Errata

- Page 38, Line 11

  Change "for all sequences $\{\psi_k\}$ and $\{\nu_k\}$ with $|\psi_k| \leq R_x$ and $|\nu_k| \leq \epsilon_r$" to "for some arbitrary sequence $\{\psi_k\}$, $|\psi_k| \leq R_x$, and for all $\{\nu_k\}$ such that $|\nu_k| \leq \epsilon_r$".

- Page 40

  Delete last sentence.

- Page 41, Theorem 2.8

  Change "(2.24) and (2.25) hold." to "(2.24) and (2.25) hold along the trajectory $\tilde{z}(k) = x(k) - z(k)$."

- Page 50, Theorem 2.9

  Change "(2.24) and (2.25) hold." to "(2.24) and (2.25) hold along the trajectory $\hat{x}(k|k) = x(k) - e(k|k)$."

- Page 67

  Delete lines 17, 18 and 19 then insert "Since the trajectory $(x_1(k), x_2(k))$ is on the unit circle, restricting ϵ_r to the range $0 < \epsilon_r < 1$ ensures $\sigma < w_3 < 3 + 4R_x$ where $\sigma > 0$ and arbitrarily small."

- Page 68

  Change " $\lambda_{\max} \leq \ldots$" to "$\lambda_{\max} < \frac{5}{r}$." Delete sentence beginning "Now the requirements …". Change sentence beginning "With this restriction …" to "The minimum eigenvalue of $\mathcal{O}(k,1)$ satisfies $\lambda_{\min} > \frac{\sigma}{2r}$." Change "$\epsilon_r = \frac{1}{\sqrt{2}} - \sigma \quad \sigma > 0$ and arbitrarily small" to "$0 < \epsilon_r < 1$". Change "$\tilde{b}_1 = \ldots$" to $b_1 = \frac{\sigma}{2r}$". Change "$b_2 = \ldots$" to "$b_2 = \frac{5}{r}$".

- Page 69, Table 3.1

  Change "maximised by r_α" to "minimised by r_α". Change "$r_\alpha = \ldots$" to "$r_\alpha = \left(\frac{5p(q+s)\sigma}{4sq}\right)^{\frac{1}{2}}$". Change "$r_\beta = \ldots$" to "$r_\beta = \left(\frac{5q_1q_2\sigma}{2}\right)^{\frac{1}{2}}$".

Approaches to Frequency Tracking and Vibration Control

Barbara Francesca La Scala

December 1994

A thesis submitted for the degree of Doctor of Philosophy of the Australian National University

Department of Systems Engineering
Research School of Information Sciences and Engineering
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Declaration

THESE doctoral studies were conducted with Dr Robert R. Bitmead as supervisor and Dr Matthew R. James and Dr Barry G. Quinn as advisors.

The work presented in this thesis is the result of original research carried out by myself, in collaboration with others, whilst enrolled in the Department of Systems Engineering as a candidate for the degree of Doctor of Philosophy. This work has not been submitted for any other degree or award in any other university or educational institution.

Barbara La Scala
December 1994

List of Publications

A number of papers resulting from this work have been submitted to refereed journals or are in preparation.

- B. F. LA SCALA, R. R. BITMEAD AND M. R. JAMES. Conditions for the Stability of the Extended Kalman Filter and their Application to the Frequency Tracking Problem. *Mathematics of Control, Signals and Systems.* Accepted for publication, June 1993.
- B. F. LA SCALA AND R. R. BITMEAD. Design of an Extended Kalman Filter Frequency Tracker. *IEEE Transactions on Signal Processing.* Accepted for publication, April 1994.
- B. F. LA SCALA, R. R. BITMEAD AND B. G. QUINN. An Extended Kalman Filter Frequency Tracker for High-Noise Environments. *IEEE Transactions on Signal Processing.* Accepted for publication, June 1994.
- B. F. LA SCALA AND R. R. BITMEAD. A Self-Tuning Regulator for Vibration Control. *In preparation.*

Two papers have been presented at conferences. Some of the material covered overlaps with that covered in the publications listed above.

- B. F. LA SCALA AND R. R. BITMEAD. Design of an Extended Kalman Filter Frequency Tracker. *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing,* Adelaide Australia, April 1994.
- B. F. LA SCALA, R. R. BITMEAD AND B. G. QUINN. An Extended Kalman Filter Frequency Tracker for High-Noise Environments. *Proceedings of the 7th IEEE SP Workshop on Statistical, Signal and Array Processing,* Quebec Canada, June 1994.

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Abstract

THIS thesis is concerned with the development and analysis of algorithms for frequency tracking and estimation. The frequency of a sinusoidal signal embedded in noise can carry important information in many areas of signal processing and control. This makes the estimation of a constant frequency or the tracking of a time-varying frequency important issues. Two problems in this area are examined. The first is the tracking of a time-varying frequency in open-loop using the extended Kalman filter (EKF). The second is the estimation of a constant frequency for the purposes of vibration control.

The extended Kalman filter was chosen as a frequency tracker because of its widespread use as a method of deriving filters for nonlinear systems. However, a thorough understanding of its behaviour and modes of failure was not available. Accordingly the stability of the EKF as an observer for nonlinear systems is examined. A new result giving sufficient conditions for bounded-input bounded-output stability of the EKF when applied to stochastic, discrete-time systems is presented. This extends previous results which were available only for continuous time and deterministic systems. The result also allows the development of theoretically supported design guidelines.

Following the stability analysis of the EKF, design guidelines for constructing EKF-based observers are presented. This section collects previously known results, as well as the new guidelines which can be derived from the new stability result. These guidelines are used to construct EKF-based frequency trackers for high strength signals as well as weak, narrowband signals. This work also illustrates the flexibility of the EKF approach to nonlinear observer design by demonstrating how the particular features of a problem

can be incorporated into the design of the filter, allowing for highly accurate estimates even at low signal-to-noise ratios.

The control problem examined is that of eliminating a vibrational disturbance from the output of a linear, time-invariant and unknown system using adaptive control. Theory is presented which shows that it is possible to design an adaptive controller which will converge to a stable controller which regulates the system. Moreover, it is shown that in this regime it is possible to estimate consistently the frequency of the disturbance. This contrasts with the well-known bias that occurs when estimating the frequency of a sinusoid in open-loop.

The results presented here extend the theoretical knowledge for nonlinear observer design in general, as well as in the particular areas of frequency tracking and estimation. All theoretical results are illustrated by numerical examples which demonstrate their conclusions.

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1

Introduction

1.1 Problem Description

IN many problems in both control and signal processing the frequency of some signal encodes important information. One obvious example is an FM radio signal where information is transmitted via modulation of the carrier frequency. In radar signal processing the aim is to recover information on the velocity and range of a target from the doppler shift in the reflected microwave signal. Monitoring the changing frequency of the waveforms produced by rotating machinery can yield information on the rate of mechanical wear, as well as providing measurements which can be used to control the plant via a feedback mechanism. Improvements in efficiency and pollution control and the elimination of "knocking" in internal combustion engines requires just such processing (Böhme and König, 1994).

Like many interesting open problems, estimating the frequency of a sinusoidal signal in noise is not an easy one. Systems which output periodic signals have the curious feature of being neither stable (in the sense of Lyapunov) nor unstable (in the sense of Lagrange). Once such a *critically stable* system has been set going it will continue to output the same sinusoidal signal forevermore, neither diminishing nor expanding in amplitude. Typically, oscillations result from nonlinear systems but their high energy fundamental modes are best modelled, analysed and controlled using critically stable linear systems.

Such systems are awkward, in-between cases which are often ignored. It is not uncommon to encounter estimation or control algorithms which consider the two cases of asymptotically stable or unstable systems, but which neglect the third, troublesome case of critically stable systems. The reason control and estimation methods based on L_p optimisation face difficulties when dealing with critically stable systems and periodic signals is that the cost function is not continuous with respect to the parameters. If a sinusoid is not precisely accounted for, then its energy contribution to the cost function is infinite. If it is correctly accounted for then it does not contribute to the cost function. For this reason L_p gain methods such as LQ, least squares and H_∞ all experience difficulties.

The problem of estimating the parameters of a constant frequency sinusoid in noise can be posed in a number of ways. It can be written as a linear, state space problem as follows

$$\begin{aligned}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\
y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + e(k).
\end{aligned}$$

Alternatively, it can be posed as the following input-output problem

$$y(k) = 2\cos(\omega)y(k-1) - y(k-2) + e(k).$$

In both cases $y(k) = \cos(k\omega + \varphi) + e(k)$ and the frequency, ω, can be recovered from the estimates of the linear parameters. There are a number of difficulties in estimating the parameters in either formulation.

Firstly, consider estimating the parameters, $\{a_i\}$, of the auto-regressive (AR) model

$$y(k) = a_1 y(k-1) + a_2 y(k-2) + \dots a_n y(k-n) + e(k)$$

where $E[e(k)] = 0$ and $E[e(k)^2] = 1$. This can be written as the linear estimation problem

$$y(k) = \phi(k-1)^T \theta + e(k) \tag{1.1}$$

where

$$\begin{aligned} \theta^T &= [a_1, a_2, \dots, a_n] \\ \phi(k-1)^T &= [y(k-1), y(k-2), \dots, y(k-n)]. \end{aligned}$$

It can also be written in state space form as

$$\theta(k+1) = \theta(k) \tag{1.2}$$

$$y(k) = \phi(k-1)^T \theta(k) + e(k). \tag{1.3}$$

The recursive least-squares (RLS) equations for (1.1) and the Kalman filter equations for (1.2)–(1.3) are identical since both estimation algorithms calculate the least-squares estimate of θ. Both algorithms are given by the set of equations

$$\begin{aligned} \hat{\theta}(k+1) &= \hat{\theta}(k) + K(k)\left(y(k) - \phi(k)^T \hat{\theta}(k)\right) \\ K(k) &= \frac{P(k)\phi(k)}{\phi(k)^T P(k)\phi(k) + 1} \\ P(k+1) &= P(k) - \frac{P(k)\phi(k)\phi(k)^T P(k)}{\phi(k)^T P(k)\phi(k) + 1} \end{aligned}$$

A known property of the RLS algorithm is that $K(k) \to 0$ as $k \to \infty$ and this property must naturally be shared by the Kalman filter algorithm when it is used to estimate parameters of autoregressive models (1.2)–(1.3). Thus, in both formulations of the constant frequency estimation problem, the least-squares estimator, however posed, will ultimately ignore the data. Standard least-squares estimation is, therefore, not a robust method for the frequency estimation problem.

The inclusion of a zero mean noise term, $w(k)$ with variance Q, in the state equation (1.2), or equivalently modifiying the RLS algorithm to prevent $\lim_{k\rightarrow\infty} K(k) = 0$, will alleviate this lack of robustness for most AR models. The cases where this performance improvement will not occur are those where $[F, Q^{\frac{1}{2}}]$ has uncontrollable modes on the unit circle. The lack of uncontrollable modes on the unit circle is a necessary condition for the maximum limiting value of $P(k)$ to be non-zero (de Souza *et al.*, 1986). Thus, the inclusion of dither or noise in the state dynamics of the frequency estimation problem will not eliminate the lack of robustness of estimators using a least-squares criterion.

Another problem in least-squares frequency estimation is that of bias. This is more clearly seen when considering the input-ouput formulation of the problem. When estimating the parameters of an AR model with unit circle zeros, equation error estimation algorithms, such as recursive least squares, will yield biased solutions (Mendel, 1973; Johnson, Jr. and Hamm, 1979). Moreover, in the case of constant frequency estimation, the bias is a function of the unknown parameter, $a_1 = 2\cos(\omega)$ (Johnson, Jr. and Hamm, 1979), which makes it difficult to eradicate.

When we extend the frequency estimation problem from merely estimating the parameter $2\cos(\omega)$, to estimating directly the frequency of the periodicity, ω, our task becomes more difficult still as the problem is now nonlinear. If the frequency varies with time the difficulty of the problem increases yet again. When there is more than one periodic component in the measured signal the issues of identification of the correct number of tones and the separation of tones closely spaced in frequency must also be taken into account. Sometimes it is not possible to measure the sinusoidal signal directly but only the combination of the sinusoid and some other signal. In fact, the sinusoid may be considered a disturbance corrupting the desired output signal.

In this thesis two general problems are considered, one an open-loop and one a closed-loop estimation problem. The first is of estimating the frequency of a single tone, sinusoidal signal in noise. This problem is termed *frequency estimation* if the frequency is constant and *frequency tracking* if the frequency varies with time. Only the open-loop frequency tracking problem is examined here although many of the results carry over to estimation. The second problem is that of eliminating a sinusoidal disturbance of unknown frequency from the output of a plant using an adaptive controller. This is a *vibration control* problem. In both cases, methods based on a least-squares criterion will be used.

In spite of the difficulties in estimating the constant frequency of a sinusoidal signal, the ubiquity of the *frequency estimation* problem has spawned a myriad of techniques to deal with it. For high signal-to-noise ratio (SNR) regimes simple, non-parametric methods such as those based on counting the zero crossings of the signal can be used. For signals with a lower SNR there is a range of more sophisticated techniques which make use of the peculiar features of a particular frequency estimation problem to improve performance. For the case of the *frequency tracking* problem there are fewer options. In both cases, the choice of which method to employ depends to a large degree on the nature of the given problem to be solved.

The next section gives a brief overview of the common methods used for tracking the frequency of a single sinusoid in noise [1]. Following this is an overview of the problem of designing adaptive controllers for linear, time-invariant plants that converge to optimal controllers. The final section gives an outline of this thesis, detailing the problems considered and the proposed methods of solution.

1.2 Common Frequency Tracking Algorithms

Probably the best known application of frequency tracking techniques is that of the demodulation of FM signals using the phase-locked loop (PLL). The application of the PLL to demodulation appears to have begun with Appleton (1922). Since it was first proposed the analog PLL has been extensively studied, see for example Viterbi (1966) or Gardner (1968). Somewhat more recently digital implementations of the PLL have also been examined (Kelly and Gupta, 1972; Polk and Gupta, 1973).

Another popular technique for frequency tracking makes use of hidden Markov models (HMMs). In this approach, at any instant the system is assumed to take one of a discrete, finite set of frequency values. The probability of the system taking on a value

[1]Many of these can be extended to the case of multiple sinusoids in noise also.

in the set at a given time instant is assumed to depend solely on the state of the system (i.e. frequency value) at the previous instant, hence the problem is Markovian. The measured output signal of the system is the sinusoid corrupted by channel noise. The HMM frequency tracker was introduced by Streit and Barrett (1990). In their algorithm the measurement sequence was also assumed to take on values from a finite set of discrete outcomes. This work has been extended by Barrett and Holdsworth (1993) to the case where the measured output is a continuous variable. The relationship between HMMs and neural networks has been explored by Adams and Evans (1994) and a neural network based frequency line tracker proposed. Hidden Markov model techniques are clearly best suited to frequency tracking problems where the set of possible frequency values is naturally discrete, such as the case of frequency-shift key (FSK) modulated signals. These methods are also applicable to cases where it is known that the frequency will vary in some given range such as in the case of sonar tracking. In such cases the frequency variable can be discretized to an appropriate degree of accuracy.

For signals whose frequency varies slowly enough with respect to the sampling rate that the frequency can be considered constant over a reasonable time period, such as sonar signals, it is possible to use block frequency estimation techniques. In this approach, the signal is divided into possibly over-lapping segments and the signal characteristics, including frequency, are treated as constant within blocks. A frequency estimation algorithm is then used on each subsequence. Such algorithms include maximum likelihood methods (Quinn and Fernandes, 1991; Starer and Nehorai, 1992), methods based on the Fourier coefficients of the blocked signal (McMahon and Barrett, 1986; Quinn, 1994) or weighted linear predictor estimators (Lank *et al.*, 1973; Kay, 1989; Lovell and Williamson, 1992; Clarkson *et al.*, 1994).

Another approach, widely discussed in the literature, is that of the use of time frequency representations (TFRs). The aim of this approach is to display the energy of the signal as a function of both time and frequency in the manner of a joint probability density for a bivariate random variable. The first moments of such representations can be used as estimators of instantaneous frequency. Unfortunately, it has been shown (Lowe, 1986) that such a joint probability density as a function of time and frequency does not exist. In spite of this, a TFR which has the spectrum and instantaneous power as its marginal densities is termed a time-frequency distribution (TFD). Such TFDs have spurious cross terms which imply that there exists energy in the signal where there is none. The use of TFRs for spectral analysis has been examined by Lovell (1990). It has been shown that the most appropriate TFR for this use is simply the short time Fourier transform (STFT) (Lovell *et al.*, 1993). In the case of the STFT the spurious cross terms coincide with the true peaks in the TFD, thus distorting the magnitude but not the location of these peaks. Other Cohen-class TFDs do not have this property. Moreover, it has been shown that estimators based on the moments of common TFDs are arithmetically equivalent to those produced by weighted linear predictor methods, are computationally more complex and have a higher variance (Kootsookos *et al.*, 1992; Lovell and Williamson, 1992). Thus TFRs, other than the short-time Fourier transform, are of little practical use.

One more technique for constructing frequency trackers, and indeed observers for nonlinear systems in general, is that of using an extended Kalman filter (EKF). As the name implies, the EKF is an extension of the well-known Kalman filter, which may be applied to linear systems, to the case of nonlinear systems. Briefly an EKF estimate is obtained by linearising a nonlinear state space model about the current state estimate and then producing an updated state estimate using the gain calculated from a Kalman filter on the resulting linearised system. While the EKF no longer possesses the optimality properties of the Kalman filter it is still a valuable technique. A derivation of the EKF and an example of applying it to the frequency tracking problem is given in Anderson and Moore (1979, Chapter 8). A more extensive examination of the EKF for frequency tracking is given in Parker and Anderson (1990). The performance of an EKF for both single and multiple tone frequency tracking is analysed using averaging analysis by James (1992).

1.3 Self-Tuning Regulators

Consider the problem illustrated in Figure 1.1 where there is a plant, P, the output of which is perturbed by some disturbance process characterised by H. Suppose the plant and disturbance process are parameterised by some unknown vector θ^0. The aim is to estimate θ^0 from the closed-loop output of the plant and to use these estimates to design an adaptive controller, C, which will reject the disturbance, v, and make the output of the plant, y, track some reference trajectory, y^*.

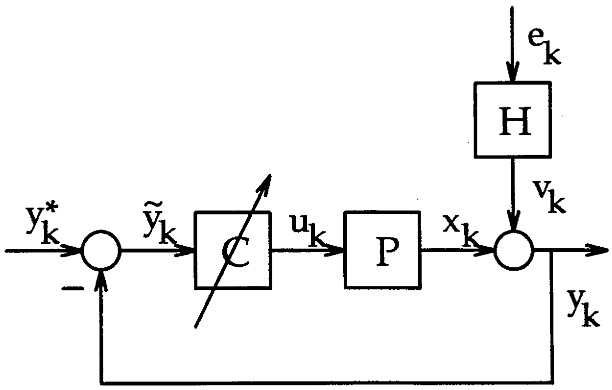

Figure 1.1: Disturbance rejection via adaptive control.

There are several questions to be answered in such a problem.

Stability The first and foremost question is that of stability of the closed-loop system. Is the computed control input, $\{u_k\}$, bounded and does it produce a bounded output, $\{y_k\}$?

Convergence The next question to consider is do the parameter estimates $\hat{\theta}_k$ converge to some limiting value $\bar{\theta}$?

Self-Optimisation If the parameter estimates converge, does the limiting adaptive controller, $C(\bar{\theta})$, satisfy the performance measure for the control law? If so, the system is said to be self-optimising.

Self-tuning If, in addition to satisfying the performance criterion, the adaptive controller, $C(\hat{\theta}_k)$, converges to the optimal controller, $C(\theta^0)$, then the system is said to be self-tuning.

Consistency If $\lim_{k\to\infty} \hat{\theta}_k = \theta^0$ then the parameter estimates are consistent and the self-tuning property follows immediately.

Systems which are self-tuning when the reference trajectory, $\{y_k^*\}$, is identically zero are termed *self-tuning regulators*. When the reference trajectory is non-zero they are termed *self-tuning trackers*.

The problem, as stated above, is impossibly general. Even when the problem is restricted to particular types of systems, control laws and estimation methods the problem is still very difficult. A well-known example of a self-tuning tracker was first proposed by Åström and Wittenmark (Åström and Wittenmark, 1973; Åström *et al.*, 1977). They posed the self-tuning regulator problem for linear time-invariant systems described by an ARMAX model, where the plant, P, was stably invertible and the disturbance transfer function, H, was stable. They employed a certainty equivalence minimum-variance controller and estimated the unknown system parameters using recursive least-squares. Experimental results suggested that this system was self-tuning. However, even for this very particular formulation of the self-tuning problem, the question of whether Åström and Wittenmark's self-tuning regulator (STR) was accurately named remained open for almost 20 years.

In order to obtain answers to the questions above a simplified version of Åström and Wittenmark's self-tuning regulator was first examined. Goodwin *et al.* (1980) showed that, for deterministic systems (that is when $v_k \equiv 0$ for all k) and when the stochastic approximation algorithm was used in place of recursive least-squares, the system was stable and self-optimising. Bittanti *et al.* (1990) showed that this form of the STR was stable and self-optimising as a regulator and as a tracker when the parameter estimation was performed using any one of a family of recursive least-squares based algorithms.

The first result for stochastic systems (that is with a non-zero, random disturbance $\{v_k\}$) was that of Becker *et al.* (1985). They showed that the form of the STR which used the stochastic approximation estimation algorithm in place of least-squares was a convergent and stable self-tuning regulator. They also showed that the parameter estimates were not consistent but converged to a random multiple of the true parameters when the optimal controller was irreducible.

The STR as posed by Åström and Wittenmark was eventually shown to be convergent, stable and self-optimising by both Radenković (1990) and Guo (1993) using different approaches. It was also shown that the parameter estimates will be consistent, and hence the STR self-tuning, if the optimal controller is irreducible (Radenković, 1990; Guo, 1993).

The problem posed by Åström and Wittenmark has now been extended to consider linear, time-invariant systems, least-squares estimation and a variety of control strategies. Becker *et al.*, Radenković and Guo used stochastic Lyapunov functions and Martingale theory in their solutions. Kumar (1990) used an alternative method of Bayesian embedding (Sternby, 1977; Rootzen and Sternby, 1984) to show that the recursive least-squares algorithm, when used in closed-loop, would converge and be consistent, independently of the control strategy used except for an exceptional set of θ^0 with Lebesgue measure zero. Moreover this result did not require assumptions on the transfer functions of the plant, P, and disturbance process, H, other than those of linearity and time-invariance. Thus it is possible to show that the STR is a self-tuning regulator and tracker for a variety of control laws provided the true system, θ^0, is not in the exceptional set. Unfortunately, the parameters for a system which has poles on the unit circle, such as systems with a sinusoidal disturbance, lie in this exceptional set (Nassiri-Toussi and Ren, 1994).

1.4 Thesis Outline

This thesis considers frequency estimation in both open and closed loop. Firstly, it examines the use of the extended Kalman filter (EKF) to estimate the time-varying frequency of a noisy sinusoidal signal. The closed loop problem considered is that of eliminating a sinusoidal disturbance from the output of a linear, time-invariant plant.

This thesis concentrates on the use of the extended Kalman filter for frequency tracking as the EKF is one of the most flexible tools for this task. By the suitable construction of a state space signal model (and possible pre-filtering of the measurement signal) an EKF tracker can be designed for a wide variety of frequency tracking problems. Before the EKF can be used intelligently it is necessary to understand the modes of failure of the EKF and their causes. Accordingly, in Chapter 2 the EKF is presented and then an overview is given of what is known of the stability of the EKF. The bulk of this chapter is then concerned with presenting a new stability result for the EKF when applied to discrete-time, stochastic systems which have nonlinear state dynamics. This theorem gives sufficient conditions for the boundedness of the errors of the EKF given a sufficiently good initial state estimate. As a corollary, this result also gives conditions under which the errors of the EKF will converge to zero when it is applied to a deterministic, discrete-time nonlinear system.

In Chapter 3 a general EKF-based frequency tracker is presented and general, *ad hoc* tuning guidelines for EKF systems are given. In addition, the EKF tracker is shown to satisfy the conditions of the result given in Chapter 2. As a result, theoretically based tuning guidelines for the EKF frequency tracker are also derived. The performance of this frequency tracker is illustrated with simulation results.

The flexibility of the EKF-based approach is illustrated in Chapter 4 by considering the problem of passive sonar tracking. In this application the frequency of the acoustic signals suffers a doppler shift which is a function of the velocity of the target. Since the sonar detectors are passive the SNR of such signals is very low (in the range -20 to -30 dB). General EKF-based frequency trackers are known to suffer from a threshold effect for signal with SNR below 5–6 dB (James, 1992). That is, the mean square error of the state estimates increases dramatically below the threshold SNR value. To overcome this problem a new state-space model is proposed which incorporates the particular characteristics of sonar signals. In this way the effective SNR of the signal is increased and the resulting EKF tracker is able to estimate the instantaneous frequency with reasonable accuracy. Simulations illustrate the performance of this low SNR frequency tracker.

A vibration control problem is considered in Chapter 5. The problem examined is that of a linear, time-invariant plant whose output is corrupted by a sinusoidal disturbance of unknown frequency. The aim is to design an adaptive, feedback control system to eliminate the disturbance and drive the plant output to zero. The adaptive control system used is that of a minimum-variance certainty equivalence controller coupled to a parameter estimator. This is an extension of Åström and Wittenmark's self-tuning regulator to the case when the disturbance process is critically stable rather than strictly stable, thus allowing for certain types of deterministic disturbances as well as purely stochastic perturbations. In this chapter a convergence proof is given for the vibration control problem using a modified least-squares algorithm. This result shows that by embedding the estimation problem in closed-loop, the known bias that occurs in frequency estimation by this method in open-loop is eliminated and the output of the plant can be regulated.

The final chapter in this thesis summarises the contributions in this work and discusses avenues for further research.

2

Stability of the Extended Kalman Filter

2.1 The Extended Kalman Filter

In this chapter the extended Kalman filter (EKF) is introduced and its stability is dicussed. We wish to determine under what conditions the estimation errors in the EKF will tend to zero (*asymptotic stability*) or be bounded for all time (*bounded-input bounded-output (BIBO) stability*). This section is devoted to an overview of the EKF and the derivation of its associated error dynamics. The following section reviews previous work on the stability of the EKF. The rest of the chapter is devoted to a new result on the stability of the EKF when applied to a stochastic, discrete-time system.

The extended Kalman filter is a tool for estimating the state of a system which is described by a nonlinear state space model. The EKF state estimates are an approximation to the mean of the conditional density of the state $\{x(k)\}$ given the measurements $\{y(1), \ldots, y(k)\}$. Jazwinski (1970) derives this conditional density for both continuous time and discrete time systems and gives several approximations to the moments of these densities, including the EKF. He discusses the effect of the neglected terms in these approximations but does not give any strict results.

The EKF is derived by linearising the signal model about the current predicted state estimate and then using the Kalman filter on this linearised system to calculate a gain matrix. This gain matrix, along with the nonlinear signal model and new signal measurement, is used to produce the filtered state estimate and then an estimate of the state at the next time instant. For a nonlinear signal model of the form

$$x(k+1) = f(x(k)) + w(k) \tag{2.1}$$

$$y(k) = h(x(k)) + v(k) \tag{2.2}$$

where $E[w(k)w(k)^T] = Q(k)$ and $E[v(k)v(k)^T] = R(k)$, the equations for the EKF are (Anderson and Moore, 1979, Chapter 8)

Measurement Update:

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K_{\hat{x}}(k)[y(k) - h(\hat{x}(k|k-1))] \tag{2.3}$$

$$P_{\hat{x}}(k|k) = [I - K_{\hat{x}}(k)H_{\hat{x}}(k)]P_{\hat{x}}(k|k-1) \tag{2.4}$$

Time Update:

$$\hat{x}(k+1|k) = f(\hat{x}(k|k)) \tag{2.5}$$

$$P_{\hat{x}}(k+1|k) = F_{\hat{x}}(k)P_{\hat{x}}(k|k)F_{\hat{x}}(k)^T + Q(k) \tag{2.6}$$

where

$$K_{\hat{x}}(k) = P_{\hat{x}}(k|k-1)H_{\hat{x}}(k)^T[H_{\hat{x}}(k)P_{\hat{x}}(k|k-1)H_{\hat{x}}(k)^T + R(k)]^{-1} \tag{2.7}$$

$$F_{\hat{x}}(k) = \left.\left[\frac{\partial f_i}{\partial x_j}\right]\right|_{x=\hat{x}(k|k)} \tag{2.8}$$

$$H_{\hat{x}}(k) = \left.\left[\frac{\partial h_i}{\partial x_j}\right]\right|_{x=\hat{x}(k|k-1)} \tag{2.9}$$

and $\hat{x}(k|k)$ is the estimate of the state at time k and $\hat{x}(k+1|k)$ is the prediction of the state at time $k+1$ using all the observations up to and including $y(k)$. The matrices $P_{\hat{x}}(k|k)$ and $P_{\hat{x}}(k+1|k)$ are approximations of the respective state estimate error covariances. [1]

[1] The notation used here explicitly shows the dependence in the EKF equations on a particular trajectory in the state space. This is non-standard but will make the later stability arguments clearer.

This thesis is concerned primarily with problems of frequency estimation and associated issues and most common models for systems with time-varying frequency are linear in either the output map (2.2) or state dynamics (2.1). Thus for the remainder of this chapter only systems with a linear output map, i.e. systems with $y(k) = H(k)x(k) + v(k)$ are considered. Stability for signal models which have linear state dynamics and a nonlinear output map can be derived in a similar fashion to the following.

Define the error in the filtered and predicted state estimates as $e(k|k)$ and $e(k|k-1)$ respectively. Thus

$$e(k|k) \triangleq x(k) - \hat{x}(k|k) \tag{2.10}$$

$$e(k|k-1) \triangleq x(k) - \hat{x}(k|k-1) \tag{2.11}$$

From (2.3)

$$e(k|k) = [I - K_{\hat{x}}(k)H(k)]e(k|k-1) - K_{\hat{x}}(k)v(k)$$

where

$$\begin{aligned} e(k+1|k) &= f(x(k)) + w(k) - f(\hat{x}(k|k)) \\ &= f(x(k)) + w(k) - f(x(k) - e(k|k)) \\ &= f(x(k)) + w(k) - f(x(k)) + \frac{\partial f}{\partial x}(x(k)) \cdot e(k|k) - \kappa_f(x(k), -e(k|k)) \\ &= \frac{\partial f}{\partial x}(x(k)) \cdot e(k|k) - \kappa_f(x(k), -e(k|k)) + w(k) \end{aligned}$$

and κ_f is the remainder term from the Taylor series expansion of f, i.e.

$$\kappa_f(a,b) = f(a+b) - f(a) - \frac{\partial f}{\partial x}(a) \cdot b.$$

Therefore

$$\begin{aligned} e(k|k) = {} & [I - K_{\hat{x}}(k)H(k)]F_x(k-1)e(k-1|k-1) \\ & -[I - K_{\hat{x}}(k)H(k)]\kappa_f(x(k-1), -e(k-1|k-1)) \\ & +[I - K_{\hat{x}}(k)H(k)]w(k-1) - K_{\hat{x}}(k)v(k) \end{aligned} \tag{2.12}$$

where

$$F_x(k) = \frac{\partial f}{\partial x}(x(k)).$$

Thus the dynamics for the filtering error of the EKF may be written as the sum of the error dynamics for the deterministic case, neglecting linearisation errors, and nonlinear perturbation terms driven by the noise processes and remainder term from the Taylor series expansion of the nonlinearity in the signal model.

In the case of a nonlinear output map and linear state dynamics the stability of the prediction error of the EKF needs to be examined. In such cases the prediction error dynamics are

$$\begin{aligned} e(k+1|k) &= F(k)[I - K_{\hat{x}}(k)H_x(k)]e(k|k-1) \\ &\quad + F(k)K_{\hat{x}}(k)\kappa_h(x(k), -e(k|k-1)) \\ &\quad - F(k)K_{\hat{x}}(k)v(k) + w(k) \end{aligned}$$

where

$$H_x(k) = \frac{\partial h}{\partial x}(x(k)).$$

In the fully nonlinear case the filtered error dynamics are given by the equation

$$\begin{aligned} e(k|k) &= [I - K_{\hat{x}}(k)H_x(k)]F_x(k)e(k-1|k-1) \\ &\quad + [I - K_{\hat{x}}(k)H_x(k)]\{w(k-1) - \kappa_f(x(k-1), -e(k-1|k-1))\} \\ &\quad - K_{\hat{x}}(k)\{v(k) + \\ &\quad \kappa_h(x(k), -F_x(k-1)e(k-1|k-1) + \kappa_f(x(k-1), -e(k-1|k-1)) - w(k))\}. \end{aligned}$$

From this equation it can be seen how the linearisation error in the state dynamics, κ_f, inflates the apparent noise in the state dynamics. Similarly, the error in the linearisation of the output map, κ_h, inflates the contribution due to "noise" in the output equation. It can also be seen how the smoothness properties of the nonlinear functions, f and h, will affect the error dynamics. If the linearisation error terms are zero then the error dynamics of the EKF reduce to those of the linear, Kalman filter. If the nonlinearites are smooth (and hence κ_f and κ_h small) the error dynamics of the EKF will be close to those of the Kalman filter. However, note that for the fully nonlinear case, the remainder terms are also effected by the state noise due to the presence of a $w(k)$ term in the argument of κ_h. Thus in this case the degree of noise in the state is also crucial to the analysis.

In the next section we will review what is already known about the dynamics of the errors of the EKF. Knowledge of these dynamics is crucial to an understanding of when and why the EKF will fail or be successful.

2.2 Previous Results

The extended Kalman filter is often used to design observers for nonlinear filtering problems in spite of the fact that there are few theoretical results to indicate when such a design will be successful. In practice it has been found that when the nonlinearities are not significant such a design will often work but, until recently, more precise guidelines have not been available. This was due to the nonlinearities and dependence of the gain calculations on previous state estimates, which make a general stability analysis of the EKF extremely difficult.

One approach to overcoming these difficulties is to concentrate on applying the EKF to a particular application. Any analysis of the performance of the EKF can then make use of the structure of the application to simplify the analysis. [2]

An alternative approach is to consider modified versions of the EKF algorithm (perhaps applicable to particular classes of nonlinear functions) which are more amenable to analysis. For *cone-bounded* functions, i.e. the class of nonlinear systems where the nonlinearities are uniformly Lipschitz in the state variable, a nonlinear observer has been proposed and analysed (Gilman and Rhodes, 1973; Rhodes and Gilman, 1975). A bounding linear system for such systems can be found. By calculating the gain using this nominal linear system the dependence of the gain on previous state estimates is removed and it is possible to derive upper and lower bounds for the estimation error covariance.

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[2] See for example Moorman and Bullock (1991) and Mehra (1971).

Safonov and Athans (1978) considered using a constant gain in the EKF. Their motivation for a constant gain EKF (CGEKF) was primarily to reduce computational complexity, however it also allowed a theoretically based stability analysis to be performed. They derived a sufficient condition for the stability of the CGEKF and described an *ad hoc*, iterative design procedure.

Song and Speyer (1985; 1986) proposed a modified gain EKF (MGEKF) for *modifiable* functions, i.e. functions for which an exact expansion of $f(a+b)$ about $(a+b)$ can be found. By incorporating this expansion, rather than the first order Taylor series approximation, in the EKF equations it is possible to derive global stability conditions for the MGEKF. Unfortunately these conditions are not verifiable analytically. Pachter and Chandler (1993) proposed a method for deriving an approximation to the appropriate expansion of $f(a+b)$ for smooth nonlinearities using a Maclaurin series expansion of f.

Ahmed and Radaideh (1994) also proposed a modified EKF (MEKF). They considered stochastic, continuous-time systems and expanded the nonlinearities about the solution of the deterministic, undriven state equation. As the solution $\{\bar{x}(t)\}$ of

$$\frac{d\bar{x}(t)}{dt} = f(t, \bar{x}(t))$$

is a known, deterministic sequence it is possible to derive upper bounds on the error between the deterministic solution and the true state, $\tilde{x}(t) = x(t) - \bar{x}(t)$. Moreover the resulting MEKF equations for $\{\tilde{x}(t)\}$ are linear in the state given the known sequence $\{\bar{x}(t)\}$. Thus their MEKF is the optimal linear estimator of $\tilde{x}$. The authors claim that as

$$E[x(t)|y(1), \ldots, y(t)] \approx \bar{x}(t) + \hat{\tilde{x}}(t) \tag{2.13}$$

where $\hat{\tilde{x}}(t)$ is the estimate of $\tilde{x}(t)$ produced by their MEKF, the MEKF is more accurate than the standard EKF. However they make no comment on the accuracy of the approximation in (2.13).

Chui *et al.* (1990) examine the case of using the EKF to estimate both the parameters as well as the state of a linear system. They propose a parallel algorithm where the states of the linear system are estimated via a linear, Kalman filter for a given sequence of parameter estimates. This sequence of parameters estimates is produced from an EKF using the state estimates from the Kalman filter. In other words, let $\hat{x}_1(k)$ be the estimate of the state of the system at time k and $\hat{x}_2(k)$ be the estimate of the parameters of the system at time k, then $\hat{x}_1(k+1)$ is given by a Kalman filter run on the linear system

$$x_1(k+1) = f_1(\hat{x}_2(k))x_1(k) + w_1(k) \tag{2.14}$$

and $\hat{x}_2(k+1)$ from an EKF run on the nonlinear system

$$x_2(k+1) = f_2(\hat{x}_1(k), x_2(k)) + w_2(k) \tag{2.15}$$

where $\{w_1(k)\}$ and $\{w_2(k)\}$ are white noise processes. The authors claim that as the estimates of $\{x_1(k)\}$ given by the Kalman filter run on the system (2.14) are the optimal estimates of $x_1(k)$ given $\{y(1), \ldots, y(k), \hat{x}_2(1), \ldots, \hat{x}_2(k)\}$, the estimates given by the coupled system (2.14)–(2.15) will be more accurate than those given by the standard EKF. This is due to the fact that the linearisation about $\hat{x}_1(k)$ in the EKF for the system given by (2.15) will be more accurate than the linearisation about the generic EKF estimate of x_1 since $\hat{x}_1(k)$ is the optimal estimator of x_1.

The first theoretically based analysis of the standard EKF in a reasonably general setting arose from Ljung's examination of the EKF as a parameter estimator (Ljung, 1979). He examined the asymptotic behaviour of the EKF when it is used to estimate the unknown parameters, as well as the state, of a linear, time-invariant system. The analysis was performed by approximating the dynamics of the errors in the EKF estimates by an ordinary differential equation and examining the limiting points of this equation. He concluded that, in general, the parameter estimates given by the EKF would be biased due to errors in modelling precisely the noise structure of the system. He proposed a modified algorithm, based on stochastic gradient descent arguments, which has improved convergence properties. The modifications are similar to the *ad hoc* process of tuning the EKF by varying the noise covariance matrices. [3]

[3]See Jazwinski (1970) for a discussion on the merits of tuning the EKF in this way.

More recently there have been several attempts to quantify the performance of the EKF for general nonlinear problems. Baras *et al.* (1988) have given conditions under which the EKF will be a locally asymptotic observer when applied to a deterministic, continuous time system. That is, they have shown that the errors in the state estimates will tend to zero if the initial error is small enough under appropriate conditions. Following from the work in Baras *et al.* (1988), Song and Grizzle (1993) have demonstrated a similar result in the discrete time case. In addition Picard (1991) has considered the case of applying the EKF to continuous time, stochastic systems with high signal-to-noise ratios. He gives conditions under which the scaled EKF filtering error will be asymptotically optimal and provides a bound for the EKF estimation error in terms of the SNR and computed error covariance.

In the following subsections the theorems of Baras *et al.* (1988), Song and Grizzle (1993) and Picard (1991) are given and then briefly discussed.

2.2.1 Deterministic, Continuous Time Systems

Baras, Bensoussan and James (1988) examined this case. Their results were the following. Consider the deterministic, nonlinear, continuous time system

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \qquad x(0) = x_0 \\ y(t) &= Hx(t) \end{aligned}$$

The extended Kalman filter for such a system is given by the equations

$$\begin{aligned} \dot{m}(t) &= f(m(t)) + N(t)H^T R^{-1}[y(t) - Hm(t)], \qquad m(0) = m_0 \\ \dot{N}(t) &= \frac{\partial f}{\partial m}(m(t))N(t) + N(t)\frac{\partial f}{\partial m}^T(m(t)) - N(t)H^T R^{-1} H N(t) + Q, \\ & \qquad N(0) = N_0 = P_0^{-1} \end{aligned}$$

Given the following definitions their results can be stated in two theorems.

Definition 2.1 *The pair* $[H, F(x)]$ *is* uniformly detectable *if there exists a bounded Borel matrix-valued function* $\Lambda(x)$ *such that*

$$\eta^T(F(x) + \Lambda(x)H)\eta \leq -\alpha_0|\eta|^2, \qquad \alpha_0 > 0$$

for all $x, \eta \in \mathbb{R}^n$.

Definition 2.2 *The pair* $[F(x), Q]$ *is* uniformly controllable *if there exists a bounded Borel matrix-valued function* $\Gamma(x)$ *such that*

$$\lambda^T(F(x) + Q\Gamma(x))\lambda \geq \beta_0|\lambda|^2, \qquad \beta_0 > 0$$

for all $x, \lambda \in \mathbb{R}^n$.

Theorem 2.1 (Theorem 7, (Baras *et al.*, 1988)) *If*

1. $[R^{-\frac{1}{2}}H, \frac{\partial f}{\partial x}(x)]$ *is uniformly detectable; and*
2. $[\frac{\partial f}{\partial x}(x), Q]$ *is uniformly controllable*

then

$$\begin{aligned} \|N(t)\| &\leq \|N_0\| + \frac{\|Q^{\frac{1}{2}}\|^2 + \|\Lambda\|^2}{2\alpha_0} \\ &\triangleq q \\ \|P(t)\| &= \|N^{-1}(t)\| \\ &\leq \|P_0\| + \frac{\|R^{-\frac{1}{2}}H\|^2 + \|\Gamma\|^2}{2\beta_0} \\ &\triangleq p \end{aligned}$$

for all $t > 0$ *where*

$$\begin{aligned} \|\Lambda\| &= \sup_{x\in\mathbf{R}^n} \|\Lambda(x)\| \\ \|\Gamma\| &= \sup_{x\in\mathbf{R}^n} \|\Gamma(x)\| \end{aligned}$$

■

Theorem 2.2 (Theorem 8, (Baras *et al.*, 1988)) *Assume*

1. $f : \mathbb{R}^n \to \mathbb{R}^n$ *is smooth with bounded first derivative;*
2. $Q \geq q_0 I$ *for some* $q_0 > 0$*;*
3. $[R^{-\frac{1}{2}}H, \frac{\partial f}{\partial x}(x)]$ *is uniformly detectable;*
4. $[\frac{\partial f}{\partial x}(x), Q]$ *is uniformly controllable; and*
5. $[H, \frac{\partial f}{\partial x}(x)]$ *is uniformly detectable.*

If

$$|x_0 - m_0| \|\partial^2 f\| < \max_{P_0, Q, R} \frac{q_0}{n^2 p^{\frac{1}{2}} \|P_0^{\frac{1}{2}}\|} \tag{2.16}$$

where

$$\|\partial^2 f\| \triangleq \sup_{x \in \mathbf{R}^n} \|\frac{\partial^2 f}{\partial x^2}(x)\|$$

then

$$|x(t) - m(t)| \leq C|x_0 - m_0| \exp(-\gamma t)$$

for all $t > 0$ *for some constants* $C > 0, \gamma > 0$. ■

2.2.2 Stochastic, Continuous Time Systems

Picard (1991) examined this case when the system has small noise. Consider the non-linear system

$$\begin{aligned} dx_t &= f(t, x_t)dt + \sqrt{\epsilon} g(t, x_t)dw_t + \sqrt{\epsilon} b(t, x_t)dv_t \\ dy_t &= h(t, x_t)dt + \sqrt{\epsilon} dv_t \end{aligned}$$

where w_t and v_t are independent Brownian motions. The extended Kalman filter for this system is given by the equations

$$\begin{aligned} m_t &= m_0 + \int_0^t f(s, m_s)ds + \int_0^t K_s(dy_s - h(s, m_s)ds) \\ K_t &= b(t, m_t) + P_t H(t)^T \end{aligned}$$

$$\begin{aligned}\dot{P}_t &= -P_t H(t)^T H(t) P_t + [F(t) - b(t, m_t) H(t)] P_t \\ &\quad + P_t [F(t) - b(t, m_t) H(t)]^T + g(t, m_t) g(t, m_t)^T\end{aligned}$$

where

$$\begin{aligned}H(t) &\triangleq \frac{\partial h}{\partial x}(t, m_t) \\ F(t) &\triangleq \frac{\partial f}{\partial x}(t, m_t)\end{aligned}$$

Make the following definitions.

Definition 2.3 *A function $f(t, x, \epsilon)$ is* almost linear *if there exists a family of matrix valued processes $F_t(\epsilon)$ such that*

$$|f(t, x, \epsilon) - f(t, m, \epsilon) - F_t(\epsilon)(x - m)| \leq \mu_\epsilon |x - m|$$

for some family of numbers $\mu_\epsilon \to 0$ as $\epsilon \to 0$.

Definition 2.4 *A function $f(t, x)$ is* strongly injective *if*

$$|f(t, x) - f(t, m)| \geq c|x - m|$$

for some $c > 0$.

The following result can then be obtained.

Theorem 2.3 (Theorem 2.2.1 (Picard, 1991)) *Assume*

1. *the functions g and b are bounded;*
2. *the functions f and h are C^1 and almost linear;*
3. *the function h is strongly injective;*
4. *gg^T is positive definite; and*
5. *the ratio of the largest and smallest eigenvalue of P_0 is bounded.*

If

$$P_0^{-\frac{1}{2}}(x_0 - m_0) = O(\sqrt{\epsilon})$$

then

$$P_t^{-\frac{1}{2}}(x_t - m_t) = O(\sqrt{\epsilon})$$

for all $t > 0$. ■

2.2.3 Deterministic, Discrete Time Systems

Song and Grizzle (1993) have examined this case. Consider the nonlinear, single-input single-output system

$$\begin{aligned} x(k+1) &= f(x(k)), \qquad x_0 \text{ unknown} \\ y(k) &= h(x(k)). \end{aligned}$$

The extended Kalman filter for this system is given by the equations

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + K(k)(y(k) - h(\hat{x}(k|k-1))) \\ P(k|k)^{-1} &= P(k|k-1)^{-1} + H(k)R^{-1}H(k) \\ \hat{x}(k+1|k) &= f(\hat{x}(k|k)) \\ P(k+1|k) &= F(k)P(k|k)F(k) + Q \\ K(k) &= P(k|k-1)H(k)[H(k)P(k|k-1)H(k) + R]^{-1} \end{aligned}$$

where

$$\begin{aligned} F(k) &\triangleq \frac{df}{dx}(\hat{x}(k|k)) \\ H(k) &\triangleq \frac{dh}{dx}(\hat{x}(k|k-1)). \end{aligned}$$

Make the following assumptions

1. There exist $\beta_1, \beta_2 > 0$ such that

$$\beta_1 I \leq \sum_{i=k-M}^{k} \Phi(i,k)^T H(k) R^{-1} H(k) \Phi(i,k) \leq \beta_2 I$$

   for some finite $M \geq 0$ and for all $k \geq M$ where

$$\Phi(k,i) \triangleq F(k-1)F(k-2)\cdots F(i);$$

2. $\frac{df}{dx}(x)$ is non-zero for each $x \in \mathbb{R}$;

3. The following bounds exist and are finite

$$\begin{aligned} |F| &\triangleq \sup_x |F(x)| \\ |F^{-1}| &\triangleq \sup_x |F(x)^{-1}| \\ |H| &\triangleq \sup_x |R^{-\frac{1}{2}} \frac{dh}{dx}(x)| \\ |\frac{d^2 f}{dx^2}| &\triangleq \sup_x |\frac{d^2 f}{dx^2}(x)| \\ |\frac{d^2 h}{dx^2}| &\triangleq \sup_x |\frac{d^2 h}{dx^2}(x)| \end{aligned}$$

4. Let

$$g(x,y) \triangleq h(x) - h(x) - \frac{dh}{dx}(x)(x-y)$$

   and suppose there exists $c < \infty$ such that

$$|g(x,y)| \leq c|\frac{d^2 h}{dx^2}||x-y|^2 \qquad \text{for all } x, y.$$

Note that the assumption of a bounding norm for $|F^{-1}|$ is unnecessary given Assumption 2. Also in the fourth assumption $g(x,y)$ is the remainder term in a first order Taylor series expansion. A bound can be found for this remainder term provided $\frac{df}{dx}$ is Lipschitz in x. Given these assumptions the following results can then be derived.

Theorem 2.4 *Given the above assumptions there exists $p, q, r, \delta > 0$ such that*

$$\begin{aligned} |P(k|k)| &\leq |P(k|k-1)| \leq q \\ |P(k|k-1)^{-1}| &\leq p \\ |P(k|k)^{-1} + F(k)Q^{-1}F(k)| &\leq r \\ |K(k)| &\leq \delta \end{aligned}$$

for all $k > 0$. ■

Theorem 2.5 (Theorem 3.2 (Song and Grizzle, 1993)) *Consider the filtering error, $e(k) = x(k) - \hat{x}(k|k-1)$, then if $|e_0|, |\frac{d^2f}{dx^2}|$ and $|\frac{d^2h}{dx^2}|$ are such that for some $\gamma > 0$*

$$\varphi(q^{\frac{1}{2}}V(0,e_0)^{\frac{1}{2}}, |\frac{d^2f}{dx^2}|, |\frac{d^2h}{dx^2}|) \leq -\gamma$$

where

$$\begin{aligned} V(k,e) &\triangleq eP(k|k-1)^{-1}e \\ \phi(e,X,Y) &\triangleq \delta cY|F| + \tfrac{1}{2}X(pq + \delta cYe)^2 \\ \varphi(e,X,Y) &\triangleq \frac{-1}{rq^2} + pe\phi(e,X,Y)(2pq|F| + \phi(e,X,Y)e) \end{aligned}$$

then

$$\lim_{k\to\infty} e(k) = 0.$$

■

2.2.4 Discussion

There are a number of common features shared by these three results. As would be expected they all require limits on the size of the nonlinearities (condition 1 and equation (2.16) for Theorem 2.2, conditions 1 and 2 for Theorem 2.3 and condition 3 for Theorem 2.5). This is a result of the fact that the derivation of the EKF is based on the assumption that the neglected, higher order terms in the Taylor series expansion of the nonlinearities in the signal model are not significant. Furthermore all three results require that the initial error be sufficiently small. Once again this is a result of the assumption that the linearisation of the signal model is sufficiently good. As this can only be true within a sufficiently small region, a global stability result for the EKF would not be possible.

In addition to conditions imposed by the assumption made when deriving the EKF all three results require that the nonlinear system satisfy *global* observability and controllability style conditions (conditions 3,4 and 5 for Theorem 2.2, condition 3 for Theorem 2.3 and condition 1 for Theorem 2.5). Such conditions are restrictive. In particular, signal models for the frequency tracking problem cannot satisfy global observability conditions as the sine and cosine functions are not one-to-one mappings. However these conditions can be weakened as they are not fundamental consequences of assumptions made when deriving the EKF.

In this rest of this chapter the remaining case of the performance of the EKF when applied to a discrete time, stochastic system is considered. (Note only the case of systems with nonlinear state dynamics but a linear output map is examined.) In addition to completing the set of results given by Baras *et al.* (1988), Song and Grizzle (1993) and Picard (1991), this result weakens some of the restrictive conditions required by previous stability analyses. In particular, the key assumption of uniformity in the bounds on the growth rate of the nonlinearities and of observability properties of the signal model used in Theorems 2.2, 2.3 and 2.5 is relaxed. In addition, an assumption of high signal-to-noise ratios is not made. Instead, the effect of the size of the noise is included explicitly in the sufficient conditions for bounding the filtering error of the EKF.

In obtaining this nonlinear perturbation result it is necessary to examine not only the dynamics of the estimator but also of the associated Riccati difference equation. The result obtained shows that the stability of the error systems depends, as would be expected, on the nature of the nonlinearities and the size of the noise processes. These results can be used to design stable, nonlinear filters.

2.3 Nonlinear Stability Theorems

2.3.1 Norms and Notation

Before presenting the required stability theorems it is necessary to define precisely the notation to be used in the remainder of the chapter.

Definition 2.5 *Let* $|\cdot|$ *be some vector norm then the corresponding* induced matrix norm *is denoted by*

$$\|A\| \triangleq \sup_{|x|=1} |Ax|$$

where A *is a matrix and* x *is an appropriately sized vector.*

Definition 2.6 *Let* $|\cdot|$ *be some vector norm then the corresponding* induced tensor norm *is denoted by*

$$\|B\| \triangleq \sup_{|x_1|,\ldots,|x_n|=1} |Bx_1 \ldots x_n|$$

where B *is a tensor of degree* n *and* $x_1, \ldots, x_n$ *are vectors of appropriate lengths.*

A property of induced matrix and tensor norms is that they are *sub-multiplicative,* i.e.

$$\|AB\| \leq \|A\|\|B\|.$$

The required nonlinear stability theorems involve the use of derivatives of vector valued functions. In the following a short-hand notation will be given for these derivatives and some chain rules for derivatives of these quantities will be derived.

Definition 2.7 *Let a column vector of length* n *be written as*

$$v \triangleq [v_i], \qquad i = 1, \ldots, n$$

where v_i *is the* i*-th element.*

Definition 2.8 *Let an $n \times m$ matrix be written as*

$$A \triangleq [A_{ij}], \qquad i = 1, \ldots, n \quad j = 1, \ldots, m$$

where A_{ij} is the (i, j)-th element.

Definition 2.9 *Let an $n \times m \times r$ tensor be written as*

$$B \triangleq [B_{ijk}], \qquad i = 1, \ldots, n \quad j = 1, \ldots, m \quad k = 1, \ldots, r$$

where B_{ijk} is the (i, j, k)-th element. Let higher order tensors be defined similarly.

Definition 2.10 *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ then*

1. $$\frac{\partial f}{\partial x} \triangleq [\frac{\partial f_i}{\partial x_j}]$$

2. $$\frac{\partial^2 f}{\partial x^2} \triangleq [\frac{\partial f_i}{\partial x_j \partial x_k}]$$

3. $$\frac{\partial^3 f}{\partial x^3} \triangleq [\frac{\partial f_i}{\partial x_j \partial x_k \partial x_l}]$$

for $i = 1, \ldots, m$ and $j, k, l = 1, \ldots, n$.

Definition 2.11 *Suppose $A : \mathbb{R}^r \rightarrow \mathbb{R}^{n \times m}$ then*

$$\frac{\partial A}{\partial x} \triangleq [\frac{\partial A_{ij}}{\partial x_k}]$$

for $i = 1, \ldots, n, j = 1, \ldots, m$ and $k = 1, \ldots, r$.

With these definitions we can now prove the following chain rules hold using the short-hand notation given above.

Lemma 2.1 *Consider $y = A(x)x$ where y is a $p \times 1$ vector, x is a $n \times 1$ vector and A is a $p \times n$ matrix which is a function of x, then*

$$\frac{\partial y}{\partial x} = A(x) + \frac{\partial A(x)}{\partial x}x$$

Proof

Immediately from the definitions

$$\begin{aligned}
\frac{\partial y_i}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{n} A(x)_{ik}x_k\right) \\
&= \sum_{k=1}^{n} A(x)_{ik}\frac{\partial x_k}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial A(x)_{ik}}{\partial x_j}x_k \\
&= \sum_{k=1}^{n} A(x)_{ik}\delta_{jk} + \sum_{k=1}^{n} \frac{\partial A(x)_{ik}}{\partial x_j}x_k
\end{aligned}$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, n$, therefore

$$\frac{\partial y}{\partial x} = A(x) + \frac{\partial A(x)}{\partial x}x$$

■

Lemma 2.2 *Consider $C(x) = A(x)B(x)$ where x is a $n \times 1$ vector, A is an $r \times p$ matrix and B is a $p \times m$ matrix then*

$$\frac{\partial C}{\partial x} = \frac{\partial A}{\partial x}B + A\frac{\partial B}{\partial x}$$

Proof

Again from the definitions

$$\frac{\partial C}{\partial x} = [\frac{\partial C_{ij}}{\partial x_k}]$$

for $i = 1, \ldots r$, $j = 1, \ldots, m$ and $k = 1, \ldots, n$ where

$$\begin{aligned}
\frac{\partial C_{ij}}{\partial x_k} &= \frac{\partial}{\partial x_k}\left(\sum_{l=1}^{p} A(x)_{il}B(x)_{lj}\right) \\
&= \sum_{l=1}^{p} \frac{\partial A(x)_{il}}{\partial x_k}B_{lj} + \sum_{l=1}^{p} A_{il}\frac{\partial B(x)_{lj}}{\partial x_k}
\end{aligned}$$

therefore

$$\frac{\partial C}{\partial x} = \frac{\partial A}{\partial x}B + A\frac{\partial B}{\partial x}$$

■

2.3.2 Stability Theorems

Recall that the EKF error equation satisfies a nonlinear equation (2.12) which has a quasi-linear homogeneous part and additive perturbation terms due to linearisation error and the noise processes. The exponential asymptotic stability of the error equations for the Kalman filter is a well known and understood property (Anderson and Moore, 1979). For the EKF in the discrete time case, Song and Grizzle (1993) have presented sufficient conditions for the exponential asymptotic stability when the signal model is deterministic. The approach used to extend these results makes use of the Total Stability Theorem of Anderson *et al.* (1986) and related results. In this section the nonlinear stability theorems necessary to prove later the stability of the error system of the EKF are developed and reviewed.

Theorem 2.6 (Total Stability Theorem) *Consider the ordinary difference equation*

$$\begin{aligned} z(k+1) &= A(k)z(k) + f(k, z(k)) + g(k, z(k)) \\ z(0) &= z_0 \in \mathbb{R}^n \end{aligned}$$

where the functions $A : \mathbb{N}^+ \to \mathbb{R}^{n \times n}$, $f : \mathbb{N}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{N}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfy the conditions:–*

for some $r > 0$ *there exists* $\zeta > 0$ *and* $\eta > 0$ *such that for all* $|z_1| \leq r$, $|z_2| \leq r$ *and* $k \in \mathbb{N}$,

1. $\sup_{k \in \mathbb{N}} |A(k)| < \infty$;
2. $f(k, 0) = 0$;
3. $|f(k, z_1) - f(k, z_2)| \leq \zeta |z_1 - z_2|$;
4. $|g(k, z_1)| \leq \eta r$;
5. $|g(k, z_1) - g(k, z_2)| \leq \eta |z_1 - z_2|$.

If the unperturbed linear system

$$\bar{z}(k+1) = A(k)\bar{z}(k) \tag{2.17}$$

is exponentially stable, i.e. the transition matrix of (2.17), $\Phi_z(k_2, k_1)$, satisfies

$$|\Phi_z(k_2, k_1)| \leq \beta\alpha^{k_2 - k_1}$$

for all $k_2 \geq k_1 \geq 0$ for some $\beta \geq 1$ and $0 \leq \alpha < 1$ then

$$|z_0| < \frac{r}{\beta} \quad \text{and} \quad \beta(\zeta + \eta) + \alpha < 1$$

imply that for $k \geq 0$

$$\begin{aligned} |z(k)| &\leq \beta(\alpha + \zeta\beta)^k |z_0| + \frac{\beta\eta r}{1 - (\alpha + \zeta\beta)} \\ &\leq r. \end{aligned}$$

■

Corollary 2.6.1 *Consider the unperturbed, homogeneous difference equation*

$$\begin{aligned} z(k+1) &= A(k)z(k) + f(k, z(k)) \\ z(0) &= z_0 \in \mathbb{R}^n \end{aligned}$$

where $A(k)$ and $f(k,z)$ satisfy the conditions of Theorem 2.6 and the transition matrix, $\Phi_z(k_2, k_1)$, of the linear system is exponentially asymptotically stable as before, then

$$|z_0| < \frac{r}{\beta} \quad \text{and} \quad \beta\zeta + \alpha < 1$$

imply that for $k \geq 0$

$$\begin{aligned} |z(k)| &\leq \beta(\alpha + \zeta\beta)^k |z_0| \\ &\leq r. \end{aligned}$$

■

Note that Theorem 2.6 applies to nonlinear equations which are composed of the sum of a linear component, a nonlinear homogeneous component and a nonlinear non-homogeneous component. Its corollary applies to the case when the non-homogeneous component is absent. The following theorem deals with the case when the linear component is absent. The EKF error dynamics are of this latter form.

Theorem 2.7 *Consider the nonlinear difference equation*

$$x(k+1) = f(x(k)) + g(x(k)) \qquad (2.18)$$
$$x(0) = x_0 \in \mathbb{R}^n$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and* $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and the associated homogeneous nonlinear equation*

$$z(k+1) = f(z(k)) \qquad z(0) = x_0 \qquad (2.19)$$

and its linearised equation

$$\bar{z}(k+1) = \frac{\partial f}{\partial x}(z(k))\bar{z}(k) \qquad \bar{z}(0) = x_0 \qquad (2.20)$$

Suppose there exists an $r_z > 0$ *such that for all* $|x_0| \leq r_z$ *:–*

C1 *the solution,* $\{z(k) : k = 0, 1, \ldots\}$*, of (2.19) satisfies*

$$|z(k)| \leq \beta_1 \alpha_1^k |x_0|$$

where $\beta_1 \geq 1$ *and* $0 < \alpha_1 < 1$*; and*

C2 *the solution,* $\{\bar{z}(k) : k = 0, 1, \ldots\}$*, of (2.20) satisfies*

$$|\bar{z}(k)| \leq \beta_2 \alpha_2^k |x_0|$$

where $\beta_2 \geq 1$ *and* $0 < \alpha_2 < 1$.

For some $r_x > \beta_1 r_z$ *define*

$$\zeta \triangleq 2 \sup_{|x| \leq r_x} \|\frac{\partial f}{\partial x}(x)\|$$

and $\eta > 0$ *such that*

$$\begin{aligned} |g(x)| &\leq \eta(r_x - \beta_1 r_z) \\ \|\frac{\partial g}{\partial x}(x)\| &\leq \tfrac{1}{2}\eta \end{aligned}$$

for all $|x| \leq r_x$. *Then*

$$\beta_2(\zeta + \eta) + \alpha_2 < 1 \qquad and \qquad |x_0| \leq r_z$$

imply

$$\begin{aligned} |x(k)| &\leq \beta_1 \alpha_1^k |x_0| + \beta_2(\alpha_2 + \zeta\beta_2)^k |x_0| + \frac{\beta_2 \eta (r_x - \beta_1 r_z)}{1 - (\alpha_2 + \zeta\beta_2)} \\ &\leq r_x \qquad by\ construction \end{aligned}$$

for all $k \geq 0$.

Proof

The proof of this result is as follows. A recursion for the difference between the solution of the homogeneous equation (2.19) and the original, perturbed equation (2.18) is derived. This recursion is then be shown to be in the form of Theorem 2.6. Applying this theorem and condition C1 yields the result.

Using Taylor's Theorem write

$$f(x + a) = f(x) + \frac{\partial f}{\partial x}(x) \cdot a + \kappa_f(x, a)$$

where κ_f is the remainder after the first order expansion of f. From (2.18) we then have

$$\begin{aligned} x(1) &= f(x_0) + g(x_0) \\ &\triangleq f(x_0) + \delta(1) \\ x(2) &= f(x(1)) + g(x(1)) \\ &= f(f(x_0) + \delta(1)) + g(x(1)) \\ &= f(f(x_0)) + \frac{\partial f}{\partial x}(f(x_0)) \cdot \delta(1) + \kappa_f(f(x_0), \delta(1)) + g(x(1)) \\ &\triangleq f^2(x_0) + \delta(2) \end{aligned}$$

$$\begin{aligned}
x(3) &= f(x(2)) + g(x(2)) \\
&= f(f^2(x_0) + \delta(2)) + g(x(2)) \\
&= f(f^2(x_0)) + \frac{\partial f}{\partial x}(f^2(x_0)) \cdot \delta(2) + \kappa_f(f^2(x_0), \delta(2)) + g(x(2)) \\
&\triangleq f^3(x_0) + \delta(3) \\
&\vdots \\
x(k) &\triangleq f^k(x_0) + \delta(k).
\end{aligned}$$

Note that the correction term, $\{\delta(k)\}$, between the solution, $\{x(k)\}$, of (2.18) and that, $\{z(k)\}$, of (2.19) obeys the recursion

$$\delta(k+1) = \frac{\partial f}{\partial x}(f^k(x_0)) \cdot \delta(k) + \kappa_f(f^k(x_0), \delta(k)) + g(x(k)).$$

Using the solution, $\{z(k)\}$, of the homogeneous equation (2.19) permits us to rewrite this as

$$\delta(k+1) = \frac{\partial f}{\partial x}(z(k)) \cdot \delta(k) + \kappa_f(z(k), \delta(k)) + g(z(k) + \delta(k)). \tag{2.21}$$

Note that (2.21) is now in precisely the form required for the Total Stability Theorem. Note also that $\delta(0)$ is zero.

Suppose $|x_0| = |z_0| \leq r_z$ and let $r_\delta = r_x - \beta_1 r_z$. From C1 and C2 we know that $\frac{\partial f}{\partial x}(z(k))$ is a bounded function and its transition matrix, $\Phi_z(k_2, k_1)$, satisfies

$$\|\Phi_z(k_2, k_1)\| \leq \beta_2 \alpha_2^{k_2 - k_1}$$

where $\beta_2 \geq 1$ and $0 \leq \alpha_2 < 1$ so the linear portion of (2.21) satisfies the conditions of the Total Stability Theorem.

Define

$$\begin{aligned}
\bar{f}(k, \delta(k)) &\triangleq \kappa_f(z(k), \delta(k)) \\
&= f(z(k) + \delta(k)) - f(z(k)) - \frac{\partial f}{\partial x}(z(k)) \cdot \delta(k)
\end{aligned}$$

then for all $|\delta_1| \leq r_\delta$, $|\delta_2| \leq r_\delta$ and $k \geq 0$

$$\bar{f}(k, 0) = 0.$$

Now note that

$$\frac{\partial \kappa_f(z,\delta)}{\partial \delta} = \frac{\partial f}{\partial x}(z+\delta) - \frac{\partial f}{\partial x}(z)$$

and for all $|\delta| \leq r_\delta$ and $|x_0| \leq r_z$

$$\begin{aligned} |z(k)+\delta| &\leq |z(k)| + |\delta| \\ &\leq \beta_1 \alpha_1^k |x_0| + r_\delta \\ &\leq \beta_1 r_z + r_\delta \\ &= r_x. \end{aligned}$$

Therefore

$$\begin{aligned} \|\frac{\partial \kappa_f(z,\delta)}{\partial \delta}\| &\leq \|\frac{\partial f}{\partial x}(z+\delta)\| + \|\frac{\partial f}{\partial x}(z)\| \\ &\leq \zeta \end{aligned}$$

and hence

$$|\bar{f}(k,\delta_1) - \bar{f}(k,\delta_2)| \leq \zeta |\delta_1 - \delta_2|.$$

Similarly let

$$\bar{g}(k, \delta(k)) \triangleq g(z(k) + \delta(k))$$

then for all $|\delta_1| \leq r_\delta$, $|\delta_2| \leq r_\delta$ and $k \geq 0$

$$\begin{aligned} |\bar{g}(k,\delta_1)| &= |g(z(k)+\delta_1)| \\ &\leq \eta r_\delta. \end{aligned}$$

Define

$$\kappa_g(k,\delta) \triangleq g(z(k)+\delta(k)) - g(z(k)) - \frac{\partial g}{\partial x}(z(k)) \cdot \delta(k)$$

then as before

$$\begin{aligned}|\bar{g}(k,\delta_1)-\bar{g}(k,\delta_2)| &= |g(z(k)+\delta_1)-g(z(k)+\delta_2)| \\ &\leq \eta|\delta_1-\delta_2|.\end{aligned}$$

Thus the nonlinear portions of (2.21) satisfy the boundedness conditions of the Total Stability Theorem provided $|x_0| \leq r_z$. Hence

$$\beta_2(\zeta+\eta)+\alpha_2<1 \quad \text{and} \quad |x_0|=|z_0|\leq r_z$$

imply that

$$\begin{aligned}|\delta(k)| &\leq \beta_2(\alpha_2+\zeta\beta_2)^k|x_0|+\frac{\beta_2\eta r_\delta}{1-(\alpha_2+\zeta\beta_2)} \\ &\leq r_\delta\end{aligned}$$

Therefore for the solution of the original equation (2.18) we have

$$\begin{aligned}|x(k)| &\leq |f^k(x_0)|+|\delta(k)| \\ &= |z(k)|+|\delta(k)| \\ &\leq \beta_1\alpha_1^k|x_0|+\beta_2(\alpha_2+\zeta\beta_2)^k|x_0|+\frac{\beta_2\eta r_\delta}{1-(\alpha_2+\zeta\beta_2)}\end{aligned}$$

whenever

$$\beta_2(\zeta+\eta)+\alpha_2<1 \quad \text{and} \quad |x_0|=|z_0|\leq r_z$$

which completes the proof. ■

2.4 EKF Stability for Stochastic, Discrete Time Systems

In this section Theorem 2.7 is applied to the EKF error dynamics. In Theorem 2.8 it is shown that condition C1 holds under conditions on the observability and controllability of the signal model given a sufficiently small initial value. In Lemma 2.3 it is shown that condition C2 holds under the same assumptions. In Lemma 2.4 an explicit equation for the bound for the noise and linearisation error based perturbation component of the error dynamics is derived and in Lemma 2.5 a similar bound is given for the nonlinear, undriven component. The stability result itself is presented in Theorem 2.9.

2.4.1 Observability and Controllability

Define the observability Gramian of $[F_z, R^{-\frac{1}{2}}H]$ as

$$\mathcal{O}(k,N) = \sum_{i=k-N}^{k} \Phi(i,k)^T H(i)^T R(i)^{-1} H(i)\Phi(i,k) \tag{2.22}$$

for some $N \geq 0$ and for all $k \geq N$, where $\Phi(k_2, k_1) = F_z(k_2-1)F_z(k_2-2)\ldots F_z(k_1)$. Similarly, define the controllability Gramian of $[F_z, Q]$ as

$$\mathcal{C}(k,N) = \sum_{i=k-N}^{k-1} \Phi(k,i+1)Q(i)^T\Phi(k,i+1)^T. \tag{2.23}$$

Assume there exists N such that for all $R_x > 0$ there exists $0 < \epsilon_r < R_x$, $a_i(R_x, \epsilon_r, N)$ and $b_i(R_x, \epsilon_r, N)$, $i = 1, 2$ such that for all sequences $\{\psi(k)\}$ and $\{\nu(k)\}$ with $|\psi(k)| \leq R_x$ and $|\nu(k)| \leq \epsilon_r$

$$a_1 I \geq \mathcal{C}(k,N) \geq a_2 I \qquad 0 < a_2 \leq a_1 < \infty \tag{2.24}$$

$$b_1 I \leq \mathcal{O}(k,N) \leq b_2 I \qquad 0 < b_1 \leq b_2 < \infty \tag{2.25}$$

where these Gramians are evaluated along the trajectory $z(k) = \psi(k) - \nu(k)$ i.e.

$$\begin{aligned} F_z(k) &= \frac{\partial f}{\partial x}(z(k)) \\ &= \frac{\partial f}{\partial x}(\psi(k) - \nu(k)). \end{aligned}$$

2.4.2 Standing Assumptions on the Signal Model

The following assumptions will be assumed to hold for the remainder of this section. The nonlinear signal model has a linear output map, $f \in C^3(\mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial f}{\partial x}(x)$ is invertible for all $x \in \mathbb{R}^n$ and for all k

$$|x(k)| \leq R_x; \tag{2.26}$$

$$\|H(k)\| \leq \rho_5; \tag{2.27}$$

$$E[w(k)w(k)^T] = Q(k) \geq \delta_1 I \quad \text{and} \quad \|w(k)\| \leq \|w\| < \infty; \tag{2.28}$$

$$E[v(k)v(k)^T] = R(k) \geq \delta_2 I \quad \text{and} \quad \|v(k)\| \leq \|v\| < \infty \tag{2.29}$$

Furthermore it is assumed that we can find N and $\epsilon_r < R_x$ such that the observability and controllability conditions (2.24) and (2.25) hold.

The EKF equations for this system are given by (2.3) – (2.7) and error dynamics of the EKF when applied to such a signal model are given by equation (2.12).

2.4.3 Signal Model Bounds

Since $f \in C^3(\mathbb{R}^n, \mathbb{R}^n)$ we can find $\rho_1, \rho_2, \rho_3 > 0$ as

$$\|\frac{\partial f}{\partial x}(x)\| \leq \rho_1 \tag{2.30}$$

$$\|\frac{\partial^2 f}{\partial x^2}(x)\| \leq \rho_2 \tag{2.31}$$

$$\|\frac{\partial^3 f}{\partial x^3}(x)\| \leq \rho_3 \tag{2.32}$$

for all $|x| \leq R_x + \epsilon_r$. Furthermore, by the continuity assumptions on f, there exists a $\rho_4 > 0$ such that

$$\|\frac{\partial f}{\partial x}(x_1) - \frac{\partial f}{\partial x}(x_2)\| \leq \rho_4 |x_1 - x_2|$$

for all $|x_1| \leq R_x$ and $|x_2| \leq R_x$ and therefore

$$\|\frac{\partial f}{\partial x}(x_1) - \frac{\partial f}{\partial x}(x_2) - \frac{\partial^2 f}{\partial x^2}(x_2) \cdot (x_1 - x_2)\| \leq \tfrac{1}{2}\rho_4 |x_1 - x_2|^2 \tag{2.33}$$

(Dennis and Schnabel, 1983, Chapter 4).

Let

$$\begin{aligned} p &= a_1 + \frac{1}{b_1} \\ q &= \frac{1}{a_2} + b_2 \\ s &= \frac{1}{a_2} + b_2 + \rho_1^2 \delta_1^{-1}. \end{aligned}$$

Consider the equations for the evolution of the EKF gain, $K_z(k)$, and the covariance matrices $P_z(k|k)$ and $P_z(k+1|k)$ along the arbitrary trajectory $z(k) = \psi(k) - \nu(k)$, which are given by the equations

$$\begin{aligned} P_z(k|k) &= [I - K_z(k)H(k)]P_z(k|k-1) \\ P_z(k+1|k) &= F_z(k)P_z(k|k)F_z(k)^T + Q(k) \\ K_z(k) &= P_z(k|k-1)H(k)^T[H(k)P_z(k|k-1)H(k)^T + R(k)]^{-1}. \end{aligned}$$

These are the same equations as those for a Kalman filter applied to the linear signal model

$$\begin{aligned} \xi(k+1) &= F_z(k)\xi(k) + w(k) \\ \Upsilon(k) &= H(k)\xi(k) + v(k) \end{aligned}$$

where $F_z(k) = \frac{\partial f}{\partial x}(\psi(k) - \nu(k))$. Thus we can use the results of Deyst and Price (1968) on the stability of the time-varying Kalman filter to obtain the bounds

$$\begin{aligned} q^{-1}I &\leq P_z(k|k) \leq pI \\ q^{-1}I &\leq P_z(k+1|k) \leq sI \end{aligned}$$

which depend on ϵ_r, R_x and N. Note that these bounds will hold for any trajectory in the ball $|z| \leq R_x + \epsilon_r$.

2.4.4 Preliminary Results

The following result gives sufficient conditions for the stability of the EKF when applied to a deterministic signal model when the linearisation errors are neglected. This theorem only requires that these properties hold in some subset of the ϵ_r-ball centred on $z = 0$.

Theorem 2.8 *Consider the nonlinear equation*

$$\begin{aligned} z(k+1) &= [I - K_{\tilde{z}}(k+1)H(k+1)]\frac{\partial f}{\partial x}(x(k)) \cdot z(k) \\ &\triangleq \bar{f}(k, z(k)) \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} \tilde{z}(k) &= x(k) - z(k) \\ K_{\tilde{z}}(k) &= P_{\tilde{z}}(k|k-1)H(k)^T[H(k)P_{\tilde{z}}(k|k-1)H(k)^T + R(k)]^{-1} \\ P_{\tilde{z}}(k+1|k) &= F_{\tilde{z}}(k)P_{\tilde{z}}(k|k)F_{\tilde{z}}(k)^T + Q(k) \\ P_{\tilde{z}}(k|k) &= [I - K_{\tilde{z}}(k)H(k)]P_{\tilde{z}}(k|k-1) \\ F_{\tilde{z}} &= \frac{\partial f}{\partial x}(x(k) - z(k)) \end{aligned}$$

Given $R_x > 0$ select N and $\epsilon_r < R_x$ such that the observability and controllability conditions, (2.24) and (2.25) hold. Let

$$\epsilon_z = \min\left\{\epsilon_r, \frac{\sqrt{2}}{\sqrt{\rho_4}}\left(-\rho_1 + \sqrt{\rho_1^2 + \frac{1}{q}(\frac{1}{sp^2} - \gamma)}\right)^{\frac{1}{2}}\right\} \tag{2.35}$$

where $0 < \gamma < \frac{1}{sp^2}$, then

$$|z(0)| < \frac{\epsilon_z}{(pq)^{\frac{1}{2}}}$$

implies

$$\begin{aligned} |z(k)| &\leq (pq)^{\frac{1}{2}}\left(1 - \frac{\gamma}{q}\right)^{\frac{k}{2}} |z(0)| \\ &\triangleq \beta\alpha^k |z(0)| \end{aligned} \tag{2.36}$$

for all $k \geq 0$. (Note that as a consequence we know that $|z(k)| \leq \epsilon_z$ for all $k \geq 0$.)

Proof

We shall prove this result using a Lyapunov stability argument. Let

$$V(k,z) = z^T P_{\tilde{z}}(k|k)^{-1} z$$

then

$$p^{-1}|z(k)|^2 \leq V(k,z(k)) \leq q|z(k)|^2$$

for all k and $|z(k)| \leq \epsilon_r$. Using the equations for the EKF and the matrix inversion lemma it can be shown that

$$\begin{aligned}
&F_{\tilde{z}}(k)^T P_{\tilde{z}}(k+1|k)^{-1} F_{\tilde{z}}(k) = \\
&\qquad P_{\tilde{z}}(k|k)^{-1} - P_{\tilde{z}}(k|k)^{-1}[P_{\tilde{z}}(k|k)^{-1} + F_{\tilde{z}}(k)^T Q(k)^{-1} F_{\tilde{z}}(k)]^{-1} P_{\tilde{z}}(k|k)^{-1} \\
&P_{\tilde{z}}(k+1|k)^{-1} K_{\tilde{z}}(k+1) H(k+1) = \\
&\qquad H(k+1)^T[H(k+1)P_{\tilde{z}}(k+1|k)H(k+1)^T + R(k+1)]^{-1} H(k+1).
\end{aligned}$$

Therefore for $|z(k)| \leq \epsilon_r$

$$\begin{aligned}
&\Delta V(k,z(k)) \\
&= V(k+1, z(k+1)) - V(k, z(k)) \\
&= z(k)^T F_x(k)^T [I - K_{\tilde{z}}(k+1)H(k+1)]^T P_{\tilde{z}}(k+1|k+1)^{-1} [I - K_{\tilde{z}}(k+1)H(k+1)] F_x(k) z(k) \\
&\quad - z(k)^T P_{\tilde{z}}(k|k)^{-1} z(k) \\
&= z(k)^T F_x(k)^T P_{\tilde{z}}(k+1|k)^{-1} F_x(k) z(k) - z(k)^T P_{\tilde{z}}(k|k)^{-1} z(k) \\
&\quad - z(k)^T F_x(k)^T H(k+1)[H(k+1)P_{\tilde{z}}(k+1|k)H(k+1)^T + R(k+1)]^{-1} H(k+1) F_x(k) z(k) \\
&\leq -z(k)^T P_{\tilde{z}}(k|k)^{-1}[P_{\tilde{z}}(k|k)^{-1} + F_{\tilde{z}}(k)^T Q(k)^{-1} F_{\tilde{z}}(k)]^{-1} P_{\tilde{z}}(k|k)^{-1} z(k) \\
&\quad + z(k)^T F_{\tilde{z}}(k)^T P_{\tilde{z}}(k+1|k)^{-1} B_{\tilde{z}}(k) z(k) \\
&\quad + z(k)^T B_{\tilde{z}}(k)^T P_{\tilde{z}}(k+1|k)^{-1} F_{\tilde{z}}(k) z(k) \\
&\quad + z(k)^T B_{\tilde{z}}(k)^T P_{\tilde{z}}(k+1|k)^{-1} B_{\tilde{z}}(k) z(k)
\end{aligned}$$

where

$$B_{\tilde{z}}(k) = \frac{\partial f}{\partial x}(x(k)) - \frac{\partial f}{\partial x}(x(k) - z(k)) - \frac{\partial^2 f}{\partial x^2}(x(k) - z(k)) \cdot z(k).$$

Therefore

$$\Delta V(k, z(k)) \leq \left\{ \frac{-1}{sp^2} + \rho_1\rho_4 q|z(k)|^2 + \tfrac{1}{4}\rho_4^2 q|z(k)|^4 \right\} |z(k)|^2.$$

Now

$$\Delta V(k, z(k)) \leq -\gamma|z(k)|^2, \qquad 0 < \gamma < \frac{1}{sp^2}$$

provided

$$\frac{-1}{sp^2} + \rho_1\rho_4 q|z(k)|^2 + \tfrac{1}{4}\rho_4^2 q|z(k)|^4 \leq -\gamma$$

which will hold when

$$|z(k)| \leq \frac{\sqrt{2}}{\sqrt{\rho_4}} \left(-\rho_1 + \sqrt{\rho_1^2 + \frac{1}{q}(\frac{1}{sp^2} - \gamma)} \right)^{\frac{1}{2}}$$

Let

$$\epsilon_z = \min \left\{ \epsilon_r, \frac{\sqrt{2}}{\sqrt{\rho_4}} \left(-\rho_1 + \sqrt{\rho_1^2 + \frac{1}{q}(\frac{1}{sp^2} - \gamma)} \right)^{\frac{1}{2}} \right\}$$

and suppose $|z(j)| \leq \epsilon_z$ for $j = 0, \ldots, k-1$ then

$$\begin{aligned} \Delta V(k-1, z(k-1)) &\leq -\gamma|z(k-1)|^2 \\ \Rightarrow V(k, z(k)) &\leq (1-\gamma p)V(k-1, z(k-1)) \\ \Rightarrow V(k, z(k)) &\leq (1-\gamma p)^k V(0, z(0)) \\ \Rightarrow p^{-1}|z(k)|^2 &\leq (1-\gamma p)^k q|z(0)|^2 \end{aligned}$$

Now since $1 - \gamma p < 1$ and $pq > 1$

$$|z(0)|^2 < \frac{\epsilon_z^2}{pq} \Rightarrow |z(k)|^2 \leq \epsilon_z^2.$$

Thus, by induction,

$$|z(0)|^2 < \frac{\epsilon_z{}^2}{pq}$$

implies

$$\Delta V(k, z(k)) \leq -\gamma|z(k)|^2$$

for all $k \geq 0$. Furthermore the bounds on the Lyapunov function give

$$\begin{aligned} |z(k)| &\leq (pq)^{\frac{1}{2}} \left(1 - \frac{\gamma}{q}\right)^{\frac{k}{2}} |z(0)| \\ &\leq \epsilon_z \end{aligned}$$

for all $k \geq 0$, which completes the proof. ∎

Now consider the linear equation

$$\bar{z}(k+1) = \frac{\partial \bar{f}}{\partial z}(k, z(k)) \cdot \bar{z}(k) \tag{2.37}$$

where $\bar{f}$ and $z(k)$ are defined in the previous theorem. Then differentiating $\bar{f}(k, z(k))$ gives

$$\begin{aligned} \frac{\partial \bar{f}}{\partial z} &= [I - K_{\bar{z}}(k+1)H(k+1)]F_x(k) \\ &\quad - \frac{\partial K_{\bar{z}}(k+1)}{\partial z} H(k+1)F_x(k)z(k). \end{aligned}$$

Therefore the equation (2.37) can be written as

$$\bar{z}(k+1) = A(k)\bar{z}(k) + \bar{f}_2(k, \bar{z}(k)) \tag{2.38}$$

where

$$A(k) = [I - K_{\bar{z}}(k+1)H(k+1)]F_x(k)$$

and

$$\bar{f}_2(k, \bar{z}) = -\frac{\partial K_{\bar{z}}(k+1)}{\partial z} H(k+1)F_x(k)z(k)\bar{z}(k)$$

With the equation in this form we can now apply Corollary 2.6.1, which gives the following lemma.

This lemma shows that the linearised, undriven portion of the EKF error dynamics is asymptotically stable when linearisation errors are neglected. It shows that the EKF error dynamics under the standing assumptions satisfy condition C2 of Theorem 2.7.

Lemma 2.3 *Consider the equation (2.38) which is the linearised, undriven portion of the EKF error dynamics, neglecting linearisation errors. If*

$$|\check{z}(0)| < \epsilon_z (pq)^{-\frac{1}{2}}$$

then the linear system

$$\check{z}(k+1) = A(k)\check{z}(k)$$

is exponentially asymptotically stable and its transition matrix, $\Phi_{\check{z}}(k_2, k_1)$, satisfies

$$\|\Phi_{\check{z}}(k_2, k_1)\| \leq (pq)^{\frac{1}{2}} \left(1 - \frac{\gamma}{q}\right)^{\frac{1}{2}k}.$$

Also, the function $\bar{f}_2$ is homogeneous and satisfies a Lipschitz condition in the ϵ_z-ball i.e.

1. $\bar{f}_2(k, 0) = 0$;
2. $\|\bar{f}_2(k, \bar{z}_1) - \bar{f}_2(k, \bar{z}_2)\| \leq \zeta_{\bar{z}} |\bar{z}_1 - \bar{z}_2|$.

for all $|\bar{z}_1| \leq \epsilon_z$, $|\bar{z}_1| \leq \epsilon_z$ and $k \geq 0$ for some $\zeta_{\bar{z}} > 0$. Therefore if

$$|\bar{z}(0)| < \epsilon_z (pq)^{-\frac{1}{2}} \qquad and \qquad (pq)^{\frac{1}{2}} \zeta_{\bar{z}} + \left(1 - \frac{\gamma}{q}\right)^{\frac{1}{2}} < 1$$

where $0 < \gamma < \frac{1}{sp^2}$, then

$$\begin{aligned} |\bar{z}(k)| &\leq (pq)^{\frac{1}{2}} \left(\left(1 - \frac{\gamma}{q}\right)^{\frac{1}{2}} + \zeta_{\bar{z}} (pq)^{\frac{1}{2}} \right)^k |\bar{z}(0)| \\ &\leq \epsilon_z \qquad \text{by construction} \end{aligned}$$

for all $k \geq 0$.

Proof

It can be shown that the linear portion of (2.38) is exponentially asymptotically stable using the same Lyapunov stability argument used in Theorem 2.8.

All the remains in order to apply Corollary 2.6.1 is to show that for all k and $|\bar{z}_1| \leq \epsilon_r$ and $|\bar{z}_2| \leq \epsilon_r$

1. $\bar{f}_2(k, 0) = 0$;

2. $\|\bar{f}_2(k, \bar{z}_1) - \bar{f}_2(k, \bar{z}_2)\| \leq \zeta_{\bar{z}}|\bar{z}_1 - \bar{z}_2|$.

Since

$$\bar{f}_2(k, \bar{z}) = -\frac{\partial K_{\bar{z}}(k+1)}{\partial z} H(k+1) F_x(k) z(k) \bar{z}(k)$$

condition (1) clearly holds.

To determine bounds for $\bar{f}_2$ note that

$$\begin{aligned}\frac{\partial K_{\bar{z}}(k+1)}{\partial z} &= \frac{\partial P_{\bar{z}}(k+1|k)}{\partial z} H(k+1)^T [H(k+1) P_{\bar{z}}(k+1|k) H(k+1)^T + R(k+1)]^{-1} \\ &\quad + P_{\bar{z}}(k+1|k) H(k+1)^T \frac{\partial}{\partial z} [H(k+1) P_{\bar{z}}(k+1|k) H(k+1)^T + R(k+1)]^{-1}.\end{aligned}$$

From equation (2.6)

$$\begin{aligned}\frac{\partial P_{\bar{z}}(k+1|k)}{\partial z} &= \frac{\partial F_{\bar{z}}(k)}{\partial z} P_{\bar{z}}(k|k) F_{\bar{z}}(k)^T + F_{\bar{z}}(k) P_{\bar{z}}(k|k) \frac{\partial F_{\bar{z}}(k)^T}{\partial z} \\ &= -\frac{\partial^2 f}{\partial x^2}(x(k) - z(k)) P_{\bar{z}}(k|k) F_{\bar{z}}(k)^T - F_{\bar{z}}(k) P_{\bar{z}}(k|k) \frac{\partial^2 f}{\partial x^2}(x(k) - z(k)).\end{aligned}$$

Thus under the assumptions

$$\begin{aligned}\left\|\frac{\partial P_{\bar{z}}(k+1|k)}{\partial z(k)}\right\| &\leq 2\rho_1\rho_2 p \\ &\triangleq \delta_p\end{aligned}$$

Furthermore

$$\begin{aligned}&\frac{\partial}{\partial z}[H(k+1) P_{\bar{z}}(k+1|k) H(k+1)^T + R(k+1)]^{-1} = \\ &\quad -[H(k+1) P_{\bar{z}}(k+1|k) H(k+1)^T + R(k+1)]^{-1} [H(k+1) \frac{\partial P_{\bar{z}}(k+1|k)}{\partial z} H(k+1)^T] \times \\ &\quad [H(k+1) P_{\bar{z}}(k+1|k) H(k+1)^T + R(k+1)]^{-1}\end{aligned}$$

hence

$$\|\frac{\partial}{\partial z}[H(k+1)P_{\bar{z}}(k+1|k)H(k+1)^T + R(k+1)]^{-1}\| \leq \rho_5^2\delta_2^{-2}\delta_p$$

and thus

$$\begin{aligned}\|\frac{\partial K_{\bar{z}}(k+1)}{\partial z}\| &\leq \delta_p\rho_5\delta_2^{-1}(1+s\rho_5^2\delta_2^{-1}) \\ &\triangleq \delta_k.\end{aligned}$$

Therefore

$$\begin{aligned}\|\bar{f}_2(k,\bar{z}_1) - \bar{f}_2(k,\bar{z}_2)\| &\leq \delta_k\rho_5\rho_1\epsilon_z|\bar{z}_1 - \bar{z}_2| \\ &\triangleq \zeta_{\bar{z}}\end{aligned} \tag{2.39}$$

hence condition (2) is also satisfied. Applying Corollary 2.6.1 completes the proof.

■

The next lemma derives a bound for the noise and linearisation error based perturbation component of the EKF error dynamics.

Lemma 2.4 *Consider the perturbation terms in the EKF error dynamics (2.12) due to the noise and linearisation errors,*

$$\bar{g}(e(k),k) \triangleq [I - K_{\bar{z}}(k+1)H(k+1)][w(k) - \kappa_f(x(k),-e(k))] - K_{\bar{z}}(k)v(k)$$

where $K(k)$ is given in Theorem 2.8. There exists $\eta > 0$ such that

- $|\bar{g}(e_1,k)| \leq \eta\epsilon_r$*; and*
- $|\bar{g}(e_1,k) - \bar{g}(e_2,k)| \leq \eta|e_1 - e_2|$

for all $|e_1| \leq \epsilon_r$, $|e_2| \leq \epsilon_r$ and $k \geq 0$.

Proof

The bounds on the error covariance matrices and the noise processes gives

$$
\begin{aligned}
&|\bar{g}(e_1, k)| \\
&\leq \; \|P_{\bar{z}}(k+1|k+1)P_{\bar{z}}(k+1|k)^{-1}[w(k) - \kappa_f(x(k), e(k))]\| \\
&\qquad +\|P_{\bar{z}}(k+1|k+1)H(k+1)R(k+1)^{-1}v(k)\| \\
&\leq \; pq(\|w\| + \tfrac{1}{2}\rho_4\epsilon_r^2) + p\rho_5\delta_2^{-1}\|v\|.
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{\partial \bar{g}}{\partial e(k)} &= -\frac{\partial K_{\bar{z}}(k+1)}{\partial e(k)}H(k+1)[w(k) - \kappa_f(x(k), -e(k))] - \frac{\partial K_{\bar{z}}(k+1)}{\partial e(k)}v(k) \\
&\quad -[I - K_{\bar{z}}(k+1)H(k+1)]\frac{\partial \kappa_f(x(k), -e(k))}{\partial e(k)}
\end{aligned}
$$

so

$$
\|\frac{\partial \bar{g}}{\partial e(k)}\| \leq \delta_k(\rho_5\|w\| + \|v\| + \tfrac{1}{2}\rho_4\rho_5\epsilon_r^2) + 2pq\rho_1
$$

Defining

$$
\eta \triangleq \max\{pq(\|w\| + \tfrac{1}{2}\rho_4\epsilon_r^2) + p\rho_5\delta_2^{-1}\|v\|, \delta_k(\rho_5\|w\| + \|v\| + \tfrac{1}{2}\rho_4\rho_5\epsilon_r^2) + 2pq\rho_1\} \qquad (2.40)
$$

completes the result. ■

The final lemma derives a bound for the nonlinear, undriven component of the EKF error dynamics.

Lemma 2.5 *Consider the function*

$$
\bar{f}(k, z(k)) = [I - K_{\bar{z}}(k+1)H(k+1)]\frac{\partial f}{\partial x}(x(k)) \cdot z(k)
$$

which is the homogeneous portion of the EKF error dynamics neglecting linearisation errors. There exists $\zeta > 0$ such that

$$
\|\frac{\partial^2 \bar{f}}{\partial z^2}(z(k))\| \leq \zeta
$$

for all $|z(k)| \leq \epsilon_r$ and $k \geq 0$.

Proof

Recall that

$$\begin{aligned}\frac{\partial \bar{f}}{\partial z} &= [I - K_{\bar{z}}(k+1)H(k+1)]F_x(k) \\ &\quad -\frac{\partial K_{\bar{z}}(k+1)}{\partial z}H(k+1)F_x(k)z(k)\end{aligned}$$

so

$$\begin{aligned}\frac{\partial^2 \bar{f}}{\partial z^2} &= -2\frac{\partial K_{\bar{z}}(k+1)}{\partial z}H(k+1)F_x(k) \\ &\quad -\frac{\partial^2 K_{\bar{z}}(k+1)}{\partial z^2}H(k+1)F_x(k)z(k)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial^2 K_{\bar{z}}(k+1)}{\partial z^2} &= \frac{\partial^2 P_{\bar{z}}(k+1|k)}{\partial x^2}H(k+1)^T[H(k+1)P_{\bar{z}}(k+1|k)H(k+1)^T + R(k+1)]^{-1} \\ &\quad +2\frac{\partial P_{\bar{z}}(k+1|k)}{\partial z}H(k+1)^T\frac{\partial}{\partial e}[H(k+1)P_{\bar{z}}(k+1|k)H(k+1)^T + R(k+1)]^{-1} \\ &\quad +P_{\bar{z}}(k+1|k)H(k+1)^T\frac{\partial}{\partial z^2}[H(k+1)P_{\bar{z}}(k+1|k)H(k+1)^T + R(k+1)]^{-1}.\end{aligned}$$

Now

$$\begin{aligned}\frac{\partial^2 P_{\bar{z}}(k+1|k)}{\partial x^2} &= \frac{\partial^3 f}{\partial x^3}(x(k)-z(k))P_{\bar{z}}(k|k)F_{\bar{z}}(k)^T \\ &\quad -2\frac{\partial^2 f}{\partial x^2}(x(k)-z(k))P_{\bar{z}}(k|k)\frac{\partial^2 f}{\partial x^2}(x(k)-z(k))^T \\ &\quad +F_{\bar{z}}(k)P_{\bar{z}}(k|k)\frac{\partial^3 f}{\partial x^3}(x(k)-z(k))\end{aligned}$$

and hence

$$\left\|\frac{\partial^2 P_{\bar{z}}(k+1|k)}{\partial x^2}\right\| \leq 2p(\rho_3\rho_1 + \rho_2^2).$$

Thus there exists a $\delta_{k2}(\rho_1, \rho_2, \rho_3, \rho_5, \delta_2, p) > 0$ such that

$$\left\|\frac{\partial^2 K_{\bar{z}}(k+1)}{\partial z^2}\right\| \leq \delta_{k2}$$

for all $k \geq 0$. Therefore

$$\|\frac{\partial^2 \bar{f}}{\partial z^2}\| \leq 2\rho_1\rho_5\delta_k + \rho_1\rho_5\delta_{k2}\epsilon_r \quad (2.41)$$

$$\triangleq \zeta \quad (2.42)$$

which completes the proof. ■

2.4.5 Main Result

The results of the previous section can be combined to give the following result for the stability of the EKF when applied to a stochastic, discrete time nonlinear system.

Theorem 2.9 (EKF Stability) *Consider the error dynamics of the EKF which are given by the equation*

$$\begin{aligned} e(k|k) &= [I - K_{\hat{x}}(k)H(k)]F_x(k-1)e(k-1|k-1) \\ &\quad + [I - K_{\hat{x}}(k)H(k)]\kappa_f(x(k-1), -e(k-1|k-1)) \\ &\quad + [I - K_{\hat{x}}(k)H(k)]w(k-1) - K_{\hat{x}}(k)v(k) \end{aligned}$$

where

$$\kappa_f(x, -e) = f(x - e) - f(x) + \frac{\partial f}{\partial x}(x) \cdot e$$

when the EKF is applied to a signal model with a linear output map which satisfies the standing assumptions (2.26) – (2.29). Select N and $0 < \epsilon_r < R_x$ such that the observability and controllability conditions (2.24) and (2.25) are satisfied. Then if

$$\beta(\zeta + \eta) + \alpha < 1 \quad (2.43)$$

$$\beta\zeta_{\bar{z}} + \alpha < 1 \quad (2.44)$$

$$|e_0| < \epsilon_z(pq)^{-\frac{1}{2}} \quad (2.45)$$

where and ϵ_z, α and β, $\zeta_{\bar{z}}$, η and ζ are given by (2.35), (2.36), (2.39), (2.40) and (2.42) respectively,

$$\begin{aligned} |e(k|k)| &\leq \beta\alpha^k|e_0| + \beta(\alpha+\zeta\beta)^k|e_0| + \frac{\beta\eta(R_x - \beta\epsilon_z)}{1-(\alpha+\zeta\beta)} \\ &\leq \epsilon_r \end{aligned}$$

for all $k \geq 0$.

Proof

To prove this result we shall appeal to Theorem 2.7. From Theorem 2.8 and (2.45) we know that the solution of

$$z(k) = [I - K_{\bar{z}}(k)H(k)]F_x(k-1)z(k), \quad z(0) = e_0$$

where $\bar{z}(k) = x(k) - z(k)$, satisfies

$$|z(k)| \leq \beta\alpha^k|z(0)|$$

and so condition C1 of Theorem 2.7 is satisfied.

From Lemma 2.3 and equations (2.44) and (2.45) we know that the solution of

$$\bar{z}(k+1) = \frac{\partial}{\partial z(k)}\{[I - K_{\bar{z}}(k)H(k)]F_x(k-1)z(k)\}\bar{z}(k), \quad \bar{z}(0) = e_0, \quad z(0) = e_0$$

will satisfy

$$|\bar{z}(k)| \leq \beta\alpha^k|\bar{z}(0)|$$

which satisfies condition C2.

Now $\beta\epsilon_z < \epsilon_r$ so with $r_x = \epsilon_r$ and $r_z = \epsilon_z$ the bounds on the nonlinear components of the error dynamics are given by Lemma 2.4 and Lemma 2.5. Noting that $\hat{x}(k|k) = x(k) - e(k|k)$ and applying Theorem 2.7 completes the proof. Note, if $|\epsilon_z| \leq 2$ then (2.44) is automatically satisfied if (2.43) holds. ∎

This theorem gives sufficient conditions for the stability of the error system of the extended Kalman filter for a general nonlinear signal model with a linear output map.

It gives coupled conditions on

- the bound on the initial error;
- bounds on the noise proceses;
- observability of the signal model state;
- controllability of the process noise with the signal model state map; and
- smoothness properties of the signal model.

Within the regime of satisfaction of the sufficient stability conditions of Theorem 2.9 there is a trade off between these requirements. The result given here represents a logical extension of linear Kalman filter theory to the nonlinear case. That is, it collapses to the linear theory where appropriate though necessarily the conditions given are more restrictive.

2.5 Conclusion

This chapter presented a new stability result for the extended Kalman filter, Theorem 2.9. This theorem gives sufficient conditions for the error system of the EKF, when applied to a discrete time system, to be asymptotically stable when there is no driving noise; and bounded-input bounded-output stable when the system is driven by noise. This result provides an understanding of EKF dynamics and limiting features. As a consequence it can be used to form the basis for design rules for the EKF, though the resulting observer will be conservative.

Theorem 2.9 relies on weaker, local conditions on the signal model than previous results which required that the signal model satisfy restrictive, global conditions. Such global conditions are not necessary in EKF analysis as it is only possible to prove local convergence results. However it is possible to weaken some of the conditions of Theorem 2.9 even further. In particular the assumptions of (local) observability and controllability can potentially be weakened to detectability and stabilizability. In addition the assumption of an invertible Jacobian of the state dynamics does not appear to be fundamental to the derivation.

A limitation of this result is the assumption of bounded noise. This cannot be weakened as there would then be a non-zero probability of the system moving outside the region where the linearisation assumptions are valid. However, this limitation is not overly restrictive as the EKF cannot be expected to work in circumstances of unbounded noise without more restrictive assumptions on the nature of the nonlinearities in the signal. In addition this result is only applicable to systems which have nonlinear state dynamics but a linear output map. The converse situation of a nonlinear measurement equation and linear state equation would satisfy a similar theorem. The extension to the fully nonlinear case will require stronger assumptions on the degree of nonlinearities in the signal and on the noise characteristics. Due to the presence of one of the noise processes in the argument of the linearisation error in this case, a stronger assumption of a sufficiently high signal-to-noise ratio will probably be required.

In the next chapter an EKF-based frequency tracker is proposed. The understanding of the dynamics of the EKF gained from Theorem 2.9 is used to tune the design of this observer. Simulation results confirm the conclusions of this theorem.

3

Design of an EKF Frequency Tracker

3.1 Introduction

IN this chapter an EKF observer is designed for the problem of tracking the time-varying frequency of a signal which has a signal-to-noise ratio (SNR) of at least 5 dB. It has been shown (James, 1992) that EKF-based frequency trackers using the noisy sinusoid signal as input to the EKF, will suffer from *thresholding* at SNRs of less than 5 dB. That is, the mean squared error of the frequency estimate increases dramatically for SNRs below the threshold value as the EKF fails to track. The problem of frequency tracking at SNRs of less than 5 dB will be considered in Chapter 4.

Studies of the behaviour of linear Kalman filtering problems (Chan *et al.*, 1984; de Souza *et al.*, 1986) indicate that stability problems are likely to arise in frequency estimation due to possible uncontrollable model modes on the unit circle. In Kalman filtering, when there are uncontrollable modes on the unit circle the Kalman gain converges to zero, ultimately causing the filter to ignore the data. Accordingly the stability of the error system of the EKF frequency tracker will be considered along with what this indicates about design. The issue of stability of the errors of the EKF is the key point in any EKF design. Unlike the linear Kalman filter where general theoretical results have long been available, most EKF designs have relied on heuristic arguments to attempt to ensure stability. A summary of these arguments will be presented in Sections 3.4.1 and 3.4.2.

In addition to these arguments, we will make use of the stability result of Chapter 2. This new result, Theorem 2.9, quantifies the stability of the EKF in terms of the degree of nonlinearity of the system, noise covariance matrices and bounds on the noise processes. This new result also provides the first theoretical analysis of EKF performance for general stochastic, discrete time, nonlinear systems. This analysis will be used to provide design guidelines for the problem of frequency tracking at high SNR.

This chapter focuses on the choice of appropriate covariance matrices to balance noise rejection with tracking at a maximal slew rate. These choices are very non-obvious and the nature of the performance penalty for over- and under-specification of noise covariances is shown. The performance of the observer is illustrated via simulation results.

3.2 Signal Model

Kalman filter design and, by implication, EKF estimator design proceeds from a state space signal model of the process to be estimated. The signal model dynamics describe a mechanism for how the process may be evolving. In Kalman filtering the signal model is linear and consists of a dynamic state equation driven by a noise process and an output measurement equation corrupted by additive noise. Kalman filter theory endeavours to construct an optimal estimator for the state given the noise covariances Q and R, where optimality is measured in terms of the error covariance. The EKF is derived for nonlinear signal models using the Kalman filter on an associated linearised system. In this case the matrices Q and R used in the EKF equations can no longer be regarded as the same values as the noise covariances of the nonlinear signal model. They are instead measures of the degree of disturbances in the associated linearised system. Our focus here will be on the appropriate selection of design Q and R values in relation to the noise covariance matrices. The inclination is to take these design values to be the same as the noise covariances driving the nonlinear equation, but it can be shown that this leads to potentially poor performance. Further, since the signal model is a construct rather than an exact description of the measurement source, robustness of the estimator to design selections in the EKF needs to be considered.

As a nonlinear signal model describing the evolution of noisy quadrature data of a slowly time-varying frequency, consider

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} \cos x_3(k) & -\sin x_3(k) & 0 \\ \sin x_3(k) & \cos x_3(k) & 0 \\ 0 & 0 & 1-\epsilon^a \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ w_3(k) \end{bmatrix} \quad (3.1)$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \quad (3.2)$$

where $\{y(k)\}$ is the 2-vector received signal, x_3 is the unknown time-varying frequency and x_1 and x_2 are the exact in-phase and quadrature signals. The parameter $\epsilon^a \in (0,1)$ determines the rate of time variation of x_3 and is chosen so that the frequency varies slowly enough that the signal appears periodic over several cycles. The signals $\{v(k)\}$ and $\{w(k)\}$ are zero mean, independent noise processes with

$$\begin{aligned} E[w(k)w(l)^T] &= Q^a\delta_{kl} \\ E[v(k)v(l)^T] &= R^a\delta_{kl} \end{aligned}$$

where δ_{kl} is the Kronecker delta function, $Q^a \geq 0$ and $R^a > 0$. The frequency of the signal, x_3, represents the state of the system that we wish to recover. The x_1 and x_2 components are noiseless transformations of the x_3 component, hence

$$Q^a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q^a \end{bmatrix}. \quad (3.3)$$

The maximal slew rate of the frequency x_3 is determined by ϵ^a and q^a. Larger ϵ^a and smaller q^a correspond to smaller variance for the x_3 component, yielding a lower slew rate.

This simple model has properties which make it useful for designing filters for quasi-periodic signals generated from a variety of applications. The cartesian formulation of the model makes it close to linear thus reducing errors due to linearisation effects when filtering, compared with the polar formulation which has the frequency and phase as state variables. Furthermore in the absence of prior knowledge of systematic variation in the frequency of the signal other than that it is slowly varying, modelling the variation in frequency via a random walk is a reasonable choice as this simple model can be tuned in a straightforward manner to allow for varying rates of change via appropriate design choices for Q^a and ϵ^a. Note that since this model has a linear output map and nonlinear state dynamics, Theorem 2.9 from Chapter 2 can be applied to derive design guidelines also.

3.2.1 Observability

A difficulty of the frequency tracking problem is that it is not possible to achieve global observability for any formulation of the discrete-time frequency tracking problem due to aliasing effects. Lack of observability increases the difficulties in state reconstruction. However consider for the moment a nonlinear system without noise, i.e.

$$x(k+1) = f(x(k)) \quad (3.4)$$

$$y(k) = h(x(k)) \quad (3.5)$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

Definition 3.1 *The system (3.4)–(3.5) is* strongly locally observable at x *if there exists a neighbourhood, U, of x : $h(f^k(x)) \neq h(f^k(\bar{x}))\ \forall\ \bar{x} \in U, \bar{x} \neq x$ and $k = 0, \ldots, n-1$.*

Definition 3.2 *Define the map $H^n : \mathbb{R}^n \to (\mathbb{R}^m)^n$*

$$H^n(x) \triangleq (h(x), h(f(x)), \ldots, h(f^{n-1}(x)))$$

The system satisfies the observability rank condition at x *if the rank of $H^n(x) = n$.*

Theorem 3.1 (Nijmeijer, (1982)) *If the system satisfies the observability rank condition at x then it is strongly locally observable at x.*

Corollary 3.1.1 (Nijmeijer, (1982)) *If the system satisfies the observability rank condition for all x then the system is strongly locally observable for all x.*

If it is possible to resolve the state locally without ambiguity in the deterministic case, it is reasonable to expect that an observer could be constructed in the stochastic environment which could successfully track the state of the system given a sufficiently good initial state estimate. The signal model, (3.1)–(3.2), proposed for the frequency tracking problem is strongly locally observable as will now be shown.

The rank of a map $H(x)$ is equal to the rank of the matrix $\frac{\partial H}{\partial x}$ evaulated at x. This makes the observability rank condition reduce to requiring that

$$\begin{aligned}
\frac{\partial H^n}{\partial x}(x_0) &= \begin{bmatrix} \frac{\partial h}{\partial x}(x_0) \\ \frac{\partial h}{\partial x}(x_1)\frac{\partial f}{\partial x}(x_0) \\ \vdots \\ \frac{\partial h}{\partial x}(x_{n-1})\frac{\partial f}{\partial x}(x_{n-2})\cdots\frac{\partial f}{\partial x}(x_0) \end{bmatrix} \\
&= \begin{bmatrix} H(x_0) \\ H(x_1)F(x_0) \\ \vdots \\ H(x_{n-1})F(x_{n-2})\cdots F(x_0) \end{bmatrix}
\end{aligned}$$

be of full rank. This is the same as requiring that if we linearise the system at each time instant around the value of the state at that time instant, then this sequence of linearised systems must be observable (in the linear systems sense) when evaluated along the trajectories of the system.

The trajectories of (3.1)–(3.2) without noise are

$$x(k) = \begin{bmatrix} x_1(0)\cos(\sum_{i=0}^{k-1} x_3(i)) - x_2(0)\sin(\sum_{i=0}^{k-1} x_3(i)) \\ x_1(0)\sin(\sum_{i=0}^{k-1} x_3(i)) + x_2(0)\cos(\sum_{i=0}^{k-1} x_3(i)) \\ (1-\epsilon)^k x_3(0) \end{bmatrix}$$

therefore

$$[\frac{\partial H^n}{\partial x}(x(0))]^T[\frac{\partial H^n}{\partial x}(x(0))] =$$
$$\begin{bmatrix} 3 & 0 & -2x_2(0) - 1 - \epsilon x_2(0) \\ 0 & 3 & 2x_1(0) + 1 - \epsilon x_1(0) \\ -2x_2(0) - 1 - \epsilon x_2(0) & 2x_1(0) + 1 - \epsilon x_1(0) & (x_1(0)^2 + x_2(0)^2)(1 + (2-\epsilon)^2) \end{bmatrix}$$

This will be of full rank except when $x_1(0) = 0$ and $x_2(0) = 0$ so the system given by (3.1) – (3.2) is strongly locally observable.

3.3 EKF Observer

Recall the EKF equations for a nonlinear signal model of the form

$$x(k+1) = f(x(k)) + w(k) \tag{3.6}$$
$$y(k) = h(x(k)) + v(k) \tag{3.7}$$

where $E[w(k)w(k)^T] = Q^a(k)$ and $E[v(k)v(k)^T] = R^a(k)$. These are given by (2.3)–(2.7).

In the case of (3.1)-(3.2) we have

$$F(k) = \left.\begin{bmatrix} \cos(\hat{x}_3) & -\sin(\hat{x}_3) & -\hat{x}_1 \sin(\hat{x}_3) - \hat{x}_2 \cos(\hat{x}_3)) \\ \sin(\hat{x}_3) & \cos(\hat{x}_3) & \hat{x}_1 \cos(\hat{x}_3) - \hat{x}_2 \sin(\hat{x}_3) \\ 0 & 0 & 1-\epsilon^d \end{bmatrix}\right|_{\hat{x}=\hat{x}(k|k)} \tag{3.8}$$

$$H(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.9}$$

In constructing our EKF frequency tracker some assumptions will be made on the structure of the noise covariance matrices used in the design. To distinguish between the noise covariance matrices assumed to be driving the signal model and those used in the construction of the observer, the former will be denoted by a superscript a indicating these are the *actual* values and the latter with a superscript d for the *design* values.

Note:

1. The EKF observer will be constructed assuming a diagonal Q of the form

$$Q^d = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_1 & 0 \\ 0 & 0 & q_2 \end{bmatrix}$$

   where $0 < q_1 \ll q_2$ instead of

$$Q^d = Q^a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q^a \end{bmatrix}$$

   as would be indicated by the signal model to ensure that the model used to construct the EKF is stabilizable. It is known that the stabilizability of $[F, Q^{\frac{1}{2}}]$ ensures that the Kalman filter is asymptotically stable in the linear filtering case. It shall be demonstrated subsequently that the design choice of Q in the EKF greater than the actual value of Q used in the signal model is, in fact, a key element in securing state estimate error bounds.

2. Assume that the noise in both channels is of the same magnitude i.e. $R^d(k) = r^d I$.

Note that the values ϵ^d, Q^d and R^d are *design parameters* and are not necessarily equal to the values given in the signal model, although the usual implication would have them so. When choosing the filter parameters Q^d and ϵ^d it is necessary to allow for not only the expected maximum slew rate of the frequency but also linearisation errors. The choice of R^d and Q^d will affect the stability properties and rate of response of the filter. If these design issues are neglected and these values are set to the estimated signal model values, the resulting filter may have undesirable properties.

The EKF is derived via Kalman filtering and linearisation but is still fundamentally nonlinear. Thus, while the performance of the Kalman filter relies mostly on the relative size of Q^d and R^d or the signal to noise ratio, as can be seen simply from the dual Linear Quadratic Regulator problem (Bitmead *et al.*, 1990), the EKF depends on the separate values of Q^d and R^d. The other feature distinguishing the EKF from the Kalman filter is that linearisation errors also must be accounted for, as we shall see. The design trade-offs in developing extended Kalman filters (and indeed Kalman filters) based on signal models are :–

- filter divergence, where the filter becomes increasingly confident of increasingly bad estimates through the computed error covariance becoming too small, leading to loss of tracking capabilities; and

- sensitivity to noise, where the parameter estimator fails to reject enough of the measurement noise process.

These effects can be tied to questions of the magnitude of the Kalman gain $K(k)$ being either too small or too large respectively. In the following sections the effect of suitable choices of Q^d, ϵ^d and R^d and how these problems might be avoided is discussed.

3.4 Heuristic Design Issues

3.4.1 General Linear Issues

To appreciate the design issues connecting guaranteed performance with computed performance assessment it is necessary to define several filters and their associated covariance functions. Consider a signal generated by a linear signal model with noise covariances Q^a and R^a.

Let $\hat{x}^o(k|k-1)$ be the state estimates produced by a Kalman filter designed for this system with $Q = Q^a$ and $R = R^a$. This estimator is the optimal state estimator, since the system is linear, and the estimates achieve the minimum covariance $P^o(k|k-1)$.

Let $\hat{x}^p(k|k-1)$ be the state estimates produced by a Kalman filter designed for the linear system with $Q = Q^d$ and $R = R^d$, which are not necessarily the same as Q^a and R^a. This is a suboptimal estimator. Its state estimates $\hat{x}^p(k|k-1)$ achieve an error covariance

$$P^p(k|k-1) = E[(x(k) - \hat{x}^p(k|k-1))(x(k) - \hat{x}^p(k|k-1))^T]$$

whereas the designer computes a "designed" covariance, $P^d(k|k-1)$ via the Riccati equation for the Kalman filter with $Q = Q^d$ and $R = R^d$. The superscript p indicates that P^p measures the performance of the filter designed with Q^d and R^d when it is run on the real data which is produced with Q^a and R^a.

The following theorem states the relationship between P^o, P^p and P^d as Q^d and R^d vary.

Theorem 3.2 *If* $Q^d \leq Q^a$ *and* $R^d \leq R^a$ *then*

$$P^d(k|k-1) \leq P^o(k|k-1) \leq P^p(k|k-1).$$

If $Q^d \geq Q^a$ *and* $R^d \geq R^a$ *then*

$$P^o(k|k-1) \leq P^p(k|k-1) \leq P^d(k|k-1).$$

■

Optimality and dual optimal control arguments can be used to establish the above result, see (Anderson and Moore, 1979; Bitmead *et al.*, 1990).

From this theorem we see that if Q^d and R^d are selected to be greater the actual values, then the achieved performance, P^p, is bounded above by the computed performance, P^d. Note that P^p is what we wish to guarantee bounded. Under-design possesses no such guarantees and there is no bound on the level of actual error. Accordingly, over-design is to be preferred with the Kalman filter and hence with the EKF. For the EKF there are further reasons to pursue still larger Q^d.

3.4.2 General Nonlinear Issues

An issue to consider is that for the EKF the solution of the Riccati equation, $P^d(k|k-1)$, is only a first order approximation to the true error covariance, $P^p(k+1|k)$. It has been shown (Jazwinski, 1970) that this approximation is an under-estimate. Consider the nonlinear function $f(x(k))$ expanded in a Taylor series about some value, x^*

$$f_i(x(k)) = f_i(x^*) + (x(k) - x^*)^T \nabla f_i + \tfrac{1}{2}(x(k) - x^*)^T \mathcal{H}_i (x(k) - x^*) + \cdots \qquad (3.10)$$

where f_i is the i-th row of $f(x(k))$, ∇f_i is the gradient and $\mathcal{H}_i$ is the Hessian.

The recursion (2.6) for $P^d(k+1|k)$ only includes terms up to first order in the expansion (3.10) and is thus a crude approximation to $P^p(k+1|k)$. If second order terms in (3.10) are taken into account the following result is obtained.

Lemma 3.1 *To second order*

$$P^p(k+1|k) = F(k)P(k|k)F(k)^T + Q(k) + \tfrac{1}{4}(P\partial^2 f)(k|k)(P\partial^2 f)(k|k)^T$$

where

$$(P\partial^2 f)(k|k) \triangleq \left[\sum_{j,k=1}^{n} p_{j,k} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\hat{x}(k|k)) \right]$$

■

This result is derived in Jazwinski (1970). This lemma shows that the linearisation itself introduces effects which increase the effective Q value. Thus the associated linearised system which is used to calculate the gain is "noisier" than the original nonlinear system.

3.5 Design Issues Arising from Stability of the Observer

The stability result of Chapter 2 can be used to show that this EKF observer for the frequency tracking problem will, in fact, be able to track the underlying frequency of its input signal subject to conditions on the evolution of the target frequency and the design parameters of the EKF. This is done by showing that the observer designed in Section 3.3 satisfies the standing assumptions on the signal model made in Chapter 2, (2.26)–(2.29) and the conditions of Theorem 2.9. This requires consideration of the properties of the received signal and also the properties of the signal model used to design the EKF observer.

Briefly the conditions our observer must satisfy are:

1. boundedness of the true state;

2. smoothness properties of the output and state dynamics;

3. bounded noise processes; and

4. observability and controllability of the signal model.

In order to obtain any meaningful results from nonlinear filtering, it is necessary that the state of the received signal be bounded for all time. It is rarely possible to derive global, nonlinear stability results and the methods employed in Chapter 2 certainly are not. However, while the signal model chosen does not ensure a bounded state, for the frequency tracking problem to be well posed we need to be sampling the received signal fast enough that $0 < x_3(k) \ll 2\pi$. Hence, in practice, the state of the signal will satisfy

$$\begin{aligned} |x(k)| &= \sqrt{x_1(k)^2 + x_2(k)^2 + x_3(k)^2} \\ &\leq \sqrt{1+4\pi^2} \\ &< 6.4 \quad \forall\, k. \end{aligned}$$

Consider now the signal model used to design the EKF. Let $||\cdot||$ be the induced Euclidean norm. Immediately we have

$$||H(k)|| = 1 \quad \text{for all } k.$$

In addition we require $f \in C^3(\mathbb{R}^n, \mathbb{R}^n)$ and $\frac{\partial f}{\partial x}$ be invertible. In our case the derivatives of f are continuous and will be bounded in a region but these bounds depend on the size of the region, that is

$$
\begin{aligned}
\|\frac{\partial f}{\partial x}(x(k) - e(k|k))\| &\leq 2(R_x + \epsilon_r), \\
\|\frac{\partial^2 f}{\partial x^2}(x(k) - e(k|k))\| &\leq \sqrt{2}(R_x + \epsilon_r), \\
\|\frac{\partial^3 f}{\partial x^3}(x(k) - e(k|k))\| &\leq \sqrt{2}(R_x + \epsilon_r)
\end{aligned}
$$

when $|x(k)| \leq R_x$ and $|e(k|k)| \leq \epsilon_r$ for some $R_x > 0$ and $\epsilon_r > 0$. Furthermore the determinant of $\frac{\partial f}{\partial x}$ is simply $1-\epsilon$ and since $0 < \epsilon < 1$ it is therefore invertible. Thus the nonlinear signal model used to design our frequency tracker is sufficiently smooth.

From the noise processes used in the design of the EKF observer, we have

$$Q(k) \geq q_1 I$$

$$R(k) = rI.$$

Suppose the noise processes are bounded by some multiple of their standard deviation i.e.

$$
\begin{aligned}
\|w\| &= c_1\sqrt{q_2} \\
\|v\| &= c_2\sqrt{r}
\end{aligned}
$$

for some constants $c_1 > 0$ and $c_2 > 0$.

All that remains in order to apply Theorem 2.9 is to show that a region can be found in which the Gramians of the EKF are positive definite and bounded. From the definitions of the Gramians, (2.22) and (2.23), and the EKF observer, the controllability Gramian $C(k,1)$ is simply $Q(k)$ and hence

$$q_2 I \geq C(k,1) \geq q_1 I$$

for any positive ϵ_r.

The observability Gramian of $[F, R^{-\frac{1}{2}}H]$ is

$$\mathcal{O}(k,1) = \begin{bmatrix} \frac{2}{r} & 0 & -\frac{1}{r}\gamma_1(k,k-1) \\ 0 & \frac{2}{r} & -\frac{1}{r}\gamma_2(k,k-1) \\ -\frac{1}{r}\gamma_1(k,k-1) & -\frac{1}{r}\gamma_2(k,k-1) & \frac{1}{r}\{m_1(k,k-1)^2 + m_2(k,k-1)^2\} \end{bmatrix}$$

where

$$\begin{aligned}
\phi(k,i) &= \sum_{l=i}^{k-1} \hat{x}_3(l) \\
\gamma_1(k,i) &= \sum_{l=i}^{k} (1-\epsilon)^{l-i}\{-\hat{x}_1(l)\sin\phi(k,l) - \hat{x}_2(l)\cos\phi(k,l)\} \\
\gamma_2(k,i) &= \sum_{l=i}^{k} (1-\epsilon)^{l-i}\{\hat{x}_1(l)\cos\phi(k,l) - \hat{x}_2(l)\sin\phi(k,l)\} \\
m_1(k,i) &= \sum_{l=i}^{k} (1-\epsilon)^{l-k}\{\hat{x}_1(l)\sin\bar{\phi}(i,l) + \hat{x}_2(l)\cos\bar{\phi}(i,l)\} \\
m_2(k,i) &= \sum_{l=i}^{k} (1-\epsilon)^{l-k}\{-\hat{x}_1(l)\cos\bar{\phi}(i,l) + \hat{x}_2(l)\sin\bar{\phi}(i,l)\} \\
\bar{\phi}(i,l) &= \phi(k,l) - \phi(k,i)
\end{aligned}$$

This has eigenvalues

$$\frac{2}{r},\ \frac{1}{2r}\{2 + w_3 \pm \sqrt{w_3^2 + 4}\}$$

where

$$\begin{aligned}
w_3 &= \hat{x}_1(k|k)^2 + \hat{x}_2(k|k)^2 \\
&= \{x_1(k) - e_1(k)\}^2 + \{x_2(k) - e_2(k)\}^2 \\
&\leq 1 + 2\epsilon_r[x_1(k) + x_2(k)] + 2\epsilon_r^2
\end{aligned}$$

and hence

$$(1 - \sqrt{2}\epsilon_r)^2 \leq w_3 \leq (1 + \sqrt{2}\epsilon_r)^2.$$

The maximum eigenvalue of $\mathcal{O}(k,1)$ therefore satisfies

$$\begin{aligned} \lambda_{\text{max}} &\leq \frac{1}{2r}\{(1+\sqrt{2}\epsilon_r)^2 + 2 + \sqrt{(1+\sqrt{2}\epsilon_r)^4 + 4}\} \\ &< \frac{3}{r}(1+\sqrt{2}\epsilon_r)^2 \end{aligned}$$

Now the requirements that ϵ_r be positive and $\mathcal{O}(k,1)$ be positive definite imply $0 < \epsilon_r < \frac{1}{\sqrt{2}}$ which ensures that w_3 can never be zero. With this restriction on ϵ_r, the minimum eigenvalue of $\mathcal{O}(k,1)$ satisfies

$$\begin{aligned} \lambda_{\text{min}} &\geq \frac{1}{2r}\{(1-\sqrt{2}\epsilon_r)^2 + 2 - \sqrt{(1-\sqrt{2}\epsilon_r)^4 + 4}\} \\ &\geq \frac{1}{2r}(1-\sqrt{2}\epsilon_r)^2\{1 - \frac{1}{4}(1-\sqrt{2}\epsilon_r)^2\} \end{aligned}$$

Thus, for the EKF frequency tracker, we can set

$R_x = 7$ $\epsilon_r = \frac{1}{\sqrt{2}} - \sigma \quad \sigma > 0$ and arbitrarily small

$\rho_1 = 2(R_x + \epsilon_r)$ $\rho_2 = \sqrt{2}(R_x + \epsilon_r)$

$\rho_3 = \sqrt{2}(R_x + \epsilon_r)$ $\rho_4 = \sqrt{2}(R_x + \epsilon_r)$

$\rho_5 = 1$

$\delta_1 = q_1$ $\delta_2 = r$

$a_1 = q_2$ $a_2 = q_1$

$b_1 = \frac{1}{2r}(1-\sqrt{2}\epsilon_r)^2\{1 - \frac{1}{4}(1-\sqrt{2}\epsilon_r)^2\}$ $b_2 = \frac{3}{r}(1+\sqrt{2}\epsilon_r)^2$

To satisfy Theorem 2.9 we need condition (2.43),

$$\beta(\zeta + \eta) + \alpha < 1$$

to hold, where α and β, η and ζ are given by the equations (2.36), (2.40) and (2.42) respectively.

Recall that the stability parameters α and β determine the rate of the decay of the linear portion of the error dynamics. The stability parameter ζ is a bound on the nonlinear portion of the dynamics and η is a bound on the term due to the noise-based perturbations and linearisation errors. The size of the parameters will depend proportionally on the size of the ball $\{e : |e| \leq \epsilon_r\}$ and, in the case of η, on the size of the noise processes. Thus, if we know the nature of the noise processes and design an EKF observer using those values, we can ensure the condition on the stability parameters is satisfied by considering a sufficiently small region.

On the other hand, we can regard r, q_1 and q_2 as design variables rather than as estimates of the true covariances of the noise processes. The first three stability parameters, α, β and ζ are functions of these design values only. The stability parameter η is a function of the properties of the noise in the received signal as well as the design values, through the bounds $||w||$ and $||v||$. Analysing the first two stability parameters as functions of the design variables gives the following table, Table 3.1. This table gives indicative results for the properties of the stability parameters with respect to each design variable.

	α	β
q_1	decreasing	decreasing
q_2	increasing	increasing
r	maximised by $r_\alpha^{(1)}$	minimised by $r_\beta^{(2)}$

Table 3.1: Stability parameters as functions of the design variables

Note:

1. $r_\alpha = \{\frac{3}{4}(1-2\epsilon_r^2)^2[1-\frac{1}{4}(1-\sqrt{2}\epsilon_r)^2]p(\frac{1}{s}+\frac{1}{q})\}^{\frac{1}{2}}$
2. $r_\beta = \{\frac{3p}{2q}(1-2\epsilon_r^2)^2[1-\frac{1}{4}(1-\sqrt{2}\epsilon_r)^2]\}^{\frac{1}{2}}$

The stability of the filter depends on making α and β sufficiently small. From this table, and the constraint that $q_1 \leq q_2$ [1], we can see that designing our EKF frequency tracker with

$$Q(k) = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_1 & 0 \\ 0 & 0 & q_2 \end{bmatrix}$$

where $q_1 = q_2 = q^d$ and q^d relatively small, will enhance the stability of the filter even though we are supposing that the first two components are obtained via a noiseless

[1] If $q_1 > q_2$ then α and β will be *increasing* functions of q_1 and *decreasing* functions of q_2.

transformation of the frequency, x_3. Note that q^d is an EKF design parameter. If this is taken to be smaller than the value indicated by the maximal slew rate of received signal, the stability parameter η will be inflated.

3.6 Choice of EKF Frequency Tracker Design Values

3.6.1 Choice of Q^d

The choice of Q^d is crucial to the performance of any EKF observer. From equation (2.6) it can be seen that the minimum value of the solution of this Riccati equation is given by Q^d. Too small a value of Q^d leads to over-confidence in the accuracy of the estimates ($P^d(k+1|k)$ too small) and consequently to the filter paying insufficient attention to new data ($K(k)$ too small). This will cause filter divergence.

For the frequency tracking problem, the values of Q^d and ϵ^d affect the stability of the filter and also determine the maximal slew rate it can detect. Even if q^a is known it is important in the design of the EKF that $||Q^d|| > ||Q^a||$.

In the past, more precise statements on the appropriate choice of Q^d have not been possible. However, Theorem 2.9 now provides the following insights. If Q^d is chosen to be positive definite, then the controllability Gramian of $[F, Q^d]$ will be bounded and positive definite. The conditions of Theorem 2.9 show that this property is one of a set of sufficient conditions for the errors of the EKF for any nonlinear system to be bounded, provided the initial error is small enough.

Also, from considering the stability parameters defined in Theorem 2.9, it was shown that choosing Q^d of the form

$$Q^d = \begin{bmatrix} q^d & 0 & 0 \\ 0 & q^d & 0 \\ 0 & 0 & q^d \end{bmatrix}$$

helped maximise the size of the region of admissible initial errors.

3.6.2 Trade-off between Q^d and ϵ^d

For the frequency tracking problem, a key design parameter is that of the maximum rate of change of the frequency (slew rate) the filter can track. As we wish to be able to track either an increasing or decreasing frequency we associate the slew rate with the variance of the frequency. In the signal model (3.1)-(3.2) the frequency variation is given by the equation

$$x_3(k+1) = (1-\epsilon^d)x_3(k) + w_3(k). \tag{3.11}$$

Thus the variance of the frequency is given by

$$\sigma^2 = \frac{q_2^d}{1-(1-\epsilon^d)^2} \tag{3.12}$$

and so the maximal slew rate of the signal that the EKF can track is determined by q_2^d and ϵ^d. Hence a large Q^d allows the EKF to track a signal with a potentially large slew rate. The drawback of a large Q^d is that even when the EKF is tracking well, if the actualslew rate is low the sensitivity to noise caused by a large Q^d causes the state estimates to fluctuate widely around the true frequency value. Setting $\epsilon^d < \epsilon^a$ alleviates this problem by allowing the value of Q^d to be decreased while still retaining the desired range for the frequency estimate. This, along with the design of Q^d and R^d, is related to measures to achieve guarantees of the degree of stability of Kalman filters (Anderson and Moore, 1979, Section 6.2).

3.6.3 Choice of R^d

The choice of R^d, for the frequency tracking problem, appears less critical than that of Q^d. The tracking ability of the EKF is relatively insensitive to the value of R. Instead, the value of R^d affects the degree of variation in the state estimates. This can be shown by considering once again the dual optimal control problem. It is well known in the control literature, Anderson and Moore (1990, Chapter 6) for example, that increasing the cost of control (i.e. increasing R^d) reduces the feedback gain and the speed of response of the controlled system. Since the feedback gain in the control problem is equal to $-K(k)^T$, reducing the magnitude of the feedback gain is equivalent to reducing the gain in the estimation problem also. If control is cheap (i.e. R^d is small) this increases the speed of response of the system by allowing higher feedback gains. The penalty of this is that the state variables may vary more widely causing excessive (and undesirable) peaking.

3.7 Simulation Results

Clearly, to design an effective EKF frequency tracker, it is essential that the choice of design parameters be conservative to ensure stability and good tracking performance. The following simulation results illustrate the importance of such a design. For these simulations the signal was not generated by the signal model. This was to emphasise the fact that the signal model is used purely for the construction of the filter. In EKF observer design we do not suppose that the signal is generated by the model. The model is chosen so that it provides an understandable mechanism for the evolution of the signal and has suitable properties which will produce an effective filter.

The values of the design parameters used and the sum of the squared error in the frequency estimate are summarised in the table below, Table 3.2.

Figure	Q^d	ϵ^d	R^d	MSE
(3.2)	0.0005	0.05	0.01	2.9236
(3.3)	0.0065	0.05	0.01	1.8317
(3.4)	0.0013	0.01	0.01	0.3416
(3.5)	0.0023	0.01	0.05	0.2853

Table 3.2: Design values

While the data set was not generated by the model it can be thought of as having been generated by the model with a variance determined by the maximum change in the frequency. The effective actual values of the parameters for this data set, assuming they were generated by the model, were

$$\begin{aligned} Q^a &= 0.0005 \\ \epsilon^a &= 0.05 \\ R^a &= 0.01. \end{aligned}$$

In each simulation $|x_3(1) - \hat{x}_3(1|0)| = 0.01$. In the figures the evolution of the normalised target frequency is given by the solid line and that of the EKF estimate by the dashed line.

Mean squared tracking error of the frequency versus Q^d is plotted in Figure 3.1. The value of Q^a in the nonlinear signal model is shown as is the best Q^d value for the EKF. It clearly illustrates that it is inappropriate to set $Q^d = Q^a$. It also shows the large penalty for too small a Q^d and the modest degradation for too large a value.

Figure 3.2 shows the effect of designing an EKF frequency tracker using the actual model values. A common assumption is that using these design values would yield good performance. As predicted in Section 3.4.2 and illustrated in Figure 3.2, this is not the case. However higher Q^d does permit tracking as depicted in Figure 3.3, where $Q^d = 0.0065$ as the linearisation errors in the associated linearised signal model are now taken in account.

It is clear from Figure 3.3 that the EKF with $Q^d > Q^a$, $\epsilon^d = \epsilon^a$ and $R^d = R^a$ is still not ideal. The filter remains unable to track when the slew rate is at its maximum and the degree of variability in the state estimates is large. Setting $\epsilon^d = 0.01$ and then adjusting Q^d down to $Q^d = 0.0013$ to maintain the same maximum slew rate permits the filter to maintain a similar tracking performance with less sensitivity to noise. This is illustrated in Figure 3.4. Finally setting $R^d = 0.05$ and increasing Q^d to $Q^d = 0.0023$ produces a filter which is able to track the frequency with less peaking. This is illustrated in Figure 3.5. Note that the MSE for this filter is less than a tenth of that for the filter using the effective actual values.

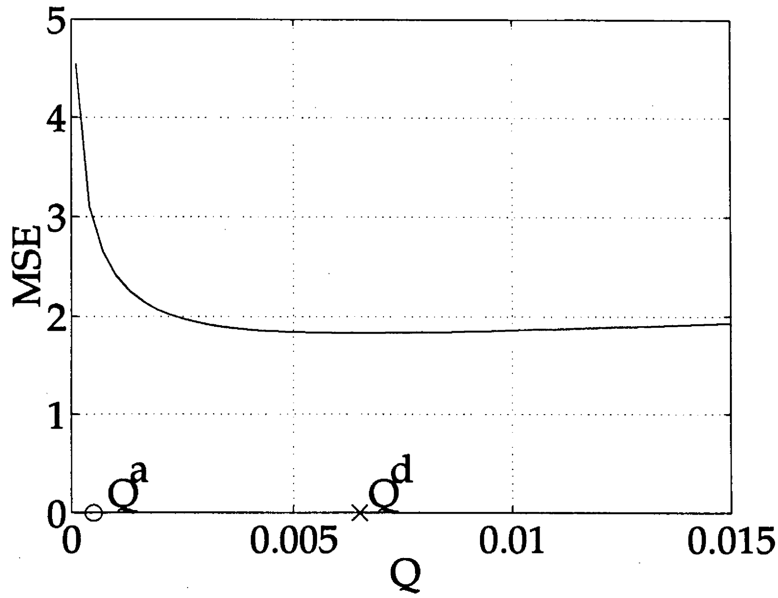

Figure 3.1: MSE of EKF frequency estimate versus Q^d

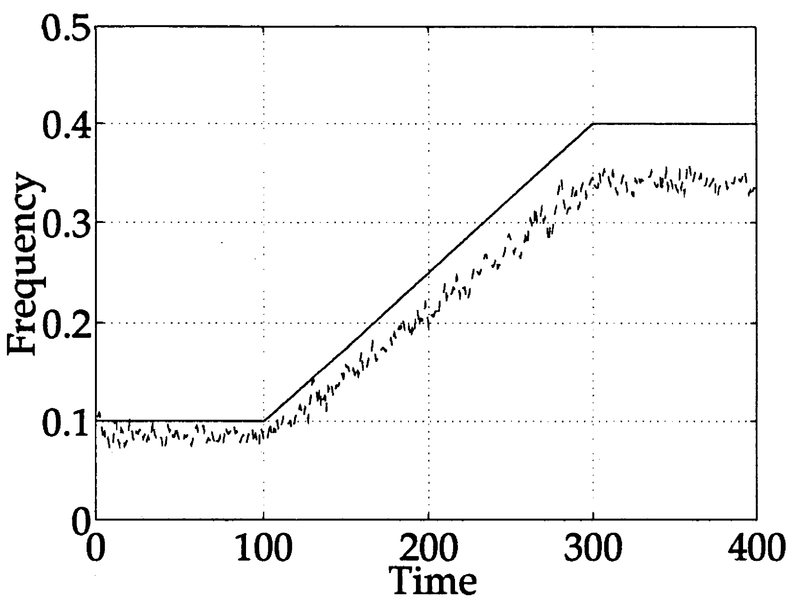

Figure 3.2: EKF frequency estimate when $Q^d = Q^a$, $\epsilon^d = \epsilon^a$ and $R^d = R^a$. The target signal is given by the solid line.

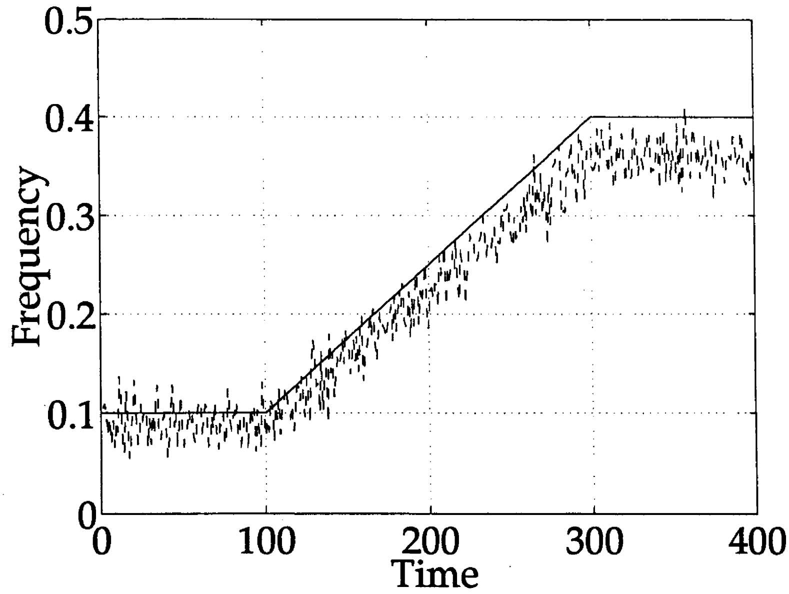

Figure 3.3: EKF frequency estimate when $Q^d > Q^a$, $\epsilon^d = \epsilon^a$ and $R^d = R^a$. The target signal is given by the solid line.

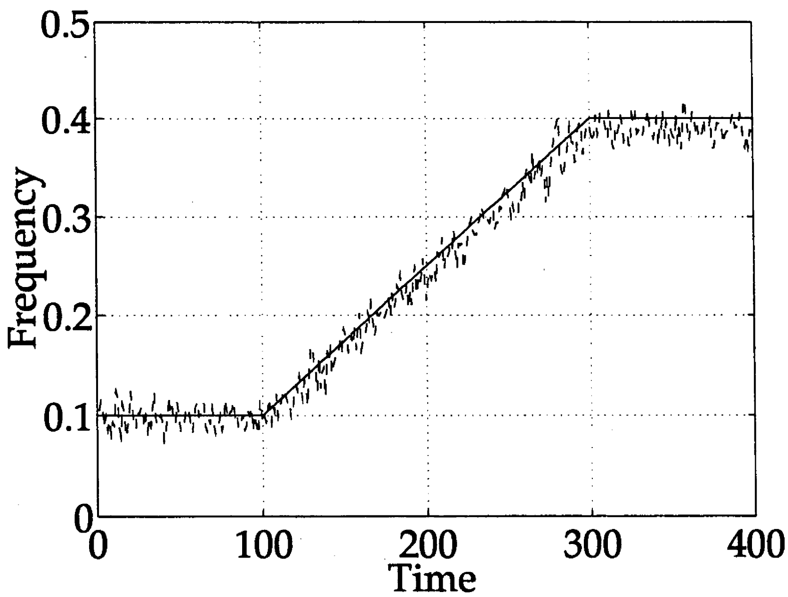

Figure 3.4: EKF frequency estimate when $Q^d > Q^a$, $\epsilon^d < \epsilon^a$ and $R^d = R^a$. The target signal is given by the solid line.

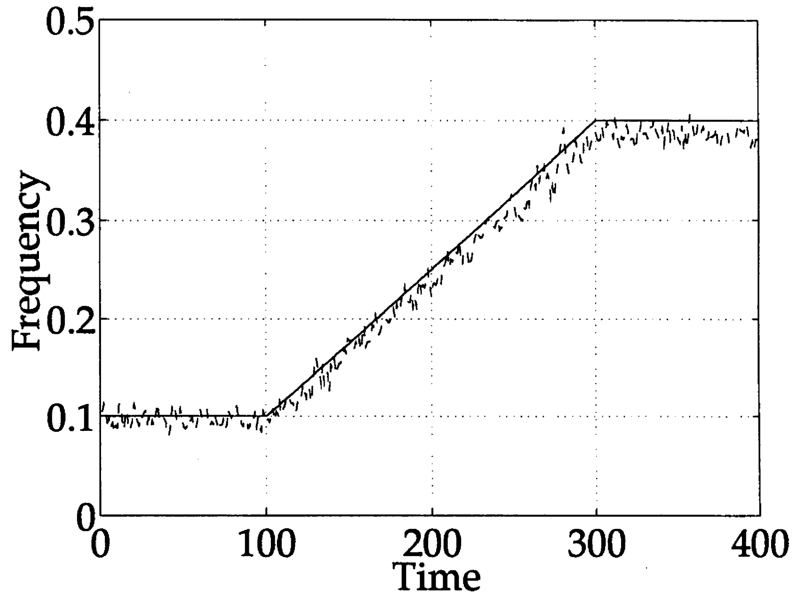

Figure 3.5: EKF frequency estimate when $Q^d > Q^a$, $\epsilon^d < \epsilon^a$ and $R^d > R^a$. The target signal is given by the solid line.

3.8 Conclusions

This chapter presented an EKF-based frequency tracker for signals with an SNR of at least 5 dB. Design guidelines for tuning this tracker were discussed. In addition to previously known heuristic tuning arguments, theoretically supported guidelines for the design of a stable EKF tracker were given. The design trade-offs between balancing noise rejection with tracking at a maximal slew rate were discussed. The nonlinearity inherent in this problem makes this trade-off non-obvious, critical to the performance of the filter and counter to the usual optimality guidelines for Kalman filtering. The performance penalties for over- and under-estimation of the noise covariances are further illustrated via simulation results. These demonstrate the importance of designing a sufficiently conservative filter and might explain why so few successful applications of extended Kalman filters have been reported.

In the next chapter an EKF frequency tracker for signals with low SNR will be given. The ability to design frequency trackers for a variety of situations by the choice of signal model illustrates the flexibility of an EKF-based approach to nonlinear observer design.

4

An EKF Frequency Tracker for High-Noise Environments

4.1 Introduction

IN this chapter, the problem of constructing an EKF-based frequency tracker for weak, narrowband signals is considered. Such signals occur in passive sonar tracking applications. This problem presents its own particular difficulties above and beyond those of the general frequency tracking problem. As a result, the general EKF-based frequency tracker designed in Chapter 3 is not appropriate. This is because, for this application, the filter designer not only has to deal with the inherent nonlinearity of the problem, but also with the difficulties caused by extremely high noise levels. Passive sonar signals typically have a signal-to-noise ratio (SNR) in the range -20 to -30 dB.

What makes the passive sonar tracking problem tractable is the *a priori* knowledge the filter designer has of the signal characteristics. These include a rough idea of the neighbourhood of the frequency sought and the very slow time variation of the signal with respect to the sampling rate. By appropriate pre-filtering of the signal this *a priori* information can be incorporated into the state-space signal model. Thus the EKF frequency tracker constructed using this signal model has a sufficiently high effective SNR that accurate tracking is achievable. The technique proposed here is a variant on methods developed by Quinn (1994) and McMahon and Barrett (1986; 1987) using block Fourier coefficients to estimate a constant frequency. It extends their techniques to the case where the frequency is slowly time-varying.

In the following sections, the appropriate pre-filtering of the signal and the resulting signal model and EKF frequency tracker are described. After this, design guidelines for this particular application are discussed. The performance of the filter is then illustrated via simulation results.

4.2 Derivation of State Space Model

By taking advantage of the known characteristics of the signal it is possible to overcome the effect of low SNR. Due to the slow time variation of the signal, the data may be divided into blocks in which the signal parameters may be considered constant. This will increase the effective SNR by averaging. The problems caused by averaging when estimating a constant frequency with the Kalman filter are avoided by taking a particular weighted average which is described below. Prior knowledge of the region in which the frequency is located is incorporated to increase the accuracy of the observer.

Consider a sinusoidal signal with slowly time-varying parameters which has been blocked into N non-overlapping segments of length T.[1] If the parameters are varying sufficiently slowly they can be considered to remain constant within a block and only vary between blocks. Such a signal can be described by the following signal model,

$$X_{t,k} = \rho_k \cos(\omega_k t + \phi + \sum_{r=1}^{k-1} \omega_r T) + \epsilon_{t+(k-1)T} \tag{4.1}$$

where $t = 0, \ldots, T-1$ is the time index within a block, $k = 1, \ldots, N$ is the block index and $\{\epsilon_t\}$ is additive noise. Prior knowledge of the frequencies of interest can be

[1]If the segments overlap then the noise processes between blocks will be correlated which is undesirable as the EKF is derived under the assumption that the noise processes are independent.

accommodated by supposing the frequency in the k-th block is given by

$$\omega_k = \frac{2\pi(M+\delta_k)}{T} \tag{4.2}$$

where $M < \frac{T}{2}$ is a known integer and $-a \le \delta_k \le b$, a, b given. In other the words the frequency is known to vary in the range

$$\left[\frac{2\pi(M-a)}{T}, \frac{2\pi(M+b)}{T}\right]. \tag{4.3}$$

Each block of data is then filtered by a discrete Fourier transform to produce a weighted average of the original signal. Defining the T-point discrete Fourier transform of a sequence $\{x_t\}$ as

$$X_r = \frac{1}{T}\sum_{t=0}^{T-1} x_t \exp\left(\frac{-\jmath 2\pi r t}{T}\right) \tag{4.4}$$

for $r = 0, \ldots, T-1$, the Fourier transform of (4.1) is given by

$$\begin{aligned} \Upsilon_{m,k} &= \frac{\rho_k}{2T}\exp\left(\jmath(\phi + 2\pi\sum_{r=1}^{k-1}\delta_r + \pi\frac{T-1}{T}(M-m+\delta_k))\right)\frac{\sin[\pi(M-m+\delta_k)]}{\sin[\frac{\pi(M-m+\delta_k)}{T}]} \quad (4.5)\\ &+\frac{\rho_k}{2T}\exp\left(\jmath(\phi + 2\pi\sum_{r=1}^{k-1}\delta_r + \pi\frac{T-1}{T}(M+m+\delta_k))\right)\frac{\sin[\pi(M+m+\delta_k)]}{\sin[\frac{\pi(M+m+\delta_k)}{T}]} \\ &+ v_{m,k} \end{aligned}$$

for $m = 0, \ldots, T-1$, where $v_{m,k}$ is the Fourier transform of the noise $\epsilon_{t+(k-1)T}$. Consider only the Fourier coefficients at the frequencies $\frac{2\pi(M-L)}{T}, \ldots, \frac{2\pi(M+L)}{T}$ where L is such that $L \ll T$ and $L \ge \max(a, b)$, then

$$\Upsilon_{m,k} \approx \frac{\rho_k}{4\pi}\exp[\jmath(\phi + 2\pi\sum_{r=1}^{k-1}\delta_r)]\frac{[\exp(-\jmath 2\pi\delta_k) - 1]}{\jmath(M-m+\delta_k)} + v_{m,k} \tag{4.6}$$

for $m = M-L, \ldots, M+L$, as the second term in $\Upsilon_{m,k}$ is $O(1)$ since $\frac{2M}{T}$ is bounded away from zero. Let

$$\bar{\rho}_k = \frac{\rho_k}{4\pi} \tag{4.7}$$

$$\varphi_k = \phi + 2\pi\sum_{r=1}^{k-1}\delta_r \tag{4.8}$$

then

$$\Upsilon_{m,k} \approx \bar{\rho}_k \exp(\jmath\varphi_k)\frac{[\exp(\jmath 2\pi\delta_k) - 1]}{\jmath(M - m + \delta_k)} + v_{m,k} \tag{4.9}$$

for $m = M - L, \ldots, M + L$.

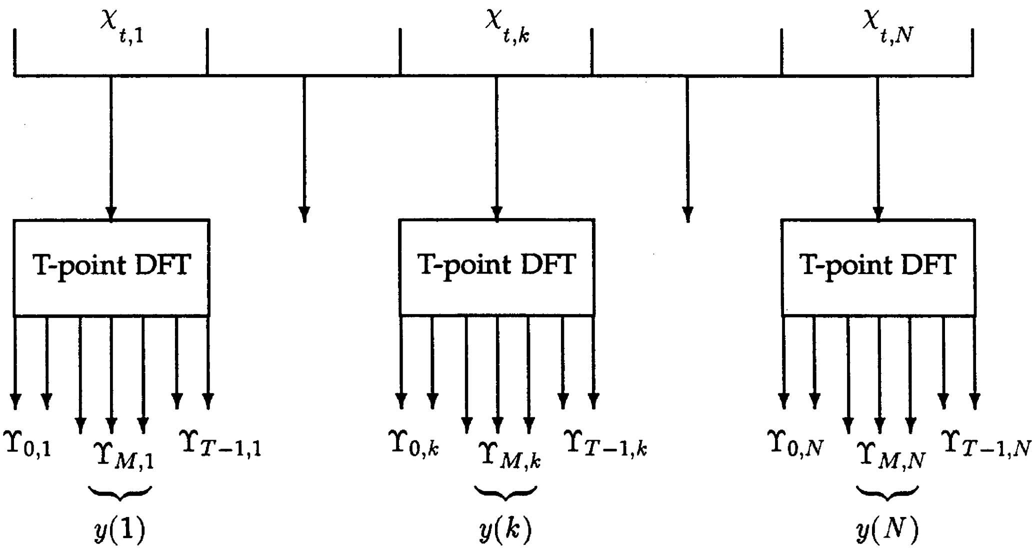

Figure 4.1: Transformation of input signal.

Figure 4.1 illustrates how the original sinusoidal signal, $\{\chi_{t,k}\}$, is blocked and then transformed to produce the measurement signal for the EKF filter, $\{y(k)\}$ in the frequency range $m = M - L, \ldots, M + L$. In the next two subsections we propose a state-space signal model which describes the evolution of the measurement signal $\{y(k)\}$ from block to block.

Another way of picturing the pre-filtering process is shown in Figure 4.2.

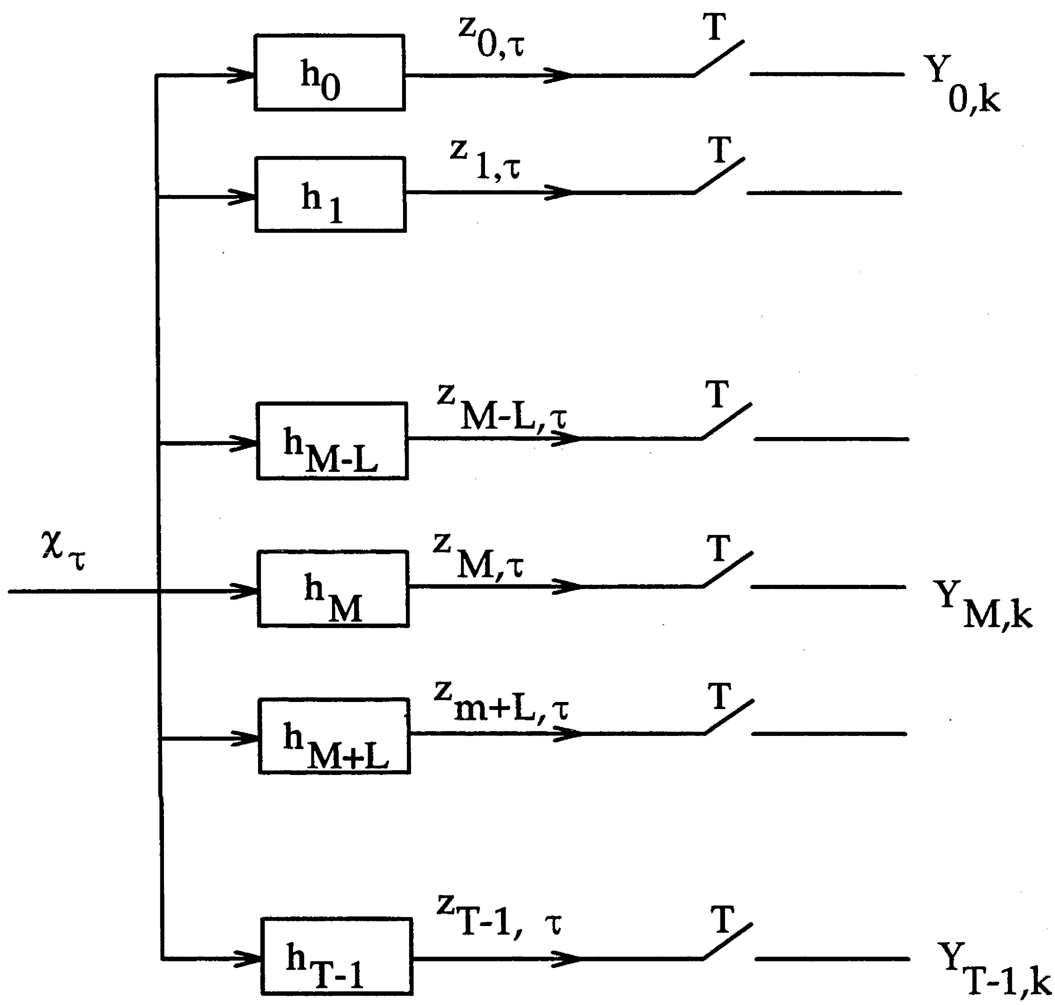

Figure 4.2: Filter bank representation of input signal pre-filtering.

In this representation the original signal, $\{\chi_{t,k}\}$, is passed through a bank of linear, FIR filters. The impulse response of each filter is

$$\begin{aligned} h_m(z) &= \frac{1}{T}[1 + \exp\left(-\jmath\frac{2\pi}{T}m\right) z^{-1} + \exp\left(-\jmath\frac{2\pi}{T}2m\right) z^{-2} \\ &\quad + \ldots + \exp\left(-\jmath\frac{2\pi}{T}(T-1)\right) z^{-(T-1)}]. \end{aligned}$$

Thus the first filter, h_0, picks out the DC component of the signal, the second, h_1, the fundamental and the remainder the harmonics of the fundamental frequency. The DFT components, $\Upsilon_{m,k} = z_{m,kT}$, are obtained by periodically sampling the output of the filter bank. The frequency response of each filter has its mainlobe centred on $\frac{m}{T}$ Hz and has a bandwidth of $\frac{1}{T}$ Hz. By selecting only those DFT components in the frequency range $m \in [M-L, M+L]$ the original, sinusoidal signal is effectively being filtered by a bandpass filter, centred on $\frac{M}{T}$ Hz, with a bandwidth of $\frac{2L+1}{T}$ Hz. It is this which contributes to the effective increase in the SNR.

In general, this type of signal filtering would produce a signal which was corrupted by correlated noise. However, as an added bonus from the DFT pre-filtering we have the following result concerning the noise processes. Assume that the noise in the original signal is ergodic, zero mean with finite variance and strictly positive spectral density (these are relatively weak assumptions). In this case its Fourier transform, $\{v_{m,k}\}$, is approximately complex Gaussian with zero mean and independent real and imaginary parts having the same variance (Hannan, 1979). The variance is given by $\frac{1}{2T}S(\omega)$ where S is the spectral density function

$$S(\omega) = \lim_{T\to\infty} \frac{1}{2T} \sum_{\tau=-T}^{T} \exp(-j\omega\tau) E[\epsilon^2_{t+(k-1)T}]. \tag{4.10}$$

4.2.1 State Equation

The equation (4.9) describes the Fourier coefficients, $\{\Upsilon_{m,k}\}$, in the frequency range of interest as functions of three variables which can be regarded as amplitude ($\bar{\rho}$), phase (φ) and frequency (δ) terms with additive, Gaussian noise. A simple state equation for such a system is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} (1-\epsilon_\rho) & 0 & 0 \\ 0 & 1 & 2\pi \\ 0 & 0 & (1-\epsilon_\delta) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} w_1(k) \\ 0 \\ w_3(k) \end{bmatrix} \tag{4.11}$$

where the states are defined as

$$x_1(k) = (\frac{\rho_k}{4\pi})^{\frac{1}{2}} \tag{4.12}$$

$$x_2(k) = \varphi_k \tag{4.13}$$

$$x_3(k) = \delta_k \tag{4.14}$$

and x_1 is defined as the square-root of the scaled amplitude to ensure its non-negativity. The frequency offset, x_3, and amplitude, x_1, are modelled as varying according to AR(1) models in the absence of any other information. Also, an AR(1) model is straightforward to tune to account for rapidly or slowly varying states.

4.2.2 Measurement Equation

As the real and imaginary parts of the additive noise are independent, the measurement equation for this state space model may be taken as

$$y(k) = \begin{bmatrix} \Re[\Upsilon_{M-L,k}] \\ \vdots \\ \Re[\Upsilon_{M+L,k}] \\ \Im[\Upsilon_{M-L,k}] \\ \vdots \\ \Im[\Upsilon_{M+L,k}] \end{bmatrix} + v(k) \tag{4.15}$$

where for $m = M - L, \dots, M + L$

$$\Re[\Upsilon_{m,k}] = \frac{x_1^2}{M - m + x_3}[\sin(x_2 + 2\pi x_3) - \sin(x_2)] \tag{4.16}$$

$$\Im[\Upsilon_{m,k}] = \frac{x_1^2}{M - m + x_3}[\cos(x_2) - \cos(x_2 + 2\pi x_3)] \tag{4.17}$$

when δ_k is not an integer, with the obvious limits taken when the denominator is zero. In this case this output vector will be zero except for two elements in the positions $(M + x_3)$ and $(M + x_3) + (2L + 1)$.

The state space model for the passive sonar tracking problem is linear in the state dynamics but has a nonlinear output map. This is the opposite case to that examined in Chapters 2 and 3. However, except for the dependence of the amplitude-like term

$$\frac{x_1^2}{M - m + x_3}$$

on the frequency offset δ, the signal model (4.11),(4.15) is similar to the standard polar form for general frequency tracking (Parker and Anderson, 1990).

4.3 EKF Observer

For our system, the signal model is given by the equations (4.11) and (4.15) which is nonlinear in the output but linear in the state. Thus for the high-noise frequency tracker the EKF filter is given by equations (2.3)–(2.7) where

$$F = \begin{bmatrix} (1-\epsilon_\rho) & 0 & 0 \\ 0 & 1 & 2\pi \\ 0 & 0 & (1-\epsilon_\delta) \end{bmatrix} \tag{4.18}$$

and H is given by the following set of equations when δ_k is not an integer

$$H(i,1) = \frac{2x_1}{L+1-i+x_3}[\sin(x_2+2\pi x_3)-\sin(x_2)] \tag{4.19}$$

$$H(i,2) = \frac{x_1^2}{L+1-i+x_3}[\cos(x_2+2\pi x_3)-\cos(x_2)] \tag{4.20}$$

$$\begin{aligned} H(i,3) &= \frac{2\pi x_1^2}{L+1-i+x_3}\cos(x_2+2\pi x_3) \\ &+ \frac{x_1^2}{(L+1-i+x_3)^2}[\sin(x_2)-\sin(x_2+2\pi x_3)] \end{aligned} \tag{4.21}$$

$$H(i+2L+1,1) = \frac{2x_1}{L+1-i+x_3}[\cos(x_2)-\cos(x_2+2\pi x_3)] \tag{4.22}$$

$$H(i+2L+1,2) = \frac{x_1^2}{L+1-i+x_3}[\sin(x_2+2\pi x_3)-\sin(x_2)] \tag{4.23}$$

$$\begin{aligned} H(i+2L+1,3) &= \frac{2\pi x_1^2}{L+1-i+x_3}\sin(x_2+2\pi x_3) \\ &+ \frac{x_1^2}{(L+1-i+x_3)^2}[\cos(x_2+2\pi x_3)-\cos(x_2)] \end{aligned} \tag{4.24}$$

for $i = 1,\ldots,2L+1$. The limiting case when δ_k is an integer will be a matrix of zeros except for the first two elements of the rows $i = M + x_3$ and $i = M + x_3 + 2L + 1$.

4.3.1 Observability

Our aim in frequency tracking translates into the reconstruction of the state of the system (4.11),(4.15) from output measurements. The observability of the system equation is then central to our ability to derive such a state estimator, in our case via Kalman filtering methods. Observability of (4.11),(4.15) is implied if the zero state is the only state yielding $K+1$ successive zero output measurements when the exogenous noise processes $\{v(k)\}$ and $\{w(k)\}$ are zero, for some $K > 0$. This, in turn, may be verified by considering the observability Gramian of the linearised system

$$\mathcal{O}(k,K) = \sum_{i=k-K}^{k} \Phi(i,k)^T H(i)^T R(i)^{-1} H(i) \Phi(i,k) \tag{4.25}$$

where $\Phi(k_2,k_1) = F(k_2-1)F(k_2-2)\dots F(k_1)$, for some $K \geq 0$ and for all $k \geq K$.

For integer δ_k the observability Gramian has a simple form which permits direct verification of observability. In this case

$$\mathcal{O}(k,1) = \begin{bmatrix} 16\pi^2[x_1(k)^2 + x_1(k-1)^2] & 0 & 0 \\ 0 & 4\pi^2[x_1(k)^4 + x_1(k-1)^4] & \frac{-1}{(1-\epsilon^d)} 8\pi^2 x_1(k-1)^4 \\ 0 & \frac{-1}{(1-\epsilon^d)} 8\pi^2 x_1(k-1)^4 & \frac{1}{(1-\epsilon^d)^2} 16\pi^3 x_1(k-1)^4 \end{bmatrix} \tag{4.26}$$

which will be of full rank provided $x_1(k) \neq 0$ for all k.

For non-integer δ_k, consider the pair of output equations for one particular value of i

$$\Re[\Upsilon_{i,k}] = \frac{x_1^2}{L+1-i+x_3}[\sin(x_2 + 2\pi x_3) - \sin(x_2)] \tag{4.27}$$

$$\Im[\Upsilon_{i,k}] = \frac{x_1^2}{L+1-i+x_3}[\cos(x_2) - \cos(x_2 + 2\pi x_3)]. \tag{4.28}$$

Over $K+1$ successive times, with $v(k)$ and $w(k)$ taken to be zero, $K+1$ successive zero outputs are possible only if $\{x_1(k-i) = 0, i = 0, \dots, K\}$ or $x_3(k-i)$ is an integer for all $i = 0, \dots, K$. The first case we rule out by assumption. The second has already been dealt with. By definition, if a system is observable from a subset of its outputs (in this case any pair) then it is observable from the complete set of outputs. The application of the full set of $2L+1$ output pairs is part of the mechanism for achieving the SNR improvement.

Our formulation contains some inherent unobservability for some particular states in that:–

- zero amplitude, $x_1 = 0$, implies zero output whatever the value of phase and frequency; and
- phase and discrete frequency are implicitly ambiguous modulo 2π.

In our reconstruction, by limiting ourselves to SNRs above -30 dB we rule out the $x_1 = 0$ case. The phase and frequency ambiguities are removed by constraining the state estimates to the appropriate intervals, which is not problematic in the absence of aliasing.

4.4 Designing the EKF

To tune the EKF, it is necessary to select appropriate design values for the matrices Q^d and R^d. The inclination is to take these EKF values to be the same as the noise covariance driving the nonlinear state space model, Q^a and R^a, but it was shown in Chapter 3 that this leads to potentially poor performance. This is because these design values need to take into account linearisation errors as well as the effect of the noise, thus it is necessary to have $Q^d > Q^a$ and $R^d > R^a$.

4.4.1 Choice of R^d

From the derivation of the state space model we know that R^a is a diagonal matrix with

$$\begin{aligned} R^a(i,i) &= R^a(i+2L+1, i+2L+1) && (4.29) \\ &= \frac{1}{2T} S\left(\frac{2\pi(M-L-1+i)}{T}\right) && (4.30) \end{aligned}$$

for $i = 1, \ldots, 2L+1$, where $S(\omega)$ is the spectral density at the frequency ω. In order to design a sufficiently conservative filter to ensure stability we therefore require

$$R^d \geq \frac{1}{2T} \max_{\omega} S(\omega) I \tag{4.31}$$

for ω in the frequency range of interest. Note if the measurement noise of the original signal, $\epsilon_{t+(k-1)T}$, is white then the equation for R^a reduces to

$$R^a = \frac{1}{2T} \sigma_\epsilon^2 I \tag{4.32}$$

where $\sigma_\epsilon^2 = E(\epsilon^2)$.

4.4.2 Choice of Q^d

For the signal model (4.11) the state noise covariance matrix is of the form

$$Q^a = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_3 \end{bmatrix} \tag{4.33}$$

However one of a set of sufficient conditions for the stability of the EKF is that $[F, Q]$ be controllable, see Section 2.4. The simplest means of ensuring this is to use a Q^d of the form

$$Q^d = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix} \tag{4.34}$$

The (scaled) amplitude is assumed to vary according to an AR(1) equation in the state space model. If the amplitude is varying slowly or is constant, $(1-\epsilon_\rho)$ should be close to 1 and q_1 small. If there is the possibility of fading channels then $(1-\epsilon_\rho)$ should be reduced and q_1 increased to allow for the greater variability. Note that the estimate of the amplitude will not be as accurate as that of the frequency as it is easier to determine frequency using the zero crossings, than it is to determine amplitude in very high noise.

The size of q_3^d determines the maximal slew rate of the signal that the EKF can track. Thus a large q_3^d allows the EKF to track a signal with a potentially large slew rate. The drawback of a large q_3^d is that even when the EKF is tracking well, if the actual slew rate is low the sensitivity to noise caused by a large q_3^d causes the state estimates to fluctuate widely around the true frequency value. Setting $\epsilon_\delta^d < \epsilon_\delta^a$ alleviates this problem by allowing the value of q_3^d to be decreased while still retaining the desired range for the frequency estimate.

Finally if q_2^d is chosen such that $0 < q_2^d \leq q_3^d$ then the evolution of the filter estimates will not be greatly altered and the controllability of $[F, Q]$ is assured.

4.4.3 Choice of L

It is clear from the formulation of the problem that the majority of the information in the EKF measurement signal (the set of Fourier coefficients for bins $m = M - L, \ldots, M + L$) is in a small number of bins centred on M. Thus there is a trade-off between the size of L and the accuracy of our knowledge of the centre frequency indexed by M. If M is known correctly, choosing a larger value of L than necessary to cover the variation in δ will reduce the efficiency of the filter by decreasing the effective SNR of the EKF measurements, as it unnecessarily increases the bandwith of the pre-filter. On the other hand if M is not known exactly L must be chosen to be large enough so that the interval $M - L, \ldots, M + L$ covers the range of variation of δ.

4.4.4 Choice of N and T

As with any application of the short-time Fourier transform (STFT), there is a trade-off between resolution in the time domain (N large) and resolution in the frequency domain (T large). The problem of resolution in the frequency domain is not as severe as in other uses of the STFT as the frequency estimates produced by the high-noise EKF frequency tracker are not restricted to the values of the Fourier frequencies. However, T must be large enough that the approximations used to derive (4.6) are valid and hence, an averaging gain is achieved to improve the SNR. On the other hand T must be small enough that the assumption that the signal parameters are constant within blocks is not unreasonable. In general, for SNRs between -20 dB and -30 dB, the value of T should be such that $T \geq 1000$.

4.5 Passive Sonar Tracking

4.5.1 Problem Description

The performance of the EKF frequency tracker for weak, narrowband signals designed above was tested on the problem of tracking a target moving past a single, stationary sonar receiver at a constant velocity, using simulated data. This problem is illustrated by Figure 4.3.

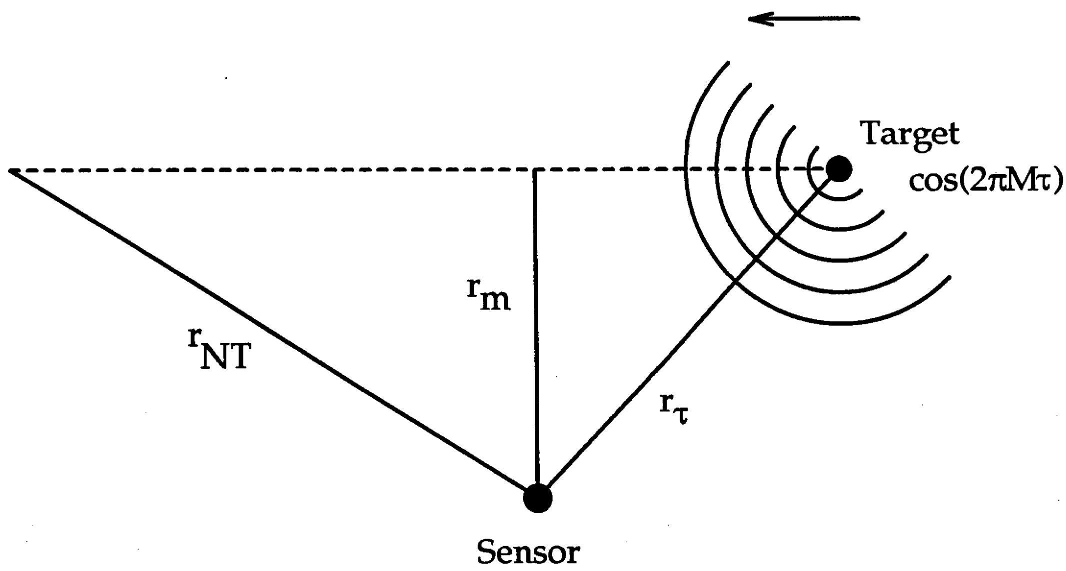

Figure 4.3: Constant velocity sonar tracking

The signal measured by the receiver was assumed to be corrupted with zero mean, Gaussian, white noise. In such a case the uncorrupted signal is given by the equation

$$X(\tau + \frac{r_\tau}{c}) = \rho_\tau \cos(2\pi M \tau)$$

where r_τ is the distance between the target and the receiver at time τ, and c is the propagation rate of the signal through water. The centre frequency, M, is measured in Hz and the sampling rate was T Hz, i.e. each time block is comprised of one second of data. All distances were normalised so that the distance between the target and receiver at the point of closest approach was $r_m = 1$. The power of the signal was modelled as decreasing in proportion to the square of the distance. The value of the SNR when the target was furthest from the receiver is given by SNR_{min}. The value of the signal-to-noise ratio when the target is closest to the receiver is given by SNR_{max} and thus the broadband, background noise has a variance of

$$\sigma^2 = 10^{-0.1\text{SNR}_{\text{max}}}.$$

Table 4.1 gives the signal characteristics for each data set.

Data Set	T	N	M	L	SNR_{min}	SNR_{max}	σ^2
1	2048	400	300	2	-10	0	1
2	2048	400	300	2	-20	-10	10
3	2048	400	300	2	-30	-20	100

Table 4.1: Simulated data sets

4.5.2 Initial State Estimates

As was illustrated by Theorem 2.9 in Chapter 2, and also by common sense, the ability of the EKF to estimate accurately the state of a system depends crucially on the accuracy of the initial state estimate. For the passive sonar tracking problem reasonably good initial estimates can be obtained using the procedure outlined in Quinn *et al.* (1994) using only the Fourier coefficients of the first block of data. Briefly, their technique is as follows. Let s_m be the Fourier coefficients of the first block of data. Then estimates of $\delta, \bar{\rho}$ and φ are obtained by minimising

$$\sum_{m=M-L}^{M+L} |s_m - \bar{\rho}\exp(\jmath\varphi)d_m(\delta)|^2$$

where

$$d_m(\delta) = \frac{\exp(\jmath 2\pi\delta) - 1}{\jmath(M - m + \delta)}.$$

The resulting estimates are given by the equations

$$\begin{aligned}
\hat{\delta} &= \hat{x}_3(1|0) \\
&= \arg\max \frac{|\sum_{m=M-L}^{M+L} s_m d_m^*(\delta)|^2}{\sum_m |d_m(\delta)|^2} && (4.35) \\
\hat{\bar{\rho}} &= \hat{x}_1(1|0)^2 \\
&= \left| \frac{\sum_m s_m d_m^*(\hat{\delta})}{\sum_m |d_m(\hat{\delta})|^2} \right| \\
\hat{\varphi} &= \hat{x}_2(1|0) \\
&= \arg\left(\frac{\sum_m s_m d_m^*(\hat{\delta})}{\sum_m |d_m(\hat{\delta})|^2} \right) && (4.36)
\end{aligned}$$

where the superscript $*$ indicates the complex conjugation. This estimator for constant δ is unbiased and asymptotically statistically efficient. The estimate of δ can be obtained by computing (4.35) for a fixed set of values in the range $[-L, L]$ or by a numerical optimisation technique.

4.5.3 Results

For each simulation the initial state estimates were obtained via the equations (4.35)–(4.36). This method is not ideal, as the variance of the estimator of δ is relatively large. In general, the technique produced sufficiently accurate initial estimates but this was not true in all cases examined. The problem of finding sufficiently accurate estimates for initialising the EKF for this scenario requires considerable further work.

The values used for the design parameters $Q^d = \text{diag}(q_1^d, q_2^d, q_3^d)$ and $R^d = \frac{1}{2T} r^d I$ are given in Table 4.2. These are not necessarily the optimal design values. They were selected by holding all but one parameter fixed and then searching for the optimal value of the remaining, free parameter for each design value in turn. This procedure was iterated two or three times until the minimising value of each parameter did not change significantly. Also given in the table are the sum of the squared errors in the estimates of the frequency offset, δ, and the power of the signal (measured by $10\log_{10}\left(\frac{\bar{\rho}^2}{\sigma^2}\right)$). In all cases the frequency offset, δ, and the scaled amplitude, $\bar{\rho}$, were assumed to vary according to random walk processes (i.e. $\epsilon_r = \epsilon_\delta = 0$).

	1	2	3
q_1^d	0.00005	0.0001	0.0001
q_2^d	0.0010	0.0010	0.0010
q_3^d	0.0025	0.0045	0.0100
r^d	12.50	20.00	125.0
MSE(power)	35.21	31.91	44.06
MSE(freq.)	0.0354	0.0327	0.0402

Table 4.2: Design values

Figures 4.4 – 4.12 display the results of the EKF tracker. A sample of the original, measured signal is given for each data set. The top trace is when the target was at the maximum distance from the receiver and the bottom trace when it was at the minimum distance. The periodicity in the data is not obvious, even for the strongest signal. Also for each data set the EKF estimates of the amplitude and frequency are plotted along with the true values. Then the errors in these estimates are plotted. These illustrate the high accuracy that is obtained by the EKF tracker even at very low signal-to-noise ratios.

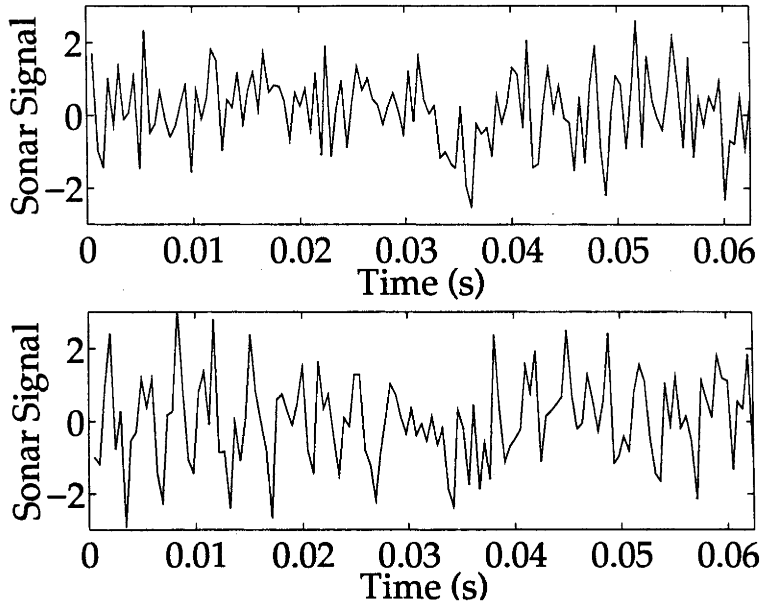

Figure 4.4: Measured sonar signal when target at greatest range (top) and closest range (bottom) for data set 1.

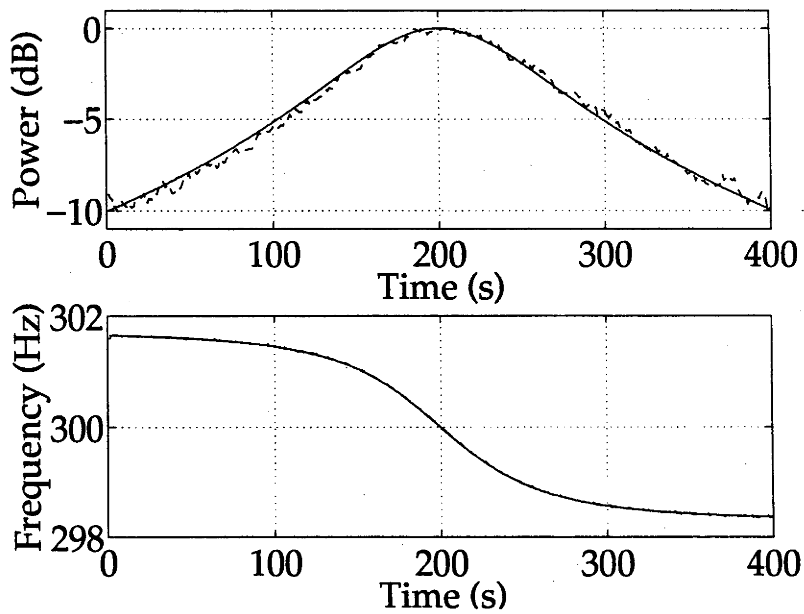

Figure 4.5: EKF estimates for data set 1. True values given by solid line.

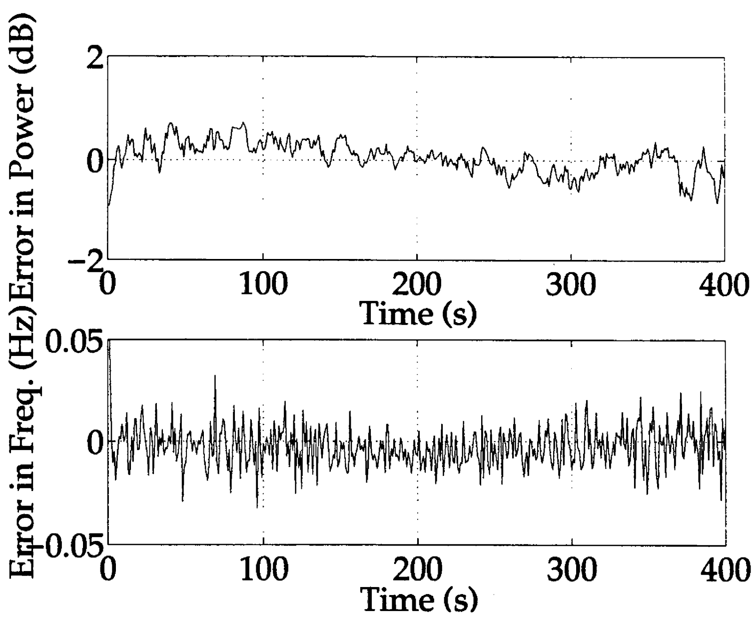

Figure 4.6: The error in the EKF estimates for data set 1.

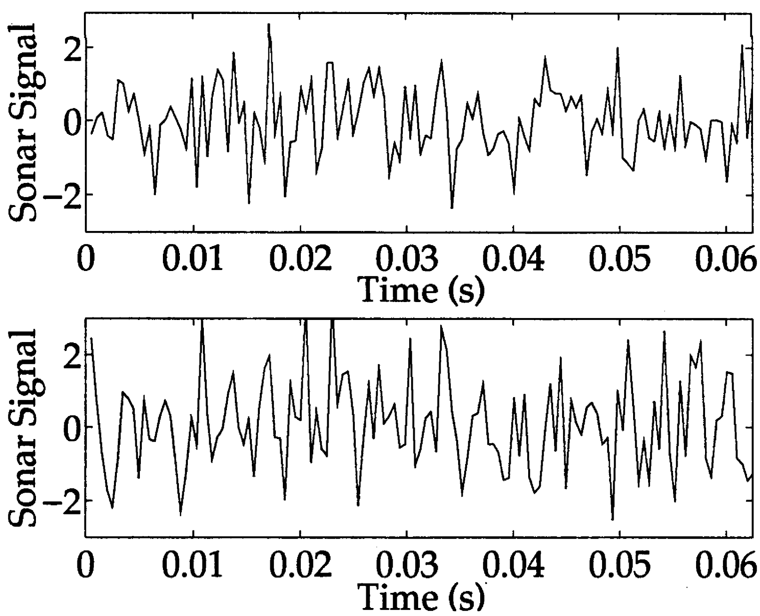

Figure 4.7: Measured sonar signal when target at greatest range (top) and closest range (bottom) for data set 2.

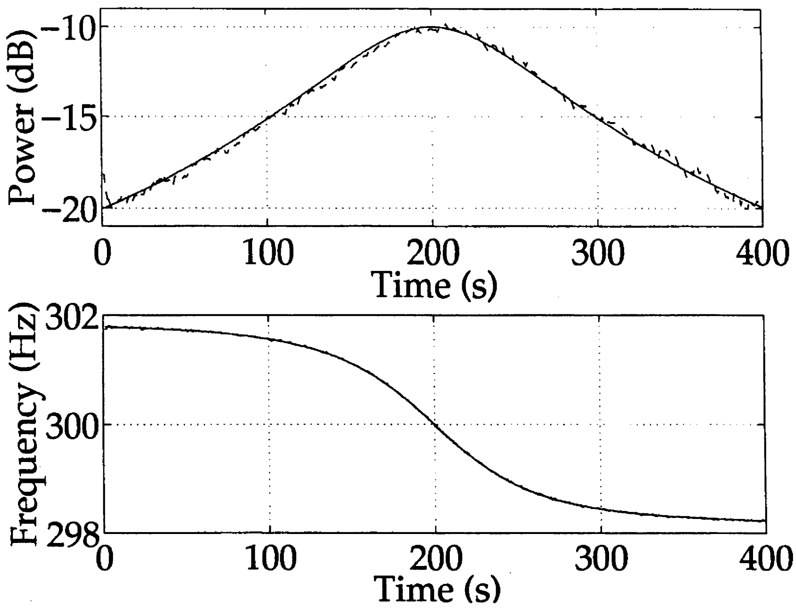

Figure 4.8: EKF estimates for data set 2. True values given by solid line.

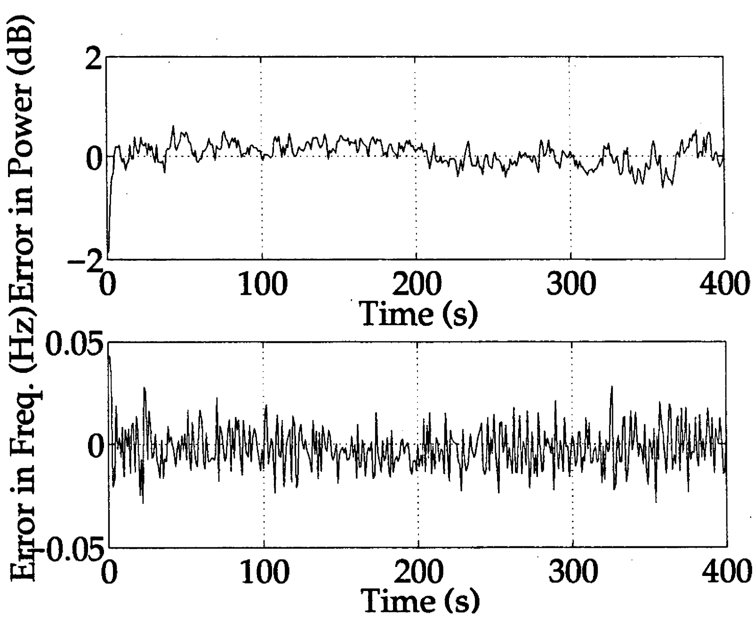

Figure 4.9: Error in the EKF estimates for data set 2.

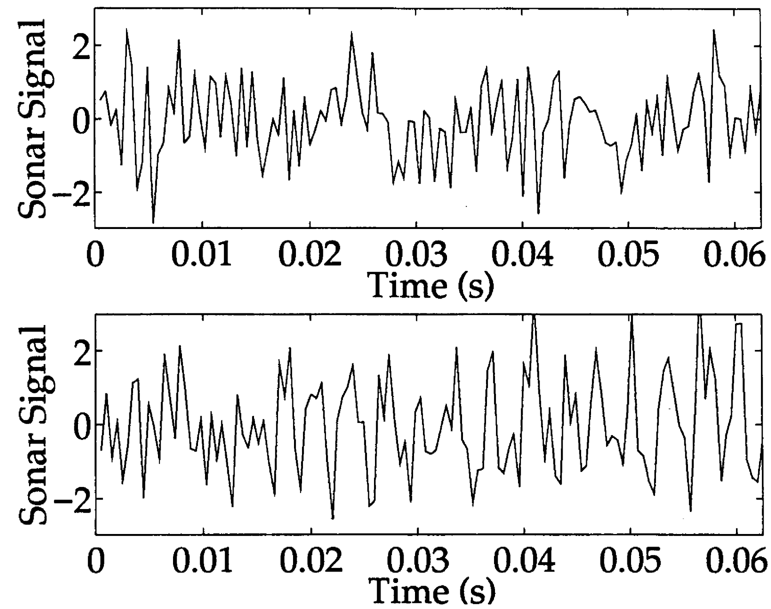

Figure 4.10: Measured sonar signal when target at greatest range (top) and closest range (bottom) for data set 3.

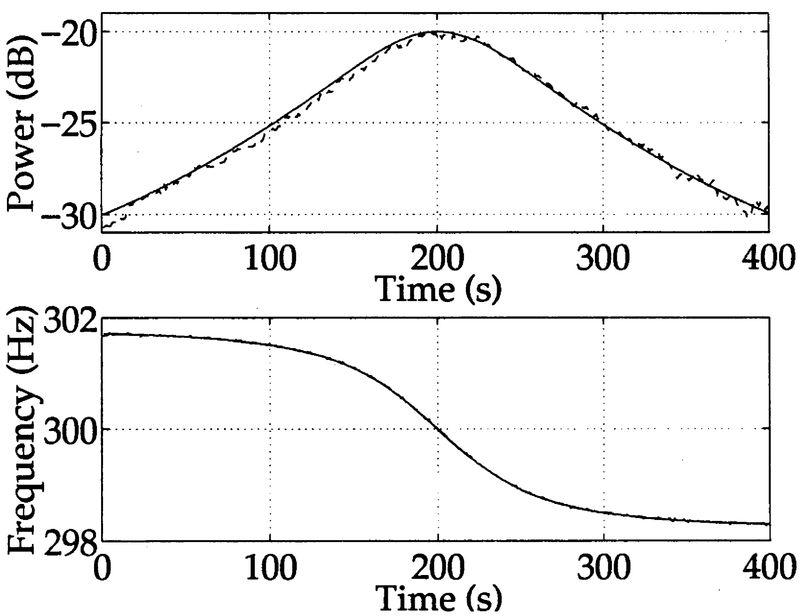

Figure 4.11: EKF estimates for data set 3. True values given by solid line.

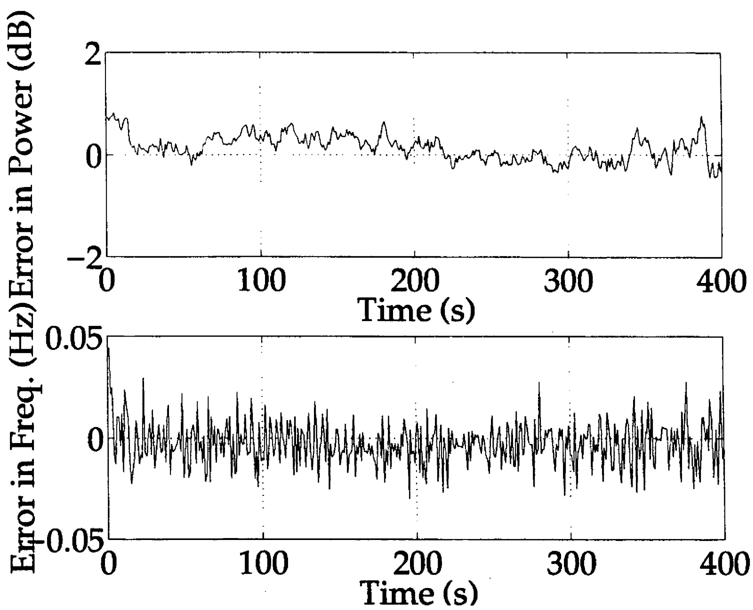

Figure 4.12: Error in the EKF estimates for data set 3.

4.6 Conclusion

In this chapter, the flexibility of the EKF approach to filter design was demonstrated. The EKF-based frequency tracker of Chapter 3 was effective only for signals which had relatively high signal-to-noise ratios. In spite of this, in this chapter an EKF tracker was designed which was effective for very weak, narrowband signals. This was done by incorporating prior knowledge of the nature of the signal into the structure of the filter by the choice of signal model. This increased the effective SNR and so allowed accurate tracking. Furthermore, the understanding of the dynamics of the errors of the EKF gained in Chapters 2 and 3 could be used to derive design guidelines applicable to this particular frequency tracker.

While the approach used in this chapter was specific to the problem of passive sonar tracking, the general methods used here and in Chapter 3 can be applied to the construction of nonlinear observers for many types of problems.

5

A Self-Tuning Regulator for Vibration Control

5.1 Introduction

THIS chapter examines a problem of active vibration control. Unlike passive control, which employs isolation and absorbtion techniques to reduce vibrations, active vibration control seeks to eliminate oscillations by generating cancelling vibrations. Such controllers are often adaptive in order to allow for changing circumstances or imperfect knowledge of the plant dynamics. The reduction of vibration in helicopters caused by the movement of the rotor blades is an important problem in their design and effective control systems offer significant benefits (see Pearson and Goodall (1994) for example).

The problem examined in this chapter is that of eliminating a sinusoidal disturbance of unknown frequency from the output of a linear, time-invariant plant with unknown parameters using an adaptive controller. Such a system has poles on the unit circle corresponding to the frequency of the disturbance and is therefore not asymptotically stable. From the internal model principle, these critically stable dynamics must be reproduced in the controller to reject successfully the disturbance. If the least-squares algorithm is used to estimate the unknown frequency in open-loop then a biased answer will be obtained. In this chapter it is shown that by estimating the unknown frequency (and other plant parameters) using a modified least-squares algorithm in closed-loop using a certainty equivalence minimum-variance controller, any bias in the frequency estimate is eliminated. The resulting adaptive controller is shown to converge to a solution which regulates the plant and which contains poles at the location of the true frequency of the disturbance.

The next section describes the plant, estimation algorithm and control law in detail and the subsequent section contains the convergence proof. The final section demonstrates the performance of the vibration control self-tuning regulator with simulation results.

5.2 Problem Description

Consider the adaptive control problem illustrated in Figure 5.1 where P is a linear, time-invariant plant which is perturbed by sinusoidal disturbance, v_k, of unknown frequency. The disturbance process is modelled by passing a white noise process through a transfer function H.

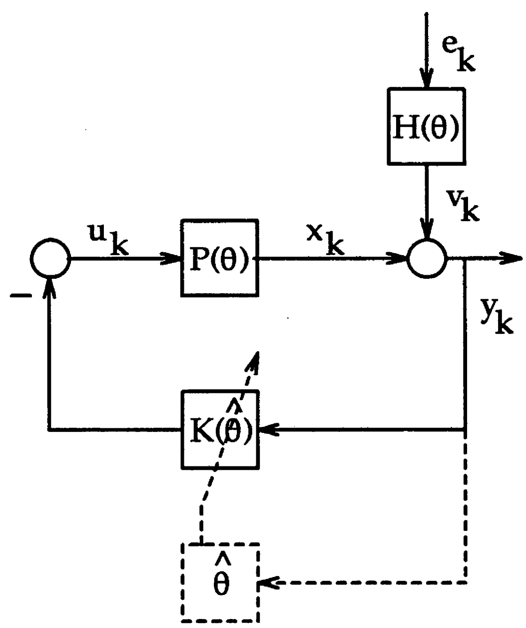

Figure 5.1: Adaptive control for vibration rejection.

The parameters of the plant and disturbance are given by the vector, θ. These unknown parameters are estimated using a modified least-squares algorithm and the estimates are used to design an adaptive minimum-variance controller, $K(\hat{\theta})$, to eliminate the effect of the disturbance on the output of the plant. The equations for the plant, disturbance process and controller are given below.

5.2.1 Plant Dynamics

The plant is a linear, time-invariant system described by the equation

$$\begin{aligned} x_k &= P u_k \\ &= \frac{B_1(q)}{A_1(q)} u_k \end{aligned}$$

where

$$\begin{aligned} B_1(q) &\doteq b'_0 q^{n-3} + b'_1 q^{n-4} + \cdots + b'_{n-3} \\ A_1(q) &= q^{n-2} + a'_1 q^{n-3} + \cdots + a'_{n-2} \end{aligned}$$

and q is the forward shift operator. To ensure the plant is sufficiently well-behaved we will make the following assumptions.

Assumptions

1. The plant is stably invertible, i.e. all the roots of $B_1(q)$ are strictly inside the unit circle.
2. The plant has unit delay, i.e. $b'_0 \neq 0$.
3. An upper bound, $\bar{n}$, is known for the order of the plant, $n - 2$.

For convenience of notation (and without loss of generality) $\bar{n}$ will be assumed to equal $n - 2$.

5.2.2 Disturbance Process

The sinusoidal disturbance process is characterised by the equation

$$\begin{aligned} v_k &= He_k \\ &= \frac{B_2(q)}{A_2(q)} e_k \end{aligned}$$

where

$$\begin{aligned} B_2(q) &= q^2 \\ A_2(q) &= 1 - 2\cos(\omega) \cdot q + q^2 \end{aligned}$$

subject to the condition that frequency of the disturbance ω is not an integer multiple of π. If the noise process, $\{e_k\}$, satisfies the conditions

$$E[e_k] = 0 \tag{5.1}$$

$$E[e_k^2] = \begin{cases} \sigma^2 & k = -2, -1 \\ 0 & k \geq 0 \end{cases}$$

$$E[e_k^4] = \mu_4 < \infty. \tag{5.2}$$

then $\{v_k\}$ is given by the second order, constant coefficient difference equation.

$$v_k = \alpha_k e_{-1} + \beta_k e_{-2}$$

where

$$\begin{aligned} \alpha_k &= \frac{\sin(k\omega)}{\sin(\omega)} \\ \beta_k &= \frac{\sin([k+1]\omega)}{\sin(\omega)}. \end{aligned}$$

Hence $\{v_k\}$ is a sinusoid with random phase and amplitude and frequency determined by ω and will be bounded for all k provided e_{-2} and e_{-1} are bounded. In the following we will assume that is the case.

5.2.3 Minimum-Variance Control

The aim of minimum-variance control is, as the name implies, to minimise the variance of the output of the plant. The performance criterion for such a system is

$$J = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E[y_{k+1}^2]$$

For a time-invariant, linear system described by the ARMAX model

$$A(q)y_k = B(q)u_k + C(q)e_k$$

the feedback control law, for stable $B(q)$, which minimises this criterion is

$$K = \frac{C(q) - A(q)}{B(q)} \tag{5.3}$$

assuming the polynomials A, B and C are known (Åström and Wittenmark, 1984). The result of using this controller is to fix $y_k = e_k$ for all k which clearly produces the minimal attainable variation in the output as all predictable variation in the output has been eliminated. If the polynomials are unknown then the controller designed using estimated values in place of the unknown values is the certainty equivalence minimum-variance controller. Note that the zeros of the plant appear in the poles of the controller. If the plant is not stably invertible then the closed-loop produced by the control law (5.3) will be unstable. In this case the internally stabilising minimum variance controller is not given by (5.3) but by an alternative formula as explained in Åström and Wittenmark (1984).

5.2.4 General System

The complete system can be described by the single ARMAX equation

$$A(q)y_k = B(q)u_k + C(q)e_k \tag{5.4}$$

where

$$
\begin{aligned}
A(q) &= A_1(q)A_2(q) \\
&= (q^{n-2} + a'_1 q^{n-3} + \cdots + a'_{n-2})(1 - 2\cos(\omega)\cdot q + q^2) \\
&= q^n + (a'_1 - 2\cos(\omega))q^{n-1} + (1 - 2\cos(\omega)a'_1 + a'_2)q^{n-2} \\
&\quad +(a'_1 - 2\cos(\omega)a'_2 + a'_3)q^{n-3} + \cdots + (a'_{n-4} - 2\cos(\omega)a'_{n-3} + a'_{n-2})q^2 \\
&\quad +(a'_{n-3} - 2\cos(\omega)a'_{n-2})q + a'_{n-2} \\
&= q^n + a_1 q^{n-1} + \cdots + a_n \\
B(q) &= A_2(q)B_1(q) \\
&= (1 - 2\cos(\omega)\cdot q + q^2)(b'_0 q^{n-3} + b'_1 q^{n-4} + \cdots + b'_{n-3}) \\
&= b'_0 q^{n-1} + (b'_1 - 2\cos(\omega)b'_0)q^{n-2} + (b'_0 - 2\cos(\omega)b'_1 - b'_2)q^{n-3} \\
&\quad +(b'_1 - 2\cos(\omega)b'_2 + b'_3)q^{n-4} + \cdots + (b'_{n-5} - 2\cos(\omega)b'_{n-4} + b'_{n-3})q^2 \\
&\quad +(b'_{n-4} - 2\cos(\omega)b'_{n-3})q + b'_{n-3}, \;\; b_0 \neq 0 \\
&= b_0 q^{n-1} + b_1 q^{n-2} + \cdots + b_{n-1} \\
C(q) &= A_1(q)B_2(q) \\
&= (q^{n-2} + a'_1 q^{n-3} + \cdots + a'_{n-2})q^2 \\
&= q^n + c_1 q^{n-1} + \cdots + c_n
\end{aligned}
$$

and $c_{n-1} = c_n = 0$. Assume the plant initial conditions are

$$u_k = y_k = v_k = 0 \;\; k \leq 0.$$

Recall that if the polynomials A, B and C are known then the minimum-variance controller is

$$
\begin{aligned}
K &= \frac{A_1(q)B_2(q) - A_1(q)A_2(q)}{A_2(q)B_1(q)} \\
&= \frac{A_1(q)\{2\cos(\omega)\cdot q - 1\}}{B_1(q)\{1 - 2\cos(\omega)\cdot q + q^2\}}
\end{aligned}
$$

The resulting ideal control law $u_k^0 = -K y_k$ causes $y_k = e_k$ so, in this case, the expression for y_k can be rewritten as follows

$$y_{k+1} = \phi_k^T \theta^0 + e_{k+1} \tag{5.5}$$

where

$$\phi_k = (y_k, y_{k-1}, \ldots, y_{k-n+1}, u_k^0, u_{k-1}^0, \ldots, u_{k-n+1}^0)^T \tag{5.6}$$

$$\theta^0 = (c_1 - a_1, \ldots, c_n - a_n, b_0, \ldots, b_{n-1})^T \tag{5.7}$$

Note that ϕ_k is known at time $k+1$ so y_{k+1} is given by a linear regression equation. In this notation the minimum-variance control law is

$$\begin{aligned} u_k^0 &= -\frac{1}{b_0}\{\sum_{i=1}^{n}(c_i - a_i)y_{k+1-i} + \sum_{i=1}^{n-1} b_i u_{k-i}^0\} \\ &= -\frac{1}{\theta_{n+1}^0}\{\sum_{i=1}^{n} \theta_i^0 y_{k+1-i} + \sum_{i=1}^{n-1} \theta_{n+1+i}^0 u_{k-i}^0\}. \end{aligned}$$

Thus the true parameter vector, θ^0, can be estimated using a technique, such as recursive least-squares, to estimate the parameters of a linear regression. At each time instant, a certainty equivalence controller can be calculated from the parameter estimates, $\hat{\theta}_k$, using the equation

$$u_k = -\frac{1}{\hat{\theta}_{n+1,k}}\{\sum_{i=1}^{n} \hat{\theta}_{i,k} y_{k+1-i} + \sum_{i=1}^{n-1} \hat{\theta}_{n+1+i,k} u_{k-i}\}. \tag{5.8}$$

where $\hat{\theta}_{i,k}$ is the i-th element of estimate of θ at time k.

5.2.5 Recursive Least Squares

The following slightly modified version of recursive least-squares from Sin and Goodwin (1982) will be used to estimate the unknown plant parameters. The modification ensures that the weighting matrix, P_k, has bounded condition number which helps prevent numerical problems. It is known as the *condition number monitoring* (CNM) algorithm. At each time instant k a new estimate of θ^0 is calculated based on the previous estimate,

$\hat{\theta}_{k-1}$ and the current output of the plant, y_k, by the following equations.

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \frac{P_{k-2}\phi_{k-1}}{1+\phi_{k-1}^T P_{k-2}\phi_{k-1}}(y_k - \phi_{k-1}^T\hat{\theta}_{k-1}) \tag{5.9}$$

$$P'_{k-1} = P_{k-2} - \frac{P_{k-2}\phi_{k-1}\phi_{k-1}^T P_{k-2}}{1+\phi_{k-1}^T P_{k-2}\phi_{k-1}}, \quad P_{-1} > 0 \tag{5.10}$$

$$\bar{r}_{k-1} = \bar{r}_{k-2}(1+\phi_{k-1}^T P_{k-2}\phi_{k-1}), \quad \bar{r}_{-1} > 0 \tag{5.11}$$

If

$$\bar{r}_{k-1}\lambda_{max}(P'_{k-1}) \le \Lambda \tag{5.12}$$

for some $0 < \Lambda < \infty$ then

$$P_{k-1} = P'_{k-1} \tag{5.13}$$

otherwise

$$P_{k-1} = \frac{\Lambda}{\bar{r}_{k-1}\lambda_{max}(P'_{k-1})} P'_{k-1} \tag{5.14}$$

This algorithm has the following properties.

1. The algorithm will perform a standard least-squares update of the parameter estimates provided the condition number of the weighting matrix is not too large. This can be seen by re-writing the equation for the weighting matrix as

$$P_{k-1} = \beta_{k-1}\left[P_{k-2} - \frac{P_{k-2}\phi_{k-1}\phi_{k-1}^T P_{k-2}}{1+\phi_{k-1}^T P_{k-2}\phi_{k-1}}\right]$$

where $0 < \beta_{k-1} \le 1$ is defined as

$$\beta_{k-1} \triangleq \frac{\Lambda}{\max(\Lambda, \bar{r}_{k-1}\lambda_{max}P'_{k-1})} = \frac{1+\alpha_{k-1}\phi_{k-1}^T P_{k-2}\phi_{k-2}}{1+\phi_{k-1}^T P_{k-2}\phi_{k-2}}$$

for some $0 \le \alpha_{k-1} \le 1$, with $\alpha_{k-1} = 1$ when $\bar{r}_{k-1}\lambda_{max}(P'_{k-1}) \le \Lambda$. That is $\beta = 1$ when the condition number is small enough and the algorithm reduces to regular least squares. When the condition number is too large β is as close to one as possible.

2. Consider the quantites R_k and r_k which are defined by the recursions.

$$\begin{aligned} R_{k-1} &= R_{k-2} + \phi_{k-1}\phi_{k-1}^T \\ r_{k-1} &= trace(R_{k-1}). \end{aligned}$$

with the initial condition $r_{-1} \geq \frac{1}{\Lambda}\bar{r}_{-1}$. With these definitions bounds for the quantities P_k and $\bar{r}_k$ can be found which are a function of the measured data. In particular, by construction

$$\bar{r}_{k-1}\lambda_{max}(P_{k-1}) \leq \Lambda.$$

Furthermore, by induction it can be shown that

$$\bar{r}_{k-1} \leq \Lambda r_{k-1}.$$

3. If a certainty equivalence minimum-variance controller is used then $\phi_{k-1}^T\hat{\theta}_{k-1} = 0$ for all k.

5.3 Properties of the Vibration Control STR

Having clearly definied the problem we now wish to determine which of the five properties of stability, convergence, self-optimality, self-tuning and consistency this system possesses. It will be shown that this system is stable, convergent and self-optimising. While consistency cannot be proven for this system, it can be shown that the poles of the controller will contain the poles due to the sinusoidal disturbance. Convergence, in general, to the correct parameter values θ^0 cannot be shown. The approach used to prove these results makes use of Martingale theory and stochastic Lyapunov functions. In the next section some preliminary theorems and lemmata are developed and reviewed. In the following section the proofs of stability and optimality (Theorem 5.2) and convergence (Corollary 5.2.1) are given.

5.3.1 Preliminary Results

Theorem 5.1 (Non-Negative Near-Supermartingale Convergence) (Robbins and Siegmund, 1971) *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ be a sequence of σ-algebras of $\mathcal{F}$. For each $k = 1, 2, \ldots$ let z_k, β_k, ξ_k and ζ_k be non-negative $\mathcal{F}_k$-measurable random variables such that*

$$E[z_{k+1}|\mathcal{F}_k] \leq z_k(1+\beta_k) + \xi_k - \zeta_k.$$

If

$$\sum_{k=1}^{\infty} \beta_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \xi_k < \infty$$

then $\lim_{k\to\infty} z_k$ exists and is finite, and

$$\sum_{k=1}^{\infty} \zeta_k < \infty.$$

■

The following lemma considers a sequence $\{y_k\}$ which is the sum of an $\mathcal{F}_{k-1}$-measurable sequence and some disturbance sequence $\{e_k\}$, and an $\mathcal{F}_{k-1}$-measurable estimate of y_k denoted by $\hat{y}_k$. It shows that if some non-decreasing, non-negative weighting sequence, $\{r_k\}$, can be found which satisfies two technical conditions, (5.15) and (5.16), then the expected error in the estimates tends to zero (result (5.17)) and the variance of the error tends to the variance of the disturbance process (result (5.19)). Thus, if a suitable weighting sequence can be found for the vibration control STR then this lemma can be used to prove that the STR is self-optimising.

Lemma 5.1 (Stochastic Key Technical Lemma, (Goodwin and Sin, 1984)) *If*

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{(\tilde{y}_k - e_k)^2}{r_{k-1}} < \infty \text{ a.s.} \tag{5.15}$$

where for some increasing sequence of σ-algebras $\mathcal{F}_{k-1}$

1. $E[e_k|\mathcal{F}_{k-1}] = 0$ a.s.

2. $E[e_k^2|\mathcal{F}_{k-1}] = \sigma^2$ a.s.

3. $E[e_k^4|\mathcal{F}_{k-1}] < \infty$ a.s.

4. $\tilde{y}_k = y_k - \hat{y}_k$ *where* $\hat{y}_k$ *and* $(y_k - e_k)$ *are* $\mathcal{F}_{k-1}$*-measurable*

5. $\{r_{k-1}\}$ *is a non-decreasing, non-negative* $\mathcal{F}_{k-1}$*-measurable sequence*

and if there exist constants K_1, K_2 *and* $\bar{N}, 0 \leq K_1 < \infty, 0 < K_2 < \infty, 0 < \bar{N} < \infty,$ *such that*

$$\frac{1}{N} r_{N-1} \leq K_1 + \frac{K_2}{N} \sum_{k=1}^{N} (\tilde{y}_k - e_k)^2 \qquad N \geq \bar{N} \text{ a.s.} \tag{5.16}$$

then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (\tilde{y}_k - e_k)^2 = 0 \text{ a.s.} \tag{5.17}$$

$$\lim_{N\to\infty} \sup \frac{1}{N} r_{N-1} < \infty \text{ a.s.} \tag{5.18}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (y_k - \hat{y}_k)^2 = \sigma^2 \text{ a.s.} \tag{5.19}$$

■

Consider the vibration control STR posed in Section 5.2. The system can be described by the equation

$$y_k = \phi_{k-1}^T \theta^0 + e_k$$

where ϕ_k is defined by equation (5.6). Define $\mathcal{F}_k$ as the σ-algebra generated by $\{y_1, \ldots, y_k, u_1, \ldots, u_k\}$ then y_k is clearly $\mathcal{F}_k$-measurable. Also e_k satisfies the conditions of the noise process of Lemma 5.1 by assumptions (5.1)–(5.2). Given the minimum-variance control law (5.8) the *a priori* estimate of y_k is $\mathcal{F}_k$-measurable and is simply

$$\begin{aligned} \hat{y}_k &= \phi_{k-1}^T \hat{\theta}_{k-1} \\ &\equiv 0 \end{aligned}$$

for all k. Thus in the notation of Lemma 5.1

$$\tilde{y}_k = E[y_k|\mathcal{F}_{k-1}] = \phi_{k-1}^T\theta^0.$$

The following lemma shows that the vibration control STR satisfies the technical condition (5.16) of the Stochastic Key Technical Lemma.

Lemma 5.2 *For the given model (5.4) with the noise assumptions (5.1)–(5.2), minimum-variance control law (5.8) and the recursive least-squares estimation algorithm (5.9)–(5.14)*

$$\frac{1}{N}\bar{r}_{N-1} \leq K_1 + K_2\frac{1}{N}\sum_{k=1}^{N} E[y_k|\mathcal{F}_{k-1}]^2 \tag{5.20}$$

for some $0 \leq K_1 < \infty$, $0 < K_2 < \infty$, *where* $\mathcal{F}_k$ *is the* σ*-algebra generated by* $\{y_1, \ldots, y_k, u_1, \ldots, u_k\}$.

Proof

Since the plant is stably invertible $\{u_k\}$ can be written as the output of an asymptotically stable linear system driven by $\{y_k\}$ and $\{v_k\}$. Hence there exist constants $c_i > 0, i = 1, \ldots, 3$ such that

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} u_k^2 &\leq c_1\frac{1}{N}\sum_{k=1}^{N} y_{k+1}^2 + c_2\frac{1}{N}\sum_{k=1}^{N} v_{k+1}^2 \\ &\leq c_1\frac{1}{N}\sum_{k=1}^{N} y_{k+1}^2 + c_3\end{aligned}$$

since $\{v_k\}$ is a bounded sequence. Now

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N} y_{k+1}^2 &= \frac{1}{N}\sum_{k=1}^{N}(E[y_{k+1}|\mathcal{F}_k] + e_{k+1})^2 \\ &= \frac{1}{N}\sum_{k=1}^{N} E[y_{k+1}|\mathcal{F}_k]^2\end{aligned}$$

by the assumptions on the noise process. Therefore since

$$r_k = r_{k-1} + \phi_k^T\phi_k$$

and

$$\phi_{k-1} = (y_{k-1}, \ldots, y_{k-n}, u_{k-1}, \ldots, u_{k-n})^T$$

we immediately have

$$\begin{aligned}\frac{1}{N}\bar{r}_{N-1} &\leq \frac{\Lambda}{N} r_{k-1} \\ &\leq K_1 + K_2 \frac{1}{N} \sum_{k=1}^{N} E[y_k|\mathcal{F}_{k-1}]^2\end{aligned}$$

■

The following lemma gives some simple results for the *a posteriori* prediction error for the vibration control STR.

Lemma 5.3 *For the given model (5.4) with the noise assumptions (5.1)–(5.2), minimum-variance control law (5.8) and the recursive least-squares estimation algorithm (5.9)–(5.14) the a posteriori prediction error* $\{\eta_k\}$ *satisfies the equations*

$$\begin{aligned}\eta_k &\triangleq y_k - \phi_{k-1}^T \hat{\theta}_k \\ &= -\phi_{k-1}^T \tilde{\theta}_k \qquad (5.21) \\ &= \frac{y_k}{1 + \phi_{k-1}^T P_{k-2} \phi_{k-1}} \qquad (5.22)\end{aligned}$$

where $\tilde{\theta}_k \triangleq \hat{\theta}_k - \theta^0$ *and* $\mathcal{F}_k$ *is the σ-algebra generated by* $\{y_1, \ldots, y_k, u_1, \ldots, u_k\}$.

Proof

From (5.5) we immediately have

$$\begin{aligned}\eta_k &= \phi_{k-1}^T \theta^0 + e_{k-1} - \phi_{k-1}^T \hat{\theta}_k \\ &= -\phi_{k-1}^T \tilde{\theta}_k\end{aligned}$$

using the noise assumptions (5.1)–(5.2) which proves (5.21).

Using the recursion for $\hat{\theta}_k$ gives

$$\eta_k = y_k - \phi_{k-1}^T \hat{\theta}_{k-1} - \frac{\phi_{k-1}^T P_{k-2} \phi_{k-1}}{1 + \phi_{k-1}^T P_{k-2} \phi_{k-1}} y_k$$

$$= \frac{y_k}{1 + \phi_{k-1}^T P_{k-2} \phi_{k-1}}$$

which proves (5.22). ∎

5.3.2 Main Result

With the results derived above it is now possible to show that the vibration control STR satisfies the remaining technical condition (5.15) of Lemma 5.1 and thus it is self-optimising. As a consequence of this proof it is also possible to derive the stability and convergence results.

Theorem 5.2 (Stability and Optimality) *For the given model (5.4) with the noise assumptions (5.1)–(5.2), minimum-variance control law (5.8) and the recursive least-squares estimation algorithm (5.9)–(5.14)*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 = 0 \text{ a.s.} \tag{5.23}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} u_k^2 < \infty \text{ a.s.} \tag{5.24}$$

Proof

Define the stochastic Lyapunov function

$$V_k \triangleq \tilde{\theta}_k^T P_{k-1}^{-1} \tilde{\theta}_k$$

where $\tilde{\theta}_k = \hat{\theta}_k - \theta^0$. From Lemma 5.3 and the recursion for $\hat{\theta}_k$ we obtain

$$\tilde{\theta}_{k-1} = \tilde{\theta}_k - P_{k-2}\phi_{k-1}\eta_k$$

therefore

$$\tilde{\theta}_{k-1}^T P_{k-2}^{-1} \tilde{\theta}_{k-1} = \tilde{\theta}_k^T P_{k-2}^{-1} \tilde{\theta}_k - 2\phi_{k-1}^T \tilde{\theta}_k \eta_k + \phi_{k-1}^T P_{k-2} \phi_{k-1} \eta_k^2$$

From the matrix inversion lemma

$$P_{k-1}^{-1} = \beta_{k-1}^{-1}[P_{k-2}^{-1} + \phi_{k-1}\phi_{k-1}^T]$$

therefore

$$\tilde{\theta}_{k-1}^T P_{k-2}^{-1}\tilde{\theta}_{k-1} = \beta_{k-1}\tilde{\theta}_k^T P_{k-1}^{-1}\tilde{\theta}_k - \tilde{\theta}_k^T \phi_{k-1}\phi_{k-1}^T\tilde{\theta}_k - 2\phi_{k-1}^T\tilde{\theta}_k\eta_k + \phi_{k-1}^T P_{k-2}\phi_{k-1}\eta_k^2$$

hence

$$\begin{aligned} \beta_{k-1}V_k &= V_{k-1} + \tilde{\theta}_k^T\phi_{k-1}\phi_{k-1}^T\tilde{\theta}_k + 2\phi_{k-1}^T\tilde{\theta}_k\eta_k - \phi_{k-1}^T P_{k-2}\phi_{k-1}\eta_k^2 \\ &= V_{k-1} - \eta_k^2 - \phi_{k-1}^T P_{k-2}\phi_{k-1}\eta_k^2 \end{aligned}$$

Thus we have

$$\begin{aligned} \beta_{k-1}\frac{\bar{r}_{k-1}}{\bar{r}_{k-2}}E[\frac{V_k}{\bar{r}_{k-1}}|\mathcal{F}_{k-1}] &= [1 + \alpha_{k-1}\phi_{k-1}^T P_{k-2}\phi_{k-1}]E[\frac{V_k}{\bar{r}_{k-1}}|\mathcal{F}_{k-1}] \\ &= \frac{V_{k-1}}{\bar{r}_{k-2}} - \frac{1}{\bar{r}_{k-2}}E[\eta_k^2|\mathcal{F}_{k-1}] - \frac{1}{\bar{r}_{k-2}}E[\phi_{k-1}^T P_{k-2}\phi_{k-1}\eta_k^2|\mathcal{F}_{k-1}] \end{aligned}$$

Define

$$M_k \triangleq \frac{V_k}{\bar{r}_{k-1}} + \sum_{j=1}^{k}\frac{\eta_j^2}{\bar{r}_{j-2}} + \sum_{j=1}^{k}\frac{\phi_{j-1}^T P_{j-2}\phi_{j-2}}{\bar{r}_{j-2}}\eta_j^2$$

then M_k is a non-negative, $\mathcal{F}_k$-measurable function. Furthermore

$$\begin{aligned} E[M_k|\mathcal{F}_{k-1}] &= E[\frac{V_k}{\bar{r}_{k-1}}|\mathcal{F}_{k-1}] + E[\frac{\eta_k^2}{\bar{r}_{j-2}}|\mathcal{F}_{k-1}] + \sum_{j=1}^{k-1}\frac{\eta_j^2}{\bar{r}_{j-2}} \\ &\quad + E[\frac{\phi_{k-1}^T P_{k-2}\phi_{k-1}}{\bar{r}_{k-2}}\eta_k^2|\mathcal{F}_{k-1}] + \sum_{j=1}^{k-1}\frac{\phi_{j-1}^T P_{j-2}\phi_{j-2}}{\bar{r}_{j-2}}\eta_j^2 \\ &\le \frac{V_{k-1}}{\bar{r}_{k-2}} - \frac{1}{\bar{r}_{k-2}}E[\eta_k^2|\mathcal{F}_{k-1}] - \frac{1}{\bar{r}_{k-2}}E[\phi_{k-1}^T P_{k-2}\phi_{k-1}\eta_k^2|\mathcal{F}_{k-1}] \\ &\quad + E[\frac{\eta_k^2}{\bar{r}_{j-2}}|\mathcal{F}_{k-1}] + \sum_{j=1}^{k-1}\frac{\eta_j^2}{\bar{r}_{j-2}} \\ &\quad + E[\frac{\phi_{k-1}^T P_{k-2}\phi_{k-1}}{\bar{r}_{k-2}}\eta_k^2|\mathcal{F}_{k-1}] + \sum_{j=1}^{k-1}\frac{\phi_{j-1}^T P_{j-2}\phi_{j-2}}{\bar{r}_{j-2}}\eta_j^2 \\ &= M_{k-1} \end{aligned}$$

By the Non-Negative Near Supermartingale Convergence Theorem, Theorem 5.1, $\{M_k\}$ converges almost surely and hence

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{\eta_k^2}{\bar{r}_{k-2}} < \infty \text{ a.s.} \tag{5.25}$$

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{\phi_{k-1}^T P_{k-2}\phi_{k-1}}{\bar{r}_{k-2}}\eta_k^2 < \infty \text{ a.s.} \tag{5.26}$$

Now

$$\begin{aligned} E[y_k|\mathcal{F}_{k-1}] &= \phi_{k-1}^T\theta^0 \\ &= \eta_k + \phi_{k-1}^T\hat{\theta}_k \\ &= \eta_k + \phi_{k-1}^T P_{k-2}\phi_{k-1}\eta_k \end{aligned}$$

therefore

$$\begin{aligned} \lim_{N\to\infty}\sum_{k=1}^{N}\frac{E[y_k|\mathcal{F}_{k-1}]^2}{\bar{r}_{k-1}} &\le 2\lim_{N\to\infty}\sum_{k=1}^{N}\frac{\eta_k^2}{\bar{r}_{k-1}} + 2\lim_{N\to\infty}\sum_{k=1}^{N}\frac{(\phi_{k-1}^T P_{k-2}\phi_{k-1})^2}{\bar{r}_{k-1}}\eta_k^2 \\ &\le 2\lim_{N\to\infty}\sum_{k=1}^{N}\frac{\eta_k^2}{r_{k-2}} + 2\Lambda\lim_{N\to\infty}\sum_{k=1}^{N}\frac{\phi_{k-1}^T\phi_{k-1}}{\bar{r}_{k-1}}\frac{\phi_{k-1}^T P_{k-2}\phi_{k-1}}{\bar{r}_{k-2}}\eta_k^2 \\ &< \infty \text{ a.s.} \end{aligned}$$

using (5.25) and (5.26). Application of the Stochastic Key Technical Lemma then proves the results. ■

Corollary 5.2.1 (Convergence) *For the given model (5.4) with the noise assumptions (5.1)–(5.2), minimum-variance control law (5.8) and the recursive least-squares estimation algorithm (5.9)–(5.14)*

$$\lim_{k\to\infty}||\hat{\theta}_k - \hat{\theta}_{k-1}||^2 = 0 \text{ a.s.} \tag{5.27}$$

Proof

From (5.22) the recursion for $\hat{\theta}_k$ can be written as

$$\hat{\theta}_k = \hat{\theta}_{k-1} + P_{k-2}\phi_{k-1}\eta_k$$

therefore

$$\begin{aligned}||\hat{\theta}_k - \hat{\theta}_{k-1}||^2 &= \phi_{k-1}^T P_{k-2}^2 \phi_{k-1} \eta_k^2 \\ &\leq \frac{\phi_{k-1}^T P_{k-2} \phi_{k-1}}{\bar{r}_{k-2}} \eta_k^2\end{aligned}$$

hence

$$\begin{aligned}\lim_{N\to\infty} \sum_{k=1}^{N} ||\hat{\theta}_k - \hat{\theta}_{k-1}||^2 &\leq \lim_{N\to\infty} \sum_{k=1}^{N} \frac{\phi_{k-1}^T P_{k-2} \phi_{k-1}}{\bar{r}_{k-2}} \eta_k^2 \\ &< \infty \text{ a.s.}\end{aligned}$$

using (5.26) which proves the result. ∎

Convergence Rate

The convergence rate of the vibration control STR described in Section 5.2 is very slow. The gain in the recursive least-squares algorithm will tend to zero because $\lim_{k\to\infty} P_k \to 0$. At the same time the magnitude of the input to the estimation algorithm, y_k, is also tending to zero. As a consequence the convergence rate decreases with k. In practice, it is necessary to use a RLS algorithm which imposes a lower, non-zero bound on the magnitude of P_k.

Consistency

These results show that the vibration control STR will converge to a stable controller which regulates the plant. As the vibration control STR eliminates the sinusoidal disturbance it follows from the internal model principle that the limiting controller must contain unit circle poles at the unknown frequency of the disturbance. Thus this STR must consistently estimate this frequency. However, it is not possible to conclude that the limiting, adaptive controller will be equal to the ideal controller, in general, for a number of reasons.

Recall that when the transfer functions of the plant and disturbance are known, the ideal controller is

$$K = \frac{A_1}{B_1}\frac{(B_2 - A_2)}{A_2}.$$

If this controller is irreducible then it can be decomposed into a compensator for the plant, P^{-1}, and an oscillator at the disturbance frequency, K_1. The system can then be represented as in Figure 5.2.

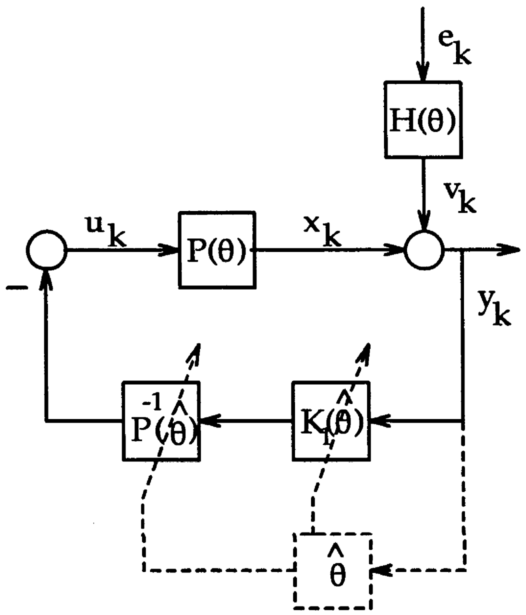

Figure 5.2: Oscillator+Compensator form of Vibration Control STR

If the adaptive controller is over-parameterised (i.e. $\bar{n} > n - 2$) then the limiting controller could have cancellations between the compensator and the oscillator and convergence to the ideal controller cannot be assumed.

Furthermore, as the only input into the system is a pure sinosoid of frequency ω, the frequency response of the limiting adaptive controller only needs to match the frequency response of the ideal controller at this frequency to eliminate the vibration. Mismatch between the estimated and ideal controller at other frequencies will have no effect. This would not be the case if the disturbance was a vibration embedded in white noise.

That the vibration control STR is not self-tuning is not unexpected. As was discussed in Chapter 2, it is well known that using least-squares to estimate the parameters of a linear system when there is a vibration present will produce a biased result. Moreover, Becket *et al.* (1985) showed that the parameter estimates produced by the form of Åström and Wittemark's STR which uses the stochastic approximation estimation algorithm, will converge to a biased answer. They showed that the limiting parameter estimates are a random multiple of the true parameter estimates when the ideal controller is irreducible. For the case of the vibration control STR a similar result can be expected. From the internal model principle we know that the limiting controller must have a denominator that has a quadratic factor, due to the vibration, which is a multiple of the same quadratic factor in the ideal controller.

If the disturbance had power at frequencies in addition to ω then this limited consistency result would be insufficient. Such a situation could arise if the frequency of the disturbance was time-varying, if the vibration contained harmonics of the fundamental frequency or if the vibration contained additive white noise. To achieve optimality in such cases the vibration control STR would have to match the ideal controller over the entire frequency range contained in the disturbance. For the case when the disturbance was a vibration with additive white noise a result similar to that of Becker *et al.* would be expected for any self-tuning adaptive vibration controller. In the case when the vibration frequency was time-varying the estimation algorithm would need to be modified, to incorporate exponential forgetting for example, to achieve a self-tuning result. In such a method the estimation estimation algorithm would operate on a sliding window of data and thus old data is discounted, allowing for variation in the parameters.

5.4 Simulations

The following simulations show the results of using a modified form of the vibration control STR in two situations. The first example is of a simple, low order, minimum phase plant of known order. In the second example the order of the plant is not known and the adaptive controller is over-parameterised.

In both cases the recursive least-squares algorithm used is the exponential forgetting and resetting algorithm (EFRA) (Salgado *et al.*, 1988) in order to achieve a reasonable convergence rate. This algorithm yields the same limiting controller as the condition number monitoring RLS algorithm (BNM algorithm) described in Section 5.2. In the EFRA algorithm, both lower and upper bounds are placed on the size of P_k. The algorithm is given by the equations

$$\begin{aligned}
\hat{\theta}_k &= \hat{\theta}_{k-1} + \frac{\alpha P_{k-2}\phi_{k-1}}{1+\phi_{k-1}^T P_{k-2}\phi_{k-1}}(y_k - \phi_{k-1}^T \hat{\theta}_{k-1}) \\
P_{k-1} &= \frac{1}{\lambda}P_{k-2} - \frac{\alpha P_{k-2}\phi_{k-1}\phi_{k-1}^T P_{k-2}}{1+\phi_{k-1}^T P_{k-2}\phi_{k-1}} + \beta I - \delta P_{k-2}^2
\end{aligned}$$

where α, β, λ and δ are constants. If the constraints

$$\begin{gathered}
0 < \gamma < \alpha < 1 \\
(\gamma - \alpha)^2 + 4\beta\delta < (1-\alpha)^2 \\
\beta > 0 \\
\delta > 0 \\
\bar{\mu} I \le P_{-1} \le \bar{\nu} I
\end{gathered}$$

hold, where

$$\begin{aligned}
\gamma &\triangleq \frac{1-\lambda}{\lambda} \\
\bar{\nu} &\triangleq \frac{\gamma}{2\delta}\left(1 + \left(1 + \frac{4\beta\delta}{\gamma^2}\right)^{\frac{1}{2}}\right) \\
\bar{\mu} &\triangleq \left(\frac{\alpha - \gamma}{2\delta}\right)\left(-1 + \left(1 + \frac{4\beta\delta}{\gamma^2}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

then

$$\bar{\mu} I \le P_k \le \bar{\nu} \qquad \text{for all } k.$$

General guidelines for choosing the constants are

- α adjusts the gain of the algorithm, typically $\alpha \in [0.1, 0.5]$;
- β is related to the minimum eigenvalue of P_k, typically $\beta \in [0, 0.01]$;
- λ is the exponential forgetting factor; typically $\lambda \in [0.9, 0.99]$; and
- δ is inversely related to the maximum eigenvalue of P_k, typically $\delta \in [0, 0.01]$.

For constants in these ranges

$$\bar{\nu} \approx \frac{\gamma}{\delta} + \frac{\beta}{\gamma}$$
$$\bar{\mu} \approx \frac{\beta}{\alpha - \gamma}$$

The simulations here used the settings

$$\alpha = 0.5 \quad \beta = 0.005$$
$$\lambda = 0.95 \quad \delta = 0.005$$

and thus

$$0.011I \leq P_k \leq 10.1I.$$

The plant model used in both simulation examples has the transfer function

$$P = \frac{B_1(z^{-1})}{A_1(z^{-1})} = \frac{3z^{-1}}{1 - 0.5z^{-1}}$$

so the plant is minimum phase. That is, it is both stable and stably invertible. The disturbance process had the transfer function

$$H = \frac{B_2(z^{-1})}{A_2(z^{-1})} = \frac{1}{1 - 2\cos(\omega)z^{-1} + z^{-2}}$$

where $\omega = \frac{\pi}{6}$ radians or 0.0833 Hz. Hence the general system was of order $n = 3$ with poles at $0.8660 \pm 0.5\jmath$. The ideal controller for this system is

$$K = \frac{A_1(z^{-1})}{B_1(z^{-1})} \frac{2\cos(\omega)z^{-1} - z^{-2}}{1 - 2\cos(\omega)z^{-1} + z^{-2}} = \frac{z^{-1}(2\cos(\omega) - (1 + \cos(\omega))z^{-1} + \frac{1}{2}z^{-2})}{z^{-1}(3 - 6\cos(\omega)z^{-1} + 3z^{-2})}$$

5.4.1 Example 1

In this example the order of the plant was known and so the controller was correctly parameterised. Figure 5.3 compares the plant output when the BNM algorithm was used with the output obtained using the EFRA algorithm. This demonstrates the slow convergence rate when the RLS gain is not bound away from zero. Both algorithms converged to a limiting adaptive controller with the correct poles as expected. The poles of the limiting adaptive controller were $0.8660 \pm 0.5000\jmath$ after $k = 250$ iterations in the EFRA case and after $k = 10,000$ iterations when using the BNM algorithm.

The EFRA parameter estimates for the controller numerator and denominator polynomials are given in Figures 5.4 and 5.5 respectively. The lack of consistency in the parameter estimates is clear. The true parameter values and the limiting estimated values are given in Table 5.1.

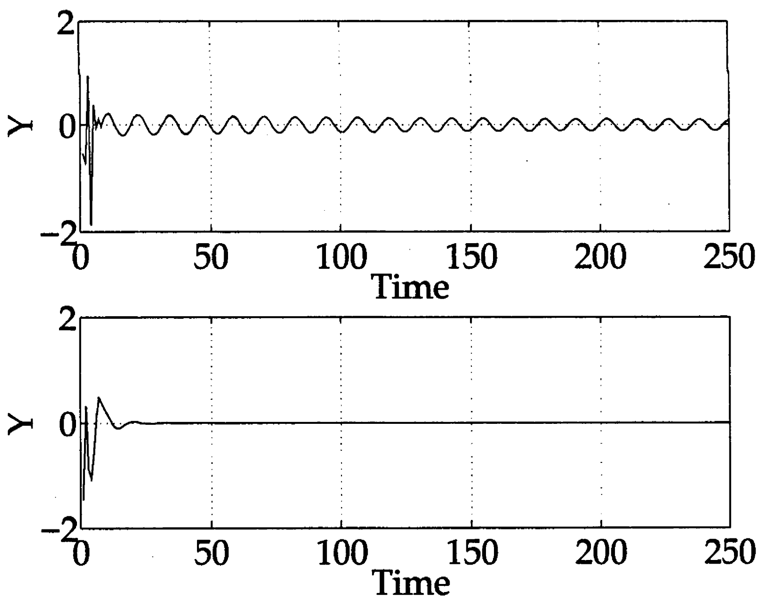

Figure 5.3: Plant output using the BNM (top) and EFRA (bottom) RLS algorithms for Example 1.

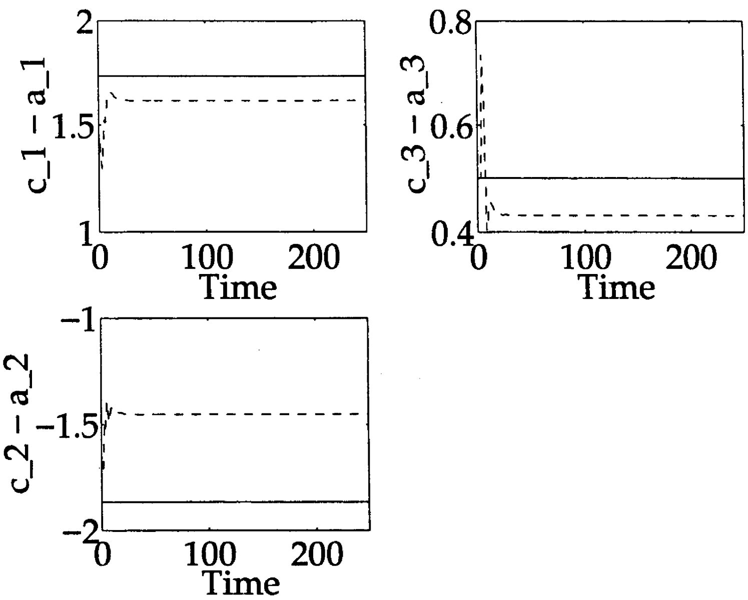

Figure 5.4: Estimates of controller numerator polynomial coefficients for Example 1. True values given by solid lines.

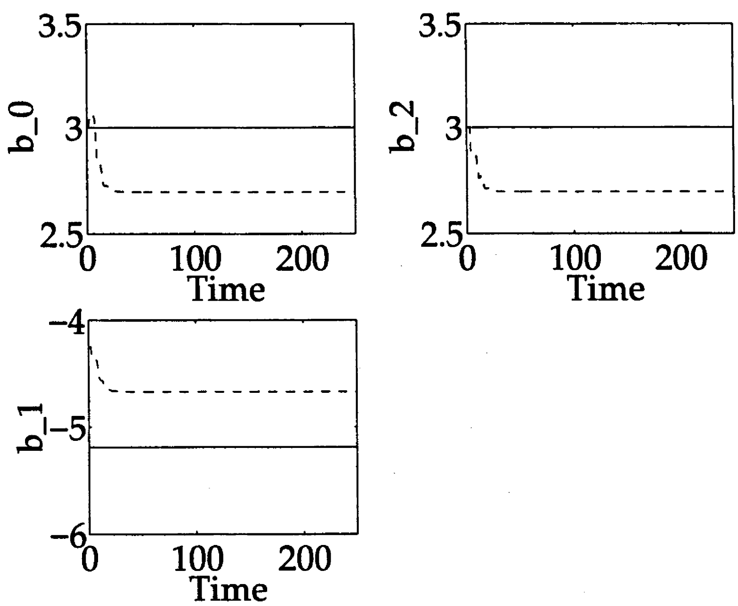

Figure 5.5: Estimates of controller denominator polynomial coefficients for Example 1. True values given by solid lines.

	True	Estimated
$c_1 - a_1$	1.7321	1.7980 ρ
$c_2 - a_2$	-1.8660	-1.6164 ρ
$c_3 - a_3$	0.5000	0.4802 ρ
b_0	3.0000	3.0000 ρ
b_1	-5.1962	-5.1962 ρ
b_2	3.0000	3.0000 ρ

Table 5.1: Parameter values for Example 1, $\rho = 1.1133$

From Table 5.1 it can be seen that the denominator of the limiting adaptive controller is a multiple of the ideal denominator, but that this is not true for the numerator. This is in contrast to the Åström and Wittenmark STR where the limiting controller is a random multiple of the entire ideal controller when this ideal controller is irreducible (Becker, Jr. *et al.*, 1985; Radenković, 1990). The vibration control STR is able to eliminate the vibration from the plant output in spite of converging to a different controller to the ideal one because, in this case, it is not necessary to match the ideal controller at all frequencies. The controller is only being excited by a signal with energy at the unknown frequency ω and thus there are an infinite number of controllers which match the ideal controller at the frequency ω and hence satisfy the performance criterion. That the limiting controller does this can be seen from the magnitude and phase response of the ideal and limiting adaptive controllers which are shown in Figures 5.6 and 5.7.

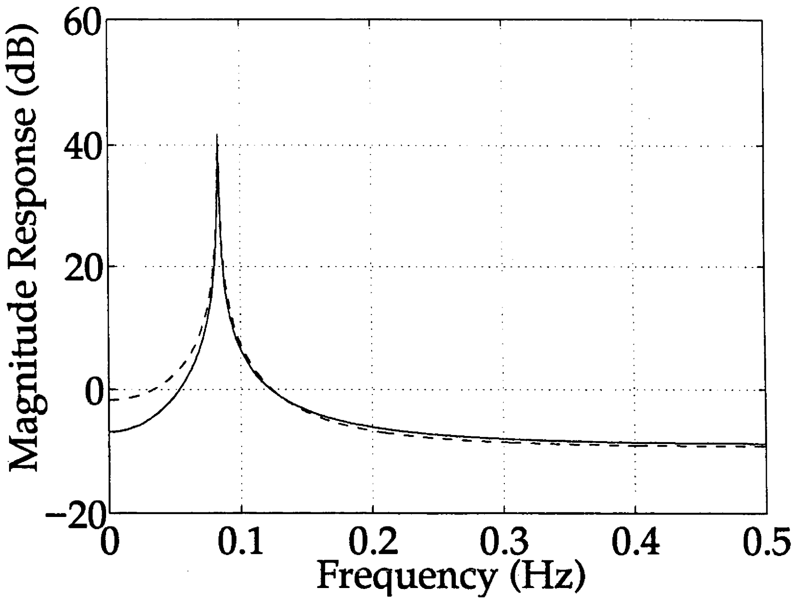

Figure 5.6: Magnitude response for limiting adaptive controller (dashed line) and ideal controller (solid line) for Example 1.

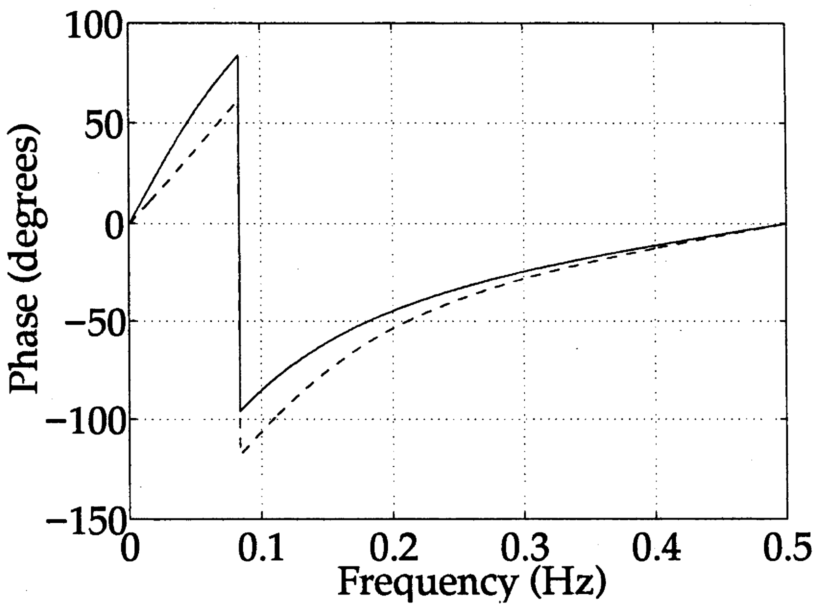

Figure 5.7: Phase response for limiting adaptive controller (dashed line) and ideal controller (solid line) for Example 1.

5.4.2 Example 2

In this example the same plant was used but the order was considered to be unknown. The adaptive controller was designed using an assumed upper bound of $n = 5$ for the order of the general system. If $\hat{\theta}_k$ was a consistent estimator of θ^0 then the extra, unnecessary parameters in the controller, $(c_4 - a_4)$, $(c_5 - a_5)$, b_3 and b_4, should converge to zero. This is not the case as can be seen from Figure 5.8. However, the limiting adaptive controller does regulate the plant, driving the output to zero, as is shown in Figure 5.9.

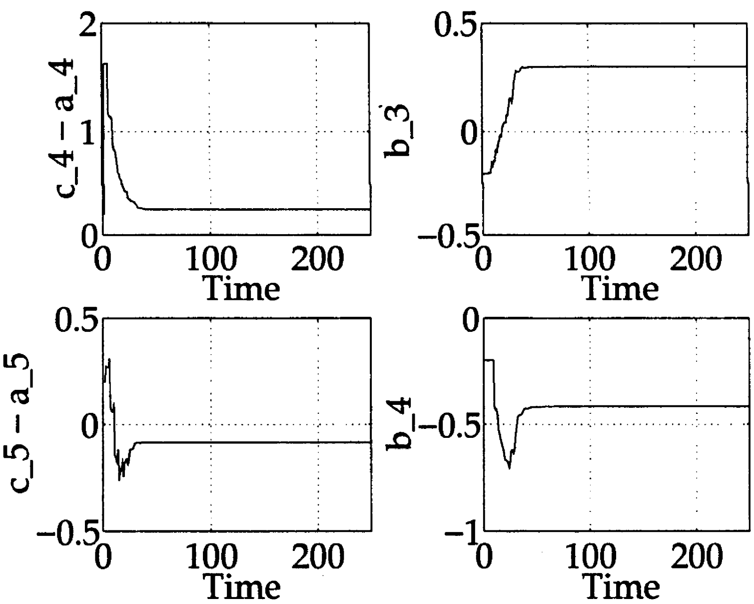

Figure 5.8: Parameter estimates for Example 2.

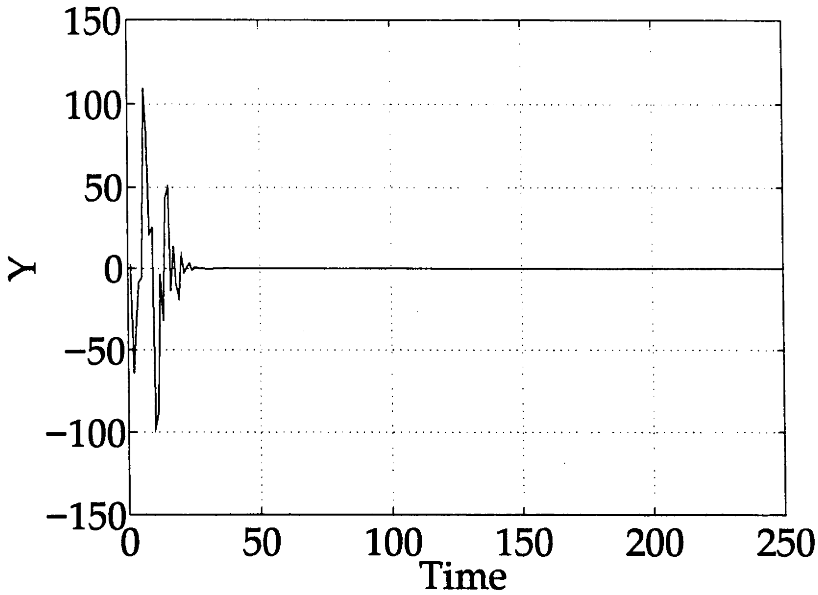

Figure 5.9: Plant output for Example 2.

The true parameter values and the limiting, estimated values are shown in Table 5.2. In this case, neither the denominator polynomial nor the numerator polynomial is a multiple of the ideal controller transfer function polynomials. This is in contrast to the case when the controller order was correctly estimated. However the poles of the limiting, adaptive controller are -3.2232, 2.2085 and $0.8660 \pm 0.5\jmath$ and hence the vibration control STR contains the poles due to the vibration as expected. Once again the limiting controller matches the ideal controller at the frequency of the vibration, ω, as in shown by Figures 5.10 and 5.11.

Parameter	$c_1 - a_1$	$c_2 - a_2$	$c_3 - a_3$	$c_4 - a_4$	$c_5 - a_5$
True	1.7321	-1.8660	0.5000	0.0000	0.0000
Estimated	2.2585	-1.9837	0.4610	0.2449	-0.0845

Parameter	b_0	b_1	b_2	b_3	b_4
True	3.0000	-5.1962	3.0000	0.0000	0.0000
Estimated	2.9604	-5.5496	3.2754	0.2984	-0.4159

Table 5.2: Parameter values for Example 2.

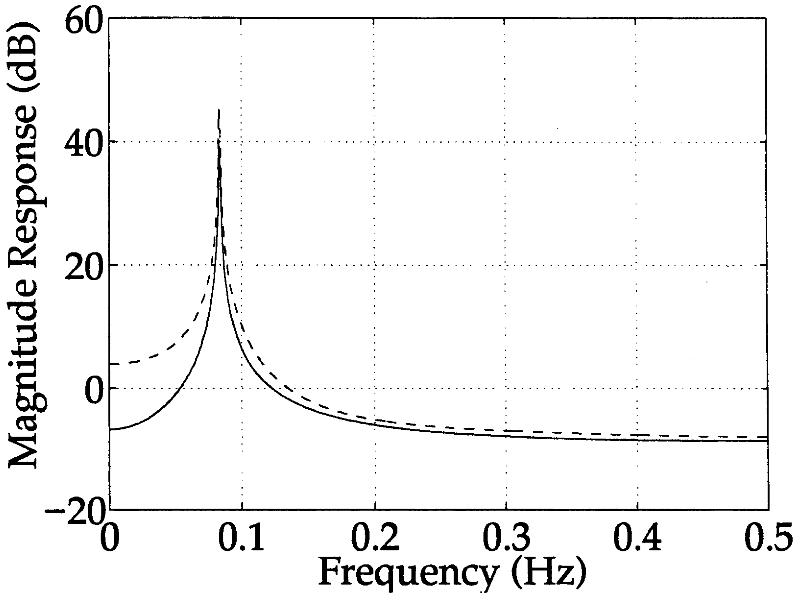

Figure 5.10: Magnitude response for limiting adaptive controller (dashed line) and ideal controller (solid line) for Example 2.

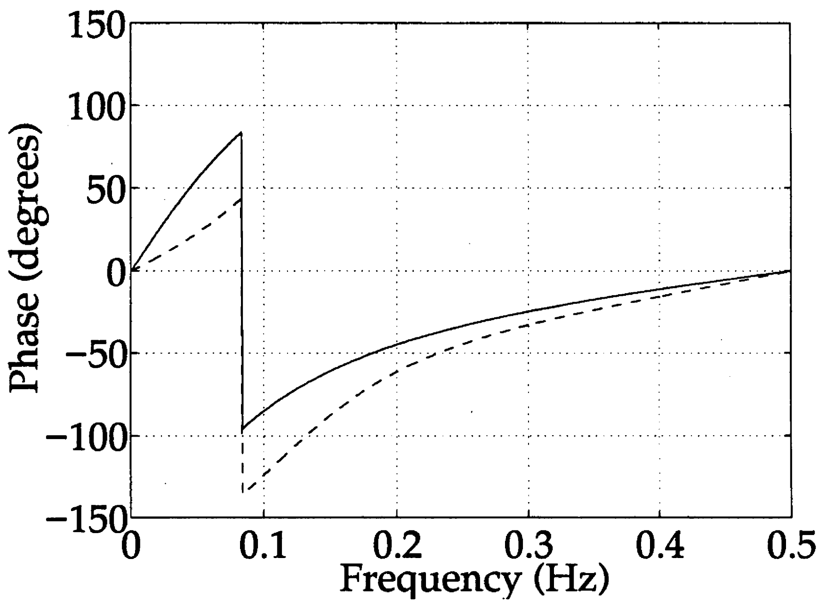

Figure 5.11: Phase response for limiting adaptive controller (dashed line) and ideal controller (solid line) for Example 2.

5.5 Conclusion

This chapter showed how the analysis for the self-tuning regulator problem posed by Åström and Wittenmark can be extended to the case when the plant disturbance is a deterministic vibration. It showed that the stability, convergence and self-optimality properties of the Åström and Wittenmark STR carry over to this new scenario, given an appropriate estimation algorithm. However the self-tuning property is not satisfied by the vibration control STR unlike the Åström and Wittenmark STR. Moreover, in general, conditions under which the parameter estimates will be consistent have not been found. However, it has been shown that the self-optimality property of the vibration control STR requires that the unit circle poles due to the vibration are consistently estimated. The properties of the vibration control STR have been illustrated with simulation results.

6

Conclusion

6.1 Summary of Major Results

THE results developed in this thesis contribute to the important problem of the estimation of the frequency of a sinusoidal signal. This area is significant because in many problems in both signal processing and control the frequency of a signal encodes useful information. This was illustrated by the choice of problems examined: the signal processing problem of tracking a time-varying frequency; and the control problem of rejecting a vibration at an unknown frequency.

In spite of the differing applications, both problems shared a number of common features. In both cases it was necessary to solve a nonlinear estimation problem. In the frequency tracking case the nonlinearity arose from two sources. The first was that the information to be recovered was a nonlinear function of the output. The second source was the nonlinearities present in the calculation of the EKF gain matrix. In the vibration control problem the quantities directly estimated (the transfer function coefficients) were linear functions of the output. However, by embedding the problem in closed-loop and using a recursive least-squares algorithm, this problem also was nonlinear.

The second common feature of both these problems was that they both relied on minimising a least-squares error criterion. It was known that using such a criterion could potentially cause difficulties. It has been shown that using the Kalman filter on a system which has uncontrollable modes on the unit circle leads to a lack of robustness. To avoid this carrying over into the EKF frequency tracker, stability theory was developed for the EKF which allowed the derivation of design guidelines for constructing robust, nonlinear filters. In the vibration control case there was a potential bias in the parameter estimates which again could have lead to a lack of robustness. However, it was shown that by embedding the estimation problem is closed-loop the effects of the known open-loop bias were eradicated.

In the vibration control problem it was possible to achieve the goal of minimising the least-squares criterion. The high degree of nonlinearity in the frequency tracking problem made an exact solution impractical. The least-squares estimator property of the Kalman filter does not carry over to the extended Kalman filter. The EKF is merely an approximation to the optimal least-squares estimator and perhaps not the most desirable one.

However, in spite of the approximate nature of the EKF, it is a widely used tool for the construction of observers for many nonlinear estimation problems in addition to the ones considered here. This means this thesis potentially contributes to a far wider area than just the frequency estimation applications examined here. The theory presented in Chapter 2 was derived without simplifying approximations based on the particular problem structure considered as is often the case when analysing EKF performance. Moreover, the methodology employed to construct the two frequency trackers presented in Chapters 3 and 4 can be used to construct any nonlinear EKF-based observer.

The contributions made by this thesis are summarised below.

- An analysis of the dynamics of the EKF and a derivation of sufficient conditions for bounding the errors of the EKF.

- Quantification of the relationship between the smoothness properties of the nonlinear system and the magnitude of the noise processes, and its implication to performance of the EKF.

- An analysis of the construction of EKF-based filters for general nonlinear systems, including theoretically supported design guidelines.
- Construction of accurate EKF-based frequency trackers for both high SNR signals and weak, narrowband signals.
- An extension of Åström and Wittenmark's self-tuning regulator to the case of deterministic disturbances.
- The construction and analysis of the properties of a self-tuning regulator for vibration control.

6.2 Future Research

The work in this thesis, while answering some questions, has raised many more, as is inevitably the case in scientific research. A list of some of these questions concludes this thesis.

6.2.1 The Extended Kalman Filter

The conditions for the stability of the extended Kalman filter, presented in Theorem 2.9, only apply to state space systems which are nonlinear solely in the state equation. The extension of this result to the fully nonlinear case is the obvious next step. To do so would require a reasonably straightforward extension of the theory and methods presented here.

In addition, the requirements of local observability and controllability could probably be weakened to detectability and stabilizability respectively. Also the assumption of an invertible Jacobian of the state dynamics does not appear to be fundamental to the derivation and could possibly be dispensed with.

Other avenues of investigation include the derivation of more accurate estimates of error covariance than the Riccati equation solutions. In particular, the derivation of bounds for the true error covariance would be a significant advance in EKF theory. In addition, extending the EKF to include smoothing could bring potential improvements in accuracy.

6.2.2 Frequency Tracking using the EKF

Future research on the use of the EKF for frequency tracker could include the following.

- The development of methods for track initialisation and termination. In particular, the development of a reliable method for initialising the high noise EKF frequency tracker is desirable. The development of accurate initialisation methods for the EKF is a crucial step in the construction of reliable nonlinear filters.
- Investigation of threshold effects for both the high and low SNR frequency trackers and the quantification of these effects in terms of slew rate and SNR.
- Fuller comparison of the cartesian high SNR frequency tracker to that of the polar form tracker.
- Comparison of the EKF based trackers to alternative frequency trackers.
- Inclusion of amplitude estimation in the high SNR frequency tracker of Chapter 3.

6.2.3 Adaptive Control for Vibration Rejection

There are two immediate extensions of the vibration control self-tuning regulator. The first is to determine under what conditions it is possible to extend the self-tuning proof to the case when the unknown frequency is time-varying or perhaps even the plant parameters are time-varying.

The second extension is to consider the case when the disturbance is the sum of a sinusoid, with possibly time-varying frequency, and white noise. Such a vibration control problem is far less idealised than the one considered here. However, analysis of the vibration control STR of Chapter 5 represents a first step towards an understanding of adaptive controllers for active vibration control.

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