t)}(\dot g(t), \dot g(t)) = 1 \quad\iff\quad u_1(t)^2 + u_2(t)^2 = 1 \quad\iff\quad H_4(p(t)) = \tfrac{1}{2}.$$

We assume that $H(p(t)) = \frac{1}{2}$; the resultant geodesics will have unit speed. The geodesic equations take the form

$$\begin{cases} \dot p = \vec{H}^{(\mathsf{SR})}(p) & \text{(vertical subsystem)} \\ \dot g = g\,\Xi(\mathbf{1},u) & \text{(horizontal subsystem).} \end{cases}$$

The vertical subsystem is exactly the equations of motion for the Hamilton-Poisson system $(\mathfrak{se}(1,1)^*_-, H_4)$:

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = p_2 p_3 \\ \dot{p}_3 = -p_1 p_2. \end{cases} \tag{5.3.1}$$

We integrated this system in section 4.2.2, and will not reproduce that effort here. However, for convenience we state a result collecting all the integral curves of $H^{(\mathsf{SR})}$.

5.3.3 Proposition. *Let $\bar{p}(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(1,1)^*$ be an integral curve of $\vec{H}^{(\mathsf{SR})}$. Suppose that $H^{(\mathsf{SR})}(\bar{p}(0)) = \frac{1}{2}$ and let $c_0 = C(\bar{p}(0)) \leq 1$.*

(a) (i) If $c_0 = 1$, then $p(t) = \big(\pm 1, 0, 0\big)$.

(ii) If $0 < c_0 < 1$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar{p}_1(t) = \sigma \operatorname{dn}(t, k) \\ \bar{p}_2(t) = -\sigma k \operatorname{cn}(t, k) \\ \bar{p}_3(t) = k \operatorname{sn}(t, k). \end{cases}$$

Here $k = \sqrt{1 - c_0}$.

(b) (i) If $c_0 = 0$ and $p_1(0) = 0$, then $p(t) = (0, 0, \pm 1)$.

(ii) If $c_0 = 0$ and $p_1(0) \neq 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma, \varsigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar{p}_1(t) = \sigma \operatorname{sech} t \\ \bar{p}_2(t) = -\sigma\varsigma \operatorname{sech} t \\ \bar{p}_3(t) = \varsigma \tanh t. \end{cases}$$

(c) If $c_0 < 0$, then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for every $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar{p}_1(t) = -\sigma k \Omega \operatorname{cn}(\Omega t, k) \\ \bar{p}_2(t) = \sigma \Omega \operatorname{dn}(\Omega t, k) \\ \bar{p}_3(t) = k \Omega \operatorname{sn}(\Omega t, k). \end{cases}$$

Here $\Omega = \sqrt{1 - c_0}$ and $k = \sqrt{\frac{1}{1 - c_0}}$.

In coordinates, the horizontal subsystem is

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 0 \\ \dot{x} & \dot{\theta}\sinh\theta & \dot{\theta}\cosh\theta \\ \dot{y} & \dot{\theta}\cosh\theta & \dot{\theta}\sinh\theta \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} (u_1 E_1 + u_2 E_3) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ p_1 & 0 & p_3 \\ 0 & p_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ p_1\cosh\theta & p_3\sinh\theta & p_3\cosh\theta \\ p_1\sinh\theta & p_3\cosh\theta & p_3\sinh\theta \end{bmatrix}. \end{aligned}$$

Comparing components on both sides, we have the following equations:

$$\begin{cases} \dot{x} = p_1 \cosh\theta \\ \dot{y} = p_1 \sinh\theta \\ \dot{\theta} = p_3. \end{cases} \tag{5.3.2}$$

Since $g(\cdot)$ starts at identity, the initial conditions are $x(0) = y(0) = \theta(0) = 0$. For convenience, we make the identification

$$(x, y, \theta) \in \mathbb{R}^3 \qquad \longleftrightarrow \qquad \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh\theta & \sinh\theta \\ y & \sinh\theta & \cosh\theta \end{bmatrix}.$$

Accordingly, we shall refer to (5.3.2) as the horizontal subsystem. Furthermore, we write geodesics $g(\cdot)$ as $g(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$.

Before integrating (5.3.2), we first describe a qualitative breakdown of cases. We seek conditions on $\dot{g}(0) = (\dot{x}(0), \dot{y}(0), \dot{\theta}(0))$ and $\ddot{g}(0) = (\ddot{x}(0), \ddot{y}(0), \ddot{\theta}(0))$ analogous to those of proposition 5.3.3. Let $c_0 = C(p(0))$. We have

$$\begin{cases} \dot{x}(0) = p_1(0) \\ \dot{y}(0) = 0 \\ \dot{\theta}(0) = p_3(0) \end{cases} \qquad \text{and} \qquad \begin{cases} \ddot{x}(0) = \dot{p}_1(0) \\ \ddot{y}(0) = \dot{p}_2(0) \\ \ddot{\theta}(0) = \dot{p}_3(0). \end{cases}$$

Consequently, using the equations of motion (5.3.1) of $H^{(\mathsf{SR})}$, we get $\ddot{\theta}(0) = -p_1(0)p_2(0) = -\dot{x}(0)p_2(0)$. If $\dot{x}(0) \neq 0$, then we can solve for $p_2(0)$. On the other hand, suppose $\dot{x}(0) = 0$. Since $\bar{g}(\cdot)$ is unit speed we have $\dot{x}(0)^2 + \dot{\theta}(0)^2 = 1$, and so $\dot{\theta}(0) \neq 0$. Then $\ddot{x}(0) - \dot{y}(0) = p_2(0)p_3(0) - p_1(0)p_3(0) = p_2(0)\dot{\theta}(0)$. Hence, we can again solve for $p_2(0)$. Let

$$\tau = \begin{cases} -\dfrac{\ddot{\theta}(0)}{\dot{x}(0)} & \text{if } \dot{x}(0) \neq 0 \\[2ex] \dfrac{\ddot{x}(0) - \dot{y}(0)}{\dot{\theta}(0)} & \text{if } \dot{x}(0) = 0. \end{cases}$$

Then $\tau = p_2(0)$, and so $c_0 = \dot{x}(0)^2 - \tau^2$. Using $\dot{x}(0) = p_1(0)$, $\dot{\theta}(0) = p_3(0)$ and $\tau = p_2(0)$, we have a total of five different qualitative cases, corresponding to those of proposition 5.3.3. Table 5.2 lists the different cases and their designations.

We shall now integrate the horizontal subsystem (5.3.2). Let $(\bar{g}(\cdot), \bar{p}(\cdot))$ be an extremal trajectory for the optimal control problem (SR), where $\bar{p}(\cdot)$ is given in proposition 5.3.3 and $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$.

Case I: $\dot{\bar{x}}(0)^2 - \tau^2 > 0$

Case I-a: $\dot{\bar{x}}(0)^2 - \tau^2 = 1$

We have $c_0 = \dot{\bar{x}}(0)^2 - \tau^2 = 1$. Accordingly, by proposition 5.3.3 $\bar{p}(\cdot)$ is constant, given by $\bar{p}(t) = (\sigma, 0, 0)$ for some $\sigma \in \{-1, 1\}$. The horizontal subsystem is thus

$$\begin{cases} \dot{\bar{x}} = \sigma \cosh\bar{\theta} \\ \dot{\bar{y}} = \sigma \sinh\bar{\theta} \\ \dot{\bar{\theta}} = 0. \end{cases}$$

Conditions		Designation
$\dot{x}(0)^2 - \tau^2 > 0$	$\dot{x}(0)^2 - \tau^2 = 1$	Case *I-a*
	$\dot{x}(0)^2 - \tau^2 < 1$	Case *I-b*
$\dot{x}(0)^2 - \tau^2 = 0$	$\dot{\bar{x}}(0) = 0$	Case *II-a*
	$\dot{\bar{x}}(0) \neq 0$	Case *II-b*
$\dot{x}(0)^2 - \tau^2 < 0$		Case *III*

Table 5.2: Qualitative breakdown of cases for the optimal control problem (SR)

These are immediately solved, to give the geodesic $\bar{g}(t) = (\sigma t, 0, 0)$. Evaluating $\dot{\bar{g}}(0)$, we have $\sigma = \operatorname{sgn}(\dot{\bar{x}}(0))$.

Case *I-b*: $\dot{\bar{x}}(0)^2 - \tau^2 < 1$

We have $c_0 = \dot{\bar{x}}(0)^2 - \tau^2$, and so $0 < c_0 < 1$. Using proposition 5.3.3, case (a), (ii), the horizontal subsystem reads

$$\begin{cases} \dot{\bar{x}} = \sigma \operatorname{dn}(t,k) \cosh \bar{\theta} \\ \dot{\bar{y}} = \sigma \operatorname{dn}(t,k) \sinh \bar{\theta} \\ \dot{\bar{\theta}} = k \operatorname{sn}(t,k). \end{cases}$$

(Here $\sigma \in \{-1, 1\}$ and $k = \sqrt{1 - c_0}$.) Separate variables in the last equation and use (A.6.15) to integrate the right-hand side. We get $\bar{\theta}(t) = k \int \operatorname{sn}(t,k)\, dt = \ln\left[\operatorname{dn}(t,k) - k \operatorname{cn}(t,k)\right] + c_{\bar{\theta}}$, with $c_{\bar{\theta}} \in \mathbb{R}$. From $\bar{\theta}(0) = 0$, it follows that $c_{\bar{\theta}} = -\ln(1-k)$. Thus $\bar{\theta}(t) = \ln\left[\frac{\operatorname{dn}(t,k) - k\operatorname{cn}(t,k)}{1-k}\right]$. The first equation of motion now becomes

$$\begin{aligned} \dot{\bar{x}} &= \sigma \operatorname{dn}(t,k) \cosh\left(\ln\left[\frac{\operatorname{dn}(t,k) - k\operatorname{cn}(t,k)}{1-k}\right]\right) \\ &= -\sigma \operatorname{dn}(t,k) \frac{\left[k\operatorname{cn}(t,k) - \operatorname{dn}(t,k)\right]^2 + (1-k)^2}{2(1-k)\left[k\operatorname{cn}(t,k) - \operatorname{dn}(t,k)\right]} \\ &= \frac{\sigma}{2(1-k)} \operatorname{dn}^2(t,k) - \frac{\sigma k}{2(1-k)} \operatorname{dn}(t,k)\operatorname{cn}(t,k) - \frac{\sigma(1-k)}{2} \frac{\operatorname{dn}(t,k)}{k\operatorname{cn}(t,k) - \operatorname{dn}(t,k)}. \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} \bar{x}(t) = \frac{\sigma}{2(1-k)} \int_0^t \operatorname{dn}^2(t,k)\, dt - \frac{\sigma k}{2(1-k)} \int_0^t \operatorname{dn}(t,k)\operatorname{cn}(t,k)\, dt \\ - \frac{\sigma(1-k)}{2} \int_0^t \frac{\operatorname{dn}(t,k)}{k\operatorname{cn}(t,k) - \operatorname{dn}(t,k)}\, dt. \end{aligned}$$

We can integrate the first term of this equation using formula (A.6.14), to get

$$\frac{\sigma}{2(1-k)} \int_0^t \operatorname{dn}^2(t,k)\, dt = \frac{\sigma}{2(1-k)} E(\operatorname{am}(t,k), k).$$

For the second integral, we use the derivative formula (A.6.1):

$$-\frac{\sigma k}{2(1-k)}\int_0^t \operatorname{dn}(t,k)\operatorname{cn}(t,k)\,dt = -\frac{\sigma k}{2(1-k)}\operatorname{sn}(t,k).$$

Lastly,

$$\begin{aligned}\frac{\operatorname{dn}(t,k)}{k\operatorname{cn}(t,k)-\operatorname{dn}(t,k)} &= \frac{\operatorname{dn}(t,k)}{k\operatorname{cn}(t,k)-\operatorname{dn}(t,k)}\cdot\frac{k\operatorname{cn}(t,k)+\operatorname{dn}(t,k)}{k\operatorname{cn}(t,k)+\operatorname{dn}(t,k)}\\ &= \frac{k\operatorname{dn}(t,k)\operatorname{cn}(t,k)+\operatorname{dn}^2(t,k)}{k^2\operatorname{cn}^2(t,k)-\operatorname{dn}^2(t,k)}\\ &= -\frac{k}{(k')^2}\operatorname{dn}(t,k)\operatorname{cn}(t,k)-\frac{1}{(k')^2}\operatorname{dn}^2(t,k).\end{aligned}$$

(We have used the square relation (A.6.6) in the final step.) Consequently,

$$\begin{aligned}&-\frac{\sigma(1-k)}{2}\int_0^t\frac{\operatorname{dn}(t,k)}{k\operatorname{cn}(t,k)-\operatorname{dn}(t,k)}\,dt\\ &\quad=\frac{\sigma k(1-k)}{2(k')^2}\int_0^t\operatorname{dn}(t,k)\operatorname{cn}(t,k)\,dt+\frac{\sigma(1-k)}{2(k')^2}\int_0^t\operatorname{dn}^2(t,k)\,dt\\ &\quad=\frac{\sigma k(1-k)}{2(k')^2}\operatorname{sn}(t,k)+\frac{\sigma(1-k)}{2(k')^2}E(\operatorname{am}(t,k),k).\end{aligned}$$

Collecting the results of these three integrals, we get $\bar{x}(t)=\frac{\sigma}{1-k^2}\left[E(\operatorname{am}(t,k),k)-k^2\operatorname{sn}(t,k)\right]$. We are left with the final equation of motion:

$$\begin{aligned}\dot{y} &= \sigma\operatorname{dn}(t,k)\sinh\left(\ln\left[\frac{\operatorname{dn}(t,k)-k\operatorname{cn}(t,k)}{1-k}\right]\right)\\ &= -\sigma\operatorname{dn}(t,k)\frac{[k\operatorname{cn}(t,k)-\operatorname{dn}(t,k)]^2-(1-k)^2}{2(1-k)\,[k\operatorname{cn}(t,k)-\operatorname{dn}(t,k)]}\\ &= \frac{\sigma}{2(1-k)}\operatorname{dn}^2(t,k)-\frac{\sigma k}{2(1-k)}\operatorname{dn}(t,k)\operatorname{cn}(t,k)+\frac{\sigma(1-k)}{2}\frac{\operatorname{dn}(t,k)}{k\operatorname{cn}(t,k)-\operatorname{dn}(t,k)}.\end{aligned}$$

We can use the preceding arguments (*i.e.*, those for $\bar{x}(\cdot)$) to integrate the right-hand side. We get $\bar{y}(t)=\frac{\sigma k}{1-k^2}\left[E(\operatorname{am}(t,k),k)-\operatorname{sn}(t,k)\right]$. Therefore we have the following expression for the geodesic $\bar{g}(\cdot)$:

$$\begin{cases}\bar{x}(t)=\dfrac{\sigma}{1-k^2}\left[E(\operatorname{am}(t,k),k)-k^2\operatorname{sn}(t,k)\right]\\ \bar{y}(t)=\dfrac{\sigma k}{1-k^2}\left[E(\operatorname{am}(t,k),k)-\operatorname{sn}(t,k)\right]\\ \bar{\theta}(t)=\ln\left[\dfrac{\operatorname{dn}(t,k)-k\operatorname{cn}(t,k)}{1-k}\right].\end{cases}$$

We now make an explicit statement regarding all sub-Riemannian geodesics for this case.

5.3.4 Proposition. *Let $g(\cdot)=(x(\cdot),y(\cdot),\theta(\cdot))$ be a unit-speed geodesic on $(\mathsf{SE}(1,1),\mathcal{D},\mathbf{g})$ such that $g(0)=\mathbf{1}$ and $0<\dot{x}(0)^2-\tau^2<1$, where $\tau=-\frac{\dot{\theta}(0)}{\dot{x}(0)}$. Then $g(t)=\bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ for every t,* *where* $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$ *is given by*

$$\begin{cases} \bar{x}(t) = \dfrac{\sigma}{1-k^2}\left[E(\mathrm{am}(t,k),k) - k^2\,\mathrm{sn}(t,k)\right] \\ \bar{y}(t) = \dfrac{\sigma k}{1-k^2}\left[E(\mathrm{am}(t,k),k) - \mathrm{sn}(t,k)\right] \\ \bar{\theta}(t) = \ln\left[\dfrac{\mathrm{dn}(t,k) - k\,\mathrm{cn}(t,k)}{1-k}\right]. \end{cases}$$

Here $k = \sqrt{1 - \dot{x}(0)^2 + \tau^2}$, $\sigma = \mathrm{sgn}(\dot{x}(0))$ *and* ρ_0 *satisfies the equation* $\mathrm{dn}(\rho_0, k) = |\dot{x}(0)|$.

PROOF. The curve $(g(\cdot), p(\cdot))$ is an extremal trajectory for (SR), for some integral curve $p(\cdot)$ of $\vec{H}^{(\mathsf{SR})}$, and corresponding to the optimal control $u(\cdot) = (p_1(\cdot), p_3(\cdot))$. Let $c_0 = C(p(0))$. We have $c_0 = \dot{x}(0)^2 - \tau^2$, and so $0 < c_0 < 1$. By proposition 5.3.3 there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every t, where $\bar{p}(\cdot)$ is given under item (a), (ii) in the statement of that proposition. Let $\rho_0 = t_0$ and $\bar{u}(\cdot) = (\bar{p}_1(\cdot), \bar{p}_3(\cdot))$. Since $(g(\cdot), p(\cdot))$ is an extremal trajectory, we have

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)) = g(t)\,\Xi(\mathbf{1}, \bar{u}(t + \rho_0)).$$

Similarly, if $\tilde{g}(t) = \bar{g}^{-1}(\rho_0)\bar{g}(t + \rho_0)$, then

$$\begin{aligned} \dot{\tilde{g}}(t) &= \bar{g}(\rho_0)^{-1}\dot{\bar{g}}(t + \rho_0) \\ &= \bar{g}(\rho_0)^{-1}\bar{g}(t + \rho_0)\,\Xi(\mathbf{1}, \bar{u}(t + \rho_0)) = \tilde{g}(t)\,\Xi(\mathbf{1}, \bar{u}(t + \rho_0)). \end{aligned}$$

Furthermore, $\tilde{g}(0) = \bar{g}(\rho_0)^{-1}\bar{g}(\rho_0) = \mathbf{1} = g(0)$. Since $t \mapsto g(t)$ and $t \mapsto \tilde{g}(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t + \rho_0)$ satisfy the same differential equation, with the same initial conditions, they both solve the same Cauchy problem, and hence are identical. Finally, we have the following horizontal subsystem for $g(\cdot)$:

$$\begin{cases} \dot{x}(t) = \bar{p}_1(t + \rho_0)\cosh\theta \\ \dot{y}(t) = \bar{p}_1(t + \rho_0)\sinh\theta \\ \dot{\theta}(t) = \bar{p}_3(t + \rho_0). \end{cases}$$

(Here $x(0) = y(0) = \theta(0) = 0$.) Consequently, $\dot{x}(0) = \bar{p}_1(\rho_0) = \sigma\,\mathrm{dn}(\rho_0, k)$. It follows that $\sigma = \mathrm{sgn}(\dot{x}(0))$ and $\mathrm{dn}(\rho_0, k) = \sigma\dot{x}(0) = |\dot{x}(0)|$. ■

Case II**:** $\dot{x}(0)^2 - \tau^2 = 0$

Case II-a**:** $\dot{x}(0) = 0$

In this case, $\bar{p}(\cdot)$ is constant, given by $\bar{p}(t) = (0, 0, \sigma)$, for some $\sigma \in \{-1, 1\}$. Indeed, $c_0 = \dot{x}(0)^2 - \tau^2 = 0$ and $\bar{p}_1(0) = \dot{x}(0) = 0$, whence $\bar{p}(\cdot)$ is constant by proposition 5.3.3. Consequently, the horizontal subsystem takes the form

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \\ \dot{\theta} = \sigma. \end{cases}$$

These are immediately solved, to give the geodesic $\bar{g}(t) = (0, 0, \sigma t)$. Evaluating $\dot{\bar{g}}(0)$, we have $\sigma = \mathrm{sgn}(\dot{\theta}(0))$.

Case II-b: $\dot{\bar{x}}(0) \neq 0$

Let $c_0 = C(\bar{p}(0)) = \bar{p}_1(0)^2 - \bar{p}_2(0)^2$. Then $c_0 = \dot{\bar{x}}(0)^2 - \tau^2 = 0$. Using proposition 5.3.3, case (b), (ii), the horizontal subsystem reads

$$\begin{cases} \dot{\bar{x}} = \sigma \operatorname{sech} t \cosh \bar{\theta} \\ \dot{\bar{y}} = \sigma \operatorname{sech} t \sinh \bar{\theta} \\ \dot{\bar{\theta}} = \varsigma \tanh t. \end{cases}$$

The final equation is immediately integrated, to give $\bar{\theta}(t) = \varsigma \int \tanh t \, dt = \varsigma \ln(\cosh t) + c_{\bar{\theta}}$, for some constant of integration $c_{\bar{\theta}} \in \mathbb{R}$. Since $\bar{\theta}(0) = 0$, it follows that $c_{\bar{\theta}} = 0$. Next, we have

$$\dot{\bar{x}} = \sigma \operatorname{sech} t \cosh\left[\ln\left(\cosh t\right)\right] = \sigma \operatorname{sech} t \frac{1}{2}(\cosh t + \operatorname{sech} t) = \frac{\sigma}{2}(1 + \operatorname{sech}^2 t).$$

Integrating both sides, the result is $\bar{x}(t) = \frac{\sigma}{2}(t + \tanh t)$. Lastly, we have

$$\dot{\bar{y}} = \sigma\varsigma \operatorname{sech} t \sinh\left[\ln\left(\cosh t\right)\right] = \sigma\varsigma \operatorname{sech} t \frac{1}{2} \sinh t \tanh t = \frac{\sigma\varsigma}{2} \tanh^2 t$$

and so $\bar{y}(t) = \frac{\sigma\varsigma}{2}(t - \tanh t)$. Therefore we have the following geodesic:

$$\begin{cases} \bar{x}(t) = \dfrac{\sigma}{2}(t + \tanh t) \\ \bar{y}(t) = \dfrac{\sigma\varsigma}{2}(t - \tanh t) \\ \bar{\theta}(t) = \varsigma \ln(\cosh t). \end{cases}$$

We now make an explicit statement regarding all sub-Riemannian geodesics for this case.

5.3.5 PROPOSITION. *Let* $g(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ *be a unit-speed geodesic on* $(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ *such that* $g(0) = \mathbf{1}$, $\dot{x}(0)^2 - \tau^2 = 0$ *and* $\dot{x}(0) \neq 0$. *Then* $g(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t + \rho_0)$ *for every* t, *where* $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$ *is given by*

$$\begin{cases} \bar{x}(t) = \dfrac{\sigma}{2}(t + \tanh t) \\ \bar{y}(t) = \dfrac{\sigma\varsigma}{2}(t - \tanh t) \\ \bar{\theta}(t) = \varsigma \ln(\cosh t). \end{cases}$$

Here $\sigma = \operatorname{sgn}(\dot{x}(0))$, $\varsigma = -\sigma \operatorname{sgn}(\tau)$ *and* ρ_0 *satisfies the equation* $\operatorname{sech} \rho_0 = |\dot{x}(0)|$.

PROOF. The curve $(g(\cdot), p(\cdot))$ is an extremal trajectory for (SR), for some integral curve $p(\cdot)$ of $\vec{H}^{(\mathsf{SR})}$, and corresponding to the optimal control $u(\cdot) = (p_1(\cdot), p_3(\cdot))$. Let $c_0 = C(p(0))$. We have $c_0 = \dot{x}(0)^2 - \tau^2 = 0$ and $\dot{p}_1(0) = \dot{x}(0) \neq 0$. By proposition 5.3.3 there exist $\sigma, \varsigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every t, where $\bar{p}(\cdot)$ is given under item (b), (ii) in the statement of that proposition. Let $\rho_0 = t_0$ and $\bar{u}(\cdot) = (\bar{p}_1(\cdot), \bar{p}_3(\cdot))$. Since $(g(\cdot), p(\cdot))$ is an extremal trajectory, we have

$$\dot{g}(t) = g(t)\, \Xi(\mathbf{1}, u(t)) = g(t)\, \Xi(\mathbf{1}, \bar{u}(t + \rho_0)).$$

Similarly, if $\tilde{g}(t) = \bar{g}^{-1}(\rho_0)\bar{g}(t + \rho_0)$, then

$$\begin{aligned} \dot{\tilde{g}}(t) &= \bar{g}(\rho_0)^{-1}\dot{\bar{g}}(t + \rho_0) \\ &= \bar{g}(\rho_0)^{-1}\bar{g}(t + \rho_0)\, \Xi(\mathbf{1}, \bar{u}(t + \rho_0)) = \tilde{g}(t)\, \Xi(\mathbf{1}, \bar{u}(t + \rho_0)). \end{aligned}$$

Furthermore, $\tilde{g}(0) = \bar{g}(\rho_0)^{-1}\bar{g}(\rho_0) = \mathbf{1} = g(0)$. Since $t \mapsto g(t)$ and $t \mapsto \tilde{g}(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ satisfy the same differential equation, with the same initial conditions, they both solve the same Cauchy problem, and hence are identical. Finally, we have the following horizontal subsystem for $g(\cdot)$:

$$\begin{cases} \dot{x}(t) = \bar{p}_1(t+\rho_0)\cosh\theta \\ \dot{y}(t) = \bar{p}_1(t+\rho_0)\sinh\theta \\ \dot{\theta}(t) = \bar{p}_3(t+\rho_0). \end{cases}$$

(Here $x(0) = y(0) = \theta(0) = 0$.) Consequently, $\dot{x}(0) = \bar{p}_1(\rho_0) = \sigma \operatorname{sech}\rho_0$ and $\tau = \bar{p}_2(\rho_0) = -\sigma\varsigma \operatorname{sech}\rho_0$. It follows that $\sigma = \operatorname{sgn}(\dot{x}(0))$, $\varsigma = -\sigma \operatorname{sgn}(\tau)$ and $\operatorname{sech}\rho_0 = \sigma\dot{x}(0) = |\dot{x}(0)|$. ■

Case III**:** $\dot{\bar{x}}(0)^2 - \tau^2 < 0$

We have $c_0 = \dot{\bar{x}}(0)^2 - \tau^2 < 0$. Accordingly, from proposition 5.3.3, the horizontal subsystem takes the form

$$\begin{cases} \dot{\bar{x}} = -\sigma k\Omega \operatorname{cn}(\Omega t, k)\cosh\bar{\theta} \\ \dot{\bar{y}} = -\sigma k\Omega \operatorname{cn}(\Omega t, k)\sinh\bar{\theta} \\ \dot{\bar{\theta}} = k\Omega \operatorname{sn}(\Omega t, k). \end{cases}$$

(Here $\sigma \in \{-1, 1\}$, $\Omega = \sqrt{1-c_0}$ and $k = \sqrt{\frac{1}{1-c_0}}$.) We can separate variables in the last equation and integrate both sides, to get $\bar{\theta}(t) = k\Omega \int \operatorname{sn}(\Omega t, k)\, dt = \ln\left[\operatorname{dn}(\Omega t, k) - k \operatorname{cn}(\Omega t, k)\right] + c_{\bar{\theta}}$ for some $c_{\bar{\theta}} \in \mathbb{R}$. (We have used (A.6.15) to integrate the right-hand side.) From the initial condition $\bar{\theta}(0) = 0$ we have $c_{\bar{\theta}} = -\ln(1-k)$. Hence $\bar{\theta}(t) = \ln\left[\frac{\operatorname{dn}(\Omega t,k) - k\operatorname{cn}(\Omega t,k)}{1-k}\right]$. Substituting the expression for $\bar{\theta}(\cdot)$ into the first equation of motion, we get

$$\begin{aligned} \dot{\bar{x}} &= -\sigma k\Omega \operatorname{cn}(\Omega t, k)\cosh\left(\ln\left[\frac{\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)}{1-k}\right]\right) \\ &= -\sigma k\Omega \operatorname{cn}(\Omega t, k)\frac{[\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)]^2 + (1-k)^2}{2(1-k)\,[\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)]} \\ &= -\frac{\sigma k\Omega}{2(1-k)}\operatorname{cn}(\Omega t, k)\,[\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)] - \frac{\sigma k\Omega}{2}\frac{(1-k)\operatorname{cn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)} \\ &= \frac{\sigma\Omega}{2(1-k)}k^2\operatorname{cn}^2(\Omega t, k) - \frac{\sigma k\Omega}{2(1-k)}\operatorname{dn}(\Omega t, k)\operatorname{cn}(\Omega t, k) - \frac{\sigma k\Omega}{2}\frac{(1-k)\operatorname{cn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)}. \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} \bar{x}(t) = {} & \frac{\sigma\Omega}{2(1-k)}\int_0^t k^2\operatorname{cn}^2(\Omega t, k)\, dt - \frac{\sigma k\Omega}{2(1-k)}\int_0^t \operatorname{dn}(\Omega t, k)\operatorname{cn}(\Omega t, k)\, dt \\ & - \frac{\sigma k\Omega(1-k)}{2}\int_0^t \frac{\operatorname{cn}(\Omega t, k)}{\operatorname{dn}(\Omega t, k) - k\operatorname{cn}(\Omega t, k)}\, dt. \end{aligned}$$

We can integrate the first term of this equation using the square relation (A.6.6) and the integral formula (A.6.14):

$$\begin{aligned} \frac{\sigma\Omega}{2(1-k)}\int_0^t k^2\operatorname{cn}^2(\Omega t, k)\, dt &= \frac{\sigma\Omega}{2(1-k)}\int_0^t \operatorname{dn}^2(\Omega t, k) - (k')^2\, dt \\ &= \frac{\sigma}{2(1-k)}E(\operatorname{am}(\Omega t, k), k) - \frac{\sigma\Omega(1+k)}{2}t. \end{aligned}$$

For the second integral, we use the derivative formula (A.6.1):

$$\frac{\sigma k\Omega}{2(1-k)}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt = \frac{\sigma k}{2(1-k)}\operatorname{sn}(\Omega t,k).$$

Lastly,

$$\begin{aligned}\frac{\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)} &= \frac{\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}\cdot\frac{\operatorname{dn}(t,k)+k\operatorname{cn}(t,k)}{\operatorname{dn}(t,k)+k\operatorname{cn}(t,k)}\\ &= \frac{k\operatorname{cn}^2(\Omega t,k)-\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)}{\operatorname{dn}^2(t,k)-k^2\operatorname{cn}^2(t,k)}\\ &= \frac{k}{(k')^2}\operatorname{cn}^2(\Omega t,k)+\frac{1}{(k')^2}\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k).\end{aligned}$$

(We have used the square relation (A.6.6) in the final step.) Consequently,

$$\begin{aligned}&\frac{\sigma k\Omega(1-k)}{2}\int_0^t \frac{\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}\,dt\\ &\quad= \frac{\sigma\Omega}{2(1+k)}\int_0^t k^2\operatorname{cn}^2(\Omega t,k)\,dt+\frac{\sigma k\Omega}{2(1+k)}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt\\ &\quad= \frac{\sigma}{2(1+k)}E(\operatorname{am}(\Omega t,k),k)-\frac{\sigma\Omega(1-k)}{2}t+\frac{\sigma k}{2(1+k)}\operatorname{sn}(\Omega t,k).\end{aligned}$$

Collecting the results of these three integrals, we get the following expression for $\bar{x}(\cdot)$:

$$\bar{x}(t) = \frac{\sigma k}{1-k^2}\left[E(\operatorname{am}(\Omega t,k),k)-\operatorname{sn}(\Omega t,k)-(1-k^2)\Omega t\right].$$

We are left with the final equation of motion:

$$\begin{aligned}\dot{y} &= -\sigma k\Omega\operatorname{cn}(\Omega t,k)\sinh\left(\ln\left[\frac{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}{1-k}\right]\right)\\ &= -\sigma k\Omega\operatorname{cn}(\Omega t,k)\frac{[\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)]^2-(1-k)^2}{2(1-k)\left[\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)\right]}\\ &= -\frac{\sigma k\Omega}{2(1-k)}\operatorname{cn}(\Omega t,k)\left[\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)\right]+\frac{\sigma k\Omega}{2}\frac{(1-k)\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}\\ &= \frac{\sigma\Omega}{2(1-k)}k^2\operatorname{cn}^2(\Omega t,k)-\frac{\sigma k\Omega}{2(1-k)}\operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)+\frac{\sigma k\Omega}{2}\frac{(1-k)\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}.\end{aligned}$$

Integrating both sides (we can employ the preceding arguments in order to integrate each term), and simplifying, we have

$$\begin{aligned}\bar{y}(t) &= \frac{\sigma\Omega}{2(1-k)}\int_0^t k^2\operatorname{cn}^2(\Omega t,k)\,dt-\frac{\sigma k\Omega}{2(1-k)}\int_0^t \operatorname{dn}(\Omega t,k)\operatorname{cn}(\Omega t,k)\,dt\\ &\qquad+\frac{\sigma k\Omega(1-k)}{2}\int_0^t \frac{\operatorname{cn}(\Omega t,k)}{\operatorname{dn}(\Omega t,k)-k\operatorname{cn}(\Omega t,k)}\,dt\\ &= \frac{\sigma}{1-k^2}\left[E(\operatorname{am}(\Omega t,k),k)-k^2\operatorname{sn}(\Omega t,k)-(1-k^2)\Omega t\right].\end{aligned}$$

Therefore we have the following expression for the geodesic $\bar{g}(\cdot)$:

$$\begin{cases} \bar{x}(t) = \dfrac{\sigma k}{1-k^2}\left[E(\mathrm{am}(\Omega t,k),k) - \mathrm{sn}(\Omega t,k) - (1-k^2)\Omega t\right] \\ \bar{y}(t) = \dfrac{\sigma}{1-k^2}\left[E(\mathrm{am}(\Omega t,k),k) - k^2\,\mathrm{sn}(\Omega t,k) - (1-k^2)\Omega t\right] \\ \bar{\theta}(t) = \ln\left[\dfrac{\mathrm{dn}(\Omega t,k) - k\,\mathrm{cn}(\Omega t,k)}{1-k}\right]. \end{cases}$$

We now make an explicit statement regarding all sub-Riemannian geodesics for this case.

5.3.6 Proposition. *Let $g(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ be a unit-speed geodesic on $(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ such that $g(0) = \mathbf{1}$ and $\dot{x}(0)^2 - \tau^2 < 0$, where*

$$\tau = \begin{cases} -\dfrac{\ddot{\theta}(0)}{\dot{x}(0)} & \text{if } \dot{x}(0) \neq 0 \\ \dfrac{\ddot{x}(0) - \ddot{y}(0)}{\dot{\theta}(0)} & \text{if } \dot{x}(0) = 0. \end{cases}$$

Then $g(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ for every t, where $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot))$ is given by

$$\begin{cases} \bar{x}(t) = \dfrac{\sigma k}{1-k^2}\left[E(\mathrm{am}(\Omega t,k),k) - \mathrm{sn}(\Omega t,k) - (1-k^2)\Omega t\right] \\ \bar{y}(t) = \dfrac{\sigma}{1-k^2}\left[E(\mathrm{am}(\Omega t,k),k) - k^2\,\mathrm{sn}(\Omega t,k) - (1-k^2)\Omega t\right] \\ \bar{\theta}(t) = \ln\left[\dfrac{\mathrm{dn}(\Omega t,k) - k\,\mathrm{cn}(\Omega t,k)}{1-k}\right]. \end{cases}$$

Here $\Omega = \sqrt{1 - \dot{x}(0)^2 + \tau^2}$, $k = \sqrt{\frac{1}{1-\dot{x}(0)^2+\tau^2}}$, $\sigma = \mathrm{sgn}(\tau)$ and ρ_0 satisfies the equation $\mathrm{dn}(\Omega\rho_0, k) = \frac{|\tau|}{\Omega}$.

Proof. The curve $(g(\cdot), p(\cdot))$ is an extremal trajectory for (SR), for some integral curve $p(\cdot)$ of $\vec{H}^{(\mathsf{SR})}$, and corresponding to the optimal control $u(\cdot) = (p_1(\cdot), p_3(\cdot))$. Let $c_0 = C(p(0))$. We have $c_0 = \dot{x}(0)^2 - \tau^2 < 0$. By proposition 5.3.3 there exist $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for every t, where $\bar{p}(\cdot)$ is given under item (a), (ii) in the statement of that proposition. Let $\rho_0 = t_0$ and $\bar{u}(\cdot) = (\bar{p}_1(\cdot), \bar{p}_3(\cdot))$. Since $(g(\cdot), p(\cdot))$ is an extremal trajectory, we have

$$\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t)) = g(t)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)).$$

Similarly, if $\tilde{g}(t) = \bar{g}^{-1}(\rho_0)\bar{g}(t+\rho_0)$, then

$$\begin{aligned} \dot{\tilde{g}}(t) &= \bar{g}(\rho_0)^{-1}\dot{\bar{g}}(t+\rho_0) \\ &= \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)) = \tilde{g}(t)\,\Xi(\mathbf{1}, \bar{u}(t+\rho_0)). \end{aligned}$$

Furthermore, $\tilde{g}(0) = \bar{g}(\rho_0)^{-1}\bar{g}(\rho_0) = \mathbf{1} = g(0)$. Since $t \mapsto g(t)$ and $t \mapsto \tilde{g}(t) = \bar{g}(\rho_0)^{-1}\bar{g}(t+\rho_0)$ satisfy the same differential equation, with the same initial conditions, they both solve the same Cauchy problem, and hence are identical. Finally, we have that $\tau = p_2(0) = \bar{p}_2(\rho_0) = \sigma\Omega\,\mathrm{dn}(\Omega\rho_0, k)$. It follows that $\sigma = \mathrm{sgn}(\tau)$ and $\mathrm{dn}(\Omega\rho_0, k) = \frac{\sigma\tau}{\Omega} = \frac{|\tau|}{\Omega}$. ∎

Conclusion

In this thesis we considered a class of invariant optimal control problems on the (three-dimensional) semi-Euclidean group. The approach was three-fold. We first considered the left-invariant control affine systems on $\mathsf{SE}(1,1)$ (this comprises the content of chapter 2). Next we treated quadratic Hamilton-Poisson systems on the (minus) Lie-Poisson space $\mathfrak{se}(1,1)^*_-$ (chapter 3 and chapter 4). Lastly, we used results from chapters 2, 3 and 4 to solve two optimal control problems, *viz.* those associated to the Riemannian and sub-Riemannian length-minimisation problems on $\mathsf{SE}(1,1)$. We discuss each chapter in some detail.

Chapter 1 is concerned with the semi-Euclidean group itself, and, in particular, the study of properties of $\mathsf{SE}(1,1)$ germane to control theory. We note that the results of this chapter are well-known. Nevertheless, a working knowledge of $\mathsf{SE}(1,1)$ is crucial for an understanding of the topics developed in later chapters.

The next chapter considers a large class of control systems evolving on $\mathsf{SE}(1,1)$. We employ a natural equivalence relation (*viz.* detached feedback equivalence) and classify all (full-rank) left-invariant control affine systems. As such, the study of such control systems on $\mathsf{SE}(1,1)$ is essentially reduced to the study of a finite list of class representatives (including two single-parameter families of representatives; see theorem 2.2.4). Furthermore, by restricting to those systems that are also controllable, this list of representatives is reduced to exactly three normal forms (see corollary 2.2.5), namely the systems

$$\Sigma^{(2,0)} : u_1E_1 + u_2E_3 \qquad \Sigma^{(2,1)}_1 : E_2 + u_1E_1 + u_2E_3 \qquad \Sigma^{(3,0)} : u_1E_1 + u_2E_2 + u_3E_3.$$

As stated in this chapter, the results of this classification have been obtained in [13, 19]. Having obtained this list of (controllable) representatives, the natural next step is to consider optimal control problems associated to each of these three systems. The (homogeneous) systems $\Sigma^{(2,0)}$ and $\Sigma^{(3,0)}$ are treated in this fashion in chapter 5.

Chapter 3 considers a class of Hamilton-Poisson systems on $\mathsf{se}(1,1)^*_-$, namely those of the form $H_{A,\mathcal{Q}}(p) = \langle p, A\rangle + \mathcal{Q}(p)$, where $A \in \mathfrak{se}(1,1)$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{se}(1,1)^*_-$. This is exactly the class of Hamilton-Poisson systems arising from optimal control problems with fixed time and quadratic cost (see section A.3.4). Specifically, the extremal controls of an optimal control problem of the form

$$\begin{gathered} \dot{g} = g\left(A + u_1B_1 + \cdots + u_\ell B_\ell\right) \\ g(0) = g_0,\ g(T) = g_1, \quad g_0, g_1 \in \mathsf{SE}(1,1),\ T > 0 \text{ fixed} \\ \mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t))\, dt \to \min \end{gathered}$$

are linearly related to the integral curves of a Hamilton-Poisson system on the Lie-Poisson space $\mathfrak{se}(1,1)^*_-$. Accordingly, the study of the extremal controls is reduced to the study of a

Hamilton-Poisson system. (Hamilton-Poisson systems also find application outside of control theory, *e.g.*, in mathematical physics.) We follow a similar approach to that used in chapter 2. We introduce an appropriate equivalence relation between Hamilton-Poisson systems, namely affine equivalence. The quadratic Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*_-$ are then classified, starting with the homogeneous systems (*i.e.*, where $A = 0$); six representatives are identified. These six representatives are then used to organise the classification of inhomogeneous systems, where we again obtain a list of representatives (including several infinite families).

Chapter 4 investigates several of the Hamilton-Poisson representatives obtained in chapter 3. Specifically, we treat all homogeneous normal forms, as well as a number of the inhomogeneous systems. (Space considerations in this thesis do not permit a treatment of all the inhomogeneous systems. Nonetheless, a similar approach to that we employ here may be followed.) For each normal form, we first consider stability. For all systems under consideration, we have performed a complete analysis of the (Lyapunov) stability nature of equilibria. Next, we find all integral curves of the associated Hamiltonian vector field. (For several systems, the integral curves are lines, and are easily determined. For these systems, the stability analysis is the main interest.) We obtain explicit expressions for all integral curves, in terms of elementary functions or Jacobi elliptic functions. Accordingly, we have essentially determined the extremal controls (up to an affine isomorphism) for a large class of optimal control problems on $\mathsf{SE}(1,1)$.

The last chapter considers the Riemannian and sub-Riemannian problems on $\mathsf{SE}(1,1)$. In order to determine the geodesics for *any* Riemannian or sub-Riemannian structure, we again follow the approach of chapter 2 and chapter 3. That is, we introduce a suitable equivalence relation (equivalence up to isometric group automorphisms and scaling) and classify all left-invariant Riemannian and sub-Riemannian structures on $\mathsf{SE}(1,1)$. We obtain a single representative for the sub-Riemannian case; a single-parameter family of representatives is identified for the Riemannian case. We then consider the Riemannian and sub-Riemannian problems for these normalised structures. In particular, we write each of these problems as an optimal control problem (with fixed time and quadratic cost). The results of chapter 4 are employed to determine the extremal controls. We are then able to integrate the geodesic equations on the group. Accordingly, we obtain explicit expressions for all geodesics. The integration of the sub-Riemannian geodesic equations replicates the results of [21] (although we have followed a different approach). Thus far, the Riemannian geodesics on $\mathsf{SE}(1,1)$ have not been explicitly determined in the literature (hence our results in this regard are original).

Lastly, we note that the classification of chapter 5 may be interpreted as a classification, under *cost equivalence* [15, 17], of all (controllable) drift-free left-invariant control affine systems on $\mathsf{SE}(1,1)$ with homogeneous cost. (Indeed, this is the approach taken in [10].) Accordingly, we have essentially determined the extremal trajectories for all such systems.

Appendix A

Review of Prerequisites

In this appendix we discuss the necessary prerequisites for the topics developed in this thesis. We provide references for all definitions and results stated. As such, we shall not give any justifications or proofs of the claims made herein (with the exception of proposition A.1.14, for which no suitable reference could be found), as they may be found in the given references. Familiarity with basic notions of differential geometry (particularly smooth manifolds), general topology and algebra is assumed.

A.1 Lie Theory

We review basic notions of Lie theory, in particular Lie groups, Lie algebras and the relationship between the two. The main references for this section are [24, 30, 26, 42]. For section A.1.4, we have also drawn upon [34] (particularly for the coajoint action and coadjoint orbits).

A.1.1 Lie groups and Lie algebras

A **Lie group** G is a smooth (*i.e.*, C^∞) manifold with a group structure, such that the multiplication and inversion maps $\mu : \mathsf{G} \times \mathsf{G} \to \mathsf{G},\ (g,h) \mapsto gh$ and $\iota : \mathsf{G} \to \mathsf{G},\ g \mapsto g^{-1}$ are smooth. In particular, a (real, finite-dimensional) **matrix Lie group** G is a closed subgroup of the general linear group $\mathsf{GL}(n, \mathbb{R})$ of $n \times n$ invertible matrices. Henceforth, we always assume we are working with a real and finite-dimensional matrix Lie group. Let H be an abstract subgroup of G. We call H a **Lie subgroup** of G if it is an immersed submanifold of G. If H is also an emdedded submanifold of G, then it is called a **closed Lie subgroup** of G. H is said to be **normal** if it is normal as an abstract subgroup of G. We have the following result.

A.1.1 Theorem. (Cartan, [24]) *Every closed subgroup of a real Lie group is a closed Lie subgroup.*

The **centre** $\mathsf{Z}(\mathsf{G})$ of G is a normal subgroup of G defined by $\mathsf{Z}(\mathsf{G}) = \{g \in \mathsf{G} : ghg^{-1}h^{-1} = \mathbf{1} \text{ for every } h \in \mathsf{G}\}$.

A Lie group **homomorphism** between Lie groups G and G' is a smooth map $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$ for every $g_1, g_2 \in \mathsf{G}$. If ϕ is bijective with a smooth inverse, then we call it a Lie group **isomorphism**. (Thus a Lie group isomorphism is a diffeomorphism that preserves the group structure.) If ϕ is a Lie group isomorphism and $\mathsf{G} = \mathsf{G}'$, then we call ϕ a Lie group **automorphism**. The group of all automorphisms of G is denoted by

$\mathsf{Aut}(\mathsf{G})$. Two distinguished Lie group automorphisms are the **left-** and **right-translation** maps $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ and $R_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto hg$, respectively.

The **semi-direct product** of Lie groups H and N is the Lie group G formed by taking the Cartesian product $\mathsf{N} \times \mathsf{H}$ (with the smooth structure of the product manifold) together with the group operation $(n_1, h_1)(n_2, h_2) = (n_1\phi(h_1)n_2, h_1h_2)$, where $\phi : \mathsf{H} \to \mathsf{Aut}(\mathsf{N})$ is some Lie group homomorphism. We write $\mathsf{G} = \mathsf{N} \rtimes \mathsf{H}$. The subsets $\{(n, \mathbf{1}) : n \in \mathsf{N}\}$ and $\{(\mathbf{1}, h) : h \in \mathsf{H}\}$ are subgroups of G, isomorphic to N and H, respectively. In particular, the subgroup N is normal in G. (H is normal in G if and only if ϕ is trivial, in which case G is simply the direct product $\mathsf{N} \times \mathsf{H}$.) A Lie group G is said to **decompose** as the semi-direct product of Lie subgroups N and H if

(*i*) N is normal in G;

(*ii*) $\mathsf{G} = \mathsf{NH}$;

(*iii*) $\mathsf{N} \cap \mathsf{H} = \{\mathbf{1}\}$.

(In this case, ϕ is defined as $\phi(h) \in \mathsf{Aut}(\mathsf{N})$, $\phi(h) : n \mapsto hnh^{-1}$ and the map $\mathsf{N} \rtimes \mathsf{H} \to \mathsf{G}$, $(n, h) \mapsto nh$ is a Lie group isomorphism.)

A (real, n-dimensional) **Lie algebra** $\mathfrak{g}$ is an n-dimensional vector space over $\mathbb{R}$ equipped with a bilinear, skew-symmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the **Lie bracket**) that satisfies the Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for every $X, Y, Z \in \mathfrak{g}$. A **Lie subalgebra** of $\mathfrak{g}$ is a subset $\mathsf{h} \subseteq \mathfrak{g}$ that is a Lie algebra in its own right. An **ideal** of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h}$ such that for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{h}$. The **centre** $\mathsf{Z}(\mathfrak{g})$ of $\mathfrak{g}$ is the ideal $\mathsf{Z}(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for every } Y \in \mathfrak{g}\}$.

Let Γ be a subset of a Lie algebra $\mathfrak{g}$. The Lie algebra **generated** by Γ is denoted $\mathsf{Lie}(\Gamma)$. That is, $\mathsf{Lie}(\Gamma)$ is the smallest Lie subalgebra of $\mathfrak{g}$ containing Γ. $\mathsf{Lie}(\Gamma)$ may be characterised as

$$\mathsf{Lie}(\Gamma) = \mathsf{span}\,\{A_1, [A_1, A_2], [A_1, [A_2, A_3]] \ldots, [A_1, [A_2, \ldots, [A_{k-1}, A_k] \cdots]] : A_i \in \Gamma,\ k \in \mathbb{N}\}\,.$$

A **homomorphism** of Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ is a linear map $\psi : \mathfrak{g} \to \mathfrak{g}'$ that preserves the Lie bracket: $\psi \cdot [X, Y] = [\psi \cdot X, \psi \cdot Y]$ for every $X, Y \in \mathfrak{g}$. If $\mathfrak{g} = \mathfrak{g}'$ and ψ is bijective, we call ψ a Lie algebra **automorphism**. The group of all automorphisms of $\mathfrak{g}$ is denoted by $\mathsf{Aut}(\mathfrak{g})$.

A.1.2 The relationship between Lie groups and Lie algebras

Let G be a (real, n-dimensional) matrix Lie group. The tangent space $T_\mathbf{1}\mathsf{G}$ of G at identity is given by $T_\mathbf{1}\mathsf{G} = \{\dot{g}(0) : g(\cdot) \text{ is a smooth curve in } \mathsf{G},\ g(0) = \mathbf{1}\}$. The tangent space at identity is isomorphic (as a vector space) to $T_g\mathsf{G}$, for any $g \in \mathsf{G}$. Indeed, we have the correspondence

$$X \in T_\mathbf{1}\mathsf{G} \qquad \longleftrightarrow \qquad T_\mathbf{1}L_g \cdot X = gX \in T_g\mathsf{G}. \tag{A.1.1}$$

We call $T_\mathbf{1}\mathsf{G}$, together with the matrix commutator $[X, Y] = XY - YX$, the **Lie algebra of** G, and denote it by $\mathfrak{g}$.

The Lie algebra of a Lie group may be characterised in terms of left-invariant vector fields. A vector field $X \in \mathsf{Vec}(\mathsf{G})$ is **left-invariant** if $T_hL_g \cdot X(h) = X(gh)$ for every $g, h \in \mathsf{G}$. (Since G is a matrix Lie group, we can write this condition in matrix form as $gX(h) = X(gh)$. Consequently, every left-invariant vector field is of the form $X(g) = gA$ for some $A \in \mathfrak{g}$.) The Lie bracket of two left-invariant vector fields is left-invariant. Indeed, we have the following result.

A.1.2 Proposition. (cf. [42]) *Let $X(g) = gA$ and $Y(g) = gB$ be left-invariant vector fields on* G*. Then $[X,Y](g) = g[A,B]$ for every $g \in \mathsf{G}$.*

The set of all left-invariant vector fields on G, together with the Lie bracket of vector fields (defined for vector fields X and Y by $[X,Y][f] = X[Y[f]] - Y[X[f]]$, $f \in C^\infty(\mathsf{G})$) forms a Lie algebra isomorphic to $\mathfrak{g}$. Indeed, we have the correspondence

$$\text{left-invariant vector field } X(g) = gA \qquad \longleftrightarrow \qquad X(\mathbf{1}) = A \in \mathfrak{g}.$$

We shall identify these two representations of the Lie algebra of a Lie group.

The Lie algebra $\mathfrak{g}$ and the Lie group G are related by the **exponential map** $\exp : \mathfrak{g} \to \mathsf{G}$, defined (for matrix Lie groups) as the power series

$$\exp X = \sum_{k=1}^{\infty} \frac{X^k}{k!}, \qquad X \in \mathfrak{g}.$$

(This series is everywhere convergent.) In general, the exponential map is not a diffeomorphism. However, we have the following result.

A.1.3 Proposition. ([24]) *The exponential map* $\exp : \mathfrak{g} \to \mathsf{G}$ *maps a certain neighbourhood of zero in the tangent algebra* $\mathfrak{g}$ *diffeomorphically onto a neighbourhood of the identity of* G*.*

Lastly, we review the relationship between normal Lie subgroups and ideals, as well as the link between Lie group homomorphisms and Lie algebra homomorphisms.

A.1.4 Theorem. (cf. [24]) *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$*.*

- (*i*) *If* H *is a Lie subgroup of* G*, then the Lie algebra of* H *is a subalgebra of* $\mathfrak{g}$*. Conversely, every subalgebra of* $\mathfrak{g}$ *is the Lie algebra of some (uniquely defined) connected Lie subgroup of* G*.*
- (*iii*) *Suppose* G *is connected. A connected Lie subgroup* H *of* G *is normal if and only if the Lie algebra of* H *is an ideal of* $\mathfrak{g}$*.*

A.1.5 Proposition. (cf. [24]) *The centre* $Z(\mathsf{G})$ *of a connected Lie group* G *is a (normal) closed Lie subgroup, whose tangent algebra coincides with the centre* $Z(\mathfrak{g})$ *of* $\mathfrak{g}$*.*

A.1.6 Theorem. (cf. [24]) *Let* G *and* G' *be Lie groups, with Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{g}'$*, respectively.*

- (*i*) *If* $\phi : \mathsf{G} \to \mathsf{G}'$ *is a Lie group homomorphism, then* $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{g}'$ *is a Lie algebra homomorphism.*
- (*ii*) *If* G *is simply connected, then for every Lie algebra homomorphism* $\psi : \mathfrak{g} \to \mathfrak{g}'$ *there exists a (unique) Lie group homomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *such that* $T_{\mathbf{1}}\phi = \psi$*.*

A.1.3 Topology of Lie groups

Let G be a Lie group. G is called **compact** if it is compact as a topological space. That is, for every open cover $\{U_\alpha\}_{\alpha \in J}$ of G there exists a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ that covers G. We say that G is **connected** if it is connected as a topological space, *i.e.*, there do not exist two nonempty open subsets U, V of G such that $U \cap V = \varnothing$ and $\mathsf{G} = U \cap V$. For Lie groups, connectedness coincides with path-connectedness.

A.1.7 Proposition. (cf. [24]) *G is connected if and only if it is path-connected,* i.e.*, for any two points $g_0, g_1 \in \mathsf{G}$, there exists a smooth curve $g(\cdot) : [0,1] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(1) = g_1$.*

G is called **simply connected** if it is connected and for every two smooth curves $g(\cdot) : [0,1] \to \mathsf{G}$ and $h(\cdot) : [0,1] \to \mathsf{G}$ with the same endpoints (*i.e.*, $g(0) = h(0)$ and $g(1) = h(1)$) there exists a continuous function $H : \mathsf{G} \times [0,1] \to \mathsf{G}$ such that $H(\cdot, 0) = g(\cdot)$ and $H(\cdot, 1) = h(\cdot)$. (That is, the curve $g(\cdot)$ can be continuously deformed into $h(\cdot)$.) To every Lie algebra is associated a (unique up to isomorphism) simply connected Lie group.

A.1.8 Theorem. ([24]) *A simply connected Lie group is determined up to an isomorphism by its Lie algebra.*

Let N be a normal closed Lie subgroup of G. The quotient G/N may be given a smooth structure such that it is a Lie group. (However, in general G/N is not a *matrix* Lie group.) Indeed, we have the following results.

A.1.9 Theorem. ([24]) *Let H be a closed Lie subgroup of a Lie group G. The set G/H of left cosets of H in G possesses a unique differentiable structure for which the canonical map $p : \mathsf{G} \to \mathsf{G}/\mathsf{H}$, $g \mapsto g\,\mathsf{H}$ is a quotient map.*

A.1.10 Theorem. ([24]) *Let N be a normal closed Lie subgroup of a Lie group G. Then the quotient group G/N with the differentiable structure of theorem A.1.9 is a Lie group.*

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. A **covering homomorphism** from G onto H is a Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ such that $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism. Equivalently, ϕ is a covering homomorphism if $\ker \phi = \{g \in \mathsf{G} : \phi(g) = \mathbf{1}\}$ is discrete. The following result implies that we may study the covering homomorphisms of connected Lie groups by studying the discrete central subgroups.

A.1.11 Proposition. ([24]) *Every discrete normal subgroup N of a connected Lie group G is contained in its centre.*

A.1.12 Theorem. ([24]) *Every connected Lie group G is isomorphic to a quotient $\widetilde{\mathsf{G}}/\mathsf{N}$, where $\widetilde{\mathsf{G}}$ is a simply connected Lie group and N a discrete normal subgroup. The pair $(\widetilde{\mathsf{G}}, \mathsf{N})$ is determined by these conditions up to an isomorphism,* i.e. *if $(\widetilde{\mathsf{G}}, \mathsf{N})$ and $(\widetilde{\mathsf{G}}', \mathsf{N}')$ are two such pairs, then there exists an isomorphism $\widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}'$, taking N to N'.*

The group $\widetilde{\mathsf{G}}$ is called the **universal covering Lie group** of G. By the previous result and theorem A.1.8, we can determine (up to isomorphism) every connected Lie group with a specified Lie algebra by finding the associated simply connected Lie group, and classifying the discrete central subgroups thereof.

A.1.4 Adjoint representations

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let V be a vector space. A **representation of** G is a Lie group homomorphism $\Psi : \mathsf{G} \to \mathsf{GL}(\mathsf{V})$, $g \mapsto \Psi_g$. (Here $\mathsf{GL}(\mathsf{V})$ is the group of invertible linear isomorphisms from V to itself.) The **orbit** $\mathcal{O}(X)$ of Ψ through the point $X \in \mathsf{V}$ is $\mathcal{O}(X) = \{\Psi_g \cdot X : g \in \mathsf{G}\}$. The orbits of V form a partition of V.

A **representation of** $\mathfrak{g}$ is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{gl}(\mathsf{V})$, $A \mapsto \psi_A$. (Here $\mathfrak{gl}(\mathsf{V})$ is the Lie algebra of $\mathsf{GL}(\mathsf{V})$.) We have the following relationship between representations of G and $\mathfrak{g}$:

A.1.13 PROPOSITION. (CF. [24]) *If* $\Psi : \mathsf{G} \to \mathsf{GL}(\mathsf{V})$ *is a representation of* G*, then the linearisation* $T_1\Psi : \mathfrak{g} \to \mathfrak{gl}(\mathsf{V})$ *is a representation of* $\mathfrak{g}$*.*

In particular, the **adjoint representations of** G **and** $\mathfrak{g}$ are defined as $\mathrm{Ad} : \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$ and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $X \mapsto \mathrm{ad}_X$, respectively. Here

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto gXg^{-1} \qquad \text{and} \qquad \mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [X, Y].$$

(The map ad can be shown to be the linearisation of Ad.) Let $\mathfrak{g}^*$ denote the dual space of $\mathfrak{g}$. The **coadjoint representations of** G **and** $\mathfrak{g}^*$ are defined as $\mathrm{Ad}^* : \mathsf{G} \to \mathsf{GL}(\mathfrak{g}^*)$, $g \mapsto \mathrm{Ad}^*_{g^{-1}}$ and $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$, $X \mapsto \mathrm{ad}^*_{-X}$, respectively. Here $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ and $\mathrm{ad}^*_{-X} : \mathfrak{g}^* \to \mathfrak{g}^*$ are the dual maps of $\mathrm{Ad}_{g^{-1}}$ and ad_{-X}, respectively. That is,

$$\left\langle \mathrm{Ad}^*_{g^{-1}}\, p, X \right\rangle = \langle p, \mathrm{Ad}_{g^{-1}} X \rangle \qquad \text{and} \qquad \left\langle \mathrm{ad}^*_{-X}\, p, Y \right\rangle = \langle p, \mathrm{ad}_{-X} Y \rangle\,.$$

We denote the orbit through $X \in \mathfrak{g}$ of the adjoint representation Ad by $\mathfrak{Orb}(X)$. Similarly, the orbit through $p \in \mathfrak{g}^*$ of the coadjoint representation Ad^* is denoted $\mathfrak{orb}(p)$.

A **bilinear form** on $\mathfrak{g}$ is a bilinear map $\mathcal{B}(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. $\mathcal{B}(\cdot,\cdot)$ is called **nondegenerate** if $\mathcal{B}(X,\cdot) = 0$ implies $X = 0$, and is called **invariant** if

$$\mathcal{B}([X,Y],Z) + \mathcal{B}(X,[Y,Z]) = 0 \tag{A.1.2}$$

for every $X, Y, Z \in \mathfrak{g}$. If G is connected, the condition (A.1.2) is equivalent to the condition that $\mathcal{B}(\mathrm{Ad}_g X, \mathrm{Ad}_g Y) = \mathcal{B}(X,Y)$ for every $g \in \mathsf{G}$ and $X, Y \in \mathfrak{g}$. The presence of a nondegenerate bilinear form on $\mathfrak{g}$ permits an identification of the adjoint and coadjoint orbits. (However, the existence of such a bilinear form is not guaranteed.)

A.1.14 PROPOSITION. *Suppose* G *is connected and* $\mathfrak{g}$ *admits a nondegenerate invariant bilinear form. Then there exists a linear isomorphism* $\psi : \mathfrak{g} \to \mathfrak{g}^*$ *such that* $\psi \cdot \mathfrak{Orb}(X) = \mathfrak{orb}(\psi \cdot X)$ *for every* $X \in \mathfrak{g}$*.*

PROOF. Denote the invariant bilinear form by $\mathcal{B}(\cdot,\cdot)$ and let $\psi : \mathfrak{g} \to \mathfrak{g}^*$ be defined by $\psi \cdot X = \mathcal{B}(X,\cdot)$. Since $\mathcal{B}(\cdot,\cdot)$ is nondegenerate, we have that ψ is injective. Furthermore, since $\dim(\mathfrak{g}) = \dim(\mathfrak{g}^*)$, it follows that ψ is a linear isomorphism. Let $g \in \mathsf{G}$ and $X \in \mathfrak{g}$. Then for every $Y \in \mathfrak{g}$ we have

$$\langle \psi \cdot \mathrm{Ad}_g X, Y \rangle = \mathcal{B}(\mathrm{Ad}_g X, Y) = \mathcal{B}(X, \mathrm{Ad}_{g^{-1}} Y) = \langle \psi \cdot X, \mathrm{Ad}_{g^{-1}} Y \rangle = \langle \mathrm{Ad}^*_{g^{-1}}(\psi \cdot X), Y \rangle.$$

Thus $\psi \cdot \mathrm{Ad}_g X = \mathrm{Ad}^*_{g^{-1}}(\psi \cdot X)$, and so $\psi \cdot \mathfrak{Orb}(X) = \mathfrak{orb}(\psi \cdot X)$. ∎

A.1.5 Classes of Lie groups and Lie algebras

We review several different classes of (connected) Lie groups and Lie algebras. The following definitions and the results that follow are drawn from [24, 30, 26, 42]. Suppose G is connected, with Lie algebra $\mathfrak{g}$. We say that G and $\mathfrak{g}$ are

(*i*) **nilpotent** if the sequence

$$\mathfrak{g}, \quad [\mathfrak{g},\mathfrak{g}], \quad [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]], \quad [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]], \quad \ldots$$

terminates in $\{0\}$ after finitely many steps. Equivalently, G and $\mathfrak{g}$ are nilpotent if the eigenvalues of ad_X are zero for every $X \in \mathfrak{g}$.

(*ii*) **completely solvable** if the eigenvalues of ad_X are real for every $X \in \mathfrak{g}$.

(*iii*) **solvable** if the sequence

$$\mathfrak{g}, \quad [\mathfrak{g},\mathfrak{g}], \quad [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]], \quad [[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]],[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]], \quad \ldots$$

terminates in $\{0\}$ after finitely many steps.

(*iv*) **simple** if the only ideals of $\mathfrak{g}$ are the trivial ideal $\{\mathbf{1}\}$ and $\mathfrak{g}$ itself.

(*v*) **semisimple** if it has no nontrivial solvable ideals.

Furthermore, G is called **exponential** if the exponential map $\exp : \mathfrak{g} \to \mathsf{G}$ is a diffeomorphism. The Lie algebra $\mathfrak{g}$ is called exponential if the simply connected Lie group with Lie algebra $\mathfrak{g}$ is exponential. Lastly, G is called **unimodular** if the left Haar measure on G is also right-invariant.

A.1.15 Proposition. *We have the following chain of implications*:

$$\mathfrak{g} \text{ is nilpotent} \quad \Rightarrow \quad \mathfrak{g} \text{ is completely solvable} \quad \Rightarrow \quad \mathfrak{g} \text{ is exponential} \quad \Rightarrow \quad \mathfrak{g} \text{ is solvable.}$$

A.1.16 Proposition. *A connected Lie group* G *is unimodular if and only if* $\mathrm{tr}\,\mathrm{ad}_X = 0$ *for every* $X \in \mathfrak{g}$.

A.1.17 Proposition. *If* $\mathfrak{g}$ *is semisimple, then it is not solvable.*

A.2 Invariant Control Systems

We review some basic notions of invariant control theory. In particular, we define the control systems considered in this thesis, as well as the admissible controls, trajectories and controllability of these systems. The main references for this section are [7, 29, 42].

A **left-invariant control system** is a pair $\Sigma = (\mathsf{G}, \Xi)$, where

(*i*) the **state space** G is a (real, finite-dimensional) connected matrix Lie group with Lie algebra $\mathfrak{g}$.

(*ii*) the **dynamics** $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is a family of left-invariant vector fields, parametrised by the **controls** $u \in \mathbb{R}^\ell$:

$$\Xi(g,u) = g\,\Xi(\mathbf{1},u), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

(Since G is a matrix Lie group, the multiplication $g\,\Xi(\mathbf{1},\cdot)$ is exactly the left-translation $T_\mathbf{1} L_g \cdot \Xi(\mathbf{1},\cdot)$, where $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$.) The map $\Xi(\mathbf{1},\cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is called the **parametrisation map**.

In classical notation, we write a control system Σ as

$$\dot{g} = \Xi(g,u) = g\,\Xi(\mathbf{1},u), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

In particular, an **ℓ-input left-invariant control affine system** is a control system with dynamics of the form $\Xi(\mathbf{1},u) = A + u_1 B_1 + \ldots + u_\ell B_\ell$. Here $A, B_1, \ldots, B_\ell$ are elements of the Lie algebra $\mathfrak{g}$ and $B_1, \ldots, B_\ell$ are linearly independent. The **trace** of Σ is $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$, *i.e.*, the affine subspace $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$. We say that Σ is **homogeneous** if $A \in \Gamma^0$, and **inhomogeneous**, otherwise. Σ is said to have **full rank** if Γ generates the entire Lie algebra, *i.e.*, the smallest Lie subalgebra $\mathsf{Lie}(\Gamma)$ of $\mathfrak{g}$ containing Γ coincides with $\mathfrak{g}$.

An **admissible control** is a piecewise continuous map $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A **trajectory** corresponding to an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = \Xi(g(t), u(t))$ for almost every $t \in [0, T]$. By the Carathéodory existence and uniqueness theorem for ordinary differential equations, trajectories must exist (at least, locally). Furthermore, by the left-invariance of Σ, left translations of trajectories are also trajectories.

Σ is called **controllable** if there exists a trajectory joining any two points of G. We have the following necessary condition for controllability.

A.2.1 PROPOSITION. (CF. [42]) *Suppose Σ is controllable. Then G is connected and Σ has full rank.*

The **attainable set from** $g_0 \in \mathsf{G}$, denoted $\mathcal{A}_{g_0}$, is defined as

$$\mathcal{A}_{g_0} = \{g(T) : g(\cdot) : [0, T] \to \mathsf{G} \text{ is a trajectory}, \ g(0) = g_0\}.$$

By left-invariance of Σ, we have $\mathcal{A}_{g_0} = g_0 \mathcal{A}_{\mathbf{1}}$. We abbreviate $\mathcal{A}_{\mathbf{1}}$ as $\mathcal{A}$. Controllability of Σ may be characterised in terms of $\mathcal{A}$.

A.2.2 PROPOSITION. ([42]) *Σ is controllable if and only if $\mathcal{A} = \mathsf{G}$.*

Lastly, for the particular case of a completely solvable, connected and simply connected Lie group G, we have the following characterisation of controllability.

A.2.3 PROPOSITION. ([42]) *Suppose G is completely solvable, connected and simply connected. Σ is controllable if and only if $\mathsf{Lie}(\Gamma^0) = \mathfrak{g}$.*

A.3 Optimal Control on Lie Groups

Optimal control theory is the natural extension of the study of control systems. It provides tools for investigating optimal solutions to a control system with specified boundary conditions, while minimising (or maximising) some (practical) cost. For our purposes, we shall only be concerned with the minimisation problem, with fixed time. The main references for this section are [7, 29, 31]. For subsections A.3.1 and A.3.2, we have also drawn upon [34, 29, 32].

Let $\Sigma = (\mathsf{G}, \Xi)$ be a left-invariant control affine system on a (real, finite-dimensional) connected matrix Lie group G. (See section A.2.) An **(invariant) optimal control problem** associated to Σ is specified by (*i*) the control system Σ, (*ii*) a positive definite quadratic form $\chi : \mathbb{R}^\ell \to \mathbb{R}$ (the **cost function**) and (*iii*) boundary conditions, *viz.* an initial state $g_0 \in \mathsf{G}$, a terminal state $g_1 \in \mathsf{G}$ and a fixed terminal time $T > 0$. We write such an optimal control problem as

$$\dot{g} = g\,\Xi(\mathbf{1}, u), \quad g(\cdot) : [0, T] \to \mathsf{G}, \ u(\cdot) : [0, T] \to \mathbb{R}^\ell \tag{A.3.1}$$

$$g(0) = g_0, \ g(T) = g_1, \quad g_0, g_1 \in \mathsf{G}, \ T > 0 \text{ fixed} \tag{A.3.2}$$

$$\mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t))\, dt \to \min. \tag{A.3.3}$$

(In fact, we may assume $g_0 = \mathbf{1}$, since this can always be arranged by using a suitable left translation.) Explicitly, we wish to minimise the functional $\mathcal{J}(\cdot)$ over trajectory-control pairs $(g(\cdot), u(\cdot))$ subject to the specified boundary conditions. Note that, in general, a solution to the optimal control problem is not guaranteed. (However, in special cases existence is assured; *e.g.*, for the length-minimisation problem in Riemannian or sub-Riemannian geometry. See section A.4.)

A.3.1 Hamiltonian formalism on symplectic and Poisson manifolds

Let M be a smooth manifold. A **symplectic structure** on M is a (smooth) nondegenerate bilinear two-form ω on M. (By nondegenerate we mean that the bilinear form $\omega_\xi : T_\xi\mathsf{M} \times T_\xi\mathsf{M} \to \mathbb{R}$ is nondenegerate for every $\xi \in \mathsf{M}$. That is, if $\omega_\xi(V, W) = 0$ for every $W \in T_\xi\mathsf{M}$, then $V = 0$.) The pair (M, ω) is called a **symplectic manifold**.

A **Poisson structure** (or **Poisson bracket**) on M is a bilinear, skew-symmetric map $\{\cdot,\cdot\} : \mathsf{C}^\infty(\mathsf{M}) \times \mathsf{C}^\infty(\mathsf{M}) \to \mathsf{C}^\infty(\mathsf{M})$ satisfying, for every $F, G, H \in \mathsf{C}^\infty(\mathsf{M})$:

(*i*) the Jacobi identity: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$;

(*ii*) $\{\cdot,\cdot\}$ is a derivation in each factor: $\{FG, H\} = F\{G, H\} + \{F, H\}G$.

(Note that $\mathsf{C}^\infty(\mathsf{M})$, together with the Poisson bracket $\{\cdot,\cdot\}$, forms a Lie algebra.) The pair $(\mathsf{M}, \{\cdot,\cdot\})$ is called a **Poisson manifold**.

Let $H \in \mathsf{C}^\infty(\mathsf{M})$ be a **Hamiltonian function** and let ω (resp. $\{\cdot,\cdot\}$) be a symplectic form (resp. Poisson bracket) on M. We associate to H a (smooth) vector field $\vec{H}$, called a **Hamiltonian vector field**, as follows. For the symplectic case, $\vec{H}$ is defined by

$$\omega_\xi(\vec{H}(\xi), V) = \mathbf{d}H(\xi) \cdot V, \qquad \xi \in \mathsf{M},\ V \in T_\xi\mathsf{M}. \tag{A.3.4}$$

In the Poisson case, $\vec{H}$ is defined by its action on smooth functions:

$$\vec{H}[F] = \{F, H\}, \qquad F \in \mathsf{C}^\infty(\mathsf{M}). \tag{A.3.5}$$

A triple $(\mathsf{M}, \{\cdot,\cdot\}, H)$, where $H \in \mathsf{C}^\infty(\mathsf{M})$ is a Hamiltonian function, is called a **Hamilton-Poisson system**. If the Poisson manifold $(\mathsf{M}, \{\cdot,\cdot\})$ is fixed, we identify a Hamilton-Poisson system with its Hamiltonian function.

Every symplectic manifold (M, ω) is a Poisson manifold. Indeed, define the Poisson bracket

$$\{F, G\}(\xi) = \omega_\xi(\vec{F}(\xi), \vec{G}(\xi)), \qquad F, G \in \mathsf{C}^\infty(\mathsf{M}).$$

Then $(\mathsf{M}, \{\cdot,\cdot\})$ is a Poisson manifold. The converse, however, does not hold. (Indeed, for a Poisson manifold to be a symplectic manifold, we require that the Poisson bracket is nondegenerate. See, *e.g.*, [34].) In the remainder of this section, we assume that we are working on a Poisson manifold $(\mathsf{M}, \{\cdot,\cdot\})$.

Given a Hamiltonian vector field $\vec{H}$ on M, an **integral curve** of $\vec{H}$ is an absolutely continuous curve $\xi(\cdot)$ that satisfies the **equations of motion**, *i.e.*, $\dot{\xi}(t) = \vec{H}(\xi(t))$. By the Carathéodory existence and uniqueness theorem for ordinary differential equations, there exists a unique solution to the Cauchy problem

$$\dot{\xi}(t) = \vec{H}(\xi(t)), \qquad \xi(0) = \xi_0 \in \mathsf{M}.$$

(As such, integral curves always exist locally.) $\vec{H}$ is said to be **complete** if the domain of every integral curve can be extended to $\mathbb{R}$.

Casimir functions are functions that Poisson-commute with every other function. That is, $C \in \mathsf{C}^\infty(\mathsf{M})$ is a Casimir function if and only if $\{C, F\} = 0$ for every $F \in \mathsf{C}^\infty(\mathsf{M})$. Equivalently, we write $\vec{C} = 0$. If C is a Casimir function, then so is $f(C)$ for any function $f : \mathbb{R} \to \mathbb{R}$. Casimir functions are constants of motion of the system. (However, nontrivial Casimir functions are not guaranteed to exist; furthermore, they may not be defined globally.)

A.3.1 Proposition. (cf. [34]) *Let C be a Casimir function on M. Then C is constant along the integral curves of every Hamiltonian vector field.*

Another constant of motion is provided by the Hamiltonian function. This result is the so-called **conservation of energy**.

A.3.2 Proposition. (cf. [34]) *Let $p(\cdot)$ be an integral curve of $\vec{H}$. Then $H(p(t))$ is constant in t.*

Suppose M admits a (global) Casimir function C and let $\vec{H}$ be a Hamiltonian vector field. Since C is a constant of motion, every integral curve $p(\cdot)$ of $\vec{H}$ develops on the level set $C^{-1}(c_0)$, where $c_0 = C(p(0))$. Similarly, H is a constant of motion (by the conservation of energy), and so $p(\cdot)$ develops on $H^{-1}(h_0)$, $h_0 = H(p(0))$. Thus $p(\cdot)$ evolves on the intersection $C^{-1}(c_0) \cap H^{-1}(h_0)$.

A.3.2 Lie-Poisson structure

Let $\mathfrak{g}$ be a (real, n-dimensional) Lie algebra, with dual space $\mathfrak{g}^*$. The **(minus) Lie-Poisson structure** (or **(minus) Lie-Poisson bracket**) on $\mathfrak{g}^*$ is defined as

$$\{F, G\}(p) = -\left\langle \mathrm{ad}^*_{\mathbf{d}F(p)}\, p, \mathbf{d}G(p) \right\rangle = -\langle p, [\mathbf{d}F(p), \mathbf{d}G(p)]\rangle.$$

Here $[\cdot, \cdot]$ denotes the Lie bracket on $\mathfrak{g}$. (As $\mathbf{d}F(p)$ and $\mathbf{d}G(p)$ are linear functions on $\mathfrak{g}^*$, they are elements of $\mathfrak{g}^{**} \cong \mathfrak{g}$.) A **Lie-Poisson space** is a pair $(\mathfrak{g}^*, \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is the minus Lie-Poisson bracket on $\mathfrak{g}^*$; we denote $\mathfrak{g}^*_- = (\mathfrak{g}^*, \{\cdot, \cdot\})$.

A **linear Poisson automorphism** is a linear isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ that preserves the Lie-Poisson bracket, *i.e.*, $\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}$ for every $F, G \in \mathsf{C}^\infty(\mathfrak{g}^*)$. The following result relates linear Poisson automorphisms to Lie algebra automorphisms.

A.3.3 Proposition. ([34]) *Let $\mathfrak{g}$ be a Lie algebra and let $\psi : \mathfrak{g} \to \mathfrak{g}$ be a linear map. The map ψ is a Lie algebra automorphism if and only if its dual $\Psi = \psi^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear Poisson automorphism.*

Let $H \in \mathsf{C}^\infty(\mathfrak{g}^*)$ be a Hamiltonian function. By equation A.3.5, we associate to H a Hamiltonian vector field $\vec{H}$. In coordinates, $\vec{H}$ is given by $\vec{H}(p) = \mathrm{ad}^*_{\mathbf{d}H(p)}\, p$. Consequently, we can write the equations of motion componentwise as $\dot{p}_i = -\langle p, [E_i, \mathbf{d}H(p)]\rangle$, $i = 1, \ldots, n$. where $(E_i)_{i=1}^n$ is a basic for $\mathfrak{g}$.

A.3.3 Symplectic structure of the coadjoint orbits

Let $\mathfrak{g}^*$ be the dual of a (finite-dimensional) Lie algebra $\mathfrak{g}$. The coadjoint orbits of G form a partition of $\mathfrak{g}^*$. (See section A.1.4.) Furthermore, each orbit $\mathfrak{orb}(p)$, $p \in \mathfrak{g}^*$ admits a symplectic structure, *i.e.*, a nondegenerate bilinear two-form. ([34] and [32] discuss this topic in detail.) In particular, this implies that the coadjoint orbits of G are even dimensional. Furthermore, the minus Lie-Poisson bracket, restricted to each coadjoint orbit $\mathfrak{orb}(p)$, is exactly the Poisson structure induced by the symplectic structure on $\mathfrak{orb}(p)$. Indeed, we have the following result.

A.3.4 Theorem. (Coadjoint Orbit Theorem, [34]) *Let G be a Lie group and let $\mathfrak{orb}(p_0)$ be the coadjoint orbit through $p_0 \in \mathfrak{g}^*$. Then*

$$\omega_p(\mathrm{ad}_X^* p, \mathrm{ad}_Y^* p) = -\langle p, [X,Y]\rangle$$

for all $p \in \mathfrak{orb}(p_0)$ and $X, Y \in \mathfrak{g}$ defines a symplectic form on $\mathfrak{orb}(p_0)$.

Furthermore, integral curves of a Hamiltonian vector field and Casimir functions are constant along the coadjoint orbits.

A.3.5 Proposition. (cf. [34]) *Let $H \in C^\infty(\mathfrak{g}^*)$ be a Hamiltonian function and $C \in C^\infty(\mathfrak{g}^*)$ a Casimir function.*

(*i*) *If $p(\cdot)$ is an integral curve of $\vec{H}$ such that $p(0) \in \mathfrak{orb}(p_0)$ for some $p_0 \in \mathfrak{g}$, then $p(t) \in \mathfrak{orb}(p_0)$ for all t.*

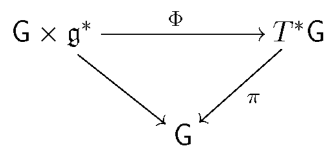
(*ii*) *$C|_{\mathfrak{orb}(p)}$ is constant for each $p \in \mathfrak{g}^*$.*

A.3.4 Pontryagin's Maximum Principle

Pontryagin's Maximum Principle provides necessary conditions for optimality of solutions to an optimal control problem (A.3.1)-(A.3.2)-(A.3.3). We state the maximum principle in the language of the Poisson (in fact, symplectic) structure on the cotangent bundle $T^*\mathsf{G}$ of G. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}^*$ denote the dual space of $\mathfrak{g}$. We have the following result.

A.3.6 Proposition. (cf. [7]) *The cotangent bundle may be trivialised from the left as $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$. That is, there exists a diffeomorphism $\Phi : \mathsf{G} \times \mathfrak{g}^* \to T^*\mathsf{G}$ such that*

(*i*) *the diagram*



commutes, i.e.*, $\pi \circ \Phi(g,p) = g$. (Here $\pi : T^*\mathsf{G} \to \mathsf{G}$ is the canonical projection.)*

(*ii*) *the map $\Phi(\cdot, g) : \mathfrak{g}^* \to T_g^*\mathsf{G}$ is a linear isomorphism for every $g \in \mathsf{G}$.*

Accordingly, we identify the cotangent bundle $T^*\mathsf{G}$ with $\mathsf{G} \times \mathfrak{g}^*$. (The tangent bundle $T\mathsf{G}$ may be trivialised in a similar fashion. To wit, we have $T\mathsf{G} = \mathsf{G} \times \mathfrak{g}$. However, we shall not require this result.)

The cotangent bundle admits a canonical symplectic structure ω. Indeed, let $\pi : T^*\mathsf{G} \to \mathsf{G}$ denote the canonical projection that sends every cotangent vector to its base point. The **tautological** one-form θ is the map $T^*\mathsf{G} \to T^*(T^*\mathsf{G})$ defined by

$$\theta : \xi \mapsto \theta_\xi \in T_\xi^*(T^*\mathsf{G}), \qquad \langle \theta_\xi, \eta\rangle = \langle \xi, T_\xi\pi \cdot \eta\rangle, \qquad \xi \in T^*\mathsf{G},\ \eta \in T_\xi(T^*\mathsf{G}).$$

Using the trivialisation $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$, we have $\xi = (g, p)$, where $g \in \mathsf{G}$ and $p \in \mathfrak{g}^*$. Thus $T_\xi(T^*\mathsf{G}) = T_g\mathsf{G} \times \mathfrak{g}^*$. Furthermore, using the identification (A.1.1), we have $\eta = (gX, q)$, where $X \in \mathfrak{g}$ and $q \in \mathfrak{g}^*$. Accordingly,

$$\begin{aligned}\langle \theta_{(g,p)}, (gX, q)\rangle &= \langle (g,p), T_{(g,p)}\pi \cdot (gX, q)\rangle \\ &= \langle (g,p), gX\rangle = \langle p, X\rangle.\end{aligned}$$

Let $\omega = -\mathbf{d}\theta$, where (the differential) $\mathbf{d}\theta$ is defined by its action on vector fields $X, Y \in \mathsf{Vec}(T^*\mathsf{G})$ as $\mathbf{d}\theta(X, Y) = X[\langle \theta, Y\rangle] - Y[\langle \theta, X\rangle] - \langle \theta, [X, Y]\rangle$. Then ω defines a (canonical) symplectic form on $T^*\mathsf{G}$. (See, *e.g.*, [34]. Note that we have followed the same sign convention $\omega = -\mathbf{d}\theta$ of [34].) As such, by equation (A.3.4) we associate to every function $H \in \mathsf{C}^\infty(T^*\mathsf{G})$ a Hamiltonian vector field $\vec{H}$. Furthermore, ω induces a Poisson structure on $T^*\mathsf{G}$.

To the optimal control problem (A.3.1)-(A.3.2)-(A.3.3), we associate a family of **control-dependent Hamiltonian functions** $(H_u^\nu)_{u \in \mathbb{R}^\ell}$, where $H_u^\nu : T^*\mathsf{G} \to \mathbb{R}$ is defined as

$$\begin{aligned}H_u^\nu(\xi) &= \langle \xi, \Xi(g, u)\rangle + \nu\,\chi(u) \\ &= \langle p, \Xi(\mathbf{1}, u)\rangle + \nu\,\chi(u), \quad \xi = (g,p) \in T^*\mathsf{G}.\end{aligned}$$

Pontryagin's Maximum Principle is stated in terms of these Hamiltonians.

PONTRYAGIN'S MAXIMUM PRINCIPLE. *Suppose that the trajectory-control pair* $(\bar{g}(\cdot), \bar{u}(\cdot))$*, defined over the interval* $[0, T]$*, is a solution for the optimal control problem* (A.3.1)-(A.3.2)-(A.3.3)*. Then there exists a curve* $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$*,* $t \in [0, T]$ *and a real number* $\nu \leq 0$ *such that the following conditions hold for almost every* $t \in [0, T]$:

$$(\nu, \xi(t)) \not\equiv (0, 0) \tag{A.3.6}$$

$$\dot{\xi}(t) = \vec{H}^\nu_{\bar{u}(t)}(\xi(t)) \tag{A.3.7}$$

$$H^\nu_{\bar{u}(t)}(\xi(t)) = \max_{u \in \mathbb{R}^\ell} H^\nu_u(\xi(t)) = \textit{constant}. \tag{A.3.8}$$

An **optimal trajectory** $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the nonautonomous vector field $\vec{H}^\nu_{\bar{u}(t)}$. A trajectory-control pair $(g(\cdot), u(\cdot))$ that satisfies the conditions (A.3.6), (A.3.7) and (A.3.8) is called an **abnormal extremal** if $\nu = 0$, and a **normal extremal**, otherwise. (In the latter case, $\nu < 0$ may be taken to have any strictly negative value. See, *e.g.*, [42].)

For the optimal control problem (A.3.1)-(A.3.2)-(A.3.3), the maximality condition (A.3.8) of the Maximum Principle eliminates the control parameter u from the family of Hamiltonians $(H_u^\nu)_{u \in \mathbb{R}^\ell}$. The result is a single, G-invariant Hamiltonian function H defined on $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$. The invariance of H permits a reduction of the Poisson structure on $T^*\mathsf{G}$ (induced by the canonical symplectic structure ω) to a Poisson structure (the (*minus*) *Lie-Poisson structure*; see section A.3.2) on the dual space $\mathfrak{g}^*$. (For details of the reduction process, see [31].) Accordingly, we obtain a Hamiltonian function H on $\mathfrak{g}^*$. The extremal controls of (A.3.1)-(A.3.2)-(A.3.3) are linearly related to the integral curves of $\vec{H}$. As such, the investigation of the extremal controls is essentially reduced to the study of the Hamilton-Poisson system $(\mathfrak{g}^*_-, H)$.

Suppose the dynamics and cost function of the optimal control problem (A.3.1)-(A.3.2)-(A.3.3) are specified by $\Xi(\mathbf{1}, u) = A + u_1 B_1 + \ldots + u_\ell B_\ell$ and $\chi(u) = u^\top Q u$, respectively, where $Q \in \mathbb{R}^{\ell \times \ell}$ is symmetric and positive definite. Form the matrix $\mathbf{B} \in \mathbb{R}^{n \times \ell}$ by taking the coordinate vector of B_i (with respect to a specified basis for $\mathfrak{g}$) for its i^{th} column. We have the following result.

A.3.8 THEOREM. ([15, 17]) *Any normal extremal trajectory-control pair* $(g(\cdot), u(\cdot))$ *of* (A.3.1)-(A.3.2)-(A.3.3) *is given by* $\dot{g}(t) = \Xi(g(t), u(t))$, $u(t) = Q^{-1}\mathbf{B}^\top p(t)^\top$. *Here* $p(\cdot)$ *is an integral curve for the Hamilton-Poisson system on* $\mathfrak{g}^*_-$ *specified by*

$$H(p) = pA + \tfrac{1}{2}\, p\, \mathbf{B}\, Q^{-1}\mathbf{B}^\top p^\top, \tag{A.3.9}$$

where p *is written as a row vector (in terms of the dual basis of* $\mathfrak{g}^*$*).*

The Hamiltonian function (A.3.9) on $\mathfrak{g}^*_-$ is called the **reduced** Hamiltonian. Furthermore, since Q is positive definite and $\mathbf{B}$ does not have full rank in general, it follows that $\mathbf{B}\, Q^{-1}\mathbf{B}^\top$ is positive semidefinite. Consequently, H is of the form

$$H(p) = H_{A,\mathcal{Q}}(p) = \langle p, A\rangle + \mathcal{Q}(p),$$

where $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{g}^*_-$.

Lastly, from theorem A.3.8, the equations of motion for the (normal) extremal trajectory-control pair $(g(\cdot), u(\cdot))$ on $T^*\mathsf{G}$ take the form

$$\begin{cases} \dot{p}(t) = \vec{H}(p(t)) \\ \dot{g}(t) = \Xi(g(t), u(t)). \end{cases}$$

The first equation is called the **vertical subsystem**, whereas the second is the **horizontal subsystem**.

A.4 Riemannian and Sub-Riemannian Geometry

Riemannian geometry is the study of manifolds admitting local notions (by means of a Riemannian metric $\mathbf{g}$ on the tangent bundle) of distance, curve length, angle, area, volume, *etc.* Sub-Riemannian geometry is a generalisation of Riemannian geometry, in that the metric $\mathbf{g}$ is restricted to a class of “admissible velocities.” We collect here some basic concepts of Riemannian and sub-Riemannian geometry. We draw from [37, 42].

Let G be a (real, finite-dimensional) connected matrix Lie group. A **sub-Riemannian structure** on G is a pair $(\mathcal{D}, \mathbf{g})$, where

(*i*) the **distribution** $\mathcal{D}$ is a smooth map that assigns to every $g \in \mathsf{G}$ a vector subspace $\mathcal{D}_g$ of $T_g\mathsf{G}$.

(*ii*) $\mathbf{g}$ is a **sub-Riemannian metric**, *i.e.*, for every $g \in \mathsf{G}$, $\mathbf{g}_g : \mathcal{D}_g \times \mathcal{D}_g \to \mathbb{R}$ is a (positive definite) inner product.

(We assume the dimension of $\mathcal{D}_g$ does not depend on g, *i.e.*, $\mathcal{D}$ has constant rank.) If $\mathcal{D}_g = T_g\mathsf{G}$ for every $g \in \mathsf{G}$, then $(\mathcal{D}, \mathbf{g})$ is called a **Riemannian structure**, and abbreviated to $\mathbf{g}$. (However, we shall continue to include Riemannian structures as a special case of sub-Riemannian structures in the following discussion.)

A sub-Riemannian structure $(\mathcal{D}, \mathbf{g})$ is said to be **left-invariant** if it is invariant under left-translations:

$$T_h L_g \cdot \mathcal{D}_h = \mathcal{D}_{gh} \qquad \text{and} \qquad \mathbf{g}_{gh}(T_h L_g \cdot X, T_h L_g \cdot Y) = \mathbf{g}_h(X, Y)$$

for every $g, h \in \mathsf{G}$ and $X, Y \in \mathcal{D}_h$. (Here $L_g : \mathsf{G} \to \mathsf{G}$ denotes the left multiplication map $L_g(h) = gh$.) For a left-invariant distribution $\mathcal{D}$ of rank ℓ, there exist left-invariant vector fields $X_1, \ldots, X_\ell$ such that $\mathcal{D}_g = \langle X_1(g), \ldots, X_\ell(g) \rangle$ for every $g \in \mathsf{G}$. Using the identification of left-invariant vector fields with elements of the Lie algebra (section A.1.2), we may (uniquely) specify a left-invariant sub-Riemannian structure $(\mathcal{D}, \mathbf{g})$ by selecting a subspace $\mathcal{D}_\mathbf{1}$ of the Lie algebra and defining an inner product $\mathbf{g}_\mathbf{1}$ on that subspace. (The structure is then extended to every point in G by means of left translations: $\mathcal{D}_g = g\,\mathcal{D}_\mathbf{1}$ and $\mathbf{g}_g(gX, gY) = \mathbf{g}_\mathbf{1}(X, Y)$, where $X, Y \in \mathcal{D}_\mathbf{1}$.) We shall assume all sub-Riemannian structures under consideration are left-invariant.

An absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ is called **horizontal** if $\dot{g}(t) \in \mathcal{D}_{g(t)}$ for almost every $t \in [0, T]$. We say that $\mathcal{D}$ is **bracket-generating** if $\mathsf{Lie}(\mathcal{D}_\mathbf{1}) = \mathfrak{g}$. We have the following result concerning bracket-generating distributions.

A.4.1 Theorem. (Chow-Rashevskii, cf. [37]) *Let $\mathcal{D}$ be a bracket-generating distribution on a connected Lie group G. Then any two points of G can be joined by a horizontal curve.*

The **length** $\ell(g(\cdot))$ and **energy** $\mathcal{J}(g(\cdot))$ of a horizontal curve $g(\cdot) : [0, T] \to \mathsf{G}$ are defined as

$$\ell(g(\cdot)) = \int_0^T \sqrt{\mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t))}\, dt \qquad \text{and} \qquad \mathcal{J}(g(\cdot)) = \int_0^T \mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t))\, dt,$$

respectively. The length of $g(\cdot)$ is invariant under reparametrisation; the energy is not. The **Carnot-Carathéodory distance** $d(g_1, g_2)$ between $g_1, g_2 \in \mathsf{G}$ is the infimum of lengths of all horizontal curves joining g_1 to g_2. That is,

$$d(g_1, g_2) = \inf \left\{ \ell(g(\cdot)) : g(\cdot) : [0, T] \to \mathsf{G} \text{ is a horizontal curve, } g(0) = g_1,\ g(T) = g_2 \right\}.$$

By the Chow-Rashevskii theorem (theorem A.4.1), $d(\cdot, \cdot)$ is well-defined, continuous and finite and induces on G the original topology. Furthermore, (G, d) is a metric space.

A **minimising geodesic** (or simply **geodesic**) of $(\mathcal{D}, \mathbf{g})$ is a horizontal curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $d(g(0), g(T)) = \ell(g(\cdot))$. The **sub-Riemannian problem** involves finding the geodesics of a given sub-Riemannian structure. (Similarly, the **Riemannian problem** concerns the Riemannian geodesics.) The (sub-)Riemannian problem may be stated as follows:

$$\begin{gathered} \dot{g}(t) \in \mathcal{D}_{g(t)}, \quad g(\cdot) : [0, T] \to \mathsf{G} \\ g(0) = g_0,\ g(T) = g_1, \quad g_0, g_1 \in \mathsf{G},\ T > 0 \text{ fixed} \\ \ell(g(\cdot)) = \int_0^T \sqrt{\mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t))}\, dt \to \min. \end{gathered}$$

Explicitly, we wish to determine those horizontal curves that minimise the length between specified points g_0 and g_1 in G in a fixed time $T > 0$. Should it exist, such a minimising curve is exactly the geodesic between g_0 and g_1. The following theorem guarantees existence (at least, locally) of minimising geodesics.

A.4.2 Theorem. ([37]) *If* G *is connected and* $\mathcal{D}$ *is bracket-generating, then any point* $g \in \mathsf{G}$ *is contained in a neighbourhood* U *such that every* $h \in U$ *can be connected to* g *by a minimising geodesic.*

We may write the sub-Riemannian (and Riemannian) problem as an optimal control problem on G. (Section A.3 discusses optimal control theory on Lie groups.). Indeed, suppose $(\mathcal{D}, \mathbf{g})$ is a bracket-generating sub-Riemannian structure on G. Since $\mathcal{D}$ is left-invariant, the condition $\dot{g}(t) \in \mathcal{D}_{g(t)}$ may be written as

$$\dot{g}(t) = g(t)(u_1(t)E_1 + \ldots + u_\ell(t)E_\ell),$$

where $u_i(\cdot) : [0, T] \to \mathbb{R}$, $i = 1, \ldots, \ell$ are admissible controls and $(E_i)_{i=1}^n$ is a basis for the Lie algebra of G. Consequently, by the left-invariance of the sub-Riemannian metric $\mathbf{g}$, we have

$$\begin{aligned}\mathbf{g}_{g(t)}(\dot{g}(t), \dot{g}(t)) &= \mathbf{g}_{g(t)}(g(t)(u_1(t)E_1 + \cdots + u_\ell(t)E_\ell), g(t)(u_1(t)E_1 + \cdots + u_\ell(t)E_\ell)) \\ &= \mathbf{g}_{\mathbf{1}}(u_1(t)E_1 + \cdots + u_\ell(t)E_\ell, u_1(t)E_1 + \cdots + u_\ell(t)E_\ell) \\ &= u_1(t)^2 + \ldots + u_\ell(t)^2.\end{aligned}$$

Lastly, we have the following result.

A.4.3 Proposition. (cf. [37]) *If the final time* $T > 0$ *is fixed, a horizontal curve* $g(\cdot) : [0, T] \to \mathsf{G}$ *is a minimiser of* $\mathcal{J}(\cdot)$ *if and only if it is a minimiser of* $\ell(\cdot)$ *and has constant speed.*

Hence, we may minimise the energy functional, rather than the length. (The energy is a smooth function, which simplifies the analysis.) Consequently, we may write the sub-Riemannian problem as the following optimal control problem:

$$\begin{gathered}\dot{g} = g(u_1E_1 + \ldots + u_\ell E_\ell), \quad g(\cdot) : [0, T] \to \mathsf{G},\ u(\cdot) : [0, T] \to \mathbb{R}^\ell \\ g(0) = g_0,\ g(T) = g_1, \quad g_0, g_1 \in \mathsf{G},\ T > 0 \text{ fixed} \\ \mathcal{J}(u(\cdot)) = \int_0^T u_1(t)^2 + \ldots + u_\ell(t)^2\, dt \to \min.\end{gathered}$$

In general, this optimal control problem admits two types of extremal trajectories, the abnormal trajectories and the normal trajectories (see section A.3.4). A geodesic $g(\cdot)$ is called **normal** (resp. **abnormal**) if it is the projection of a normal (resp. abnormal) extremal of the above optimal control problem. (There are no abnormal extremals for the Riemannian problem, hence all geodesics are normal. See *e.g.*, [37].)

A.5 Stability of Dynamical Systems

We review some concepts of (nonlinear) stability theory and spectral stability on Poisson manifolds (see section A.3.1), and cite a result for proving the nonlinear stability of systems. Our definitions in this section draw from [39, 34].

Let $(\mathsf{M}, \{\cdot, \cdot\})$ be a (smooth) Poisson manifold and let $X \in \mathsf{Vec}(\mathsf{M})$ be a smooth vector field. An **equilibrium point** of X is a point $z_e \in \mathsf{M}$ such that $X(z_e) = 0$. The unique integral curve $\xi(\cdot)$ of X starting at the point z_e is constant, *i.e.*, $\xi(t) = z_e$ for all t. The stability analysis of equilibria concerns the behaviour of integral curves starting near the equilibrium points. We say that an equilibrium point z_e of X is (*i*) **Lyapunov stable** (or simply **stable**) if for every neighbourhood U of z_e there exists a neighbourhood $V \subseteq U$ of z_e such that, for every integral curve $\xi(\cdot)$ of X with $\xi(0) \in V$, we have $\xi(t) \in U$ for all $t > 0$.

(*ii*) **Lyapunov unstable** (or simply **unstable**) if it is not Lyapunov stable. That is, there exists a neighbourhood U of z_e such that, for every neighbourhood $V \subseteq U$ of z_e, there exists an integral curve $\xi(\cdot)$ of X with $\xi(0) \in V$ and $\xi(t_1) \notin U$ for some $t_1 > 0$.

(*iii*) **spectrally stable** if all eigenvalues of the linearised dynamical system $\mathbf{D}\vec{X}(z_e)$ have non-positive real parts.

(*iv*) **spectrally unstable** if it is not spectrally stable.

In this thesis, we shall not be concerned with spectral stability *per se*. However, the following result permits us to prove Lyapunov instability by means of spectral instability.

A.5.1 PROPOSITION. (CF. [34]) *If an equilibrium point is (Lyapunov) stable, then it is spectrally stable.*

Let $H \in \mathsf{C}^\infty(\mathsf{M})$ be a Hamiltonian function. The **energy-Casimir method** (see, *e.g.*, [34]) provides sufficient conditions for stability of an equilibrium point $z_e \in \mathsf{M}$ of $\vec{H}$. To wit, suppose there exists a constant of motion C such that $\mathbf{d}(H+C)(z_e) = 0$ and $\mathbf{d}^2(H+C)(z_e)$ is positive definite. Then z_e is stable. (By a constant of motion we mean a function $C \in \mathsf{C}^\infty(\mathsf{M})$ such that $\{C, H\} = 0$.) The authors of [38] have extended the energy-Casimir method, to the effect that we need only check the definiteness of $\mathbf{d}^2(H+C)(z_e)$ on a certain subspace. We state the result here.

A.5.2 PROPOSITION. (CF. [38]) *Let $\vec{H}$ be a Hamiltonian vector field on M corresponding to a Hamiltonian function $H \in \mathsf{C}^\infty(\mathsf{M})$. Let z_e be an equilibrium point of $\vec{H}$ and $C_1, \ldots, C_k \in \mathsf{C}^\infty(\mathsf{M})$ conserved quantities of $\vec{H}$,* i.e., *$\{C_i, H\} = 0$ $(i = 1, \ldots, k)$. Assume there exist constants $\lambda_0, \ldots, \lambda_k \in \mathbb{R}$ such that*

$$\mathbf{d}(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(z_e) = 0$$

and the quadratic form

$$\mathbf{d}^2(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(z_e)\big|_{W \times W}$$

is positive definite with

$$W = \ker \mathbf{d}H(z_e) \cap \ker \mathbf{d}C_1(z_e) \cap \ldots \cap \ker \mathbf{d}C_k(z_e).$$

Then z_e is (Lyapunov) stable.

A.6 Jacobi Elliptic Functions

The Jacobi elliptic functions are a class of elliptic functions obtained by inverting certain elliptic integrals. These functions find application in numerous areas of mathematics, particularly geometry and mechanics. The following exposition draws from [8, 35].

A.6.1 Definition and basic properties

Let $k \in (0,1)$ and $k' = \sqrt{1-k^2}$ (k is called the **modulus**, whereas k' is called the **complementary modulus**). The Jacobi elliptic functions $\operatorname{sn}(\cdot,k)$, $\operatorname{cn}(\cdot,k)$ and $\operatorname{dn}(\cdot,k)$ are defined to be the solutions to the following initial value problem:

$$\begin{cases} \dot{x} = yz \\ \dot{y} = -zx \\ \dot{z} = -k^2 xy \end{cases} \qquad \begin{cases} \operatorname{sn}(0,k) = x(0) = 0 \\ \operatorname{cn}(0,k) = y(0) = 1 \\ \operatorname{dn}(0,k) = z(0) = 1. \end{cases}$$

From the above system of differential equations, we get the following derivative formulae:

$$\frac{d}{dt}\operatorname{sn}(t,k) = \operatorname{cn}(t,k)\operatorname{dn}(t,k) \tag{A.6.1}$$

$$\frac{d}{dt}\operatorname{cn}(t,k) = -\operatorname{dn}(t,k)\operatorname{sn}(t,k) \tag{A.6.2}$$

$$\frac{d}{dt}\operatorname{dn}(t,k) = -k^2\operatorname{sn}(t,k)\operatorname{cn}(t,k). \tag{A.6.3}$$

Limiting $k \to 0$ or $k \to 1$, we recover the usual trigonometric and hyperbolic functions, respectively. Indeed, as $k \to 0$, we have $\operatorname{sn}(\cdot,k) \to \sin(\cdot)$, $\operatorname{cn}(\cdot,k) \to \cos(\cdot)$ and $\operatorname{dn}(\cdot,k) \to 1$. As $k \to 1$, $\operatorname{sn}(\cdot,k) \to \tanh(\cdot)$, $\operatorname{cn}(\cdot,k) \to \operatorname{sech}(\cdot)$ and $\operatorname{dn}(\cdot,k) \to \operatorname{sech}(\cdot)$.

In addition to the three basic Jacobi elliptic functions $\operatorname{sn}(\cdot,k)$, $\operatorname{cn}(\cdot,k)$ and $\operatorname{dn}(\cdot,k)$, we define the following reciprocals and ratios (these are written in the so-called "Glaisher notation"):

$$\begin{aligned}
\operatorname{ns}(\cdot,k) &= \frac{1}{\operatorname{sn}(\cdot,k)} & \operatorname{nc}(\cdot,k) &= \frac{1}{\operatorname{cn}(\cdot,k)} & \operatorname{nd}(\cdot,k) &= \frac{1}{\operatorname{dn}(\cdot,k)} \\
\operatorname{sc}(\cdot,k) &= \frac{\operatorname{sn}(\cdot,k)}{\operatorname{cn}(\cdot,k)} & \operatorname{sd}(\cdot,k) &= \frac{\operatorname{sn}(\cdot,k)}{\operatorname{dn}(\cdot,k)} & \operatorname{cs}(\cdot,k) &= \frac{\operatorname{cn}(\cdot,k)}{\operatorname{sn}(\cdot,k)} \\
\operatorname{cd}(\cdot,k) &= \frac{\operatorname{cn}(\cdot,k)}{\operatorname{dn}(\cdot,k)} & \operatorname{ds}(\cdot,k) &= \frac{\operatorname{dn}(\cdot,k)}{\operatorname{sn}(\cdot,k)} & \operatorname{dc}(\cdot,k) &= \frac{\operatorname{dn}(\cdot,k)}{\operatorname{cn}(\cdot,k)}.
\end{aligned}$$

The Jacobi elliptic functions can also be defined in terms of the inverse of a particular elliptic integral. The **elliptic integral of the first kind** is defined as

$$F(\varphi,k) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

For a fixed modulus k, the **complete** elliptic integral of the first kind is the constant $K = F(\frac{\pi}{2},k)$. Similarly, the **elliptic integral of the second kind** is

$$E(\varphi,k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2 t}\; dt$$

and the **complete** elliptic integral of the second kind is $E = E(\frac{\pi}{2},k)$. The Jacobi **amplitude function** $\operatorname{am}(\cdot,k)$ is defined to be the inverse of $F(\cdot,k)$, *i.e.*, $\operatorname{am}(\cdot,k) = F(\cdot,k)^{-1}$. Then we have

$$\operatorname{sn}(\cdot,k) = \sin\operatorname{am}(\cdot,k) \qquad \operatorname{cn}(\cdot,k) = \cos\operatorname{am}(\cdot,k) \qquad \operatorname{dn}(\cdot,k) = \sqrt{1-k^2\sin^2\operatorname{am}(\cdot,k)}.$$

The basic elliptic functions $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$ and $\mathrm{dn}(\cdot, k)$ have the following periodicity and parity properties: $\mathrm{sn}(\cdot, k)$ and $\mathrm{cn}(\cdot, k)$ have period $4K$, whereas $\mathrm{dn}(\cdot, k)$ has period $2K$. On the other hand, $\mathrm{sn}(\cdot, k)$ is an odd function and $\mathrm{cn}(\cdot, k)$, $\mathrm{dn}(\cdot, k)$ are even functions.

Lastly, we present several formulae for the Jacobi elliptic functions, as well as the elliptic integral $E(\cdot, k)$. (See [8] for a more comprehensive collection of formulae; we state only those used in this thesis.) Firstly, we have the following square relations:

$$k^2 \,\mathrm{sn}^2(x, k) + \mathrm{dn}^2(x, k) = 1 \tag{A.6.4}$$

$$\mathrm{sn}^2(x, k) + \mathrm{cn}^2(x, k) = 1 \tag{A.6.5}$$

$$k^2 \,\mathrm{cn}^2(x, k) - \mathrm{dn}^2(x, k) + (k')^2 = 0. \tag{A.6.6}$$

Secondly, the integral formulae for elliptic integrals

$$\int_b^x \frac{dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} = \frac{1}{a}\,\mathrm{nd}^{-1}\left(\frac{x}{b}, \frac{\sqrt{a^2-b^2}}{a}\right) \qquad (b \leq x \leq a) \tag{A.6.7}$$

$$\int_x^a \frac{dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{x}{a}, \frac{\sqrt{a^2-b^2}}{a}\right) \qquad (b \leq x \leq a) \tag{A.6.8}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \frac{1}{a}\,\mathrm{dc}^{-1}\left(\frac{x}{a}, \frac{b}{a}\right) \qquad (b < a \leq x) \tag{A.6.9}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 + b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{nc}^{-1}\left(\frac{x}{a}, \frac{b}{\sqrt{a^2+b^2}}\right) \qquad (a \leq x) \tag{A.6.10}$$

as well as the following (elementary) formulae for "degenerate" elliptic integrals:

$$\int \frac{dt}{a^2 + t^2} = \frac{1}{a}\tan^{-1}\left(\frac{t}{a}\right) \tag{A.6.11}$$

$$\int \frac{dt}{a^2 - t^2} = \frac{1}{a}\coth^{-1}\left(\frac{t}{a}\right) \qquad (t^2 > a^2) \tag{A.6.12}$$

$$\int \frac{dt}{a^2 - t^2} = \frac{1}{a}\tanh^{-1}\left(\frac{t}{a}\right). \qquad (t^2 < a^2) \tag{A.6.13}$$

Finally, we have the following two integral formulae for $\mathrm{dn}^2(\cdot, k)$ and $\mathrm{sn}(\cdot, k)$ (these are used for finding expressions for the Riemannian and sub-Riemannian geodesics on $\mathsf{SE}(1, 1)$; see chapter 5):

$$\int_0^x \mathrm{dn}^2(t, k)\, dt = E(\mathrm{am}(x, k), k) \tag{A.6.14}$$

$$\int \mathrm{sn}(t, k)\, dt = \frac{1}{k}\ln\left[\mathrm{dn}(t, k) - k\,\mathrm{cn}(t, k)\right]. \tag{A.6.15}$$

A.6.2 Reduction to standard form

In this thesis we use the Jacobi elliptic functions to solve differential equations of the form

$$\dot{x}^2 = X_1 X_2 = (a_1 x^2 + 2b_1 x + c_1)(a_2 x^2 + 2b_2 x + c_2),$$

where $a_i, b_i, c_i \in \mathbb{R}$ $(i = 1, 2)$. We assume that the roots of either quadratic do no interlace. Separating variables, we have

$$\frac{dx}{\sqrt{X_1 X_2}} = \frac{dx}{\sqrt{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)}} = \pm dt. \tag{A.6.16}$$

We shall discuss how to rewrite the integral on the left-hand side in a form to which one of the integral formulae given in section A.6.1 will apply. Consider the polynomial $X_1 - \lambda X_2$. This will be a perfect square (in x) if and only if

$$(a_1 - \lambda a_2)(c_1 - \lambda c_2) = (b_1 - \lambda b_2)^2. \tag{A.6.17}$$

Denote the roots of (A.6.17) by λ_1, λ_2. Then there exist $r_1, r_2 \in \mathbb{R}$ such that

$$X_1 - \lambda_1 X_2 = (a_1 - \lambda_1 a_2)(x - r_1)^2 \qquad \text{and} \qquad X_1 - \lambda_2 X_2 = (a_1 - \lambda_2 a_2)(x - r_2)^2.$$

(The roots λ_1 and λ_2 are real and distinct, unless $a_1b_2 = a_2b_1$, whereupon $X_1 = a_1(x-r_1)^2+B_1$ and $X_2 = a_2(x - r_2)^2 + B_2$.) Solving for X_1 and X_2, we can express

$$X_1X_2 = \left[A_1(x - r_1)^2 + B_1(x - r_2)^2\right]\left[A_2(x - r_1)^2 + B_2(x - r_2)^2\right].$$

Here

$$A_1 = \frac{\lambda_1(a_1 - \lambda_1 a_2)}{\lambda_2 - \lambda_1}, \qquad B_1 = -\frac{\lambda_1(a_1 - \lambda_2 a_2)}{\lambda_2 - \lambda_1}, \qquad A_2 = \frac{a_1 - \lambda_1 a_2}{\lambda_2 - \lambda_1}, \qquad B_2 = -\frac{a_1 - \lambda_2 a_2}{\lambda_2 - \lambda_1}.$$

Thus we have the integral

$$\frac{dx}{\sqrt{X_1X_2}} = \frac{dx}{\sqrt{\left[A_1(x - r_1)^2 + B_1(x - r_2)^2\right]\left[A_2(x - r_1)^2 + B_2(x - r_2)^2\right]}}. \tag{A.6.18}$$

Assume $A_1, A_2 \neq 0$. Make the change of variables $u = \frac{x-r_1}{x-r_2}$. We have $x = \frac{r_2u-r_1}{u-1}$ and $dx = \frac{(x-r_2)^2}{r_1-r_2}\,du$. Accordingly, (A.6.18) becomes

$$\begin{aligned}
&\frac{(r_1 - r_2)\,du}{(u-1)^2\sqrt{\left[A_1(\frac{r_2u-r_1}{u-1} - r_1)^2 + B_1(\frac{r_2u-r_1}{u-1} - r_2)^2\right]\left[A_2(\frac{r_2u-r_1}{u-1} - r_1)^2 + B_2(\frac{r_2u-r_1}{u-1} - r_2)^2\right]}} \\
&= \frac{(r_1 - r_2)\,du}{(u-1)^2\sqrt{\frac{(r_1-r_2)^4}{(u-1)^4}(A_1u^2 + B_1)(A_2u^2 + B_2)}} \\
&= \frac{du}{(r_1 - r_2)\sqrt{(A_1u^2 + B_1)(A_2u^2 + B_2)}} \\
&= \frac{du}{(r_1 - r_2)\sqrt{\sigma A_1A_2}\sqrt{\sigma\left(u^2 + \frac{B_1}{A_1}\right)\left(u^2 + \frac{B_2}{A_2}\right)}}.
\end{aligned}$$

(Here $\sigma = \operatorname{sgn}(A_1A_2)$.) Consequently, the separable equation (A.6.16) may be written as

$$\frac{du}{\sqrt{\sigma\left(u^2 + \frac{B_1}{A_1}\right)\left(u^2 + \frac{B_2}{A_2}\right)}} = \pm(r_1 - r_2)\sqrt{\sigma A_1A_2}\,dt.$$

The left-hand side of this equation is now in a suitable form for the use of the integral formulae (A.6.7), (A.6.8), (A.6.9), (A.6.10) or similar. ([8] provides a more comprehensive list of integral formulae that may be employed.)

Appendix B

Mathematica Code

In this chapter we list the MATHEMATICA code that was developed for many of the calculations performed in this thesis.

Text in bold is the actual MATHEMATICA code. The non-bold text in a smaller size is output from the preceding block of code. (However, in several places we have suppressed the output. In particular, we do not display any of the graphical output.)

B.1 Semi-Euclidean Group

```
MF:=MatrixForm;
FS:=FS;
```

Plot the Casimir function $C(p) = p_1^2 - p_2^2$, as well as the intersection of the Casimir level set with a specified level set of the given Hamiltonian function

```
Ca[c_, h_]:=
  ParametricPlot3D[
    Which[
      Abs[c] < 10^-5, {
      {θ, θ, z},
      {θ, -θ, z}
    },
      c > 0, {
      { √c Cosh[θ], √c Sinh[θ], z},
      {- √c Cosh[θ], - √c Sinh[θ], z}
    },
      c < 0, {
      { √-c Sinh[θ], √-c Cosh[θ], z},
      {- √-c Sinh[θ], - √-c Cosh[θ], z}
    }],
  {θ, -3, 3}, {z, -3, 3}, Mesh → 4];

Int[c_, h_, hfn_]:=Module[{Int1, Int2},
  Int1 = Which[
    Abs[c] < 10^-5,
      {θ, θ, z}/.Solve[hfn[θ, θ, z] == h, {z}],
    c > 0,
      { √c Cosh[θ], √c Sinh[θ], z}
      /.Solve[hfn[ √c Cosh[θ], √c Sinh[θ], z]
        == h, {z}],
    c < 0,
      { √-c Sinh[θ], √-c Cosh[θ], z}
      /.Solve[hfn[ √-c Sinh[θ], √-c Cosh[θ], z]
        == h, {z}]
  ];
  Int2 = Which[
    Abs[c] < 10^-5,
      {θ, -θ, z}/.Solve[hfn[θ, -θ, z] == h, {z}],
    c > 0,
      {- √c Cosh[θ], - √c Sinh[θ], z}
      /.Solve[hfn[- √c Cosh[θ], - √c Sinh[θ], z]
        == h, {z}],
    c < 0,
      {- √-c Sinh[θ], - √-c Cosh[θ], z}
      /.Solve[hfn[- √-c Sinh[θ], - √-c Cosh[θ], z]
        == h, {z}]
  ];
  ParametricPlot3D[{Int1, Int2}, {θ, -3, 3},
      PlotStyle → Directive[Thick, Black]]
];

Opts = {Axes → True, BoxRatios → {1, 1, 1},
     PlotRange → {{-3, 3}, {-3, 3}, {-3, 3}},
     Boxed → False, ImageSize → Small,
     AxesLabel → {"E1*", "E2*", "E3*"},
     LabelStyle → Directive[Medium],
     AxesEdge → {{1, -1}, {1, -1}, {1, -1}},
     FaceGrids → {{-1, 0, 0}, {0, -1, 0}, {0, 0, -1}},
     TicksStyle → Directive[Medium],
     ViewVertical → {0, 0, 1},
     ViewPoint → {π, π/2, π/1.1}};

lie[X_, Y_]:=Simplify[X.Y - Y.X];
soph[X_, Y_]:=Simplify@Minv@lie[M@@X, M@@Y];
```

$$m[\text{x\_},\text{y\_},\theta\_]:=\begin{pmatrix}1&0&0\\x&\text{Cosh}[\theta]&\text{Sinh}[\theta]\\y&\text{Sinh}[\theta]&\text{Cosh}[\theta]\end{pmatrix};$$

$$M[\text{x\_},\text{y\_},\theta\_]:=\begin{pmatrix}0&0&0\\x&0&\theta\\y&\theta&0\end{pmatrix};$$

```
minv[MM_]:={MM[[2,1]], MM[[3,1]], ArcSinh[MM[[2,3]]]};
Basis = M@@IdentityMatrix[3][[#]]&/@Range[3];
{E1,E2,E3} = Basis;

TBase = Table[E_i, {i,3}];
CRs = Partition[TBase.Minv@lie[
        Basis[[#[[1]]]],
        Basis[[#[[2]]]]]&
    /@Tuples[Range[3],2],3];
TableForm[CRs, TableHeadings -> {TBase, TBase}]
```

	E_1	E_2	E_3
E_1	0	0	$-E_2$
E_2	0	0	$-E_1$
E_3	E_2	E_1	0

```
exp = MatrixExp[M[x,y,θ]];
exp//FS//MF

Limit[exp, θ -> 0]//Simplify//MF
```

$$\begin{pmatrix}1&0&0\\\frac{y(-1+\text{Cosh}[\theta])+x\text{Sinh}[\theta]}{\theta}&\text{Cosh}[\theta]&\text{Sinh}[\theta]\\\frac{x(-1+\text{Cosh}[\theta])+y\text{Sinh}[\theta]}{\theta}&\text{Sinh}[\theta]&\text{Cosh}[\theta]\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\x&1&0\\y&0&1\end{pmatrix}$$

```
ad[x_,y_,θ_]:=
  With[{X = M[x,y,θ]},
  (Minv@lie[X,#]&/@{E1,E2,E3})^T//Simplify];
Ad[x_,y_,θ_]:=
  With[{g = m[x,y,θ]},
  (Minv[g.#.Inverse[g]]&/@{E1,E2,E3})^T
     //Simplify];

ad[x,y,θ]//MF
Ad[x,y,θ]//MF
```

$$\begin{pmatrix}0&\theta&-y\\\theta&0&-x\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}\text{Cosh}[\theta]&\text{Sinh}[\theta]&-y\\\text{Sinh}[\theta]&\text{Cosh}[\theta]&-x\\0&0&1\end{pmatrix}$$

```
Eigenvalues[ad[x,y,θ]]
```

$\{0,-\theta,\theta\}$

B.1.1 Automorphisms

$$\Psi[\text{x\_},\text{y\_},\text{v\_},\text{w\_},\text{k\_}]:=\begin{pmatrix}x&y&v\\\varsigma y&\varsigma x&w\\0&0&\varsigma\end{pmatrix};$$

```
X = Minv[x1E1 + x2E2 + x3E3];
Y = Minv[y1E1 + y2E2 + y3E3];
ψ = Ψ[x,y,v,w,ς];
ψ.soph[X,Y] - soph[ψ.X,ψ.Y]
    //Simplify[#, ς^2 == 1]&//MF
```

$$\begin{pmatrix}0\\0\\0\end{pmatrix}$$

```
MF/@Simplify@Eigenvectors[Ψ[x,y,v,w,1]]
MF/@Simplify@Eigenvectors[Ψ[x,y,v,w,-1]]
```

$$\left\{\begin{pmatrix}\frac{v-vx+wy}{1-2x+x^2-y^2}\\\frac{w-wx+vy}{1-2x+x^2-y^2}\\1\end{pmatrix},\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix}\frac{v-vx-wy}{-1+x^2-y^2}\\\frac{w+wx+vy}{-1+x^2-y^2}\\1\end{pmatrix},\begin{pmatrix}\frac{-x+\sqrt{x^2-y^2}}{y}\\1\\0\end{pmatrix},\begin{pmatrix}-\frac{x+\sqrt{x^2-y^2}}{y}\\1\\0\end{pmatrix}\right\}$$

B.1.2 Adjoint orbits

```
AdjOpts = {Axes -> True, BoxRatios -> {1,1,1},
    PlotRange -> {{-π,π},{-π,π},{-π,π}},
    Boxed -> False, ImageSize -> Small,
    LabelStyle -> Directive[Medium],
    AxesEdge -> {{1,-1},{1,-1},{1,-1}},
    FaceGrids -> {{-1,0,0},{0,-1,0},{0,0,-1}},
    TicksStyle -> Directive[Medium],
    AxesLabel -> {"E1","E2","E3"},
    ViewVertical -> {0,0,1},
    ViewPoint -> {π, π, 100531/20000}};
```

B.1.2.1 $X = 0$

```
g = m[v,w,ϑ];
X = M[0,0,0];
```

$$\begin{pmatrix}0\\0\\0\end{pmatrix}$$

```
AdjType1 = Show[ListPointPlot3D[{{0,0,0}},
    PlotStyle -> Directive[Black,
      PointSize[Medium]]], AdjOpts]
```

B.1.2.2 $\theta = 0$, $x^2 \neq y^2$

```
g = m[v,w,ϑ];
X = M[x,y,0];
```

$$\begin{pmatrix}x\text{Cosh}[\vartheta]+y\text{Sinh}[\vartheta]\\y\text{Cosh}[\vartheta]+x\text{Sinh}[\vartheta]\\0\end{pmatrix}$$

```
AdjType2 = Show[
Map[v ↦ With[{x = v[[1]], y = v[[2]]},
    ParametricPlot3D[{xCosh[ϑ] + ySinh[ϑ],
    yCosh[ϑ] + xSinh[ϑ], 0}, {ϑ,-5,5},
    PlotStyle -> Directive[Thick, Black]]],
    {{-2.5,-1},{-1.5,-0.5},{-1,-2.5},
    {-0.5,-1.5},{0.5,1.5},{1,2.5},{1.5,0.5},
    {2.5,1}}], AdjOpts]
```

B.1.2.3 $\theta = 0$, $x - y = 0$ (so $x^2 = y^2$)

```
g = m[v,w,ϑ];
X = M[x,x,0];
```

$$\begin{pmatrix}x(\text{Cosh}[\vartheta]+\text{Sinh}[\vartheta])\\x(\text{Cosh}[\vartheta]+\text{Sinh}[\vartheta])\\0\end{pmatrix}$$

```
AdjType3a = Show[ Map[x ↦
    ParametricPlot3D[{xt, xt, 0}, {t, 0.1, 3},
```

```
    PlotStyle → Directive[Thick, Black]],
    {−3, 3}], AdjOpts]
```

B.1.2.4 $\theta = 0$, $x + y = 0$ (so $x^2 = y^2$)

```
g = m[v, w, ϑ];
X = M[x, −x, 0];
```

$$\begin{pmatrix} x(\mathrm{Cosh}[\vartheta] - \mathrm{Sinh}[\vartheta]) \\ x(-\mathrm{Cosh}[\vartheta] + \mathrm{Sinh}[\vartheta]) \\ 0 \end{pmatrix}$$

```
AdjType3b = Show[ Map[x ↦
    ParametricPlot3D[{xt, −xt, 0}, {t, 0.1, 3},
    PlotStyle → Directive[Thick, Black]],
    {−3, 3}], AdjOpts]
```

B.1.2.5 $\theta \neq 0$

```
g = m[v, w, ϑ];
X = M[x, y, θ];
```

$$\begin{pmatrix} -w\theta + x\mathrm{Cosh}[\vartheta] + y\mathrm{Sinh}[\vartheta] \\ -v\theta + y\mathrm{Cosh}[\vartheta] + x\mathrm{Sinh}[\vartheta] \\ \theta \end{pmatrix}$$

```
AdjType4 = Show[ Map[θ ↦
    ParametricPlot3D[{t, s, θ}, {s, −π, π},
    {t, −π, π}, Mesh → 5,
    PlotStyle → Directive[Nest[Darker,
      RGBColor[154, 209, 255], −θ − 1]]],
    {−3, −2, −1}], AdjOpts]
```

B.1.2.6 Plot all adjoint orbits together

```
Show[AdjType1, AdjType2, AdjType3a, AdjType3b,
    AdjType4, AdjOpts]
```

B.1.3 Coadjoint orbits

```
CoadjOpts = {Axes → True, BoxRatios → {1, 1, 1},
    PlotRange → {{−π, π}, {−π, π}, {−π, π}},
    Boxed → False, ImageSize → Small,
    LabelStyle → Directive[Medium],
    AxesEdge → {{1, −1}, {1, −1}, {1, −1}},
    FaceGrids → {{−1, 0, 0}, {0, −1, 0}, {0, 0, −1}},
    TicksStyle → Directive[Medium],
    AxesLabel → {"E1", "E2", "E3"},
    ViewVertical → {0, 0, 1},
    ViewPoint → {π, π, 100531/20000}};
```

B.1.3.1 $x = y = 0$

$$\{0,0,\theta\}.\begin{pmatrix} \mathrm{Cosh}[\vartheta] & \mathrm{Sinh}[\vartheta] & -w \\ \mathrm{Sinh}[\vartheta] & \mathrm{Cosh}[\vartheta] & -v \\ 0 & 0 & 1 \end{pmatrix} //\mathrm{MF}$$

$$\begin{pmatrix} 0 \\ 0 \\ \theta \end{pmatrix}$$

```
CoadjType1 = Show[
    ListPointPlot3D[{{0, 0, −3}, {0, 0, −2},
    {0, 0, −1}, {0, 0, 0}, {0, 0, 1}, {0, 0, 2}, {0, 0, 3}},
    PlotStyle → Directive[Black,
      PointSize[Medium]]], CoadjOpts]
```

B.1.3.2 $x^2 \neq y^2$

$$\{x,y,\theta\}.\begin{pmatrix} \mathrm{Cosh}[\vartheta] & \mathrm{Sinh}[\vartheta] & -w \\ \mathrm{Sinh}[\vartheta] & \mathrm{Cosh}[\vartheta] & -v \\ 0 & 0 & 1 \end{pmatrix} //\mathrm{MF}$$

$$\begin{pmatrix} x\mathrm{Cosh}[\vartheta] + y\mathrm{Sinh}[\vartheta] \\ y\mathrm{Cosh}[\vartheta] + x\mathrm{Sinh}[\vartheta] \\ -wx - vy + \theta \end{pmatrix}$$

```
CoadjType2 = Show[
    Map[u ↦ With [{x = u[[1]], y = u[[2]]} ,
    ParametricPlot3D[{xCosh[ϑ] + ySinh[ϑ],
      yCosh[ϑ] + xSinh[ϑ], t}, {ϑ, −3, 3}, {t, −π, π},
      Mesh → 6]],
    {{−2.5, −1}, {−1.5, −0.5}, {−1, −2.5},
    {−0.5, −1.5}, {0.5, 1.5}, {1, 2.5}, {1.5, 0.5},
    {2.5, 1}}], CoadjOpts]
```

B.1.3.3 $x - y = 0$ (so $x^2 = y^2$)

$$\{x,x,\theta\}.\begin{pmatrix} \mathrm{Cosh}[\vartheta] & \mathrm{Sinh}[\vartheta] & -w \\ \mathrm{Sinh}[\vartheta] & \mathrm{Cosh}[\vartheta] & -v \\ 0 & 0 & 1 \end{pmatrix} //\mathrm{MF}$$

$$\begin{pmatrix} x(\mathrm{Cosh}[\vartheta] + \mathrm{Sinh}[\vartheta]) \\ x(\mathrm{Cosh}[\vartheta] + \mathrm{Sinh}[\vartheta]) \\ -vx - wx + \theta \end{pmatrix}$$

```
CoadjType3a = Show[ Map[x ↦
    ParametricPlot3D[{xt, xt, s}, {t, 0.1, 3},
      {s, −π, π}, Mesh → 5], {−3, 3}], CoadjOpts]
```

B.1.3.4 $x + y = 0$ (so $x^2 = y^2$)

$$\{x,-x,\theta\}.\begin{pmatrix} \mathrm{Cosh}[\vartheta] & \mathrm{Sinh}[\vartheta] & -w \\ \mathrm{Sinh}[\vartheta] & \mathrm{Cosh}[\vartheta] & -v \\ 0 & 0 & 1 \end{pmatrix} //\mathrm{MF}$$

$$\begin{pmatrix} x(\mathrm{Cosh}[\vartheta] - \mathrm{Sinh}[\vartheta]) \\ x(-\mathrm{Cosh}[\vartheta] + \mathrm{Sinh}[\vartheta]) \\ vx - wx + \theta \end{pmatrix}$$

```
CoadjType3b = Show[ Map[x ↦
    ParametricPlot3D[{xt, −xt, s}, {t, 0.1, 3},
      {s, −π, π}, Mesh → 5], {−3, 3}], CoadjOpts]
```

B.1.3.5 Plot all coadjoint orbits together

```
Show[CoadjType1, CoadjType2, CoadjType3a,
    CoadjType3b, CoadjOpts]
```

B.2 $\mathfrak{L}$-Equivalence

B.2.1 (1,1)-affine subspaces

B.2.1.1 $E_3^*(\Gamma) \neq \{0\}$

$$\mathtt{With}\left[\left\{\Gamma = \begin{pmatrix} \mathrm{a1} & \mathrm{b1} \\ \mathrm{a2} & \mathrm{b2} \\ 0 & 1 \end{pmatrix}\right\},\right.$$

$$\begin{pmatrix}1&0&-\mathtt{b1}\\0&1&-\mathtt{b2}\\0&0&1\end{pmatrix}.\Gamma\texttt{//MF}\Big]$$

$$\begin{pmatrix}\mathrm{a1}&0\\\mathrm{a2}&0\\0&1\end{pmatrix}$$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}\mathtt{a1}&0\\\mathtt{a2}&0\\0&1\end{pmatrix}\right\},\right.$$
$$\left.\begin{pmatrix}\frac{\mathtt{a1}}{\mathtt{a1}^2-\mathtt{a2}^2}&-\frac{\mathtt{a2}}{\mathtt{a1}^2-\mathtt{a2}^2}&0\\-\frac{\mathtt{a2}}{\mathtt{a1}^2-\mathtt{a2}^2}&\frac{\mathtt{a1}}{\mathtt{a1}^2-\mathtt{a2}^2}&0\\0&0&1\end{pmatrix}.\Gamma\texttt{//Simplify//MF}\right]$$

$$\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}$$

B.2.1.2 $E_3^*(\Gamma)=\{0\}$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}\mathtt{a1}&\mathtt{b1}\\\mathtt{a2}&\mathtt{b2}\\\mathtt{a3}&0\end{pmatrix}\right\},\right.$$
$$\left.\begin{pmatrix}1&0&-\frac{\mathtt{a1}}{\mathtt{a3}}\\0&1&-\frac{\mathtt{a2}}{\mathtt{a3}}\\0&0&1\end{pmatrix}.\Gamma\texttt{//MF}\right]$$

$$\begin{pmatrix}0&\mathrm{b1}\\0&\mathrm{b2}\\\mathrm{a3}&0\end{pmatrix}$$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}0&\mathtt{b1}\\0&\mathtt{b2}\\\mathtt{a3}&0\end{pmatrix}\right\},\right.$$
$$\begin{pmatrix}\frac{\mathtt{b1}}{\mathtt{b1}^2-\mathtt{b2}^2}&-\frac{\mathtt{b2}}{\mathtt{b1}^2-\mathtt{b2}^2}&0\\-\frac{\mathtt{Sign[a3]b2}}{\mathtt{b1}^2-\mathtt{b2}^2}&\frac{\mathtt{Sign[a3]b1}}{\mathtt{b1}^2-\mathtt{b2}^2}&0\\0&0&\mathtt{Sign[a3]}\end{pmatrix}.\Gamma$$
$$\left.\texttt{//Simplify//MF}\right]$$

$$\begin{pmatrix}0&1\\0&0\\\mathrm{a3Sign[a3]}&0\end{pmatrix}$$

B.2.1.3 $\Gamma_{2,\alpha}^{(1,1)}$ is not 𝔏-equivalent to $\Gamma_{2,\alpha'}^{(1,1)}$

$$\texttt{With}\left[\left\{\Gamma1=\begin{pmatrix}0&1\\0&0\\\alpha&0\end{pmatrix},\Gamma2=\begin{pmatrix}0&1\\0&0\\\alpha\mathtt{p}&0\end{pmatrix}\right\},\right.$$
$$\left.\texttt{MF/@}\left\{\begin{pmatrix}x&y&v\\\varsigma y&\varsigma x&w\\0&0&\varsigma\end{pmatrix}.\Gamma1,\Gamma2\right\}\texttt{//Simplify}\right]$$

$$\left\{\begin{pmatrix}v\alpha&x\\w\alpha&y\varsigma\\\alpha\varsigma&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\\\alpha\mathrm{p}&0\end{pmatrix}\right\}$$

B.2.2 (2,1)-affine subspaces

B.2.2.1 $E_3^*(\Gamma)\neq\{0\}$, $E_1+E_2\notin\Gamma^0$ and $E_1-E_2\notin\Gamma^0$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}\mathtt{a1}&\mathtt{b1}&1\\\mathtt{a2}&\mathtt{b2}&\sigma\\0&1&0\end{pmatrix}\right\},\right.$$
$$\begin{pmatrix}1&0&-\mathtt{b1}\\0&1&-\mathtt{b2}\\0&0&1\end{pmatrix}.\Gamma\texttt{//Simplify}\left[\#,\sigma^2==1\right]\&$$
$$\left.\texttt{//MF}\right]$$

$$\begin{pmatrix}\mathrm{a1}&0&1\\\mathrm{a2}&0&\sigma\\0&1&0\end{pmatrix}$$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}\mathtt{a1}&0&1\\\mathtt{a2}&0&\sigma\\0&1&0\end{pmatrix}\right\},\right.$$
$$\begin{pmatrix}\frac{\mathtt{a1}}{\mathtt{a1}^2-\mathtt{a2}^2}&-\frac{\mathtt{a2}}{\mathtt{a1}^2-\mathtt{a2}^2}&0\\-\frac{\sigma\mathtt{a2}}{\mathtt{a1}^2-\mathtt{a2}^2}&\frac{\sigma\mathtt{a1}}{\mathtt{a1}^2-\mathtt{a2}^2}&0\\0&0&\sigma\end{pmatrix}.\Gamma$$
$$\left.\texttt{//Simplify}\left[\#,\sigma^2==1\right]\&\texttt{//MF}\right]$$

$$\begin{pmatrix}1&0&\frac{\mathrm{a1}-\mathrm{a2}\sigma}{\mathrm{a1}^2-\mathrm{a2}^2}\\0&0&\frac{\mathrm{a1}-\mathrm{a2}\sigma}{\mathrm{a1}^2-\mathrm{a2}^2}\\0&\sigma&0\end{pmatrix}$$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}\mathtt{a1}&1&0\\\mathtt{a2}&0&0\\0&0&1\end{pmatrix}\right\},\right.$$
$$\left.\begin{pmatrix}\frac{1}{\mathtt{a2}}&0&0\\0&\frac{1}{\mathtt{a2}}&0\\0&0&1\end{pmatrix}.\Gamma\texttt{//Simplify//MF}\right]$$

$$\begin{pmatrix}\frac{\mathrm{a1}}{\mathrm{a2}}&\frac{1}{\mathrm{a2}}&0\\1&0&0\\0&0&1\end{pmatrix}$$

B.2.2.2 $E_3^*(\Gamma)\neq\{0\}$, $E_1+E_2, E_1-E_2\in\Gamma^0$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}\mathtt{a1}&\mathtt{b1}&\mathtt{c1}\\\mathtt{a2}&\mathtt{b2}&\mathtt{c2}\\0&0&1\end{pmatrix}\right\},\right.$$
$$\left.\begin{pmatrix}1&0&-\mathtt{c1}\\0&1&-\mathtt{c2}\\0&0&1\end{pmatrix}.\Gamma\texttt{//MF}\right]$$

$$\begin{pmatrix}\mathrm{a1}&\mathrm{b1}&0\\\mathrm{a2}&\mathrm{b2}&0\\0&0&1\end{pmatrix}$$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}\mathtt{a1}&\mathtt{b1}&0\\\mathtt{a2}&\mathtt{b2}&0\\0&0&1\end{pmatrix}\right\},\right.$$
$$\begin{pmatrix}\frac{\mathtt{b1}}{\mathtt{b1}^2-\mathtt{b2}^2}&-\frac{\mathtt{b2}}{\mathtt{b1}^2-\mathtt{b2}^2}&0\\-\frac{\mathtt{b2}}{\mathtt{b1}^2-\mathtt{b2}^2}&\frac{\mathtt{b1}}{\mathtt{b1}^2-\mathtt{b2}^2}&0\\0&0&1\end{pmatrix}.\Gamma$$
$$\left.\texttt{//Simplify//MF}\right]$$

$$\begin{pmatrix}\frac{\mathrm{a1b1}-\mathrm{a2b2}}{\mathrm{b1}^2-\mathrm{b2}^2}&1&0\\\frac{\mathrm{a2b1}-\mathrm{a1b2}}{\mathrm{b1}^2-\mathrm{b2}^2}&0&0\\0&0&1\end{pmatrix}$$

B.2.2.3 $E_3^*(\Gamma)=\{0\}$

$$\texttt{With}\left[\left\{\Gamma=\begin{pmatrix}0&1&0\\0&0&1\\\mathtt{a3}&0&0\end{pmatrix}\right\},\right.$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \text{Sign[a3]} & 0\\ 0 & 0 & \text{Sign[a3]}\end{pmatrix}.\Gamma\text{//MF}\Big]$$

$$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & \text{Sign[a3]}\\ \text{a3Sign[a3]} & 0 & 0\end{pmatrix}$$

B.2.2.4 $\Gamma^{(2,1)}_{3,\alpha}$ is not $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{3,\alpha'}$

$$\text{With}\left[\left\{\Gamma1 = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \alpha & 0 & 0\end{pmatrix}, \Gamma2 = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \alpha\text{p} & 0 & 0\end{pmatrix}\right\},\right.$$
$$\left.\text{MF/@}\left\{\begin{pmatrix}x & y & v\\ \varsigma y & \varsigma x & w\\ 0 & 0 & \varsigma\end{pmatrix}.\Gamma1, \Gamma2\right\}\text{//Simplify}\right]$$

$$\left\{\begin{pmatrix}v\alpha & x & y\\ w\alpha & y\varsigma & x\varsigma\\ \alpha\varsigma & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \alpha\text{p} & 0 & 0\end{pmatrix}\right\}$$

B.3 Classification of QHP Systems

B.3.1 Homogeneous systems

$$Q = \begin{pmatrix}\text{a1} & \text{b1} & \text{b2}\\ \text{b1} & \text{a2} & \text{b3}\\ \text{b2} & \text{b3} & \text{a3}\end{pmatrix};$$
$$K = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix};$$
$$\text{pp} = \{p_1, p_2, p_3\};$$

B.3.1.1 $a_3 = 0$ and $a_1 + a_2 = 0$

Since $a_3 = 0$ we have $b_2 = b_3 = 0$ (principal minors must be nonnegative, since Q is PSD).

$$\text{Block}[\{\text{a1} = 0, \text{a2} = 0, \text{a3} = 0, \text{b1} = 0, \text{b2} = 0, \text{b3} = 0\}, Q\text{//MF}]$$

$$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Hence equivalent to H_0.

B.3.1.2 $a_3 = 0$ and $4b_1{}^2 \neq (a_1 + a_2)^2$

Since $a_3 = 0$ we have $b_2 = b_3 = 0$ (principal minors must be nonnegative, since Q is PSD).

$$\text{Block}[\{\text{a3} = 0, \text{b2} = 0, \text{b3} = 0\}, \text{With}\left[\psi = \begin{pmatrix}x & 1 & 0\\ 1 & x & 0\\ 0 & 0 & 1\end{pmatrix}\right\}, \psi.Q.\psi^\top\text{//MF}]]$$

$$\begin{pmatrix}\text{a2} + \text{b1}x + x(\text{b1} + \text{a1}x) & \text{b1} + \text{a1}x + x(\text{a2} + \text{b1}x) & 0\\ \text{b1} + \text{a2}x + x(\text{a1} + \text{b1}x) & \text{a1} + \text{b1}x + x(\text{b1} + \text{a2}x) & 0\\ 0 & 0 & 0\end{pmatrix}$$

The solution of the off-diagonal equation is given below:

$$\text{Solve}[\text{b1} + \text{a2}x + x(\text{a1} + \text{b1}x) == 0, \{x\}]$$

$$\left\{\left\{x \to \frac{-\text{a1}-\text{a2}-\sqrt{(\text{a1}+\text{a2})^2-4\text{b1}^2}}{2\text{b1}}\right\}, \left\{x \to \frac{-\text{a1}-\text{a2}+\sqrt{(\text{a1}+\text{a2})^2-4\text{b1}^2}}{2\text{b1}}\right\}\right\}$$

Let a_1' and a_2' denote the elements on the diagonal. If $a_1' = 0$ and $a_2' = 0$, then equivalent to H_0. Otherwise, equivalent to H_1.

$$\text{Block}[\{\text{a1} = \text{a1p}, \text{a2} = \text{a2p}, \text{a3} = 0, \text{b1} = 0, \text{b2} = 0, \text{b3} = 0\},$$
$$\text{With}\left[\psi = \begin{pmatrix}\frac{1}{\sqrt{\text{a1p}+\text{a2p}}} & 0 & 0\\ 0 & \frac{1}{\sqrt{\text{a1p}+\text{a2p}}} & 0\\ 0 & 0 & 1\end{pmatrix}\right\},$$
$$\left\{\tfrac{1}{2}\psi.Q.\psi^\top + \tfrac{1}{2}\left(\tfrac{\text{a2p}}{\text{a1p}+\text{a2p}}\right)K\text{//Simplify//MF}, \tfrac{1}{2}\text{pp}.\psi.Q.\psi^\top.\text{pp} + \tfrac{1}{2}\left(\tfrac{\text{a2p}}{\text{a1p}+\text{a2p}}\right)\text{pp}.K.\text{pp}\text{//Simplify}\right\}]]$$

$$\left\{\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \frac{p_1^2}{2}\right\}$$

B.3.1.3 $4b_1{}^2 = (a_1 = a_2)^2$ and $a_1 + a_2 > 0$

$$\text{Block}\left[\{\text{a3} = 0, \text{b1} = \tfrac{1}{2}\sigma(\text{a1} + \text{a2}), \text{b2} = 0, \text{b3} = 0\},\right.$$
$$\text{With}\left[\psi = \begin{pmatrix}\sqrt{\frac{2}{\text{a1}+\text{a2}}} & 0 & 0\\ 0 & \sigma\sqrt{\frac{2}{\text{a1}+\text{a2}}} & 0\\ 0 & 0 & \sigma\end{pmatrix}\right\},$$
$$\left\{\tfrac{1}{2}\psi.Q.\psi^\top - \tfrac{1}{2}\left(\tfrac{\text{a1}-\text{a2}}{\text{a1}+\text{a2}}\right)K\text{//Simplify}[\#, \sigma^2 == 1]\&\text{//MF},\right.$$
$$\left.\tfrac{1}{2}\text{pp}.\psi.Q.\psi^\top.\text{pp} - \tfrac{1}{2}\left(\tfrac{\text{a1}-\text{a2}}{\text{a1}+\text{a2}}\right)\text{pp}.K.\text{pp}\text{//Simplify}[\#, \sigma^2 == 1]\&\right\}]]$$

$$\left\{\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}, \frac{1}{2}(p_1 + p_2)^2\right\}$$

Hence equivalent to H_2.

B.3.1.4 $a_3 > 0$

$$\text{With}\left[\psi = \begin{pmatrix}1 & 0 & -\frac{\text{b2}}{\text{a3}}\\ 0 & 1 & -\frac{\text{b3}}{\text{a3}}\\ 0 & 0 & 1\end{pmatrix}\right\}, \psi.Q.\psi^\top\text{//Simplify//MF}]$$

$$\begin{pmatrix}\text{a1} - \frac{\text{b2}^2}{\text{a3}} & \text{b1} - \frac{\text{b2b3}}{\text{a3}} & 0\\ \text{b1} - \frac{\text{b2b3}}{\text{a3}} & \text{a2} - \frac{\text{b3}^2}{\text{a3}} & 0\\ 0 & 0 & \text{a3}\end{pmatrix}$$

Denote

$$\begin{pmatrix}a_1' & b_1' & 0\\ b_1' & a_2' & 0\\ 0 & 0 & \alpha\end{pmatrix} = \begin{pmatrix}\text{a1} - \frac{\text{b2}^2}{\text{a3}} & \text{b1} - \frac{\text{b2b3}}{\text{a3}} & 0\\ \text{b1} - \frac{\text{b2b3}}{\text{a3}} & \text{a2} - \frac{\text{b3}^2}{\text{a3}} & 0\\ 0 & 0 & \text{a3}\end{pmatrix}.$$

Now notice that α is left unchanged by transformations of the form $\psi\text{Q}\psi^\top$:

```
Block[{a1 = a1p, a2 = a2p, b1 = b1p, b2 = 0, b3 = 0,
    a3 = α},
  With[{ψ = ( x   y   v
              ςy  ςx  w
              0   0   ς )},
    (ψ.Q.ψ^T)[[3,3]]//Simplify[#, ς^2 == 1] &
    //MF]]
```

α

H_{Q_1} is equivalent to H_3

```
Block[{a1 = 0, a2 = 0, a3 = α},
      b1 = 0, b2 = 0, b3 = 0,,
    1/(2α) pp.Q.pp//Simplify//MF]
```

$\frac{p_3^2}{2}$

H_{Q_2} is equivalent to $H_{Q_2'}$, where $Q_2' = \operatorname{diag}\left(\frac{1}{\alpha}, 0, 1\right)$, which is in turn equivalent to H_4.

```
Block[{a1 = 1/α, a2 = 0, a3 = 1,
    b1 = 0, b2 = 0, b3 = 0, },
  With[{ψ = ( √α  0   0
              0   √α  0
              0   0   1 )},
    ψ.Q.ψ^T//Simplify//MF]]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

H_{Q_3} is equivalent to $H_{Q_3'}$, where $Q_3' = \begin{pmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} & 0 \\ \frac{1}{\alpha} & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is in turn equivalent to H_5.

```
Block[{a1 = 1/α, a2 = 1/α, a3 = 1,
    b1 = 1/α, b2 = 0, b3 = 0},
  With[{ψ = ( √α  0   0
              0   √α  0
              0   0   1 )},
    ψ.Q.ψ^T//Simplify//MF]]
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Verification that representatives are distinct

```
Module[{CheckEquiv, GetEqns, Hvec},
  GetEqns[e_] :=
    Thread[DeleteCases[Flatten[
      CoefficientList[e, {p1, p2, p3}], 0] == 0]];
  CheckEquiv[Hvec_, Gvec_] := With[{
      ψ = ( ψ11  ψ12  ψ13
            ψ21  ψ22  ψ23
            ψ31  ψ32  ψ33 )},
    Reduce[Append[GetEqns[ψ.Hvec − (Gvec
      /.Thread[{p1, p2, p3} → {p1, p2, p3}.ψ])],
      Det[ψ] ≠ 0], Flatten[ψ]]];
  Hvec = {
      {0, 0, −p1p2},
      {0, 0, −(p1 + p2)^2},
      {p2p3, p1p3, 0},
      {p2p3, p1p3, −p1p2},
      {p2p3, p1p3, −(p1 + p2)^2}
  };
  If[CheckEquiv[Hvec[[#[[1]]]], Hvec[[#[[2]]]]],
      False,
    Print[H#[[1]], '' not A-equivalent to '',
        H#[[2]]],
    Print[H#[[1]], '' A-equivalent to '',
        H#[[2]]]]&
  /@Subsets[{1, 2, 3, 4, 5}, {2}];]
```

```
H1 not A-equivalent to H2
H1 not A-equivalent to H3
H1 not A-equivalent to H4
H1 not A-equivalent to H5
H2 not A-equivalent to H3
H2 not A-equivalent to H4
H2 not A-equivalent to H5
H3 not A-equivalent to H4
H3 not A-equivalent to H5
H4 not A-equivalent to H5
```

B.3.2 Linear Poisson symmetries

$$H_0 = 0;$$

$$H_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$H_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$H_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix};$$

$$H_4 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix};$$

$$H_5 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix};$$

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

B.3.2.1 $H_0(p) = 0$

Clearly, any automorphism preserves H_0 up to dilations and addition of the Casimir.

B.3.2.2 $H_1(p) = \frac{1}{2}{p_1}^2$

```
With[{ψ = Ψ[x, y, v, w, ς]},
    2ψ.H1.ψ^T//Simplify[#, ς^2 == 1]&//MF]
```

$\begin{pmatrix} x^2 & \varsigma xy & 0 \\ \varsigma xy & y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Therefore we must have either $x = 0$ or $y = 0$.

Suppose $y = 0$.

$\texttt{With}[\{y = 0\},$
$\quad \texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad\quad \frac{1}{x^2}\psi.H_1.\psi^\top \texttt{//Simplify//MF}]]$

$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Suppose $x = 0$.

$\texttt{With}[\{x = 0\},$
$\quad \texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad\quad \frac{1}{y^2}\left(\psi.H_1.\psi^\top + \frac{y^2}{2}K\right)$
$\quad\quad \texttt{//Simplify}\left[\#, \varsigma^2 == 1\right] \texttt{\&//MF}]]$

$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Therefore ψ is of the form $\psi^{(1)} = \begin{pmatrix} x & 0 & v \\ 0 & \varsigma x & w \\ 0 & 0 & \varsigma \end{pmatrix}$ or $\psi^{(1)} = \begin{pmatrix} 0 & y & v \\ \varsigma y & 0 & w \\ 0 & 0 & \varsigma \end{pmatrix}$.

B.3.2.3 $H_2(p) = \frac{1}{2}(p_1 + p_2)^2$

$\texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad 2\psi.H_2.\psi^\top \texttt{//Simplify}\left[\#, \varsigma^2 == 1\right] \texttt{\&//MF}]$

$\begin{pmatrix} (x+y)^2 & \varsigma(x+y)^2 & 0 \\ \varsigma(x+y)^2 & (x+y)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Therefore we must have $\varsigma = 1$.

$\texttt{With}[\{\varsigma = 1\},$
$\quad \texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad\quad \frac{1}{(x+y)^2}\psi.H_2.\psi^\top \texttt{//Simplify//MF}]]$

$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Therefore ψ is of the form $\psi^{(2)} = \begin{pmatrix} x & y & v \\ y & x & w \\ 0 & 0 & 1 \end{pmatrix}$.

B.3.2.4 $H_3(p) = \frac{1}{2}{p_3}^2$

$\texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad 2\psi.H_3.\psi^\top \texttt{//Simplify}\left[\#, \varsigma^2 == 1\right] \texttt{\&//MF}]$

$\begin{pmatrix} v^2 & vw & \varsigma v \\ vw & w^2 & \varsigma w \\ \varsigma v & \varsigma w & 1 \end{pmatrix}$

Therefore we must have $v = w = 0$.

$\texttt{With}[\{v = 0, w = 0\},$
$\quad \texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad\quad \psi.H_3.\psi^\top \texttt{//Simplify}\left[\#, \varsigma^2 == 1\right] \texttt{\&//MF}]]$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

Therefore ψ is of the form $\psi^{(3)} = \begin{pmatrix} x & y & 0 \\ \varsigma y & \varsigma x & 0 \\ 0 & 0 & \varsigma \end{pmatrix}$.

B.3.2.5 $H_4(p) = \frac{1}{2}\left({p_1}^2 + {p_3}^2\right)$

$\texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad 2\psi.H_4.\psi^\top \texttt{//Simplify}\left[\#, \varsigma^2 == 1\right] \texttt{\&//MF}]$

$\begin{pmatrix} v^2 + x^2 & vw + \varsigma xy & \varsigma v \\ vw + \varsigma xy & w^2 + y^2 & \varsigma w \\ \varsigma v & \varsigma w & 1 \end{pmatrix}$

Therefore we must have $v = w = 0$ and either $x = 0$ or $y = 0$.

Suppose $y = 0$.

$\texttt{With}[\{v = 0, w = 0, y = 0\},$
$\quad \texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad\quad \psi.H_4.\psi^\top \texttt{//Simplify}\left[\#, \varsigma^2 == 1\&\&x^2 == 1\right] \&$
$\quad\quad \texttt{//MF}]]$

$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

Therefore x must satisfy $x^2 = 1$.

Suppose $x = 0$.

$\texttt{With}[\{v = 0, w = 0, x = 0\},$
$\quad \texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad\quad \psi.H_4.\psi^\top + \frac{1}{2}K$
$\quad\quad \texttt{//Simplify}\left[\#, \varsigma^2 == 1\&\&y^2 == 1\right] \texttt{\&//MF}]]$

$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

Therefore y must satisfy $y^2 = 1$.

Therefore ψ is of the form $\psi^{(4)} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm\varsigma & 0 \\ 0 & 0 & \varsigma \end{pmatrix}$ or $\psi^{(4)} = \begin{pmatrix} 0 & \pm 1 & 0 \\ \pm\varsigma & 0 & 0 \\ 0 & 0 & \varsigma \end{pmatrix}$.

B.3.2.6 $H_5(p) = \frac{1}{2}\left((p_1 + p_2)^2 + {p_3}^2\right)$

$\texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad 2\psi.H_5.\psi^\top \texttt{//Simplify}\left[\#, \varsigma^2 == 1\right] \texttt{\&//MF}]$

$\begin{pmatrix} v^2 + (x+y)^2 & vw + \varsigma(x+y)^2 & \varsigma v \\ vw + \varsigma(x+y)^2 & w^2 + (x+y)^2 & \varsigma w \\ \varsigma v & \varsigma w & 1 \end{pmatrix}$

Therefore we must have $v = w = 0$, $\varsigma = 1$ and $(x + y)^2 = 1$.

$\texttt{With}[\{v = 0, w = 0, \varsigma = 1\},$
$\quad \texttt{With}[\{\psi = \Psi[x, y, v, w, \varsigma]\},$
$\quad\quad \psi.H_5.\psi^\top$
$\quad\quad \texttt{//Simplify}\left[\#, \varsigma^2 == 1\&\&(x+y)^2 == 1\right] \&$
$\quad\quad \texttt{//MF}]]$

$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

Therefore ψ is of the form $\psi^{(5)} = \begin{pmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where$(x + y)^2 = 1$.

B.3.3 Reduction of elements in $\mathfrak{se}(1,1)$

$$\Psi0[\mathtt{x\_,y\_,v\_,w\_,}\varsigma\mathtt{\_}] := \begin{pmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{pmatrix};$$

$$\Psi1[\mathtt{x\_,0,v\_,w\_,}\varsigma\mathtt{\_}] := \begin{pmatrix} x & 0 & v \\ 0 & \varsigma x & w \\ 0 & 0 & \varsigma \end{pmatrix};$$

$$\Psi1[\mathtt{0,y\_,v\_,w\_,}\varsigma\mathtt{\_}] := \begin{pmatrix} 0 & y & v \\ \varsigma y & 0 & w \\ 0 & 0 & \varsigma \end{pmatrix};$$

$$\Psi2[\mathtt{x\_,y\_,v\_,w\_,1}] := \begin{pmatrix} x & y & v \\ y & x & w \\ 0 & 0 & 1 \end{pmatrix};$$

$$\Psi3[\mathtt{x\_,y\_,0,0,}\varsigma\mathtt{\_}] := \begin{pmatrix} x & y & 0 \\ \varsigma y & \varsigma x & 0 \\ 0 & 0 & \varsigma \end{pmatrix};$$

$$\Psi4[\mathtt{x\_,0,0,0,}\varsigma\mathtt{\_}] := \begin{pmatrix} x & 0 & 0 \\ 0 & \varsigma x & 0 \\ 0 & 0 & \varsigma \end{pmatrix}; \; (*\; x \in \{-1,1\} \;*)$$

$$\Psi4[\mathtt{0,y\_,0,0,}\varsigma\mathtt{\_}] := \begin{pmatrix} 0 & y & 0 \\ \varsigma y & 0 & 0 \\ 0 & 0 & \varsigma \end{pmatrix}; \; (*\; y \in \{-1,1\} \;*)$$

$$\Psi5[\mathtt{x\_,y\_,0,0,1}] := \begin{pmatrix} x & y & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; (*\; (x+y)^2 = 1 \;*)$$

$A = \{\mathbf{a1}, \mathbf{a2}, \mathbf{a3}\};$

B.3.3.1 Reduction under $\psi^{(0)}$

$a_3 = 0$ and ${a_1}^2 \neq {a_2}^2$

Assumps $= \{\mathbf{a3} == 0, \mathbf{a1}^2 \neq \mathbf{a2}^2\}$;
$\psi = \Psi0\left[\frac{\mathbf{a1}}{\mathbf{a1}^2-\mathbf{a2}^2}, -\frac{\mathbf{a2}}{\mathbf{a1}^2-\mathbf{a2}^2}, 0, 0, 1\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\mathrm{a1}}{\mathrm{a1}^2-\mathrm{a2}^2} & \frac{\mathrm{a2}}{-\mathrm{a1}^2+\mathrm{a2}^2} & 0 \\ \frac{\mathrm{a2}}{-\mathrm{a1}^2+\mathrm{a2}^2} & \frac{\mathrm{a1}}{\mathrm{a1}^2-\mathrm{a2}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$a_3 = 0$ and ${a_1}^2 = {a_2}^2$ (so $a_1 = a$, $a_2 = \pm a$, $a \neq 0$)

Assumps $= \{\mathbf{a3} == 0, \mathbf{a1} == a, \mathbf{a2} == \sigma a, \sigma^2 == 1, a \neq 0\}$;
$\psi = \Psi0\left[\frac{1}{a}, 0, 0, 0, \sigma\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\mathrm{a1}} & 0 & 0 \\ 0 & \frac{\sigma}{\mathrm{a1}} & 0 \\ 0 & 0 & \sigma \end{pmatrix}\right\}$$

$a_3 \neq 0$

Assumps $= \{\mathbf{a3} \neq 0\}$;
$\psi = \Psi0\left[1, 0, -\frac{\mathbf{a1}}{\mathbf{a3}}, -\frac{\mathbf{Sign[a3]a2}}{\mathbf{a3}}, \mathbf{Sign[a3]}\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 0 \\ 0 \\ \mathrm{a3Sign[a3]} \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\frac{\mathrm{a1}}{\mathrm{a3}} \\ 0 & \mathrm{Sign[a3]} & -\frac{\mathrm{a2Sign[a3]}}{\mathrm{a3}} \\ 0 & 0 & \mathrm{Sign[a3]} \end{pmatrix}\right\}$$

B.3.3.2 Reduction under $\psi^{(1)}$

$a_3 = 0$ and $a_2 = 0$ ($\Rightarrow a_1 \neq 0$)

Assumps $= \{\mathbf{a3} == 0, \mathbf{a2} == 0, \mathbf{a1} \neq 0\}$;
$\psi = \Psi1\left[\frac{1}{\mathbf{a1}}, 0, 0, 0, 1\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\mathrm{a1}} & 0 & 0 \\ 0 & \frac{1}{\mathrm{a1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$a_3 = 0$, $a_2 \neq 0$ and $a_1 = 0$

Assumps $= \{\mathbf{a3} == 0, \mathbf{a1} == 0, \mathbf{a2} \neq 0\}$;
$\psi = \Psi1\left[0, \frac{1}{\mathbf{a2}}, 0, 0, 1\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\mathrm{a2}} & 0 \\ \frac{1}{\mathrm{a2}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$a_3 = 0$, $a_2 \neq 0$ and $a_1 \neq 0$

Assumps $= \{\mathbf{a3} == 0, \mathbf{a2} \neq 0, \mathbf{a1} \neq 0\}$;
$\psi = \Psi1\left[0, \frac{1}{\mathbf{a2}}, 0, 0, \mathbf{Sign}\left[\frac{\mathbf{a1}}{\mathbf{a2}}\right]\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 1 \\ \frac{\mathrm{a1Sign[a1]}}{\mathrm{a2Sign[a2]}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\mathrm{a2}} & 0 \\ \frac{\mathrm{Sign[a1]}}{\mathrm{a2Sign[a2]}} & 0 & 0 \\ 0 & 0 & \frac{\mathrm{Sign[a1]}}{\mathrm{Sign[a2]}} \end{pmatrix}\right\}$$

$a_3 \neq 0$

Assumps $= \{\mathbf{a3} \neq 0\}$;
$\psi = \Psi1\left[1, 0, -\frac{\mathbf{a1}}{\mathbf{a3}}, -\frac{\mathbf{Sign[a3]a2}}{\mathbf{a3}}, \mathbf{Sign[a3]}\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 0 \\ 0 \\ \mathrm{a3Sign[a3]} \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\frac{\mathrm{a1}}{\mathrm{a3}} \\ 0 & \mathrm{Sign[a3]} & -\frac{\mathrm{a2Sign[a3]}}{\mathrm{a3}} \\ 0 & 0 & \mathrm{Sign[a3]} \end{pmatrix}\right\}$$

B.3.3.3 Reduction under $\psi^{(2)}$

$a_3 = 0$ and ${a_1}^2 \neq {a_2}^2$

Assumps $= \{\mathbf{a3} == 0, \mathbf{a1}^2 \neq \mathbf{a2}^2\}$;
$\psi = \Psi2\left[\frac{\mathbf{a1}}{\mathbf{a1}^2-\mathbf{a2}^2}, -\frac{\mathbf{a2}}{\mathbf{a1}^2-\mathbf{a2}^2}, 0, 0, 1\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\mathrm{a1}}{\mathrm{a1}^2-\mathrm{a2}^2} & \frac{\mathrm{a2}}{-\mathrm{a1}^2+\mathrm{a2}^2} & 0 \\ \frac{\mathrm{a2}}{-\mathrm{a1}^2+\mathrm{a2}^2} & \frac{\mathrm{a1}}{\mathrm{a1}^2-\mathrm{a2}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$a_3 = 0$ and ${a_1}^2 = {a_2}^2$ (so $a_1 = a$, $a_2 = \pm a$, $a \neq 0$)

Assumps $= \{\mathbf{a3}{==}0, \mathbf{a1} == a, \mathbf{a2} == \sigma a, \sigma^2 == 1\}$;
$\psi = \Psi2\left[\frac{1}{a}, 0, 0, 0, 1\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 1 \\ \frac{\mathrm{a2}}{\mathrm{a1}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\mathrm{a1}} & 0 & 0 \\ 0 & \frac{1}{\mathrm{a1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$a_3 \neq 0$

Assumps $= \{\mathbf{a3} \neq 0\}$;
$\psi = \Psi2\left[1, 0, -\frac{\mathbf{a1}}{\mathbf{a3}}, -\frac{\mathbf{a2}}{\mathbf{a3}}, 1\right]$;
MatrixForm/@$\{\psi.A, \psi\}$
//Simplify[#, Assumps]&

$$\left\{\begin{pmatrix} 0 \\ 0 \\ \mathrm{a3} \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\frac{\mathrm{a1}}{\mathrm{a3}} \\ 0 & 1 & -\frac{\mathrm{a2}}{\mathrm{a3}} \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

B.3.3.4 Reduction under $\psi^{(3)}$

$a_3 = 0$ and ${a_1}^2 \neq {a_2}^2$

$$\texttt{Assumps} = \left\{\mathbf{a3} == 0, \mathbf{a1}^2 \neq \mathbf{a2}^2\right\};$$
$$\psi = \Psi 3\left[\frac{\mathbf{a1}}{\mathbf{a1}^2-\mathbf{a2}^2}, -\frac{\mathbf{a2}}{\mathbf{a1}^2-\mathbf{a2}^2}, 0, 0, 1\right];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}\frac{\mathrm{a1}}{\mathrm{a1}^2-\mathrm{a2}^2} & \frac{\mathrm{a2}}{-\mathrm{a1}^2+\mathrm{a2}^2} & 0\\ \frac{\mathrm{a2}}{-\mathrm{a1}^2+\mathrm{a2}^2} & \frac{\mathrm{a1}}{\mathrm{a1}^2-\mathrm{a2}^2} & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

$a_3 = 0$ and ${a_1}^2 = {a_2}^2$ (so $a_1 = a$, $a_2 = \pm a$, $a \neq 0$)

$$\texttt{Assumps} = \left\{\mathbf{a3} == 0, \mathbf{a1} == a, \mathbf{a2} == \sigma a, \sigma^2 == 1\right\};$$
$$\psi = \Psi 3\left[\frac{1}{a}, 0, 0, 0, \sigma\right];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{\mathrm{a1}} & 0 & 0\\ 0 & \frac{\sigma}{\mathrm{a1}} & 0\\ 0 & 0 & \sigma\end{pmatrix}\right\}$$

$a_3 \neq 0$ and ${a_1}^2 \neq {a_2}^2$

$$\texttt{Assumps} = \left\{\mathbf{a3} \neq 0, \mathbf{a1}^2 \neq \mathbf{a2}^2\right\};$$
$$\psi = \Psi 3\left[\frac{\mathbf{a1}}{\mathbf{a1}^2-\mathbf{a2}^2}, -\frac{\mathbf{a2}}{\mathbf{a1}^2-\mathbf{a2}^2}, 0, 0, \mathbf{Sign[a3]}\right];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}1\\0\\\mathrm{a3Sign[a3]}\end{pmatrix}, \begin{pmatrix}\frac{\mathrm{a1}}{\mathrm{a1}^2-\mathrm{a2}^2} & \frac{\mathrm{a2}}{-\mathrm{a1}^2+\mathrm{a2}^2} & 0\\ \frac{\mathrm{a2Sign[a3]}}{-\mathrm{a1}^2+\mathrm{a2}^2} & \frac{\mathrm{a1Sign[a3]}}{\mathrm{a1}^2-\mathrm{a2}^2} & 0\\ 0 & 0 & \mathrm{Sign[a3]}\end{pmatrix}\right\}$$

$a_3 \neq 0$ and ${a_1}^2 = {a_2}^2$ ($a_1 = a$, $a_2 = \pm a$, $a \neq 0$)

$$\texttt{Assumps} = \left\{\mathbf{a3} \neq 0, \mathbf{a1} == a, \mathbf{a2} == \sigma a, \sigma^2 == 1\right\};$$
$$\psi = \Psi 3\left[\frac{1}{a}, 0, 0, 0, \sigma\right];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}1\\1\\\mathrm{a3}\sigma\end{pmatrix}, \begin{pmatrix}\frac{1}{\mathrm{a1}} & 0 & 0\\ 0 & \frac{\sigma}{\mathrm{a1}} & 0\\ 0 & 0 & \sigma\end{pmatrix}\right\}$$

$a_3 \neq 0$ and $a_1 = a_2 = 0$

$$\texttt{Assumps} = \{\mathbf{a3} \neq 0, \mathbf{a1} == 0, \mathbf{a2} == 0\};$$
$$\psi = \Psi 3[1, 0, 0, 0, \mathbf{Sign[a3]}];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}0\\0\\\mathrm{a3Sign[a3]}\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & \mathrm{Sign[a3]} & 0\\ 0 & 0 & \mathrm{Sign[a3]}\end{pmatrix}\right\}$$

B.3.3.5 Reduction under $\psi^{(4)}$

$a_3 = 0$ and $a_2 \neq 0$ and $a_1 = 0$

$$\texttt{Assumps} = \{\mathbf{a3} == 0, \mathbf{a2} \neq 0, \mathbf{a1} == 0\};$$
$$\psi = \Psi 4[1, 0, 0, 0, \mathbf{Sign[a2]}];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}0\\\mathrm{a2Sign[a2]}\\0\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & \mathrm{Sign[a2]} & 0\\ 0 & 0 & \mathrm{Sign[a2]}\end{pmatrix}\right\}$$

$a_3 = 0$ and $a_2 \neq 0$ and $a_1 \neq 0$

$$\texttt{Assumps} = \{\mathbf{a3} == 0, \mathbf{a2} \neq 0, \mathbf{a1} \neq 0\};$$
$$\psi = \Psi 4[\mathbf{Sign[a1]}, 0, 0, 0, \mathbf{Sign[a1a2]}];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}\mathrm{a1Sign[a1]}\\\mathrm{a2Sign[a1]Sign[a1a2]}\\0\end{pmatrix}, \begin{pmatrix}\mathrm{Sign[a1]} & 0 & 0\\ 0 & \mathrm{Sign[a1]Sign[a1a2]} & 0\\ 0 & 0 & \mathrm{Sign[a1a2]}\end{pmatrix}\right\}$$

$a_3 = 0$ and $a_2 = 0$ ($\Rightarrow a_1 \neq 0$)

$$\texttt{Assumps} = \{\mathbf{a3} == 0, \mathbf{a2} == 0, \mathbf{a1} \neq 0\};$$
$$\psi = \Psi 4[0, \mathbf{Sign[a1]}, 0, 0, 1];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}0\\\mathrm{a1Sign[a1]}\\0\end{pmatrix}, \begin{pmatrix}0 & \mathrm{Sign[a1]} & 0\\ \mathrm{Sign[a1]} & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

$a_3 \neq 0$ and $a_2 \neq 0$

$$\texttt{Assumps} = \{\mathbf{a3} \neq 0, \mathbf{a2} \neq 0\};$$
$$\psi = \Psi 4[\mathbf{Sign[a2a3]}, 0, 0, 0, \mathbf{Sign[a3]}];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}\mathrm{a1Sign[a2a3]}\\\mathrm{a2Sign[a3]Sign[a2a3]}\\\mathrm{a3Sign[a3]}\end{pmatrix}, \begin{pmatrix}\mathrm{Sign[a2a3]} & 0 & 0\\ 0 & \mathrm{Sign[a3]Sign[a2a3]} & 0\\ 0 & 0 & \mathrm{Sign[a3]}\end{pmatrix}\right\}$$

$a_3 \neq 0$ and $a_2 = 0$ and $a_1 = 0$

$$\texttt{Assumps} = \{\mathbf{a3} \neq 0, \mathbf{a2} == 0, \mathbf{a1} == 0\};$$
$$\psi = \Psi 4[1, 0, 0, 0, \mathbf{Sign[a3]}];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\texttt{//}\ \texttt{Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}0\\0\\\mathrm{a3Sign[a3]}\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & \mathrm{Sign[a3]} & 0\\ 0 & 0 & \mathrm{Sign[a3]}\end{pmatrix}\right\}$$

$a_3 \neq 0$ and $a_2 = 0$ and $a_1 \neq 0$

$$\texttt{Assumps} = \{\mathbf{a3} \neq 0, \mathbf{a2} == 0, \mathbf{a1} \neq 0\};$$
$$\psi = \Psi 4[0, \mathbf{Sign[a1a3]}, 0, 0, \mathbf{Sign[a3]}];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\ \texttt{//Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}0\\\mathrm{a1Sign[a3]Sign[a1a3]}\\\mathrm{a3Sign[a3]}\end{pmatrix}, \begin{pmatrix}0 & \mathrm{Sign[a1a3]} & 0\\ \mathrm{Sign[a3]Sign[a1a3]} & 0 & 0\\ 0 & 0 & \mathrm{Sign[a3]}\end{pmatrix}\right\}$$

B.3.3.6 Reduction under $\psi^{(5)}$

${a_1}^2 \neq {a_2}^2$

$$\texttt{Assumps} = \left\{\mathbf{a1}^2 \neq \mathbf{a2}^2\right\};$$
$$\psi = \Psi 5\left[\frac{\mathbf{Sign[a1+a2]a1}}{\mathbf{a1-a2}}, -\frac{\mathbf{Sign[a1+a2]a2}}{\mathbf{a1-a2}}, 0, 0, 1\right];$$
$$\texttt{MatrixForm/@}\{\psi.A, \psi\}\texttt{//}\ \texttt{Simplify[\#, Assumps]\&}$$
$$\left\{\begin{pmatrix}(\mathrm{a1}+\mathrm{a2})\mathrm{Sign}[\mathrm{a1}+\mathrm{a2}]\\0\\\mathrm{a3}\end{pmatrix}, \begin{pmatrix}\frac{\mathrm{a1Sign[a1+a2]}}{\mathrm{a1-a2}} & -\frac{\mathrm{a2Sign[a1+a2]}}{\mathrm{a1-a2}} & 0\\ -\frac{\mathrm{a2Sign[a1+a2]}}{\mathrm{a1-a2}} & \frac{\mathrm{a1Sign[a1+a2]}}{\mathrm{a1-a2}} & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

${a_1}^2 = {a_2}^2$ (so $a_1 = a$, $a_2 = \pm a$, $a \neq 0$)

```
Assumps = {a1 == a, a2 == σa, σ^2 == 1, a ≠ 0};
ψ = Ψ5[σSign[a], 0, 0, 0, 1];
MatrixForm/@{ψ.A, ψ}//
Simplify[#, Assumps]&
```

$$\left\{\begin{pmatrix}\text{a2Sign[a1]}\\ \text{a1Sign[a1]}\\ \text{a3}\end{pmatrix}, \begin{pmatrix}\sigma\text{Sign[a1]} & 0 & 0\\ 0 & \sigma\text{Sign[a1]} & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

$a_1 = a_2 = 0$

```
Assumps = {a1 == 0, a2 == 0};
ψ = Ψ5[x, y, 0, 0, 1];
MatrixForm/@{ψ.A, ψ}//
Simplify[#, Assumps]&
```

$$\left\{\begin{pmatrix}0\\ 0\\ \text{a3}\end{pmatrix}, \begin{pmatrix}x & y & 0\\ y & x & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$$

B.3.4 Inhomogeneous systems

```
Minv[MP_, A_]:=Module[{ss, z1, z2, z3},
    ss = Solve[A ==
        z1MP[{1, 0, 0}]+
        z2MP[{0, 1, 0}]+
        z3MP[{0, 0, 1}], {z1, z2, z3}];
    {z1, z2, z3}/.ss[[1]]
];

Hvec[MP_, Hmax_]:=Hvec[MP, Hmax, pp]
Hvec[MP_, Hmax_, v_]:=
  Module[{DHmax, Base, De1, De2, De3, ht},
      DHmax = D[Hmax, {{p1, p2, p3}}];
      Base = {
        MP[{1, 0, 0}],
        MP[{0, 1, 0}],
        MP[{0, 0, 1}]
      }
      De1 = FS[Minv[MP,
          -lie[Base[[1]], DHmax.Base]]. {p1, p2, p3}];
      De2 = FS[Minv[MP,
          -lie[Base[[2]], DHmax.Base]]. {p1, p2, p3}];
      De3 = FS[Minv[MP,
          -lie[Base[[3]], DHmax.Base]]. {p1, p2, p3}];
      ht = {De1, De2, De3}
            /.Thread[{p1, p2, p3} → {p1, p2, p3}];
      ht/.Thread[{p1, p2, p3} → v]
];
ψ = (ψ11 ψ12 ψ13
     ψ21 ψ22 ψ23
     ψ31 ψ32 ψ33);
pp = {p1, p2, p3};
qq = {q1, q2, q3};

GetEqualities[expr_]:=
    Thread[DeleteCases[Flatten@
        CoefficientList[expr, pp], 0] == 0];
CheckEquiv[HH_, GG_, Assumps_, pp_, ψ_, q_,
        MP_, Solv_, Reduc_]:=
  Module[{A, Vars, Eqns},
      A = GetEqualities/@(ψ.Hvec[MP, HH, pp]−
      Hvec[MP, GG, ψ.pp + q]);
      Vars = Flatten[{ψ, q}];
      Eqns = Flatten[{A, Det[ψ] ≠ 0}];
      Which[
        Solv&&Reduc,
          {Solve[Eqns, Vars],
           Simplify[Reduce[Eqns, Vars], Assumps]},
        Solv,
          Solve[Eqns, Vars],
        Reduc,
          Simplify[Reduce[Eqns, Vars], Assumps]]];

CheckEquiv[HH_, GG_, Assumps_, Solv_, Reduc_]:=
    CheckEquiv[HH, GG, Assumps, pp, ψ, qq, M60,
        Solv, Reduc];
CheckEquiv[HH_, GG_, Assumps_]:=
    CheckEquiv[HH, GG, Assumps, pp, ψ, qq, M60,
        True, True];
CheckEquiv[HH_, GG_]:=
    CheckEquiv[HH, GG, {}, pp, ψ, qq, M60, True, True];
```

B.3.4.1 $H_{A,Q}(p) = pA + H_0(p)$

- $G_1(p) = p_1$
- $G_2(p) = p_1 + p_2$
- $G_{3,\alpha}(p) = \alpha p_3$ $(\alpha > 0)$

G_1 is equivalent to G_2

```
With[{
    H = p1,
    G = p1 + p2},
CheckEquiv[H, G]]
```

G_1 is not equivalent to $G_{3,\alpha}$

```
With[{
    H = p1,
    G = αp3,
    Assumps = {α > 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{3,\alpha}$ is unique for unique values of $\alpha > 0$

```
With[{
    H = α1p3,
    G = α2p3,
    Assumps = {α1 > 0, α2 > 0, α1 ≠ α2}},
CheckEquiv[H, G, Assumps, False, True]]
```

B.3.4.2 $H_{A,Q}(p) = pA + H_1(p)$

- $G_{1,\beta}(p) = p_1 + \beta p_3 + \frac{1}{2}{p_1}^2$ $(\beta \geq 0)$
- $G_{2,\alpha}(p) = \alpha p_3 + \frac{1}{2}{p_1}^2$ $(\alpha > 0)$
- $G_{1,\beta}$ is equivalent to $G_{1,1}$ when $\beta > 0$

```
With[{
    H = p1 + βp2 + 1/2 p1^2,
    G = p1 + p2 + 1/2 p1^2,
    Assumps = {β > 0}},
CheckEquiv[H, G, Assumps]]
```

$G_{1,0}$ is not equivalent to $G_{1,1}$

```
With[{
    H = p1 + 1/2 p1^2,
    G = p1 + p2 + 1/2 p1^2,
    Assumps = {}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{1,\lambda}$ is not equivalent to $G_{2,\alpha}$ (where $\lambda \in \{0,1\}$)

```
With[{
    H = p1 + λp2 + 1/2 p1^2,
    G = αp3 + 1/2 p1^2,
    Assumps = {α > 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{2,\alpha}$ is unique for unique values of $\alpha > 0$

```
With[{
    H = α1p3 + 1/2 p1^2,
    G = α2p3 + 1/2 p1^2,
    Assumps = {α1 > 0, α2 > 0, α1 ≠ α2}},
CheckEquiv[H, G, Assumps, False, True]]
```

B.3.4.3 $H_{A,Q}(p) = pA + H_2(p)$

- $G_1(p) = p_1 + \frac{1}{2}(p_1 + p_2)^2$
- $G_{2,\sigma}(p) = p_1 + \sigma p_2 + \frac{1}{2}(p_1 + p_2)^2$ ($\sigma \in$ {-1.1})
- $G_{3,\delta}(p) = \delta p_3 + \frac{1}{2}(p_1 + p_2)^2$ ($\delta \neq 0$)

G_1 is equivalent to $G_{2,-1}$

```
With[{
    H = p1 + 1/2 (p1 + p2)^2,
    G = p1 - p2 + 1/2 (p1 + p2)^2} ,
CheckEquiv[H, G]]
```

G_1 is not equivalent to $G_{2,1}$

```
With[{
    H = p1 + 1/2 (p1 + p2)^2,
    G = p1 + p2 + 1/2 (p1 + p2)^2},
CheckEquiv[H, G, {}, False, True]]
```

$G_{3,\delta}$ is unique for unique values of $\delta \neq 0$

```
With[{
    H = δ1p3 + 1/2 (p1 + p2)^2,
    G = δ2p3 + 1/2 (p1 + p2)^2,
    Assumps = {δ1 ≠ 0, δ2 ≠ 0, δ1 ≠ δ2}},
CheckEquiv[H, G, Assumps, False, True]]
```

B.3.4.4 $H_{A,Q}(p) = pA + H_3(p)$

- $G_{1,\beta}(p) = p_1 + \beta p_3 + \frac{1}{2}p_3^2$ ($\beta \geq 0$)
- $G_{2,\gamma}(p) = p_1 + p_2 + \gamma p_3 + \frac{1}{2}p_3^2$ ($\gamma \in \mathbb{R}$)
- $G_{3,\alpha}(p) = \alpha p_3 + \frac{1}{2}p_3^2$ ($\alpha > 0$)

$G_{1,\beta}$ is equivalent to $G_{1,0}$

```
With[{
    H = p1 + βp3 + 1/2 p3^2,
    G = p1 + 1/2 p3^2,
    Assumps = {β > 0}},
CheckEquiv[H, G, Assumps]]
```

$G_{2,\gamma}$ is equivalent to $G_{2,0}$

```
With[{
    H = p1 + p2 + γp3 + 1/2 p3^2,
    G = p1 + p2 + 1/2 p3^2,
    Assumps = {γ ≠ 0}},
CheckEquiv[H, G, Assumps, True, False]]
```

$G_{3,\alpha}$ is equivalent to $G_{3,0} = H_3$

```
With[{
    H = αp3 + 1/2 p3^2,
    G = 1/2 p3^2,
    Assumps = {α > 0}},
CheckEquiv[H, G, Assumps]]
```

$G_{3,\alpha}$ is equivalent to $G_{3,1}$

```
With[{
    H = αp3 + 1/2 p3^2,
    G = p3 + 1/2 p3^2,
    Assumps = {α > 0}},
CheckEquiv[H, G, Assumps]]
```

$G_{1,0}$ is not equivalent to $G_{2,0}$

```
With[{
    H = p1 + p3^2,
    G = p1 + p2 + 1/2 p3^2},
CheckEquiv[H, G, {}, False, True]]
```

$G_{1,0}$ is not equivalent to $G_{3,1}$

```
With[{
    H = p1 + 1/2 p3^2,
    G = p3 + 1/2 p3^2},
CheckEquiv[H, G, {}, False, True]]
```

$G_{2,0}$ is not equivalent to $G_{3,1}$

```
With[{
    H = p1 + p2 + 1/2 p3^2,
    G = p3 + 1/2 p3^2},
CheckEquiv[H, G, {}, False, True]]
```

B.3.4.5 $H_{A,Q}(p) = pA + H_4(p)$

- $G_{1,\alpha,\beta}(p) = \beta p_1 + \alpha p_2 + \frac{1}{2}(p_1^2 + p_3^2)$ ($\alpha > 0$, $\beta \geq 0$)
- $G_{2,\alpha.\beta,\gamma}(p) = \gamma p_1 + \beta p_2 + \alpha p_3 + \frac{1}{2}(p_1^2 + p_3^2)$ ($\alpha > 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$)

$G_{2,\alpha,\beta,\gamma}$ is equivalent to $G_{2,0,\beta,\gamma}$

```
With[{
    H = γp1 + βp2 + αp3 + 1/2 (p1^2 + p3^2) ,
    G = γp1 + βp2 + 1/2 (p1^2 + p3^2) ,
    Assumps = {α > 0, β ≥ 0, γ ∈ ℝ}},
CheckEquiv[H, G, Assumps, True, False]]
```

$G_{2,0,\beta,\gamma}$ is equivalent to $G_{2,0,\beta,-\gamma}$

```
With[{
    H = γp1 + βp2 + 1/2 (p1^2 + p3^2) ,
    G = -γp1 + βp2 + 1/2 (p1^2 + p3^2) ,
    Assumps = {β ≥ 0, γ ∈ ℝ}},
CheckEquiv[H, G, Assumps, True, False]]
```

$G_{2,0,\beta_1,\beta_2}$ ($\beta_1 > 0$) is equivalent to $G_{1,\alpha,\beta}$, where $\alpha = \beta_1 > 0$ and $\beta = \beta_2 \geq 0$

```
With[{
    H = β1p1 + β2p2 + 1/2 (p1^2 + p3^2),
    G = β2p1 + β1p2 + 1/2 (p1^2 + p3^2),
    Assumps = {β1 > 0, β2 ≥ 0}},
CheckEquiv[H, G, Assumps, True, False]]
```

$G_{1,\alpha,0}$ is equivalent to $G_{3,\alpha}(p) = \alpha p_1 + \frac{1}{2}(p_1{}^2 + p_3{}^2)$

```
With[{
    H = αp2 + 1/2 (p1^2 + p3^2),
    G = αp1 + 1/2 (p1^2 + p3^2),
    Assumps = {α > 0}},
CheckEquiv[H, G, Assumps, True, False]]
```

G_{1,α_1,α_2} is equivalent to G_{1,α_2,α_1}

```
With[{
    H = α1p1 + α2p2 + 1/2 (p1^2 + p3^2),
    G = α2p1 + α1p2 + 1/2 (p1^2 + p3^2),
    Assumps = {α1 > 0, α2 > 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{3,\alpha}(p) = \alpha p_1 + \frac{1}{2}(p_1{}^2 + p_3{}^2)$ is a unique representative for unique values of $\alpha > 0$

```
With[{
    H = α1p1 + 1/2 (p1^2 + p3^2),
    G = α2p1 + 1/2 (p1^2 + p3^2),
    Assumps = {α1 > 0, α2 > 0, α1 ≠ α2}},
CheckEquiv[H, G, Assumps, False, True]]
```

G_{1,α_1,α_2} is a unique representative for unique values of $\alpha_1 \geq \alpha_2 > 0$

```
With[{
    H = α1p1 + α2p2 + 1/2 (p1^2 + p3^2),
    G = α3p1 + α4p2 + 1/2 (p1^2 + p3^2),
    Assumps = {α1 ≥ α2 > 0, α3 ≥ α4 > 0,
        α1 ≠ α3||α2 ≠ α4}},
CheckEquiv[H, G, Assumps, False, True]]
```

G_{1,α_1,α_2} is not equivalent to $G_{3,\alpha}$

```
With[{
    H = α1p1 + α2p2 + 1/2 (p1^2 + p3^2),
    G = αp1 + 1/2 (p1^2 + p3^2),
    Assumps = {α1 ≥ α2 > 0, α > 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

B.3.4.6 $H_{A,Q}(p) = pA + H_5(p)$

- $G_{1,\alpha,\gamma}(p) = \alpha p_1 + \gamma p_3 + \frac{1}{2}((p_1 + p_2)^2 + p_3^2)$ $(\alpha > 0, \gamma \in \mathbb{R})$
- $G_{2,\alpha,\delta,\gamma}(p) = \delta p_1 + \alpha p_2 + \gamma p_3 + \frac{1}{2}((p_1 + p_2)^2 + p_3^2)$ $(\alpha > 0, \delta \neq 0, \gamma \in \mathbb{R})$

$G_{1,\alpha,\gamma}$ is equivalent to $G_{1,\alpha,0}$

```
With[{
    H = αp1 + γp3 + 1/2 ((p1 + p2)^2 + p3^2),
    G = αp1 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α > 0, γ ∈ ℝ}},
CheckEquiv[H, G, Assumps, True, False]]
```

$G_{2,\alpha,\delta,\gamma}$ is equivalent to $G_{2,\alpha,\delta,0}$

```
With[{
    H = δp1 + αp2 + γp3 + 1/2 ((p1 + p2)^2 + p3^2),
    G = δp1 + αp2 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α > 0, δ ≠ 0, γ ∈ ℝ}},
CheckEquiv[H, G, Assumps, True, False]]
```

$G_{2,\alpha,\delta,0}$ $(\alpha^2 \neq \delta^2)$ is equivalent to $G_{1,|\delta+\alpha|}$

```
With[{
    H = δp1 + αp2 + 1/2 ((p1 + p2)^2 + p3^2),
    G = σ(δ + α)p1 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α > 0, δ ≠ 0, δ + α ≠ 0}},
CheckEquiv[H, G, Assumps, True, True]]
```

$G_{2,\alpha,\delta,0}$ $(\alpha^2 \neq \delta^2)$ is equivalent to $G_{3,\delta'}(p) = \delta' p_1 + \frac{1}{2}((p_1 + p_2)^2 + p_3{}^2)$, where $\delta' = \alpha + \delta \neq 0$.

```
With[{
    H = δp1 + αp2 + 1/2 ((p1 + p2)^2 + p3^2),
    G = δpp1 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α > 0, δ ≠ 0, δp == α + δ, δp ≠ 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{3,\delta}(p) = \delta p_1 + \frac{1}{2}((p_1 + p_2)^2 + p_3{}^2)$ $(\delta \neq 0)$ is equivalent to $G_{1,\alpha}$, where $\alpha = |\delta| > 0$. NB: if $\delta > 0$ then $G_{2,\delta} = G_{1,\alpha}$ (where $\alpha = \delta > 0$), hence we only need to show that $G_{2,\delta} = G_{2,-\delta} = G_{1,\alpha}$ (where $\alpha = -\delta > 0$)

```
With[{
    H = δp1 + 1/2 ((p1 + p2)^2 + p3^2),
    G = −δp1 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {δ ≠ 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{3,\delta}(p) = \delta p_1 + \frac{1}{2}((p_1 + p_2)^2 + p_3{}^2)$ $(\delta \neq 0)$ is equivalent to $G_{3,-\delta}$ (and so $G_{3,\alpha} = G_{1,\alpha}$)

```
With[{
    H = δp1 + 1/2 ((p1 + p2)^2 + p3^2),
    G = −δp1 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {δ ≠ 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{2,\alpha,\delta,0}$ $(\alpha = -\delta)$ is equivalent to $G_4(p) = p_1 - p_2 + \frac{1}{2}((p_1 + p_2)^2 + p_3^2)$.

```
With[{
    H = δp1 + αp2 + 1/2 ((p1 + p2)^2 + p3^2),
    G = p1 − p2 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α > 0, δ ≠ 0, α == −δ}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{2,\alpha,\delta,0}$ $(\alpha = \delta)$ is equivalent to $G_{5,\alpha}(p) = \alpha(p_1 + p_2) + \frac{1}{2}((p_1 + p_2)^2 + p_3{}^2)$ (no calculations necessary, since $\alpha = \delta$)

$G_{1,\alpha}$ is not equivalent to $G_4(p) = p_1 - p_2 + \frac{1}{2}((p_1 + p_2)^2 + p_3^2)$

```
With[{
    H = αp1 + 1/2 ((p1 + p2)^2 + p3^2),
    G = p1 − p2 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α > 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{1,\alpha}$ is not euivalent to $G_{5,\alpha}(p) = \alpha(p_1 + p_2) + \frac{1}{2}((p_1 + p_2)^2 + p_3{}^2)$ $(\alpha > 0)$

```
With[{
    H = α1p1 + 1/2 ((p1 + p2)^2 + p3^2),
    G = α2 (p1 + p2) + 1/2 ((p1 + p2)^2 + p3^2),
```

```
    Assumps = {α1 > 0, α2 > 0}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_4(p) = p_1 - p_2 + \frac{1}{2}\left((p_1+p_2)^2 + p_3{}^2\right)$ is not equivalent to $G_{5,\alpha}(p) = \alpha(p_1+p_2) + \frac{1}{2}\left((p_1+p_2)^2 + p_3{}^2\right)$ $(\alpha > 0)$

```
With[{
    H = p1 - p2 + 1/2 ((p1 + p2)^2 + p3^2),
    G = α (p1 + p2) + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α1 > 0, α2 > 0, α1 ≠ α2}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{1,\alpha}$ is unique for unique values of $\alpha > 0$

```
With[{
    H = α1p1 + 1/2 ((p1 + p2)^2 + p3^2),
    G = α2p1 + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α1 > 0, α2 > 0, α1 ≠ α2}},
CheckEquiv[H, G, Assumps, False, True]]
```

$G_{5,\alpha}(p) = \alpha(p_1+p_2) + \frac{1}{2}\left((p_1+p_2)^2 + p_3{}^2\right)$ is unique for unique values of $\alpha > 0$

```
With[{
    H = α1 (p1 + p2) + 1/2 ((p1 + p2)^2 + p3^2),
    G = α2 (p1 + p2) + 1/2 ((p1 + p2)^2 + p3^2),
    Assumps = {α1 > 0, α2 > 0, α1 ≠ α2}},
CheckEquiv[H, G, Assumps, False, True]]
```

B.4 Stability and Integration of Hamilton-Poisson Systems

B.4.1 The system H_1

The equilibrium states are:

```
Reduce[-p1p2 == 0, {p1, p2, p3}, Reals]
p1 == 0||p2 == 0
```

The states $e_1^{\eta,\mu} = (\eta, 0, \mu)$, $\eta \neq 0$, are unstable:

```
p[t_]:={η, δ, μ - δηt};

Norm[p[0] - {η, 0, μ}]^2//FS[#, δ > 0&&
    {t, η} ∈ Reals]&
Norm[p[t]]^2//FS[#, δ > 0&&{t, η, μ} ∈ Reals]&
Limit[%, t → ∞,
    Assumptions → {δ > 0, η ≠ 0, η ∈ Reals}]
```

δ^2

$\delta^2 + \eta^2 + (-t\delta\eta + \mu)^2$

∞

The states $e_1^{0,\mu} = (0, 0, \mu)$ are unstable:

```
p[t_]:={1/2 δ, √3/2 δ, μ - √3/4 δ^2 t};

Norm[p[0] - {0, 0, μ}]^2//FS[#, δ > 0&&t ∈ Reals]&
Norm[p[t]]^2//FS[#, δ > 0&&{t, μ} ∈ Reals]&
Limit[%, t → ∞, Assumptions → {δ > 0}]
```

δ^2

$\delta^2 + \left(-\frac{1}{4}\sqrt{3}\,t\delta^2 + \mu\right)^2$

∞

The states $e_2^{\nu,\mu} = (0, \nu, \mu)$ are unstable:

```
p[t_]:={δ, ν, μ - δνt};

Norm[p[0] - {0, ν, μ}]^2//FS[#, δ > 0&&t ∈ Reals]&
Norm[p[t]]^2//FS[#, δ > 0&&{t, ν, μ} ∈ Reals]&
Limit[%, t → ∞,
    Assumptions → {δ > 0, ν ≠ 0, ν ∈ Reals}]
```

δ^2

$\delta^2 + \nu^2 + (\mu - t\delta\nu)^2$

∞

B.4.2 The system H_2

The equilibrium states are:

```
Reduce[-(p1 + p2)^2 == 0, {p1, p2, p3}, Reals]
p2 == -p1
```

Hence we have the equilibrium states $e_1^{\eta,\mu} = (\eta, -\eta, \mu)$, where $\eta, \mu \in \mathbb{R}$.

The states $e_1^{\eta,\mu}$ are unstable:

```
p[t_]:={δ + η, δ - η, μ - 4δ^2 t};

Norm[p[0] - {η, -η, μ}]//FS[#, δ > 0&&t ∈ Reals]&
Norm[p[t]]^2//FS[#, δ > 0&&{t, η, μ} ∈ Reals]&
Limit[%, t → ∞, Assumptions → {δ > 0}]
```

$\sqrt{2}\,\delta$

$(\delta - \eta)^2 + (\delta + \eta)^2 + \left(-4t\delta^2 + \mu\right)^2$

∞

B.4.3 The system H_3

B.4.3.1 Stability

The equilibrium states are:

```
Reduce[p2p3 == 0&&p1p3 == 0, {p1, p2, p3}, Reals]

(p1 == 0&&p2 == 0)||p3 == 0
```

The states $e_1^{\eta,\mu} = (\eta, \mu, 0)$, $\eta \neq -\mu$ are unstable

```
p[t_]:=
    {ηCosh[δt] + μSinh[δt], ηSinh[δt] + μCosh[δt], δ};
p'[t][[1]] - p[t][[2]]p[t][[3]]//Simplify
p'[t][[2]] - p[t][[1]]p[t][[3]]//Simplify
p'[t][[3]]//Simplify

Norm[p[0] - {η, μ, 0}]^2//FS[#, δ > 0]&
Norm[p[t]]^2//FS[#, δ > 0&&{t, μ, η} ∈ Reals]&
Limit[%, t → ∞, Assumptions → {δ > 0}]
    //FS[#, (η + μ)^2 > 0]&
```

0

0

0

δ^2

$\delta^2 + \left(\eta^2 + \mu^2\right)\mathrm{Cosh}[2t\delta] + 2\eta\mu\mathrm{Sinh}[2t\delta]$

∞

The states $e_1^{\eta,-\eta} = (\eta, -\eta, 0)$, $\eta \neq 0$ are unstable

```
p[t_]:={ηExp[δt], -ηExp[δt], -δ};

Norm[p[0] - {η, -η, 0}]^2//FS[#, δ > 0]&
Norm[p[t]]^2//FS[#, δ > 0&&{t, η} ∈ Reals]&
```

```
Limit[%, t → ∞,
    Assumptions → {δ > 0, η ≠ 0, η ∈ Reals}]
```

δ^2

$\delta^2 + 2e^{2t\delta}\eta^2$

∞

The states $e_1^{0,0} = (0,0,0)$ is unstable

```
p[t_]:={δExp[δt], δExp[δt], δ};

Norm[p[0] − {0,0,0}]//FS[#, δ > 0&&t ∈ Reals]&
Norm[p[t]]^2//FS[#, δ > 0&&t ∈ Reals]&
Limit[%, t → ∞, Assumptions → {δ > 0}]
```

$\sqrt{3}\delta$

$\left(1 + 2e^{2t\delta}\right)\delta^2$

∞

The states $e_2^\nu = (0,0,\nu)$ are (spectrally) unstable

```
D[{p2p3, p1p3, 0}, {{p1, p2, p3}}]
    /.{p1 → 0, p2 → 0, p3 → ν}//Eigenvalues
```

$\{0, -\nu, \nu\}$

B.4.3.2 Visualisation

```
Ha[c_, h_]:=ContourPlot3D[1/2 z^2 == h,
    {x, −3, 3}, {y, −3, 3}, {z, −3, 3}, Mesh → 4];
Int[c_, h_]:=Int[c, h, {p1, p2, p3} ↦ 1/2 p3^2];

Manipulate[{
    Show[Ca[c, h], Ca[−c, h], Ha[c, h], Int[c, h],
        Int[−c, h], Opts],
    Show[
      ParametricPlot3D[{0, 0, ν}, {ν, −3, 3},
        PlotStyle → Directive[Thick, Red]],
      ParametricPlot3D[{η, μ, 0}, {η, −3, 3},
        {μ, −3, 3}, PlotStyle → Directive[Red],
        Mesh → 4],
      Int[c, h], Int[−c, h], Opts]},
{{c, 1}, −3, 3}, {{h, 1}, 0, 3}]
```

B.4.4 The system H_4

B.4.4.1 Stability

The equilibrium states are:

```
Reduce[p2p3 == 0&&p1p3 == 0&& − p1p2 == 0]
```

(p2 == 0&&p1 == 0)||(p3 == 0&&p1 == 0)||(p3 == 0&&p2 == 0)

```
H4 = 1/2 (p1^2 + p3^2);
Ca = p1^2 − p2^2;
```

The state $e_1^0 = (0,0,0)$ is stable:

```
MF/@{
    D[2H4 − 1/2 Ca, {{p1, p2, p3}}],
    D[D[2H4 − 1/2 Ca, {{p1, p2, p3}}], {{p1, p2, p3}}]
}/.{p1 → 0, p2 → 0, p3 → 0}
```

$$\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\right\}$$

The states $e_1^\mu = (\mu,0,0)$, $\mu \neq 0$ are stable:

```
MF/@{
  D[λ0H4 + λ1Ca, {{p1, p2, p3}}],
  D[D[λ0H4 + λ1Ca, {{p1, p2, p3}}], {{p1, p2, p3}}]
}/.{p1 → μ, p2 → 0, p3 → 0}
 /.{λ0 → −2λ1}/.{λ1 → −1/2}

kerH4 = D[H4, {{p1, p2, p3}}]
    /.{p1 → μ, p2 → 0, p3 → 0};
kerCa = D[Ca, {{p1, p2, p3}}]
    /.{p1 → μ, p2 → 0, p3 → 0};
Reduce[kerH4.{x, y, z} == 0&&kerCa.{x, y, z} == 0]
    //Simplify[#, μ ≠ 0]&
```

$$\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}$$

$x == 0$

The states $e_2^\nu = (0,\nu,0)$ are stable:

```
MF/@{
  D[λ0H4 + λ1Ca, {{p1, p2, p3}}],
  D[D[λ0H4 + λ1Ca, {{p1, p2, p3}}], {{p1, p2, p3}}]
}/.{p1 → 0, p2 → ν, p3 → 0}/.{λ0 → 1, λ1 → 0}

kerH4 = D[H4, {{p1, p2, p3}}]
    /.{p1 → 0, p2 → ν, p3 → 0};
kerCa = D[Ca, {{p1, p2, p3}}]
    /.{p1 → 0, p2 → ν, p3 → 0};
Reduce[kerH4.{x, y, z} == 0&&kerCa.{x, y, z} == 0]
    //Simplify[#, ν ≠ 0]&
```

$$\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$$

$y == 0$

The states $e_3^\nu = (0,0,\nu)$ are (spectrally) unstable:

```
D[{p2p3, p1p3, −p1p2}, {{p1, p2, p3}}]
    /.{p1 → 0, p2 → 0, p3 → ν}//Eigenvalues
```

$\{0, -\nu, \nu\}$

B.4.4.2 Integration: case $c_0 > 0$

```
Ω = Sqrt[2h0];
k = Sqrt[1 − c0/(2h0)];
p1[t_]:=σΩJacobiDN[Ωt, k^2]
p2[t_]:= − σΩkJacobiCN[Ωt, k^2];
p3[t_]:=ΩkJacobiSN[Ωt, k^2];

p1'[t] − p2[t]p3[t]//FS[#, σ^2 == 1]&
p2'[t] − p1[t]p3[t]//FS[#, σ^2 == 1]&
p3'[t] + p1[t]p2[t]//FS[#, σ^2 == 1]&
```

0

0

0

B.4.4.3 Integration: case $c_0 = 0$

```
Ω = Sqrt[2h0];
p1[t_]:=σΩSech[Ωt]
p2[t_]:= − ςσΩSech[Ωt];
p3[t_]:=ςΩTanh[Ωt];

p1'[t] − p2[t]p3[t]//FS[#, σ^2 == 1&&ς^2 == 1]&
```

```
p2'[t] − p1[t]p3[t]//FS[#, σ² == 1&&ς² == 1] &
p3'[t] + p1[t]p2[t]//FS[#, σ² == 1&&ς² == 1] &
0
0
0
```

B.4.4.4 Visualisation

```
Ha[c_, h_]:=
    ParametricPlot3D[{√(2h) Cos[θ], y, √(2h) Sin[θ]},
        {y, −3, 3}, {θ, −3, 3}, Mesh → 4];
Int[c_, h_]:=
    Int[c, h, {p1, p2, p3} 1/2 (p1² + p3²)];

Manipulate[{
    Show[Ca[c, h], Ha[c, h], Int[c, h], Opts],
    Show[
      ParametricPlot3D[{0, 0, ν}, {ν, −3, 3},
        PlotStyle → Directive[Thick, Red]],
      ParametricPlot3D[{μ, 0, 0}, {μ, −3, 3},
        PlotStyle → Directive[Thick, Blue]],
      ParametricPlot3D[{0, μ, 0}, {μ, −3, 3},
        PlotStyle → Directive[Thick, Blue]],
      ListPointPlot3D[{{0, 0, 0}},
        PlotStyle → Directive[Blue,
        PointSize[Large]]],
      Int[c, h], Opts]},
{{c, 1}, −3, 3}, {{h, 2}, 0, 3}]
```

B.4.5 The system H_5

B.4.5.1 Stability

The equilibrium states are:

```
Reduce[p2p3 == 0&&p1p3 == 0&&
    −(p1 + p2)² == 0]
(p2 == 0&&p1 == 0)||(p3 == 0&&p1 == −p2)
```

The state $e_1^0 = (0, 0, 0)$ is unstable:

```
p[t_]:={δExp[δt], −δExp[δt], −δ};

Norm[p[0] − {0, 0, 0}]//FS[#, δ > 0]&
Norm[p[t]]²//FS[#, δ > 0&&t ∈ Reals]&
Limit[%, t → ∞, Assumptions → {δ > 0}]
√3 δ
(1 + 2e^(2tδ)) δ²
∞
```

The states $e_1^\mu = (\mu, -\mu, 0)$, $\mu \neq 0$ are unstable:

```
p[t_]:={μExp[δt], −μExp[δt], −δ};

Norm[p[0] − {μ, −μ, 0}]//FS[#, δ > 0&&μ ∈ Reals]&
Norm[p[t]]²//FS[#, δ > 0&&t ∈ Reals&&μ ∈ Reals]&
Limit[%, t → ∞,
    Assumptions → {δ > 0, μ ≠ 0, μ ∈ Reals}]
δ
δ² + 2e^(2tδ) μ²
∞
```

The states $e_2^\nu = (0, \nu, 0)$ are (spectrally) unstable:

```
D[{p2p3, p1p3, −(p1 + p2)²}, {{p1, p2, p3}}]
    /.{p1 → 0, p2 → 0, p3 → ν}//Eigenvalues
{0, −ν, ν}
```

B.4.5.2 Integration: case $c_0 > 0$

```
Ω = √(2h0);

p1[t_]:=σ 1/(2Ω) (Ω²Sech[Ωt] + c0Cosh[Ωt]);
p2[t_]:=σ 1/(2Ω) (Ω²Sech[Ωt] − c0Cosh[Ωt]);
p3[t_]:= − ΩTanh[Ωt];

p1'[t] − p2[t]p3[t]//FS[#, σ² == 1] &
p2'[t] − p1[t]p3[t]//FS[#, σ² == 1] &
p3'[t] + (p1[t] + p2[t])²//FS[#, σ² == 1] &
0
0
0
```

B.4.5.3 Integration: case $c_0 = 0$, $p_1 - p_2 = 0$

```
Ω = √(2h0);

q1[t_]:=σΩ/2 Sech[Ωt];
q2[t_]:=σΩ/2 Sech[Ωt];
q3[t_]:= − ΩTanh[Ωt];

q1'[t] − q2[t]q3[t]//FS[#, σ² == 1] &
q2'[t] − q1[t]q3[t]//FS[#, σ² == 1] &
q3'[t] + (q1[t] + q2[t])²//FS[#, σ² == 1] &
0
0
0
```

B.4.5.4 Integration: case $c_0 = 0$, $p_1 + p_2 = 0$

```
Ω = √(2h0);

s1[t_]:=ςExp[σΩt];
s2[t_]:= − ςExp[σΩt];
s3[t_]:= − σΩ;

s1'[t] − s2[t]s3[t]//FS[#, σ² == 1&&ς² == 1] &
s2'[t] − s1[t]s3[t]//FS[#, σ² == 1&&ς² == 1] &
s3'[t] + (s1[t] + s2[t])²
    //FS[#, σ² == 1&&ς² == 1] &
0
0
0
```

B.4.5.5 Visualisation

```
Ha[c_, h_]:=ParametricPlot3D[
    {1/2(√(2h) Cos[θ] + z),
```

```
    1/2( √(2h) Cos[θ] − z),
    √(2h) Sin[θ]}, {θ, −3, 3}, {z, −10, 10}, Mesh → 4];
Int[c_, h_] :=
    Int [c, h, {p1, p2, p3} ↦ 1/2 ((p1 + p2)^2 + p3^2)] ;

Manipulate[{
    Show[Ca[c, h], Ca[−c, h], Ha[c, h], Int[c, h],
      Int[−c, h], Opts],
    Show[
      ParametricPlot3D[{0, 0, ν}, {ν, −3, 3},
        PlotStyle → Directive[Thick, Red]],
      ParametricPlot3D[{μ, −μ, 0}, {μ, −3, 3},
        PlotStyle → Directive[Thick, Red]],
      Int[c, h], Int[−c, h], Opts]},
{{c, 1}, −3, 3}, {{h, 2}, 0, 3}]
```

B.4.6 The system $H_1^{(0)}$

The equilibrium states are:

```
Reduce [−p2 == 0]
 p2 == 0
```

The states $e_1^{\eta,\mu} = (\eta, 0, \mu)$ are unstable:

```
p[t_] := {η, δ, μ − δt};

Norm[p[0] − {η, 0, μ}]
    //FS[#, {η, μ} ∈ Reals&&δ > 0]&
Norm[p[t]]^2//FS[#, {η, μ, t} ∈ Reals&&δ > 0]&
Limit[%, t → ∞,
    Assumptions → {{η, μ} ∈ Reals, δ > 0}]
 δ
 δ^2 + η^2 + (−tδ + μ)^2
 ∞
```

B.4.7 The system $H_{2,\alpha}^{(0)}$

B.4.7.1 Stability

The equilibrium states are:

```
Reduce [αp2 == 0&&αp1 == 0]
 p1 == 0&&p2 == 0
```

The states $e_1^{\mu} = (0, 0, \mu)$ are (spectrally) unstable:

```
D[{αp1, αp2, 0}, {{p1, p2, p3}}]
    /.{p1 → 0, p2 → 0, p3 → μ}//Eigenvalues
 {0, α, α}
```

B.4.7.2 Visualisation

```
Ha[c_, h_, α_] :=
ContourPlot3D[αz == h, {x, −3, 3}, {y, −3, 3},
    {z, −3, 3}, Mesh → 4];
Int[c_, h_, α_?NumberQ] := Int[c, h, {p1, p2, p3} ↦ αp3];

Manipulate[{
    Show[Ca[c, h], Ca[−c, h], Ha[c, h, α],
      Int[c, h, α], Int[−c, h, α], Opts],
    Show[
      ParametricPlot3D[{0, 0, μ}, {μ, −3, 3},
        PlotStyle → Directive[Thick, Red]],
      Int[c, h, α], Int[−c, h, α], Opts]},
{{c, 1}, −3, 3}, {{h, 0}, 0, 3}, {{α, 1}, 0, 3}]
```

B.4.8 The system $H_1^{(1)}$

The equilibrium states are:

```
Reduce[−p2(1 + p1) == 0]
 p1 == −1||p2 == 0
```

The states $e_1^{\eta,\mu} = (\eta, 0, \mu)$, $\eta \neq -1$ are unstable:

```
p[t_] := {η, δ, μ − δ(1 + η)t};

Norm[p[0] − {η, 0, μ}]
    //FS[#, {η, μ} ∈ Reals&&δ > 0]&
Norm[p[t]]^2//FS[#, {η, μ, t} ∈ Reals&&δ > 0]&
Limit[%, t → ∞,
    Assumptions → {{η, μ} ∈ Reals, δ > 0, η ≠ −1}]

 δ
 δ^2 + η^2 + (−tδ(1 + η) + μ)^2
 ∞
```

The states $e_1^{-1,\mu} = (-1, 0, \mu)$ are unstable:

```
p[t_] := {δ − 1, δ, μ − δ^2 t};

Norm[p[0] − {−1, 0, μ}]//FS[#, μ ∈ Reals&&δ > 0]&
Norm[p[t]]^2//FS[#, {μ, t} ∈ Reals&&δ > 0]&
Limit[%, t → ∞,
    Assumptions → {μ ∈ Reals, δ > 0}]
 √2 δ
 (−1 + δ)^2 + δ^2 + (−tδ^2 + μ)^2
 ∞
```

The states $e_2^{\nu,\mu} = (-1, \nu, \mu)$ are unstable:

```
p[t_] := {δ − 1, ν, μ − νδt};

Norm[p[0] − {−1, ν, μ}]//FS[#, {ν, μ} ∈ Reals&&
    δ > 0]&
Norm[p[t]]^2//FS[#, {ν, μ, t} ∈ Reals&&δ > 0]&
Limit[%, t → ∞,
    Assumptions → {{ν, μ} ∈ Reals, δ > 0, ν ≠ 0}]
 δ
 (−1 + δ)^2 + ν^2 + (μ − tδν)^2
 ∞
```

B.4.9 The system $H_2^{(1)}$

The equilibrium states are:

```
Reduce[p1 + p2 + p1p2 == 0]//Simplify
 1 + p1 ≠ 0&&p1 + p2 + p1p2 == 0
```

The states $e_1^{\eta,\mu} = \left(\eta, -\frac{\eta}{\eta+1}, \mu\right)$ are unstable:

```
p[t_] := {η, δ − η/(1+η), μ − δ(1 + η)t};

Norm [p[0] − {η, −η/(1+η), μ}] //FS[#, δ > 0&&
    η ≠ −1]&
Norm[p[t]]^2//FS[#, {η, μ, t} ∈ Reals&&δ > 0&&
    η ≠ −1]&
```

```
Limit[%, t → ∞,
    Assumptions → {{η, μ} ∈ Reals, δ > 0, η ≠ −1}]
```

δ

$\eta^2+\left(-1+\delta+\frac{1}{1+\eta}\right)^2+(-t\delta(1+\eta)+\mu)^2$

∞

B.4.10 The system $H^{(1)}_{3,\alpha}$

B.4.10.1 Stability

The equilibrium states are:

```
Reduce[αp2 == 0&&αp1 == 0&& − p1p2 == 0
    &&α > 0]//Simplify[#, α > 0]&
```

p1 == 0&&p2 == 0

The states $e^{\mu}_1=(0,0,\mu)$ are (spectrally) unstable:

```
D[{αp2, αp1, −p1p2}, {{p1, p2, p3}}]
    /.{p1 → 0, p2 → 0, p3 → μ}//Eigenvalues
```

$\{0,-\alpha,\alpha\}$

B.4.10.2 Visualisation

```
Ha[c_, h_, α_]:=
    ContourPlot3D[αz + 1/2 x^2 == h, {x, −5, 5},
        {y, −5, 5}, {z, −5, 5}, Mesh → 4];
Int[c_, h_, α_?NumberQ]:=
    Int [c, h, {p1, p2, p3} ↦ αp3 + 1/2 p1^2] ;

Manipulate[{
    Show[Ca[c, h], Ca[−c, h], Ha[c, h, α],
      Int[c, h, α], Int[−c, h, α], Opts],
    Show[
      ParametricPlot3D[{0, 0, μ}, {μ, −3, 3},
        PlotStyle → Directive[Thick, Red]],
      Int[c, h, α], Int[−c, h, α], Opts]},
{{c, 1}, −3, 3}, {{h, 2}, 0, 3}, {{α, 1}, 0, 3}]
```

B.4.11 The system $H^{(2)}_1$

The equilibrium states are:

```
Reduce [−p2 − (p1 + p2)^2 == 0]
```

$\left(\mathrm{p1}==-\frac{1}{4}\&\&\mathrm{p2}==-\frac{1}{4}\right)\|\left(\mathrm{p1}>-\frac{1}{4}\right.$
$\&\&\left(\mathrm{p2}==\frac{1}{2}(-1-2\mathrm{p1})-\frac{1}{2}\sqrt{1+4\mathrm{p1}}\right.$
$\left.\left.\|\mathrm{p2}==\frac{1}{2}(-1-2\mathrm{p1})+\frac{1}{2}\sqrt{1+4\mathrm{p1}}\right)\right)$

The states $e^{\eta,\mu}=(\eta,\epsilon,\mu)$ are unstable:

```
p[t_]:= {η, ϵ − σδ, μ − (ϵ − σδ)t − (η + ϵ − σδ)^2 t} ;

Norm[p[0] − {η, ϵ, μ}]
    //FS[#, η ≥ −1/4&&δ > 0&&Abs[σ] == 1]&
Norm[p[t]]^2//FS[#, η ≥ −1/4&&δ > 0&&
    {t, μ, ϵ} ∈ Reals]&
Limit[%, t → ∞,
    Assumptions → {η ≥ −1/4&&δ > 0&&
    {μ, σ} ∈ Reals}]
    /.{ϵ → −1/2(1 + 2η + σ √(1 + 4η))}//Expand
    //FS[#, σ^2 == 1&&δ > 0&&η ≥ −1/4]&
```

δ

$\eta^2+(\epsilon-\delta\sigma)^2+\left(\mu-t(\epsilon+\eta-\delta\sigma)^2+t(-\epsilon+\delta\sigma)\right)^2$

∞

B.4.12 The system $H^{(2)}_2$

The equilibrium states are:

```
Reduce[(p1 + p2)(1 + p1 + p2) == 0]//Simplify
```

1 + p1 + p2 == 0||p1 + p2 == 0

The states $e^{\eta,\mu}_1=(\eta,-1-\eta,\mu)$ are unstable:

```
p[t_]:={η, −δ − 1 − η, μ − δ(δ + 1)t};

Norm[p[0] − {η, −1 − η, μ}]
    //FS[#, {η, μ} ∈ Reals&&δ > 0]&
Norm[p[t]]^2//FS[#, {η, μ, t} ∈ Reals&&δ > 0]&
Limit[%, t → ∞,
    Assumptions → {{η, μ} ∈ Reals, δ > 0}]
```

δ

$\eta^2+(1+\delta+\eta)^2+(-t\delta(1+\delta)+\mu)^2$

∞

The states $e^{\eta,\mu}_2=(\eta,-\eta,\mu)$ are unstable:

```
p[t_]:={η, δ − η, μ − δ(δ + 1)t};

Norm[p[0] − {η, −η, μ}]//FS[#, {η, μ} ∈ Reals&&
    δ > 0]&
Norm[p[t]]^2//FS[#, {η, μ, t} ∈ Reals&&δ > 0]&
Limit[%, t → ∞,
    Assumptions → {{η, μ} ∈ Reals, δ > 0}]
```

δ

$(\delta-\eta)^2+\eta^2+(-t\delta(1+\delta)+\mu)^2$

∞

B.4.13 The system $H^{(2)}_{3,\delta}$

B.4.13.1 Stability

The equilibrium states are:

```
Reduce[δp2 == 0&&δp1 == 0
    && − (p1 + p2)^2 == 0&&δ ≠ 0]
    //Simplify[#, δ ≠ 0]&
```

p1 == 0&&p2 == 0

The states $e^{\mu}_1=(0,0,\mu)$ are (spectrally) unstable:

```
D [{δp2, δp1, −(p1 + p2)^2} , {{p1, p2, p3}}]
    /.{p1 → 0, p2 → 0, p3 → μ}//Eigenvalues
```

$\{0,-\delta,\delta\}$

B.4.13.2 Visualisation

```
Ha[c_, h_, δ_]:=
  ContourPlot3D [δz + 1/2(x + y)^2 == h, {x, −5, 5},
    {y, −5, 5}, {z, −5, 5}, Mesh → 4];
Int[c_, h_, δ_?NumberQ]:=
  Int [c, h, {p1, p2, p3}δp3 + 1/2(p1 + p2)^2] ;

Manipulate[{
    Show[Ca[c, h], Ca[−c, h], Ha[c, h, δ],
      Int[c, h, δ], Int[−c, h, δ], Opts],
```

```
Show[
  ParametricPlot3D[{0,0,μ},{μ,−3,3},
    PlotStyle → Directive[Thick,Red]],
  Int[c,h,δ],Int[−c,h,δ],Opts],
Show[Ca[c,h],Ca[−c,h],Ha[c,h,−δ],
  Int[c,h,−δ],Int[−c,h,−δ],Opts],
Show[
  ParametricPlot3D[{0,0,μ},{μ,−3,3},
    PlotStyle → Directive[Thick,Red]],
  Int[c,h,−δ],Int[−c,h,−δ],Opts]},
{{c,1},−3,3},{{h,0},0,3},{{δ,1},0.1,3}]
```

B.4.14 The system $H_1^{(3)}$

B.4.14.1 Stability

The equilibrium states are:

```
Reduce[p2p3 == 0&&p1p3 == 0&& − p2 == 0]
```

(p2 == 0&&p1 == 0)||(p3 == 0&&p2 == 0)

The states $e_1^\mu = (\mu, 0, 0)$, $\mu > 0$ are stable:

```
H = p1 + 1/2 p3^2;
Ca = p1^2 − p2^2;
MF/@{
  D[λ0H31 + λ1Ca,{{p1,p2,p3}}],
  D[D[λ0H31 + λ1Ca,{{p1,p2,p3}}],{{p1,p2,p3}}]
}/.{p1 → μ,p2 → 0,p3 → 0}
/.{λ0 → −2μλ1}/.{λ1 → −1/2}

kerH = D[H,{{p1,p2,p3}}]
   /.{p1 → μ,p2 → 0,p3 → 0};
kerCa = D[Ca,{{p1,p2,p3}}]
   /.{p1 → μ,p2 → 0,p3 → 0};
Reduce[kerH{x,y,z} == 0&&kerCa.{x,y,z} == 0]
   //Simplify[#,μ ≠ 0]&
```

$$\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}-1&0&0\\0&1&0\\0&0&\mu\end{pmatrix}\right\}$$

$x == 0$

The state $e_1^0 = (0, 0, 0)$ is unstable:

```
p[t_]:={−2/t^2, 2/t^2, 2/t};
p'[t][[1]] − p[t][[2]]p[t][[3]]//Simplify
p'[t][[2]] − p[t][[1]]p[t][[3]]//Simplify
p'[t][[3]] + p[t][[2]]//Simplify

Limit[Norm[p[t]]^2,t → −∞,Assumptions → {μ < 0}]
Limit[Norm[p[t]]^2,t → 0,Assumptions → {μ < 0}]
```

0
0
0
0
∞

The states $e_1^\mu = (\mu, 0, 0)$, $\mu < 0$ are unstable:

```
p[t_]:={μ(1 + 2Csch[t √(−μ)]^2),
   −2μCoth[t √(−μ)]Csch[t √(−μ)],
   2 √(−μ) Csch[t √(−μ)]};
p'[t][[1]] − p[t][[2]]p[t][[3]]//Simplify
p'[t][[2]] − p[t][[1]]p[t][[3]]//Simplify
p'[t][[3]] + p[t][[2]]//Simplify

Limit[Norm[p[t] − {μ,0,0}]^2,t → −∞,
Assumptions → {μ < 0}]
```

0
0
0
0
∞

The states $e_2^\nu = (0, 0, \nu)$ are (spectrally) unstable:

```
D[{p2p3,p1p3,−p2},{{p1,p2,p3}}]
   /.{p1 → 0,p2 → 0,p3 → ν}//Eigenvalues
```

$\{0, -\nu, \nu\}$

B.4.14.2 Integration: case $c_0 > 0$, $h_0 > \sqrt{c_0}$, $\bar{p}_1(t) \leq -\sqrt{c_0}$

```
a1 = 1;b1 = 0;c1 = −c0;
a2 = 0;b2 = −1;c2 = 2h0;

X1 = a1x^2 + 2b1x + c1;
X2 = a2x^2 + 2b2x + c2;

Solve[(a1 − λa2)(c1 − λc2) − (b1 − λb2)^2 == 0,λ]
   //Simplify;
λ1 = −h0 − √(h0^2 − c0);
λ2 = −h0 + √(h0^2 − c0);

(a1 − λ1a2)(c1 − λ1c2) − (b1 − λ1b2)^2//Simplify
(a1 − λ2a2)(c1 − λ2c2) − (b1 − λ2b2)^2//Simplify
```

0
0

```
Solve[(X1−λ1X2)/(a1−λ1a2) == 0,x] //Simplify;
Solve[(X1−λ2X2)/(a1−λ2a2) == 0,x] //Simplify;

r1 = −λ1;
r2 = −λ2;

A1 = λ2(a1−λ1a2)/(λ2−λ1)//Simplify;
A2 = (a1−λ1a2)/(λ2−λ1)//Simplify;
B1 = λ1(a1−λ2a2)/(λ1−λ2)//Simplify;
B2 = (a1−λ2a2)/(λ1−λ2)//Simplify;

A1(x − r1)^2 + B1(x − r2)^2//Simplify
A2(x − r1)^2 + B2(x − r2)^2//Simplify

{A1,B1,A2,B2}//Simplify
```

$-\mathrm{c0} + x^2$

$2(\mathrm{h0} - x)$

$$\left\{\frac{1}{2} - \frac{\mathrm{h0}}{2\sqrt{-\mathrm{c0}+\mathrm{h0}^2}}, \frac{1}{2}\left(1 + \frac{\mathrm{h0}}{\sqrt{-\mathrm{c0}+\mathrm{h0}^2}}\right), \frac{1}{2\sqrt{-\mathrm{c0}+\mathrm{h0}^2}}, -\frac{1}{2\sqrt{-\mathrm{c0}+\mathrm{h0}^2}}\right\}$$

```
δ = √(h0^2 − c0);
```

$a = \sqrt{\frac{h0+\delta}{h0-\delta}}$;
$b = 1$;
$\Omega = \sqrt{h0+\delta}$;
$k = \sqrt{\frac{2\delta}{h0+\delta}}$;
$kp = \sqrt{\frac{h0-\delta}{h0+\delta}}$;
$p1[t\_] := \frac{(\delta+h0)JacobiDN[\Omega t,k^2]+(\delta-h0)}{JacobiDN[\Omega t,k^2]-1}$;
$p2[t\_] := \sigma 2\delta \frac{JacobiCN[\Omega t,k^2]}{JacobiDN[\Omega t,k^2]-1}$;
$p3[t\_] := \sigma \sqrt{2\delta}\, k \frac{JacobiSN[\Omega t,k^2]}{JacobiDN[\Omega t,k^2]-1}$;

B.4.14.3 Integration: case $c_0 > 0$, $h_0 > \sqrt{c_0}$, $\bar{p}_1(t) \geq \sqrt{c_0}$

$a1 = 1; b1 = 0; c1 = -c0;$
$a2 = 0; b2 = -1; c2 = 2h0;$

$X1 = a1x^2 + 2b1x + c1;$
$X2 = a2x^2 + 2b2x + c2;$
Solve $[(a1 - \lambda a2)(c1 - \lambda c2) - (b1 - \lambda b2)^2 == 0, \lambda]$//Simplify;
$\lambda 1 = -h0 - \sqrt{-c0 + h0^2}$;
$\lambda 2 = -h0 + \sqrt{-c0 + h0^2}$;

$(a1 - \lambda 1a2)(c1 - \lambda 1c2) - (b1 - \lambda 1b2)^2$//Simplify
$(a1 - \lambda 2a2)(c1 - \lambda 2c2) - (b1 - \lambda 2b2)^2$//Simplify
0
0
Solve $\left[\frac{(X1-\lambda 1X2)}{a1-\lambda 1a2} == 0, x\right]$ //Simplify;
Solve $\left[\frac{(X1-\lambda 2X2)}{a1-\lambda 2a2} == 0, x\right]$ //Simplify;

$r1 = -\lambda 1;$
$r2 = -\lambda 2;$

$A1 = \frac{\lambda 2(a1-\lambda 1a2)}{\lambda 2-\lambda 1}$//Simplify;
$A2 = \frac{a1-\lambda 1a2}{\lambda 2-\lambda 1}$//Simplify;
$B1 = \frac{\lambda 1(a1-\lambda 2a2)}{\lambda 1-\lambda 2}$//Simplify;
$B2 = \frac{a1-\lambda 2a2}{\lambda 1-\lambda 2}$//Simplify;

$A1(x - r1)^2 + B1(x - r2)^2$//Simplify
$A2(x - r1)^2 + B2(x - r2)^2$//Simplify

{A1, B1, A2, B2}//Simplify
$-c0 + x^2$
$2(h0 - x)$
$$\left\{\frac{1}{2} - \frac{h0}{2\sqrt{-c0+h0^2}}, \frac{1}{2}\left(1 + \frac{h0}{\sqrt{-c0+h0^2}}\right), \frac{1}{2\sqrt{-c0+h0^2}}, -\frac{1}{2\sqrt{-c0+h0^2}}\right\}$$
$\delta = \sqrt{h0^2 - c0}$;
$a = -\sqrt{\frac{h0+\delta}{h0-\delta}}$;
$b = -1$;

$\Omega = \sqrt{h0+\delta}$;

$k = \sqrt{\frac{2\delta}{h0+\delta}}$;
$kp = \sqrt{\frac{h0-\delta}{h0+\delta}}$;
$p1[t\_] := \sqrt{c0}\frac{kpJacobiDN[\Omega t,k^2]+1}{kp+JacobiDN[\Omega t,k^2]}$;
$p2[t\_] := k^2\sqrt{c0}\frac{JacobiSN[\Omega t,k^2]}{kp+JacobiDN[\Omega t,k^2]}$;
$p3[t\_] := k\sqrt{2\delta}\frac{JacobiCN[\Omega t,k^2]}{kp+JacobiDN[\Omega t,k^2]}$;

B.4.14.4 Integration: case $c_0 > 0$, $h_0 = \sqrt{c_0}$

$\delta = \sqrt{h0^2 - c0}$;
$\Omega = \sqrt{h0+\delta}$;
$k = \sqrt{\frac{2\delta}{h0+\delta}}$;

Limit$\left[\frac{(\delta+h0)JacobiDN[\Omega t,k^2]+(\delta-h0)}{JacobiDN[\Omega t,k^2]-1}, h0 \to \sqrt{c0},\right.$
Assumptions $\to \left.\{h0 > \sqrt{c0}, c0 > 0\}\right]$;
Limit$\left[\sigma 2\delta\frac{JacobiCN[\Omega t,k^2]}{JacobiDN[\Omega t,k^2]-1}, h0 \to \sqrt{c0},\right.$
Assumptions $\to \left.\{h0 > \sqrt{c0}, c0 > 0\}\right]$;
Limit$\left[\sigma\sqrt{2\delta}\, k\frac{JacobiSN[\Omega t,k^2]}{JacobiDN[\Omega t,k^2]-1}, h0 \to \sqrt{c0},\right.$
Assumptions $\to \left.\{h0 > \sqrt{c0}, c0 > 0\}\right]$;

$\Omega = \sqrt{h0}$;

$p1[t\_] := -h0\left(1 + 2Tan\left[\sqrt{h0}t\right]^2\right)$;
$p2[t\_] := -\sigma 2h0Sec\left[\sqrt{h0}t\right] Tan\left[\sqrt{h0}t\right]$;
$p3[t\_] := 2\sigma\sqrt{h0}Sec\left[\sqrt{h0}t\right]$;

$p1'[t] - p2[t]p3[t]$//Simplify[#, h0 > 0&&c0 > 0&&
h0 == $\sqrt{c0}$&&σ^2 == 1]&
$p2'[t] - p1[t]p3[t]$//Simplify[#, h0 > 0&&c0 > 0&&
h0 == $\sqrt{c0}$&&σ^2 == 1]&
$p3'[t] + p2[t]$//Simplify[#, h0 > 0&&c0 > 0&&
h0 == $\sqrt{c0}$&&σ^2 == 1]&
0
0
0

B.4.14.5 Integration: case $c_0 > 0$, $-\sqrt{c_0} < h_0 < \sqrt{c_0}$

$\delta = \sqrt{2\left(c0 - \sqrt{c0}h0\right)}$;

$a1 = 0; b1 = \frac{1}{2}; c1 = -\sqrt{c0}$;
$a2 = -2; b2 = h0 - \sqrt{c0}; c2 = 2\sqrt{c0}h0$;

$X1 = a1x^2 + 2b1x + c1;$
$X2 = a2x^2 + 2b2x + c2;$

Solve $[(a1 - \lambda a2)(c1 - \lambda c2) - (b1 - \lambda b2)^2 == 0, \lambda]$;
$\lambda 1 = \frac{1}{-6\sqrt{c0}+2h0+4\delta}$;

$\lambda 2=\frac{1}{-6\sqrt{c0}+2h0-4\delta}$;

(a1 − λ1a2)(c1 − λ1c2) − (b1 − λ1b2)^2//Simplify
(a1 − λ2a2)(c1 − λ2c2) − (b1 − λ2b2)^2//Simplify

0

0

Solve $\left[\frac{(X1-\lambda 1X2)}{a1-\lambda 1a2}==0,x\right]$ //Simplify;
Solve $\left[\frac{(X1-\lambda 2X2)}{a1-\lambda 2a2}==0,x\right]$ //Simplify;

$r1=\sqrt{c0}-\delta$;
$r2=\sqrt{c0}+\delta$;

$A1=\frac{\lambda 2(a1-\lambda 1a2)}{\lambda 2-\lambda 1}$//Simplify;
$A2=\frac{a1-\lambda 1a2}{\lambda 2-\lambda 1}$//Simplify;
$B1=\frac{\lambda 1(a1-\lambda 2a2)}{\lambda 1-\lambda 2}$//Simplify;
$B2=\frac{a1-\lambda 2a2}{\lambda 1-\lambda 2}$//Simplify;

{A1, B1, A2, B2}//Simplify
A1$(x-r1)^2+$B1$(x-r2)^2$//Simplify
A2$(x-r1)^2+$B2$(x-r2)^2$//Simplify

$\left\{\frac{1}{4\delta},-\frac{1}{4\delta},\frac{-3\sqrt{c0}+h0-2\delta}{2\delta},-\frac{-3\sqrt{c0}+h0+2\delta}{2\delta}\right\}$

$-\sqrt{c0}+x$

$2(h0-x)(\sqrt{c0}+x)$

$b=\sqrt{\frac{\lambda 2}{\lambda 1}}$;
$a=1$;

$\Omega=\frac{1}{2}\sqrt{6\sqrt{c0}-2h0+4\delta}$;
$k=2\sqrt{\frac{\delta}{3\sqrt{c0}-h0+2\delta}}$;

p1[t_]:= $\frac{(\delta+\sqrt{c0})\text{JacobiDN}[\Omega t,k^2]+(\delta-\sqrt{c0})}{\text{JacobiDN}[\Omega t,k^2]-1}$;
p2[t_]:= $\sigma k\sqrt{\delta(\delta+2\sqrt{c0})}\frac{\text{JacobiCN}[\Omega t,k^2]\sqrt{\text{JacobiDN}[\Omega t,k^2]+1}}{\sqrt{\text{JacobiDN}[\Omega t,k^2]+\text{kp}}(\text{JacobiDN}[\Omega t,k^2]-1)}$;
p3[t_]:= $\sigma\sqrt{2(\delta+\sqrt{c0}-h0)}\frac{\sqrt{\text{JacobiDN}[\Omega t,k^2]+\text{kp}}\sqrt{1-\text{JacobiDN}[\Omega t,k^2]}}{(\text{JacobiDN}[\Omega t,k^2]-1)}$;

B.4.14.6 Integration: case $c_0>0$, $h_0=-\sqrt{c_0}$

Limit$\left[\frac{(\delta+\sqrt{c0})\text{JacobiDN}[\Omega t,k^2]+(\delta-\sqrt{c0})}{\text{JacobiDN}[\Omega t,k^2]-1},\right.$
h0 → $-\sqrt{c0}$, Assumptions → {c0 > 0},
Direction → +1$\left.\right]$
/. $\{\sqrt{c0}\to -h0, c0^{1/4}\to\sqrt{-h0}\}$
/. $\{\sqrt{-h0}\to\omega\}$//FS[#, h0 < 0]&
//FS[#, ω > 0&&t ∈ Reals]&
Limit$\left[\sigma k\sqrt{\delta(\delta+2\sqrt{c0})}\frac{\text{JacobiCN}[\Omega t,k^2]\sqrt{\text{JacobiDN}[\Omega t,k^2]+1}}{\sqrt{\text{JacobiDN}[\Omega t,k^2]+\text{kp}}(\text{JacobiDN}[\Omega t,k^2]-1)}\right.$
h0 → $-\sqrt{c0}$, Assumptions → {c0 > 0},
Direction → +1$\left.\right]$
/. $\{\sqrt{c0}\to -h0, c0^{1/4}\to\sqrt{-h0}\}$
/. $\{\sqrt{-h0}\to\omega\}$//FS[#, h0 < 0]&
//FS[#, ω > 0&&t ∈ Reals]&
Limit$\left[\sigma\sqrt{2(\delta+\sqrt{c0}-h0)}\frac{\sqrt{\text{JacobiDN}[\Omega t,k^2]+\text{kp}}\sqrt{1-\text{JacobiDN}[\Omega t,k^2]}}{(\text{JacobiDN}[\Omega t,k^2]-1)},\right.$
h0 → $-\sqrt{c0}$, Assumptions → {c0 > 0},
Direction → +1$\left.\right]$
/. $\{\sqrt{c0}\to -h0, c0^{1/4}\to\sqrt{-h0}\}$
/. $\{\sqrt{-h0}\to\omega\}$//FS[#, h0 < 0]&
//FS[#, ω > 0&&t ∈ Reals]&

$h0+2h0\text{Csch}[t\omega]^2$

$-2h0\sigma\text{Coth}[t\omega]\text{Csch}[t\omega]$

$2\sigma\omega\text{Csch}[t\omega]$

$\Omega=\sqrt{-h0}$;

p1[t_]:=h0 $(1+2\text{Csch}[\Omega t]^2)$;
p2[t_]:= $\sigma 2\Omega^2\text{Coth}[\Omega t]\text{Csch}[\Omega t]$;
p3[t_]:= $\sigma 2\Omega\text{Csch}[\Omega t]$;

p1′[t] − p2[t]p3[t]//Simplify [#, σ^2 == 1] &
p2′[t] − p1[t]p3[t]//Simplify [#, σ^2 == 1] &
p3′[t] + p2[t]//Simplify [#, σ^2 == 1] &

0

0

0

B.4.14.7 Integration: case $c_0>0$, $h_0<-\sqrt{c_0}$

$\delta=\sqrt{h0^2-c0}$;

a1 = 1; b1 = 0; c1 = −c0;
a2 = 0; b2 = −1; c2 = 2h0;

X1 = a1x^2 + 2b1x + c1;
X2 = a2x^2 + 2b2x + c2;

Solve [(a1 − λa2)(c1 − λc2) − (b1 − λb2)^2 == 0, λ]
//Simplify;
λ1 = −h0 − δ;
λ2 = −h0 + δ;

(a1 − λ1a2)(c1 − λ1c2) − (b1 − λ1b2)^2//Simplify

0

0

Solve $\left[\frac{(X1-\lambda 1X2)}{a1-\lambda 1a2}==0,x\right]$ //Simplify;
Solve $\left[\frac{(X1-\lambda 2X2)}{a1-\lambda 2a2}==0,x\right]$ //Simplify;

r1 = −λ1;
r2 = −λ2;

$$\mathtt{A1} = \frac{\lambda2(\mathtt{a1}-\lambda1\mathtt{a2})}{\lambda2-\lambda1}\mathtt{//Simplify;}$$
$$\mathtt{A2} = \frac{\mathtt{a1}-\lambda1\mathtt{a2}}{\lambda2-\lambda1}\mathtt{//Simplify;}$$
$$\mathtt{B1} = \frac{\lambda1(\mathtt{a1}-\lambda2\mathtt{a2})}{\lambda1-\lambda2}\mathtt{//Simplify;}$$
$$\mathtt{B2} = \frac{\mathtt{a1}-\lambda2\mathtt{a2}}{\lambda1-\lambda2}\mathtt{//Simplify;}$$

$$\mathtt{A1}(x-\mathtt{r1})^2+\mathtt{B1}(x-\mathtt{r2})^2\mathtt{//Simplify}$$
$$\mathtt{A2}(x-\mathtt{r1})^2+\mathtt{B2}(x-\mathtt{r2})^2\mathtt{//Simplify}$$

$$\{\mathtt{A1},\mathtt{B1},\mathtt{A2},\mathtt{B2}\}\mathtt{//Simplify}$$

$-\mathrm{h0}^2+x^2+\delta^2$

$2(\mathrm{h0}-x)$

$\left\{\frac{-\mathrm{h0}+\delta}{2\delta},\frac{\mathrm{h0}+\delta}{2\delta},\frac{1}{2\delta},-\frac{1}{2\delta}\right\}$

$$b=-\sqrt{\frac{\mathbf{h0}+\delta}{\mathbf{h0}-\delta}};$$
$$a=-1;$$

$$\Omega=\sqrt{\delta-\mathbf{h0}};$$
$$k=\sqrt{\frac{\mathbf{h0}+\delta}{\mathbf{h0}-\delta}};$$

$$\mathtt{p1[t\_]:=}\frac{(\mathbf{h0}+\delta)\mathtt{JacobiCN}[\Omega t,k^2]+(\mathbf{h0}-\delta)\mathtt{JacobiDN}[\Omega t,k^2]}{\mathtt{JacobiDN}[\Omega t,k^2]+\mathtt{JacobiCN}[\Omega t,k^2]};$$
$$\mathtt{p2[t\_]:=}\sigma\frac{2\delta}{\mathtt{JacobiDN}[\Omega t,k^2]+\mathtt{JacobiCN}[\Omega t,k^2]};$$
$$\mathtt{p3[t\_]:=}-\sigma\frac{\sqrt{2\delta}\,\mathtt{kpJacobiSN}[\Omega t,k^2]}{\mathtt{JacobiDN}[\Omega t,k^2]+\mathtt{JacobiCN}[\Omega t,k^2]};$$

B.4.14.8 Integration: case $c_0 = 0$, $h_0 > 0$

$$\Omega=\sqrt{\frac{\mathbf{h0}}{2}};$$

$$\mathtt{p1[t\_]:=}-2\Omega^2\mathtt{Csch}[\Omega t]^2;$$
$$\mathtt{p2[t\_]:=}\sigma 2\Omega^2\mathtt{Csch}[\Omega t]^2;$$
$$\mathtt{p3[t\_]:=}\sigma 2\Omega\mathtt{Coth}[\Omega t];$$

$$\mathtt{p1}'[t]-\mathtt{p2}[t]\mathtt{p3}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$
$$\mathtt{p2}'[t]-\mathtt{p1}[t]\mathtt{p3}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$
$$\mathtt{p3}'[t]+\mathtt{p2}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$

0

0

0

B.4.14.9 Integration: case $c_0 = 0$, $h_0 = 0$

$$\mathtt{p1[t\_]:=}-\frac{2}{t^2};$$
$$\mathtt{p2[t\_]:=}\sigma\frac{2}{t^2};$$
$$\mathtt{p3[t\_]:=}\sigma\frac{2}{t};$$

$$\mathtt{p1}'[t]-\mathtt{p2}[t]\mathtt{p3}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$
$$\mathtt{p2}'[t]-\mathtt{p1}[t]\mathtt{p3}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$
$$\mathtt{p3}'[t]+\mathtt{p2}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$

0

0

0

B.4.14.10 Integration: case $c_0 = 0$, $h_0 < 0$

$$\Omega=\sqrt{-\frac{\mathbf{h0}}{2}};$$

$$\mathtt{p1[t\_]:=}-2\Omega^2\mathtt{Sec}[\Omega t]^2;$$
$$\mathtt{p2[t\_]:=}-\sigma 2\Omega^2\mathtt{Sec}[\Omega t]^2;$$
$$\mathtt{p3[t\_]:=}\sigma 2\Omega\mathtt{Tan}[\Omega t];$$

$$\mathtt{p1}'[t]-\mathtt{p2}[t]\mathtt{p3}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$
$$\mathtt{p2}'[t]-\mathtt{p1}[t]\mathtt{p3}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$
$$\mathtt{p3}'[t]+\mathtt{p2}[t]\mathtt{//Simplify}\ [\#,\sigma^2==1]\ \&$$

0

0

0

B.4.14.11 Integration: case $c_0 < 0$

$$\delta=\sqrt{\mathbf{h0}^2-\mathbf{c0}};$$

$$\mathbf{a1}=1;\mathbf{b1}=0;\mathbf{c1}=-\mathbf{c0};$$
$$\mathbf{a2}=0;\mathbf{b2}=-1;\mathbf{c2}=2\mathbf{h0};$$

$$\mathtt{X1}=\mathbf{a1}x^2+2\mathbf{b1}x+\mathbf{c1};$$
$$\mathtt{X2}=\mathbf{a2}x^2+2\mathbf{b2}x+\mathbf{c2};$$

$$\mathtt{Solve}\left[(\mathbf{a1}-\lambda\mathbf{a2})(\mathbf{c1}-\lambda\mathbf{c2})-(\mathbf{b1}-\lambda\mathbf{b2})^2==0,\lambda\right]\mathtt{//Simplify;}$$
$$\lambda1=-\mathbf{h0}-\delta;$$
$$\lambda2=-\mathbf{h0}+\delta;$$

$$(\mathbf{a1}-\lambda1\mathbf{a2})(\mathbf{c1}-\lambda1\mathbf{c2})-(\mathbf{b1}-\lambda1\mathbf{b2})^2\mathtt{//Simplify}$$
$$(\mathbf{a1}-\lambda2\mathbf{a2})(\mathbf{c1}-\lambda2\mathbf{c2})-(\mathbf{b1}-\lambda2\mathbf{b2})^2\mathtt{//Simplify}$$

0

0

$$\mathtt{Solve}\left[\frac{(\mathtt{X1}-\lambda1\mathtt{X2})}{\mathbf{a1}-\lambda1\mathbf{a2}}==0,x\right]\mathtt{//Simplify;}$$
$$\mathtt{Solve}\left[\frac{(\mathtt{X1}-\lambda2\mathtt{X2})}{\mathbf{a1}-\lambda2\mathbf{a2}}==0,x\right]\mathtt{//Simplify;}$$

$$\mathtt{r1}=-\lambda1;$$
$$\mathtt{r2}=-\lambda2;$$

$$\mathtt{A1} = \frac{\lambda2(\mathtt{a1}-\lambda1\mathtt{a2})}{\lambda2-\lambda1}\mathtt{//Simplify;}$$
$$\mathtt{A2} = \frac{\mathtt{a1}-\lambda1\mathtt{a2}}{\lambda2-\lambda1}\mathtt{//Simplify;}$$
$$\mathtt{B1} = \frac{\lambda1(\mathtt{a1}-\lambda2\mathtt{a2})}{\lambda1-\lambda2}\mathtt{//Simplify;}$$
$$\mathtt{B2} = \frac{\mathtt{a1}-\lambda2\mathtt{a2}}{\lambda1-\lambda2}\mathtt{//Simplify;}$$

$$\mathtt{A1}(x-\mathtt{r1})^2+\mathtt{B1}(x-\mathtt{r2})^2\mathtt{//Simplify;}$$
$$\mathtt{A2}(x-\mathtt{r1})^2+\mathtt{B2}(x-\mathtt{r2})^2\mathtt{//Simplify;}$$

$$\{\mathtt{A1},\mathtt{B1},\mathtt{A2},\mathtt{B2}\}\mathtt{//Simplify}$$

$-\mathrm{c0}+x^2$

$2(\mathrm{h0}-x)$

$\left\{\frac{-\mathrm{h0}+\delta}{2\delta},\frac{\mathrm{h0}+\delta}{2\delta},\frac{1}{2\delta},-\frac{1}{2\delta}\right\}$

$$a=-1;$$
$$b=\sqrt{\frac{\delta+\mathbf{h0}}{\delta-\mathbf{h0}}};$$

```
Ω = Sqrt[2δ];
k = Sqrt[(δ+h0)/(2δ)];
p1[t_]:= ((h0+δ)JacobiCN[Ωt,k^2]+(h0-δ))/(JacobiCN[Ωt,k^2]+1);
p2[t_]:= σ (Ω^2 JacobiDN[Ωt,k^2])/(JacobiCN[Ωt,k^2]+1);
p3[t_]:= -σ (Ω JacobiSN[Ωt,k^2])/(JacobiCN[Ωt,k^2]+1);
```

B.4.14.12 Visualisation

```
Ha[c_,h_]:=
    ParametricPlot3D[{h - 1/2 z^2, y, z}, {y,-π,π},
      {z,-π,π}, Mesh -> 4];
Int[c_,h_]:=
    Int[c,h,{p1,p2,p3}p1 + 1/2 p3^2];

Manipulate[{
    Show[Ca[c,h],Ha[c,h],Int[c,h],Opts],
    Show[Int[c,h],
      ListPointPlot3D[{{0,0,0}},
        PlotStyle -> Directive[Red,
        PointSize[Large]]],
      ParametricPlot3D[{0,0,ν},{ν,-5,5},
        PlotStyle -> Directive[Thick,Red]],
      ParametricPlot3D[{μ,0,0},{μ,-5,0},
        PlotStyle -> Directive[Thick,Red]],
      ParametricPlot3D[{μ,0,0},{μ,0,5},
        PlotStyle -> Directive[Thick,Blue]],
      Opts]
},{{c,-1},-3,3},{{h,2},-3,3}]
```

B.4.15 The system $H_2^{(3)}$

B.4.15.1 Stability

The equilibrium states are:

```
Reduce[p2p3 == 0&&p1p3 == 0&&(p1 + p2) == 0]
(p2 == 0&&p1 == 0)||(p3 == 0&&p1 == -p2)
```

The state $e_1^0 = (0,0,0)$ is unstable:

```
p[t_]:={δExp[δt], -δExp[δt], -δ};

Norm[p[0] - {0,0,0}]//FS[#,δ > 0]&
Norm[p[t]]^2//FS[#,δ > 0&&t ∈ Reals]&
Limit[%,t -> ∞,Assumptions -> {δ > 0}]
```

$\sqrt{3}\,\delta$

$\left(1+2e^{2t\delta}\right)\delta^2$

∞

The states $e_1^\mu = (\mu,-\mu,0)$, $\mu \neq 0$ are unstable:

```
p[t_]:={μExp[δt], -μExp[δt], -δ};

Norm[p[0] - {μ,-μ,0}]//FS[#,δ > 0]&
Norm[p[t]]^2//FS[#,δ > 0&&{t,μ} ∈ Reals]&
Limit[%,t -> ∞,
    Assumptions -> {δ > 0,μ ≠ 0,μ ∈ Reals}]
```

δ

$\delta^2+2e^{2t\delta}\mu^2$

∞

The states $e_2^\nu = (0,0,\nu)$ are (spectrally) unstable:

```
D[{p2p3,p1p3,-(p1 + p2)},{{p1,p2,p3}}]
    /.{p1 -> 0,p2 -> 0,p3 -> ν}//Eigenvalues
```

$\{0,-\nu,\nu\}$

B.4.15.2 Integration: case $c_0 > 0$, $h_0 > 0$

```
Ω = Sqrt[h0/2];

p1[t_]:= -1/(4Ω^2) (4Ω^4 Csch[Ωt]^2 + c0Sinh[Ωt]^2);
p2[t_]:= -1/(4Ω^2) (4Ω^4 Csch[Ωt]^2 - c0Sinh[Ωt]^2);
p3[t_]:= -2ΩCoth[Ωt];

p1'[t] - p2[t]p3[t]//Simplify[#,h0 > 0]&
p2'[t] - p1[t]p3[t]//Simplify[#,h0 > 0]&
p3'[t] + p1[t] + p2[t]//Simplify[#,h0 > 0]&
```

0

0

0

```
q1[t_]:= 1/(4Ω^2) (4Ω^4 Sech[Ωt]^2 + c0Cosh[Ωt]^2);
q2[t_]:= 1/(4Ω^2) (4Ω^4 Sech[Ωt]^2 - c0Cosh[Ωt]^2);
q3[t_]:= -2ΩTanh[Ωt];

q1'[t] - q2[t]q3[t]//Simplify
q2'[t] - q1[t]q3[t]//Simplify
q3'[t] + q1[t] + q2[t]//Simplify
```

0

0

0

B.4.15.3 Integration: case $c_0 > 0$, $h_0 = 0$

```
Ω = Sqrt[h0/2];

Limit[{-1/(4Ω^2) (4Ω^4 Csch[Ωt]^2 + c0Sinh[Ωt]^2),
    -1/(4Ω^2) (4Ω^4 Csch[Ωt]^2 - c0Sinh[Ωt]^2),
    -2ΩCoth[Ωt]},
  h0 -> 0, Assumptions -> {c0 > 0}]
```

$\left\{-\frac{4+c0t^4}{4t^2}, \frac{-4+c0t^4}{4t^2}, -\frac{2}{t}\right\}$

```
s1[t_]:= -(4+c0t^4)/(4t^2);
s2[t_]:= -(4-c0t^4)/(4t^2);
s3[t_]:= -2/t;

s1'[t] - s2[t]s3[t]//Simplify
s2'[t] - s1[t]s3[t]//Simplify
s3'[t] + s1[t] + s2[t]//Simplify
```

0

0

0

B.4.15.4 Integration: case $c_0 > 0$, $h_0 < 0$

```
Ω = Sqrt[-h0/2];

p1[t_]:= - 1/(4Ω^2) (4Ω^4 Sec[Ωt]^2 + c0 Cos[Ωt]^2);
p2[t_]:= - 1/(4Ω^2) (4Ω^4 Sec[Ωt]^2 - c0 Cos[Ωt]^2);
p3[t_]:=2Ω Tan[Ωt];

p1'[t] - p2[t]p3[t]//Simplify[#,h0 < 0]&
p2'[t] - p1[t]p3[t]//Simplify[#,h0 < 0]&
p3'[t] + p1[t] + p2[t]//Simplify[#,h0 < 0]&
0
0
0
```

B.4.15.5 Integration: case $c_0 = 0$, $h_0 > 0$

```
Ω = Sqrt[h0/2];

p1[t_]:= - Ω^2 Csch[Ωt]^2;
p2[t_]:= - Ω^2 Csch[Ωt]^2;
p3[t_]:= - 2Ω Coth[Ωt];

p1'[t] - p2[t]p3[t]//Simplify[#,h0 > 0]&
p2'[t] - p1[t]p3[t]//Simplify[#,h0 > 0]&
p3'[t] + p1[t] + p2[t]//Simplify[#,h0 > 0]&
0
0
0
q1[t_]:=Ω^2 Sech[Ωt]^2;
q2[t_]:=Ω^2 Sech[Ωt]^2;
q3[t_]:= - 2Ω Tanh[Ωt];

q1'[t] - q2[t]q3[t]//Simplify
q2'[t] - q1[t]q3[t]//Simplify
q3'[t] + q1[t] + q2[t]//Simplify
0
0
0
w1[t_]:=σ Exp[-ς Sqrt[2h0] t];
w2[t_]:= - σ Exp[-ς Sqrt[2h0] t];
w3[t_]:=ς Sqrt[2h0];

w1'[t] - w2[t]w3[t]
    //Simplify[#,h0 > 0&&σ^2 == 1&&ς^2 == 1]&
w2'[t] - w1[t]w3[t]
    //Simplify[#,h0 > 0&&σ^2 == 1&&ς^2 == 1]&
w3'[t] + w1[t] + w2[t]
    //Simplify[#,h0 > 0&&σ^2 == 1&&ς^2 == 1]&

0
0
0
```

B.4.15.6 Integration: case $c_0 = 0$, $h_0 = 0$

```
p1[t_]:= - 1/t^2;
p2[t_]:= - 1/t^2;
p3[t_]:= - 2/t;

p1'[t] - p2[t]p3[t]//Simplify
p2'[t] - p1[t]p3[t]//Simplify
p3'[t] + p1[t] + p2[t]//Simplify
0
0
0
```

B.4.15.7 Integration: case $c_0 = 0$, $h_0 < 0$

```
Ω = Sqrt[-h0/2];

p1[t_]:= - Ω^2 Sec[Ωt]^2;
p2[t_]:= - Ω^2 Sec[Ωt]^2;
p3[t_]:=2Ω Tan[Ωt];

p1'[t] - p2[t]p3[t]//Simplify
p2'[t] - p1[t]p3[t]//Simplify
p3'[t] + p1[t] + p2[t]//Simplify
0
0
0
```

B.4.15.8 Visualisation

```
Ha[c_,h_]:=ParametricPlot3D[
    {1/2 (z + h - 1/2 x^2), 1/2 (h - 1/2 x^2 - z), x},
    {x, -3, 3}, {z, -10, 10}, Mesh -> 4];
Int[c_,h_]:=
    Int[c, h, {p1, p2, p3}p1 + p2 + 1/2 p3^2];

Opts = Join[{ViewPoint -> {62831/100000, 11781/6250, 11781/6250}},
    Opts];

Manipulate[{
    Show[Ca[c,h], Ha[c,h], Int[c,h], Opts],
    Show[Int[c,h],
      ParametricPlot3D[{0, 0, ν}, {ν, -3, 3},
        PlotStyle -> Directive[Thick, Red]],
      ParametricPlot3D[{μ, -μ, 0}, {μ, -3, 3},
        PlotStyle -> Directive[Thick, Red]],
      Opts]
}, {{c, 1}, -3, 3}, {{h, 1}, -3, 3}]
```

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