Non-Perturbative Quantum Gravity: The Loop Representation

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PREFACE

This dissertation is the result of my own original research. Any contributions made by others are acknowledged explicitly by appropriate references. The contents presented in Chapters 1 and 2 are review material. Chapters 3 and 7 are due primarily to the efforts of my own unpublished research whilst any other material present therein which are contributed by others are duly acknowledged as such.

The contents of Chapter 4 are based on a paper published jointly by my supervisor Dr M. Anderson and myself as:

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ABSTRACT

The loop representation theory of quantum gravity which was developed in the late 80's by Rovelli and Smolin is a rather novel approach towards unravelling the strands of puzzle that weave about the quantum aspects of Einstein's theory of general relativity. In this thesis, certain aspects of this theory will be explored and in particular, the theory will be set forth on a rigorous mathematical foundation.

Several issues arising from the loop representation of quantum gravity will be addressed. Briefly, they are: (i) to establish a relationship between knot states and the states of 3-geometries; (ii) to show the existence of a diffeomorphism-invariant multi-loop measure on the space of multi-loops; (iii) constructing a gauge-invariant promeasure on the space of Ashtekar connection 1-forms; (iv) the issue of implementing the reality condition in the loop representation and the question of a physical inner product on the space of multi-loop states.

The relationship between a judiciously chosen subset of loops defined on a fixed compact Riemannian 3-manifold and the geometry of the 3-manifold will be established in this thesis. Loosely put, the subset of loops chosen is a denumerable set of loops that are piecewise geodesic with respect to a fixed 3-metric and such that the base points of the chosen loops form a dense subset in the 3-manifold. The existence of a diffeomorphism-invariant multi-loop measure is demonstrated in some depth and the construction of a gauge-invariant promeasure is also given.

The multi-loop space will be constructed in detail and its basic topological properties analysed. Moreover, the existence of a manifold structure on the loop space will be sketched and light is shed on the inadmissibility of a manifold structure on the multi-loop space. The space of the multi-loop functionals will also be briefly studied. The issue regarding the determination of the action of the Hermitian conjugates of the quantum T^n-operators on the multi-loop functionals will be broached. And furthermore, the reality-conditions in the loop representation as well as the possible construction of a physical inner product for the multi-loop states will be tentatively delineated. Finally, an attempt will also be made to endow the multi-loop space with a generalised differentiable structure.

CONTENTS

CHAPTER I

QUANTUM GRAVITY: AN OVERVIEW

1.1. Introduction

It is known since early this century that general relativity is incompatible with quantum theory. The incompatibility is indeed more profound than the fact that gravity is perturbatively non-renormalisable in the covariant quantisation scheme. It ultimately lies in the rôle in which space and time play in general relativity and quantum theory. This is a rather subtle issue and is undoubtedly the main culprit that defies various quantisation approaches to gravity. A less conceptually subtle issue is, of course, the non-renormalisability of gravity. This is more of a technical issue than a conceptual one. It arises from the attempt to depict gravity as another field defined on Minkowski space-time. Here, the problem encountered is primarily due to the presence of a dimensionful coupling constant—the Gravitational constant—(resulting from the Principle of Equivalence) that prevents the construction of a predictive quantum theory of gravity. Indeed, the advent of quantum field theory led invariably to valiant attempts in quantising Einstein's theory of gravitation. All of which proved futile. Perhaps Isham [25, p. 8] was on the right track all along when he remarked that rather than quantising gravity, one should seek a quantum theory which yields general relativity as its classical limit. But then, the main obstruction here is the lack of a starting point to construct such a quantum theory.

By assuming that quantum theory is the underlying principle governing the behaviour of nature at the fundamental level, it is then almost inevitable that a quantum theory of gravitation should exist.[1] Perhaps a more pertinent question to be raised at this juncture is the following: why quantise gravity in the first place? First, there are issues in quantum cosmology—such as the quantum effects of black holes due to their intense gravitational fields—which cannot be fully addressed without a consistent theory of quantum gravity. Second, it is hoped that a theory

[1] In this thesis, it will be assumed tacitly that quantum laws are the fundamental laws that govern at the microscopic level.

of quantum gravity will clear up various enigmatic questions such as the structure of space-time at a microscopic level, causality (and hence the arrow of time), and possibly even account for the presence of singularities in classical space-times [36, Chapter 8, p. 256] established by Hawking and Penrose. These questions provide rather strong incentives for constructing a theory of quantum gravity.

An early effort at quantising gravity was made by Rosenfeld in 1930 [48, 49]; needless to say, he headed rapidly into insurmountable technical difficulties! This is hardly surprising since it is now well known that pure gravity is perturbatively non-renormalisable at the 2-loop level and non-renormalisable at the 1-loop level when coupled with matter fields. Indeed, a simple power counting argument will quickly predict the non-renormalisability of gravity. In the early 1960's, Weinberg studied the quantum aspects of general relativity within the framework of S-matrix theory [61, 62], but his work was hindered by hideous non-linearities encountered in Einstein's field equations. His task was continued by Boulware and Deser [22] who showed in detail that, provided that the long range interactions of gravity are mediated by massless spin-2 particles, in the S-matrix formulation, general relativity is indeed the classical limit of the quantum theory. However, their calculations were done in the low-frequency domain.

In a paper by 't Hooft [57], it was demonstrated that pure gravity is 1-loop renormalisable but when coupled with matter, the theory ceases to make sense perturbatively. Specifically, Deser and Nieuwenhuizen showed that the Einstein-Maxwell fields diverge at the 1-loop level [27] and the quantised Einstein-Dirac system also diverges at the 1-loop level [26]. In a recent paper by van de Ven [58], the 2-loop non-renormalisability of covariant quantum gravity was proved explicitly. And to make matters even worse, aside from the technical issues of non-renormalisability, more conceptually profound questions posed—just to mention a few—by Wheeler regarding measurement [24, p. 224], and the issue of causality—*cf.* for example, references [15, 40]—must also be explained in a satisfactory manner by any candidate theory of quantum gravity.

An initial motivation for quantising gravity lay in the hope that it might eliminate the divergences that exist in quantum field theory—unfortunately, not only is such a hope dashed, but using perturbative methods gravity cannot be renormalised. This clearly suggests that the conventional means of quantising gravity, that is, the use of (perturbative) covariant quantisation, is not the right approach; or perhaps quantum theory is ultimately not a complete theory but merely an approximate theory describing the behaviour of nature at the fundamental level. Having said this much, this speculative note will not be pursued any further in this dissertation. However, the failure of gravity to be quantised perturbatively does not necessarily mean that a theory of quantum gravity fails to exist.

Quantum field theory demands that the background metric of space-time be fixed and that Poincaré-invariance be preserved.[2] Moreover, it assumes the smoothness of the underlying space-time manifold. In quantum gravity, the metric itself becomes a dynamical variable and the gauge group is no longer the Poincaré group but the group of smooth diffeomorphisms. Also, it is worthwhile pointing out that quantum gravity, should it exist, ought to determine (or at least, predict) the structure of space-time at the Planck scale and below—assuming the smoothness of space-time certainly defeats this very purpose. Furthermore, the presence of quantum fluctuations of space-time geometry might well destroy its smooth structure. Indeed, a number of researchers in this field, Penrose [30, p. 4] or [47, p. 31] in particular, are quite convinced that the smoothness of space-time geometry at very small distances must be sacrificed. Some researchers go a step further and toy with the idea that perhaps even topology itself ought to be quantised, whatever such a statement might imply. At least, the motivation for such an observation is that *perhaps*, at the Planck scale, fluctuations in the spatial topology (of space-time) might occur, resulting in a space-time foam structure. For an account of space-time foams, refer to Hawking's paper [35]. Initial moves towards topological quantisation was initiated in a rigorous way by Isham *et al.* [39]. A rather eloquent (*and* convincing) argument outlining the need for a non-perturbative approach to gravity can be found in a monograph by Ashtekar [1, p. 3]; consult also references [55, §1], [31, p. 327] and [4].

1.2. Supergravity Theories

It should be pointed out that perturbative covariant quantisation of gravity (which failed to succeed anyway!) and the Ashtekar's quantisation programme are not the only means of tackling the problem of quantising gravity. There are others besides those two such as the Keluza-Klein theory which currently seems to have gone out of favour amongst researchers working in the mainstream of quantum gravity. Probably the two most well known ones are supergravity and superstring theory. Incidentally, they were also candidates for a Unified Field theory. Curiously

[2] This is required in order for energy and momentum to be conserved locally.

enough, string theory was originally conceived to provide an explanation for the behaviour of hadrons and not to quantise gravity!

Supersymmetry is the underlying principal ingredient in supergravity and superstrings. Roughly, it describes a transformation between bosonic fields and fermionic fields. Indeed, supersymmetry can only be implemented if space-time is curved! An heuristic argument outlining the equivalence between the presence of gravity and the implementation of local supersymmetry can be found in [59, p. 201]. This fact alone is suggestive that perhaps quantising gravity requires the unification of fundamental forces of nature. An excellent review article on supergravity can be found in reference [59].

In supergravity theories, each bosonic field has its fermionic counterpart (and vice versa). The fermionic partner of gravitational field is a spin $\frac{3}{2}$ field called the *gravitino*. If there are $n \leqq 8$ gravitinos, the theory is called an $N = n$ *supergravity* theory. $N = 0$ corresponds to general relativity theory. If $N > 8$, fields of spin $\frac{5}{2}$ (and higher) enter into the picture and this includes several spin 2 fields as well. However, the coupling of spin $\frac{5}{2}$ to gravity and to fields of different spins are known to be inconsistent, and no satisfactory coupling of fields with spins greater than 2 exists. Hence, N cannot be greater than 8.

In $N = 1$ supergravity theory, bosons and fermions (which occur in pairs) form irreducible representations of a supersymmetric algebra[3]—these are the spin $(2, \frac{1}{2})$ doublets (*i.e.*, the graviton-gravitino system), the spin $(1, \frac{1}{2})$ doublets (the photon-neutrino system) and the spin $(0, \frac{1}{2})$ doublets. It is a feature of the theory that as many matter doublets may be added to the spin $(2, \frac{3}{2})$ doublet as desired: in doing so, say, by adding one or more spin $(1, \frac{3}{2})$ doublets to spin $(2, \frac{3}{2})$ doublets, one obtains the extended $(N = 2, \ldots, 8)$ supergravity theories. These theories possess N Fermi-Bose symmetries (plus the usual space-time symmetries of course), $\frac{1}{2}N(N-1)$ spin 1 real vector fields and fields of lower spins. Moreover, they also have a global $U(N)$ group whereby the fermions rotate into themselves, and an $O(N)$ subgroup which rotates bosonic fields into themselves. In this way, the graviton—in N-extended supergravity theories—is replaced by a new superparticle whose "polarizations" yield gravitons, quarks, photons, gravitinos, leptons. This unification of particles into one superparticle leads to the unification of forces.

The ultra-violet divergences appearing in supergravity theories seem to be much better behaved. For instance, the infinities in the S-matrix in the first and sec-

[3]Very briefly, this is an algebra with both commutation and anticommutation brackets.

ond order quantum corrections cancel due to the symmetry between bosonic and fermionic fields. Nonetheless, even the presence of supersymmetry is not sufficient to guarantee finiteness at all loops—at least, there are no conclusive proofs that supergravity is perturbatively renormalisable [32]. Indeed, there are strong reasons to suspect that in 4-dimensional space-time, supergravity theories will diverge at the 3-loop level [43]. Hence, it too is not a particularly successful theory of quantum gravity. Moreover, only N-extended matter may be coupled to N-extended supergravity.

1.3. Superstring Theory

Superstrings paint a more optimistic picture than supergravity theories. However, one now requires a 10 dimensional space-time with supersymmetry built in. In spite of that, gravity is a necessary ingredient in order for a consistent quantum theory of superstrings to exist. From this viewpoint, strings as fundamental quanta are strongly supported by the presence of gravity. An introduction to Superstrings can be found in reference [25, p. 301] by Schwarz or Kaku [42]. Hitherto, it is the only candidate for a Unified Field Theory. Supergravity is now understood to be the low-energy limit of superstring theory. More on this matter will be broached in the next paragraph.

In the theory of Superstrings, the fundamental objects are extended 1-dimensional objects called *strings*. The strings can either be open (*i.e.*, a curve) or closed (*i.e.*, a loop). In short, this extension enables ultra-violet divergences appearing in the Feynman diagrams to be removed. There are two basic types of string theory: the *type I* superstring theory, wherein the strings are unoriented, and *type II* in which the strings are oriented. The latter is also known as *heterotic superstrings*. Type II closed superstring theories have $N = 2$ supersymmetry and hence contain $N = 8$ supergravity modelled on a 4-dimensional space-time as a limiting case. Informally, supergravity lies in the zero-mass sector of closed superstring theory. There, supergravity is quadratically divergent at the 1-loop level whereas its corresponding superstring theory is finite. Strings can interact by joining two ends (for open strings), or by breaking at an "interior" point (in the case of a loop) to form an open string. The latter is demanded by causality simply because two ends of a string must "decide" to interact at once without determining first whether they belong to the same string or not.

The inclusion of supersymmetry to string theory means that, aside from general relativity and Yang-Mills theory being included in it, supergravity and GUT are also included in this theory! However, in spite of such grandiose achievements, perturbative approach to superstring theory is plagued with problems [42, p. 285]. Only three major problems will be listed here:

(i) the low energy mass spectrum is still wrong;
(ii) the theory cannot select the true vacuum amongst the host of possible conformal field theories;
(iii) although supersymmetry is preserved to all orders in perturbative theory, it must be broken down in the low energy régime.

To address these problems, researchers turn towards a non-perturbative approach to superstring theory. Also, note that for bosonic string theory, the entire sum of the perturbative expansion diverges [33, 34]. The Ashtekar loop programme takes a more modest turn: it only seeks to formulate a consistent theory of quantum gravity without any thought of unifying the fundamental forces. And more importantly, the approach is non-perturbative from the outset! Indeed, the problems encountered by superstring theory, which is hitherto the sole candidate for a "proper" Unified Field theory, points towards a non-perturbative approach. A second important point to observe here is that the Ashtekar programme asserts that the gravitational field can be quantised on its own without any other fields, whereas in superstring theory, the very presence of supersymmetry necessitates the unification of forces in order to produce a consistent theory of quantum gravity. Quite a strong contrast indeed!

1.4. Non-perturbative Canonical Quantum Gravity

In this section, a cursory account of the canonical quantisation of gravity, together with the strengths and shortcomings of the Ashtekar quantisation programme, will be sketched. To condense the historical development of quantum gravity, it is enough to point out that from the late 1940's up to the mid-1950's, Bergmann embarked on a quest to canonically quantise field theories which are covariant under general coordinate transformations [18, 19, 20, 21]; here general relativity is of course a particular case those theories. He began by doing away with a space-time metric and considered instead a more fundamental field from which the Lagrangian of the theory was constructed. He quickly discovered that the system possessed constraints. Although his quantisation programme was not successfully completed, he nonetheless laid some important ground work for later researchers. In 1966, a comprehensive analysis of canonical quantum gravity was eventually carried out by DeWitt [28, 29].

In the canonical formalism of general relativity, covariance is violated and space-time is split into space and time. The resulting classical configuration space is the space of Riemannian 3-geometries. More of this will be discussed in Chapter 2. Here, it will suffice to note that the resulting phase space of the gravitational system is constrained. That is, the physical trajectories in the phase space lie on a constraint surface defined by the Hamiltonian constraint and the diffeomorphism constraints. Upon canonically quantising this classical system, the physical states lie precisely in the kernel of both the quantum Hamiltonian and diffeomorphism constraint operators. In fact, this is only true for the case when the spatial 3-dimensional slice is chosen to be compact; in the non-compact case, the wavefunctionals must also satisfy an additional Schrödinger equation [25, Eqn (6.1.4), p. 79]. In this thesis, only the spatially compact case will be considered. Unfortunately, due to the intractability of the quantum Hamiltonian constraint equation arising from involuted non-linearities, not a single explicit solution is known. This equation is known as the *Wheeler-DeWitt equation*,[4] and the wavefunctional that satisfies it is known broadly as the *wavefunction of the universe.*

Approximate solutions were of course found, but this involved truncating the Wheeler-DeWitt equation so that only a finite number of degrees of freedom are retained (instead of an infinite number of degrees of freedom in the full equation); this gave rise to the theory of baby universes—the *mini-superspace approximation.* At best, such solutions offer researchers a myopic insight into the convoluted nature of gravity. However, it should be remarked that even if the Wheeler-DeWitt equation can be solved, there remains the question of interpreting the solutions.

Loosely put, the wavefunctionals describe the physical states of space-time as probability amplitudes of possible histories. But this implies at once that the concept of time seems to have vanished in this picture; that is, there is the unpalatable absence of dynamics, of evolution, of time. This disturbing dissonance is seemingly overcome by identifying part of the geometry as an "intrinsic" time; then, the Wheeler-DeWitt equation is interpreted as encoding information that relates to how a wavefunctional changes with respect to this newly introduced notion of "time". But alas, by introducing a physical inner product on the Hilbert space of physical states, the integration integrates over "time" as well! Hence, the problem of time is really not resolved at all. Time, however it might be interpreted here, is

[4] More accurately, the sum of the diffeomorphism and Hamiltonian constraint equation is known as the *Wheeler-DeWitt equation.*

treated very differently from quantum theory. See Isham [25, §6, p. 78] for a lucid but laconic account relating to the problem of time in this canonical formulation and other related problems arising from quantising in the canonical formalism.

It should be pointed out that the riddle of timelessness only occurs for the spatially compact case. When the spatial slice of space-time is non-compact, time is defined by a Schrödinger evolution equation. For a lively assessment of the canonical approach, refer to [31, §2, p. 330] by Ashtekar. Before concluding this sorry tale, a brief word must be mentioned on the Hartle-Hawking functional integral approach to Wheeler-DeWitt equation. Aside from commenting that it yields, heuristically at least,[5] explicit solutions to the Wheeler-DeWitt equation, it fails to provide any information whatsoever at the instant of creation. Also, there is the confounded issue of time cropping up time and time again! It is thus a fervent hope that the problem of time will be resolved with the formulation of a consistent theory of quantum gravity.

If (2+1)-quantum gravity was not mentioned earlier, then it is simply because it is essentially an open book! Much work has been done on it. In particular, (2+1)-quantum gravity is often used as a toy-model for the seemingly intractable (3+1)-quantum gravity. For more details, see for example reference [63] by Witten—as well as a complementary paper by Moncrief [45] who made some constructive criticisms regarding the conclusions drawn by Witten in his paper—and more recent ones such as [44, 11], or a somewhat refreshing article by Waelbroeck [60] to name just a few out of the plethora of literatures on (2+1)-quantum gravity.

This section will end with some comments on Ashtekar's approach to quantising gravity: the connection representation and the loop representation of gravity. A concise summary and motivation for Ashtekar's alternative Hamiltonian formulation of general relativity [2, 3]—which initiated what is now known as the *Ashtekar quantisation programme*—can be found in the introduction of Chapter 2. It should suffice to mention here that Ashtekar's formulation of "complex" general relativity [3] led immediately to the connection representation of quantum gravity—the general relativity formulated in [3] is really "real" general relativity in terms of a *complex* and a real variable, the Ashtekar connection and its conjugate momentum respectively.

An advantage of formulating general relativity in terms of connections (the Ashtekar connection 1-forms) and their conjugates—these are the soldering forms;

[5] The infinite-dimensional measure involved in the integral is not rigorously defined.

i.e., "square roots" of metrics—is that that the conjugate variable need not be invertible! This differs greatly from general relativity which demands that the metric be non-degenerate. An obvious conclusion to be drawn from Ashtekar's formulation is that it yields solutions that are more general than those obtained via Einstein's field equations. It is perhaps a somewhat tantalising speculation that Ashtekar's formulation will yield a profound insight into the relation between signature changes in the space-time metric and the changes in spatial topology of space-time, and perhaps even more interestingly, how these affect quantum gravity. An instructive preliminary analysis regarding spatial topological changes and the degeneracies of Lorentzian metrics can be found in an article by Horowitz [38]. A related comment, if somewhat premature at this stage as it pertains to the loop representation to be mentioned shortly below, relates to an intriguing paper by Smolin [56]: he demonstrated that, using the loop representation of quantum gravity, the spatial topological changes effected by creating or annihilating a special class of wormholes—what he calls *minimalist wormholes*, which are created by identifying pairs of distinct points on the spatial 3-manifold—is equivalent to general relativity coupled to a single Weyl fermion field!

Another positive spin-off from Ashtekar's formulation of general relativity is that in the connection representation, the Hamiltonian constraint is greatly simplified—indeed, to the extent that some nontrivial solutions can now be found: they are just the Wilson loops. Unfortunately, Wilson loops are not invariant under diffeomorphisms. For more details, see [46, p. 12–13] or [41, §7, p. 333]. This startling hitch led to the development of the loop representation of quantum gravity by Rovelli and Smolin [52]. In the loop representation, solutions to *all* the quantum constraints were found—refer to [41, 52] again.

The loop representation theory was applied to free Maxwell theory with resounding success [12]. It was later applied to linearised quantum gravity [13] and was shown to correctly reproduce gravitons. Applications were also made to (2+1)-dimensional quantum gravity primarily on tori [44] using the connection as well as the loop representation—the Dirac transformation reveals that they are all equivalent. In the case of (2+1)-quantum gravity, the loop representation yields a combinatorial picture whereas the connection representation depicts a "timeless" one. Of course, going over to (3+1)-quantum gravity is a different matter altogether. There are no local degrees of freedom in (2+1)-dimensions (due to the vanishing of Weyl tensor), whereas this is no longer the case with (3+1)-gravity. For other work on (2+1)-quantum gravity in the loop representation, refer to papers by Ashtekar *et al.* [6, 11].

Unfortunately, like most theories in the real world, the loop representation of quantum gravity is not free of problems. There are a number of unresolved issues. One of the problems of the loop representation was discovered by Brügmann and Pullin [23, §4, p. 239]. They noticed with some consternation that solutions of the quantum Hamiltonian operator represented by products of Wilson loops were also annihilated by a metric determinant operator in terms of the Ashtekar variables. It follows as a corollary that the solutions will also satisfy the Hamiltonian constraints for arbitrary cosmological constant! A concise account can be found in [46, p. 13–14].

Another disturbing problem of the loop representation lies in the physical interpretation of the theory. Attempts have been made at interpreting the theory in terms of knots and weaves by Rovelli and Smolin [53, 10]. See also references [64, 65, 66] by Zegwaard. Also, a physical inner product on the multi-loop states is not known: this is a problem that is intimately tied with the physical interpretation of the theory. Moreover, there is the pressing issue of defining physical observables [5, 40, 51]. Once again, all of these issues are intertwined; plus, the fact that very little is known about classical observables in general relativity does very little by way of lighting a path for ardent researchers.

In spite of this setback, Smolin [53, 54] has constructed a number of interesting observables in quantum gravity: a surface area operator, a volume operator and an operator that measures the "length" of a 1-form on the spatial slice of space-time. The spectacular results arising from the first two operators are that area and volume in quantum gravity are quantised in some multiple of the Planck area and Planck volume respectively! This seems to vindicate the conjecture that the structure of space-time is discrete at the Planck scale—a conjecture that was established heuristically by Rovelli [49, §4, p. 1648]. Along this note, Rovelli and Smolin [54] also constructed a physical Hamiltonian operator (with a cosmological term included) which acts in essence by breaking and rejoining the points of intersections of loops in different ways. Moreover, it is also finite as well as diffeomorphism-invariant. Hope is expressed that the Hamiltonian operator might encode the full contents of Einstein's field equations in a diffeomorphism-invariant manner.

Returning to other obstacles present in the theory, there are technical matters such as the construction of a measure on the space of Ashtekar connection

1-forms—preliminary studies have been made by Ashtekar *et al.* [7, 8, 9, 16][6] and by Baez [16, 17]. The construction of a diffeomorphism-invariant measure on the multi-loop space is another issue that needs addressing: this is a problem related to the absence of a physical inner product to date. On a whole, the future to the Ashtekar quantisation programme is not as bleak as it seems, and aside from its mathematical beauty, it is at present, a novel approach towards a non-perturbative quantum gravity that has yet to reach an *impasse.* Indeed, recently, further progress in the connection representation is made. Ashtekar *et al.* [10] performed a detailed study of diffeomorphism-invariant theories in the connection representation and they found complete solutions to the Gauss and diffeomorphism constraints for the following class of theories in the connection representation: the Husain-Kuchař model, Riemannian general relativity and Chern-Simons theories. Furthermore, they were able to endow the space of such states with a Hilbert space structure, where the inner product of the Hilbert space is compatible with the reality conditions imposed on the theories.

1.5. Summary of Thesis

In Chapter 2, Ashtekar's Hamiltonian formulation of general relativity will be reviewed and the loop representation presented in an informal setting. In Chapter 3, the topological structure of the multi-loop space will be examined. It will be established there that the loop space is second countable and it moreover admits a manifold structure. Unfortunately, it will also be shown that the multi-loop space does not admit any manifold structure although it is second countable and metrizable. The space of the multi-loop functionals will also be studied briefly and the action of the quantum T^0-operator on the multi-loop functionals will be discussed at length.

An exact relationship between the knot classes of a subset of $\aleph_0$-loops (multi-loops with denumerably infinite loop components) and the 3-geometries defined on a compact 3-manifold will be described in Chapter 4, and in Chapter 5, a diffeomorphism-invariant measure on the space of multi-loops will be constructed. Moreover, questions regarding the Hermitian conjugates of the quantum T^n-operators will also be discussed therein. This then followed by the construction of a gauge-invariant promeasure on the space of Ashtekar connection 1-forms described in the following chapter. The construction differs somewhat from that developed

[6] They constructed a diffeomorphism-invariant promeasure on the space of connections.

by Ashtekar and Lewandowski [8], and so provides an alternative construction. Unfortunately, the promeasure was not constructed to be diffeomorphism-invariant, unlike the construction carried out in reference [8]. Finally, in the concluding chapter, the loop representation will be briefly reviewed from a more rigorous perspective and issues relating to the implementation of the reality conditions in the loop representation will be touched upon.

To conclude this introductory chapter, some conventions used throughout this thesis will be defined below.

(1) Σ will always denote a compact, orientable, smooth, closed Riemannian 3-manifold;

(2) a *Riemannian 3-metric* q on Σ is defined as a *positive-definite* (*i.e.*, has signature $(+,+,+)$), symmetric, covariant 2-tensor field on Σ;

(3) the signature of a (smooth) *Lorentzian 4-metric* g is taken to be $(-,+,+,+)$,

(4) units will be chosen so that the speed of light c and the Gravitational constant G are set to unity for notational convenience, although at times they will be written down explicitly to highlight certain points.

The term diffeomorphic will mean smoothly diffeomorphic unless explicitly stated and the Einstein summation convention will be used throughout: *i.e.*, a sum is implied whenever identical upper and lower indices are encountered. $\mathrm{Diff}(\Sigma)$ will denote the topological group of 3-diffeomorphisms endowed with the compact C^∞-topology—refer to §A of the Appendix for a description of this topology—and $I \stackrel{\text{def}}{=} [0,1]$.

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CHAPTER II

THE ASHTEKAR QUANTISATION PROGRAMME

2.1. Introduction

In this chapter, a non-perturbative quantisation of canonical gravity in terms of the Ashtekar connections and loops will be reviewed.[1] In this thesis, only vacuum general relativity will be considered: *i.e.*, vacuum Einstein's field equations. The traditional Hamiltonian approach to general relativity, begins with a gravitational phase space defined in terms of a Riemannian 3-metric and its conjugate momentum. However, under this pair of canonical variables, the constraints of general relativity were non-polynomial in their dependence on the 3-metric. This led to technical difficulties in finding solutions that satisfy the constraints. This problem, together with works on conical singularities of the reduced phase space of spatially compact space-times by Arms *et al.* [1], and a connection 1-form introduced by Sen [20], motivated Ashtekar [3] to construct what is now referred to as the *Ashtekar variables.*

In essence, Ashtekar shifted the emphasis of traditional canonical formulation of general relativity from the metric representation to the connection representation. Recall briefly that in the metric representation, the fundamental (canonical) variables are the 3-metric and its conjugate momentum (a covector in the cotangent bundle over the space of Riemannian 3-metrics), whilst in the connection representation, the canonical pair is the connection 1-form and its conjugate momentum. The advantages arising from this shift in viewpoint are many. However, for the purpose of this introduction, it will suffice to highlight the main benefits of such an approach. For a more detailed explanation, refer to references [2, p. 19], [16] and of course, Ashtekar's original article [3] on the new Hamiltonian formulation of general relativity. Some of the major advantages are listed below.

(a) The constraint equations in Ashtekar's new variables are much simpler in

[1] The word non-perturbative used here should be interpreted in the following context: the theory does not rely on a fixed classical background metric, and the gravitational field is quantised in full; in particular, gravity is not treated as a perturbed field about a fixed classical background metric (which is not quantised!) and then quantised perturbatively.

appearance: they are polynomial in their dependence on the new "canonical" pair of variables (A, E), where A is the Ashtekar connection 1-form and E is its conjugate momentum. To wit, the Hamiltonian constraint is quadratic in E, whilst the other constraints are linear in E.[2]

(b) The constrained phase space of general relativity may be imbedded into the phase space of complex Yang-Mills theory and hence bringing Einstein's theory of gravity in line with the theories that describe fundamental interactions of nature—gauge theories. Explicitly, every initial datum (A, E) of Einstein's theory is also an initial datum for Yang-Mills theory—it satisfies a vector and a scalar constraint (in addition to the Gauss constraint mentioned in (a) which is satisfied by the Yang-Mills theory).[3]

(c) The constraints do not depend on the inverse E^{-1} of the conjugate momentum E; consequently, Ashtekar's formulation is an extension of Einstein's theory of gravity as degenerate metrics are also possible solutions under Ashtekar's formalism. Hence, restricting the metrics to be non-degenerate in Ashtekar's formalism yields precisely general relativity. This has possible implications in quantum gravity where perhaps the change in signature of the metric might become significant, and more importantly, it might also play a crucial rôle in the study of singularities of classical space-time and the topological changes in the spatial slice of space-time.

Ashtekar's new variables not only had a profound impact on quantum gravity, they also provided a deeper insight into the classical solutions of Einstein's field equations. For instance, his variables led to an alternative characterization of half-flat solutions to Einstein's field equations [4].[4] It relies essentially on the fact that the Ashtekar connection A can be either the potential ${}^{-}A$ for the self-dual part of the Weyl tensor or the potential ${}^{+}A$ for the anti-self-dual part of the Weyl tensor. The self-dual solutions are then obtained by setting ${}^{+}A \equiv 0$, and vice versa—see reference [4].

The use of loops in physics is not a new idea. A brief historical account can be found in [16, p. 1635] and the references cited therein. Suffice to note that Jacobson and Smolin [11] discovered nontrivial solutions to the Hamiltonian con-

[2] In this formulation, an additional constraint, the Gauss constraint, is introduced due to the additional degrees of freedom introduced by the formalism. However, the structure of this constraint is not complicated: it is linear in the conjugate momentum.

[3] More details concerning the relation between the Einstein and Yang-Mills equations can be found in a paper by Mason and Newman [12].

[4] Recall that a 4-metric is *half-flat* if its Riemann tensor is proportional to its dual.

straint of general relativity in the connection representation and this in turn motivated Rovelli and Smolin [17] to construct the loop representation of quantum gravity. And because the the loop formalism automatically captures SU(2) gauge invariance, the additional constraint—the Gauss constraint—present in the connection representation is eliminated in the loop representation. Furthermore, in the loop representation, all solutions of the diffeomorphism constraints are known: they are nothing but loop functionals defined on the space of equivalent classes of loops, where two loops are said to be *equivalent* if they are related by a smooth orientation-preserving diffeomorphism; that is, if the two loops are knotted in the same way.

Wilson loops and the conjugate momentum of the Ashtekar connection 1-forms play an essential rôle in the theory of loop representation. By taking the traces of suitable combinations of the complexified SU(2) holonomies and the conjugate momenta, a class of observables called the T-observables are obtained. The constraints of general relativity can then be recast in terms of suitably defined limits of these T-observables. In short, this yields the loop representation. Unfortunately, the physical interpretation of the loop representation is far from trivial. For a comprehensive (but intuitive) insight into how the way loops are knotted to give rise to gravity, refer to a comprehensive review article by Rovelli [16, §4, p. 1648].

2.2. Ashtekar's Hamiltonian Formulation

In this section, the traditional *canonical formulation* (or the *ADM formalism*) of general relativity will be outlined in order to motivate Ashtekar's Hamiltonian formulation of general relativity [3]. Incidentally, ADM stands for Arnowitt-Deser-Misner. For more details regarding the ADM formalism, consult reference [9, p. 138] for a detailed exposition by Fischer and Marsden, or to reference [8, Chapter 7, p. 226] for the initial value formulation of general relativity. The ADM formalism given below is based on a laconic exposition by Romano [15, §2, p. 765].

Let (X, g) be a smooth, globally hyperbolic, Lorentzian 4-manifold (which is both space- and time-orientable), and $i : \Sigma \hookrightarrow X$ be a spacelike smooth imbedding—that is, $q \overset{\text{def}}{=} i^*g$ is a Riemannian 3-metric on Σ. Then, Σ is a Cauchy surface for X [5, theorem 1, p. 88]. In fact, it may be assumed without any loss of generality that X is diffeomorphic to $\Sigma \times \mathbb{R}$. Let $t : X \to \mathbb{R}$ be a smooth function defining a spacelike foliation (of codimension 1) such that for each fixed $\lambda \in \mathbb{R}$, the 3-surface $\Sigma_\lambda \overset{\text{def}}{=} \{\, x \in X \mid t(x) = \lambda \,\}$ is diffeomorphic to Σ. Such a foliation exists as X is globally hyperbolic. The vector field v_t tangent to t identifies points on Σ_λ for different λ's and it defines evolution via the Lie derivative $\mathcal{L}_{v_t}$ along the integral curve of v_t.

2.2.1. Remark. There are strong reasons to support the restriction of the topology of Σ to be compact. If Σ were not compact, strong conditions must be imposed on X in order for it to admit a Cauchy spacelike surface [5, §IV, p. 94].

Let n be a normalised, timelike vector field—that is, normalised relative to the Lorentzian 4-metric g on X: $g_x(n(x), n(x)) = -1\ \forall x \in X$—and set $q^a_b \stackrel{\text{def}}{=} \delta^a_b + n^a n_b$. This defines a projection operator onto Σ_λ for each fixed $\lambda \in \mathbb{R}$. If $g_\lambda \stackrel{\text{def}}{=} g|\Sigma_\lambda$, then the induced metric q_λ on Σ_λ is given by

$$q_{\lambda ab} \stackrel{\text{def}}{=} q^k_{\lambda a} q^l_{\lambda b} g_{\lambda kl} = g_{\lambda ab} + n_{\lambda a} n_{\lambda b}.$$

The timelike vector field v_t may be decomposed as

$$v^a_t = N n^a + N^a,$$

where $N \stackrel{\text{def}}{=} -n^a v^b_t q_{ab}$ is the *lapse function* which determines the infinitesimal deformation of Σ_λ to $\Sigma_{\lambda+\delta\lambda}$ in X, and $N^a \stackrel{\text{def}}{=} q^a_b v^b_t$ is the *shift vector* and is responsible for generating a 1-parameter family of 3-diffeomorphisms on Σ_λ. For notational simplicity, identify Σ with its image in X in all that follows. The *Einstein-Hilbert action* $S_{\text{EH}} \stackrel{\text{def}}{=} \int_X \sqrt{-\det g}\, {}^4R$, where ${}^4R$ is the scalar curvature with respect to the Lorentzian 4-metric g, can be written in terms of the induced Riemannian 3-metric q on Σ as

$$S_{\text{EH}} = \int \mathrm{d}t \int_\Sigma (\det q)^{\frac{1}{2}} N(R + K_{ab}K^{ab} - K^2) + \text{surface integral},$$

where R is the scalar curvature of q, $K_{ab} \stackrel{\text{def}}{=} -\frac{1}{2} q^k_a q^l_b (\mathcal{L}_n g)_{kl}$ is the extrinsic curvature of Σ and $K \stackrel{\text{def}}{=} K_{ab} q^{ab}$. For more details, see reference [15, §2, p. 765].

Let Γ^+_2 denote the space of Riemannian 3-metrics q on Σ and $T^*\Gamma^+_2$ its cotangent bundle.[5] In the ADM formalism, the evolution of the initial data $(q, \tilde{p}) \in T^*\Gamma^+_2$, where $\tilde{p} \stackrel{\text{def}}{=} \delta L_{\text{EH}}/\delta \mathcal{L}_{v_t} q$ and $L_{\text{EH}} \stackrel{\text{def}}{=} (\det q)^{\frac{1}{2}} N(R + K_{ab}K^{ab} - K^2)$ is the Einstein-Hilbert Lagrangian, is studied. However, in order to satisfy Einstein's field equations, not every point in $T^*\Gamma^+_2$ is accessible: there exist constraints. These constraints—the *diffeomorphism* and *Hamiltonian constraints* respectively—are

$$C^b(q, \tilde{p}) \stackrel{\text{def}}{=} D_a \tilde{p}^{ab} = 0, \tag{2.2.1}$$

$$C(q, \tilde{p}) \stackrel{\text{def}}{=} \tilde{p}^{ab}\tilde{p}_{ab} - \frac{1}{2}\tilde{p}^2 - \frac{\det q}{G^2} R = 0, \tag{2.2.2}$$

[5] Γ^+_2 is a smooth infinite-dimensional Fréchet manifold [19, pp. 267–269].

where D is the Levi-Civita connection of q.

Upon canonically quantising this using the metric representation—*i.e.*, where the wavefunctions are essentially functionals $\Psi[q]$ of the 3-metric q—two major barriers are encountered: to wit, the complexity of the non-linear scalar constraint (2.2.2) and the problem of factor ordering. An in-depth discussion can be found in a paper written by DeWitt [6] in the late 1960's. This hurdle encountered in canonical quantisation led Ashtekar [3] to construct what is now known as the *Ashtekar (canonical) variables.* In terms of these variables, great simplification to the constraint equations are achieved when restricted to the self-dual solutions.

The Ashtekar variables will be constructed below. First of all, let Σ be simply a 3-manifold with *no* particular (Riemannian) 3-metric specified on it. Let $(\mathrm{SU}(2),\phi_{\mathrm{s}})$ denote the double cover of SO(3) and let $\tilde{\xi} = (P_{\tilde{\xi}}, p_{\tilde{\xi}}, \Sigma, \mathrm{SU}(2))$ and $\xi = (P_\xi, p_\xi, \Sigma, \mathrm{SO}(3))$ be the principal SU(2)- and SO(3)-bundle over Σ respectively. Next, observe from Stiefel's Theorem [13, Ex. 12-B, p. 148] that every compact, orientable 3-manifold is parallelisable. In particular, Σ admits a spin structure and spinor fields thus exist on Σ.[6] Hence, if a Riemannian metric is fixed on Σ so that a reduction of the frame bundle over Σ to the bundle of orthonormal frames on Σ is specified, then a *spin structure* $\tilde{\psi} : P_{\tilde{\xi}} \to P_\xi$—$\tilde{\psi}(u \cdot g) = \tilde{\psi}(u) \cdot \phi_{\mathrm{s}}(g)$, where $g \in \mathrm{SU}(2)$, $u \in P_{\tilde{\xi}}$—can indeed be defined on Σ. However, because a Riemannian structure is not specified *a priori* on Σ, the concept of SU(2) spinors will be introduced via the Infeld-van der Waerden fields σ to be defined below.

Set W to be a 2-dimensional complex vector space with W^* its dual space. Then, relative to a fixed basis of W (and the dual basis of W^*), the elements of $W \otimes W^*$ may be regarded as 2×2 Hermitian matrices. More precisely, if $\overline{W}$ denotes the complex conjugate of W, then the elements of $W \otimes \overline{W}$ may be regarded as 2×2 Hermitian matrices (with 4 independent real variables). Let G be a positive-definite, Hermitian, bilinear form on W—$\bar{\psi}^{A'} G_{A'A} \psi^A > 0 \ \forall\, \psi^A \neq 0$, with $\psi^A \in W$ and $\overline{\psi^A} \stackrel{\text{def}}{=} \bar{\psi}^{A'} \in \overline{W}$—and let ϵ be a (fixed) nowhere vanishing 2-form[7] on W:

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, $\overline{W}$ is linearly isomorphic to W^* under the map $\bar{\psi}^{A'} \mapsto \psi_A$ given by

$$\epsilon^{BC} \bar{\psi}^{A'} G_{A'C} \epsilon_{BA} = \epsilon^{BC} \psi_C \epsilon_{BA} = \psi^B \epsilon_{BA} = \psi_A.$$

[6] These are just cross sections of vector bundles over Σ associated with the principal Spin(3)-bundle $\tilde{\xi}$ of Σ, where $\mathrm{Spin}(3) \equiv \mathrm{SU}(2)$.

[7] Observe that as $\dim_{\mathbb{C}} W = 2$, the space of 2-forms is 1-dimensional and hence all 2-forms ϵ are proportional to one another: for if (e_1, e_2) forms a basis for W, then all 2-forms are of the form $ce^1 \wedge e^2$, where $c = \epsilon_{12} - \epsilon_{21} \in \mathbb{R}$.

Hence, $W \otimes W^* \approx W \otimes \overline{W}$. Refer to reference [2, p. 287] or [3, p. 1601] for a concise account of SU(2) spinors. Let $V_+ \subset W \otimes W^*$ consist of elements which, when considered as complex 2×2 matrices, are traceless and Hermitian—that is, let $V_+ = \{\, \lambda \in M_{2\times 2}(\mathbb{C}) \mid \operatorname{tr}\lambda = 0,\ \lambda^\dagger = \lambda \,\}$—and let $\{e_i\}_{i=1}^3$ be a fixed basis of V_+. Then, the vector space $V_- \stackrel{\text{def}}{=} \operatorname{span}\langle\, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \,\rangle$, where $\tilde{e}_k = \mathrm{i} e_k$, $\mathrm{i} = \sqrt{-1}$ and $\{e_k\}_{k=1}^3$ is the dual basis of $\{e^i\}_{i=1}^3$—$\langle e^i, e_j \rangle = \delta^i_j$—is the space of traceless, anti-Hermitian, complex 2×2 matrices. That is, $V_- \equiv \mathfrak{su}(2)$, the SU(2) Lie algebra.

2.2.2. Remarks. 1. Define a map $\tilde{\dagger} : W \to W$ by $(\lambda^{\tilde{\dagger}})^A \stackrel{\text{def}}{=} -\epsilon^{AB} G_{BA'} \bar{\lambda}^{A'}$, where G is a positive-definite bilinear Hermitian form given above. Then, $(\lambda^{\tilde{\dagger}})^A \lambda_A \geqq 0$—and is equal to zero *iff* $\lambda_A = 0$—and $(\lambda^{\tilde{\dagger}\tilde{\dagger}})^A = -\lambda^A$. Call λ^A $\tilde{\dagger}$*-Hermitian* if $\lambda^{\tilde{\dagger}} = \lambda$.

2. Some possible notational conflict will arise from the term *Hermiticity* used here and that used in the cited literatures: the symbol $\dagger$ introduced above coincides with the usual sense of conjugate transpose of a matrix whilst the symbol $\tilde{\dagger}$ defined above coincides with Ashtekar's $\dagger$ symbol [2, 3]. In other words, $\tilde{\dagger}$ defines Hermiticity of spinors whereas $\dagger$ defines Hermiticity of matrices.

Now, consider the tensor bundle $(T\Sigma \otimes \mathfrak{su}(2), p, \Sigma)$ over Σ and define $\mathcal{C}$ to be the space of smooth cross sections—the *Infeld-van der Waerden fields*—$\sigma : \Sigma \to T\Sigma \otimes \mathfrak{su}(2)$ satisfying:

(1) for each $x \in \Sigma$ and σ, $\sigma(x)$ induces a linear isomorphism $\mathfrak{su}(2) \approx T_x\Sigma$ defined by $\lambda \mapsto -\sigma(x)\cdot\lambda \stackrel{\text{def}}{=} \lambda^a(x)\partial_a$, where $-\sigma(x)\cdot\lambda \stackrel{\text{def}}{=} -\operatorname{tr}(\sigma(x)^a\lambda)\partial_a = -\sigma(x)^a{}_A{}^B \lambda_B{}^A \partial_a = \lambda^a(x)\partial_a \in T_x\Sigma$,

(2) $-\operatorname{tr}(\sigma^a\sigma^b) \stackrel{\text{def}}{=} q^{ab}$, where q^{ab} is the inverse matrix of the Riemannian 3-metric q_{ab} defined in the natural basis on Σ.

The elements of $\mathcal{C}$ are called the *SU(2) soldering forms* on Σ.

2.2.3. Remark. Note trivially that global sections σ exist on Σ as it is parallelisable—σ determines a spin structure on Σ. Write the components of σ as $\sigma^a{}_A{}^B$. Then, the inverse $\sigma(x)^{-1}$ of $\sigma(x)$ will be written as $\sigma(x)_{aA}{}^B$.

Let $C^\infty_{\mathrm{cs}}(\Sigma, T\Sigma \otimes \mathfrak{su}(2))$ denote the space of smooth cross sections of the bundle $(T\Sigma \otimes V_-, p, \Sigma)$, endowed with the compact C^∞-topology. Then, it follows from [19, §7.2, pp. 259–260] that $C^\infty_{\mathrm{cs}}(\Sigma, T\Sigma \otimes \mathfrak{su}(2))$ is a smooth Fréchet manifold. Furthermore, let $S_2\Sigma$ be the bundle space of symmetric covariant 2-tensors on Σ and let $C^\infty_{\mathrm{cs}}(\Sigma, S_2\Sigma)$ be the space of smooth cross sections on the tensor bundle equipped with the compact C^∞-topology. Then, the space Γ_2^+ of (smooth) Riemannian metrics on Σ is an open convex cone in $C^\infty_{\mathrm{cs}}(\Sigma, S_2\Sigma)$ [7, p. 1001]. Hence, by property (2) of σ, $\mathcal{C} \subset C^\infty_{\mathrm{cs}}(\Sigma, T\Sigma \otimes \mathfrak{su}(2))$ is open and hence a smooth manifold. As a brief reminder, the compact C^r-topology is generated by a subbase consisting of open sets $N(f; (\varphi, U), (\psi, V), K, \varepsilon)$ of the form

$$\{\, g \in \mathrm{C}^r : g(K) \subset V \text{ and } \|D^k f(x) - D^k g(x)\| < \varepsilon,\ 0 \leqq k \leqq r,\ \forall x \in \varphi(K) \,\},$$

where $K \subset U$ is compact, (φ, U), (ψ, V) are charts of Σ and $D^k h(x)$ denotes $D^k(\psi \circ h \circ \varphi^{-1})(x)$ in abused notation. For more details, see [10, p. 34] or [19].

Now, let $T^*\mathcal{C}$ denote the cotangent bundle space over $\mathcal{C}$ and denote an element in $T^*\mathcal{C}$ by $(\sigma, \tilde{M})$, where it is shown in [3, p. 1592] that $\tilde{M}$ is a densitized $\mathfrak{su}(2)$-valued 1-form of weight 1 on $\mathcal{C}$ based at σ. It is related to $\tilde{p}$ by $\tilde{p}^{ab} = -\operatorname{tr} \tilde{M}^{(a}\sigma^{b)}$. Let $\tilde{\xi}[W] = (B, p_B, \Sigma, \mathrm{SU}(2), W)$ be a complex vector bundle of rank 2 over Σ—with typical fibre W—associated with the principal SU(2)-bundle $\tilde{\xi}$. In this section, notations consistent with reference [3] will be used.

Fix an element $(\sigma, \tilde{M}) \in T^*\mathcal{C}$ and define an SU(2) connection 1-form ω_σ on $P_{\tilde{\xi}}$ such that the covariant derivative D induced by ω_σ on B is compatible with σ: $\mathrm{D}_a \sigma^b{}_A{}^B = 0$ (with respect to each chart U_α)—that is, the SU(2) spin connection coefficients $\Gamma_{aA}{}^B$ of ω_σ satisfy $\Gamma_{aA}{}^B = \frac{1}{2}\sigma_b{}^{EB}(\sigma^c{}_{AE}\Gamma^b_{ca} + \partial_a \sigma^b{}_{AE})$, where Γ^b_{ca} are the Christoffel symbols of $q = -\operatorname{tr}\sigma \cdot \sigma$. Next, following [3, p. 1593], define a local $\mathfrak{su}^{\mathbb{C}}(2)$-valued connection 1-form A_α—the *Ashtekar connection 1-form*—of the complex vector bundle B, on each (B-trivialising) chart U_α of Σ, by

$$^{\pm}A_\alpha \stackrel{\mathrm{def}}{=} G^{-1} s_\alpha^* \omega_\sigma \pm \tfrac{\mathrm{i}}{\sqrt{2}} G^{-1} \Pi_\alpha,$$

where $\mathfrak{su}^{\mathbb{C}}(2)$ is the complexification[8] of $\mathfrak{su}(2)$, G is the Gravitational constant, $s_\alpha : U_\alpha \to P_{\tilde{\xi}}$ a smooth cross section on U_α, and $\Pi = \Pi(\sigma, \tilde{M})$ is related to the extrinsic curvature K_{ab} of Σ by $K_{ab} = -\operatorname{tr}\Pi_{(a}\sigma_{b)}$. Explicitly, it is defined [3] by

$$\Pi_{aA}{}^B \stackrel{\mathrm{def}}{=} G(\det q)^{-\frac{1}{2}}(\tilde{M}_{aA}{}^B + \tfrac{1}{2}\operatorname{tr}(\tilde{M}_b \sigma^b)\sigma_{aA}{}^B).$$

The Ashtekar connection 1-form ^{+}A may be regarded as the anti-self-dual potential for the Weyl tensor ^3C on Σ and ^{-}A as the self-dual potential [3, Eqn (19′), p. 1600]:

$$^3C^{ab} \stackrel{\mathrm{def}}{=} -G\ \operatorname{tr}(^{\pm}F_{cd}\sigma^a)\epsilon^{cdb} = -\sqrt{2}G(E_{ab} \mp \mathrm{i}B_{ab}),$$

where $^{\pm}F_{ab}$ is the curvature of $^{\pm}A_a$, $E_{ab} \stackrel{\mathrm{def}}{=} C_{acbd}n^c n^d$ is the electric and $B_{ab} \stackrel{\mathrm{def}}{=} {}^*C_{acbd}n^c n^d$ the magnetic part (relative to Σ) of the 4-dimensional Weyl tensor C.

[8]If V is a vector space, then the elements of $V^{\mathbb{C}}$ are precisely $u + \mathrm{i}v$, where $u, v \in V$.

For an in-depth study, see reference [4]. In the following analysis, only the self-dual potential will be considered inasmuch as canonical quantum gravity can be formulated with either the self-dual or the anti-self-dual potential [3, 16]. Thus, in view of this restriction, denote ${}^{-}A$ for convenience by A.

2.2.4. Remark. Since Σ is parallelisable and orientable, the principal SO(3)-bundle ξ is trivial: $P_\xi \cong \Sigma \times \mathrm{SO}(3)$. Furthermore, $P_{\tilde{\xi}}$ is also trivial. To establish this, fix a metric and an orientation on Σ so that P_ξ may be regarded as a reduction of the frame bundle over Σ. The metric determines a spin structure $(\mathrm{SU}(2), \tilde{\psi})$ on Σ. Let $\tilde{s} : \Sigma \longrightarrow P_\xi$ be a (global) cross section of ξ defined by $x \mapsto (x, e)$, where $e \stackrel{\text{def}}{=} \mathrm{id}_{\mathrm{SO}(3)}$. Then, $\tilde{\psi}^{-1}(x, e) = \{e_+(x), e_-(x)\} \subset P_{\tilde{\xi}}$. On setting $s(x) = e_+(x)\ \forall x \in \Sigma$, it is clear that $\tilde{\psi} \circ s \equiv \tilde{s}$ on Σ and s is thus the desired (global) cross section of $\tilde{\xi}$. Hence, $P_{\tilde{\xi}} \cong \Sigma \times \mathrm{SU}(2)$ and the Ashtekar connections A are thus globally defined on Σ.

It should be pointed out that in the process of enlarging the phase space $T^*\Gamma_2^+$ to $T^*\mathcal{C}$ (so as to include spinor fields), in addition to the two sets of constraint equations defined by equations (2.2.1) and (2.2.2), a new set of constraints is imposed on $T^*\mathcal{C}$, namely

$$\operatorname{tr} \tilde{M}^{[a}\sigma^{b]} = 0, \tag{2.2.3}$$

simply because at each point $x \in \Sigma$, q is a six-component field whereas σ is a nine-component field. Thus, 3 additional degrees of freedom exist at each point x by enlarging the original phase space to $T^*\mathcal{C}$. Notice that when constraint (2.2.3) is satisfied, $\Pi_a = K_{ab}\sigma^b$.

The canonical transformations generated by constraints (2.2.3) correspond precisely to SU(2) transformations. Now, it turns out that on $T^*\mathcal{C}$, each set $\{\, {}^{\pm}A(x) \mid x \in \Sigma \,\}$ and $\{\, \tilde{\sigma}(x) \mid x \in \Sigma \,\}$ forms a complete set of commuting variables with respect to the Poisson bracket induced by a symplectic structure on $T^*\mathcal{C}$, where $\tilde{\sigma} \stackrel{\text{def}}{=} (\det q)^{\frac{1}{2}}\sigma$ and $q = -\operatorname{tr}(\sigma \cdot \sigma)$:

$$\{{}^{+}A(x), {}^{+}A(y)\} = 0 = \{{}^{-}A(x), {}^{-}A(y)\} \quad \text{and} \quad \{\tilde{\sigma}(x), \tilde{\sigma}(y)\} = 0\ \forall x, y \in \Sigma.$$

The details of this can be found in [3]. In particular, the Poisson bracket of ${}^{\pm}A$ and $\tilde{\sigma}$ is

$$\{{}^{\pm}A_b^{MN}(x), \tilde{\sigma}^a_{AB}(y)\} = \pm\frac{\mathrm{i}}{\sqrt{2}}\delta^a_b \delta_A{}^{(M}\delta_B{}^{N)}\delta^3(x, y). \tag{2.2.4}$$

Thus, in view of equation (2.2.4), ${}^{\pm}A$ and $\tilde{\sigma}$ are *canonically conjugate* to one another—cf. reference [3, p. 1594].[9] In this thesis, $A \stackrel{\text{def}}{=} {}^{-}A$ and $\tilde{\sigma}$ will be taken

[9] In the strict sense of the term, this is not true since ${}^{\pm}A$ are complex whereas $\tilde{\sigma}$ is real.

as the fundamental variables, with the 3-metric q being a *derived* quantity. In summary, Ashtekar obtained a canonical transformation $(\sigma, \tilde{M}) \mapsto (\tilde{\sigma}, A)$. Since A contains information about the pair $(\sigma, \tilde{M})$, the constraint equations, when expressed in terms of the pair $(\tilde{\sigma}, A)$, are very much more appealing: indeed, the constraint dependence on the latter pair is at most quadratic.

When the constraints (2.2.3), (2.2.1) and (2.2.2) are expressed in terms of the Ashtekar variables $(A, \tilde{\sigma})$, they are, respectively,

$$C_1(\tilde{\sigma}, A) = \mathcal{D}_a \tilde{\sigma}^a = 0, \tag{2.2.5}$$

$$C_{2b}(\tilde{\sigma}, A) = \operatorname{tr}(\tilde{\sigma}^a F_{ab}) = 0, \tag{2.2.6}$$

$$C_3(\tilde{\sigma}, A) = \operatorname{tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) = 0, \tag{2.2.7}$$

where $\mathcal{D}$ is the covariant derivative induced by the Ashtekar connection A—$\mathcal{D}_a \psi_M = \partial_a \psi_M + G A_{aM}{}^N \psi_N$—and $F_{abM}{}^N \overset{\text{def}}{=} 2\partial_{[a} A_{b]M}{}^N + G[A_a, A_b]_M{}^N$ is the curvature of A. More precisely, to obtain *functions* on the new Ashtekar phase space, the constraints (2.2.5–2.2.7) must be smeared with appropriate test fields:

$$\int_\Sigma \operatorname{tr}(\lambda \cdot \mathcal{D}_a \tilde{\sigma}^a) = 0, \tag{2.2.5$'$}$$

$$\int_\Sigma \operatorname{tr}(N^a \tilde{\sigma}^b F_{ab}) = 0, \tag{2.2.6$'$}$$

$$\int_\Sigma N \operatorname{tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) = 0, \tag{2.2.7$'$}$$

where λ is a $\tilde{\dagger}$-Hermitian traceless field on Σ, N^a is a complex vector field (the shift vector) and N is a scalar density of weight -1 (the lapse function).

Note that in order for the pair $(A, \tilde{\sigma})$ to yield general relativity, they must satisfy not only constraints (2.2.6) and (2.2.7), but also two extra conditions: (i) $\tilde{\sigma}$ is $\tilde{\dagger}$-Hermitian (but is anti-Hermitian, when considered as a matrix), (ii) $\Pi_{aA}{}^B$ is $\tilde{\dagger}$-Hermitian. Furthermore, observe that by eliminating the $\tilde{\dagger}$-Hermiticity conditions, all the fields become $\mathfrak{sl}(2, \mathbb{C})$-valued and hence yield *complex general relativity!*

2.3. The Self-Dual Representation

Let $C^\infty_{\text{cs}}(\Sigma, P_{\tilde{\xi}})$ be the space of smooth cross sections over the principal SU(2)-bundle $\tilde{\xi}$ and set

$$\mathcal{A} = \left\{ G^{-1} s^* \omega - \tfrac{i}{\sqrt{2}} G^{-1} \Pi(\sigma, M) \,\middle|\, \omega \in \tilde{\mathcal{A}}_C,\ s \in C^\infty_{\text{cs}}(\Sigma, P_{\tilde{\xi}}) \text{ and } (\sigma, M) \in S_C \right\}$$

to be the space of Ashtekar connections on Σ, where $S_{\mathcal{C}} \subset T^*\mathcal{C}$ is the constraint surface in $T^*\mathcal{C}$ defined by constraints (2.2.5)–(2.2.7), and $\tilde{\mathcal{A}}_{\mathcal{C}}$ is the set of connection 1-forms ω on $P_{\tilde{\xi}}$ such that for each ω, $\exists\, \sigma_\omega \in \mathcal{C}$ wherein the covariant derivative induced by ω is compatible with ω_σ. Then, denoting $\Gamma_{\mathcal{A}}$ to be the phase space over $\mathcal{A}$ consisting of pairs $(A, \tilde{\sigma})$ with $\mathcal{A}$ treated as the configuration space, by selecting a natural complex polarization over the Hermitian line-bundle of $\Gamma_{\mathcal{A}}$, *quantum states* correspond to suitable polarized cross sections of the Hermitian line bundle over $\Gamma_{\mathcal{A}}$; these cross sections may be represented by complex functionals Ψ on $\mathcal{A}$ that are *holomorphic*: that is, Ψ satisfies

$$\frac{\delta \Psi[A]}{\delta A_a^\dagger(x)} = 0.$$

This is known in the literature as the *self-dual representation*. More precisely, let $\mathcal{S} \subset C^\infty(\Gamma_{\mathcal{A}}, \mathbb{R})$ be a suitably chosen complex vector space. Then, the self-dual representation is the map $\lambda : \mathcal{S} \to H$ given by $(\lambda(f)\Psi)[A] \stackrel{\text{def}}{=} (\hat{f}\Psi)[A]$, where H is the vector space of complex functionals $\Psi = \Psi[A]$ and $\hat{f}$ is the corresponding quantum operator associated with the classical observable f. The completion of a subset $\mathcal{H} \subset H$ (equipped with a suitable inner product) of these functionals annihilated by the quantum constraints then constitutes the *physical state space* of quantum gravity. The description given here is of course done in a rather blasé fashion. The term "self-dual" arises from the fact that the Ashtekar connection A is the self-dual potential for the Weyl tensor discussed above in §2.2.

Upon quantising gravity via these new variables $(A, \tilde{\sigma})$, the quantum operator $\hat{A}(x)$ is treated as a multiplicative operator on $\mathcal{H}$ and $\hat{E}(x)$, where $E(x) \stackrel{\text{def}}{=} \tilde{\sigma}(x)$ (for typesetting convenience), is the operator $-i\hbar\delta/\delta A_a(x)$:

$$\hat{A}_a(x)\Psi[A] \stackrel{\text{def}}{=} A_a(x)\Psi[A] \quad \text{and} \quad \hat{E}^a(x)\Psi[A] \stackrel{\text{def}}{=} -i\hbar\frac{\delta\Psi[A]}{\delta A_a(x)}.$$

The classical constraints (2.2.5)–(2.2.7), with the ordering that E is placed on the right of A, then becomes:

$$C_1(\hat{A}(x), \hat{E}(x))\Psi[A] = \mathcal{D}_a \frac{\delta\Psi[A]}{\delta A_a(x)} = 0, \tag{2.3.1}$$

$$C_{2a}(\hat{A}(x), \hat{E}(x))\Psi[A] = \operatorname{tr} F_{ab}(x)\frac{\delta\Psi[A]}{\delta A_b(x)} = 0, \tag{2.3.2}$$

$$C_3(\hat{A}(x), \hat{E}(x))\Psi[A] = \operatorname{tr} F_{ab}(x)\frac{\delta^2\Psi[A]}{\delta A_a(x)\delta A_b(x)}, \tag{2.3.3}$$

It can be shown that constraint (2.3.1) generates infinitesimal SU(2) gauge transformations, constraint (2.3.2) generates (orientation-preserving) diffeomorphisms and the last constraint is responsible for "time" evolution of the initial data (A, E) in $\Gamma_{\mathcal{A}}$. In all cases, the constraints were shown to be of first class in detail by Ashtekar [2, 3].

To conclude this section, the motivation for choosing the above operator ordering will be sketched. The full details can be found in [11, §2.3, p. 308]. See also reference [14, §3.2, p. 12] by Nicolai and Matschull. So, briefly, the ordering imposed on constraint (2.3.1) implies that $\Psi[A]$ is invariant under infinitesimal SU(2) transformations; that on constraint (2.3.2) ensures that the constraint generates, on gauge invariant functionals, infinitesimal 3-diffeomorphisms rather than diffeomorphisms with a divergent term. And finally, the ordering in constraint (2.3.3) guarantees that the algebra of the constraints be consistent (*i.e.*, no anomalous c-number terms and so forth).

2.4. The Loop Representation

In this section, loop variables—developed by Rovelli and Smolin [17]—will be introduced. The motivation for introducing loop variables arose from a result in an article by Jacobson and Smolin [11, §7.1, p. 333] wherein a class of solutions to the scalar constraint (in the self-dual representation) were determined and because the SU(2)-gauge invariance of the theory is captured by the loop formalism. More of these will be covered later. The main element of the construction of the loop variables is the complexified SU(2) holonomy defined by the Ashtekar connection 1-form A around the loop γ:

$$U[\gamma, A] \stackrel{\text{def}}{=} \mathcal{P}e^{\int_0^1 A_a(\gamma(t))\dot{\gamma}^a(t)\mathrm{d}t},$$

where $\mathcal{P}$ is the path-ordering operator around γ in the line integral; that is,

$$U[\gamma, A] = \sum_{n=0}^{\infty} \int_0^1 \mathrm{d}t_n \int_0^{t_n} \mathrm{d}t_{n-1} \cdots \int_0^{t_2} \mathrm{d}t_1 A_{a_n}(\gamma(t_n))\dot{\gamma}^{a_n}(t_n) \ldots A_{a_1}(\gamma(t_1))\dot{\gamma}^{a_1}(t_1).$$

Classical observables—the *T-observables*—are then build up from $U[\gamma, A]$ and $E(\gamma(t))$.

The first observable, the T^0*-observable*, is nothing but the *Wilson loop*:

$$T[\gamma, A] \stackrel{\text{def}}{=} \operatorname{tr} U[\gamma, A].$$

A general T^n*-observable* is obtained by taking the trace of the alternating matrix products of $U[\gamma, A]$ and $E(\gamma(s))$ and it also depends on n fixed points,

$\gamma(s_1), \ldots, \gamma(s_n)$, on the loop γ, where $0 < s_1 < \cdots < s_n \leqq 1$. Explicitly, let $T_0^{a_1 \ldots a_n}[\gamma, A](s_1, \ldots, s_n)$, defined by

$$\operatorname{tr}(E^{a_1}(\gamma(s_1))U[\gamma, A](s_1, s_2)E^{a_2}(\gamma(s_2))U[\gamma, A](s_2, s_1) \ldots E^{a_n}(\gamma(s_n))U[\gamma, A](s_n, s_1)),$$

be an observable that depends on the ordering placed on the loop parameters $s_1, \ldots, s_n$, where $U[\gamma, A](s, t) \stackrel{\text{def}}{=} \mathcal{P}e^{\int_{\gamma(s)}^{\gamma(t)} A}$ is the parallel propagator for spinors along γ from $\gamma(s)$ to $\gamma(t)$. Then, its corresponding $T^{a_1 \ldots a_n}$ observable is a T^n-observable that is independent on the ordering of s_i; that is, $T^{a_1 \ldots a_n}[\gamma, A](s_1, \ldots s_n)$ is defined by

$$\sum_P \theta(s_{P(1)} - s_{P(2)}) \ldots \theta(s_{P(n)} - s_{P(1)}) T_0^{a_{P(1)} \ldots a_{P(n)}}[\gamma, A](s_{P(1)}, \ldots, s_{P(n)}),$$

where P is a (fixed) permutation of $\{1, \ldots, n\}$,

$$\theta(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1$, $T^a[\gamma, A](s) = \operatorname{tr}(U[\gamma, A]E^a(\gamma(s)))$. Note that if $\gamma' = \gamma \circ f$, where f is an orientation-preserving diffeomorphism on I—i.e., a *reparametrisation*—then $T^{a_1 \ldots a_n}[\gamma, A](s_1, \ldots, s_n) = T^{a_1 \ldots a_n}[\gamma', A](f^{-1}(s_1), \ldots, f^{-1}(s_n))$. In other words, the T^n observables for $n \geqq 1$ are reparametrisation covariant, and the T^n-observables thus depend only on the geometric points on the loop γ. In the light of this observation, the T^n-observable may be denoted by $T^{a_1 \ldots a_n}[\gamma, A](x_1, \ldots, x_n)$, where $x_i = \gamma(s_i)$ are n fixed points on γ in Σ. Moreover, as reparametrisations are orientation-preserving, the natural linear order $\leqq$ on I induces, in a natural way, an ordering $\preccurlyeq$ on the set of x_i's by

$$x_i \preccurlyeq x_j \text{ if } s_i \leqq s_j, \text{ where } \gamma(s_k) = x_k.$$

It can be verified that the T^0-observables are invariant under reparametrisations.

Some comments are now due. First, A may be regarded as an element of $\mathfrak{sl}(2, \mathbb{C})$. Then, by virtue of the following algebraic identity of $\mathfrak{sl}(2, \mathbb{C})$,

$$\delta_A{}^B \delta_C{}^D + \epsilon_{AC} \epsilon^{DB} = \delta_A{}^D \delta_C{}^B,$$

the T^0-observables can be shown [11, §5.1, pp.324–325] to satisfy the following *spinor identity*:

$$T[\gamma * \eta, A] + T[\gamma * \eta_-, A] = T[\gamma, A]T[\eta, A],$$

where $\gamma * \eta$ is defined by

$$\gamma * \eta(t) = \begin{cases} \gamma(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2}, \\ \eta(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

and $\eta_-(t) \stackrel{\text{def}}{=} \eta(1-t)$ is the reverse orientation of a loop η. Furthermore, the T^0-observables also satisfy the *retracing identity*:

$$T[\gamma * \zeta * \zeta_-, A] = T[\gamma, A],$$

where $\zeta : I \to \Sigma$ here denotes a non-self-intersecting curve that starts from a point on the loop γ. Indeed, these two identities imply [17, p. 105] that the T^0-observables also preserve the following two relations:

$$\begin{aligned} T[\gamma, A]T[\eta, A] &= T[\gamma * \zeta * \eta * \zeta_-, A] + T[\gamma * \zeta * \eta_- * \zeta_-, A], \\ T[\gamma_1 * \gamma_2, A] &= T[\gamma_1 * \eta, A]T[\gamma_2 * \eta_-, A] - T[\gamma_1 * \eta * (\gamma_2)_- * \eta, A], \end{aligned}$$

where γ and η are arbitrary loops, ζ is a curve that connects γ and η, and $\gamma_1 * \gamma_2$ is a loop defined by two curves γ_i, $i = 1, 2$: $\gamma_1(0) = \gamma_2(1)$ and $\gamma_1(1) = \gamma_2(0)$. The last two relations permit the reduction of the loop space by excising loops of the form $\gamma * \eta * \eta_-$ from it. The Poisson Brackets of the T^0-observables commute: $\{ T[\gamma, A], T[\eta, A] \} = 0 \ \forall \gamma, \eta$. The Poisson brackets between T^n- and T^m-observables were worked out explicitly in reference [17, §2.3, p. 101]. They possess the general structure of $\{T^n, T^m\} \sim T^{n+m-1}$, where $n < m$. That is, the set $\mathcal{T} = \{ T^n \mid n \in \mathbb{N} \}$ of T^n-observables form a closed graded Poisson algebra. The Poisson brackets between T^n and T^m actually contained singularities of the form

$$\Delta^a[\gamma, \eta](x) \equiv \Delta^a[\gamma, \eta](s) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^1 \delta^3(\gamma(s), \eta(t))\dot{\eta}^a(t)\mathrm{d}t, \tag{2.4.1}$$

where $x = \gamma(s)$. However, these can be eliminated by introducing suitable smearing fields [17, §2.4, p. 103]. This will be described in detail in §7.2 of the final chapter.

Notice that the T^n-observables are non-local operators in the sense that for $n > 1$, $T^{a_1 \ldots a_n}[\gamma, A, E](s_1, \ldots, s_n)$ takes values in $T_{\gamma(s_1)}\Sigma \otimes \cdots \otimes T_{\gamma(s_n)}\Sigma$, where in general, $\gamma(s_i) \neq \gamma(s_j)$ for $i \neq j$. In particular, T^n-observables, for $n > 1$, are not conventional tensors as such; that is, they are not contravariant n-tensors defined pointwise on Σ. Also, note trivially that the presence of the trace in the definition of the T^n-observables renders them invariant under $SU(2)$-gauge transformations. Finally, due to the trace operator, it is easy to see that $T^{a_1 \ldots a_n}[\gamma, A, E](s_1, \ldots, s_n)$ transforms under a smooth diffeomorphism $\phi \in \mathrm{Diff}^+(\Sigma)$ to $T^{a_1 \ldots a_n}[\phi \circ \gamma, A, E](s_1, \ldots, s_n)$ given by

$$J^{-1}(s_1) \ldots J^{-1}(s_n) \partial_{b_1} \phi^{a_1}(\gamma(s_1)) \ldots \partial_{b_n} \phi^{a_n}(\gamma(s_n)) T^{b_1 \ldots b_n}[\gamma, A, E](s_1, \ldots, s_n),$$

where $J(s_i) \stackrel{\text{def}}{=} J(\gamma(s_i))$ for $i = 1, \ldots, n$, is the Jacobian of ϕ at $\gamma(s_i)$ and they arise on account of E being a vector density of weight one: $E(x) = (\det q(x))^{\frac{1}{2}} \sigma(x)$, where $q(x) = -\operatorname{tr}(\sigma(x) \cdot \sigma(x))$ is the 3-metric defined by the triad σ.

2.4.1. Example. There is a simple diagrammatic way of working out the Poisson relations. Let γ, η be two loops such that $\gamma(s_\gamma) = x = \eta(s_\eta)$ for some $s_\gamma, s_\eta \in I$. Then, there are two ways of joining γ, η at x after breaking them. In essence, $\{\cdot, \cdot\}$ acts on the pair (γ, η) of loops by breaking them at the point x and then rejoining them in two possible ways, based at x: $\gamma *_x \eta$ and $\gamma *_x \eta_-$, where $\gamma *_x \eta \stackrel{\text{def}}{=} \gamma_x * \eta_x$

$$\gamma_x(t) \stackrel{\text{def}}{=} \begin{cases} \gamma(t + s_\gamma) & \text{for } 0 \leqq t \leqq 1 - s_\gamma, \\ \gamma(t - 1 + s_\gamma) & \text{for } 1 - s_\gamma \leqq t \leqq 1, \end{cases}$$

and

$$\eta_x(t) \stackrel{\text{def}}{=} \begin{cases} \eta(t + s_\eta) & \text{for } 0 \leqq t \leqq 1 - s_\eta, \\ \eta(t - 1 + s_\eta) & \text{for } 1 - s_\eta \leqq t \leqq 1. \end{cases}$$

Diagrammatically, $\{T^a[\gamma, A](x), T[\eta, A]\}$ is given by

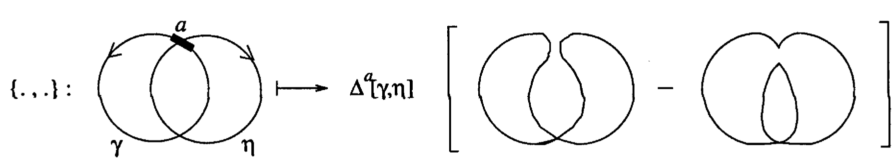

Figure 2.4.1 (a). The Poisson Brackets of T^1 and T^0.

and $\{T^{ab}[\gamma, A](t_1, t_2), T^c[\eta, A](s)\}$ is given by

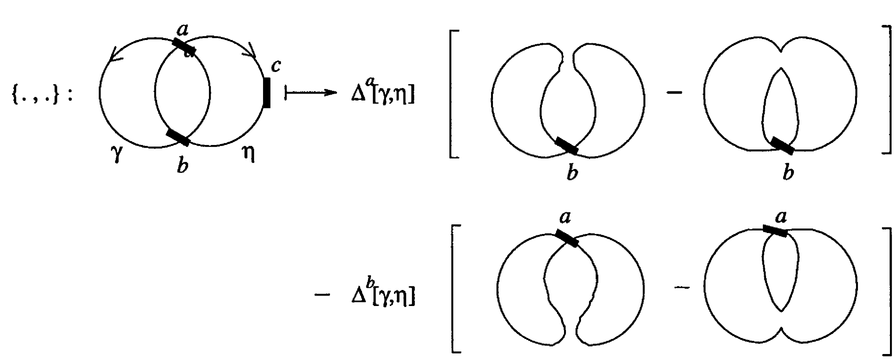

Figure 2.4.1 (b). The Poisson Brackets of two T^1-observables.

In summary, the action of the *grasp operator*—*i.e.*, the action of the Poisson Brackets—at the point where η intersects γ (depending on the orientation of γ and η) are:

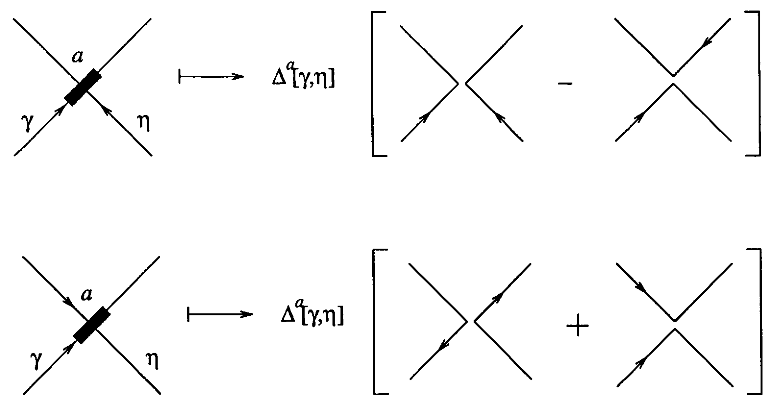

Figure 2.4.1 (c). The action of the grasp operator.

where the arrows within square brackets indicate the *final* orientation of the loops.
□

The loop representation can now be concisely outlined. First, let $\mathcal{M}$ denote the *multi-loop space* of Σ; that is, each element of $\mathcal{M}$ is just a countable subset $\{\gamma^1, \ldots, \gamma^n\}$ of loops in Σ, called an *n-loop*, with the constant loops in Σ identified. The topological structure of $\mathcal{M}$ will be given in some detail in the next chapter. For now, the description here will be carried out at an informal level. Let $\mathcal{M}'$ denote the topological dual of $\mathcal{M}$, where $\mathcal{M}$ is endowed with some suitable topology—once again, refer to the following chapter for more details. Then, define the quantum operator $\hat{T}[{}^1\gamma, A]$ corresponding to $T[{}^1\gamma, A]$ by $(\hat{T}[{}^1\gamma, A]\Psi)[\eta] \stackrel{\text{def}}{=} \Psi[{}^1\gamma \cup \eta]$, where ${}^1\gamma$ is a 1-loop (*i.e.*, a loop) and $\eta \in \mathcal{M}$ is an n-loop; furthermore, $\Psi \in \mathcal{M}'$ is assumed to vanish on $\aleph_0$-loops; that is, on loops with a countably infinite number of loop components.[10]

The quantum T^1-operator $\hat{T}^a[\gamma, A](x)$ is defined by

$$(\hat{T}^a[\gamma, A](x)\Psi)[\eta] \stackrel{\text{def}}{=} \hbar \sum_{\epsilon} (-1)^{n(\epsilon)} \Delta^a[\gamma, \eta](x) \Psi[(\gamma *_x \eta)^\epsilon],$$

where x is a point on γ, $\epsilon \in \{+, -\}$, the crossings of the form given by the first term in Figure 2.4.1 (c) of example 2.4.1 is denoted by $+$ whilst the second term is denoted by $-$, $n(\epsilon)$ is the number of arrows that need reversing at the intersection to maintain a consistent orientation, and

$$(\gamma *_x \eta)^+ \stackrel{\text{def}}{=} \gamma_x * \eta_x \quad \text{and} \quad (\gamma *_x \eta)^- \stackrel{\text{def}}{=} \gamma_x * (\eta_-)_x.$$

Referring to Figure 2.4.1 (c), for the first instance, $n(+) = 0$, $n(-) = 1$, whilst the latter yields $n(+) = 2$ and $n(-) = 0$.

[10] This condition is essentially equivalent to the restriction made by Rovelli and Smolin in their formulation of the loop representation wherein they assumed the multi-loop space $\mathcal{M}$ to consist only of n-loops where $n < \aleph_0$.

In general, a quantum T^n-observable acts on Ψ by

$$(\hat{T}^{a_1\dots a_n}[\gamma, A](x_1,\dots,x_2)\Psi)[\eta] = \hbar^n \sum_{\epsilon_1}\cdots\sum_{\epsilon_n}(-1)^{n(\epsilon)}\Delta^{a_1}[\gamma,\eta](x_1)\dots$$
$$\Delta^{a_n}[\gamma,\eta](x_n)\Psi[(\gamma *_{x_\gamma} \eta)^{\epsilon_1\dots\epsilon_n}],$$

where $x_\gamma = \{x_1,\dots,x_n\}$, $\epsilon_i = +,-$ for $i = 1,\dots,n$, x_i are the n points where η intersects γ—otherwise $(\hat{T}^n[\gamma, A]\Psi)[\eta] = 0$—and $(\gamma *_{x_\gamma} \eta)^{\epsilon_1\dots\epsilon_n}$ corresponds to 2^n ways of rejoining n points (the points of intersection) simultaneously. Lastly, $n(\epsilon) \stackrel{\text{def}}{=} n(\epsilon_1,\dots,\epsilon_n)$ is equal to the number of loop segments resulting from breaking and linking at the intersections which orientation requires reversing in order for the segments to be parametrised in a consistent manner. Refer to Figure 2.4.1 (c) for clarification.

2.4.2. Example. The case for $n = 2$ will be worked out explicitly in this example. Without any loss of generality, it may be supposed that η intersects γ at x and y; otherwise, the action of $\hat{T}^2$ on Ψ at η,

$$(\hat{T}^{ab}[\gamma, A](x,y)\Psi)[\eta] = \hbar^2 \sum_{\epsilon_1}\sum_{\epsilon_2}(-1)^{n(\epsilon_1,\epsilon_2)}\Delta^{ab}[\gamma,\eta](x,y)\Psi[(\gamma *_{xy} \eta)^{\epsilon_1\epsilon_2}],$$

where $\Delta^{ab}[\gamma,\eta](x,y) \stackrel{\text{def}}{=} \Delta^a[\gamma,\eta](x)\Delta^b[\gamma,\eta](y)$ for typesetting convenience, vanishes identically. This is easily seen from equation (2.4.1). Evaluating the expression yields

(2.4.2)
$$(\hat{T}^{ab}[\gamma, A](x,y)\Psi)[\eta] = \hbar^2\Delta^a[\gamma,\eta](x)\Delta^b[\gamma,\eta](y)\times$$
$$\Big\{(-1)^{n(+,+)}\Psi[(\gamma *_{xy} \eta)^{++}] + (-1)^{n(-,-)}\Psi[(\gamma *_{xy} \eta)^{--}] +$$
$$(-1)^{n(+,-)}\Psi[(\gamma *_{xy} \eta)^{+-}] + (-1)^{n(-,+)}\Psi[(\gamma *_{xy} \eta)^{-+}]\Big\}.$$

To evaluate $n(\epsilon_1,\epsilon_2)$, suppose for concreteness and simplicity that γ, η are coordinate circles,[11] both of which are oriented in the counter-clockwise direction. Then, using Figure 2.4.1 (c), $n(+,+) = 2 = n(-,-)$ and $n(+,-) = 2 = n(-,+)$. The loops $(\gamma *_{xy} \eta)^{\epsilon_1\epsilon_2}$ are represented diagrammatically as follows:

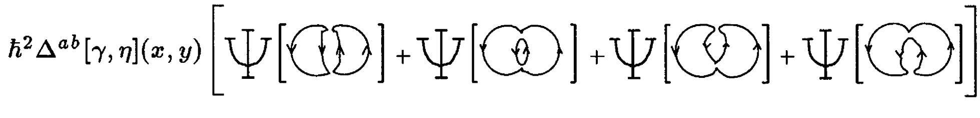

Figure 2.4.2. The action of $\hat{T}^1$ on $\Psi[\eta]$.

[11] A loop $\gamma : I \to \Sigma$ is said to be a *coordinate circle* if $\exists$ a chart (U,ϕ) such that $\gamma(I) \subset U$ and $\phi \circ \gamma$ is a circle on a 2-plane in $\mathbb{R}^3$.

The arrows on the loops indicate the original orientation of the loops γ and η, which are assumed to be counter-clockwise. The values of $n(\epsilon_1, \epsilon_2)$ are then determined by the number of arrows that need reversing: here, for the first term in the above figure, two arrows, one on each loop segment, require reversing in order for the loops to maintain a consistent orientation. The first term of equation (2.4.2) corresponds to the first summand in Figure 2.4.2, and so on. This is the loop $(\gamma *_{xy} \eta)^{++}$, and its structure will be spelt out below. The remaining three other loops can be worked out from the above loop diagrams along a similar vein.

Let s_1, s_2 and t_1, t_2 be γ- and η-parameters at x, y respectively. That is, $\gamma(s_1) = x = \eta(t_1)$ and $\gamma(s_2) = y = \eta(t_2)$ with $x \preccurlyeq y$. Then, $(\gamma *_{xy} \eta)^{++} = \zeta_1 * \zeta_2$, where

$$\zeta_1(t) = \begin{cases} \gamma(2(s_2 - s_1)t + s_1) & \text{for } 0 \leqq t \leqq \frac{1}{2}, \\ \eta(1 - (2(t_2 - t_1)t - t_2 + 2t_1)) & \text{for } \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

$$\zeta_2(t) = \begin{cases} \gamma_{12}(1 - 2t) & \text{for } 0 \leqq t \leqq \frac{1}{2}, \\ \eta_{12}(2t - 1) & \text{for } \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

ζ_i are two loops based at $\gamma(s_1)$, $\gamma_{12} = \gamma_1 * \gamma_2$, with $\gamma_1(t) = \gamma(2(1 - s_2)t + s_2)$ and $\gamma_2(t) = \gamma(2s_1 t - s_1)$, and $\eta_{12} = \eta_1 * \eta_2$ with $\eta_1(t) = \eta(2(1 - t_2)t + t_2)$ and $\eta_2(t) = \eta(2t_1 t - t_1)$.

The second loop $(\gamma *_{xy} \eta)^{--} = \zeta'_1 * \zeta'_2$, where $\zeta'_1 = \gamma'_1 * \eta'_1$ and $\zeta'_2 = (\gamma'_2)_- * (\eta'_2)_-$ and γ'_i, η'_i are defined by

$$\gamma'_1(t) = \gamma((s_2 - s_1)t + s_1), \;\; \eta'_1(t) = \eta((1 - t_2)t + t_2) * \eta(t_1 t),$$

and

$$\gamma'_2(t) = \gamma(s_1 t) * \gamma((1 - s_2)t + s_2), \;\; \eta'_2(t) = \eta((t_2 - t_1)t + t_1).$$

The convention adopted here in defining the loops ζ_i in the case of n intersections is that each loop ζ_i be based at the $\preccurlyeq$-smallest point $\gamma(s_j)$, $j = 1, \ldots, n-1$, that lies on it. For $n = 2$, the loops ζ_i are based at $\gamma(s_1)$. As a final comment, the loops drawn in Figure 2.4.2 are really *one* single loop with self-intersections occurring at x and y. Gaps in the loops were deliberately drawn purely for the purpose of indicating the various ways in which they can be joined again. $\square$

This section will conclude with some remarks concerning the diffeomorphism and Hamiltonian constraints expressed in terms of the loop variables and the restrictions placed on the loop functionals Ψ. The restrictions are:

(1) $\Psi[\gamma]$ should remain invariant under loop-reparametrisations and inversions $(\gamma \to \gamma_-)$;

(2) $\Psi[\gamma \cup \eta] - \Psi[(\gamma * \eta)^{+}] - \Psi[(\gamma * \eta)^{-}] = 0$;

(3) $\Psi[\gamma * \eta * \eta_{-}] = \Psi[\gamma]$, where η is a curve beginning at a point on γ.

Consult reference [21, §2.2] or reference [17, §3.3]. Condition (1) follows from the invariance of the holonomy $U[\gamma, A]$ under reparametrisations. Conditions (2) and (3) follow from equations (2.4.1)–(2.4.2). Refer to [17, Eqns (661)–(67b), p. 115] for more details. The discussion carried out extends to multi-loops in the obvious way. There is actually a final condition to be imposed on the loop functionals. It relates to the "zero loop". This condition will be covered in the next chapter where the zero loop and the multi-loop space will be defined.

Given a multi-loop $\gamma = \{\gamma^1, \dots, \gamma^n\}$, the Diff($\Sigma$)-action on $\mathcal{M}$, $\mathrm{Diff}(\Sigma) \times \mathcal{M} \to \mathcal{M}$, is defined by

$$(f, \gamma) \mapsto f \cdot \gamma \stackrel{\text{def}}{=} \{ f \circ \gamma^1, \dots, f \circ \gamma^n \}.$$

This induces a natural linear representation L of Diff(Σ) on $\mathcal{M}'$ by

$$(L(f)\Psi)[\eta] \stackrel{\text{def}}{=} \Psi[f^{-1} \cdot \eta].$$

Now, let v be a vector field on Σ generated by a one-parameter group f_t of diffeomorphisms on Σ. Then, the generators $D(v)$ of L are defined as

$$(D(v)\Psi)[\eta] \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}(L(f_t)\Psi)|_{t=0}[\eta].$$

These generators satisfy $[D(v), D(u)] = D([v, u])$ and it is sketched in reference [17, p. 118] that $D(v)$ may be identified with the smeared form of constraint (2.3.2):

$$\int_{\Sigma} v^a(x) C_{2a}(x) \mathrm{d}^3 x.$$

Observe in passing that if Ψ is a functional on the knot classes of Σ, that is, $\Psi[\gamma] = \Psi[\gamma']$ whenever γ and γ' belong to the same knot class, then Ψ trivially satisfies the diffeomorphism constraint $D(v)\Psi = 0$.

The Hamiltonian constraint is constructed as follows. Fix a chart (U_α, ϕ_α) of Σ and consider a coordinate circle $\gamma^\delta : I \to \Sigma$ based at $x \in \Sigma$, where $\phi_\alpha \circ \gamma^\delta$ is a circle of radius $\delta \in (0,1)$ in $\mathbb{R}^3$ lying on a 2-plane. By defining $C_3^\delta(x) \stackrel{\text{def}}{=} T^{[ab]}[\gamma^\delta, A](\delta^2, 1)$, it can be shown—see reference [2, pp. 244–245] or [17, p. 106]—that $C_3^\delta(x) = \delta^2 \operatorname{tr}(E^b(x) E^a(x) F_{ab}(x)) + \mathrm{o}(\delta^2)$, and hence,

$$C_3(A, E) = \lim_{\delta \to 0} \frac{1}{\delta^2} C_3^\delta(x).$$

Indeed, it can also be shown[12] that

$$\det q(x) q^{ab}(x) = -\frac{1}{2} \lim_{\delta \to 0} T^{ab}[\gamma^\delta, A](s_{\delta 1}, s_{\delta 2}),$$

where $s_{\delta i} \to 0$ as $\delta \to 0$.

[12] Ibid.

2.5. Discussion

In this final section, a link between the self-dual representation and the loop representation, and some remarks pertaining to some solutions of the quantum constraints will be made. Prior to these however, some comments regarding multi-loops will be given. The loops γ in $\Psi[\gamma]$ may be replaced with multi-loops $\{\gamma^1, \ldots, \gamma^n\}$ in the loop representation discussed in §2.4. Thus, denoting an n-loop $\{\gamma^1, \ldots, \gamma^n\}$ by ${}^n\gamma$, $(\hat{T}^{a_1 \ldots a_n}[{}^1\gamma, A](x_1, \ldots, x_n)\Psi)[{}^n\gamma]$ is defined by

$$\hbar^n \sum_{\epsilon_1} \cdots \sum_{\epsilon_n} (-1)^{n(\epsilon)} \Delta^{a_1}[{}^1\gamma, {}^n\eta](x_1) \ldots \Delta^{a_n}[{}^1\gamma, {}^n\eta](x_n) \Psi[({}^1\gamma *_{x_\gamma} {}^n\eta)^{\epsilon_1 \ldots \epsilon_n}],$$

where $\Delta^a[{}^1\gamma, {}^n\eta](x) \overset{\text{def}}{=} \prod_{i=1}^n \Delta^a[{}^1\gamma, \eta^i](x)$, ${}^n\eta = \{\eta^1, \ldots, \eta^n\}$ and ${}^1\gamma *_{x_\gamma} {}^n\eta \overset{\text{def}}{=} \{{}^1\gamma *_{x_\gamma} \eta^1, \ldots, {}^1\gamma *_{x_\gamma} \eta^n\}$. Restriction (3)—$\Psi[\gamma * \eta * \eta_-] = \Psi[\gamma]$—on the loop functionals in §2.4 extends trivially to multi-loop functionals by

$$\Psi[{}^1\gamma * {}^1\eta * {}^1\eta_-, {}^2\gamma, \ldots] = \Psi[{}^1\gamma, {}^2\gamma, \ldots].$$

A bridge between the self-dual and the loop representation is made via the map $\mathcal{F} : \mathcal{H}' \to \mathcal{M}'$ to be defined below [17, §4.5, p. 126]. First of all, observe that operators $\hat{O}$ on $\mathcal{H}$ are associated with operators $\hat{O}^*$ on $\mathcal{H}'$—the *conjugate self-dual representation*—by $(\hat{O}^*\Phi)(\tilde{\Psi}) \overset{\text{def}}{=} \Phi(\hat{O}\tilde{\Psi})$. Then, defining a delta-distribution δ_{A_0} by $\delta_{A_0}(\tilde{\Psi}) \overset{\text{def}}{=} \tilde{\Psi}[A_0]$, where $A_0 \in \mathcal{A}$ is fixed, it follows [17, Eqn (90), p. 123] that $(\hat{T}^*[\gamma, A]\delta_{A_0})(\tilde{\Psi}) = (T[\gamma, A_0]\delta_{A_0})(\tilde{\Psi})$, and hence, $T[\gamma, A_0]$ is the eigenvalue of $\hat{T}^*[\gamma, A]\delta_{A_0}$. The emphasis here is made on the T^0-observables inasmuch as they form a maximal set of commuting observables that are well-defined in *both* representations:

$$\hat{T}[\gamma, A]\Psi_i = c_i[\gamma, A]\Psi_i, \quad i = \text{s}, \ell,$$

where Ψ_s (Ψ_ℓ) is a state vector in the self-dual (loop) representation.

Now, because $\hat{T}[\gamma, A]$ is diagonal in the self-dual representation, $(\hat{T}[\gamma, A]\tilde{\Psi})[A'] = T[\gamma, A]\tilde{\Psi}[A']$. As an aside, $\hat{T}[\gamma, A]$ may be interpreted [17, p. 123] as a creation operator which creates excitations of the connection localised along the loop γ. In the conjugate representation, $\hat{T}^*[\gamma, A]$ becomes the annihilation operator. It is also sketched in some detail in reference [17, §4.4, pp. 124–125] that $(\hat{T}[\gamma, A]\Psi)[\eta] = T[\gamma, A]\Psi[\eta]$.

Without going into any great length, $\mathcal{F} : \mathcal{H}' \to \mathcal{M}'$, given by

$$\Phi \mapsto \Phi(T[\gamma, \cdot]),$$

sends T^0-eigenvectors in $\mathcal{H}'$ to its corresponding eigenvectors in $\mathcal{M}'$. Some properties of $\mathcal{F}$ will be tersely mentioned without proofs—see reference [17, §4.5, p. 126] for details. First, the T^n-operators that act on Ψ_{s} and Ψ_ℓ are $\mathcal{F}$-equivalent, where a loop operator $\hat{T}_\ell$ and a self-dual operator $\hat{T}_{\mathrm{s}}$ are said to be *$\mathcal{F}$-equivalent* if $\mathcal{F} \circ \hat{T}_{\mathrm{s}}^* = \hat{T}_\ell \circ \mathcal{F}$. Note trivially that the observable $T[\gamma, A]$ may be regarded either as a (multi-)loop functional or a connection functional; for by fixing A, $T[\gamma, A] = \Psi_\ell[\gamma]$ and by fixing γ instead, $T[\gamma, A] = \Psi_{\mathrm{s}}[A]$.

Second, the left inverse of $\mathcal{F}$ exists: $\mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{id}$. Finally, because the self-dual internal gauge constraint is $\mathcal{F}$-equivalent to the null-operator in the loop space (as it annihilates the holonomy), it follows that the internal gauge constraint holds when going from the self-dual representation to the loop representation (via $\mathcal{F}$). Hence, only the diffeomorphism and the Hamiltonian constraints remain in the loop representation.

The mapping $\mathcal{F}$ admits a heuristic integral representation. It is effected by the following transform—the *Rovelli-Smolin transform*—which resembles the Fourier transform:

$$\Psi[\gamma] = \int_{\mathcal{A}[\mathrm{SU}(2)]} \overline{T[\gamma, A]} \tilde{\Psi}[A] \mathrm{d}\mu(A),$$

where μ is a measure on the space $\mathcal{A}[\mathrm{SU}(2)]$ of Ashtekar connections modulo the SU(2)-gauge orbits. More of $\mathcal{A}[\mathrm{SU}(2)]$ and μ will be said in a later chapter.

To conclude this chapter, a brief word about the solutions of the diffeomorphism and the Hamiltonian loop constraints will be said. It was shown in reference [17, §5, p. 132] that multi-loop functionals Ψ which satisfy

(1) $\Psi[\eta] = 0 \ \forall \eta \in \mathcal{M}$ such that $\exists\, \eta^i \in \eta$ that is either not smooth, or possessing self-intersections, and

(2) $\Psi[\eta] = \Psi[\eta'] \ \forall \eta, \eta' \in L(\eta)$, where $L(\eta)$ is the link class of η in Σ; *i.e.*, $L(\eta) = \{\, f \cdot \eta \mid f \text{ is a smooth ambient isotopy} \,\}$ and $f \cdot \eta \stackrel{\text{def}}{=} \{\, f \circ \eta^1, \ldots, f \circ \eta^n \,\}$,

satisfy both the loop diffeomorphism and Hamiltonian constraints. Indeed, Ψ when extended to multi-loops $\gamma \in \mathcal{M}$ that contain loop components of the form $\gamma^i * \zeta^i * \eta^i * \zeta^i_-$ or $\gamma^i * \zeta^i * \eta^i_- * \zeta^i_-$—where γ^i, η^i and ζ^i are smooth, and ζ^i connects η to γ—are also annihilated by the loop constraints. Thus, in conclusion, a class of solutions, *i.e.*, physical states in the loop representation, is obtained and they consist essentially of multi-loop functionals on the set of smooth non-intersecting multi-loops. In spite of this remarkable achievement, the physical interpretation of the loop representation still proves to be somewhat elusive. However recently, Rovelli and Smolin made further advances in the loop representation by constructing a basis on the space of multi-loops that are labelled by a generalisation of Penrose's spin networks. The details can be found in reference [18]. The basis of linearly independent states constructed by Rovelli and Smolin solved the long standing issue of over-completeness in the loop representation arising from the Mendalstam relations [18, §2, Eqns (2)–(3)].

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CHAPTER III

THE STRUCTURE OF THE MULTI-LOOP SPACE

3.1. Introduction

In this chapter, the topological structure of the multi-loop space $\mathcal{M}$ and the multi-loop functionals defined on it will be studied. Recall that in this thesis, Σ denotes a smooth, closed, compact, orientable, Riemannian 3-manifold. Another notation to be introduced is the following: if (X, ρ) is a metric space, then the symbol $B_\varepsilon(x)$ will denote an open ε-ball centred about x in X. Somewhat loosely worded, the multi-loop space of Σ is just the set of countable subsets of loops in Σ, where the constant loops in Σ are identified with a single point. The multi-loop space used in the loop representation is again a quotient of this space under certain equivalence relations such as reparametrisations. The details can be found in reference [9, §2.2], and the resulting quotient space is called a *non-parametric loop space* by Smolin. However, these equivalence relations will not be enforced because of the resulting complexity of the quotient topology on the quotient space and also because it will be convenient to work with parametrised loops. Instead, these conditions will be imposed on the multi-loop functionals. Since a topology will be needed on the multi-loop space in order to construct suitable measures and to define continuous functionals on it, it would be fruitful to construct a suitable topology on the multi-loop space—this, at least, is the motivation for analysing the topological structure of the multi-loop space.

It should be mentioned in passing that the multi-loop space as defined by Rovelli and Smolin [7, p. 107] is none other than the topological sum of the n-loop spaces with a point representing the zero loop—the set of n-loops is just a set whose elements are sets of n loops. In this chapter, the multi-loop space will be made precise and its topological properties studied. It will be shown in §3.2 below that the multi-loop space as depicted in [7] is metrizable as well as second countable. These two properties of the multi-loop space will be used in a later chapter to construct a measure on it. In the third section, the space of multi-loop functionals will be constructed and it will be seen that the space is just a direct sum of the space of n-loop functionals. In the final section, the action of the quantum T^0-operator on the multi-loop functionals will be analysed.

3.2. The Topological Structure Of $\mathcal{M}$

In this section, the properties of the multi-loop space of Σ will be probed. Concisely, a *loop* is a continuous map $\gamma : I \to \Sigma$ such that $\gamma(0) = \gamma(1)$ and $\gamma(I)$ is homeomorphic to the unit circle S^1, where $I \stackrel{\text{def}}{=} [0,1]$. In this thesis, the definition of a loop will be generalized to include a finite number of self-intersections. As such, the above definition of a loop will be referred to as a *standard loop*.

Let $\gamma_s : I \to \Sigma$ be a standard loop in Σ and let $\{x_1, \dots, x_n\}$ be a finite set of points on $\gamma_s(I)$ which may also be the empty set. Furthermore, let $\gamma_s(I)[x_1, \dots, x_n]$ denote the quotient space of $\gamma_s(I)$ such that the collection of points $\{x_{i_1}, \dots, x_{i_{m(i)}}\}$ are identified (with one another) for each $i = 1, \dots, p \leqq n$, where $\sum_{i=1}^{p} m(i) = n$, $\{x_1, \dots, x_n\} = \bigcup_{i=1}^{p} \{x_{i_1}, \dots, x_{i_{m(i)}}\}$, and $\{x_{i_1}, \dots, x_{i_{m(i)}}\} \cap \{x_{j_1}, \dots, x_{j_{m(j)}}\} = \varnothing \ \forall i \neq j$. Then, informally, a loop associated with γ_s is a continuous map $\gamma : I \to \Sigma$ such that $\gamma(I)$ is homeomorphic to $\gamma_s(I)[x_1, \dots, x_n]$. Roughly, this extends standard loops to loops with a finite number of self-intersections. The formal definition is given below.

3.2.1. Definition. Let $\gamma : I \to \Sigma$ be a continuous map with $\gamma(0) = \gamma(1)$ and $J = \{t_1, \dots, t_n\} \subset I$ be a finite subset (possibly empty) such that

(1) for each distinct pair $s, t \in (0,1) - J$, $\gamma(s) \neq \gamma(t)$;

(2) there is a unique (finite) partition $\mathcal{P}(J)$ of J—that is, $\mathcal{P}(J) = \{\, J_\alpha : \alpha = 1, \dots, m < n \,\}$, $J = \bigcup_\alpha J_\alpha$ and $J_\alpha \cap J_\beta = \varnothing \ \forall \alpha \neq \beta$—satisfying for each $J_\alpha \in \mathcal{P}(J)$, $\gamma(s) = \gamma(t) \ \forall s, t \in J_\alpha$, and $\gamma(s) \neq \gamma(t)$ whenever $s \in J_\alpha$, $t \in J_\beta$ and $\alpha \neq \beta$;

(3) $\gamma(I)$ is homeomorphic to $\gamma_s(I)[x_1, \dots, x_n]$, where $x_i = \gamma(t_i)$ for $i = 1, \dots, n$.

Then, γ is called a *loop* in Σ associated with γ_s.

An important point to note is that loops belonging to the set of Peano spaces, which exist by virtue of the Hahn-Mazurkiewicz Theorem [8, p. 343], are not included in the definition of loops given here; in particular, a "closed" Peano space $p : I \to \Sigma$, where $p(0) = p(1)$ and $p(I) = \Sigma$, is not considered to be loop in this thesis. This will ensure that pathological issues, if any, that might arise from the loop representation by the existence of loops whose images are the entire 3-manifold Σ will not be present. This is the reason why standard loops were defined (to be homeomorphic to S^1) in the first place, as the collection of "loops" defined by the set $\{\gamma : I \to \Sigma \mid \gamma(0) = \gamma(1),\ \gamma \text{ continuous}\}$ would be far too large, and in particular, would include Peano spaces whose initial and final points coincide.

The above generalisation of loops is *not* quite the whole story! In the loop representation, the domain of definition of the loop functionals also contain "loops" consisting of two loops connected by a curve.[1] Explicitly, these "loops" are of the form $\gamma\hat{*}c\hat{*}\eta$, where γ, η are loops in Σ and $c : I \to \Sigma$ is a curve in Σ that connects $\gamma(t_1)$ in γ to $\eta(t_2)$ in η for some $t_1, t_2 \in I$:

$$\gamma\hat{*}c\hat{*}\eta(t) \stackrel{\text{def}}{=} \begin{cases} \gamma(6t_1t) & \text{for } 0 \leqq t \leqq \frac{1}{6}, \\ c(6t-1) & \text{for } \frac{1}{6} \leqq t \leqq \frac{2}{6}, \\ \eta(6(1-t_2)t + 3t_2 - 2) & \text{for } \frac{2}{6} \leqq t \leqq \frac{3}{6}, \\ \eta(3t_2(2t-1)) & \text{for } \frac{3}{6} \leqq t \leqq \frac{4}{6}, \\ c(5-6t) & \text{for } \frac{4}{6} \leqq t \leqq \frac{5}{6}, \\ \gamma(6(1-t_1)t + 6t_1 - 5) & \text{for } \frac{5}{6} \leqq t \leqq 1. \end{cases}$$

Strictly of course, $\gamma\hat{*}c\hat{*}\eta$ should be denoted by $\gamma\hat{*}_{t_1}c\hat{*}_{t_2}\eta$. Hence, mappings of the form $\gamma\hat{*}c\hat{*}\eta$, where η may just be the reverse curve $c_-(t) \stackrel{\text{def}}{=} c(1-t)$, and more generally, $\gamma_1\hat{*}c_1\hat{*}\gamma_2\hat{*}c_2\hat{*}\ldots\hat{*}c_{n-1}\hat{*}\gamma_n$—where γ_i are loops and c_i are curves joining γ_i to γ_{i+1}—will also be included in the definition of a loop. And finally, just when it is reasonable to expect that the tale of the loop definition should end happily ever after, one discovers that a zero loop—a constant loop—is also included in the recipe of the loop representation!

Let $\mathcal{L}_0 \stackrel{\text{def}}{=} \{\gamma \in \tilde{L}_\Sigma \mid \gamma(I) = x_\gamma,\ x_\gamma \in \Sigma\}$ be the set of constant loops. Then, the *loop space* Ω_Σ of Σ in this thesis is defined to be the set

$$\Omega_\Sigma \stackrel{\text{def}}{=} \{\gamma : I \to \Sigma \mid \gamma \text{ is a loop}\} \cup \mathcal{L}_0.$$

This space is endowed with the compact-open topology. Recall that this topology is generated by the subbase consisting of open sets of the form

$$M(U, W) = \{\gamma \in \Omega_\Sigma \mid \gamma(U) \subset W,\ U \subseteq I \text{ compact},\ W \subset \Sigma \text{ open}\}.$$

Observe from [5, Theorem 4.2.17, p. 263] that the compact-open topology on Ω_Σ is compatible with the metric topology induced by the following metric:

$$\hat{d}(\gamma, \eta) \stackrel{\text{def}}{=} \sup_{t \in I} d(\gamma(t), \eta(t)),$$

[1] Here, a *curve* in Σ is defined to be a topological imbedding $c : I \to \Sigma$; in particular, c does not possess any self-intersections.

where d is the distance function on Σ induced by a fixed Riemannian metric q on it. Furthermore, note trivially that if q_1 and q_2 are any two admissible Riemannian metrics on Σ, then the d_1-topology and the d_2-topology on Σ coincide, where d_i is the distance function induced by q_i. In particular, $\hat{d}_1$ is equivalent to $\hat{d}_2$. Hence, there is no loss of generality in fixing an admissible Riemannian metric on Σ when defining the metrizable topology on Ω_Σ.

A word regarding the "zero loop" should perhaps be made. In [7, p. 107], a zero loop was included in the definition of the multi-loop space. Because the topology of the space was not specified *a priori*, it is unclear whether the zero loop is an isolated point or not. In this thesis, the zero loop—which is essentially obtained by identifying all the constant loops with a single point—is not an isolated point by construction. The construction of the multi-loop space to be given below is based on a rather concise description given in [7]. And whilst no topology were constructed for the multi-loop space by the authors of reference [7], a topology on the multi-loop space will be constructed here in some detail.

Let $(J, \leqq) \subset (I, \leqq)$ be a linearly ordered subset that is at most countably infinite and that satisfies $t_i < t_{i+1}\ \forall\, i = 1, \ldots, n \leqq \aleph_0$, where $J = \{\, t_i \mid i = 1, \ldots, n\,\}$. Then, $(J, \leqq)$ is said to be *discretely ordered.*

3.2.2. Definition. Let M be a smooth manifold and $f : I \to M$ be continuous. Then f is said to be *piecewise smooth* if there exists a discretely ordered subset $J \subset I$ such that f is smooth on $I - J$. That is, f is smooth on $(t_i, t_{i+1})\ \forall\, i = 1, \ldots, n \leqq \aleph_0$, where $J = \{\, t_i \mid i = 1, \ldots, n\,\}$.

Let $\tilde{\mathcal{L}}_\Sigma \subset \Omega_\Sigma$ be the set of piecewise smooth loops, $\overline{\mathbb{R}}_+ \overset{\text{def}}{=} [0, +\infty]$ and $\mathcal{L}_0$ the set of constant loops; that is, loops γ satisfying $\gamma(t) = \gamma(s)\ \forall\, s, t \in I$. A (topological) metric can be constructed on $\tilde{\mathcal{L}}_\Sigma$ in the following way. Fix a finite atlas $\mathfrak{A} = \{(U_\alpha, \varphi_\alpha)\}$ on Σ and define $\hat{d}' : \tilde{\mathcal{L}}_\Sigma \times \tilde{\mathcal{L}}_\Sigma \to \overline{\mathbb{R}}_+$ by

$$\hat{d}'(\gamma, \eta) \overset{\text{def}}{=} \text{ess }\sup\{\, \|D^\ell\gamma(t) - D^\ell\eta(t)\| : t \in I,\ \ell \geqq 1\,\},$$

where sup ranges over all the relevant (finite) charts, $D^\ell\gamma(t)$ denotes—in abused notation—the ℓth differential of γ at t, and *ess* means that the expression $\|D^\ell\gamma(t) - D^\ell\eta(t)\|$ is defined on I *apart* from a *finite* number (possibly zero) of points $\{\, t_1, \ldots, t_n\,\} \subset I$ wherein γ or η are *not* differentiable. Evidently, $\rho(\gamma, \eta) \overset{\text{def}}{=} \hat{d}(\gamma, \eta) + \hat{d}'(\gamma, \eta)$ defines a metric on $\tilde{\mathcal{L}}_\Sigma$. From here on, $\tilde{\mathcal{L}}_\Sigma$ will be endowed with the ρ-topology.

3.2.3. Remark. The ρ-topology above does not depend on the choice of (admissible) finite atlas. To see this, let $\overline{\mathfrak{A}}$ denote the maximal atlas of Σ and define a subbasic open set in $\tilde{\mathcal{L}}_\Sigma$ by

$$N_\delta(\gamma;(U_\alpha,\varphi_\alpha),K) \stackrel{\text{def}}{=} \{\,\eta \in \tilde{\mathcal{L}}_\Sigma \mid \eta(K) \subset U_\alpha,\ \hat{d}_K(\gamma,\eta) + \hat{d}'_{\alpha K}(\gamma,\eta) < \delta\,\},$$

where $K \subset I$ is compact, $\hat{d}_K(\gamma,\eta) = \sup\{\, d(\gamma(t),\eta(t)) \mid t \in K\,\}$, $\gamma(K) \subset U_\alpha$, and $\hat{d}'_{\alpha K}$ is just $\hat{d}'$ 'restricted' to those η's satisfying $\eta(K) \subset U_\alpha$; more precisely, $\hat{d}'_{\alpha K}(\gamma,\eta)$ is defined by

$$\text{ess}\ \sup\{\,\|D^\ell\varphi_\alpha \circ \gamma(t) - D^\ell\varphi_\alpha \circ \eta(t)\| : t \in K \text{ compact},\ \ell \geqq 1\,\},$$

where each η satisfies $\eta(K) \subset U_\alpha$. This topology is equivalent to the ρ-topology introduced above on $\tilde{\mathcal{L}}_\Sigma$. For given any ρ-open δ-ball $B_\delta(\gamma)$, consider $N(\gamma;\varepsilon) = \bigcap_{i=1}^n N_\varepsilon(\gamma;(U_{\alpha_i},\varphi_{\alpha_i}),K_i)$, where $I = \bigcup_{i=1}^n K_i$. Then, clearly, for a suitable choice of charts $(U_{\alpha_i},\varphi_{\alpha_i})$ and taking $\varepsilon \ll \delta$, $N(\gamma;\varepsilon) \subset B_\delta(\gamma)$ can always be satisfied. Conversely, given $N_\varepsilon(\gamma;(U,\varphi),K)$, there clearly exists some $\delta > 0$ such that $B_\delta(\gamma) \subset N_\varepsilon(\gamma;(U,\varphi),K)$. In particular, given any finite intersection $N(\gamma;\varepsilon)$ of subbasic open sets of γ, there $\exists\,\delta > 0$ such that $B_\delta(\gamma)$ is contained within it. Hence, the two topologies are compatible, as claimed. Since no dependence on the particular choice of finite atlas were invoked, the initial assertion made above follows.

A central result needed in the construction of a measure on $\tilde{\mathcal{L}}_\Sigma$ will be established below; namely that $\tilde{\mathcal{L}}_\Sigma$ is second countable in the ρ-topology. Observe first of all that because $(\Omega_\Sigma, \hat{d})$ is second countable with respect to the compact-open topology [2, corollary 4.2.18, p. 263], so is $(\tilde{\mathcal{L}}_\Sigma, \hat{d}|\tilde{\mathcal{L}}_\Sigma)$ with respect to its subspace topology. Hence, there exists a countable $\hat{d}$-dense subset D in $\tilde{\mathcal{L}}_\Sigma$. The dense subset D will be fixed in all that follows.

3.2.4. Lemma. *For each $n,m,k \in \mathbb{N}$ and $\gamma \in D$, define $\tilde{D}_\gamma(m,n,k)$ to be the set of all loops $\eta \in \tilde{\mathcal{L}}_\Sigma$ such that*

(1) $\eta(0) = \gamma(0)$,
(2) $\hat{d}(\gamma,\eta) < \frac{1}{m}$,
(3) $\frac{k}{n+1} \leqq \hat{d}'(\gamma,\eta) \leqq \frac{k}{n}$.

Then, $S[\rho] \stackrel{\text{def}}{=} \bigcup_{m,n,k\in\mathbb{N}} \bigcup_{\gamma\in D} \tilde{D}_\gamma(m,n,k)$ is ρ-dense in $\tilde{\mathcal{L}}_\Sigma$.

Proof. Given any $\eta \in \tilde{\mathcal{L}}_\Sigma$, it will suffice to establish a sequence $\{\xi_i\}_i$ in $S[\rho]$ that ρ-converges to η in $\tilde{\mathcal{L}}_\Sigma$. Now, since D is a countable dense subset in $\tilde{\mathcal{L}}_\Sigma$, there exists a sequence $\{\gamma_i\}_i$ in D such that γ_i $\hat{d}$-converges to η. So, consider the sequence of sets $\{\tilde{D}_{\gamma_i}m,n,k) \mid m,n,k,i \in \mathbb{N}\}$. Set $s_{\eta i} = \hat{d}'(\gamma_i, \eta)$ and choose $k_i, n_i \in \mathbb{N}$ such that $k_i/(n_i+1) \leqq s_{\eta i} \leqq k_i/n_i$. Fix any $\varepsilon > 0$ and choose $m_i, k_i > 2/\varepsilon$. Then, from the definition of $\tilde{D}_\gamma(m,n,k)$—using property (3)—$\exists\, \tilde{\xi}_i \in \tilde{D}_{\gamma_i}(m_i,n_i,k_i)$ such that $\hat{d}'(\tilde{\xi}_i,\eta) < 1/m_i$. Since $\hat{d}(\tilde{\xi}_i,\eta) < 1/m_i$, by property (2), it follows at once that $\rho(\tilde{\xi}_i,\eta) = \hat{d}(\tilde{\xi}_i,\eta) + \hat{d}'(\tilde{\xi}_i,\eta) < 2/m_i < \varepsilon$. Hence, from the arbitrariness of $\varepsilon > 0$, there exists a sequence $\{\tilde{\xi}\}_i$ in $S[\rho]$ which ρ-converges to η, as asserted. □

3.2.5. Proposition. *$(\tilde{\mathcal{L}}_\Sigma, \rho)$ is second countable.*

Proof. The notations introduced in Lemma 3.2.4 above will be used in the proof sketched below. Fix $\gamma \in D$ and let $J_{k,n} = \mathbb{Q} \cap [\frac{k}{n+1}, \frac{k}{n}]$ and set $\Lambda_m = \mathbb{Q} \cap [0, \frac{1}{m})$. For each $t \in J_{k,n}$ and $\alpha \in \Lambda_m$, let $D^\alpha_m(\gamma;t) \subset \tilde{D}_\gamma(m,n,k)$ be a countable subset consisting of loops η satisfying

(a) $\hat{d}'(\gamma,\eta) = t$,
(b) $D^\alpha_m(\gamma;t)$ is $\hat{d}$-dense in the set $\{\zeta \in \tilde{\mathcal{L}}_\Sigma \mid \hat{d}(\gamma,\zeta) = \alpha\}$.

The existence of property (b) follows trivially from the fact that $(\tilde{\mathcal{L}}_\Sigma, \hat{d}|\tilde{\mathcal{L}}_\Sigma)$ is second countable. Let $D_m(\gamma;t) \stackrel{\text{def}}{=} \bigcup_{\alpha\in\Lambda_m} D^\alpha_m(\gamma;t)$. Then, by definition, $D_m(\gamma;t)$ is countable. Set $D_\gamma(m,n,k) \stackrel{\text{def}}{=} \bigcup_{t\in J_{k,n}} D_m(\gamma;t) \cup \{\gamma\}$.

Claim: $D[\rho] \stackrel{\text{def}}{=} \bigcup_{\gamma\in D} \bigcup_{n,m,k\in\mathbb{N}} D_\gamma(m,n,k)$ is ρ-dense in $\tilde{\mathcal{L}}_\Sigma$.

It is clear from the construction that $D[\rho]$ is denumerable as it is a countable union of countable sets. Notice also that given any $\delta > 0$ and $\eta \in \tilde{\mathcal{L}}_\Sigma$, $\exists\, \gamma \in D$ such that $\hat{d}(\gamma,\eta) < \delta$ as D is $\hat{d}$-dense in $\tilde{\mathcal{L}}_\Sigma$.

To verify the claim, it will suffice to show that for each $\eta \in \tilde{\mathcal{L}}_\Sigma$, there exists a sequence $\{\zeta_i\}_i$ in $D[\rho]$ such that $\zeta_i \to \eta$ as $i \to \infty$ in the ρ-topology, for invoking lemma 2.2 will complete the proof. So, fix $\eta \in \tilde{\mathcal{L}}_\Sigma$ and for each $i > 0$, consider $\gamma_i \in D$ such that $\hat{d}(\gamma_i,\eta) < 1/2i$. Set $s_{\eta i} = \hat{d}'(\gamma_i,\eta)$. Now, given any $\zeta \in \tilde{D}_\gamma(m,n,k)$ and $\delta > 0$, it will be established below that that $\exists\, \zeta_\delta \in D_{\gamma'}(m',n',k')$, for some $m',n',k' \in \mathbb{N}$ and $\gamma' \in D$, such that $\rho(\zeta,\zeta_\delta) < \delta$.

First, recall that the $\hat{d}$-denseness of D implies the existence of a sequence $\{\gamma_i\}_i$ in D such that γ_i $\hat{d}$-converges to ζ. Hence, for i sufficiently large, it is always possible to choose $m',n',k' \in \mathbb{N}$ such that $\zeta \in \tilde{D}_{\gamma_i}(m',n',k')$, where $k'/n' < \frac{1}{4}\delta$, $m' > 4/\delta$. By definition, for any $t \in J_{k',n'}$ and $\alpha \in \Lambda_{m'}$ with i large enough, $\exists\, \tilde{\zeta} \in D^\alpha_{m'}(\gamma_i;t)$ such that $\rho(\tilde{\zeta},\gamma_i) = \hat{d}(\tilde{\zeta},\gamma_i) + \hat{d}'(\tilde{\zeta},\gamma_i) < \frac{1}{4}\delta + \frac{1}{4}\delta = \frac{1}{2}\delta$ by properties (b) and (a). Hence, by taking i to be sufficiently large so that $\hat{d}(\gamma_i,\zeta) < \frac{1}{4}\delta$, the above choices of m',n',k' imply that $\hat{d}'(\gamma_i,\zeta) < \frac{1}{4}\delta$. Whence, $\rho(\tilde{\zeta},\zeta) \leqq \rho(\tilde{\zeta},\gamma_i) + \rho(\gamma_i,\zeta) <$

$\frac{1}{2}\delta + \frac{1}{2}\delta = \delta$, as claimed. As an aside, note that $D_\gamma(m,n,k)$ is not to be confused with $\tilde{D}_\gamma(m,n,k)$; the former is denumerable whereas the latter is not.

Now, by Lemma 3.2.4, $\exists\,\{\tilde{\xi}_i\}_i$ in $S[\rho]$ such that $\tilde{\xi}_i$ ρ-converges to η. However, from the observation made in the previous paragraph, for each $\varepsilon > 0$ and $\tilde{\xi}_i \in \tilde{D}_{\gamma_i}(m_i,n_i,k_i)$, $\exists\,\bar{\xi}_i \in D_{\gamma_i^\varepsilon}(m_i^\varepsilon,n_i^\varepsilon,k_i^\varepsilon)$ such that $\hat{d}'(\bar{\xi}_i,\tilde{\xi}_i) < \varepsilon$. Hence, a sequence in $D[\rho]$ can be constructed from $\{\tilde{\xi}_i\}_i$ in $S[\rho]$ to obtain the desired ρ-convergent sequence. Explicitly, let $\{\zeta_{ij}\}_{j=1}^\infty$ be a sequence in $D[\rho]$ such that $\lim\limits_{j\to\infty}\zeta_{ij} = \tilde{\xi}_i$ for each i. In particular, it is possible to choose the sequence such that for some sufficiently large $N > 0$, $\hat{d}'(\zeta_{ij},\eta) < 1/2i$ whenever $j \geqq N$ for each i. Hence, for each i, $\rho(\zeta_{ij},\eta) < 1/2i + 1/2i = 1/i$ whenever $j \geqq N$ and, consequently, the sequence $\{\hat{\zeta}_i\}_i$, where $\hat{\zeta}_i \equiv \zeta_{iN}$, in $D[\rho]$ ρ-converges to η, and $\tilde{\mathcal{L}}_\Sigma$ is thus second countable as claimed. $\square$

Now, define an equivalence relation $\tilde{\mathcal{R}} \subset \tilde{\mathcal{L}}_\Sigma \times \tilde{\mathcal{L}}_\Sigma$ on $\tilde{\mathcal{L}}_\Sigma$ by $\tilde{\mathcal{R}} = \{\,(\gamma,\eta) \mid \gamma,\eta \in \mathcal{L}_0\,\}$. Let $\mathcal{L}_\Sigma \overset{\text{def}}{=} \tilde{\mathcal{L}}_\Sigma/\tilde{\mathcal{R}}$ denote the quotient space, $\tilde{\pi} : \tilde{\mathcal{L}}_\Sigma \to \mathcal{L}_\Sigma$ the natural map, and set $0_\Sigma = \tilde{\pi}(\gamma)\ \forall\,\gamma \in \mathcal{L}_0$. By construction, $\tilde{\pi}|(\tilde{\mathcal{L}}_\Sigma - \mathcal{L}_0) = \mathrm{id}_{\tilde{\mathcal{L}}_\Sigma}|(\tilde{\mathcal{L}}_\Sigma - \mathcal{L}_0)$. Consequently, the neighbourhood system of $\mathcal{L}_\Sigma$ will be completely determined if the neighbourhood base at 0_Σ is known. It is clear from the definition that for each neighbourhood N_{0_Σ} of 0_Σ in $\mathcal{L}_\Sigma$, $\tilde{\pi}^{-1}(N_{0_\Sigma})$ is a neighbourhood of $\mathcal{L}_0$ in $\tilde{\mathcal{L}}_\Sigma$. Hence, it follows at once that the neighbourhood system of 0_Σ is the following base (at 0_Σ):

$$\mathcal{N}_{0_\Sigma} = \left\{ \bigcup_{\gamma\in\mathcal{L}_0} \tilde{\pi}(N_\gamma) \,\middle|\, N_\gamma \in \mathcal{N}_\gamma \right\},$$

where $\mathcal{N}_\gamma$ is the neighbourhood base of γ.

3.2.6. Lemma. *$\mathcal{L}_0$ is closed and nowhere dense in $\tilde{\mathcal{L}}_\Sigma$.*

Proof. Let $\{\gamma_n\}_n$ be a sequence in $\mathcal{L}_0$ which converges to $\gamma_0 \in \tilde{\mathcal{L}}_\Sigma$. By definition, $\gamma_n(I) = x_n \in \Sigma\ \forall n$ and Σ is compact Hausdorff together imply that $\exists\,x_0 \in \Sigma$ and a subsequence $\{x_{n_k}\}_k \subset \{x_n\}_n$ such that $x_{n_k} \to x_0$. Hence, $\forall\,\varepsilon > 0, \exists\,N_\varepsilon > 0$ such that $\rho(\gamma_n,\gamma_0) \equiv \sup_{t\in I} d(x_n,\gamma_0(t)) + \text{ess sup}\{\,\|D^\ell\gamma_0(t)\| : t \in I,\ \ell \geqq 1\,\} < \varepsilon$ whenever $n > N_\varepsilon$. However, this implies at once that $\gamma_0(t) \equiv x_0$ on I; thus, $\mathcal{L}_0$ is closed.

Finally, to complete the proof, suppose that the interior $\mathcal{L}_0^\circ \neq \varnothing$. Then, for any fixed $\eta \in \mathcal{L}_0^\circ$, there exists a δ-ball $B_\delta(\eta) \subset \mathcal{L}_0^\circ$ for some $\delta > 0$. However, it follows from the definition of ρ that this clearly cannot be true: for there certainly exists a non-constant loop $\hat{\gamma}$ such that $0 < \hat{d}(\hat{\gamma},\eta) \leqq \frac{1}{4}\delta$ (say) and $0 < \text{ess sup}\{\,\|D^\ell\hat{\gamma}(t)\| :$

$t \in I,\, \ell \geqq 1\} \leqq \frac{1}{4}\delta$. Hence, $B_\delta(\eta) \not\subset \mathcal{L}_0$, which is a contradiction. Consequently, the interior $\mathcal{L}_0^\circ \equiv \varnothing$, as required. □

3.2.7. Lemma. *$\tilde{\pi} : \tilde{\mathcal{L}}_\Sigma \to \mathcal{L}_\Sigma$ is closed.*

Proof. To establish this claim, it is enough to show that $\tilde{\pi}$ maps closed neighbourhoods C_η of $\eta \in \mathcal{L}_0$ into closed neighbourhoods of 0_Σ. Invoking the quotient topology, it will suffice to verify that $\tilde{\pi}^{-1} \circ \tilde{\pi}(C_\eta)$ is closed in $\tilde{\mathcal{L}}_\Sigma$. Since $\tilde{\pi}^{-1} \circ \tilde{\pi}(C_\eta) = C_\eta \cup \mathcal{L}_0$ by definition, Lemma 3.2.6 yields the assertion. □

Notice however that $\tilde{\pi}$ is not an open map. For given any neighbourhood N_η of $\eta \in \mathcal{L}_0$ satisfying $\mathcal{L}_0 \not\subset N_\eta$, $\tilde{\pi}^{-1} \circ \tilde{\pi}(N_\eta) = N_\eta \cup \mathcal{L}_0$ which is neither closed nor open (by Lemma 3.2.5). Nevertheless, for each neighbourhood N of $\mathcal{L}_0$, $\tilde{\pi}(N)$ is a neighbourhood of 0_Σ by definition.

3.2.8. Proposition. *$\mathcal{L}_\Sigma$ is metrizable and second countable.*

Proof. To verify that $\mathcal{L}_\Sigma$ is metrizable, it will suffice to show that $\mathcal{L}_\Sigma$ is first countable [5, Theorem 4.4.17, p. 285] by Lemma 3.2.7; and from the definition of $\tilde{\pi}$, it is enough to verify that 0_Σ has a countable neighbourhood base. Let $\mathfrak{B}_\eta = \{\, B_{\frac{1}{n}}(\eta) \mid n \in \mathbb{N}\,\}$ be a countable neighbourhood base at $\eta \in \mathcal{L}_0$ in $\tilde{\mathcal{L}}_\Sigma$ and recall that $\bigcup_{\eta \in \mathcal{L}_0} \tilde{\pi}(B_{\frac{1}{n}}(\eta))$ is a neighbourhood of 0_Σ in $\mathcal{L}_\Sigma$. The collection $\mathfrak{B}_{0_\Sigma}$ defined by

$$\mathfrak{B}_{0_\Sigma} = \left\{ \bigcup_{\eta \in \mathcal{L}_0} \tilde{\pi}(B_{\frac{1}{n}}(\eta)) \,\middle|\, n \in \mathbb{N} \right\}$$

is clearly a countable neighbourhood base of 0_Σ. Hence, $\mathcal{L}_\Sigma$ is metrizable, as required.

The second countability of $\mathcal{L}_\Sigma$ follows trivially from the definition of $\tilde{\pi}$ and the second countability of $\tilde{\mathcal{L}}_\Sigma$. For let $\mathfrak{B}(\tilde{\mathcal{L}}_\Sigma)$ be a countable open base of $\tilde{\mathcal{L}}_\Sigma$ and let $\mathfrak{B}_0(\tilde{\mathcal{L}}_\Sigma) \subset \mathfrak{B}(\tilde{\mathcal{L}}_\Sigma)$ such that $\forall\, G \in \mathfrak{B}_0(\tilde{\mathcal{L}}_\Sigma)$, $G \cap \mathcal{L}_0 = \varnothing$. Then, $\tilde{\pi}(G) = G$ is open in $\mathcal{L}_\Sigma$ $\forall\, G \in \mathfrak{B}_0(\tilde{\mathcal{L}}_\Sigma)$ and hence $\tilde{\pi}(\mathfrak{B}_0(\tilde{\mathcal{L}}_\Sigma)) \cup \mathfrak{B}_{0_\Sigma}$ forms an open countable base for $\mathcal{L}_\Sigma$, where $\tilde{\pi}(\mathfrak{B}_0(\tilde{\mathcal{L}}_\Sigma)) \overset{\text{def}}{=} \{\, \tilde{\pi}(G) \mid G \in \mathfrak{B}_0(\tilde{\mathcal{L}}_\Sigma)\,\}$, as required. □.

Now, let $\mathcal{L}_\Sigma^\infty$ be the countably infinite Cartesian product of $\mathcal{L}_\Sigma$ and given any element $\gamma \in \mathcal{L}_\Sigma^\infty$, let $[\gamma] = \{\gamma^1, \gamma^2, \dots\}$ denote the set of the components of $\gamma = (\gamma^i)_{i=1}^\infty$. Define an equivalence relation $\mathcal{R} \subset \mathcal{L}_\Sigma^\infty \times \mathcal{L}_\Sigma^\infty$ on $\mathcal{L}_\Sigma^\infty$ by

$$\mathcal{R} = \{\, (\gamma, \eta) \in \mathcal{L}_\Sigma^\infty \times \mathcal{L}_\Sigma^\infty : [\gamma] = [\eta] \,\}$$

and let $\hat{M} \stackrel{\text{def}}{=} \mathcal{L}_\Sigma^\infty/\mathcal{R}$ be the quotient space (endowed with the quotient topology) with $\pi : \mathcal{L}_\Sigma^\infty \to \hat{M}$ the natural map. This map is defined explicitly by

$$\pi : (\gamma^1, \gamma^2, \dots) \mapsto \{\gamma^1, \gamma^2, \dots\}.$$

3.2.9. Lemma. *The map $\pi : \mathcal{L}_\Sigma^\infty \to \hat{M}$ is both open and closed.*

Proof. Let $d : \mathcal{L}_\Sigma \times \mathcal{L}_\Sigma \to \mathbb{R}_+$ be a metric compatible with the quotient topology on $\mathcal{L}_\Sigma$. Fix some $\gamma_0 \in \mathcal{L}_\Sigma^\infty$ and consider, without loss of generality, a neighbourhood of γ_0 of the form $N_\varepsilon(\gamma_0) = \prod_{i=1}^\infty N_i$, where $N_i = B_\varepsilon(\gamma_0^i)$ is a d-open ε-ball in $\mathcal{L}_\Sigma$ for $i = 1, \dots, n$, and $N_i = \mathcal{L}_\Sigma \; \forall i > n$. Then, it is clear from the definition of π that

$$\pi^{-1} \circ \pi(N_\varepsilon(\gamma_0)) = \bigcup_\sigma \prod_{i=1}^\infty N_{\sigma(i)} \cup G_\varepsilon(\gamma_0),$$

where σ is a permutation of $\mathbb{N}$ and $G_\varepsilon(\gamma_0)$ is the union of open subsets of the form

$$\prod_1^{m_1} \mathcal{L}_\Sigma \times \prod_{i=1}^{n_1} \tilde{N}_{1i} \times \prod_1^{m_2} \mathcal{L}_\Sigma \times \cdots \times \prod_{i=1}^{n_\ell} \tilde{N}_{\ell i} \times \prod_1^\infty \mathcal{L}_\Sigma$$

with (i) $n_i, m_i < \aleph_0$ for $i = 1, \dots, \ell < \aleph_0$, (ii) $\tilde{N}_{ki} \in \{ B_\varepsilon(\gamma_0^j) \mid j = 1, \dots, n \}$ for $k = 1, \dots, \ell$ and each i, where $\tilde{N}_{k_j i_j} \equiv B_\varepsilon(\gamma_0^j) \; \forall j = 1, \dots, n$ for at least n of $\tilde{N}_{ki}$. Consequently, the fact that $\pi^{-1} \circ \pi(N_\varepsilon(\gamma_0))$ is open implies that π must also be open from the definition of the quotient topology.

Finally, it remains to show that π is closed. Let $N \subset \mathcal{L}_\Sigma^\infty$ be closed and set $\tilde{N} = \pi^{-1} \circ \pi(N)$. Let $\{\gamma_n\}_n$ be a sequence in $\tilde{N}$ which converges to γ_0 in $\mathcal{L}_\Sigma^\infty$. Then, the continuity of π implies at once that $\pi(\gamma_n) \to \pi(\gamma_0)$. It is clear from the definition of π that $\forall n, \exists \eta_n \in N \cap \pi^{-1}(\pi(\gamma_n))$ such that $\{\eta_n\}_n$ is a convergent sequence in $\mathcal{L}_\Sigma^\infty$. To see this, it is enough to note that $\gamma_n \to \gamma_0 \Rightarrow \gamma_n^i \to \gamma_0^i$ for each fixed i. Hence, by choosing $\eta_n^i = \gamma_n^{i'}$ for each i so that $\eta_n = (\eta_n^i)_{i=1}^\infty \in N$ for each n, yields the desired sequence. So, N is closed implies that $\eta_0 \in N$. That is, $\pi(\gamma_0) \equiv \pi(\eta_0) \in \tilde{N}$, and π is thus closed. □

Furthermore, as metrizability is an invariant under a surjective closed-and-open mapping [5, Theorem 4.4.18, p. 285], Lemma 3.2.9 yields the following corollary.

3.2.10. Corollary. *$\hat{M}$ is a metrizable space.* □

3.2.11. Remark. The *n-loop space* $\mathcal{M}_n$ of Σ is defined by $\mathcal{M}_n = \{ \gamma \in \hat{M} \mid \gamma = \{\gamma^1, \dots, \gamma^n\} \}$, where $\gamma^i \in \mathcal{L}_\Sigma$ for each i (and $\gamma^i \neq \gamma^j \; \forall i \neq j$). Clearly, $\hat{M} = \bigcup_{n=1}^\infty \mathcal{M}_n \cup \mathcal{M}_\infty$ (as sets), and by definition, $\mathcal{M}_n \cap \mathcal{M}_m = \varnothing \; \forall n \neq m$;

moreover, for each fixed $m > 1$, $\mathcal{M}_\ell \subset \partial\mathcal{M}_m \ \forall \ell < m$. It is easy to see that $\overline{\mathcal{M}}_n = \bigcup_{i=1}^n \mathcal{M}_i \ \forall n > 0$, and in particular, $\overline{\mathcal{M}}_\infty = \hat{M}$; that is, $\mathcal{M}_\infty$ is a dense subset of $\hat{M}$. Finally, it is also clear that $\hat{M}$ is second countable as π is open and $\mathcal{L}_\Sigma^\infty$ is second countable. As a consequence, $\mathcal{M}_\infty$ is both second countable and metrizable.

Now, as with the case of $\hat{M}$, let $\mathcal{R}_n = \{(\gamma, \eta) \in \mathcal{L}_\Sigma^n \times \mathcal{L}_\Sigma^n : [\gamma] = [\eta]\}$ define an equivalence relation on $\mathcal{L}_\Sigma^n$ and let $\pi_n : \mathcal{L}_\Sigma^n \to \mathcal{L}_\Sigma^n/\mathcal{R}_n \equiv \bigcup_{k=1}^n \mathcal{M}_k$, given by $(\gamma^1, \dots, \gamma^n) \mapsto \{\gamma^1, \dots, \gamma^n\}$, denote the natural map. Then, π_n is both open and closed. By Remark 3.2.11, it is clear that $\overline{\mathcal{M}}_n \equiv \mathcal{L}_\Sigma^n/\mathcal{R}_n$, and in particular, it is not difficult to verify that $\bigcup_{k=1}^{n-1} \mathcal{M}_k$ is closed in $\overline{\mathcal{M}}_n$. Hence, $\mathcal{M}_n$ is open in $\overline{\mathcal{M}}_n \ \forall n > 0$. Since π_n is both open and closed, $\mathcal{L}_\Sigma^n/\mathcal{R}_n$ is metrizable, and hence $\mathcal{M}_n$ is also, for each n. From the construction, $\mathcal{M}_1$ is homeomorphic to $\mathcal{L}_\Sigma$ and hence $\mathcal{M}_1 \equiv \mathcal{L}_\Sigma$ as $\pi_1 = \mathrm{id}_{\mathcal{L}_\Sigma}$. Furthermore, for each n, $\mathcal{M}_n$ is second countable.

In summary, the above analyses reveal that each n-loop space $\mathcal{M}_n$, for $1 \leqq n \leqq \infty$, is both metrizable and second countable. The motivation for constructing the space $\hat{M}$ is essentially to determine the topological properties of $\mathcal{M}_\infty$. In [7, p. 107], the *multi-loop space* $\mathcal{M}$ was defined to be the sum of the n-loop spaces $\mathcal{M}_n$, for $n < \infty$. However, in this thesis, the $\aleph_0$-loop space $\mathcal{M}_\infty$ will be included because in the following chapter, an exact relation between a subset of $\aleph_0$-loops and the space of Riemannian 3-geometries is shown to exist; this in turn suggests that there might be functionals on $\mathcal{M}_\infty$ which are physical states of gravity in the loop representation. So, following [7, p. 107], define $\mathcal{M}$ to be the topological sum of $\mathcal{M}_n$'s; that is, $\mathcal{M} \stackrel{\text{def}}{=} \bigoplus_{n=1}^\infty \mathcal{M}_n \oplus \mathcal{M}_\infty$, where relative to the sum topology, $D \subset \mathcal{M}$ is open *iff* $D \cap \mathcal{M}_n$ is open in $\mathcal{M}_n \ \forall n$. Since the (countable) sum operation preserves metrizability and second countability [5, Theorem 4.2.1, p. 258], $\mathcal{M}$ is again metrizable and second countable.

3.2.12. Remark. Rayner [6, §2, p. 652] outlined two alternative ways of constructing the multi-loop space; however, the constant loops were excluded from the construction and no topologies were specified. Moreover, the two multi-loop spaces constructed were not equivalent nor indeed do they coincide as sets. And on a slightly different note, it is of interest to note that Di Bartolo *et al.* [4] introduced a set of coordinates on the space of non-parametric loops—*i.e.*, the space of equivalence classes of closed oriented paths. Consult reference [4] for more details. They then showed that the space admits an infinite-dimensional manifold structure. However, it should be pointed out that the space they introduced is different from $\tilde{\mathcal{L}}_\Sigma$ constructed above; moreover, the non-parametric loop space admits a group structure whereas $\tilde{\mathcal{L}}_\Sigma$ does not.

3.3. Manifold Structure of the Loop Space

In this short section, the manifold structure of $\tilde{\mathcal{L}}_\Sigma$ will be briefly sketched and the failure of $\mathcal{M}_n$, for each n, to admit differentiable structures will be clarified. Let $(T\Sigma, p_\Sigma, \Sigma)$ denote the tangent bundle over Σ and consider a real vector space $T(\gamma)$, for each $\gamma \in \tilde{\mathcal{L}}_\Sigma$, defined by $T(\gamma) \stackrel{\text{def}}{=} \{\, u : I \to T\Sigma \mid p_\Sigma \circ u = \gamma \,\}$. Observe trivially that as p_Σ is smooth, u must be piecewise smooth. Now, define $\|u\|_\gamma$ by

$$\|u\|_\gamma \stackrel{\text{def}}{=} \sup_{t\in I} \|u(t)\|_{\gamma(t)} + \operatorname{ess}\ \sup_{k\geqq 1,\, t\in I} \|u^{(k)}(t)\|.$$

Notice that $\operatorname{ess}\ \sup_{k\geqq 1,\, t\in I} \|u^{(k)}(t)\| < \infty$ follows from the construction of $\tilde{\mathcal{L}}_\Sigma$.

3.3.1. Proposition. *$(T(\gamma), \|\cdot\|_\gamma)$ is a Banach space.*

Proof. It will suffice to show that $(T(\gamma), \|\cdot\|_\gamma)$ is complete. So, suppose that $\{u_n\}_n$ is a sequence in $T(\gamma)$ such that $\sum_n \|u_n\|_\gamma < \infty$. By definition, $\exists\, J_n \subset I$ for each u_n such that J_n is countably linearly ordered. In particular, $J = \bigcup_n J_n$ is countably linearly ordered. Hence, $\sum_{n=1}^\infty u_n$ is smooth on $I - J$ by construction. Since

$$\begin{aligned}
\left\| \sum_n u_n \right\|_\gamma &\stackrel{\text{def}}{=} \sup_{t\in I} \left\| \sum_n u_n(t) \right\|_{\gamma(t)} + \operatorname{ess}\ \sup_{k\geqq 1, t\in I} \left\| \left(\sum_n u_n(t) \right)^{(k)} \right\| \\
&\leqq \sum_n \sup_{t\in I} \|u_n(t)\|_{\gamma(t)} + \sum_n \operatorname{ess}\ \sup_{k\geqq 1, t\in I} \|u_n^{(k)}(t)\| \\
&\equiv \sum_n \left(\sup_{t\in I} \|u_n(t)\|_{\gamma(t)} + \operatorname{ess}\ \sup_{k\geqq 1, t\in I} \|u_n^{(k)}(t)\| \right) \\
&= \sum_n \|u_n\|_\gamma < \infty
\end{aligned}$$

by assumption, it follows at once that $u \in T(\gamma)$ and $(T(\gamma), \|\cdot\|_\gamma)$ is thus complete. □

3.3.2. Theorem. *$\tilde{\mathcal{L}}_\Sigma$ is a Banach manifold.*

Proof (Sketch). Fix $\gamma \in \tilde{\mathcal{L}}_\Sigma$ and set $x_t = \gamma(t)$. Consider $\lambda_t > 0$ such that $\exp_{x_t} : B_{\lambda_t}(0) \cong B_{\lambda_t}(x_t)$ for each $t \in I$, where exp is the exponential map $\exp : T\Sigma \to \Sigma$. Set $\lambda_\gamma = \inf\{\, \lambda_t \mid t \in I \,\}$, and consider a λ_γ-open ball $B_{\lambda_\gamma}(0) \subset T(\gamma)$ and a map $\psi_\gamma : B_{\lambda_\gamma}(0) \to \tilde{\mathcal{L}}_\Sigma$ defined by

$$u \mapsto \exp_\gamma^{-1}(u),$$

where $\eta(t) \stackrel{\text{def}}{=} \exp_{\gamma(t)}^{-1}(u(t))$ is a piecewise smooth loop. Then, $D_{\lambda_\gamma}(\gamma) \stackrel{\text{def}}{=} \psi_\gamma(B_{\lambda_\gamma}(0))$ is open. To verify this, suppose that $D_{\lambda_\gamma}(\gamma)$ is not open. Then, $\exists\, \eta \in D_{\lambda_\gamma}(\gamma)$, where $u \in B_{\lambda_\gamma}(0)$ and $\eta = \exp_\gamma(u)$, such that for any $\delta > 0$, $B_\delta(\eta) \not\subset D_{\lambda_\gamma}(\gamma)$. In particular, for any decreasing sequence $\{\delta_n\}_n$ with $\delta_n > 0$ and $\delta_n \to 0$ as $n \to \infty$, $\exists\, \tau_n \in B_{\delta_n}(\eta)$ such that $\|v_n\|_\gamma > \lambda_\gamma$, where $v_n \stackrel{\text{def}}{=} \psi_\gamma^{-1}(\tau_n)$. However, for each $n > 0$, $\tau_n \in B_{\delta_n}(\eta)$ implies that $\operatorname{ess\,sup}_{k \geqq 1, t \in I} \|D^k\eta(t) - D^k\tau_n(t)\| = \operatorname{ess\,sup}_{k \geqq 1, t \in I} \|D^k \exp_\gamma(u - v_n)\| < \delta_n$ for each n. This in turn implies that $\exists\, N > 0$ large enough such that $\operatorname{ess\,sup}_{k \geqq 1, t \in I} \|u^{(k)}(t) - v_n^{(k)}(t)\| < \varepsilon\ \forall n > N$, where $\varepsilon = \frac{1}{2}(\lambda_\gamma - \|u\|_\gamma)$, and hence, for $n > 0$ sufficiently large, $\|v_n\|_\gamma < \lambda_\gamma$ which contradicts the assumption. Hence, for each $\eta \in D_{\lambda_\gamma}(\gamma)$, $\exists\, \delta > 0$ such that $B_\delta(\eta) \subset D_{\lambda_\gamma}(\gamma)$, and $D_{\lambda_\gamma}(\gamma)$ is thus open. Finally, it is easy to see that ψ_γ is smooth and it maps $B_{\lambda_\gamma}(0)$ bijectively onto $D_{\lambda_\gamma}(\gamma)$. $\square$

It is rather unfortunate that $\mathcal{L}_\Sigma$ is not a manifold, for 0_Σ does not possess a neighbourhood homeomorphic to any open neighbourhood about the 0 of any topological vector space. Thus, $\mathcal{L}_\Sigma$ is a manifold with a *cusp* at 0_Σ. In particular, the spaces $\mathcal{M}_n$ and $\mathcal{M}$ do not possess manifold structures. However, it is not very difficult to verify that for each $n \in \mathbb{N}$, $\mathcal{M}_n$ is the union of an open subset which admits a manifold structure and a nowhere dense subset which is not a manifold. In spite of this setback, the concept of derivatives can still be defined on the multi-loop spaces. Preliminary attempts will be delineated in §B of the Appendix. The primary motivation here being the desire to express the loop Hamiltonian constraint in terms of derivatives defined on the loop space in order to gain a deeper insight into the loop Hamiltonian constraint. It should however be noted here that various authors [2, 3] have attempted to find alternative expressions for the quantum loop constraints. Attention should be drawn in particular to Blencowe [2] who derived a generalised loop Hamiltonian constraint so that the resulting Hamiltonian operator is defined on loop functionals that do not vanish on self-intersecting loops.

3.4. The Multi-loop Functionals

The aim of this section is to provide the necessary foundational work for the determination of the Hermitian conjugates of the T-operators. Note in hindsight—*cf.* §5.4 or reference [6, p. 656]—that the explicit form of a physical inner product is surprisingly not needed for the determination of the Hermitian conjugates of the T-operators. The Hermitian conjugates of the T^n-operators for $n \geqq 1$ will not be covered in this section; they will be considered in §5.4 instead.

From here on, given a functional Ψ on the n-loop space, the unwieldy notation $\Psi[\{\gamma^1,\ldots,\gamma^n\}]$ will be written as $\Psi[\gamma^1,\ldots,\gamma^n]$ for simplicity. Let $\mathcal{M}_1' \stackrel{\text{def}}{=} \{\psi : \mathcal{M}_1 \to \mathbb{C} \mid \psi \text{ continuous}\}$ be the topological dual of $\mathcal{M}_1$. Then, $\mathcal{M}_1'$ can be trivially made into a $\mathbb{C}$-linear space in the following manner. Given any $\gamma \in \mathcal{M}_1$ and $\psi, \phi \in C_1$, define addition and scalar multiplication on C_1 by

(1) $(\psi + \phi)[\gamma] \equiv \psi[\gamma] + \phi[\gamma]$,
(2) $\theta[\gamma] = 0$, where θ is the zero functional on $\mathcal{M}_1$,
(3) $(c\psi)[\gamma] = c\psi[\gamma], \forall\, c \in \mathbb{C}$.

Note also that $\mathcal{M}_1'$ possesses a natural algebraic structure defined by the multiplication $\cdot$ as follows:

$$\psi \cdot \phi[\gamma] \equiv \psi[\gamma]\phi[\gamma], \forall\, \gamma \in \mathcal{M}_1,\ \psi, \phi \in \mathcal{M}_1'.$$

A subset $\mathcal{M}_2^* \subset \mathcal{M}_2'$ of 2-loop functional will be constructed via a symmetric tensor $\otimes^{\mathcal{M}}$ operation to be defined below. Let $\psi, \phi \in \mathcal{M}_1'$ and define the usual symmetric tensor product $\psi \otimes \phi$ of ψ and ϕ by

$$(\psi \otimes \phi)[\{\gamma^1,\gamma^2\}] \equiv \tfrac{1}{2}(\psi[\gamma^1]\phi[\gamma^2] + \phi[\gamma^1]\psi[\gamma^2]),$$

where $\gamma^1, \gamma^2 \in \mathcal{M}_1$. In particular, $(\psi \otimes \psi)[\{\gamma^1,\gamma^2\}] = \psi[\gamma^1]\psi[\gamma^2]$. Thus, $\psi \otimes \phi$ is a continuous functional on the Cartesian product $\mathcal{M}_1 \times \mathcal{M}_1$. Unfortunately, it is strictly not a functional on $\mathcal{M}_2$ simply because the quantity $\Psi[\gamma,\gamma]$ is not defined for a given functional Ψ of $\mathcal{M}_2$ (as $\{\gamma,\gamma\} \equiv \{\gamma\} \notin \mathcal{M}_2$). To rectify this problem, consider the map $\hat{\epsilon}_2 : \mathcal{M}_1 \times \mathcal{M}_1 \to \{0,1\}$ given by

$$\hat{\epsilon}_2[\gamma,\eta] \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \gamma \neq \eta, \\ 0 & \text{if } \gamma = \eta. \end{cases}$$

From this, it is clear that if $\Psi \in (\mathcal{M}_1 \times \mathcal{M}_1)'$, then $\hat{\epsilon}_2 \cdot \Psi \in \mathcal{M}_2'$. This motivates the following definition of an $\mathcal{M}$-*tensor* $\otimes^{\mathcal{M}}$: given a pair of loop functionals $\psi, \phi \in \mathcal{M}_1'$,

$$\psi \otimes^{\mathcal{M}} \phi \stackrel{\text{def}}{=} \hat{\epsilon}_1 \wedge \hat{\epsilon} \cdot \psi \otimes \phi,$$

where $\hat{\epsilon}_1 \wedge \hat{\epsilon}_1 \stackrel{\text{def}}{=} \hat{\epsilon}_2$. In general, define $\psi_1 \otimes^{\mathcal{M}} \ldots \otimes^{\mathcal{M}} \psi_n \stackrel{\text{def}}{=} \overbrace{\hat{\epsilon}_1 \wedge \cdots \wedge \hat{\epsilon}_1}^{n \text{ times}} \cdot \psi_1 \otimes \cdots \otimes \psi_n$, where

$$\overbrace{\hat{\epsilon}_1 \wedge \cdots \wedge \hat{\epsilon}_1}^{n \text{ times}}[\gamma^1,\ldots,\gamma^n] \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \gamma^i \neq \gamma^j \ \forall\, i \neq j, \\ 0 & \text{otherwise}, \end{cases}$$

with $i, j = 1, \ldots, n$. Denote $\hat{\epsilon}_1 \wedge \cdots \wedge \hat{\epsilon}_1$ by $\hat{\epsilon}_n$. Furthermore, note out of interest that $\hat{\epsilon}_n$ can be decomposed into $\hat{\epsilon}_2$'s as follows:

$$\hat{\epsilon}_n[\gamma^1,\ldots,\gamma^n] = \hat{\epsilon}_2[\gamma^1,\gamma^2]\ldots\hat{\epsilon}_2[\gamma^1,\gamma^n]\hat{\epsilon}_2[\gamma^2,\gamma^3]\ldots\hat{\epsilon}_2[\gamma^2,\gamma^n]\ldots\hat{\epsilon}_2[\gamma^{n-1},\gamma^n].$$

Now, define $\hat{\mathcal{M}}'_2$ to be the space spanned by elements of the form $\psi \otimes^{\mathcal{M}} \phi$, where $\psi, \phi \in \mathcal{M}'_1$. That is to say,

$$\hat{\mathcal{M}}'_2 \stackrel{\text{def}}{=} \left\{ \sum_{i_1, i_2} a_{i_1 i_2} \psi_{i_1} \otimes^{\mathcal{M}} \psi_{i_2} \,\middle|\, a_{i_1 i_2} \in \mathbb{C},\ \psi_{i_1}, \psi_{i_2} \in \mathcal{M}'_1 \right\},$$

where $a_{i_1 i_2} = 0$ for all but a *finite* number of $i_1, i_2 \in \mathbb{N}$.

Then, functionals belonging to the space $\hat{\mathcal{M}}^*_2$ defined by

$$\hat{\mathcal{M}}^*_2 \stackrel{\text{def}}{=} \left\{ \sum_{n=1}^{\infty} c_n \tilde{\Psi}_n \in \mathcal{M}'_2 \,\middle|\, c_n \in \mathbb{C},\ \tilde{\Psi}_n \in \hat{\mathcal{M}}'_2\ \forall n \in \mathbb{N} \right\}$$

will be considered below. It is highly unlikely that this space will coincide with the entire space $\mathcal{M}'_2$ of all continuous functionals on $\mathcal{M}_2$. However, for the purpose of this section, this fact will not matter.

It should thus be obvious by now that $\hat{\mathcal{M}}^*_n$ can be defined inductively from $\mathcal{M}'_1$'s. First, note trivially that for each functional $\Psi_n \in (\mathcal{M}^n_1)'$, $\hat{\epsilon}_n \cdot \Psi_n \in \mathcal{M}'_n$. Hence, let

$$\hat{\mathcal{M}}'_n \stackrel{\text{def}}{=} \left\{ \sum_{i_1, \ldots, i_n} a_{i_1 \ldots i_n} \psi_{i_1} \otimes^{\mathcal{M}} \cdots \otimes^{\mathcal{M}} \psi_{i_n} \,\middle|\, a_{i_1 \ldots i_n} \in \mathbb{C},\ \psi_{i_j} \in \mathcal{M}'_1 \right\},$$

where $a_{i_1 \ldots i_n} = 0$ for all but a finite number of $i_1, \ldots, i_n \in \mathbb{N}$. Here, the n symmetric tensors of ψ_{i_j}'s are defined in the usual way:

$$(\psi_{i_1} \otimes \cdots \otimes \psi_{i_n})[\{\gamma^1, \ldots, \gamma^n\}] = \frac{1}{n!} \sum_{\sigma} \psi_{\sigma(i_1)}[\gamma^1] \cdots \psi_{\sigma(i_n)}[\gamma^n],$$

where $\sum_\sigma$ is the sum over all permutations σ of $\{1, \ldots, n\}$ for each $n > 0$. In this way, one arrives at the definition of the space $\hat{\mathcal{M}}^*_n$:

$$\hat{\mathcal{M}}^*_n \stackrel{\text{def}}{=} \left\{ \sum_{k=1}^{\infty} c_k \Psi_k \in \mathcal{M}'_n \,\middle|\, c_k \in \mathbb{C},\ \Psi_k \in \hat{\mathcal{M}}'_n\ \forall k \in \mathbb{N} \right\}.$$

Note in passing that the spaces $\hat{\mathcal{M}}^*_n$ could be restricted further by considering only functionals that satisfy the supremum norm

$$\|\Psi\|_n \stackrel{\text{def}}{=} \sup\{\, |\Psi[\gamma]| : \gamma \in \mathcal{M}_n \,\}$$

so that they become Banach algebras under the uniform norm topology induced by $\|\cdot\|_n$. However, this will not be done here as it will not have any physical significance when it comes to singling out the relevant subspaces of physical interest. As such, no further restrictions will be placed on $\hat{\mathcal{M}}_n^*$ other than what is already imposed on it. And indeed, for the sake of simplicity, attention will be restricted to $\hat{\mathcal{M}}_n^*$'s instead of $\mathcal{M}_n'$'s in all that follows.

For completeness, the space $\hat{\mathcal{M}}_\infty^*$ will be constructed although it will not be used in §5.4. Given a finite set of loop functionals $\psi_i \in \mathcal{M}_1'$, for $i = 1, \dots, n$, define $\psi_1 \hat{\oplus} \dots \hat{\oplus} \psi_n$ on $\mathcal{M}_1^n$ by

$$(\psi_1 \hat{\oplus} \dots \hat{\oplus} \psi_n)[\gamma^1, \dots, \gamma^n] \stackrel{\text{def}}{=} \psi_1[\gamma^1] + \cdots + \psi_n[\gamma^n].$$

Observe that $\psi_1 \hat{\oplus} \dots \hat{\oplus} \psi_n$ is not, in general, a symmetric functional on $\mathcal{M}_1^n$. Let $\Psi = \{ \psi_j \in \mathcal{M}_1' \mid j \in \mathbb{N} \}$ be a denumerable set of loop functionals that satisfies the following two conditions:

(1) $\lim\limits_{n \to \infty} (\ln \psi_{\sigma(1)} \hat{\oplus} \dots \hat{\oplus} \ln \psi_{\sigma(n)})$ is well-defined on $\mathcal{M}_1^\infty$ for each $\sigma \in \Lambda_\Psi$,

(2) for each fixed $\gamma = (\gamma^n)_{n=1}^\infty \in \mathcal{M}_\infty$, the set $\{ |\widehat{\bigoplus}_{n=1}^\infty \ln(\psi_{\sigma(n)}[\gamma^n])| : \sigma \in \Lambda_\Psi \}$ of numbers is bounded by some constant $K_\gamma > 0$,

where Λ_Ψ is the set of permutations σ of the set Ψ and $\sigma(\psi_n)$ is denoted by $\psi_{\sigma(n)}$ for notational clarity.

Condition (1) guarantees that for each $\gamma \in \mathcal{M}_\infty$ and $\sigma \in \Lambda_\Psi$, the sum $\lim\limits_{n \to \infty} (\psi_{\sigma(1)}[\gamma^1] + \cdots + \psi_{\sigma(n)}[\gamma^n]) < \infty$ and condition (2) ensures formally at least that the infinite tensor product $\lim\limits_{n \to \infty} (\psi_1 \otimes^{\mathcal{M}} \dots \otimes^{\mathcal{M}} \psi_n)[\gamma] < \infty$ for each fixed $\gamma \in \mathcal{M}_\infty$. Continuing with this informal construction, let $\mathcal{F}$ denote the set of infinite $\mathcal{M}$-tensor products of the elements of Ψ's, where each set Ψ satisfies criteria (1) and (2). The space $\hat{\mathcal{M}}_\infty^*$ can now be defined as

$$\hat{\mathcal{M}}_\infty^* \stackrel{\text{def}}{=} \left\{ \sum_{n=1}^\infty c_n \cdot \Psi_n \in \mathcal{M}_\infty' \,\middle|\, c_n \in \mathbb{C},\ \Psi_n \in \mathcal{F}\ \forall n \in \mathbb{N} \right\},$$

Finally, extend the definition of $\Psi \in \hat{\mathcal{M}}_n^*$ for each n so that $\Psi[\gamma] \equiv 0\ \forall \gamma \in \mathcal{M}_k$ whenever $k \neq n$, and define $\hat{\mathcal{M}}^* = \bigoplus_n \hat{\mathcal{M}}_n^* \oplus \hat{\mathcal{M}}_\infty^*$ to be the direct sum of $\hat{\mathcal{M}}_n^*$'s. Hence, unconstraint gravitational states in the loop representation are just direct sums of n-loop functionals: $\Psi = \sum_n a_n \Psi_n$, where $a_n = 0$ for all but a finite set of a_n's and $\Psi_n \in \hat{\mathcal{M}}_n^*$ for each n.

3.5. Discussion

This chapter will close with a brief discussion on the quantum T^0-operator $\hat{T}[\gamma, A]$. Rovelli and Smolin defined the T^0-operator to act on a multi-loop functional Ψ by $\hat{T}[\gamma, A]\Psi \stackrel{\text{def}}{=} \Psi[\gamma, \cdot\,]$. The significance of this definition will be elaborated somewhat below.

An intuitive grasp of the definition of the quantum T^0-operator can be found in a paper on the loop representation of the quantum Maxwell field by Ashtekar and Rovelli [1, §3.3, p. 1134]. That having been said, this operator will be exmined in greater depth. Recall that an n-loop functional Ψ_n is a totally symmetric functional of the form $\Psi_n = \Psi_n[\gamma^1, \ldots, \gamma^n]$, where $\{\gamma^1, \ldots, \gamma^n\} \in \mathcal{M}_n$. This merely follows from the definition of $\mathcal{M}_n$ which is essentially the set of all subsets of n loops. However, this also means in particular that $\Psi[\gamma^1, \ldots, \gamma^n]$ is *not* defined whenever $\gamma^i = \gamma^j$ for some $i \neq j$; that is, if $\{\gamma^1, \ldots, \gamma^n\} = \{\gamma^{i_1}, \ldots, \gamma^{i_{n-1}}\}$, for instance. Hence, strictly, the map $\hat{T}[\gamma^1, A] : \mathcal{M}'_n \to \mathcal{M}'_{n-1}$ should be defined by

$$(3.5.1) \qquad (\hat{T}[\gamma^1, A]\Psi_n)[\eta] = \begin{cases} \Psi_n[\gamma^1, \eta] & \text{if } \gamma^1 \notin \eta, \\ 0 & \text{if } \gamma^1 \in \eta, \end{cases}$$

where $\gamma^1 \in \mathcal{M}_1$, $\Psi_n \in \mathcal{M}'_n$ is an n-loop functional and $\eta \in \mathcal{M}_{n-1}$.

In this section, $\hat{T}^0$ will be restricted to the spaces $\hat{\mathcal{M}}^*_n$ constructed in the previous section. Explicitly, if $\Phi = \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}\phi_{i_1} \otimes^{\mathcal{M}} \ldots \otimes^{\mathcal{M}} \phi_{i_n}$, then $\hat{T}[\gamma, A]\Phi$ is

$$\begin{aligned}
\hat{T}[\gamma, A]\Phi &= \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}\hat{T}[\gamma, A]\phi_{i_1} \otimes^{\mathcal{M}} \ldots \otimes^{\mathcal{M}} \phi_{i_n} \\
&= \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}\frac{1}{n!}\hat{T}[\gamma, A]\sum_{\sigma}\hat{\epsilon}_n \cdot \phi_{\sigma(i_1)}\cdots\phi_{\sigma(i_n)} \\
&= \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}\frac{1}{n!}\Bigg\{\phi_{i_1}[\gamma]\sum_{\sigma_1}\hat{\epsilon}_{n-1}\cdot\phi_{\sigma_1(i_2)}\cdots\phi_{\sigma_1(i_n)} + \cdots + \\
&\qquad\qquad \phi_{i_n}[\gamma]\sum_{\sigma_n}\hat{\epsilon}_{n-1}\cdot\phi_{\sigma_n(i_1)}\cdots\phi_{\sigma_n(i_{n-1})}\Bigg\},
\end{aligned}$$

where σ_k is the permutation of the $(n-1)$-element set $\{\, i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n \,\}$. Thus,

$$\begin{aligned}
\hat{T}[\gamma]\Phi &= \sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}\frac{1}{n}\sum_{k=1}^{n}\phi_{i_k}[\gamma]\frac{1}{(n-1)!}\sum_{\sigma_k}\hat{\epsilon}_{n-1}\cdot\phi_{\sigma_k(i_{k,1})}\cdots\phi_{\sigma_k(i_{k,n-1})} \\
&= \frac{1}{n}\sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}\sum_{k=1}^{n}\phi_{i_k}[\gamma]\phi_{i_{k,1}} \otimes^{\mathcal{M}} \ldots \otimes^{\mathcal{M}} \phi_{i_{k,n-1}},
\end{aligned}$$

with $\{\, i_{k,1}, \ldots, i_{k,n-1} \,\} = \{\, i_1, \ldots, i_n \,\} - \{i_k\}$.

It should be noted that for each pair (γ, A), the $\mathbb{C}$-linear map $\hat{T}[\gamma, A] : \hat{\mathcal{M}}^*_n \to \hat{\mathcal{M}}^*_{n-1}$ is neither an epimorphism nor a monomorphism. The fact that $\hat{T}[\gamma, A]$ is not an epimorphism follows immediately from (3.5.1): any $\Psi_{n-1} \in \hat{\mathcal{M}}^*_{n-1}$ such that Ψ_{n-1} is nowhere zero on $\mathcal{M}_{n-1}$ cannot be of the form $\hat{T}[\gamma, A]\Psi_n$ for any pair

(γ, A) and $\Psi_n \in \hat{\mathcal{M}}_n^*$. That $\hat{T}[\gamma, A]$ is not a monomorphism is also obvious from its definition: consider two distinct n-loop functionals Ψ_n and Φ_n such that they coincide on the subset consisting of all loops of the form $\{\gamma\} \cup \eta \;\forall \eta \in \mathcal{M}_{n-1}$, where $\gamma \in \mathcal{M}_1$ is fixed, but differ elsewhere on $\mathcal{M}_n$. Then, by definition, $\Psi_n \neq \Phi_n$ although $\hat{T}[\gamma, A]\Psi_n = \hat{T}[\gamma, A]\Phi_n$.

This section will conclude with a small remark on the eigenvalue equation[2]

$$(\hat{T}[\gamma]\Psi_i)[\{\eta^1, \ldots, \eta^{i-1}\}] = c_i[\gamma]\Psi_{i-1}[\{\eta^1, \ldots, \eta^{i-1}\}]$$

—here, meant in the sense that for some $\Psi \in \hat{\mathcal{M}}^*$, $\Psi|\mathcal{M}_n \equiv \Psi_n \;\forall n > 0$—can only be solved by elements in $\hat{\mathcal{M}}_n^*$ of the form

$$\Psi_n = \psi \otimes^{\mathcal{M}} \ldots \otimes^{\mathcal{M}} \psi \text{ (n factors)}, \tag{3.5.2}$$

for some $\psi \in \mathcal{M}_1^*$. See Equation (94) of [7, p. 124]. In other words, the present formulation of $\hat{\mathcal{M}}^*$ is entirely consistent with the formal considerations given in [7]. In summary, is clear from the brief discussion above that any multi-loop functional of the form given by Equation (3.5.1) is an "eigenstate" of the T^0-operator.

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[2]See [7, Eqns (81) and (93)]. This terminology stems perhaps from the slight notational abuse: to wit, $\hat{T}[\gamma]\Psi_i[\{\eta\}] \equiv \Psi_i[\gamma \cup \{\eta\}] = c_i[\gamma]\Psi_i[\{\eta\}]$ to use the notations of [7, Eqn (93)] (where $\{\eta\}$ denotes a collection of loops). This notation may cause a little confusion as the domains of $\Psi_i \in \hat{\mathcal{M}}_i^*$ and $\Psi_{i-1} \in \hat{\mathcal{M}}_{i-1}^*$ have empty intersection by definition: $\mathcal{M}_n \cap \mathcal{M}_k = \varnothing \;\forall n \neq k$.

CHAPTER IV

KNOTS AND CLASSICAL 3-GEOMETRIES

4.1. Introduction

In reference [5, p. 1661], Rovelli sketched a proof showing how a special collection of n-loops[1] which he called *weaves*, are related to the flat 3-metric. He then made a rather fascinating conjecture that perhaps there exists a relationship between n-loops, for $n < \infty$, and 3-metrics. The relationship between n-loops, for $n < \infty$, and 3-metrics will not be solved in this chapter (and so, it may still remain an open question to date); however, what will be established in this chapter is that there exists a precise relationship between 3-geometries and a subset of $\aleph_0$-knots, where an n-*knot* is defined to be an equivalence class of n-loops under (smooth) ambient isotopies—*cf.* §4.3. The approach given here is rather different from that outlined by Rovelli [5].

To delineate Rovelli's construction in brief, he set out to define a map from the graviton states derived from the linearised theory [1] into the knot states of the full theory [6]. This entailed the introduction of a lattice spacing—the distance between parallel non-intersecting curves—on the 3-manifold to define the weaves. He then made the following ansatz: suppose that ψ is a graviton state of the linearised loop representation and Ψ is a knot state of the full theory. Then, $\psi[0_\Sigma, 0_\Sigma, 0_\Sigma] = \Psi[\Delta]$ whenever $\psi = \mathcal{M}\Psi$, where $\mathcal{M}$ is a map [5, p. 1658] defined up to first order in the Gravitational constant G that relates Ψ to ψ, Δ is a weave and 0_Σ is the zero loop. In this way, after some effort, he obtained his conclusion regarding weaves and flat metrics; this, in turn, motivated his conjecture. Here, no assumptions of a lattice spacing will be made and the results are thus purely "topological". The work given here is based on reference [7].

The attention here will be focused on a compact, Riemannian 3-manifold. The fact that such a 3-manifold is separable is crucial in the construction: this, at least, explains why $\aleph_0$-loops are used rather than n-loops for $n < \infty$. The main interest

[1] Recall that an n-*loop* $\gamma \stackrel{\text{def}}{=} \{\gamma^1, \ldots, \gamma^n\}$ is just a subset of the loop space consisting of n loops; *i.e.*, γ^i, for each $i = 1, \ldots, n$, are (distinct) closed curves in Σ.

in Rovelli's conjecture is that it will provide a tentative physical interpretation of the loop representation of quantum gravity [6]: it yields a possible insight into the interweaving of topology and geometry at the quantum level. More will be said in §4.5.

Some notations introduced in previous chapters will be briefly recalled again. In all that follows, the spatial (Riemannian) 3-manifold, denoted by Σ, is assumed to be smooth, orientable, closed and compact; $\mathbb{R}_+ \stackrel{\text{def}}{=} \{ s \in \mathbb{R} \mid s \geqq 0 \}$ and $I \stackrel{\text{def}}{=} [0,1]$. Lastly, let $\text{Diff}^+(\Sigma)$ denote the group of smooth, orientation-preserving, diffeomorphisms on Σ. An overview of this chapter runs as follows: in section 2, the required notations and definitions will be introduced and in section 3, the property of the space of $\aleph_0$-knots of a subset of $\aleph_0$-loops will be examined. This space will establish the sought for correspondence between topology and geometry. In section 4, a variant of the Rovelli conjecture will be formulated precisely and then verified. Finally, in section 5, some speculations—which hopefully will prove to be illuminating rather than be a gross caricature of reality—regarding the results established in section 4 will be outlined.

4.2. Preliminary Definitions and Notations

The space $\tilde{\mathcal{L}}_\Sigma$ of piecewise smooth loops defined in §3.2 will be considered here instead of its quotient space $\mathcal{L}_\Sigma$. As a brief reminder, $\tilde{\mathcal{L}}_\Sigma$ is endowed with a topology induced by a metric $d_\Omega : \tilde{\mathcal{L}}_\Sigma \times \tilde{\mathcal{L}}_\Sigma \to \mathbb{R}_+$ defined by

$$d_\Omega(\gamma, \eta) \stackrel{\text{def}}{=} \sup_{t \in I} \hat{d}(\gamma(t), \eta(t)),$$

where $\hat{d}$ is a distance function on Σ compatible with its topology.

The concept of a q-geodesic, where q is an admissible Riemannian metric on Σ, will now be introduced. These curves will provide a basis for constructing a countably infinite set of loops that relates to the geometry of Σ. The motivation springs from the observation that because Σ is a separable metric space, its geometry can be reproduced by wisely choosing a countable set of loops such that the closure of the union of their images is precisely the underlying 3-manifold Σ.

A curve in Σ is said to be a *q-geodesic* if it is a geodesic in Σ relative to the Riemannian metric q. Also, if γ, η are curves such that $\gamma(1) = \eta(0)$, then define $\gamma * \eta$ by

$$\gamma * \eta(t) \stackrel{\text{def}}{=} \begin{cases} \gamma(2t) & \text{for } 0 \leqq t \leqq \frac{1}{2}, \\ \eta(2t-1) & \text{for } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

4.2.1. Definition. Let $\gamma \in \tilde{\mathcal{L}}_\Sigma$. Then, γ is said to be a *piecewise geodesic loop* if there exists a Riemannian metric q on Σ and n smooth q-geodesics $\gamma_1, \ldots, \gamma_n : I \to \Sigma$, $1 \leqq n < \infty$, such that $\gamma = \gamma_1 * \cdots * \gamma_n$.[2]

Let Γ_2^+ denote the space of (smooth) Riemannian metrics on Σ (endowed with the compact C^∞-topology[3]) and $D_\Sigma \subset \Sigma$ a countably dense subset of Σ. Now, define $\tilde{\mathcal{M}}_\infty[q]$, for each $q \in \Gamma_2^+$, to be the set of all countably infinite multi-loops $\gamma = \{\gamma^i \mid i \in \mathbb{N}\}$ satisfying the following two properties:

(1) for each i, $\gamma^i \in \tilde{\mathcal{L}}_\Sigma$ is a piecewise (affinely parametrised) q-geodesic loop in Σ,

(2) the subset γ is in bijective correspondence with D_Σ under the map $\gamma^i \mapsto \gamma^i(0)$.

Finally, set $\tilde{\mathcal{M}}_\infty[\Gamma_2^+] = \bigcup_{q \in \Gamma_2^+} \tilde{\mathcal{M}}_\infty[q]$. An immediate consequence of the definition is the following two observations. Suppose $\gamma \in \tilde{\mathcal{M}}_\infty[q] \cap \tilde{\mathcal{M}}_\infty[q']$. Let $\Gamma(q)$ and $\Gamma(q')$ be the Riemannian connections of q and q' respectively. Fix an admissible atlas $\{(U_\alpha, \psi_\alpha)\}_\alpha$ on Σ. Then, with respect to each chart U_α,

$$(\ddot{\gamma}^i_\alpha)^\ell + \Gamma_\alpha(q)^\ell_{kj}(\dot{\gamma}^i_\alpha)^k(\dot{\gamma}^i_\alpha)^j \overset{\text{a.e.}}{=} 0 \text{ and } (\ddot{\gamma}^i_\alpha)^\ell + \Gamma_\alpha(q')^\ell_{kj}(\dot{\gamma}^i_\alpha)^k(\dot{\gamma}^i_\alpha)^j \overset{\text{a.e.}}{=} 0$$

on $\gamma^i(I) \cap U_\alpha$ for each i (no summation over α, obviously), where $F(t) \overset{\text{a.e.}}{=} 0$ means $F(t) = 0$ on I *apart* from a *finite* number of points in I. Hence, $(\Gamma_\alpha(q)^\ell_{kj} - \Gamma_\alpha(q')^\ell_{kj})(\dot{\gamma}^i_\alpha)^k(\dot{\gamma}^i_\alpha)^j \overset{\text{a.e.}}{=} 0 \;\forall \gamma^i \in \gamma$ and α. Thus, by property (2), $\Gamma(q)^\ell_{kj}(x) \equiv \Gamma(q')^\ell_{kj}(x)$ on a dense subset of Σ simply because $\overline{\bigcup\{\gamma^i(I) \mid \gamma^i \in \gamma\}} \equiv \Sigma$ by property (2). Hence, invoking the continuity of $\Gamma(h)$ for $h = q, q'$, it follows at once that $\Gamma(q) \equiv \Gamma(q')$ on Σ. Now, with respect to a local coordinate basis, $\Gamma(q)^i_{kj} = \frac{1}{2}q^{ih}(\partial_k q_{hj} + \partial_j q_{hk} - \partial_h q_{kj})$ (and likewise for q'); consequently, q and q' are related homothetically; that is, $\exists\, c > 0$ constant such that $q' = cq$.[4] More generally, q, q' are related by some coordinate transformation, as is shown below.

Let $f : \Sigma \to \Sigma$ be a smooth diffeomorphism, where $\Sigma = (\Sigma, q)$ and set $\Sigma_f = f(\Sigma) \overset{\text{def}}{=} (\Sigma, (f^{-1})^*q)$. Clearly, if $\gamma : I \to \Sigma$ is a q-geodesic, then $\gamma : I \to \Sigma_f$ is an $(f^{-1})^*q$-geodesic in Σ_f and conversely, by symmetry (as isometries map geodesics into geodesics). Hence, in view of these two observations, each $\gamma \in \tilde{\mathcal{M}}_\infty[\Gamma_2^+]$ is

[2] Note that each γ_i in γ is still a q-geodesic with respect to its new parametrisation $[\frac{i-1}{n}, \frac{i}{n}]$, as the reparametrisation $\gamma_i(t) \to \gamma_i(nt - i + 1) \equiv \gamma|[\frac{i-1}{n}, \frac{i}{n}]$ is clearly an affine transformation. Another important point to note is that the definition of "piecewise" given here is strictly finite; that is, $n < \aleph_0$ unlike the notion of "piecewise" introduced in Definition 3.2.2 wherein $n = \aleph_0$ is allowed.

[3] The compact C^∞-topology is defined in §A of the Appendix.

[4] Note trivially that as q, q' are positive-definite, $c < 0$ is not an admissible solution.

assigned to a unique 3-geometry of Σ, where the space of *3-geometries* is defined to be the quotient space $\mathcal{Q} \stackrel{\text{def}}{=} \Gamma_2^+/\operatorname{Diff}^+(\Sigma)$. Recall that each element $[q] \in \mathcal{Q}$ is defined by $[q] = \{ f^*q \mid f \in \operatorname{Diff}^+(\Sigma) \}$. Let $\pi_+ : \Gamma_2^+ \to \mathcal{Q}$ denote the natural projection. Then, π_+ is open [2, §3.1, p. 317] and $\mathcal{Q}$ is a second countable, metrizable space [2, Theorem 1, p. 318].

As a converse remark, notice that if Σ were not separable or that $\gamma_q = \{ \gamma_q^i \mid i \in \mathbb{N} \}$ were not chosen to satisfy property (2), γ_q need not uniquely determine $[q] \in \mathcal{Q}$. For want of a better term, call $\tilde{\mathcal{M}}_\infty[\Gamma_2^+]$ the space of *piecewise geodesic $\aleph_0$-loops*. A suitable topology can be defined on this space. This will be done below.

Let L_∞ be the set of affinely parametrised, piecewise geodesic loops in Σ and let L_Σ^∞ denote the countably infinite set-theoretic product of L_Σ. Define an equivalence relation $R_\Sigma \subset L_\Sigma^\infty \times L_\Sigma^\infty$ by

$$R_\Sigma \stackrel{\text{def}}{=} \{ (\gamma, \gamma') \subset L_\Sigma^\infty \times L_\Sigma^\infty : [\gamma] = [\gamma'] \},$$

where $[\eta] \stackrel{\text{def}}{=} \{ \eta^i \mid i \in \mathbb{N} \}$ is just the set of components of $\eta \stackrel{\text{def}}{=} (\eta^i)_{i=1}^\infty$. Let $\pi_\Sigma : L_\Sigma^\infty \to M_\Sigma \stackrel{\text{def}}{=} L_\Sigma^\infty / R_\Sigma$ denote the natural map. Then clearly, as a subset, $\tilde{\mathcal{M}}_\infty[\Gamma_2^+] \subset M_\Sigma$.

Now, let $M_\infty \subset L_\Sigma^\infty$ be a subset satisfying

(i) for each $\gamma \stackrel{\text{def}}{=} (\gamma^i)_{i=1}^\infty$, $\gamma^i \neq \gamma^j \ \forall\, i \neq j$,

(ii) $\pi_\Sigma(M_\infty) = \tilde{\mathcal{M}}_\infty[\Gamma_2^+]$.

It is clear from the definition of M_∞ that there exists a family of subsets $M_\sigma \subset M_\infty$ such that

(a) $M_\infty = \bigcup_\sigma M_\sigma$,

(b) $M_\sigma \cap M_{\sigma'} = \varnothing \ \forall\, \sigma \neq \sigma'$,

(c) $\pi_\Sigma | M_\sigma : M_\sigma \to \tilde{\mathcal{M}}_\infty[\Gamma_2^+]$ is a (set-theoretic) bijection.

Let $h_\sigma \stackrel{\text{def}}{=} \pi_\Sigma | M_\sigma$ and for each $\gamma \in \tilde{\mathcal{M}}_\infty[\Gamma_2^+]$, set $\gamma_\sigma = h_\sigma^{-1}(\gamma) \in M_\sigma$.[5] Now, fix a finite atlas $\mathfrak{A}$ on Σ. Then, for any pair $\gamma, \eta \in M_\sigma$, let $d_\sigma(\gamma, \eta) \stackrel{\text{def}}{=} \sup_i d_\Omega(\gamma^i, \eta^i) + \sup_i d'_\Omega(\gamma^i, \eta^i)$, where

$$d'_\Omega(\gamma^i, \eta^i) \stackrel{\text{def}}{=} \text{ess } \sup\{ \|D^k\gamma^i(t) - D^k\eta^i(t)\| : t \in I,\ k \geqq 1 \}$$

with sup running over all relevant (finite) charts $(U, \varphi) \in \mathfrak{A}$, ess denoting that the expression $\|D^k\gamma^i(t) - D^k\eta^i(t)\|$ is *not* defined only on a *finite* (possibly zero) set of points in I wherein γ^i and η^i are not differentiable, and $D^k\gamma^i(t)$ denotes the kth differential of γ^i at t in abused notation.

[5] The subscript σ on γ_σ will be omitted should no confusion arise from the context.

Unfortunately, a slight complication could arise from the current choice of M_σ's: to wit, d_σ need not be well-defined at each pair of points in M_σ. Equivalently, the set $\tilde{\mathcal{M}}_\infty[\Gamma_2^+]$ may be far too large for d_σ to be a well-defined function on $M_\sigma \times M_\sigma$. To avoid this potential embarrassment, consider a maximal subset $\mathcal{M}_\infty[\Gamma_2^+] \subset \tilde{\mathcal{M}}_\infty[\Gamma_2^+]$ such that by replacing $\tilde{\mathcal{M}}_\infty[\Gamma_2^+]$ in the defining sets of properties (i) – (ii) and (a) – (c) with $\mathcal{M}_\infty[\Gamma_2^+]$, each pair of points $\gamma, \gamma' \in M_\sigma$ for each σ satisfies $d_\sigma(\gamma, \gamma') < \infty$. In this way, an impending disaster is averted and d_σ is well-defined on $M_\sigma \times M_\sigma$ for each σ. It is routine then to verify that d_σ is indeed a metric. In all that follows, M_σ will be endowed with the d_σ-topology. Moreover, M_σ's will also be chosen so that they satisfy the following additional property:

$$\text{(d)} \qquad \forall\, \delta > 0 \text{ and } \gamma \in M_\sigma,\ \exists\, \eta \in M_\sigma \text{ such that } d_\sigma(\gamma, \eta) < \delta.$$

Note that from here on, whenever the defining properties (i) – (d) of M_σ's are mentioned, it will be tacitly assumed that $\tilde{\mathcal{M}}_\infty[\Gamma_2^+]$ is replaced with $\mathcal{M}_\infty[\Gamma_2^+]$ so that d_σ is a well-defined metric on M_σ for each σ.[6]

4.2.2. Proposition. *The d_σ-topology does not depend on the particular choice of (admissible) finite atlas $\mathfrak{A}$ of Σ and hence is a well-defined topology.*

Proof. To demonstrate this claim, let $\overline{\mathfrak{A}}$ denote the maximal atlas of Σ and define a topology on M_σ to be generated by subbasic open sets $N_\varepsilon(\gamma; (U_{\alpha(i)}, \varphi_{\alpha(i)})_{i=1}^\infty, K)$ in M_σ—to be constructed below—where $K \subset I$ is compact, $\gamma^i(K) \subset U_{\alpha(i)}$ and $(U_{\alpha(i)}, \varphi_{\alpha(i)}) \in \overline{\mathfrak{A}}\ \forall i$. Denote $\{\, \alpha(i) \mid 1 \leqq i \leqq \infty \,\}$ by α and $(U_{\alpha(i)}, \varphi_{\alpha(i)})_i$ by $(U, \varphi)_\alpha$ for notational simplicity, and let

$$d'_{\sigma\alpha K}(\gamma^i, \eta^i) \stackrel{\text{def}}{=} \text{ess sup}\{\, \|D^k\varphi_{\alpha(i)} \circ \gamma^i(t) - D^k\varphi_{\alpha(i)} \circ \eta^i(t)\| : t \in K,\ k \geqq 1 \,\}$$

whenever $\gamma^i(K), \eta^i(K) \subset U_{\alpha(i)}\ \forall i$. Then, for a fixed $\gamma \in M_\sigma$ such that $\gamma^i(K) \subset U_{\alpha(i)}\ \forall i$, let $N_\varepsilon(\gamma; (U, \varphi)_\alpha, K) \stackrel{\text{def}}{=} \{\, \eta \in M_\sigma \mid d_{\sigma\alpha K}(\gamma, \eta) < \varepsilon,\ \eta^i(K) \subset U_{\alpha(i)}\ \forall i \,\}$, where

$$d_{\sigma\alpha K}(\gamma, \eta) \stackrel{\text{def}}{=} \sup_i d_\Omega(\gamma^i, \eta^i) + \sup_i d'_{\sigma\alpha K}(\gamma^i, \eta^i).$$

That this topology is equivalent to the d_σ-topology on M_σ may be seen as follows.

First, given any d_σ-open δ-ball $B_\delta(\gamma)$ about γ in M_σ, consider a finite (compact) covering $\{K_i\}_{i=1}^n$ of I: $I = \bigcup_{i=1}^n K_i$, K_i compact. Set $N_\varepsilon(\gamma) = \bigcap_{i=1}^n N_\varepsilon(\gamma; (U, \varphi)_{\alpha_i}, K_i)$, where $(U, \varphi)_{\alpha_i} = \{\, (U_{\alpha_i(l)}, \varphi_{\alpha_i(l)}) \mid l = 1, 2, \ldots \,\}$ and

[6] A more rigorous presentation could certainly be given; however then, the entire analysis will get laborously tedious and out of hand: a situation to be clearly avoided!

$\gamma^l(K_i) \subset U_{\alpha_i(l)}$ $\forall i = 1, \ldots, n$. Clearly, from the definition of the subbasic neighbourhoods of γ and the finiteness of $\mathfrak{A}$, choosing $\varepsilon < \delta$ to be sufficiently small, there exists a suitable choice of compact covering $\{K_i\}_i$ of I and charts $(U_{\alpha_i(l)}, \varphi_{\alpha_i(l)}) \in \overline{\mathfrak{A}}$ for each $l \geqq 1$ and $i = 1, \ldots, n$, such that $N_\varepsilon(\gamma) \subset B_\delta(\gamma)$. Conversely, given any subbasic neighbourhood $N_\varepsilon(\gamma; (U, \varphi)_\alpha, K)$, it is obvious from property (d) above that there exists some $\delta > 0$ small enough such that $B_\delta(\gamma) \subset N_\varepsilon(\gamma; (U, \varphi)_\alpha, K)$. Hence, it follows at once that given any finite intersection of subbasic neighbourhoods of γ, there exists some $\delta > 0$ so that $B_\delta(\gamma)$ is contained within it. Consequently, the two topologies are equivalent, as claimed. In particular, the d_σ-topologies on M_σ defined relative to any two finite atlases of Σ are equivalent. Hence, in this sense, the d_σ-topology is well-defined as it does not depend on the choice of *finite* atlas $\mathfrak{A}$ on Σ. $\square$

A topology on $\mathcal{M}_\infty[\Gamma_2^+]$ can now be constructed. First, let $\mathcal{F}$ denote the family of subsets $M_\sigma \in M_\infty$ satisfying properties (a) – (d). It is evident that each pair of spaces $M_\sigma, M_{\sigma'} \in \mathcal{F}$ are homeomorphic—define $h_{\sigma\sigma'} : M_\sigma \to M_{\sigma'}$ by $\gamma_\sigma \mapsto \gamma_{\sigma'}$, where $h_\sigma(\gamma_\sigma) = \gamma = h_{\sigma'}(\gamma_{\sigma'})$. The existence of the homeomorphism $h_{\sigma\sigma'}$ follows immediately from properties (i), (c) and (d). Hence, it is possible to endow $\mathcal{M}_\infty[\Gamma_2^+]$ with a topology τ_∞ so that each $h_\sigma : M_\sigma \to \mathcal{M}_\infty[\Gamma_2^+]$ defines a homeomorphism: $h_\sigma(\tau_\sigma) = \tau_\infty$, where τ_σ is the d_σ-topology on M_σ. This will be the topology defined on $\mathcal{M}_\infty[\Gamma_2^+]$. Denote $\mathcal{M}_\infty[\Gamma_2^+]$ with this topology by $\mathcal{M}_\infty^{\mathcal{F}}[\Gamma_2^+]$. As an aside, if M_∞ is given the sum topology, $M_\infty \stackrel{\text{def}}{=} \bigoplus_\sigma M_\sigma$, then $h : M_\infty \to \mathcal{M}_\infty^{\mathcal{F}}[\Gamma_2^+]$ given by $h|M_\sigma \stackrel{\text{def}}{=} h_\sigma$ defines a continuous open surjection.

To conclude this section, it remains to verify that the topology on $\mathcal{M}_\infty^{\mathcal{F}}[\Gamma_2^+]$ is independent of the choice of the family $\mathcal{F}$, and hence implying at once that the topology on $\mathcal{M}_\infty[\Gamma_2^+]$ is well-defined once again.

4.2.3. Lemma. *Let $M, M' \subset M_\infty$ such that the mappings $h \stackrel{\text{def}}{=} \pi|M : M \to \mathcal{M}_\infty$ and $h' \stackrel{\text{def}}{=} \pi|M' : M' \to \mathcal{M}_\infty$ are bijective. If M and M' are endowed with the d-topology, where given $\gamma, \eta \in M_\infty$,*

$$d(\gamma, \eta) \stackrel{\text{def}}{=} \sup_i d_\Omega(\gamma^i, \eta^i) + \sup_i d'_\Omega(\gamma^i, \eta^i),$$

and satisfy property (d), then M and M' are homeomorphic.[7]

Proof. Suppose that $M \neq M'$; otherwise, there is nothing to proof. Let $i : M \to M'$ be the composition $i = h'^{-1} \circ h$. By definition, i is bijective. Fix $\gamma \in M$ and let

[7]Observe trivially that the metric d_σ is just the restriction $d|M_\sigma \times M_\sigma$.

$B_\varepsilon(\gamma) \subset M$ be an open ε-ball about γ. Set $D_\varepsilon(\gamma) = i(B_\varepsilon(\gamma))$ and fix a point $\eta' \in D_\varepsilon(\gamma)$, where $\eta' = i(\eta)$ for some $\eta \in B_\varepsilon(\gamma)$. If $\forall\, \delta > 0$, $B_\delta(\eta') \not\subset D_\varepsilon(\gamma)$, then for each $\delta > 0$, $\exists\, \tau' \in B_\delta(\eta')$ such that $\sup_i d_\Omega(\gamma^i, \tau^i) + \sup_i d'_\Omega(\gamma^i, \tau^i) > \varepsilon$, where $\tau = i^{-1}(\tau')$. This clearly cannot be satisfied by taking $\delta > 0$ to be sufficiently small (which is possible by (d)). Hence, $D_\varepsilon(\gamma)$ is open in M' and i is thus an open mapping. Since i is a bijection, invoking symmetry[8] yields the continuity of i. Hence, i defines a homeomorphism, as claimed. □

4.2.4. Corollary. *For any pair of families $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfying properties (a) – (d), $\mathcal{M}_\infty^{\mathcal{F}_1}[\Gamma_2^+] \cong \mathcal{M}_\infty^{\mathcal{F}_2}[\Gamma_2^+]$.*

Proof. By Lemma 4.2.3, $M_{1\sigma} \cong M_{2\lambda}$ $\forall\, M_{1\sigma} \in \mathcal{F}_1$ and $M_{2\lambda} \in \mathcal{F}_2$. Since $\mathcal{M}_\infty^{\mathcal{F}_1}[\Gamma_2^+] \cong M_{1\sigma}$ and $\mathcal{M}_\infty^{\mathcal{F}_2}[\Gamma_2^+] \cong M_{2\lambda}$ by definition, it follows at once that $\mathcal{M}_\infty^{\mathcal{F}_1}[\Gamma_2^+] \cong \mathcal{M}_\infty^{\mathcal{F}_2}[\Gamma_2^+]$. □

In view of this cororllary, $\mathcal{M}_\infty^{\mathcal{F}}[\Gamma_2^+]$ may be denoted by $\mathcal{M}_\infty[\Gamma_2^+]$ since the topology does not depend on the particular choice of $\mathcal{F}$—in particular, a family $\mathcal{F}$ may be fixed without any loss of generality. Hence, in the analysis to be carried out in the next section, a family $\mathcal{F}$ satisfying properties (a) – (d) is understood to be fixed once and for all.

4.3. The Space of $\aleph_0$-Knots of $\mathcal{M}_\infty[\Gamma_2^+]$

First of all, some notations and elementary properties of the space of equivalence $\aleph_0$-loop classes will be established. Let $\mathcal{G}_{\mathrm{a}}^+$ be the set of (smooth) orientation-preserving, ambient isotopies on Σ. That is, $\mathcal{G}_{\mathrm{a}}^+ \subset C^\infty(\Sigma \times I, \Sigma \times I)$ is the following set:

$$\{\, F : \Sigma \times I \to \Sigma \times I \mid F(x,t) \stackrel{\text{def}}{=} (F_t(x), t),\ F_0 = \mathrm{id}_\Sigma,\ F_t \in \mathrm{Diff}^+(\Sigma)\ \forall\, t \in I \,\}$$

and define composition $\circ$ on $\mathcal{G}_{\mathrm{a}}^+$ by

$$(F' \circ F) : (x,t) \mapsto (F'_t \circ F_t(x), t).$$

Then, clearly, $F' \circ F \in \mathcal{G}_{\mathrm{a}}^+$ and $1_{\Sigma\times I} \stackrel{\text{def}}{=} \mathrm{id}_\Sigma \times \mathrm{id}_I \in \mathcal{G}_{\mathrm{a}}^+$. It is straight forward to check that $\langle \mathcal{G}_{\mathrm{a}}^+, \circ \rangle$ forms a group under $\circ$, where the inverse F^{-1} of $F = (F_t, \mathrm{id}_I)$ is defined to be $(F_t^{-1}, \mathrm{id}_I)$. In particular, $\circ$ is compatible with the compact C^∞-topology on $\mathcal{G}_{\mathrm{a}}^+$—*cf.* [3, Ex. 9, p. 64].

[8] That is, repeating the same argument for i^{-1}.

Now, because $\mathrm{Diff}^+(\Sigma)$ is the subgroup of smooth diffeomorphisms that are connected to the identity id_Σ of Σ, it follows that $\mathcal{G}_{\mathrm{a}}^+$ is closed in $C^\infty(\Sigma \times I, \Sigma \times I)$ with respect to the compact C^∞-topology. To verify this claim, it will suffice to consider a sequence $\{F_n\}_n$ in $\mathcal{G}_{\mathrm{a}}^+$ that converges to $F_0 \in C^\infty(\Sigma \times I, \Sigma \times I)$. By assumption, $F_n \to F_0 \Rightarrow F_n(\cdot, t) \to F_0(\cdot, t)$ uniformly on Σ for each fixed t. In particular, since $F_n(\cdot, 0) = \mathrm{id}_\Sigma \;\; \forall n$, the uniform convergence $F_n(\cdot, 0) \to F_0(\cdot, 0)$ implies at once that $F_0(\cdot, 0) \equiv \mathrm{id}_\Sigma$, as desired.

If $\gamma, \eta \in \tilde{\mathcal{L}}_\Sigma$ are any pair of loops and γ is ambient isotopic to η under some $F \in \mathcal{G}_{\mathrm{a}}^+$, denote this by $F : \gamma \simeq \eta$. Now, given any pair of $\aleph_0$-loops $\gamma, \eta \in \mathcal{M}_\infty[\Gamma_2^+]$, define an equivalence relation R generated by $\simeq$ on $\mathcal{M}_\infty[\Gamma_2^+]$ as follows:

$$\gamma \simeq \eta \quad \Longleftrightarrow \quad \exists\, F \in \mathcal{G}_{\mathrm{a}}^+ \text{ such that } F \cdot \gamma = \eta,$$

where $F \cdot \gamma \stackrel{\mathrm{def}}{=} \{\, F_1 \circ \gamma^1, F_1 \circ \gamma^2, \dots \}$ and $F : \gamma^i \simeq \eta^i \; \forall i$. Then, the space $\mathcal{K}[\Gamma_2^+]$ of equivalence classes of $\aleph_0$-loops in $\mathcal{M}_\infty[\Gamma_2^+]$ is defined to be the quotient space $\mathcal{M}_\infty[\Gamma_2^+]/\mathcal{G}_{\mathrm{a}}^+$. Henceforth, for simplicity, the term *(piecewise geodesic)* $\aleph_0$*-knot* will mean an element of the quotient space $\mathcal{K}[\Gamma_2^+]$; that is, an $\aleph_0$-knot denotes an equivalence class of $\aleph_0$-loops under a smooth, orientation-preserving, ambient isotopy. The space $\mathcal{K}[\Gamma_2^+]$ will be called the $(\aleph_0, \Gamma_2^+)$*-knot space* of $\mathcal{M}_\infty[\Gamma_2^+]$. Let $\kappa_\infty : \mathcal{M}_\infty[\Gamma_2^+] \to \mathcal{K}[\Gamma_2^+]$ denote the natural map, where $\mathcal{K}[\Gamma_2^+]$ is endowed with the quotient topology.

4.3.1. Lemma. *The natural projection* $\kappa_\infty : \mathcal{M}_\infty[\Gamma_2^+] \to \mathcal{K}[\Gamma_2^+]$ *is open.*

Proof. A sketch of the proof will be given. To see that κ_∞ is an open mapping, it is enough to note that for each open subset $N \subset \mathcal{M}_\infty[\Gamma_2^+]$,

$$\kappa_\infty^{-1} \circ \kappa_\infty(N) = \bigcup_{F \in \mathcal{G}_{\mathrm{a}}^+} F \cdot N,$$

where $F \cdot N = \{\, F \cdot \gamma \mid \gamma \in N \,\}$. Since $F \cdot N$ is open in $\mathcal{M}_\infty[\Gamma_2^+]$, the quotient topology implies that $\kappa_\infty^{-1} \circ \kappa_\infty(N)$, and hence κ_∞, must also be open. $\square$

4.3.2. Proposition. $\mathcal{K}[\Gamma_2^+]$ *is Hausdorff.*

Proof. By Lemma 4.3.1, it will suffice to show that the equivalence relation R generated by $\simeq$ is closed in $\mathcal{M}_\infty[\Gamma_2^+] \times \mathcal{M}_\infty[\Gamma_2^+]$ [4, Theorem 11, p. 98]. Let $\{(\gamma_n, \eta_n)\}_n$ be a sequence in R which converges in $\mathcal{M}_\infty[\Gamma_2^+] \times \mathcal{M}_\infty[\Gamma_2^+]$ to (γ_0, η_0). By definition, $\exists$ a sequence $\{F_n\}_n$ in $\mathcal{G}_{\mathrm{a}}^+$ such that $F_n : \gamma_n \simeq \eta_n$ for each n. So, $(\gamma_n, F_n \cdot \gamma_n) \to (\gamma_0, \eta_0) \Rightarrow F_n \cdot \gamma_n \to \eta_0$ and $\gamma_n \to \gamma_0$, and hence implying that

$\{F_n\}_n$ is a convergent sequence in $\mathcal{G}_a^+$. Consequently, $\mathcal{G}_a^+$ is closed implies that $F_n \to F_0 \in \mathcal{G}_a^+$ for some F_0. Whence, $\eta_0 \equiv F_0 \cdot \gamma_0$ and R is thus closed, as desired. $\square$

In the interest of simplicity, call $\gamma \in \mathcal{M}_\infty[\Gamma_2^+]$ a *piecewise* $(\aleph_0, q)$*-geodesic loop* whenever the 3-metric q is required to be specified.

4.3.3. Lemma. *Let* $\gamma, \tilde{\gamma} \in \mathcal{M}_\infty[\Gamma_2^+]$ *be piecewise* $(\aleph_0, q)$*- and* $(\aleph_0, \tilde{q})$*-geodesic loops respectively. If* $\gamma \simeq \tilde{\gamma}$*, then* $\exists\, f \in \mathrm{Diff}^+(\Sigma)$ *such that* $q = f^*\tilde{q}$.

Proof. Let $F \in \mathcal{G}_a^+$ be an ambient isotopy of γ and $\tilde{\gamma}$: $F \cdot \gamma = \tilde{\gamma}$. Then, evidently, $\tilde{\gamma}$ is a piecewise $(\aleph_0, (F_1^{-1})^* q)$-geodesic. However, $\tilde{\gamma}$ is also a piecewise $(\aleph_0, \tilde{q})$-geodesic; hence, by §4.2 (2), $\exists\, f \in \mathrm{Diff}^+(\Sigma)$ such that $\tilde{q} = f^* q$, as required. $\square$

4.4. $\aleph_0$-Knots and Classical Geometry

In this section, the relationship between the equivalence classes of $\aleph_0$-loops in Σ and the (classical) geometries admissible on Σ will be studied. This correspondence can be easily sought by simply noting that each element in $\mathcal{M}_\infty[\Gamma_2^+]$ corresponds to a unique 3-geometry $[q]$ of Σ by construction. The modified form of Rovelli's Conjecture can now be formulated.

4.4.1. Theorem. *There exists a continuous, open surjection* $\hat{\chi} : \mathcal{M}_\infty[\Gamma_2^+] \to \mathcal{Q}$ *given by* $\gamma_q \mapsto [q]$*, where* γ_q *is a (piecewise)* $(\aleph_0, q)$*-geodesic loop and* $q \in [q]$.

Proof (Sketch). First, $\hat{\chi}$ is well-defined from the definition of $\mathcal{M}_\infty[\Gamma_2^+]$. Second, the surjective property of $\hat{\chi}$ is also clear. Third, in this proof, Γ_2^+ will be identified with its image under the (topological) imbedding $j^\infty : \Gamma_2^+ \hookrightarrow C(\Sigma, J^\infty[p_\Sigma])$, where $J^\infty[p_\Sigma]$ is the C^∞-jet bundle of cross sections of $S_2^+\Sigma$, and $S_2^+\Sigma$ is the bundle of symmetric covariant 2-tensors over Σ.[9] So, $\mathcal{Q} \equiv j^\infty\Gamma_2^+ / \mathrm{Diff}^+(\Sigma)$ and $\pi_+ : j^\infty\Gamma_2^+ \to \mathcal{Q}$.

Now, fix some $\gamma_0 \in \mathcal{M}_\infty[\Gamma_2^+]$ and let $N(q_0) = \bigcap_{i=1}^n M(K_i, (\pi_\Sigma^{n_i})^{-1}(U^{n_i}))$ be a neighbourhood of q_0 in Γ_2^+, where $n_i \in \mathbb{N}$, $n < \infty$ and $q_0 \in \hat{\chi}(\gamma_0) = [q_0]$ is a representative of the q_0-equivalence class. Set $N([q_0]) = \pi_+(N(q_0))$. Then, $\tilde{N}([q_0]) \stackrel{\text{def}}{=} \pi_+^{-1}(N([q_0])) = \bigcup\{\, f^* \circ N(q_0) \mid f \in \mathrm{Diff}^+(\Sigma) \,\}$, where $f^* \circ N(q_0) \stackrel{\text{def}}{=} \{\, f^* q \mid q \in N(q_0) \,\}$. Let $D_\varepsilon(\gamma_0)$ be an ε-neighbourhood of γ_0 defined by $B_\varepsilon(h_\sigma^{-1}(\gamma_0)) = h_\sigma^{-1}(D_\varepsilon(\gamma_0))\ \forall\, \sigma$. Then, $\forall\, \eta \in D_\varepsilon(\gamma_0)$, $d_\Omega(\gamma_{0\sigma}^i, \eta_\sigma^i) + d'_\Omega(\gamma_{0\sigma}^i, \eta_\sigma^i) < \varepsilon\ \forall\, i$ and σ, where $h_\sigma^{-1}(\gamma) \stackrel{\text{def}}{=} \gamma_\sigma$.

[9] The notations used here—the C^∞-jets and compact C^∞-topology—can be found in the Appendix, §A.

Next, observe from the definition that

$$(\ddot{\gamma}^i)^\ell + \Gamma(q)^\ell_{kj}(\dot{\gamma}^i)^k(\dot{\gamma}^i)^j \overset{\text{a.e.}}{=} 0 \ \forall i \in \mathbb{N} \text{ and } \ell = 1,2,3, \tag{4.4.1}$$

where $\Gamma(q)$ is a Riemannian connection determined by the 3-metric q (with the connection coefficients written with respect to the natural frame for simplicity). So, by choosing $\varepsilon > 0$ to be sufficiently small, and by fixing *any* σ—and setting $\gamma^i_0 = \gamma_0^{\sigma(i)}$, $\eta^i = \eta^{\sigma(i)}$—it follows that $|\ddot{\eta}^i - \ddot{\gamma}^i_0| < \varepsilon$ and $|\dot{\eta}^i - \dot{\gamma}^i_0| < \varepsilon$ (almost everywhere), and in particular, using (4.4.1),

$$\begin{aligned}
&\left|(\ddot{\gamma}^i_0)^\ell + \Gamma(q_\eta)^\ell_{kj}(\dot{\gamma}^i_0)^k(\dot{\gamma}^i_0)^j\right| \\
\overset{\text{a.e.}}{=}\ &\left|(\ddot{\eta}^i + \mathcal{O}(\varepsilon))^\ell + \Gamma(q_\eta)^\ell_{kj}(\dot{\eta}^i + \mathcal{O}(\varepsilon))^k(\dot{\eta}^i + \mathcal{O}(\varepsilon))^j\right| \\
\overset{\text{a.e.}}{=}\ &\left|(\ddot{\eta}^i)^\ell + \Gamma(q_\eta)^\ell_{kj}(\dot{\eta}^i)^k(\dot{\eta}^i)^j + \mathcal{O}(\varepsilon)\right| \\
\sim\ &\mathcal{O}(\varepsilon) \text{ a.e. on } I,
\end{aligned}$$

where $q_\eta \in \hat{\chi}(\eta)$. Whence, $|(\ddot{\gamma}^i_0)^\ell + \Gamma(q_\eta)^\ell_{kj}(\dot{\gamma}^i_0)^k(\dot{\gamma}^i_0)^j| \overset{\text{a.e.}}{=} |(\ddot{\gamma}^i_0)^\ell + \Gamma(q_0)^\ell_{kj}(\dot{\gamma}^i_0)^k(\dot{\gamma}^i_0)^j - (\ddot{\gamma}^i_0)^\ell - \Gamma(q_\eta)^\ell_{kj}(\dot{\gamma}^i_0)^k(\dot{\gamma}^i_0)^j| = |(\Gamma(q_0) - \Gamma(q_\eta))^\ell_{kj}(\dot{\gamma}^i_0)^k(\dot{\gamma}^i_0)^j| \sim \mathcal{O}(\varepsilon)$ a.e. (from above) $\forall i \in \mathbb{N} \Rightarrow |\Gamma(q_\eta)^\ell_{kj} - \Gamma(q_0)^\ell_{kj}|$ is small on Σ for each fixed ℓ, k, j whenever $\varepsilon > 0$ is small enough by appealing to property (2) in §4.2 and the continuity of Γ. Thus, from the equality $\Gamma(q)^\ell_{kj} \equiv \frac{1}{2}q^{\ell h}(\partial_k q_{hj} + \partial_j q_{hk} - \partial_h q_{kj})$ (in the natural frame), it follows that $\exists\, f \in \mathrm{Diff}^+(\Sigma)$ such that f^*q_η and q_0, together with their kth derivatives, must be close to one another. In particular, $f^*q_\eta(K_i) \subset (\pi^{n_i}_\Sigma)^{-1}(U^{n_i})\ \forall i = 1, \ldots, n$. So, f^*q_η and hence q_η must both belong to $\tilde{N}([q_0])$ for $\varepsilon > 0$ sufficiently small. Whence, $\hat{\chi}(D_\varepsilon(\gamma_0)) \subset N([q_0])$, and the continuity of $\hat{\chi}$ follows.

Finally, to conclude this proof, observe that for any $\gamma \in \mathcal{M}_\infty[\Gamma^+_2]$, $\hat{\chi}^{-1} \circ \hat{\chi}(\gamma) = \{\, f \circ \gamma \mid f \in \mathrm{Diff}^+(\Sigma)\,\}$, where $f \circ \gamma \overset{\text{def}}{=} \{\, f \circ \gamma^1, f \circ \gamma^2, \ldots\}$. Hence, for any ε-neighbourhood $D_\varepsilon(\gamma)$,

$$\hat{\chi}^{-1} \circ \hat{\chi}(D_\varepsilon(\gamma)) = \bigcup_{f \in \mathrm{Diff}^+(\Sigma)} f \circ D_\varepsilon(\gamma),$$

and $\hat{\chi}$ is thus open, as desired. $\square$

In spite of the divergent approach given here with Rovelli's original idea, the following corollary could perhaps be christened as the *weak Rovelli conjecture* inasmuch as the notion of relating knots with geometry originated from Rovelli [5].

4.4.2. Corollary (Weak Rovelli Conjecture). *The map $\hat{\chi}$ induces a continuous, open surjection $\chi : \mathcal{K}[\Gamma_2^+] \to \mathcal{Q}$ given by $[\gamma_q] \mapsto \hat{\chi}(\gamma_q)$, where $\gamma_q \in \kappa_\infty^{-1}([\gamma_q])$ is any fixed representative.*

Proof. This map χ is well-defined by Lemma 4.3.3. The result now follows immediately from Theorem 4.4.1, Lemma 4.3.1 and the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{M}_\infty[\Gamma_2^+] & \xrightarrow{\hat{\chi}} & \mathcal{Q} \\ {\scriptstyle \kappa_\infty}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{K}[\Gamma_2^+] & \xrightarrow{\chi} & \mathcal{Q}. \end{array}$$

□

4.4.3. Remark. It is worthwhile pointing out that none of the results obtained thus far will be affected by relaxing condition (2) of §4.2 to the following weaker condition:

(2′) the map $\gamma \to D_\Sigma$ given by $\gamma^i \mapsto \gamma^i(0)$ is a surjection.

Two comments regarding Theorem 4.4.1 and its corollary are now in order. First, it is certainly evident that if Σ be separable (which, here, it is in any case!), then it is sufficient to characterized its 3-geometries by the $\aleph_0$-loops in $\mathcal{M}_\infty[\Gamma_2^+]$ since, by construction, $\overline{\{\, \gamma^i(0) \mid i \in \mathbb{N}\,\}} \equiv \Sigma$, whereas using this construction, n-loops, for $n < \infty$, are not sufficient to determine the 3-geometry uniquely (as might well be expected): *cf.* §4.2.5 for a detailed account.

Second, it has been established elsewhere—*cf.* for example, [6, §5.1, p. 132] using the diffeomorphism constraints of general relativity (in the loop representation)—that functionals on $\mathcal{L}_\Sigma$ which describe gravitational states are constant on the $\mathcal{G}_{\mathrm{a}}^+$-orbits of $\mathcal{L}_\Sigma$: $\psi[\gamma] = \psi[\gamma']\ \forall \gamma, \gamma' \in [\gamma]$, where $\psi : \mathcal{L}_\Sigma \to \mathbb{C}$ is a loop functional. This can be easily seen from §2.2.4:

$$\psi[\gamma'] = \psi[\gamma]\ \forall \gamma, \gamma' \in [\gamma] \Rightarrow (D(v)\psi)[\gamma] \overset{\mathrm{def}}{=} \lim_{t \to 0} \frac{\psi[\phi_t^{-1} \circ \gamma] - \psi[\gamma]}{t} \equiv 0,$$

where $\phi_t \in \mathrm{Diff}^+(\Sigma)\ \forall t$ and v is a vector field on Σ that generates the 1-parameter group of diffeomorphisms $\{\phi_t\}_t$, and $D(v)$ is the diffeomorphism constraint in the loop representation defined in §2.2.4 by $(D(v)\Psi)[\eta] = \frac{\mathrm{d}}{\mathrm{d}t}(\Psi[\phi_t^{-1} \circ \eta])|_{t=0}$.

However, surprisingly enough, this condition follows immediately from Corollary 4.4.2. This can be easily seen as follows. Functionals on Γ_2^+ that describe gravitational states are those which are invariant under $\mathrm{Diff}^+(\Sigma)$: *i.e.*, they are essentially functionals on $\mathcal{Q}$ in the metric representation. Let $C(\mathcal{Q}, \mathbb{C})$ be the set of continuous functionals on $\mathcal{Q}$ and let $C(\mathcal{K}[\Gamma_2^+], \mathbb{C})$ be the set of functionals on $\mathcal{K}[\Gamma_2^+]$. Then, $\forall\, \tilde{\Psi} \in C(\mathcal{Q}, \mathbb{C})$, $\tilde{\Psi} \circ \chi \in C(\mathcal{K}[\Gamma_2^+], \mathbb{C})$; that is, $\chi^*(C(\mathcal{Q}, \mathbb{C})) \subset C(\mathcal{K}[\Gamma_2^+], \mathbb{C})$, and the assertion thus follows. This concludes the classical description of $\aleph_0$-knots and their relationship with 3-geometries.

4.5. Discussion.

In this final section, a possible physical interpretation—albeit a highly speculative one!—regarding knots and gravity will be sketched. As was pointed out before, the separability of Σ guarantees that $\hat{\chi}$ in Theorem 4.4.1 remains well-defined. Furthermore, as classically, gravity—or equivalently, the 4-metric—of space-time is determined by the distribution of matter in the universe via Einstein's field equations, gravity is a "global" concept. In this sense, if n-loops can describe gravity in any way, then, provided that space-time be separable, loops that will best describe it are $\aleph_0$-loops. Indeed, a judicious choice of $\aleph_0$-loops—such as those given in the preceding sections—enables one to recover the underlying Riemannian 3-manifold Σ simply because $\overline{\{\, \gamma_q^i(0) \mid i \in \mathbb{N} \,\}} = \Sigma$, and $\hat{\chi}(\gamma_q) = [q]$. In the light of this observation, it is not unreasonable to conclude that gravity is the result of the way 3-space (and hence, space-time) is knotted, where (Σ, q) is said to be $[\gamma]$-*knotted* if $\chi([\gamma]) = [q]$. And since χ is not one-one, Σ can be knotted in two $\mathcal{G}_{\mathrm{a}}^+$-inequivalent ways and yet give rise to the same gravitational configuration (determined by χ). In short, having determined $\mathcal{M}_\infty[\Gamma_2^+]$ from Σ, each element in $\mathcal{M}_\infty[\Gamma_2^+]$ contains the necessarily information to reconstruct Σ. Along this note, Rovelli [5, §4.2.3, p. 1660] delineated an interesting argument showing how gravitational states in the loop representation may be interpreted as the way loops are entangled with a given weave. His construction leads to a fascinating conclusion that the structure of space-time is discrete at the Planck scale!

To conclude with an untampered speculative note on the quantum aspect of a knot $[\gamma]$, a *knot state* $|[\gamma]\rangle$ might be heuristically interpreted to correspond to the pair $[(\Sigma, q)]$, where $[(\Sigma, q)] \stackrel{\text{def}}{=} \{\, (\Sigma, q) \mid q \in \chi([\gamma]) \,\}$. In particular, $|[\gamma]\rangle$ is associated with a particular 3-geometry $\chi([\gamma])$. Thus, $|[\gamma]\rangle$ corresponds to the global degrees of freedom of gravity: and since gravitons are associated with the local degrees of freedom of gravity, it has *no direct* relationship with a knot state. This is of course expected as gravitons are essentially linearised gravitational states. In the full quantum theory, it is quite reasonable to expect that $|[\gamma]\rangle$ will not span a Hilbert space due to the highly non-linear nature of gravity and the violation of the asymptotic freedom condition. Hence, a knot state most probably cannot be interpreted in the usual quantum field theoretic sense in that it lies in some Hilbert space, although it is tempting to conjecture that the knot states lie in some $\aleph_0$-dimensional smooth Kähler manifold.

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CHAPTER V

DIFFEOMORPHISM-INVARIANT MULTI-LOOP MEASURE

5.1. Introduction

The question regarding the existence of a measure on the multi-loop space of a smooth, Riemannian 3-manifold that is independent of the underlying metric structure of the manifold and invariant under smooth diffeomorphisms on it was raised in [3, p. 336]. It plays an important rôle in the loop representation theory of quantum gravity. Indeed, it was used as a heuristic device to go from the self-dual representation theory of quantum gravity to the loop representation theory:

$$\tilde{\psi}[A] = \int \mathrm{d}\mu(\gamma) h[\gamma, A] \psi[\gamma],$$

where $h[\gamma, A]$ is the trace of the complexified SU(2) holonomy, $\mathrm{d}\mu$ is a loop measure and $\tilde{\psi}$, ψ are, respectively, the connection and the loop functionals. In this chapter, the existence of a diffeomorphism-invariant loop measure will be settled for the case when the Riemannian 3-manifold is smooth, orientable, closed and compact.

The purpose of constructing a measure on the multi-loop space is—very briefly—to define a physical inner product (ultimately!) on the space of multi-loop functionals. Indeed, it is also interesting to note in passing that Rayner [4] got around this problem—the lack of a multi-loop measure—by using a discrete sum rather than an integral in his construction of an inner product on the space of multi-loop functionals. Another physical motivation for proving the existence of a diffeomorphism-invariant measure on the multi-loop space arises from an interesting result of Jacobson and Smolin [3, p. 337]: it states that *given* the existence of a diffeomorphism-invariant multi-loop measure, there exists a space of physical quantum states of the gravitational field on Σ which is spanned by a basis in one-to-one correspondence with the knot classes of Σ.

The definition of a measure will be recalled here. For more details, refer to reference [2]. Let X be a non-empty set. A *σ-algebra* $\mathcal{A}$ of X is a subset of the power set 2^X of X satisfying

(a) $X \in \mathcal{A}$,

(b) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$, where $A^c \stackrel{\text{def}}{=} X - A$, and

(c) if $\{A_n\}_{n=1}^\infty$ is any sequence in $\mathcal{A}$, then $\bigcup_{i=1}^n A_i \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a *measurable space.* If $U \subset X$ is an element of $\mathcal{A}$, then U is said to be $\mathcal{A}$-measurable. If $\mathcal{F} \subset 2^X$ is any subset, then the σ-algebra *generated* by $\mathcal{F}$ is the smallest σ-algebra that contains $\mathcal{F}$ as a subset. This σ-algebra is unique. Finally, a *Borel σ-algebra* of a topological space is the σ-algebra generated by the collection of all open subsets of the space.

5.1.1. Definition. Let X be a non-empty set and $\mathcal{A}$ a σ-algebra of X. Then, a *measure* μ on X—or more precisely, a measure on $\mathcal{A}$—is a non-negative set-function $\mu : \mathcal{A} \to [0, +\infty]$ such that

(1) $\mu(\varnothing) = 0$,

(2) μ is *σ-additive*; that is, $\mu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$ whenever the sequence $\{A_n\}_n$ is disjoint.

The pair $(X, \mathcal{A}, \mu)$ is called a *measure space.*

If condition (2) above only holds for finitely many disjoint subsets $A_1, \ldots, A_n$, then μ is said to be a *finitely additive measure.* In this thesis, a measure will always mean a σ-additive measure.

The contents of this chapter are based on reference [5]. A cursory survey of this chapter runs as follows. In section 2, a diffeomorphism-invariant measure on the space of piecewise smooth loops of Σ will be constructed. This will lead to the existence proof of a diffeomorphism-invariant measure on the multi-loop space in section 3. A brief sketch of the Hermitian conjugates of the quantum T^n-operators will be made in the last section. It will be shown that whilst the Hermitian conjugate of the T^0-operator is indeed independent of the choice of inner products, the same is not true for the T^n-operators, where $n > 0$—there is an implicit dependence on the type of inner product chosen.

5.2. Diffeomorphism-Invariant Borel Measure on $\mathcal{L}_\Sigma$

The existence of an outer-regular Borel measure on $\tilde{\mathcal{L}}_\Sigma$ will be established in this section. From this, an outer-regular, diffeomorphism-invariant Borel measure on $\mathcal{L}_\Sigma$ will be constructed. The notations used here will be identical to that used in Chapter 2. In particular, $\tilde{\mathcal{L}}_\Sigma$ denotes the space of piecewise-smooth loops on Σ and $\mathcal{L}_\Sigma$ denotes the quotient space $\tilde{\mathcal{L}}_\Sigma$ after identifying all the constant loops in it with a single point.

Let $\tilde{\mathcal{L}}^*_\Sigma = \{\psi : \tilde{\mathcal{L}}_\Sigma \to \mathbb{R} \mid \psi \text{ continuous}\}$ and $\mathcal{K}(\tilde{\mathcal{L}}_\Sigma) \subset \tilde{\mathcal{L}}^*_\Sigma$ be a subset of (bounded) loop functionals such that for each $\psi \in \mathcal{K}(\tilde{\mathcal{L}}_\Sigma)$, $\exists\, \delta > 0$ and $\gamma_\delta \in \tilde{\mathcal{L}}_\Sigma$ satisfying $\operatorname{supp}(\psi) \subset B_\delta(\gamma_\delta)$. Also, let $\mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma) \stackrel{\text{def}}{=} \{\psi \in \mathcal{K}(\tilde{\mathcal{L}}_\Sigma) \mid \psi \geqq 0\}$ and set

$$N(B_\delta(\gamma_\delta)) \stackrel{\text{def}}{=} \{\psi \in \mathcal{K}(\tilde{\mathcal{L}}_\Sigma) \mid \operatorname{supp}(\psi) \subset B_\delta(\gamma_\delta)\}.$$

The space $\mathcal{K}(\tilde{\mathcal{L}}_\Sigma)$ will be endowed with the topology induced by the sup-norm: $\|\psi\| \stackrel{\text{def}}{=} \sup\{|\psi(\gamma)| : \gamma \in \tilde{\mathcal{L}}_\Sigma\}$. Finally, a linear functional I on $\mathcal{K}(\tilde{\mathcal{L}}_\Sigma)$ is said to be *positive* if $I(\psi) \geqq 0\ \forall\, \psi \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma)$.

5.2.1. Lemma. *For each $\delta > 0$ and $\gamma_\delta \in \tilde{\mathcal{L}}_\Sigma$, $\sup\{|\psi(\gamma)| : \gamma \in \tilde{\mathcal{L}}_\Sigma, \psi \in N(B_\delta(\gamma_\delta))\} < \infty$.*

Proof. Suppose the contrary. Then, for each $n > 0$, $\exists\, \psi_n \in N(B_\delta(\gamma_\delta))$ such that $\|\psi_n\| > n$. Choose $\gamma_n \in \tilde{\mathcal{L}}_\Sigma - \operatorname{supp}(\psi_n)$ so that $\rho(\gamma_n, \operatorname{supp}(\psi_n))$ is small, and choose some small $\delta_n > 0$ satisfying $\operatorname{supp}(\psi_n) \subset B_{\delta_n}(\gamma_n)$. Since ψ_n is continuous (by definition), given any $\varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ such that

$$\gamma \in B_{\delta_\varepsilon}(\gamma_n) \Rightarrow |\psi_n(\gamma) - \psi_n(\gamma_n)| \equiv |\psi_n(\gamma)| < \varepsilon. \tag{5.2.1}$$

Clearly, by taking $n > 0$ to be sufficiently large, and setting $\varepsilon = n$, may take $\delta_\varepsilon = \delta_n$. However then, $\|\psi_n\| > n$ implies that (5.2.1) cannot be satisfied, yielding a contradiction. Hence, the Lemma follows. □

5.2.2. Corollary. *Let I be a linear functional on $\mathcal{K}(\tilde{\mathcal{L}}_\Sigma)$. Then, $\sup\{|I(\psi)| : \psi \in N(B_\delta(\gamma_\delta))\} < \infty$ for each $\delta > 0$.* □

5.2.3. Lemma. *Let $G = \bigcup_i G_i$ be open, $\psi \in \mathcal{K}(\tilde{\mathcal{L}}_\Sigma)$ and $\operatorname{supp}(\psi) \subset G$. Then, $\exists\, \psi_i \in \mathcal{K}(\tilde{\mathcal{L}}_\Sigma)$ with $\operatorname{supp}(\psi_i) \subset G_i$ such that $\psi = \sum_i \psi_i$. In particular, if $\psi \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma)$, then $\psi_i \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma)$ exists for each i.*

Proof. Given ψ above, set $C = \operatorname{supp}(\psi)$. Since $\{G_i\}_i$ is an open covering of C and C is paracompact (as it is closed and metrizable), there exists a partition of unity $\{e_i\}_i$ subordinate to $\{G_i\}_i$: $\sum_i e_i(\gamma) = 1$, $0 \leqq e_i(\gamma) \leqq 1$ and $\operatorname{supp}(e_i) \subset G_i\ \forall i$. Set $\psi_i = \psi e_i$, where $\psi_i(\gamma) \stackrel{\text{def}}{=} \psi(\gamma)e_i(\gamma)$. Then, by construction, $\operatorname{supp}(\psi_i) \subset G_i$ and $\psi = \psi \sum_i e_i \equiv \sum_i \psi_i$, as asserted. The second statement follows trivially from the fact that $e_i \geqq 0$ for each i. □

5.2.4. Theorem. *$\tilde{\mathcal{L}}_\Sigma$ admits an outer-regular Borel measure.*

Proof. Fix a positive linear functional I on $\mathcal{K}(\tilde{\mathcal{L}}_\Sigma)$ and define μ^* on the collection of open subsets U of $\tilde{\mathcal{L}}_\Sigma$ in the following way:

$$\mu^*(U) \stackrel{\text{def}}{=} \sup\{I(\psi) \mid \psi \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma), \operatorname{supp}(\psi) \subset U\}. \tag{5.2.2}$$

This is well-defined by Corollary 5.2.2. Next, extend μ^* to the power set $2^{\tilde{\mathcal{L}}_\Sigma}$ of $\tilde{\mathcal{L}}_\Sigma$ by

$$\mu^*(A) \overset{\text{def}}{=} \inf\{\, \mu^*(U) \mid A \subseteq U,\, U \text{ open} \,\}. \tag{5.2.3}$$

Claim: $\mu^* : 2^{\tilde{\mathcal{L}}_\Sigma} \to \mathbb{R}_+$ is non-trivial.

To establish this assertion, it will suffice to show—from (5.2.3) and the fact that $\tilde{\mathcal{L}}_\Sigma$ is second countable (Proposition 3.2.4)—that μ^* is non-trivial on the collection of open balls in $\tilde{\mathcal{L}}_\Sigma$. Recall that the second countability of $\tilde{\mathcal{L}}_\Sigma$ implies that every open subset in $\tilde{\mathcal{L}}_\Sigma$ can be expressed as a countable union of open balls. So, to this end, consider any open ε-ball $B_\varepsilon(\gamma)$ in $\tilde{\mathcal{L}}_\Sigma$ for some fixed $\gamma \in \tilde{\mathcal{L}}_\Sigma$. Set $\delta_1 = \frac{1}{2}\varepsilon$ and $\delta_2 = \frac{3}{4}\varepsilon$. Since $\tilde{\mathcal{L}}_\Sigma$ is a metric space it is normal and hence, by Urysohn's Lemma, $\exists\, \hat{\psi} : \tilde{\mathcal{L}}_\Sigma \to [0,1]$ continuous such that

$$\hat{\psi}[\eta] = \begin{cases} 1 & \forall \eta \in \overline{B_{\delta_1}(\gamma)}, \\ 0 & \forall \eta \in B_{\delta_2}(\gamma)^{\mathrm{c}}, \end{cases}$$

where $B_{\delta_2}(\gamma)^{\mathrm{c}} \overset{\text{def}}{=} \tilde{\mathcal{L}}_\Sigma - B_{\delta_2}(\gamma)$. In particular, $\operatorname{supp}(\hat{\psi}) \subset B_\varepsilon(\gamma)$ by construction and $\|\hat{\psi}\| = 1$. Hence, $\mu^*(B_\varepsilon(\gamma)) \geqq I(\hat{\psi}) > 0$ and μ^* is thus a non-trivial function on $2^{\tilde{\mathcal{L}}_\Sigma}$.

It is evident from the definition that $\mu^*(\varnothing) = 0$ and $\mu^*(A) \leqq \mu^*(B)$ whenever $A \subseteq B$. Now, let $\{G_i\}_i$ be a sequence of open subsets in $\tilde{\mathcal{L}}_\Sigma$ and set $G = \bigcup_i G_i$. By Lemma 5.2.3, $\operatorname{supp}(\psi) \subset G \Rightarrow \exists\, \psi_i \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma)$ such that $\operatorname{supp}(\psi_i) \subset G_i$ and $\psi = \sum_i \psi_i$. Hence,

$$\begin{aligned} \mu^*(G) &= \sup\{\, I(\psi) \mid \psi \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma),\, \operatorname{supp}(\psi) \subset G \,\} \\ &= \sup\left\{ \sum_i I(\psi_i) \,\middle|\, \psi = \sum_i \psi_i,\, \psi_i \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma),\, \operatorname{supp}(\psi_i) \subset G_i \right\} \\ &\leqq \sum_i \mu^*(G_i). \end{aligned}$$

Next, let $A = \bigcup_i A_i \in 2^{\tilde{\mathcal{L}}_\Sigma}$. By (5.2.3), $\exists\, G_i$ open such that $A_i \subset G_i$ and $\mu^*(G_i) \leqq \mu^*(A_i) + \varepsilon/2^i$ for each i. Consequently, $\mu^*(A) \leqq \mu^*(G) \leqq \sum_i \mu^*(G_i) \leqq \sum_i \mu^*(A_i) + \varepsilon$, and so, the arbitrariness of $\varepsilon > 0$ implies that μ^* is an outer measure. Note that outer-regularity follows immediately from (5.2.3).

To complete the proof, it is enough to show that Borel sets are μ^*-measurable. For then, restricting μ^* to the Borel σ-algebra $\mathfrak{B}(\tilde{\mathcal{L}}_\Sigma)$ of $\tilde{\mathcal{L}}_\Sigma$ yields the desired (Borel) measure on $\tilde{\mathcal{L}}_\Sigma$ [2, Theorem 1.3.4, p. 18]. That is, one must show that for each Borel subset G,

$$\mu^*(A \cap G) + \mu^*(A \cap G^{\mathrm{c}}) = \mu^*(A) \quad \forall\, A \in 2^{\tilde{\mathcal{L}}_\Sigma} \text{ with } \mu^*(A) < \infty,$$

where $G^c = \tilde{\mathcal{L}}_\Sigma - G$. Now, because $\tilde{\mathcal{L}}_\Sigma$ is separable, it is sufficient to verify that $B_\delta(\gamma)$ is μ^*-measurable for any $\delta > 0$ and $\gamma \in \tilde{\mathcal{L}}_\Sigma$.[1] And since μ^* is σ-subadditive (as it is an outer measure), it is enough to establish that $\forall\, A \in 2^{\tilde{\mathcal{L}}_\Sigma}$ with $\mu^*(A) < \infty$, $\mu^*(A) \geqq \mu^*(A \cap B_\delta(\gamma)) + \mu^*(A \cap B_\delta(\gamma)^c)$. Using (5.2.3), choose an open subset D, $\mu^*(D) < \infty$, so that $A \subset D$ and $\mu^*(D) \leqq \mu^*(A) + \frac{1}{3}\varepsilon$. Using (5.2.2), consider $\psi_\varepsilon \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma)$ such that $\operatorname{supp}(\psi_\varepsilon) \subset D \cap B_\delta(\gamma)$ and $\mu^*(D \cap B_\delta(\gamma)) \leqq I(\psi_\varepsilon) + \frac{1}{3}\varepsilon$. Set $C_\varepsilon = \operatorname{supp}(\psi_\varepsilon)$. Then, clearly, $\exists\, \phi_\varepsilon \in \mathcal{K}_+(\tilde{\mathcal{L}}_\Sigma)$ such that $\operatorname{supp}(\phi_\varepsilon) \subset D \cap C_\varepsilon^c$ and $\mu^*(D \cap C_\varepsilon^c) \leqq I(\phi_\varepsilon) + \frac{1}{3}\varepsilon$ (as C_ε^c is open and $\mu^*(D) < \infty$). Whence, $\mu^*(D \cap B_\delta(\gamma)) + \mu^*(D \cap B_\delta(\gamma)^c) \leqq I(\psi_\varepsilon) + I(\phi_\varepsilon) + \frac{2}{3}\varepsilon$. However, by construction, $A \cap B_\delta(\gamma) \subseteq D \cap B_\delta(\gamma)$ and $D \cap B_\delta(\gamma)^c \subseteq D \cap C_\varepsilon^c$. Consequently,

$$\mu^*(A \cap B_\delta(\gamma)) + \mu^*(A \cap B_\delta(\gamma)^c) \leqq I(\psi_\varepsilon) + I(\phi_\varepsilon) + \tfrac{2}{3}\varepsilon \equiv I(\psi_\varepsilon + \phi_\varepsilon) + \tfrac{2}{3}\varepsilon.$$

Since $\operatorname{supp}(\psi_\varepsilon) \subset D \cap B_\delta(\gamma)$ and $\operatorname{supp}(\phi_\varepsilon) \subset D \cap C_\varepsilon^c$, $\operatorname{supp}(\psi_\varepsilon) \cup \operatorname{supp}(\phi_\varepsilon) \subset D$ and hence $I(\psi_\varepsilon + \phi_\varepsilon) \leqq \mu^*(D)$. Thus,

$$\mu^*(A \cap B_\delta(\gamma)) + \mu^*(A \cap B_\delta(\gamma)^c) \leqq I(\psi_\varepsilon + \phi_\varepsilon) + \tfrac{2}{3}\varepsilon \leqq \mu^*(A) + \varepsilon$$

implies, from the arbitrariness of $\varepsilon > 0$, that $B_\delta(\gamma)$ is μ^*-measurable, and the result thus follows. □

5.2.5. Corollary. *$\mathcal{L}_\Sigma$ admits an outer-regular Borel measure μ.*

Proof. Fix an outer-regular Borel measure $\tilde{\mu}$ on $\tilde{\mathcal{L}}_\Sigma$ (induced by some positive linear functional $\tilde{I}$—*cf.* the proof of Theorem 5.2.4). Since for any subset $A \subset \tilde{\mathcal{L}}_\Sigma$,

$$\tilde{\pi}^{-1} \circ \tilde{\pi}(A) = \begin{cases} A \cup \mathcal{L}_0 & \text{if } \mathcal{L}_0 \cap A \neq \varnothing, \\ A & \text{if } \mathcal{L}_0 \cap A = \varnothing, \end{cases}$$

and $\tilde{\mu}(\mathcal{L}_0) = 0$ by (5.2.3) as it has empty interior, it follows that $\mu : \mathfrak{B}(\mathcal{L}_\Sigma) \to \overline{\mathbb{R}}_+$ given by $\mu(G) \overset{\text{def}}{=} \tilde{\mu}(\tilde{\pi}^{-1}(G))$ is a well-defined outer-regular Borel measure on $\mathcal{L}_\Sigma$. □

This section will conclude with the construction of a $\mathrm{Diff}^+(\Sigma)$-invariant (outer-regular) Borel measure on $\mathcal{L}_\Sigma$, where $\mathrm{Diff}^+(\Sigma)$ is the set of smooth, orientation-preserving diffeomorphisms on Σ. Let $R_\Sigma \subset \mathcal{L}_\Sigma \times \mathcal{L}_\Sigma$ be an equivalence relations generated by $\sim$ in the following way:

$$\gamma \sim \eta \quad \Longleftrightarrow \quad \exists\, f \in \mathrm{Diff}^+(\Sigma) \text{ such that } \eta = f \circ \gamma.$$

[1] That this is the case follows trivially from the fact that for a metric space, separability is equivalent to second countability. This in turn implies that every open subset in $\tilde{\mathcal{L}}_\Sigma$ can be expressed as a countable union of open balls.

Let $L_\Sigma^+ \stackrel{\text{def}}{=} \mathcal{L}_\Sigma / R_\Sigma$ denote the quotient space and $\pi_\Sigma : \mathcal{L}_\Sigma \to L_\Sigma^+$ denote the natural map $\gamma \mapsto [\gamma]$, where $[\gamma] \stackrel{\text{def}}{=} \{\, f \circ \gamma \mid f \in \mathrm{Diff}^+(\Sigma) \,\}$ and $f \circ 0_\Sigma \stackrel{\text{def}}{=} 0_\Sigma \ \forall f \in \mathrm{Diff}^+(\Sigma)$. It is easy to see that π_Σ is an open map, for if $G \subset \tilde{\mathcal{L}}_\Sigma$ is open, then $\pi_\Sigma^{-1} \circ \pi_\Sigma(G) = \bigcup_{f \in \mathrm{Diff}^+(\Sigma)} f \cdot G$, where $f \cdot G \stackrel{\text{def}}{=} \{\, f \circ \gamma \mid \gamma \in G \,\}$, is also open as $f \cdot G$ is open for each f. Note that because $\mathcal{L}_\Sigma$ is second countable, the second countability of L_Σ^+ follows immediately from the openness of π_Σ. Finally, note also that a $\mathrm{Diff}^+(\Sigma)$-invariant measure on $\mathcal{L}_\Sigma$ is equivalent to a measure on L_Σ^+.

5.2.6. Lemma. *Let $B_\varepsilon(\gamma) \subset \tilde{\mathcal{L}}_\Sigma$ for any $\varepsilon > 0$ and $\gamma \in \tilde{\mathcal{L}}_\Sigma$ and set $D_{\varepsilon,\gamma} = \pi_\Sigma^{-1} \circ \pi_\Sigma(B_\varepsilon(\gamma))$. Let $S = \{\, \varepsilon_\eta \mid \eta \in D_{\varepsilon,\gamma} \,\}$ be a set of positive numbers, where $\varepsilon_\eta \stackrel{\text{def}}{=} \sup\{\, \delta : B_\delta(\eta) \subset D_{\varepsilon,\gamma} \,\}$. Then, S is bounded.*

Proof. Suppose that S is not bounded. This means that there exists an unbounded increasing sequence $\{\varepsilon_n\}_n$ in S such that $B_{\varepsilon_n}(\eta_n) \subset D_{\varepsilon,\gamma}\ \forall n$ by definition, where $\lim_{n\to\infty} \varepsilon_n = \infty$. However, because for any $\xi \in \tilde{\mathcal{L}}_\Sigma$, $\lim_{r\to\infty} B_r(\xi) = \tilde{\mathcal{L}}_\Sigma$—since given any $\eta \in \tilde{\mathcal{L}}_\Sigma$, $\exists N > 0$ sufficiently large such that $\eta \in B_r(\xi)$ whenever $r > N$—it follows from the finiteness of $\varepsilon > 0$ that such an increasing sequence cannot exists. Hence, S must be bounded. $\square$

5.2.7. Theorem. *There exists an outer-regular Borel measure on L_Σ^+.*

Proof. Let μ denote an outer-regular Borel measure on $\mathcal{L}_\Sigma$ and define $\hat{\mu}^*$ on the set of open subsets of L_Σ^+ as follows. For each open subset $D \subset L_\Sigma^+$, consider the pair $(\gamma, \varepsilon_\gamma)$, where $\gamma \in \pi_\Sigma^{-1}(D)$ and ε_γ is the largest positive real number (possibly ∞) such that $B_{\varepsilon_\gamma}(\gamma) \subset \pi_\Sigma^{-1}(D)$. Let $S(D) = \{\, (\gamma, \varepsilon_\gamma) \mid \gamma \in \pi_\Sigma^{-1}(D) \,\}$. Then,

$$\hat{\mu}^*(D) \stackrel{\text{def}}{=} \sup\{\, \mu(B_{\varepsilon_\gamma}(\gamma)) \mid (\gamma, \varepsilon_\gamma) \in S(D) \,\}$$

is a well-defined function on the set of all open subsets in L_Σ^+ by lemma 3.6. Next, extend $\hat{\mu}^*$ to the power set $2^{L_\Sigma^+}$ of L_Σ^+ by

$$\hat{\mu}^*(A) \stackrel{\text{def}}{=} \inf\{\, \mu^*(U) \mid A \subseteq U,\ U \text{ open} \,\}.$$

It is obvious that $\hat{\mu}^* : 2^{L_\Sigma^+} \to \overline{\mathbb{R}}_+$ is a well-defined but non-trivial function by construction.

It is not difficult to show that $\hat{\mu}^*$ is an outer measure. Indeed, to this end, it is enough to verify that $\hat{\mu}^*$ is σ-subadditive since $\hat{\mu}^*(\varnothing) = 0$ and $\hat{\mu}^*(A) \leqq \hat{\mu}^*(B)\ \ \forall A \subseteq B$. So, first, let $\{G_i\}_i$ be a sequence of open subsets in L_Σ^+. Then, it is evident that $\hat{\mu}^*(\bigcup_i G_i) = \sup\{\, \mu(B_{\varepsilon_\gamma}(\gamma)) \mid (\gamma, \varepsilon_\gamma) \in S(\bigcup_i G_i) \,\} \leqq$

$\sup\{\sum_i \mu(B_{\varepsilon_i}(\gamma_i)) \mid (\gamma_i, \varepsilon_i) \in S(G_i)\} \leqq \sum_i \sup\{\mu(B_{\varepsilon_\gamma}(\gamma)) \mid (\gamma, \varepsilon_\gamma) \in S(G_i)\} = \sum_i \hat{\mu}^*(G_i)$. Now let $\{A_i\}_i$ be a sequence of subsets in L_Σ^+. Fix any $\varepsilon > 0$ and consider open subsets $G_i \subset L_\Sigma^+$ such that $A_i \subset G_i$ and $\hat{\mu}^*(G_i) \leqq \hat{\mu}^*(A_i) + \varepsilon/2^i$ for each i. Then, by construction, $\hat{\mu}^*(\bigcup_i A_i) \leqq \hat{\mu}^*(\bigcup_i G_i) \leqq \sum_i \hat{\mu}^*(G_i) \leqq \sum_i \hat{\mu}^*(A_i) + \varepsilon$. Consequently, $\varepsilon > 0$ is arbitrary implies that $\hat{\mu}^*$ is σ-subadditive and hence an outer measure. By construction, $\hat{\mu}^*$ is outer-regular.

To conclude this proof, it remains to verify that Borel subsets are $\hat{\mu}^*$-measurable; for then, restricting $\hat{\mu}^*$ to the Borel σ-algebra of L_Σ^+ will provide the desired measure. In particular, it is enough to show that for any open subset D of L_Σ^+,

$$\hat{\mu}^*(A) \geqq \hat{\mu}^*(A \cap D) + \hat{\mu}^*(A \cap D^c) \quad \forall A \in 2^{L_\Sigma^+} \text{ with } \hat{\mu}^*(A) < \infty.$$

Fix any $\varepsilon > 0$ and choose an open subset $G \subset L_\Sigma^+$, where $\hat{\mu}^*(G) < \infty$, such that $A \subset G$ and $\hat{\mu}^*(G) \leqq \hat{\mu}^*(A) + \frac{1}{2}\varepsilon$.

Note first of all that given any $\varepsilon > 0$, there exist open subsets $U'_\varepsilon, U_\varepsilon$ satisfying

$$U'_\varepsilon \subset G \cap D^c \subset U_\varepsilon \quad \text{and} \quad \hat{\mu}^*(U_\varepsilon) \leqq \hat{\mu}^*(U'_\varepsilon) + \tfrac{1}{2}\varepsilon.$$

Hence, for the given $\varepsilon > 0$, $\hat{\mu}^*(G \cap D^c) \leqq \hat{\mu}^*(U_\varepsilon) \leqq \hat{\mu}^*(U'_\varepsilon) + \frac{1}{2}\varepsilon$. Furthermore, since $\pi_\Sigma^{-1}(U'_\varepsilon) \cap \pi_\Sigma^{-1}(G \cap D) = \varnothing$ and $U'_\varepsilon \subset G \cap D^c$, it follows that $\pi_\Sigma^{-1}(U'_\varepsilon) \cup \pi_\Sigma^{-1}(G \cap D) \subset \pi_\Sigma^{-1}(G)$ and hence $\hat{\mu}^*(U'_\varepsilon) + \hat{\mu}^*(G \cap D) \leqq \hat{\mu}^*(G)$. Whence, the fact that $A \cap D \subset G \cap D$ and $A \cap D^c \subset G \cap D^c$ imply that $\hat{\mu}^*(A \cap D) + \hat{\mu}^*(A \cap D^c) \leqq \hat{\mu}^*(G) + \frac{1}{2}\varepsilon \leqq \hat{\mu}^*(A) + \varepsilon$, and the arbitrariness of $\varepsilon > 0$ thus yields the assertion. □

5.3. A Diff$^+(\Sigma)$-Invariant Measure on $\mathcal{M}$.

The construction of a Diff$^+(\Sigma)$-invariant measure on $\mathcal{M}_n$ will be made first for $n > 1$ and then followed by a diffeomorphism-invariant measure on $\mathcal{M}$. Observe that as $\mathcal{M}_1$ is identical to $\mathcal{L}_\Sigma$, an outer-regular Borel measure trivially exists on it. By theorem 5.2.7, a Diff$^+(\Sigma)$-invariant measure exists on $\mathcal{L}_\Sigma$ and hence on $\mathcal{M}_1$. Denote this Diff$^+(\Sigma)$-invariant measure on $\mathcal{M}_1$ by ν_1.

5.3.1. Theorem. *For each $n > 0$, an outer-regular,* Diff$^+(\Sigma)$*-invariant Borel measure exists on the n-loop space $\mathcal{M}_n$.*

Proof. Recall from §3.2 that $\pi_n : \mathcal{L}_\Sigma^n \to \bigcup_{i=1}^n \mathcal{M}_i$ is a continuous, closed and open surjection, and that $\mathcal{M}_n \subset \bigcup_{i=1}^n \mathcal{M}_i$ is an open subset. Also, the case for $n = 1$ has already been established above. Let $\mathfrak{B}_n$ be the Borel σ-algebra of $\mathcal{L}_\Sigma^n$

generated by $\mathfrak{B}_1 \times \cdots \times \mathfrak{B}_1$ (n-times), where $\mathfrak{B}_1$ is the Borel σ-algebra of $\mathcal{L}_\Sigma$ and let $\tilde{\mu}_n : \tilde{\mathfrak{B}}_n \to \overline{\mathbb{R}}_+$ denote the $\mathrm{Diff}^+(\Sigma)$-invariant product measure on $\mathcal{L}_\Sigma^n$ induced from ν_1: $\tilde{\mu}_n(B_1 \times \cdots \times B_n) \stackrel{\text{def}}{=} \nu_1(B_1)\ldots\nu_1(B_n)$. Define ν_n^* on the set of open subsets in $\mathcal{M}_n$ in a similar way to $\hat{\mu}^*$ in the proof of theorem 5.2.7. That is, given an open subset $D \subset \mathcal{M}_n$, let $S(D) = \{((\gamma_1, \varepsilon_i), \ldots, (\gamma_n, \varepsilon_n)) \mid (\gamma_1, \ldots, \gamma_n) \in \pi_\Sigma^{-1}(D)\}$, where $\varepsilon_1, \ldots, \varepsilon_n$ are the largest positive numbers (possibly ∞) for the given n-tuple $(\gamma_1, \ldots, \gamma_n) \in \pi_\Sigma^{-1}(D)$ satisfying $B_{\varepsilon_1}(\gamma_1) \times \cdots \times B_{\varepsilon_n}(\gamma_n) \subset \pi_\Sigma^{-1}(D)$. Then, define ν_n^* by

$$\nu_n^*(G) = \sup\{\, \tilde{\mu}_n(B_{\varepsilon_1}(\gamma_1) \times \cdots \times B_{\varepsilon_n}(\gamma_n)) \mid ((\gamma_1, \varepsilon_1), \ldots, (\gamma_n, \varepsilon_n)) \in S(D) \,\}$$

and extend ν_n^* to the power set $2^{\mathcal{M}_n}$ of $\mathcal{M}_n$ by

$$\nu_n^*(A) = \inf\{\, \nu_n^*(G) \mid A \subseteq G,\ G \text{ open} \,\}.$$

The set-function ν_n^* is well-defined by lemma 5.2.6 and the verification that $\nu_n \stackrel{\text{def}}{=} \nu_n^*|\mathfrak{B}_n$, where $\mathfrak{B}_n$ is the Borel σ-algebra of $\mathcal{M}_n$, is indeed an outer-regular Borel measure on $\mathcal{M}_n$ is identical to the proof of theorem 5.2.7. Last, notice that ν_n is $\mathrm{Diff}^+(\Sigma)$-invariant by construction as ν_1 is $\mathrm{Diff}^+(\Sigma)$-invariant. $\square$

The construction of an outer-regular Borel measure ν_∞ on $\mathcal{M}_\infty$ requires some modification. An infinite product measure cannot be constructed from ν_1 on $\mathcal{M}_\infty$ as ν_1 is not a bounded measure on $\mathcal{M}_1$. However, observe from the proof of theorem 5.2.4 that $\tilde{\mathcal{L}}_\Sigma$ may be replaced with any second countable metrizable space. In particular, the theorem holds for $\mathcal{M}_\infty$ as it is second countable and metrizable—see remark 3.2.10. Hence, an outer-regular Borel measure μ_∞ on $\mathcal{M}_\infty$ exists. Now, let $R_\infty \subset \mathcal{M}_\infty \times \mathcal{M}_\infty$ be an equivalence relation generated by $\sim$ in the following way:

$$\gamma \sim \eta \quad \Longleftrightarrow \quad \exists\, f \in \mathrm{Diff}^+(\Sigma) \text{ such that } \eta^i = f \circ \gamma^i \ \forall\, i$$

(by reordering the elements of η if necessary). That is, $f \cdot \gamma = \eta$, where $f \cdot \gamma \stackrel{\text{def}}{=} \{\, f \circ \gamma^i \mid \gamma^i \in \gamma \}$. Let $L_\infty^+ \stackrel{\text{def}}{=} \mathcal{M}_\infty / R_\infty$ denote the quotient space and $\pi_\infty : \mathcal{M}_\infty \to L_\infty^+$ denote the quotient map. It is not difficult to check that π_∞ is open. Then, in the light of the proof of theorem 5.2.7, it is possible to construct a non-trivial outer-regular Borel measure on L_∞^+ induced from the measure on $\mathcal{M}_\infty$ in an identical fashion. To wit, if μ_∞ is an outer-regular Borel measure on $\mathcal{M}_\infty$, then $\hat{\mu}_\infty^* : 2^{\mathcal{M}_\infty} \to \overline{\mathbb{R}}_+$ defined in the identical way as $\hat{\mu}^*$ was defined in the proof of theorem 5.2.7 yields a well-defined outer-regular Borel measure on L_∞^+. Since a measure on L_∞^+ is equivalent to a $\mathrm{Diff}^+(\Sigma)$-invariant measure on $\mathcal{M}_\infty$, the following proposition is established.

5.3.2. Proposition. *There exists an outer-regular,* $\mathrm{Diff}^+(\Sigma)$*-invariant Borel measure* ν_∞ *on* $\mathcal{M}_\infty$. □

The principal result of this paper is now due and, with it, this paper will be brought to a timely conclusion.

5.3.3. Theorem. $\mathcal{M} = \bigoplus_{n=1}^{\infty} \mathcal{M}_n \oplus \mathcal{M}_\infty$ *has an outer-regular Borel measure that is invariant under* $\mathrm{Diff}^+(\Sigma)$.

Proof. Let $\mathfrak{B}$ be the Borel σ-algebra of $\mathcal{M}$ and define $\nu : \mathfrak{B} \to \overline{\mathbb{R}}_+$ by

$$\nu(G) = \sum_{n=1}^{\infty} \nu_n(G \cap \mathcal{M}_n).$$

Then, it is enough to show that ν defines a measure since the outer-regularity of ν follows trivially from the outer-regularity of ν_n for each $n \leqq \infty$. Clearly, $\nu(\varnothing) \equiv 0$. Now, suppose that $\bigcup_i G_i \in \mathfrak{B}$ is any disjoint union; then, $\nu(\bigcup_i G_i) = \sum_n \nu_n(\bigcup_i G_i \cap \mathcal{M}_n) = \sum_n \sum_i \nu_n(G_i \cap \mathcal{M}_n) \equiv \sum_i \sum_n \nu_n(G_i \cap \mathcal{M}_n) = \sum_i \nu(G_i)$. So, ν is a measure, as required. □

5.3.4. Remark. The constructed measures did not depend on a particular choice of (Riemannian) 3-metric of Σ in anyway—this is, of course, a crucial point in quantum gravity. It is also clear that the constructed measures are by no means unique.

From the definition of ν on $\mathcal{M}$, it is clear that not every open subset in $\mathcal{M}$ is ν-finite. However, it is obvious that if $G = G_{n_1} \cup \cdots \cup G_{n_i} \subset \mathcal{M}$ is a finite union of open subsets $G_{n_j} \subset \mathcal{M}_{n_j}$, and $\nu_{n_j}(G_{n_j}) < \infty$ for each n_j, then $\nu(G) < \infty$. However, this minor point regarding the construction of ν need not be of any concern since the measure that defines the physical inner product would be of the form $c[\gamma]\,\mathrm{d}\nu(\gamma)$, where $c[\gamma]$ is some suitable weight determined by the Hermiticity condition imposed on the observables—in particular, by the T-observables: $\langle \psi \,, T^n \phi\rangle = \langle (T^n)^\dagger \psi \,, \phi\rangle$ (and hence by the reality conditions). And most important of all, this factor would render the physical measure $c[\gamma]\,\mathrm{d}\nu(\gamma)$ finite: $\langle \psi \,, \psi\rangle < \infty$ for each unnormalised physical state $|\psi\rangle$. The crucial point to be emphasized here is the existence of a diffeomorphism-invariant multi-loop measure.

This section will end with a summary of the main results obtained. The following results—independent of the 3-metric on Σ—were established: (i) the existence of a $\mathrm{Diff}^+(\Sigma)$-invariant, outer-regular Borel measure on $\mathcal{M}_n$, for each $n \geqq 1$, and (ii) an outer-regular, $\mathrm{Diff}^+(\Sigma)$-invariant Borel measure on $\mathcal{M}$.

5.4. Discussion

In this final section, the Hermitian conjugates of the quantum T^n-operators will be explored.[2] And apart from making some tentative remarks regarding the question of an inner product, this question will be covered in the final chapter where the issue of the reality conditions in the loop representation will be broached.

First, the Hermitian conjugate of the T^0-operator will be analysed and then followed by the Hermitian conjugates of the T^n-operators. As a preliminary observation, one might suspect at first glance that the explicit form of the action of the Hermitian conjugate $\hat{T}^{\tilde{\dagger}}[\gamma, A]$ on Ψ_n depends on the explicit choice of inner products used. Surprisingly, this is not the case, as will be demonstrated below.

Now, suppose an inner product is defined on $\hat{\mathcal{M}}^*_n$, or at least, on a non-trivial subspace in it and consider the following requirement for Hermitian conjugacy:

$$\langle \hat{T}^{\tilde{\dagger}}[\gamma, A]\Psi \,, \Phi\rangle = \langle \Psi \,, \hat{T}[\gamma, A]\Phi\rangle. \tag{5.4.1}$$

It is clear from (5.4.1) that the right-hand side is well-defined if $\Psi \in \mathcal{M}'_{n-1}$ whenever $\Phi \in \mathcal{M}'_n$; hence, in order for the left-hand side of the inner product to be well-defined, it is required that $\hat{T}^{\tilde{\dagger}}[\gamma, A]$ be a "raising" quantum operator. That is, for the left-hand side of the equality, it is required that $\hat{T}^{\tilde{\dagger}}[\gamma, A]\Psi \in \mathcal{M}'_n$ for consistency as $\Phi \in \mathcal{M}'_n$. This immediately suggests that $\hat{T}^{\tilde{\dagger}}[\gamma, A]$ acts on an $(n-1)$-loop functional Ψ_{n-1} by

$$(\hat{T}^{\tilde{\dagger}}[\gamma^1, A]\Psi_{n-1})[\eta] \stackrel{\text{def}}{=} \begin{cases} \Psi_{n-1}[\eta - \{\gamma^1\}] & \text{if } \gamma^1 \in \eta, \\ 0 & \text{if } \gamma^1 \notin \eta, \end{cases}$$

where $\eta \in \mathcal{M}_n$ and $\gamma^1 \in \mathcal{M}_1$. Thus far, the definition of $\hat{T}^{\tilde{\dagger}}[\gamma, A]\Psi$ is consistent with that given by Rayner [4, p. 656]. Indeed, the astonishing result here is that the definition of the Hermitian conjugate of the T^0-operator did not depend on the explicit definition of the inner product at all: it merely followed from a consistency requirement. From §3.5, the explicit form of $\hat{T}^{\tilde{\dagger}}[\gamma, A]\Psi_n$, where $\Psi_n \in \hat{\mathcal{M}}^*_n$, can indeed be constructed. To do this, define a *loop Kronecker function* $\tilde{\delta}_\gamma : \mathcal{M}_1 \to \{0,1\}$ by

$$\tilde{\delta}_\gamma[\eta] \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \gamma = \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is evident that the map $\hat{T}^{\tilde{\dagger}}[\gamma, A] : \hat{\mathcal{M}}^*_{n-1} \to \hat{\mathcal{M}}^*_n$ is just given by $\hat{T}^{\tilde{\dagger}}[\gamma, A]\Psi_n \equiv \Psi_n \otimes^{\mathcal{M}} \tilde{\delta}_\gamma$.

[2] Strictly, to be consistent with the notations introduced in Remark 2.2.2, the term Hermitian here should really be $\tilde{\dagger}$-Hermitian. However, for convenience, $\tilde{\dagger}$ will be left out.

Next, the Hermitian conjugate of the quantum T^1-operator will be considered and then the generalisation to the T^n-operator will be made. For simplicity, the unsmeared operators will be used in the following analysis. As a brief reminder, $(\hat{T}^a[\gamma, A]\Psi_n)[\gamma_n]$, where $\Psi_n \in \mathcal{M}'_n$ and $\gamma_n \in \mathcal{M}_n$, is defined by

$$\hat{T}^a[\gamma, A](s)\Psi[\gamma_n] = -\hbar\Delta^a[\gamma, \gamma_n](s)\Psi[\gamma * \gamma_n] + \hbar\Delta^a[\gamma, \gamma_n](s)\Psi[\gamma * \gamma_{n-}],$$

where (i) $\gamma_n \in \mathcal{M}_n$, $\gamma_n = \{\gamma^1, \ldots, \gamma^n\}$ and $\gamma * \gamma_n$ is the abbreviation for $\{\gamma * \gamma^1, \ldots, \gamma * \gamma^n\}$; (ii) $\gamma_{n-} = \{\gamma^1_-, \ldots, \gamma^n_-\}$, (iii) s corresponds to the point where γ intersects γ^i (at $\gamma(s)$) in the definition of $\gamma * \gamma^i$, for each i; (iv) the distribution $\Delta^a[\gamma, \gamma_n](s) \stackrel{\text{def}}{=} \sum_{i=1}^n \Delta^a[\gamma, \gamma^i](s)$, where $\Delta^a[\gamma, \gamma^i](s)$ given by

$$\Delta^a[\gamma, \gamma^i](s) = \frac{1}{2}\int_0^1 \delta^3(\gamma(s), \gamma^i(t))(\dot{\gamma}^i)^a(t)\mathrm{d}t.$$

Note in hindsight that the concept of a distribution is needed in order to define the Hermitian conjugate of the T-operators. The details of a distribution can be found in [1, p. 437]. In this section, the term "distribution" should be taken with a grain of salt simply because the multi-loop spaces are not locally compact nor indeed is the concept of derivatives defined on them, whilst distributions (in the strict sense) are functionals belonging to the dual of the space of smooth funtions with compact support. A rigorous development would lead one far afield from the original aim of this section. As such, one shall be content with an informal presentation given here and realise that the definition of distributions introduced below are strictly directed at a heuristic level.

It will be assumed here for concreteness that the inner product $\langle \cdot, \cdot \rangle_n$ is defined by the measure $c_n[\gamma]\, \mathrm{d}\nu_n(\gamma)$ for each n, where c_n left unspecified in this section, such that there exists a non-empty subspace in $\hat{\mathcal{M}}^*_n$ whose elements have finite norms with respect to the inner product. In fact, for the convenience of the following discussion, it will be assumed that $\hat{\mathcal{M}}^*_n$ can be endowed with an inner product defined by ν_n for each n.

5.4.1. Definition. Let δ_γ be a distribution on $\mathcal{M}_1$ that satisfies

(1) $\langle \delta_\gamma, \phi \rangle_1 = \phi[\gamma]\ \forall \phi \in \mathcal{M}'_1$ and $\gamma \in \mathcal{M}_1$,

(2) $\langle \delta_\gamma, \phi \rangle_1 = \langle \bar{\delta}_\gamma, \phi \rangle_1$,

where the inner product $\langle \cdot, \cdot \rangle_1$ is defined with respect to ν_1. Then δ_γ is called a *Dirac ν_1-distribution* on $\mathcal{M}_1$.

5.4.2. Remark. Let $\gamma_n = \{\gamma^1, \ldots, \gamma^n\}$ and set $\delta^n_{\gamma_n} \stackrel{\text{def}}{=} \delta_{\gamma^1} \otimes^{\mathcal{M}} \ldots \otimes^{\mathcal{M}} \delta_{\gamma^n}$. Then, $\langle \delta^n_{\gamma_n}, \Psi_n \rangle_n = \Psi_n[\gamma_n]$. To see this, it is enough to note that $\delta^n_{\gamma_n}[\eta_n] \equiv 0$ whenever

$\eta_n \neq \gamma_n$ by definition, where $\eta_n \in \mathcal{M}_n$. Call $\delta^n_{\gamma_n}$ the *Dirac ν_n-distribution* on $\mathcal{M}_n$. Finally, it should be pointed out that the inner products defined here have no physical significance; and indeed, their introduction were carried out in a rather cavalier fashion.

It should also be emphasized that just by assuming the existence of an inner product on $\hat{\mathcal{M}}^*_n$ without any specification on the type of inner product chosen, explicit forms of the Hermitian conjugate of the T^n-operators can be defined! However, a clearer analysis will destroy the illusion that the Hermitian conjugates of the T^n-operators for $n > 0$ can be defined independently of the choice of inner products—this will be seen shortly below. Briefly, this misleading conclusion arises from the observation (see below) that the definition of δ_γ or $\delta^n_{\gamma_n}$ defined above *depends* implicitly on the precise choice of inner product used.

A heuristic sketch for finding the Hermitian conjugate $\hat{T}^a[\gamma, A]^{\bar{\dagger}}(s)$ of $\hat{T}^a[\gamma, A](s)$ will now be illustrated. Let $\psi, \phi \in \mathcal{M}^*_1$. Then,

$$\begin{aligned}\langle \psi\, , \hat{T}^a[\gamma, A](s)\phi\rangle = &- \hbar \int \bar{\psi}[\eta]\Delta^a[\gamma, \eta](s)\phi[\gamma * \eta]\mathrm{d}\nu_1(\eta)\\ &+ \hbar \int \bar{\psi}[\eta]\Delta^a[\gamma, \eta](s)\phi[\gamma * \eta_-]\mathrm{d}\nu_1(\eta).\end{aligned}$$

However, $\int \bar{\psi}[\eta]\Delta^a[\gamma, \eta]\delta_{\gamma*\eta}[\lambda]\phi[\lambda]\mathrm{d}\nu_1(\lambda)\mathrm{d}\nu_1(\eta) = \int \bar{\psi}[\eta]\Delta^a[\gamma, \eta]\phi[\gamma * \eta]\mathrm{d}\nu_1(\eta)$. Thus, this suggests that $\hat{T}^a[\gamma, A]^{\bar{\dagger}}(s)$ should be defined by

$$\begin{aligned}(\hat{T}^a[\gamma, A]^{\bar{\dagger}}(s)\bar{\psi})[\eta] = &- \hbar \int \bar{\psi}[\lambda]\Delta^a[\gamma, \lambda](s)\delta_{\gamma*\lambda}[\eta]\, \mathrm{d}\nu_1(\lambda)\\ &+ \hbar \int \bar{\psi}[\lambda]\Delta^a[\gamma, \lambda](s)\delta_{\gamma*\lambda_-}[\eta]\, \mathrm{d}\nu_1(\lambda).\end{aligned}$$

5.4.3. Proposition. *Let $\Phi \in \hat{\mathcal{M}}^*_n$, $\eta, \lambda \in \mathcal{M}_n$ and $\gamma \in \mathcal{M}_1$. Then, the map $\hat{T}^a[\gamma, A]^{\bar{\dagger}}(s)$ defined on $\hat{\mathcal{M}}^*_n$ by*

$$\begin{aligned}(\hat{T}^a[\gamma, A]^{\bar{\dagger}}(s)\bar{\Phi})[\eta] = &- \hbar \int \bar{\Phi}[\lambda]\Delta^a[\gamma, \lambda](s)\delta^n_{\gamma*\lambda}[\eta]\, \mathrm{d}\nu_n(\lambda)\\ &+ \hbar \int \bar{\Phi}[\lambda]\Delta^a[\gamma, \lambda](s)\delta^n_{\gamma*\lambda_-}[\eta]\, \mathrm{d}\nu_n(\lambda)\end{aligned}$$

defines the Hermitian conjugate of $\hat{T}^a[\gamma, A](s)$.[3]

[3] Strictly, the T^n-operators and their Hermitian conjugates do not define maps from $\hat{\mathcal{M}}^*$ into itself unless they are smeared with suitable smearing functions. This will be ignored in this informal section in the interest of simplicity.

Proof. Let $\Psi, \Phi \in \mathcal{M}_n^*$ and set $I = \langle \Psi, \hat{T}^a[\gamma, A](s)\Phi \rangle$, $I^{\dagger} = \langle \hat{T}^a[\gamma, A]^{\dagger}\Psi, \Phi \rangle$. Then,

$$\begin{aligned} I^{\dagger} &= -\hbar \int \Delta^a[\gamma, \lambda](s)\bar{\Psi}[\lambda]\delta^n_{\gamma*\lambda}[\eta]\Phi[\eta]\mathrm{d}\nu_n(\lambda)\mathrm{d}\nu_n(\eta) \\ &\quad + \hbar \int \Delta^a[\gamma, \lambda](s)\bar{\Psi}[\lambda]\delta^n_{\gamma*\lambda_-}[\eta]\Phi[\eta]\mathrm{d}\nu_n(\lambda)\mathrm{d}\nu_n(\eta) \\ &= -\hbar \int \Delta^a[\gamma, \lambda](s)\bar{\Psi}[\lambda]\Phi[\gamma * \lambda]\mathrm{d}\nu_n(\lambda) \\ &\quad + \hbar \int \Delta^a[\gamma, \lambda](s)\bar{\Psi}[\lambda]\Phi[\gamma * \lambda_-]\mathrm{d}\nu_n(\lambda) \\ &\equiv I. \quad \square \end{aligned}$$

It is not difficult to deduce that the corresponding Hermitian conjugate of $\hat{T}^{a_1\ldots a_n}[\gamma, A]^{\dagger}(s_1, \ldots, s_n)$ has the following form:

$$\hat{T}^{a_1\ldots a_n}[\gamma, A]^{\dagger}(s_1, \ldots, s_n)\Psi[\eta] = -\hbar^n \sum_{\epsilon}(-1)^{n(\epsilon)} \int \Psi[\lambda]\Delta^{a_1}[\gamma, \lambda](s_1)\ldots$$
$$\Delta^{a_n}[\gamma, \lambda](s_n)\delta^k_{(\gamma*\lambda)^{\epsilon}}[\eta]\mathrm{d}\nu_k(\lambda),$$

where $\Psi \in \mathcal{M}_k^*$, $\lambda, \eta \in \mathcal{M}_m$ and $\epsilon = \epsilon_1 \ldots \epsilon_n$ for typesetting convenience. For more details regarding the notations used, refer to §2.4.

A short comment on the physical inner product will be made before closing this chapter. First, recall from the last paragraph of §3.4 that a multi-loop state is really the direct sum of n-loop functionals. This means that any inner product defined on the space of multi-loop functionals is of the form

$$\langle \cdot, \cdot \rangle = \sum_{n=1}^{\infty} \langle \cdot, \cdot \rangle_n,$$

where each inner product $\langle \cdot, \cdot \rangle_n$ is determined by the measure $c_n[\gamma]\,\mathrm{d}\nu_n(\gamma)$ with c_n being some predetermined functional on $\mathcal{M}_n$.

Second, observe that the equality in (5.4.1) is really of the form $\langle \cdot, \cdot \rangle_{n-1} = \langle \cdot, \cdot \rangle_n$: the inner products are defined on two distinct spaces! So, a necessary step towards finding the correct physical inner product is the determination of a sequence of n-loop functionals $\{c_n\}_n$ such that

$$\int_{\mathcal{M}_{n-1}} \mathrm{d}\nu_{n-1}(\gamma)c_{n-1}[\gamma]\bar{\Psi}[\gamma](\hat{T}[\gamma^1, A]\Phi)[\gamma] = \int_{\mathcal{M}_n} \mathrm{d}\nu_n(\gamma)c_n[\gamma](\hat{T}^{\dagger}[\gamma^1, A]\bar{\Psi})[\gamma]\Phi[\gamma]$$

for each n. However, a closer scrutiny will quickly reveal that the inner product is identically equal to zero! To see this, it will suffice to note that $\hat{T}^{\dagger}[\gamma^1, A]\bar{\Psi}_{n-1} = \bar{\Psi}_{n-1} \otimes^{\mathcal{M}} \delta_{\gamma^1}$. By definition then, $\hat{T}^{\dagger}[\gamma^1, A]\bar{\Psi}_{n-1}$ is zero outside of the subset

$$S(\gamma^1) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{M}_n \mid \gamma^1 \in \gamma\}.$$

However, $S(\gamma^1)$ is nowhere dense in $\mathcal{M}_n$ (as it is homeomorphic to $\mathcal{M}_{n-1}$) and hence ν_1-null; consequently, the inner product collapses to zero and no information regarding the functional form of c_n can be retrived from it. There is however one saving grace: these T-observables are not physical observables.

In conclusion, the determination of $\{c_n\}_n$ and hence the physical inner product requires the explicit implementation of the reality conditions in the loop representation. An attempt towards implementing the reality conditions in the loop representation will be outlined in Chapter 7.

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CHAPTER VI

A PROMEASURE ON THE SPACE OF ASHTEKAR CONNECTION 1-FORMS

6.1. Introduction

A promeasure (also known as a *cylindrical measure*) on the space $\mathcal{A}$ of Ashtekar connection 1-forms defined on Σ will be constructed in this chapter. This in turn will be used to construct a promeasure on the SU(2) moduli space $\mathcal{A}[\mathrm{SU}(2)]$—the space of SU(2)-gauge orbits of $\mathcal{A}$. Any two representatives of a coset in $\mathcal{A}[\mathrm{SU}(2)]$ are related by an SU(2)-gauge transformation. Like the multi-loop measure, the main interest in the construction of a promeasure of $\mathcal{A}[\mathrm{SU}(2)]$ lies in its application to the loop representation of quantum gravity—see, for example, [10, Eqn (3.40), p. 1644] and its accompanying footnote, or references [11] and [12, §6]. Very briefly, given a promeasure μ on $\mathcal{A}$, a relation between the functionals $\tilde{\Psi}[A]$ on $\mathcal{A}$ and the functionals $\Psi[\gamma]$ on the space of piecewise smooth loops of Σ can be obtained via

$$\Psi[\gamma] = \int_{\mathcal{A}} \overline{h[\gamma, A]}\tilde{\Psi}[A]\, \mathrm{d}\mu(A),$$

where $h[\gamma, A]$ is the trace of the complexified SU(2) holonomy. A second reason for constructing a measure lies in the attempt to define a physical inner product on the physical state space of the self-dual representation theory of quantum gravity.

Attempts at constructing promeasures on the space of connection 1-forms were also made by various researchers: primarily Ashtekar, Lewandowski and Baez [1, 2, 3, 5, 6]. Attention should be drawn to reference [2] in particular, where Ashtekar and Lewandowski constructed diffeomorphism-invariant promeasures on the space of connections and where the family of finite dimensional spaces defining the promeasure is a family of compact Hausdorff spaces. However, the programme is still not quite complete although a lot of advances have been made: for instance, in reference [4], a detailed exposition of the construction of a physical inner product for a class of diffeomorphism-invariant theories is given.

This chapter is based on the work done in reference [13]. It provides an alternative but possibly simpler approach towards the construction of a promeasure on the space of Ashtekar 1-forms. The projective family of spaces used here is a family of finite dimensional manifolds. A drawback of the construction described in this chapter is that the promeasure is not diffeomorphism-invariant.

The definition of a promeasure taken from [7, p. 576] will be briefly recalled here. For more details, refer to [7, §D, p. 573]. Let X be a locally convex, Hausdorff topological vector space and

$$F(X) \stackrel{\text{def}}{=} \{\, H \leqslant X \mid H \text{ a closed subspace with } \dim(X/H) < \infty \,\}$$

be the set of closed, finite codimensional subspaces H in X that is partially ordered by the set-inclusion $\subset$ relation. Then, $F(X)$ generates a *projective system* (or an *inverse system*) $(X/H, p_{VW})$ of finite dimensional quotient spaces of X via

$$p_W : X \to X/W,\ p_{VW} : X/W \to X/V, \text{ and } p_V = p_{WV} \circ p_W,$$

where $W, V \in F(X)$, $W \subset V$, and p_W, p_{VW} are canonical mappings.

6.1.1. Definition. A *promeasure* (or *cylindrical measure*) on X is a family $\mu = \{\, \mu_V \mid V \in F(X) \,\}$ of bounded measures μ_V on X/V such that

(1) $\mu_V = \mu_W \circ p_{VW}^{-1}$ whenever $W \subset V$, and $W, V \in F(X)$, and
(2) $\mu_V(X/V) = \mu_W(X/W)$ $\forall\, W, V \in F(X)$—$\mu_V(X/V)$ is called the *total mass* of the promeasure μ and is denoted by $\mu(X)$.

Let $\mathrm{Cyl}(X)$ be the *cylindrical σ-algebra* of X; that is, $\mathrm{Cyl}(X)$ is a σ-algebra generated by *cylindrical subsets* $p_V^{-1}(B)$ $\forall\, V \in F(X)$ and Borel subsets $B \subseteq X/V$. Then, a promeasure μ of X defines a finitely additive measure $\tilde{\mu} : \mathrm{Cyl}(X) \to \overline{\mathbb{R}}_+$ via $\tilde{\mu}(\tilde{B}) \stackrel{\text{def}}{=} \mu_V(B)$, where $\tilde{B} = p_V^{-1}(B)$. Note that $\tilde{\mu}$ is in general not σ-additive simply because cylindrical subsets do not necessarily have to be inverse images of Borel subsets from a fixed quotient space X/V, for some fixed $V \in F(X)$—V is allowed to vary.

The contents of this chapter are organised as follows. In section 2, the definition of the Ashtekar connection 1-forms on Σ will be briefly reviewed and some topological aspects of the space $\mathcal{A}$ of Ashtekar connection 1-forms will be studied. In section 3, a promeasure on $\mathcal{A}$ will be constructed. This is followed by the construction of a promeasure on $\mathcal{A}[\mathrm{SU}(2)]$ in section 4. In the final section, some possible ways of constructing a diffeomorphism-invariant promeasure on $\mathcal{A}[\mathrm{SU}(2)]$ will be sketched, where ideas from Ashtekar *et al.* [2] and Baez [6] will be tentatively imported.

6.2. The Space of Ashtekar Connections

The space of Ashtekar connection 1-forms will be briefly recalled. The details can be found in §2.2. Let $(T\Sigma \otimes \mathfrak{su}(2), p, \Sigma)$ be a tensor bundle over Σ and define $\mathcal{C}$ to be the space of smooth cross sections $\sigma : \Sigma \to T\Sigma \otimes \mathfrak{su}(2)$ satisfying:

(1) for each $x \in \Sigma$ and σ, $\sigma(x)$ induces a linear isomorphism $\mathfrak{su}(2) \approx T_x\Sigma$ defined by $\lambda \mapsto -\sigma(x) \cdot \lambda \overset{\text{def}}{=} \lambda^a(x)\partial_a$, where $-\sigma(x) \cdot \lambda \overset{\text{def}}{=} -\operatorname{tr}(\sigma(x)^a\lambda)\partial_a = -\sigma(x)^a{}_A{}^B\lambda_B{}^A\partial_a = \lambda^a(x)\partial_a \in T_x\Sigma$,

(2) $-\operatorname{tr}(\sigma^a\sigma^b) \overset{\text{def}}{=} q^{ab}$, where q_{ab} is a Riemannian 3-metric on Σ in the natural basis.

Then, the Ashtekar connection 1-form when restricted to the constraint surface of the phase space of general relativity is given by

$$A = s^*\omega_\sigma - \tfrac{\mathrm{i}}{\sqrt{2}}K(\sigma),$$

where $s : \Sigma \to P_\xi$ is a smooth cross section of the principal SU(2)-bundle $\xi = (P_\xi, \Sigma, \mathrm{SU}(2))$, ω_σ is an SU(2) connection 1-form on P_ξ compatible with σ, and K is the extrinsic curvature of Σ with $K(\sigma)_{aA}{}^B \overset{\text{def}}{=} K_{ab}\sigma^b_A{}^B$. Finally, recall that A is an $\mathfrak{su}^{\mathbb{C}}$-valued 1-form on Σ.

Now, observe that as $\mathfrak{su}^{\mathbb{C}}(2) \approx \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, it follows from the definition of $A \in \mathcal{A}$ that $A(x)$ may be identified with an element in $T^*_x\Sigma \otimes \mathfrak{su}(2) \oplus T^*_x\Sigma \otimes \mathfrak{su}(2)$ for each $x \in \Sigma$. Hence, the identification $\mathcal{A} \subset C^\infty_{\text{cs}}(\Sigma, E(\Sigma))$ can be made, where $E(\Sigma) \overset{\text{def}}{=} T^*\Sigma \otimes \mathfrak{su}(2) \oplus T^*\Sigma \otimes \mathfrak{su}(2)$ and $C^\infty_{\text{cs}}(\Sigma, E(\Sigma))$ is the space of C^∞-cross sections of the tensor bundle $E(\Sigma) \to \Sigma$. Let $\overline{\mathfrak{A}}(\Sigma)$ denote the maximal atlas of Σ and $\overline{\mathfrak{A}}(E(\Sigma))$ that of $E(\Sigma)$. Given $\varepsilon > 0$, $r \in \mathbb{N}$ and $K \subset U$ compact, let $N^r_\varepsilon(A; (U,\varphi), (V,\psi), K)$ denote the set of elements $A' \in \mathcal{A}$ such that

$$\sup\{\, \|D^k\psi \circ A \circ \varphi^{-1}(x) - D^k\psi \circ A' \circ \varphi^{-1}(x)\| : x \in \varphi(K),\ 0 \leqq k \leqq r\,\} < \varepsilon.$$

Let $\mathcal{T}_r$ be the set of all subsets $N^r_\varepsilon(A; (U,\varphi), (V,\psi), K)$ for each $A \in \mathcal{A}$, $\varepsilon > 0$, $(U,\varphi) \in \overline{\mathfrak{A}}(\Sigma)$ and $(V,\psi) \in \overline{\mathfrak{U}}(E(\Sigma))$. Then, $\bigcup_r \mathcal{T}_r$ forms a subbase for the compact C^∞-topology of $\mathcal{A}$.

6.2.1. Proposition. *$\mathcal{A}$ is second countable and completely metrizable.*

Proof. First, recall that a countable Cartesian product of second countable, completely metrizable spaces is again second countable and completely metrizable. Let $\hat{\mathcal{A}}_r$ denote the space of Ashtekar connections that are of class C^r (endowed with the compact C^r-topology). Then, $\hat{\mathcal{A}}_{r+1} \subset \hat{\mathcal{A}}_r$ $\forall r \in \mathbb{N}_0 \overset{\text{def}}{=} \mathbb{N} \cup \{0\}$, and hence,

$\{(\hat{\mathcal{A}}_r, i^r_{r-1}, \mathbb{N}_0)\}$ defines an inverse sequence, where $i^r_{r-1} : \hat{\mathcal{A}}_r \hookrightarrow \hat{\mathcal{A}}_{r-1}$ is the inclusion map. Since Σ is compact and $E(\Sigma)$ is second countable and completely metrizable, so is $\hat{\mathcal{A}}_0$. Furthermore, because $\hat{\mathcal{A}}_r$ is second countable and completely metrizable for each r as well, it follows that $\mathcal{A} \equiv \varprojlim \hat{\mathcal{A}}_r$, the inverse limit of $\hat{\mathcal{A}}_r$, is second countable and completely metrizable, as the inverse limit $\mathcal{A}$ is a closed subset of $\prod_{r\in\mathbb{N}_0} \hat{\mathcal{A}}_r$ (which is second countable and completely metrizable). $\square$

It will be shown next that $\mathcal{A}$ is an infinite-dimensional manifold. Fix $A_0 \in \mathcal{A}$ and define $T(A_0) \overset{\text{def}}{=} \{ v : \Sigma \to TE(\Sigma) \mid \pi_{E(\Sigma)} \circ v = A_0 \}$, where $(TE(\Sigma), \pi_{E(\Sigma)}, E(\Sigma))$ is the tangent bundle of $E(\Sigma)$. Let $\exp : TE(\Sigma) \to E(\Sigma)$ be the exponential map. Then, for each $x \in \Sigma$, there exists a neighbourhood $B_{\delta_x}(A_0(x))$ of $A_0(x)$ in $E(\Sigma)$ for some $\delta_x > 0$ such that $\exp_{A_0(x)} : B_{\delta_x}(0_x) \cong B_{\delta_x}(A_0(x))$, where $B_{\delta_x}(0_x)$ is a δ_x-ball about the zero vector 0_x in the tangent space $T_{A_0(x)}E(\Sigma)$, and $B_{\delta_x}(A_0(x))$ is a δ_x-ball in $E(\Sigma)$—both of which are defined relative to the respective metrics compatible with the topologies of $T_{A_0(x)}E(\Sigma)$ and $E(\Sigma)$.

Let $\{K_i\}_{i=1}^n$ be a (finite) compact covering of Σ, where $K_i \subset U_i$ lies in a chart of Σ for each i and set $N^r_\varepsilon(A_0) = \bigcap_{i=1}^n N^r_\varepsilon(A_0; (U_i, \varphi_i), (V, \psi_i), K_i)$, with $\varepsilon = \inf\{ \delta_x : x \in \Sigma \}$. Define $\Psi_{A_0} : N^r_\varepsilon(0) \to N^r_\varepsilon(A_0)$ by $v \mapsto A \overset{\text{def}}{=} \exp_{A_0} v$, where $A(x) = \exp_{A_0(x)} v(x)$ on Σ, and $N^r_\varepsilon(0) = \bigcap_{i=1}^n N^r_\varepsilon(0; (U_i, \varphi_i), K_i)$. It is clear that Ψ_{A_0} is injective: $\exp_{A_0(x)} v(x) = \exp_{A_0(x)} v'(x) \Rightarrow v(x) = v'(x)$ for each $x \in \Sigma$. To see that Ψ_{A_0} is surjective, let $A \in N^r_\varepsilon(A_0)$. By construction, since $\exp^{-1}_{A_0(x)} A(x) \in N^r_\varepsilon(0_x)$ for each $x \in \Sigma$, set $v \equiv \exp^{-1}_{A_0} A$. Then, $\Psi_{A_0}(v) = A$ and Ψ_{A_0} is thus surjective and hence a bijection. The continuity of Ψ_{A_0} follows trivially from the continuity of $\exp_{A_0(x)}$. Likewise, the continuity of $\Psi^{-1}_{A_0}$ follows from the continuity of $\exp^{-1}_{A_0(x)}$ and Ψ_{A_0} is thus a homeomorphism. Finally, if (U_A, Ψ_A) and $(U_{A'}, \Psi_{A'})$ are two overlapping charts, then the smoothness of $\exp_{A(x)}$ for each x implies that $\Psi_{A'} \circ \Psi^{-1}_A$ is also smooth. Hence, $\mathcal{A}$ is a smooth manifold modelled on $V_{\mathcal{A}}$, where $V_{\mathcal{A}}$ is a linear space smoothly diffeomorphic to $T(A)$ for any $A \in \mathcal{A}$. Denote the tangent space $T(A)$ at A by $T_A\mathcal{A}$ from here on.

6.2.2. Remark. Since $T^*\Sigma \approx \Sigma \times \mathbb{R}^3$ and Σ, $\mathbb{R}^3$ and $\mathfrak{su}(2)$ are geodesically complete, so is $E(\Sigma)$. Hence, $\exp_A$ is defined on $T_A\mathcal{A}$ and in particular, there exists a neighbourhood in $\mathcal{A}$ that is homeomorphic to $T_A\mathcal{A}$ under Ψ_A.

6.3. A Promeasure on $\mathcal{A}$

Let $D_\Sigma \subset \Sigma$ be a countably dense subset of Σ and let $c[D_\Sigma]$ be a countable cover of D_Σ such that $\forall G \in c[D_\Sigma]$, $G \subset D_\Sigma$ and $|G| < \aleph_0$. Set $c[D_\Sigma] = \{ G_{\alpha n} \mid \alpha, n \in$

$\mathbb{N}\}$. Each $G_{\alpha n}$ defines an equivalence relation $R_{\alpha n} \subset \mathcal{A} \times \mathcal{A}$ by

$$A \sim A' \iff A(x_i) = A'(x_i) \ \forall\, x_i \in G_{\alpha n}.$$

Let $p_{\alpha n} : \mathcal{A} \to \mathcal{A}_{\alpha n} \stackrel{\text{def}}{=} \mathcal{A}/R_{\alpha n}$ denote the natural map and endow $\mathcal{A}_{\alpha n}$ with the quotient topology.

6.3.1. Lemma. *$p_{\alpha n} : \mathcal{A} \to \mathcal{A}_{\alpha n}$ is an open and closed mapping.*

Proof. Fix $A_0 \in \mathcal{A}$ and let $N^r_\varepsilon(A_0) = N^r_\varepsilon(A_0; (U, \varphi), (V, \psi), K)$ for some $\varepsilon > 0$. Set $\tilde{N}^r_\varepsilon(A_0) = p^{-1}_{\alpha n} \circ p_{\alpha n}(N^r_\varepsilon(A_0))$, and fix $B \in \tilde{N}^r_\varepsilon(A_0)$. Then, by definition, $\exists\, B_0 \in N^r_\varepsilon(A_0)$ such that $B_0(x_i) = B(x_i)$ $\forall\, x_i \in G_{\alpha n}$. Choose $\delta < \varepsilon$ and $s > r$ so that $\hat{N}^s_\delta(B_0) \stackrel{\text{def}}{=} \bigcap_{i=1}^n N^s_\delta(B_0; (U_i, \phi_i), (V_i, \psi_i), K_i)$—where $K_i \subset U_i$ and $\{K_i\}_{i=1}^n$ is a finite compact covering of Σ—is contained in $N^r_\varepsilon(A_0)$, and set $\delta' = \frac{1}{2}\delta$ and $s' = 2s$. Recall that for any $B' \in \hat{N}^{s'}_{\delta'}(B) \stackrel{\text{def}}{=} \bigcap_{i=1}^n N^{s'}_{\delta'}(B; (U_i, \phi_i), (V'_i, \psi'_i), K_i)$, $B' = s'^*\omega_{\sigma'} - \frac{\mathrm{i}}{\sqrt{2}} K'(\sigma')$ and $K(\sigma)_{aA}{}^B \stackrel{\text{def}}{=} K_{ab}\sigma^b{}_A{}^B$.

To show that $\hat{N}^{s'}_{\delta'}(B) \subset \tilde{N}^r_\varepsilon(A_0)$, it is enough to verify that for each $B' \in \hat{N}^{s'}_{\delta'}(B)$, $\exists\, \hat{B} \in \hat{N}^s_\delta(B_0)$ such that $p_{\alpha n}(B') = p_{\alpha n}(\hat{B})$. Since $\sup\{\, \|D^k B'(x) - D^k B(x)\| \,:\, 0 \leqq k \leqq s', x \in \Sigma\} < \delta'$ (in grossly abused notation), the equality $B(x_i) = B_0(x_i)$ on $G_{\alpha n} \Rightarrow \|B_0(x_i) - B'(x_i)\| < \delta'$ on $G_{\alpha n}$ for any $B' \in \hat{N}^{s'}_{\delta'}(B)$. Hence, for any $\hat{B} \stackrel{\text{def}}{=} \hat{s}^*\omega_{\hat{\sigma}} - \frac{\mathrm{i}}{\sqrt{2}}\hat{K}(\hat{\sigma}) \in \hat{N}^r_{\delta'}(B_0)$ and $B' \in \hat{N}^{s'}_{\delta'}(B)$, $\|\hat{B}(x_i) - B'(x_i)\| < 2\delta' \equiv \delta$ on $G_{\alpha n}$. Since Σ is parallelisable, for any distinct finite number of points $x_1, \dots, x_n \in \Sigma$ and any fixed $B' \in \hat{N}^{s'}_{\delta'}(B)$, it is always possible to choose suitable (global) cross sections $\hat{K}$, $\hat{\sigma}$, $\hat{s}$ of appropriate tensor bundles over Σ so that $\hat{s}^*\omega_{\hat{\sigma}}(x_i) = s'^*\omega_{\sigma'}(x_i)$ and $\hat{K}(\hat{\sigma})(x_i) = K'(\sigma')(x_i)$ $\forall\, i = 1, \dots, n < \infty$. Consequently, for each $B' \in \hat{N}^{s'}_{\delta'}(B)$, $\exists\, \hat{B} \in \hat{N}^r_\delta(B_0)$ such that $B'(x_i) = \hat{B}(x_i)$ on $G_{\alpha n}$. Hence, $\hat{N}^{s'}_{\delta'}(B) \subset \tilde{N}^r_\varepsilon(A_0)$ and $p_{\alpha n}$ is thus open from the definition of the quotient topology.

Finally, to show that $p_{\alpha n}$ is closed, it will suffice to show that given any point $[A] \in \mathcal{A}_{\alpha n}$ and any neighbourhood M of $p^{-1}_{\alpha n}([A])$ in $\mathcal{A}$, there exists a neighbourhood $N[A]$ of $[A]$ such that $p^{-1}_{\alpha n}(N[A]) \subset M$. So, given $[A]$, let M be a neighbourhood of $p^{-1}_{\alpha n}([A])$. For simplicity, denote $p_{\alpha n}(A)$ and $p^{-1}_{\alpha n} \circ p_{\alpha n}(A)$ by the same symbol $[A]$.

First, recall from the proof of the open property of $p_{\alpha n}$ above that given any pair $B, B' \in [A]$, there exist neighbourhoods V_B and $V_{B'}$ of B and B' respectively in $\mathcal{A}$ satisfying

(1) $\forall\, \hat{B}' \in V_{B'}$, $\exists\, \hat{B} \in V_B$ such that $p_{\alpha n}(\hat{B}') = p_{\alpha n}(\hat{B})$, and
(2) $\forall\, \hat{B} \in V_B$, $\exists\, \hat{B}' \in V_{B'}$ such that $p_{\alpha n}(\hat{B}) = p_{\alpha n}(\hat{B}')$.

Hence, given a neighbourhood V_B of B, there exist neighbourhoods $V_{B'}$ of $B' \in [B]$ such that $p_{\alpha n}^{-1} \circ p_{\alpha n}(V_B) = \bigcup\{ V_{B'} \mid B' \in [B] \}$. So, for any $B \in [A]$ and any neighbourhood V_B of B in M, if $p_{\alpha n}^{-1} \circ p_{\alpha n}(V_B) \not\subset M$, then, because $p_{\alpha n}^{-1} \circ p_{\alpha n}(V_B) = \bigcup\{ V_{B'} \mid B' \in [A] \}$—where, for each $B' \in [A]$, the pair $(V_B, V_{B'})$ satisfies properties (1) and (2)—$\exists\, V_{\hat{B}} \in \{ V_{B'} \mid B' \in [A] \}$, for some $\hat{B} \in [A]$, such that $V_{\hat{B}} \not\subset M$. However, since V_B is an arbitrary neighbourhood of B in M and M is open, this is not possible. Otherwise, this implies that $\exists\, \hat{B} \in [A]$ such that each neighbourhood $N_{\hat{B}}$ of $\hat{B}$ satisfy $N_{\hat{B}} \not\subset M$, which is clearly impossible as $\hat{B} \in M$. Thus, for any fixed $B \in [A]$, there exists a neighbourhood $N_B \subset M$ of B such that $p_{\alpha n}^{-1} \circ p_{\alpha n}(N_B) \subset M$ and $p_{\alpha n}(N_B)$ is thus the required neighbourhood of $[A]$ in $\mathcal{A}_{\alpha n}$. So, $p_{\alpha n}$ is also closed. $\square$

Now, given any $A \in \mathcal{A}$, let $T_{[A]}\mathcal{A} \stackrel{\text{def}}{=} \bigcup\{ T_A\mathcal{A} : A \in [A] \}$, where the space is endowed with the subspace topology on the tangent bundle $T\mathcal{A}$. Then, as before, each $G_{\alpha n} \in c[D_\Sigma]$ induces an equivalence relation $\hat{R}_{\alpha n} \subset T_{[A]}\mathcal{A} \times T_{[A]}\mathcal{A}$ on $T_{[A]}\mathcal{A}$ by

$$v \sim v' \iff v(x_i) = v'(x_i)\ \forall\, x_i \in G_{\alpha n}.$$

Let $T_{[A]}\mathcal{A}_{\alpha n} \stackrel{\text{def}}{=} T_{[A]}\mathcal{A}/\hat{R}_{\alpha n}$ denote the quotient space (endowed with the quotient topology) and $p_{\alpha n*} : T_{[A]}\mathcal{A} \to T_{[A]}\mathcal{A}_{\alpha n}$ the natural map. Then, by construction, $p_{\alpha n*}(T_A\mathcal{A}) = T_{[A]}\mathcal{A}_{\alpha n}$ for each $A \in [A]$. By definition, $p_{\alpha n*}$ an open mapping.

6.3.2. Lemma. *$T_{[A]}\mathcal{A}_{\alpha n}$ is a finite-dimensional linear space.*

Proof. First of all, observe that $T_{[A]}\mathcal{A}_{\alpha n}$ can be endowed with a vector space structure: define $\tilde{+}$ and $\cdot$ by

$$[v]\tilde{+}[u] \stackrel{\text{def}}{=} [v+u] \quad \text{and} \quad c \cdot [v] \stackrel{\text{def}}{=} [cv]\ \forall\, c \in \mathbb{R}.$$

These operations are well-defined since $p_{\alpha n*}(u+v) = [u+v] = \{ u'+v' : u|G_{\alpha n} = u'|G_{\alpha n} \text{ and } v|G_{\alpha n} = v'|G_{\alpha n} \} = p_{\alpha n*}(u)\tilde{+}p_{\alpha n*}(v)$, and $p_{\alpha n*}(cu) = \{ cu' : u'|G_{\alpha n} = u|G_{\alpha n} \} = c \cdot p_{\alpha n*}(u)\ \forall\, c \in \mathbb{R}$.

To check that this vector space structure is compatible with the quotient topology, it must be verified that $\tilde{+} : T_{[A]}\mathcal{A}_{\alpha n} \times T_{[A]}\mathcal{A}_{\alpha n} \to T_{[A]}\mathcal{A}_{\alpha n}$ and $\cdot : \mathbb{R} \times T_{[A]}\mathcal{A}_{\alpha n} \to T_{[A]}\mathcal{A}_{\alpha n}$ are both continuous. Because $(p_{\alpha n*}|T_A\mathcal{A})(T_A\mathcal{A}) = T_{[A]}\mathcal{A}_{\alpha n}$, there is no loss of generality in fixing some $A \in [A]$ and considering $T_A\mathcal{A}$ instead of $T_{[A]}\mathcal{A}$ in the discussion that follows. So, consider the following diagram:

$$\begin{array}{ccc} T_A\mathcal{A} \times T_A\mathcal{A} & \xrightarrow{+} & T_A\mathcal{A}_{\alpha n} \\ {\scriptstyle p_{\alpha n*} \times p_{\alpha n*}}\big\downarrow & & \big\downarrow{\scriptstyle p_{\alpha n*}} \\ T_{[A]}\mathcal{A}_{\alpha n} \times T_{[A]}\mathcal{A}_{\alpha n} & \xrightarrow{\tilde{+}} & T_{[A]}\mathcal{A}_{\alpha n}. \end{array}$$

It is clear from the definition that the diagram commutes: $\tilde{+} \circ (p_{\alpha n*} \times p_{\alpha n*}) = p_{\alpha n*} \circ +$. Hence, the continuity of $\tilde{+}$ follows from the surjectivity of $p_{\alpha n*} \times p_{\alpha n*}$ and the continuity of $p_{\alpha n*}$, $+$ and $p_{\alpha n*} \times p_{\alpha n*}$. In a similar way, it can be verified easily that $\cdot$ is continuous. Whence, $T_{[A]}\mathcal{A}_{\alpha n}$ is a linear space.

Set $V_{\alpha n} = \prod_{i=1}^n (T^*_{x_i}\Sigma \otimes \mathfrak{su}(2)) \oplus (T^*_{x_i}\Sigma \otimes \mathfrak{su}(2))$, which is linearly isomorphic to $\mathbb{R}^{18n}$ (and hence finite-dimensional), and define $L_{\alpha n} : T_{[A]}\mathcal{A}_{\alpha n} \to V_{\alpha n}$ by

$$[v] \mapsto (v(x_1), \ldots, v(x_n)),$$

where v is any representative of $[v]$ and $(T^*_{x_i}\Sigma \otimes \mathfrak{su}(2)) \oplus (T^*_{x_i}\Sigma \otimes \mathfrak{su}(2))$ is identified with its tangent space for each i. By definition, $L_{\alpha n}$ is linear and injective as $v|G_{\alpha n} \equiv u|G_{\alpha n} \Leftrightarrow [v] = [u]$. Fix $u = (u^i)_{i=1}^n \in V_{\alpha n}$. It is obvious that $\exists\, v \in T_A\mathcal{A}$ such that $v(x_i) = u^i\ \forall\, i = 1, \ldots, n$ as $T_A\mathcal{A}$ is a vector space. Hence, $L_{\alpha n}([v]) = u$ and $L_{\alpha n}$ is thus onto. So, $L_{\alpha n}$ defines an isomorphism, as required. $\square$

6.3.3. Theorem. *$\mathcal{A}_{\alpha n}$ is a finite-dimensional, paracompact, Hausdorff, second countable manifold modelled on $V_{\alpha n}$.*

Proof. Note that as metrizability is an invariant under closed surjective mappings [8, Theorem 4.4.18, p. 285], by Lemma 6.3.1, $\mathcal{A}_{\alpha n}$ is metrizable and hence paracompact and Hausdorff. Lastly, because $p_{\alpha n}$ is open and $\mathcal{A}$ is second countable by Proposition 6.2.1, so is $\mathcal{A}_{\alpha n}$.

Given $[A] \in \mathcal{A}_{\alpha n}$, fix a representative A of $[A]$ and consider the following diagram:

$$\begin{array}{ccc} T_A\mathcal{A} & \xrightarrow{p_{\alpha n*}} & T_{[A]}\mathcal{A}_{\alpha n} \\ {\scriptstyle \Psi_A}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_{A*}} \\ \mathcal{A} & \xrightarrow{p_{\alpha n}} & \mathcal{A}_{\alpha n}, \end{array}$$

where $\Psi_{A*} : [v] \mapsto [\Psi_A v]$ and $p_{\alpha n*}$ is understood to be the restriction $p_{\alpha n*}|T_A\mathcal{A}$. Note firstly that Ψ_{A*} is well-defined; that is, it does not depend on the choice of representative A in $[A]$. For suppose $(U'_{A'}, \Psi^{-1}_{A'})$ and (U_A, Ψ^{-1}_A) are two overlapping charts with $[A] = [A']$. Choose some $v \in T_A\mathcal{A}$ and $v' \in T_{A'}\mathcal{A}$ such that $\Psi_A(v) =$

$\tilde{A} = \Psi_{A'}(v')$ for some $\tilde{A} \in U_A \cap U'_{A'}$. Then, $\Psi_{A*}([v]) = [\Psi_A(v)] = [\Psi_{A'}(v')] = \Psi_{A'*}([v'])$, as expected.

Since $\Psi_{A*} \circ p_{\alpha n*}(v) = \Psi_{A*}([v]) = [\Psi_A v]$, for $v \in T_A\mathcal{A}$, and $p_{\alpha n} \circ \Psi_A(v) = p_{\alpha n*}(\Psi_A v) = [\Psi_A v]$, the diagram commutes. Furthermore, as $p_{\alpha n}$, Ψ_A and $p_{\alpha n*}$ are open and $p_{\alpha n}$ is surjective, the commutativity of the diagram implies that Ψ_{A*} is also open. Likewise, Ψ_{A*} is continuous. Set $N([A]) = \Psi_{A*} \circ p_{\alpha n*}(T_A\mathcal{A})$; then Ψ_{A*} maps $T_{[A]}\mathcal{A}_{\alpha n} (= p_{\alpha n*}(T_A\mathcal{A}))$ bijectively onto $N([A])$. To verify this, only the injective property of Ψ_{A*} needs any verification since Ψ_{A*} maps $T_{[A]}\mathcal{A}_{\alpha n}$ onto $N([A])$ by construction. If $\Psi_{A*}([v]) = \Psi_{A*}([u])$, then $[\Psi_A v] = [\Psi_A u]$ and hence, the fact that Ψ_A is injective implies at once that $[v] \equiv [u]$, and Ψ_{A*} is thus injective. Consequently, Ψ_{A*}^{-1} maps $N([A])$ homeomorphically onto $T_{[A]}\mathcal{A}_{\alpha n}$ and the pair $(N([A]), \Psi_{A*}^{-1})$ thus constitutes a chart at $[A]$. □

For any $G_{\alpha n}, G_{\alpha' n'} \in c[D_\Sigma]$ with $G_{\alpha' n'} \subset G_{\alpha n}$ and $n' < n$, let $p^{\alpha n}_{\alpha' n'} : \mathcal{A}_{\alpha n} \to \mathcal{A}_{\alpha' n'}$ denote the natural map $[A]_{\alpha n} \mapsto [A]_{\alpha' n'}$, where $[A]_{\beta m} \stackrel{\text{def}}{=} \{ A' \mid A'(x_i) = A(x_i) \ \forall\, x_i \in G_{\beta m} \}$. This map is clearly a continuous surjection. It is not difficult to see that it is also an open map as $p^{\alpha n}_{\alpha' n'} \circ p_{\alpha n} = p_{\alpha' n'}$ and $p_{\alpha n}$ is onto. Order $c[D_\Sigma]$ with $\preccurlyeq$ by $G' \preccurlyeq G \Leftrightarrow G' \subseteq G$. Then, $\{(\mathcal{A}_{\alpha n}, p^{\alpha n}_{\alpha' n'}) \mid G_{\alpha' n'} \preccurlyeq G_{\alpha n},\ n' \leqq n,\ n', n \in \mathbb{N}\}$ defines an inverse sequence.

Suppose $m < n$ and $G_{\alpha(m)m} \subset G_{\alpha(n)n}$. Define $\hat{p}^{\alpha(n)n}_{\alpha(m)m} : V_{\alpha(n)n} \to V_{\alpha(m)m}$ by

$$\hat{p}^{\alpha(n)n}_{\alpha(m)m} \stackrel{\text{def}}{=} L_{\alpha(m)m} \circ (p^{\alpha(n)n}_{\alpha(m)m})_* \circ L^{-1}_{\alpha(n)n},$$

where $(p^{\alpha(n)n}_{\alpha(m)m})_* : T_{[A]}\mathcal{A}_{\alpha(n)n} \to T_{[A]}\mathcal{A}_{\alpha(m)m}$ is a map induced by $p^{\alpha(n)n}_{\alpha(m)m} : \mathcal{A}_{\alpha(n)n} \to \mathcal{A}_{\alpha(m)m}$. In all that follows, $(p^{\alpha(n)n}_{\alpha(m)m})_*$ will be identified with $\hat{p}^{\alpha(n)n}_{\alpha(m)m}$ and $T_{[A]}\mathcal{A}_{\alpha(n)n}$ with $V_{\alpha(n)n}$ for each $[A] \in \mathcal{A}_{\alpha(n)n}$.

Let $V \stackrel{\text{def}}{=} \varprojlim V_{\alpha(n)n}$ denote the inverse limit of $V_{\alpha(n)n}$, where $V_{\alpha(n)n} \approx \mathbb{R}^{18n}$ for each n. Then, V is a locally convex Hausdorff space. By [7, Theorem, p. 582], given a positive definite quadratic form $Q : V' \to \mathbb{R}$ on the topological dual V' of V, there exists a unique Gaussian promeasure $\hat{\lambda}$ of variance Q. That is, its Fourier transform is $\mathrm{e}^{-Q/2}$. So, let $\hat{\lambda} = \{(\hat{\lambda}_{\alpha(n)n}, V_{\alpha(n)n})\}$ denote the Gaussian promeasure on V of variance Q: that is, $\hat{\lambda}_{\alpha(n)n}$ is a bounded measure on $V_{\alpha(n)n}$, $\hat{\lambda}_{\alpha(m)m} = \hat{\lambda}_{\alpha(n)n} \circ \hat{p}^{\alpha(m)m}_{\alpha(n)n}$, where $m < n$, $G_{\alpha(m)m} \subset G_{\alpha(n)n}$ and $\hat{\lambda}_{\alpha(n)n}(V_{\alpha(n)n}) = \hat{\lambda}_{\alpha(m)m}(V_{\alpha(m)m})\ \forall\, n, m$.

6.3.4. Lemma. *Let $m < n$ and (U_{ni}, ϕ_{ni}) be a chart in $\mathcal{A}_{\alpha n}$ about the point $[A]_n$. Then, $U_{mi} \stackrel{\text{def}}{=} p^{\alpha n}_{\alpha m}(U_{ni})$ is a chart in $\mathcal{A}_{\alpha m}$ about the point $[A]_m = p^{\alpha n}_{\alpha m}([A]_n)$.*

Proof. First, let $\phi_{mi} : \tilde{U}_{mi} \cong V_{\alpha m}$ be a chart in $\mathcal{A}_{\alpha m}$ about $[A]_m$ so that the following diagram commutes:

$$\begin{tikzcd} V_{\alpha n} \arrow[r, "\hat{p}^{\alpha n}_{\alpha m}"] \arrow[d, "\phi^{-1}_{ni}"'] & V_{\alpha m} \arrow[d, "\phi^{-1}_{mi}"] \\ \mathcal{A}_{\alpha n} \arrow[r, "p^{\alpha n}_{\alpha m}"] & \mathcal{A}_{\alpha m}. \end{tikzcd}$$

Second, observe trivially from the definition that $U_{ni} = \phi^{-1}_{ni}(V_{\alpha n})$ and $\hat{p}^{\alpha n}_{\alpha m}(V_{\alpha n}) = V_{\alpha m}$. Hence, the commutativity of the diagram implies at once that $U_{mi} \stackrel{\text{def}}{=} p^{\alpha n}_{\alpha m}(U_{ni}) = p^{\alpha n}_{\alpha m} \circ \phi^{-1}_{ni}(V_{\alpha n}) = \phi^{-1}_{mi} \circ \hat{p}^{\alpha n}_{\alpha m}(V_{\alpha n}) = \tilde{U}_{mi}$, as claimed. □

Note that if $\{(U_{ni}, \phi_{ni})\}$ is an atlas on $\mathcal{A}_{\alpha n}$ with an associated partition of unity $\{\theta_{ni}\}$ subordinate to it, then there is a partition of unity $\{\theta_{mi}\}$ subordinate to $\{(p^{\alpha n}_{\alpha m}(U_{mi}), \phi_{mi})\}$ such that $\theta_{ni} = \theta_{mi} \circ p^{\alpha n}_{\alpha m}$.

6.3.5. Lemma. *Let $m < n$ and (U_{ki}, ϕ_{ki}) be a chart in $\mathcal{A}_{\alpha k}$ about $[A]_k$ for $k = m, n$, where $U_{mi} = p^{\alpha n}_{\alpha m}(U_{ni})$, and set $[A]_m = p^{\alpha n}_{\alpha m}([A]_n)$. Then, for any subset $D_m \subset \mathcal{A}_{\alpha m}$, $\hat{p}^{\alpha m}_{\alpha n} \circ \phi_{mi}(D_m \cap U_{mi}) = \phi_{ni}(p^{\alpha m}_{\alpha n}(D_m) \cap U_{ni})$ for each i.*

Proof. Let $[v]_n \in \hat{p}^{\alpha m}_{\alpha n} \circ \phi_{mi}(D_m \cap U_{mi})$. Then, $\exists\, [A]_n \in U_{ni}$ such that $\phi_{ni}([A]_n) = [v]_n$. Since $p^{\alpha n}_{\alpha m}([A]_n) = p^{\alpha n}_{\alpha m} \circ \phi^{-1}_{ni}([v]_n) \in U_{mi} \cap D_m$ by definition, it follows immediately that $[A]_n \in U_{ni} \cap p^{\alpha m}_{\alpha n}(D_m)$. Hence, $[v]_n \in \phi_{ni}(p^{\alpha m}_{\alpha n}(D_m) \cap U_{ni})$ and $\hat{p}^{\alpha m}_{\alpha n} \circ \phi_{mi}(D_m \cap U_{mi}) \subseteq \phi_{ni}(p^{\alpha m}_{\alpha n}(D_m) \cap U_{ni})$.

Conversely, suppose $[u]_n \in \phi_{ni}(p^{\alpha m}_{\alpha n}(D_m) \cap U_{ni})$. Then, $\exists\, [A']_n \in U_{ni}$ such that $\phi_{ni}([A']_n) = [u]_n$, and $p^{\alpha n}_{\alpha m}([A']_n) = [A']_m \in D_m \cap U_{mi}$ by definition, as $U_{mi} = p^{\alpha n}_{\alpha m}(U_{ni})$. Hence, $\phi_{mi}([A']_m) \in \phi_{mi}(U_{mi} \cap D_m)$ and $[u]_n \in \hat{p}^{\alpha m}_{\alpha n} \circ \phi_{mi}(U_{mi} \cap D_m)$, yielding the converse set-inequality $\phi_{ni}(p^{\alpha m}_{\alpha n}(D_m) \cap U_{ni}) \subseteq \hat{p}^{\alpha m}_{\alpha n} \circ \phi_{mi}(D_m \cap U_{mi})$, as desired. □

6.3.6. Theorem. *$\hat{\lambda}$ induces a promeasure on $\mathcal{A}$.*

Proof. Let $\hat{p}_{\alpha(n)n} : V \to V_{\alpha(n)n}$ be the canonical projection $v \mapsto [v]_{\alpha(n)n}$. Then, $\hat{p}_{\alpha(n)n}$ defines a *transposition* $\hat{p}^{\mathrm{t}}_{\alpha(n)n} : V'_{\alpha(n)n} \to V'$ given by $\langle \hat{p}^{\mathrm{t}}_{\alpha(n)n}(f), v\rangle = \langle f, \hat{p}_{\alpha(n)n}(v)\rangle_{\alpha(n)n}$, where V' and $V'_{\alpha(n)n}$ are the topological duals of V and $V_{\alpha(n)n}$ respectively, $v \in V$, $f \in V'_{\alpha(n)n}$, and $Q_n \stackrel{\text{def}}{=} Q \circ \hat{p}^{\mathrm{t}}_{\alpha(n)n} : V_{\alpha(n)n} \to \mathbb{R}$ defines a positive-definite quadratic form on $V_{\alpha(n)n}$, with $V'_{\alpha(n)n}$ identified with $V_{\alpha(n)n}$.

Choose a basis in $V_{\alpha(n)n}$ so that $Q_n(x) = \sum_{i=1}^N a_{n,i}(x^i)^2$, where $a_{n,i} \in \mathbb{R}$ are constants, $N = 18n$, and set $\mathrm{d}\hat{\lambda}_{a_{n,i}}(x^i) \stackrel{\text{def}}{=} (2\pi a_i)^{-\frac{1}{2}} \mathrm{e}^{-(x^i)^2/2a_{n,i}}\, \mathrm{d}x^i$. Then,

$$\hat{\lambda}_{\alpha(n)n} = \hat{\lambda}_{a_{n,1}} \otimes \cdots \otimes \hat{\lambda}_{a_{n,N}}$$

defines a Gaussian measure on $V_{\alpha(n)n}$ of variance Q_n. Let $\{(U_{ni}, \phi_{ni})\}$ be an atlas on $\mathcal{A}_{\alpha(n)n}$ such that there exists a partition of unity $\{\theta_{ni}\}_i$ subordinate to it. Note trivially that the atlas may be taken to be countable as $\mathcal{A}_{\alpha(n)n}$ is a Lindelöf space.[1] Given a chart (U_{ni}, ϕ_{ni}), define an N-form ω_{ni} on U_{ni} by

$$\omega_{ni}(x) \stackrel{\text{def}}{=} (\mathrm{d}\hat{\lambda}_{a_{n,1}} \wedge \cdots \wedge \mathrm{d}\hat{\lambda}_{a_{n,N}})(\phi_{ni}(x)).$$

Then, $(\theta_{ni} \cdot \omega_{ni})(x) \stackrel{\text{def}}{=} \theta_{ni}(x)\omega_{ni}(x)$ is well-defined on $\mathcal{A}_{\alpha(n)n}$ as $\theta_{ni}(x) \equiv 0\ \forall x \notin U_{ni}$, and hence $(\theta_{ni} \cdot \omega_{ni})(x) = 0$ on U_{nj} for $j \neq i$, irrespective of whether ω_{ni} is defined on U_{nj}. Thus, $\omega_n \stackrel{\text{def}}{=} \sum_i \theta_{ni} \cdot \omega_{ni}$ is a well-defined N-form on $\mathcal{A}_{\alpha(n)n}$.

Now, observe that given a smooth diffeomorphism f on $\mathcal{A}$, it induces a smooth diffeomorphism $\tilde{f}$ on $\mathcal{A}_{\alpha(n)n}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "p_{\alpha(n)n}"'] & \mathcal{A} \arrow[d, "p_{\alpha(n)n}"] \\
\mathcal{A}_{\alpha(n)n} \arrow[r, "\tilde{f}"] & \mathcal{A}_{\alpha(n)n}.
\end{tikzcd}$$

To verify this, let $\tilde{f}$ be a map on $\mathcal{A}_{\alpha(n)n}$ so that the diagram commutes: $p_{\alpha(n)} \circ f = \tilde{f} \circ p_{\alpha(n)n}$. To see that $\tilde{f}$ is injective, suppose $\tilde{f}([A]_n) = \tilde{f}([A']_n)$ for some $[A]_n, [A']_n \in \mathcal{A}_{\alpha(n)n}$. Then, there exists $\tilde{A}, \tilde{A}' \in \mathcal{A}$ such that $p_{\alpha(n)n}(\tilde{A}) = [A]_n$ and $p_{\alpha(n)n}(\tilde{A}') = [A']_n$. From the commutativity of the diagram, $p_{\alpha(n)n} \circ f(\tilde{A}) = \tilde{f} \circ p_{\alpha(n)n}(\tilde{A}) = \tilde{f}([A]_n)$ and $p_{\alpha(n)n} \circ f(\tilde{A}') = \tilde{f} \circ P_{\alpha(n)n}(\tilde{A}') = \tilde{f}([A']_n)$ imply that $[f(\tilde{A})]_n = [f(\tilde{A}')]_n$, where $[A]_n \stackrel{\text{def}}{=} p_{\alpha(n)n}(A)$. Whence, the fact that f is injective yields the equality $[A]_n = [A']_n$. The surjectivity of $\tilde{f}$ is equally trivial to verify: given $[\tilde{B}]_n \in \mathcal{A}_{\alpha(n)n}$, $\exists\, B \in \mathcal{A}$ such that $p_{\alpha(n)n} \circ f(B) = [\tilde{B}]$. Since $p_{\alpha(n)n} \circ f(B) = \tilde{f} \circ p_{\alpha(n)n}(B)$, set $[B]_n = p_{\alpha(n)n}(B)$. Then, $\tilde{f}([B]_n) = [\tilde{B}]_n$ and $\tilde{f}$ is thus onto. Finally, that $\tilde{f}$ is a homeomorphism follows from the surjectivity of $p_{\alpha(n)n}$ together with the continuity and the open property of $p_{\alpha(n)n}$ and f. Likewise, the smoothness of $\tilde{f}$ follows from the smoothness of $p_{\alpha(n)n}$ and f.

It follows from the above discussion that given a vector field v and a 1-form w on $\mathcal{A}$, the scalar $\langle v\,, w\rangle$ on $\mathcal{A}$ defines a scalar $\langle v_n\,, w_n\rangle$ on $\mathcal{A}_{\alpha(n)n}$ via $\langle v\,, w\rangle = \langle v_n\,, w_n\rangle \circ p_{\alpha(n)n}$, where v_n and w_n are suitably chosen vector and covector fields respectively on $\mathcal{A}_{\alpha(n)n}$ (as a diffeomorphism f on $\mathcal{A}$ induces a diffeomorphism $\tilde{f}$ on $\mathcal{A}_{\alpha(n)n}$). Fix a fiducial point $A_0 \in \mathcal{A}$ once and for all and let ϱ denote the set of all admissible and hence unbounded equivalent (topological) metrics $\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_+$

[1]It follows from Theorem 6.3.1 that $\mathcal{A}_{\alpha(n)n}$ is paracompact and second countable. This in turn implies that it is Lindelöf; *i.e.*, every open covering of $\mathcal{A}_{\alpha(n)n}$ has a countable subcovering.

on $\mathcal{A}$ compatible with its underlying topology. Next, consider a vector field v and a 1-form w on $\mathcal{A}$—which are not nowhere zero in general—such that $\langle v\,,w\rangle$ is unbounded on $\mathcal{A}$ in the following sense: given any sequence $\{A_n\}_n$ in $\mathcal{A}$,

$$\lim_{n\to\infty} |\langle v\,,w\rangle(A_n)| = \infty \text{ whenever } \lim_{n\to\infty} \rho(A_0, A_n) = \infty \ \forall\, \rho \in \boldsymbol{\varrho}.$$

That is, $|\langle v\,,w\rangle|$ is unbounded on $\mathcal{A}$ with respect to A_0; and for each n, choose a vector field v_n and a 1-form w_n so that $\langle v\,,w\rangle = \langle v_n\,,w_n\rangle \circ p_{\alpha(n)n}$.

Let $\{(U_{ni}, \phi_{ni})\}$ be an atlas on $\mathcal{A}_{\alpha(n)n}$ associated with a partition of unity $\{\theta_{ni}\}_i$, and let $\{(U_{mi}, \phi_{mi})\}$ be an atlas on $\mathcal{A}_{\alpha(m)m}$ for $m < n$, where $p^{\alpha(n)n}_{\alpha(m)m}(U_{ni}) = U_{mi}$ (which is possible by Lemma 6.3.4), with its associated partition of unity $\{\theta_{mi}\}_i$ satisfying $\theta_{ni} = \theta_{mi} \circ p^{\alpha(n)n}_{\alpha(m)m}$. If $D_m \subset \mathcal{A}_{\alpha(m)m}$ is any subset, define

$$S_m[D_m \cap U_{mi}] \stackrel{\text{def}}{=} \sup\{\, |\langle v_m\,,w_m\rangle(A)| : A \in D_m \cap U_{mi}\,\}.$$

The "set-function" S_n is well-defined as $\langle v_n\,,w_n\rangle$ is continuous on $\mathcal{A}_{\alpha(n)n}$, and for each i, $\exists\, \delta_i > 0$ such that $U_{ni} \subset B_{\delta_i}(A_0) = \{\, A \in \mathcal{A}_{\alpha(n)n} \mid \rho_{\alpha(n)n}(A, A_0) < \delta_i \,\}$ for some $\delta_i > 0$ large enough, where $\rho_{\alpha(n)n}$ is a metric on $\mathcal{A}_{\alpha(n)n}$ compatible with its quotient topology. Note that S_n is of course unbounded on the set $\{\, U_{ni} : i \in \mathbb{N}\}$ by construction. Furthermore, the fact that $\langle v\,,w\rangle = \langle v_n\,,w_n\rangle \circ p_{\alpha(n)n}$ $\forall\, n$ and $p_{\alpha(m)m} = p^{\alpha(n)n}_{\alpha(m)m} \circ p_{\alpha(n)n}$ for $m < n$ will imply that $\langle v_n\,,w_n\rangle = \langle v_m\,,w_m\rangle \circ p^{\alpha(n)n}_{\alpha(m)m}$, and

$$\begin{aligned} S_n[p^{\alpha(m)m}_{\alpha(n)n}(D_m) \cap U_{ni}] &= \sup\{\, |\langle v_n\,,w_n\rangle(A)| : A \in p^{\alpha(m)m}_{\alpha(n)n}(D_m) \cap U_{ni}\,\} \\ &= \sup\{\, |\langle v_m\,,w_m\rangle(A)| : A \in p^{\alpha(n)n}_{\alpha(m)m}(p^{\alpha(m)m}_{\alpha(n)n}(D_m) \cap U_{ni})\,\} \\ &= \sup\{\, |\langle v_m\,,w_m\rangle(A)| : A \in D_m \cap p^{\alpha(n)n}_{\alpha(m)m}(U_{ni})\,\} \\ &= \sup\{\, |\langle v_m\,,w_m\rangle(A)| : A \in D_m \cap U_{mi}\,\} \\ &= S_m[D_m \cap U_{mi}]. \end{aligned}$$

As a corollary, setting $D_m = U_{mi}$ yields $S_n[U_{ni}] = S_m[U_{mi}]$ for each $m, n > 0$.

Now, define an N-form $\tilde{\omega}_n$ on $\mathcal{A}_{\alpha(n)n}$ by

$$\tilde{\omega}_n \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \mathrm{e}^{-S_n[U_{ni}]}\theta_{ni} \cdot \omega_{ni},$$

and choose the pair (v, w) on $\mathcal{A}$ and hence (v_n, w_n) on $\mathcal{A}_{\alpha(n)n}$ so that the sum below is bounded:

$$\left| \sum_{i=1}^{\infty} \mathrm{e}^{-S_n[U_{ni}]} \int_{U_{ni}} \omega_{ni} \right| < \infty.$$

This means in particular that $\sum_i \mathrm{e}^{-S_n[U_{ni}]} \int_{U_{ni}} \theta_{ni}^2 \cdot \omega_{ni}$ is bounded as $|\theta_{ni}| \leqq 1$ for each i. Then, $\tilde{\omega}_n$ defines a bounded measure $\lambda_{\alpha(n)n}$ on $\mathcal{A}_{\alpha(n)n}$ by

$$\lambda_{\alpha(n)n}(D) = \int_D \tilde{\omega}_n \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \int_{D\cap V_{ni}} \tilde{\theta}_{ni} \cdot \tilde{\omega}_n,$$

where $\{\tilde{\theta}_{ni}\}$ is a partition of unity subordinate to the atlas $\{V_{ni}\}$ and D is a subset in $\mathcal{A}_{\alpha(n)n}$ such that for each i, $\phi_{ni}(D \cap V_{ni})$ is $\hat{\lambda}_{\alpha(n)n}$-measurable.

This definition is clearly independent of the choice of partition of unity $\{\tilde{\theta}_{ni}\}$ chosen. For let $\{\tilde{\theta}'_{ni}\}$ be another partition of unity subordinate to the atlas $\{V'_{ni}\}$. Then, $\sum_i \tilde{\theta}_{ni}(A) = 1 = \sum_i \tilde{\theta}'_{ni}(A)$ for each $A \in \mathcal{A} \Rightarrow \int_{\mathcal{A}} \sum_i \tilde{\theta}_{ni} \cdot \tilde{\omega}_n = \int_{\mathcal{A}} \sum_i \tilde{\theta}'_{ni} \cdot \tilde{\omega}_n \Rightarrow \sum_i \int_{\mathcal{A}} \tilde{\theta}_{ni} \cdot \tilde{\omega}_n = \sum_i \int_{\mathcal{A}} \tilde{\theta}'_{ni} \cdot \tilde{\omega}_n$. This is not to be confused with $\{\theta_{ni}\}$ which is part of the definition of the 1-form $\tilde{\omega}_n$. Since $\tilde{\theta}_{ni}$ is arbitrary, one may set $\tilde{\theta}_{ni} = \theta_{ni}$ without any loss of generality and hence take $V_{ni} = U_{ni}$ for each i. Note that the definition of $\lambda_{\alpha(n)n}$ assumes that $\mathcal{A}_{\alpha n}$ is orientable. If it is not, then construct ω_n to be an odd form [7, p. 212] and denote it by the same symbol.

The system $\lambda \stackrel{\text{def}}{=} \{(\lambda_{\alpha(n)n}, \mathcal{A}_{\alpha(n)n})\}$ defines a promeasure on $\mathcal{A}$. To verify this final part, it is enough to check that for $m < n$, $\lambda_{\alpha(m)m} = \lambda_{\alpha(n)n} \circ p_{\alpha(n)n}^{\alpha(m)m}$. Fix a measurable subset $D \subset \mathcal{A}_{\alpha(m)m}$. Then,

$$\begin{aligned}
\lambda_{\alpha(m)m}(D) &= \sum_i \mathrm{e}^{-S_m[U_{mi}]} \int_{D\cap U_{mi}} \theta_{mi}^2 \cdot \omega_{mi} \\
&= \sum_i \mathrm{e}^{-S_m[U_{mi}]} \int_{\phi_{mi}(D\cap U_{mi})} (\theta_{mi} \circ \phi_{mi}^{-1})^2 \, \mathrm{d}\hat{\lambda}_{\alpha(m)m} \\
&= \sum_i \mathrm{e}^{-S_n[U_{ni}]} \int_{\hat{p}_{\alpha(n)n}^{\alpha(m)m} \circ \phi_{mi}(D_m\cap U_{mi})} (\theta_{mi} \circ \phi_{mi}^{-1} \circ \hat{p}_{\alpha(m)m}^{\alpha(n)n})^2 \, \mathrm{d}\hat{\lambda}_{\alpha(n)n} \\
&= \sum_i \mathrm{e}^{-S_n[U_{ni}]} \int_{\phi_{ni}(p_{\alpha(n)n}^{\alpha(m)m}(D)\cap U_{ni})} (\theta_{ni} \circ \phi_{ni}^{-1})^2 \, \mathrm{d}\hat{\lambda}_{\alpha(n)n} \\
&\equiv \sum_i \mathrm{e}^{-S_n[U_{ni}]} \int_{p_{\alpha(n)n}^{\alpha(m)m}(D)\cap U_{ni}} \theta_{ni}^2 \cdot \omega_{ni} \\
&= \lambda_{\alpha(n)n} \circ p_{\alpha(n)n}^{\alpha(m)m}(D),
\end{aligned}$$

where $\hat{\lambda}_{\alpha(m)m} = \hat{\lambda}_{\alpha(n)n} \circ \hat{p}_{\alpha(n)n}^{\alpha(m)m}$ was used in the third equality, and $\theta_{ni} = \theta_{mi} \circ p_{\alpha m}^{\alpha n}$ and $p_{\alpha(m)m}^{\alpha(n)n} \circ \phi_{ni}^{-1} = \phi_{mi}^{-1} \circ \hat{p}_{\alpha(m)m}^{\alpha(n)n}$ together with Lemma 6.3.5 in the fourth equality. From this, it follows as an immediate consequence that $\lambda_{\alpha(n)n}(\mathcal{A}_{\alpha(n)n}) = \lambda_{\alpha(m)m}(\mathcal{A}_{\alpha(m)m})\ \forall n, m$. $\square$

6.4. A Promeasure on the Ashtekar Moduli Space

Let $\mathrm{Diff}(P_\xi)$ denote the group of smooth diffeomorphisms (endowed with the compact C^∞-topology) on the bundle space P_ξ and let $\mathcal{G} \subset \mathrm{Diff}(P_\xi)$ be the set of elements $f \in \mathrm{Diff}(P_\xi)$ satisfying

(1) $f(ug) = f(u)g \ \forall\, u \in P_\xi$ and $g \in \mathrm{SU}(2)$,

(2) $\pi_\xi \circ f = \pi_\xi$, where $\pi_\xi : P_\xi \to \Sigma$ is the bundle projection.

Note that the bundle is actually trivial and hence $P_\xi \cong \Sigma \times \mathrm{SU}(2)$ (which is compact).

Identify P_ξ with $\Sigma \times \mathrm{SU}(2)$ and let $p_2 : \Sigma \times \mathrm{SU}(2) \to \mathrm{SU}(2)$ be the projection of the second factor, $(x, g) \mapsto g$. Set $\phi_f \stackrel{\text{def}}{=} p_2 \circ f$ for any $f \in \mathcal{G}$. Define an equivalence relation $\mathcal{R} \subset \mathcal{A} \times \mathcal{A}$ on $\mathcal{A}$ by

$$A \sim A' \iff A'(x) = \phi_f(x)^{-1} A(x) \phi_f(x) + \phi_f(x)^{-1}\, \mathrm{d}\phi_f(x),$$

and denote A' by A^{ϕ_f}. Then, the SU(2) gauge-equivalence class $\hat{A}$ of A is the set $\{\, A^{\phi_f} \mid f \in \mathcal{G} \,\}$. Let $\pi_{\mathrm{SU}(2)} : \mathcal{A} \to \mathcal{A}[\mathrm{SU}(2)]$ be the natural map and $\mathcal{A}[\mathrm{SU}(2)]$ be given the quotient topology. Observe that ϕ_f induces a homeomorphism $\Phi_f : \mathcal{A} \to \mathcal{A}$ by $A \mapsto A^{\phi_f}$ and hence, if $N^r_\delta(A)$ is any neighbourhood of A in $\mathcal{A}$, then $\Phi_f(N^r_\delta(A)) \stackrel{\text{def}}{=} \{\, A^{\phi_f} \mid A \in N^r_\delta(A) \,\}$ is open. Consequently, $\pi_{\mathrm{SU}(2)}$ is an open mapping as

$$\pi^{-1}_{\mathrm{SU}(2)} \circ \pi_{\mathrm{SU}(2)}(N^r_\delta(A)) = \bigcup_{f \in \mathcal{G}} \Phi_f(N^r_\delta(A)).$$

6.4.1. Proposition. *$\mathcal{A}[SU(2)]$ is Hausdorff.*

Proof. Since $\pi_{\mathrm{SU}(2)}$ is open, to verify this proposition it suffices to show that $\mathcal{R}$ is closed in $\mathcal{A} \times \mathcal{A}$. Let $\{(A_n, A_n^{\phi_{f_n}})\}_n$ be a sequence in $\mathcal{R}$ which converges to $(B_0, \tilde{B}_0) \in \mathcal{A} \times \mathcal{A}$. So, $A_n \to B_0$ and $A_n^{\phi_{f_n}} \to \tilde{B}_0$ in $\mathcal{A}$ as $n \to \infty$ with respect to the compact C^∞-topology. Thus, given any $\varepsilon > 0$, $r \in \mathbb{N}$ and a finite compact covering $\mathcal{K} \stackrel{\text{def}}{=} \{K_i\}_{i=1}^n$ of Σ, where $K_i \subset U_i$, set $N^r_\varepsilon(B_0; \mathcal{K}) = \bigcap_{i=1}^n N^r_\varepsilon(B_0; (U_i, \phi_i), (V_i, \psi_i), K_i)$. Then, $\exists\, N_{\varepsilon,r} > 0$ so that $A_n \in N^r_\varepsilon(B_0; \mathcal{K})$ whenever $n > N_{\varepsilon,r}$. However, because $A_n^{\phi_{f_n}} \to \tilde{B}_0$ as $n \to \infty$, it follows at once that $A_n^{\phi_{f_n}} \in \pi^{-1}_{\mathrm{SU}(2)} \circ \pi_{\mathrm{SU}(2)}(N^r_\varepsilon(B_0; \mathcal{K}))$ whenever $n > N_{\varepsilon,r}$. Consequently, $\tilde{B}_0 \in \pi^{-1}_{\mathrm{SU}(2)} \circ \pi_{\mathrm{SU}(2)}(N^r_\varepsilon(B_0; \mathcal{K}))$. Now, suppose that $\tilde{B}_0 \notin \pi^{-1}_{\mathrm{SU}(2)} \circ \pi_{\mathrm{SU}(2)}(B_0)$. This means that $\exists\, \hat{\varepsilon} > 0$ sufficiently small and some $\hat{r} > 0$ large enough so that $\tilde{B}_0 \notin \pi^{-1}_{\mathrm{SU}(2)} \circ \pi_{\mathrm{SU}(2)}(N^{\hat{r}}_{\hat{\varepsilon}}(B_0; \mathcal{K}))$; however, from the assumption of convergence $(A_n, A_n^{\phi_{f_n}}) \to (B_0, \tilde{B}_0)$, $\exists\, \hat{N} > 0$ such that $A_n, A_n^{\phi_{f_n}} \in \pi^{-1}_{\mathrm{SU}(2)} \circ$

$\pi_{\mathrm{SU}(2)}(N_{\hat{\varepsilon}}^{\hat{r}}(B_0;\mathcal{K}))\ \forall n > \hat{N}$, but $\tilde{B}_0 \notin \pi_{\mathrm{SU}(2)}^{-1} \circ \pi_{\mathrm{SU}(2)}(N_{\hat{\varepsilon}}^{\hat{r}}(B_0;\mathcal{K}))$, which is a contradiction. Hence, $\tilde{B}_0 \in \pi_{\mathrm{SU}(2)}^{-1} \circ \pi_{\mathrm{SU}(2)}(B_0)$, and $\mathcal{R}$ is thus closed, as required. □

To construct a promeasure on $\mathcal{A}[\mathrm{SU}(2)]$, one must define spaces $\mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$, the analogues of $\mathcal{A}_{\alpha n}$. To this end, observe first of all that if $A(x_i) = B(x_i)$ for $i = 1, \ldots, n$, then $A^{\phi_f}(x_i) = B^{\phi_f}(x_i)$ for each $i = 1, \ldots, n$ and $f \in \mathcal{G}$. Hence, each $G_{\alpha n} \in c[D_\Sigma]$ generates an equivalence relation $R_{\alpha n} \subset \mathcal{A}[\mathrm{SU}(2)] \times \mathcal{A}[\mathrm{SU}(2)]$ by

$$\hat{A} \sim \hat{B} \iff A(x_i) = B(x_i) \text{ on } G_{\alpha n},$$

where $A \in \hat{A}$ and $B \in \hat{B}$ are any fixed representatives such that $A^{\phi_f}(x_i) = B^{\phi_f}(x_i)$ on $G_{\alpha n}$ for each $f \in \mathcal{G}$. Denote the quotient space by $\mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ and $\tilde{p}_{\alpha n}$: $\mathcal{A}[\mathrm{SU}(2)] \to \mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ its natural map. Moreover, define a map $\pi_{\alpha n} : \mathcal{A}_{\alpha n} \to \mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "p_{\alpha n}"] \arrow[d, "\pi_{\mathrm{SU}(2)}"'] & \mathcal{A}_{\alpha n} \arrow[d, "\pi_{\alpha n}"] \\
\mathcal{A}[\mathrm{SU}(2)] \arrow[r, "\tilde{p}_{\alpha n}"] & \mathcal{A}_{\alpha n}[\mathrm{SU}(2)];
\end{tikzcd}$$

that is, $\pi_{\alpha n} \circ p_{\alpha n} = \tilde{p}_{\alpha n} \circ \pi_{\mathrm{SU}(2)}$. It is easy to verify that $\pi_{\alpha n}$ is a continuous surjection.

A promeasure on $\mathcal{A}[\mathrm{SU}(2)]$ can now be constructed. First, $p_{\beta m}^{\alpha n} : \mathcal{A}_{\alpha n} \to \mathcal{A}_{\beta m}$ induces a map $\tilde{p}_{\beta m}^{\alpha n} : \mathcal{A}_{\alpha n}[\mathrm{SU}(2)] \to \mathcal{A}_{\beta m}[\mathrm{SU}(2)]$ given by $[\hat{A}]_{\alpha n} \mapsto [\hat{A}]_{\beta m}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}_{\alpha n} \arrow[r, "p_{\beta m}^{\alpha n}"] \arrow[d, "\pi_{\alpha n}"'] & \mathcal{A}_{\beta m} \arrow[d, "\pi_{\beta m}"] \\
\mathcal{A}_{\alpha n}[\mathrm{SU}(2)] \arrow[r, "\tilde{p}_{\beta m}^{\alpha n}"] & \mathcal{A}_{\beta m}[\mathrm{SU}(2)].
\end{tikzcd}$$

So, define $\mu_{\alpha n}$ by $\mu_{\alpha n} \stackrel{\text{def}}{=} \lambda_{\alpha n} \circ \pi_{\alpha n}^{-1}$. Then, $\mu_{\alpha n}$ defines a measure on $\mathcal{A}_{\alpha n}[\mathrm{SU}(2)]$ as $\lambda_{\alpha n}$ is a bounded measure. Thus $\mu \stackrel{\text{def}}{=} \{(\mathcal{A}_{\alpha(n)n}[\mathrm{SU}(2)], \mu_{\alpha(n)n})\}$ defines a promeasure on $\mathcal{A}[\mathrm{SU}(2)]$. To verify this, recall that $\lambda_{\alpha(m)m} = \lambda_{\beta(n)n} \circ p_{\alpha(n)n}^{\beta(m)m}$, and for each n, $\lambda_{\alpha(n)n}$ is bounded. Hence, the relation $\pi_{\beta(m)m} \circ p_{\beta(m)m}^{\alpha(n)n} = \tilde{p}_{\beta(m)m}^{\alpha(n)n} \circ \pi_{\alpha(n)n}$ together with the surjectivity of the maps imply that $\mu_{\beta(m)m} = \lambda_{\beta(m)m} \circ \pi_{\beta(m)m}^{-1} = \lambda_{\alpha(n)n} \circ p_{\alpha(n)n}^{\beta(m)m} \circ \pi_{\beta(m)m}^{-1} = \mu_{\alpha(n)n} \circ \tilde{p}_{\alpha(n)n}^{\beta(m)m}$, as required.

6.5. Discussion

This chapter will end with a tentative sketch of the construction of a promeasure on $\mathcal{A}$ that is simultaneously SU(2) gauge-invariant and $\mathrm{Diff}^+(\Sigma)$-invariant. The construction will in fact turn out to be amazingly simple. First, some notations will be introduced. Let $\hat{\pi}_\Sigma : \mathcal{A} \to \mathcal{A}^\Sigma$ denote the quotient map, where $\mathcal{A}^\Sigma \stackrel{\text{def}}{=} \mathcal{A}/\mathrm{Diff}^+(\Sigma)$ is the space of $\mathrm{Diff}^+(\Sigma)$-equivalent connection 1-forms:

$$A \sim A' \iff A' = f^*A \text{ for some } f \in \mathrm{Diff}^+(\Sigma).$$

Now, fix any $f \in \mathrm{Diff}^+(\Sigma)$ and consider $A, B \in \mathcal{A}$ such that $A(x_i) = B(x_i)\ \forall i = 1, \ldots, n$. Since under a coordinate transformation induced by f, $A_a(x_i) \to A'_a(y_i) = \frac{\partial x^b(x_i)}{\partial f^a(x_i)} A_b(x_i)$ and $B_a(x_i) \to B'_a(y_i) = \frac{\partial x^b(x_i)}{\partial f^a(x_i)} B_b(x_i)$ for each $i = 1, \ldots, n$, where $x_i = f(y_i)$, it follows at once that $f^*A(y_i) = f^*B(y_i)$ for each i. Hence, the following equivalence relation $\sim$ on $\mathcal{A}_{\alpha n}$ given by

$$[A]_{\alpha n} \sim [A']_{\alpha n} \iff \exists f \in \mathrm{Diff}^+(\Sigma) \text{ such that } f^*A(x_i) = A'(x_i)\ \forall i = 1, \ldots, n,$$

where A (resp. A') is any representative of $[A]_{\alpha n}$ (resp. $[A']_{\alpha n}$) is well-defined.

Denote the coset of $[A]_{\alpha n}$ under $\sim$ by $[A]^\Sigma_{\alpha n}$ and the quotient space $\mathcal{A}_{\alpha n}/\mathrm{Diff}^+(\Sigma)$ by $\mathcal{A}^\Sigma_{\alpha n}$. Furthermore, let $\hat{\pi}_{\alpha n} : \mathcal{A}_{\alpha n} \to \mathcal{A}^\Sigma_{\alpha n}$ denote the quotient map. Then, the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{p_{\alpha n}} & \mathcal{A}_{\alpha n} & \xrightarrow{p^{\alpha n}_{\alpha m}} & \mathcal{A}_{\alpha m} \\ \downarrow{\scriptstyle \hat{\pi}_\Sigma} & & \downarrow{\scriptstyle \hat{\pi}_{\alpha n}} & & \downarrow{\scriptstyle \hat{\pi}_{\alpha m}} \\ \mathcal{A}^\Sigma & \xrightarrow{p'_{\alpha n}} & \mathcal{A}^\Sigma_{\alpha n} & \xrightarrow{(p')^{\alpha n}_{\alpha m}} & \mathcal{A}^\Sigma_{\alpha m}, \end{array}$$

where $p'_{\alpha n}$ (resp. $(p')^{\alpha n}_{\alpha m}$) is a projection induced by $p_{\alpha n}$ (resp. $p^{\alpha n}_{\alpha m}$). Finally, given $f \in \mathrm{Diff}^+(\Sigma)$ and an SU(2) gauge transformation ϕ_g, where $g \in \mathcal{G}$, set $f^*D \stackrel{\text{def}}{=} \{ f^*A \mid A \in D \}$ and $D^{\phi_g} = \{ A^{\phi_g} \mid A \in D \}$, where $D \subset \mathcal{A}$ and $A^{\phi_g} = \phi_g^{-1} A \phi_g + \phi_g^{-1}\, \mathrm{d}\phi_g$.

6.5.1. Remark. It is easy to see that given any subset D of $\mathcal{A}_{\alpha n}$ such that $D \neq \mathcal{A}_{\alpha n}$, $\hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D) \neq \mathcal{A}_{\alpha n}$, where $\hat{\Pi}_{\alpha n} \stackrel{\text{def}}{=} \hat{\pi}^{-1}_{\alpha n} \circ \hat{\pi}_{\alpha n}$ and $\Pi_{\alpha n} \stackrel{\text{def}}{=} \pi^{-1}_{\alpha n} \circ \pi_{\alpha n}$.

6.5.2. Lemma. *For each $D \subseteq \mathcal{A}_{\alpha n}$, $\hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D) = \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$.*

Proof. Now, given $f^*(A^\phi) \in \hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D)$, where $A \in D$ and $D \subseteq \mathcal{A}_{\alpha n}$, $f^*(A^\phi) = (\phi \circ f)^{-1} f^*A(\phi \circ f) + (\phi \circ f)^{-1} f^*\, \mathrm{d}\phi$. So, set $B = f^*A$ and $\varphi = \phi \circ f$. Then,

$\mathrm{d}\varphi = f^* \,\mathrm{d}\phi$ and

$$\begin{aligned} f^*(A^\phi) &= (\phi \circ f)^{-1} f^* A (\phi \circ f) + (\phi \circ f)^{-1} f^* \,\mathrm{d}\phi \\ &= \varphi^{-1} B \varphi + \varphi^{-1} \,\mathrm{d}\varphi \\ &= B^\varphi \in \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D). \end{aligned}$$

Hence, $\hat{\Pi}_\alpha \circ \Pi_{\alpha n}(D) \subseteq \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$.

Conversely, consider any element $(f^* A)^\phi \in \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$, where $A \in D$, set $\varphi = \phi \circ f^{-1}$. Then, $\mathrm{d}\varphi = (f^{-1})^* \,\mathrm{d}\phi$ and

$$\begin{aligned} (f^* A)^\phi &= \phi^{-1} f^* A \phi + \phi^{-1} \,\mathrm{d}\phi \\ &= \phi^{-1} f^* A \phi + \phi^{-1} f^* \circ (f^{-1})^* \,\mathrm{d}\phi \\ &= f^*(\varphi^{-1} A \varphi + \varphi^{-1} \,\mathrm{d}\varphi) \\ &= f^*(A^\varphi) \in \hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D). \end{aligned}$$

Hence, $\hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}(D) \subseteq \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}(D)$ and the assertion thus follows. □

6.5.3. Theorem. *$\mathcal{A}_{\alpha n}$ admits a (bounded) measure $\nu_{\alpha n}$ that is simultaneously* $\mathrm{Diff}^+(\Sigma)$*-invariant as well as SU(2) gauge-invariant.*

Proof. Set $\nu_{\alpha n} \stackrel{\text{def}}{=} \lambda_{\alpha n} \circ \hat{\Pi}_{\alpha n} \circ \Pi_{\alpha n}$. By remark 6.5.1, $\nu_{\alpha n}$ is not a "trivial" measure in the sense that the equality $\nu_{\alpha n}(D) = \nu_{\alpha n}(\mathcal{A}_{\alpha n})$ for each measurable D with non-empty interior will not hold in general. And by lemma 6.5.2, $\nu_{\alpha n} = \lambda_{\alpha n} \circ \Pi_{\alpha n} \circ \hat{\Pi}_{\alpha n}$. Hence, for each $\lambda_{\alpha n}$-measurable subset $D \subset \mathcal{A}_{\alpha n}$, $\nu_{\alpha n}(f^*(D^\phi)) = \nu_{\alpha n}(D) = \nu_{\alpha n}((f^* D)^\phi)$ $\forall f \in \mathrm{Diff}^+(\Sigma)$ and any SU(2) gauge transformation ϕ, and it is thus the desired $\mathrm{Diff}^+(\Sigma)$- and SU(2) gauge-invariant promeasure on $\mathcal{A}_{\alpha n}$. □

The following result is now immediate and it will conclude this paper.

6.5.4. Corollary. *A* $\mathrm{Diff}^+(\Sigma)$*-invariant and SU(2) gauge-invariant promeasure ν exists on $\mathcal{A}$.*

Proof. Set $\nu = \{(\nu_{\alpha(n)n}, \mathcal{A}_{\alpha(n)n})\}$. Then, the proof that ν is a promeasure follows from the commutativity of the following two diagrams:

$$\begin{array}{ccc} \mathcal{A}_{\alpha(n)n} & \longrightarrow & \mathcal{A}_{\alpha(m)m} \\ \downarrow & & \downarrow \\ \mathcal{A}^{\mathrm{SU}(2)}_{\alpha(n)n} & \longrightarrow & \mathcal{A}^{\mathrm{SU}(2)}_{\alpha(m)m} \end{array} \qquad \begin{array}{ccc} \mathcal{A}_{\alpha(n)n} & \longrightarrow & \mathcal{A}_{\alpha(m)m} \\ \downarrow & & \downarrow \\ \mathcal{A}^{\Sigma}_{\alpha(n)n} & \longrightarrow & \mathcal{A}^{\Sigma}_{\alpha(m)m}, \end{array}$$

for each $m < n$, where $\mathcal{A}^{\mathrm{SU}(2)}_{\alpha k} \stackrel{\text{def}}{=} \mathcal{A}_{\alpha k}[\mathrm{SU}(2)]$ for typesetting convenience. □

In the previous section, the existence of an SU(2) gauge-invariant promeasure on $\mathcal{A}$ was demonstrated. This chapter will close with an alternative (but somewhat speculative) construction of a diffeomorphism-invariant promeasure on $\mathcal{A}[\mathrm{SU}(2)]$. The results developed by Ashtekar *et al.* in reference [2] applies to a projective family of compact Hausdorff spaces whereas the projective family of spaces introduced above are non-compact. Perhaps the concise ideas expressed along the lines introduced by Ashtekar *et al.* in reference [2, §3.3, Eqns (3.13a) – (3.13c)] will yield a diffeomorphism-invariant promeasure. The details, of course, are yet to be worked out.[2]

The ideas regarding an alternative construction of a diffeomorphism-invariant promeasure on $\mathcal{A}[\mathrm{SU}(2)]$ revolve around a result obtained by Baez [6, §4, Theorem 5]. Using notations consistent with Baez [6], define $\mathcal{A}_\gamma$ to be the set of spinor propagators along a path γ defined by the Ashtekar connections $A \in \mathcal{A}$. There are subtleties involved such as the requirement that the path γ be piecewise real-analytic (as opposed to piecewise smooth); however, for the sake of brevity, these technicalities will be ignored here. Only the main ideas will be of interest.

It is known that $\mathrm{Diff}(\Sigma)$ is a locally Fréchet C^∞-group [9]. Conceivably, it is not very difficult to modify the theory of promeasures on locally convex linear spaces to manifolds modelled on locally convex linear spaces to construct promeasure on the manifolds. So, suppose that a non-trivial promeasure ρ can be constructed via a projective family of left Haar measures defined on a family of finite dimensional projective topological groups of $\mathrm{Diff}^+(\Sigma)$. Then, appealing to Theorem 5 of article [6] by Baez, the convolution $\rho * \lambda$ defines a $\mathrm{Diff}^+(\Sigma)$-invariant promeasure on $\mathcal{A}$, where λ is a promeasure on $\mathcal{A}$ and the *convolution* $\rho * \lambda$ is defined [6] by

$$(\rho * \lambda)(f) \stackrel{\mathrm{def}}{=} \int_{\mathrm{Diff}^+_\phi(\Sigma)\times\mathcal{A}_\phi} f(gA)\,\mathrm{d}\rho(g)\,\mathrm{d}\lambda(A),$$

where ϕ is an embedded graph [6, §3] in Σ, $\mathcal{A}_\phi \stackrel{\mathrm{def}}{=} \prod_{\gamma\in E(\phi)} \mathcal{A}_\gamma$ is the Cartesian product of (finite) edges γ of the graph ϕ, $E(\phi)$ denotes the (finite) set of edges of ϕ, and $\mathrm{Diff}^+_\phi(\Sigma) \cong \prod_{x\in V(\phi)} \mathrm{Diff}^+(\Sigma)$ with $V(\phi)$ being the (finite) set of distinct vertices of ϕ. Having obtained a $\mathrm{Diff}^+(\Sigma)$-invariant promeasure on $\mathcal{A}$, one can then project the promeasure down to $\mathcal{A}[\mathrm{SU}(2)]$ which was shown in detail in the previous section.

It is immediately obvious here that the burden of constructing a $\mathrm{Diff}^+(\Sigma)$-invariant promeasure on $\mathcal{A}[\mathrm{SU}(2)]$ is now shifted towards the task of constructing

[2] Keeping fingers crossed for good measure!

a suitable promeasure on $\mathrm{Diff}^+(\Sigma)$ and then modifying the analysis given in reference [6] so that the results there applies to the ideas delineated somewhat tersely here. Assuredly, the entire analysis sketched here is delivered in a rather cavalier and rushed manner; however, hopefully, the essence of the ideas to be conveyed is not lost in the flurry!

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Author's Note: During the period when I constructed a promeasure on the space of Ashtekar connections, I was unaware of the work done by Ashtekar and Lewandowski [2] regarding the construction of a promeasure on the space of connection 1-forms.

CHAPTER VII

SPECULATIONS AND CONCLUDING REMARKS

7.1. Introduction

In this final chapter, an attempt at expressing the reality conditions

$$E^a = (E^a)^{\tilde{\dagger}} \quad \text{and} \quad A_a + A_a^{\tilde{\dagger}} = 2\Gamma_a,$$

where Γ_a is the spin-connection coefficients, directly in terms of the loop variables will be made. For more details regarding the reality conditions, consult references [1, Chpt 8, p. 101], [2, Chpt 7, p. 102], [4, §3.1.3, p. 1636] and [3, p. 305].

Briefly, the way the reality conditions relate to the physical inner product can be explained as follows. Consider a gauge-invariant observable $P[\gamma, A]$, which will be assumed here to be a function of both the loop as well as the Ashtekar connection 1-form. Suppose that it commutes (weakly) with the diffeomorphism and Hamiltonian constraints. Furthermore, suppose that an inner product on the space of multi-loop functionals is also defined. Then, because $P[\gamma, A]$ is a physical observable, $\hat{P}[\gamma, A]$ must be Hermitian with respect to the physical inner product of the theory. Hence, any potential physical inner product must satisfy

$$\langle \hat{P}^{\tilde{\dagger}}[\gamma, A]\psi \mid \phi\rangle = \langle \psi \mid \hat{P}[\gamma, A]\phi\rangle.$$

The next criterion that the inner product must satisfy is that it must be defined in such a way that $A_a + A_a^{\tilde{\dagger}} = 2\Gamma_a$ also holds.

It is not at all obvious how the reality condition can be implemented in the loop representation directly. This will be the contents of section 3. In section 2, the smearing of the quantum T^n-operators will be described. The final section will summarise the major results obtained in this thesis.

7.2. The Loop Representation Revisited

In this section, the smeared version of the $\hat{T}$-operators will be constructed. The construction here is based on a rather brief description given in reference [5, §3.6,

p. 118]. A word regarding the Mandelstam identities [2, p. 274] will also be mentioned. Let $\hat{\mathcal{M}}_1^*$ be a subset of $\mathcal{M}_1'$ spanned by loop functionals ψ satisfying

(1) $\psi[\gamma] = \psi[\gamma']$ whenever $\gamma' \sim \gamma$,[1]
(2) $\psi[\gamma * \eta * \eta_-] = \psi[\gamma]\ \forall \gamma, \eta \in \mathcal{M}_1$,
(3) $\psi[0_\Sigma * \gamma] = 2\psi[\gamma]\ \forall \gamma \in \mathcal{M}_1$,

where γ, γ' are loops and η is a curve based at a point on γ.

In the construction of $\hat{\mathcal{M}}_n^*$ for $n > 1$ given in §3.4, replace $\mathcal{M}_1'$ with $\hat{\mathcal{M}}_1^*$, and consider the resulting smaller subspaces $\hat{\mathcal{M}}_n^*$ arising from the replacement of $\mathcal{M}_1'$ with $\hat{\mathcal{M}}_1^*$ in all the discussion that follows. Then, it is clear from the definition of $\hat{\mathcal{M}}_1^*$ that each $\Psi_n \in \hat{\mathcal{M}}_n^*$ satisfies

(a) $\Phi_n[\gamma_n] = \Phi_n[\gamma_n']$, where $\eta_n \stackrel{\text{def}}{=} \{\eta^1, \ldots, \eta^n\}$ and $\gamma_n \sim \gamma_n'$;
(b) $\Phi_n[\gamma^1 * \eta^1 * \eta_-^1, \gamma^2, \ldots, \gamma^n] = \Phi_n[\gamma^1, \ldots, \gamma^n]$,
(c) $\Phi_n[0_\Sigma * \gamma^1, \ldots, 0_\Sigma * \gamma^n] = 2\Phi_n[\gamma^1, \ldots, \gamma^n]$.

A final non-linear condition will be imposed on the elements of $\hat{\mathcal{M}}_n^*$. This non-linear condition is a consequence of the spinor identity $T[\gamma * \eta, A] + T[\gamma * \eta_-, A] = T[\gamma, A]T[\eta, A]$ mentioned in Chapter 2:

(d) $\Phi_n[\gamma * \eta, \ldots] + \Phi_n[\gamma_- * \eta, \ldots] - \Phi_{n+1}[\gamma, \eta, \ldots] = 0$.

From these, it can be shown that $\Phi[\gamma] = \Phi[\gamma_-]$ and hence Φ's are also invariant under orientation-reversing reparametrisations.

What was not mentioned in §2.4—or at least, deferred to this section—was the Mandelstam identities. They are a set of conditions, often "non-linear" ones, imposed on the Wilson loops. They will not be given here but can be found, for example, in a well written article [2, p. 274] by Loll on the loop representation. It should suffice to mention here that these conditions must also be included in the loop representation theory although, things being the way they often are in the real world, they greatly complicate the theory. Fortunately however, Rovelli and Smolin recently overcame the hurdles imposed by the Mandelstam identities in the loop representation—the details of which can be found in reference [6].

Returning to the main topic of this section which is to construct a rigorous smearing procedure for the T-operators, the idea of a loop with "fattened" regions will now be defined. Given a loop $\gamma \in \mathcal{M}_1$, let $\mathcal{P}_n(I, \gamma)$ denote the set of pairs $(P, J(P))$, where $P = \{0 \leqq s_1 < \cdots < s_n < 1\}$ is a partition of I, and $J(P)$ is the

[1] $\gamma' \sim \gamma$ denotes that γ' is a reparametrisation of γ.

finite set $\{ J(s_i) : i = 1, \dots, n \}$ of closed intervals satisfying

(i) $J(s_i) \subset I$ is a closed interval,
(ii) $J(s_i) \cap J(s_j) = \varnothing \ \forall i \neq j$,
(iii) if $s_1 = 0$, then $J(s_1) = J(0) \stackrel{\text{def}}{=} [0, \varepsilon_+] \cup [\varepsilon_-, 1]$ with $[0, \varepsilon_+] \cap J(s_2) = \varnothing$ and $[\varepsilon_-, 1] \cap J(s_n) = \varnothing$ for some $\varepsilon_\pm \in I$.

For any pair $(P, J) \in \mathcal{P}_n(I, \gamma)$, consider a continuous mapping $\gamma_n : I^2 \times I \to \Sigma$ that satisfies the following:

(1) for each $\sigma \in I^2$, $\gamma_{n,\sigma} \stackrel{\text{def}}{=} \gamma_n(\sigma, \cdot\,)$ is a loop in Σ;
(2) $\gamma_n(\frac{1}{2}, \frac{1}{2}, \cdot\,) = \gamma$;
(3) $\gamma_n(\sigma, u) = \gamma(u) \ \forall (\sigma, u) \in I^2 \times (I - \bigcup_{i=1}^n J(s_i))$;
(4) suppose that $J(s_i) = [s_i^-, s_i^+]$. Then, $\gamma_n(\sigma, \cdot\,)|J(s_i)$ is a curve in Σ that connects $\gamma(s_i^-)$ to $\gamma(s_i^+)$ for each $\sigma \in I^2$;
(5) $\gamma_n(I^2 \times J(s_i)) \cap \gamma_n(I^2 \times J(s_j)) = \varnothing \ \forall i \neq j$ unless $\gamma(s_i) = \gamma(s_j)$.

Call γ_n a *2-parameter congruence* associated with (P, J).

Let $\Lambda^1(\Sigma)$ be the space of smooth 1-forms on Σ and $D_n(\Sigma, I, \gamma)$ be the space spanned by n-forms $\hat{f}$ along γ_n of the form

$$\hat{f}(\gamma_n(\sigma_1, s_1)), \dots, \gamma_n(\sigma_n, s_n)) = \hat{f}_1(\gamma_n(\sigma_1, s_1)) \wedge \cdots \wedge \hat{f}_n(\gamma_n(\sigma_n, s_n)),$$

where $\hat{f}_i(\gamma_n(\sigma_i, s_i)) \stackrel{\text{def}}{=} e_i^n(s_i) \cdot f_i(\gamma_n(\sigma_i, s_i))$, $f_i \in \Lambda^1(\Sigma)$ for each i, and $e_i^n : I \to I$ is a smooth function with support $\operatorname{supp}(e_i^n) = \overline{J}(s_i)$ for each i. The functions e_i^n ensure that $E^{a_i}(\gamma(s))$ remains within the intervals $J(s_i)$. See [5, §3.6, p. 118] for an equivalent formulation.

Now, given a $\hat{T}$-operator $\hat{T}^n[\gamma, A](s_1, \dots, s_n) = \hat{T}^{a_1 \dots a_n}[\gamma, A](s_1, \dots, s_n)$, its *smeared* version $\hat{T}^n[\gamma_n, A](\hat{f})$ is defined by

$$\int \mathrm{d}^2\sigma_1 \dots \mathrm{d}^2\sigma_n \int \mathrm{d}s_1 \dots \mathrm{d}s_n \hat{f}(\gamma_n(\sigma_1, s_1), \dots, \gamma_n(\sigma_n, s_n)) \cdot \hat{T}^n[\gamma, A](s_1, \dots, s_n),$$

where γ_n is a 2-parameter congruence associated with some $(P, J) \in \mathcal{P}_n(I, \gamma)$ and $\hat{f} \in D_n(\Sigma, I, \gamma)$. To simplify notations, the Ashtekar 1-forms in the $\hat{T}$-operators will be assumed fixed and not spelt out explicitly in the $\hat{T}$-operators.

3.4.1. Example. The smeared commutator between $\hat{T}^1[\gamma_1](f)$ and $\hat{T}[\eta]$ will be worked out below, where γ_1 is a 2-parameter congruence associated with $(P, J) \in$

$\mathcal{P}_1(I,\gamma)$ and $f \in D_1(\Sigma, I, \gamma)$.

$$
\begin{aligned}
&([\hat{T}^1[\gamma_1](f), \hat{T}[\eta]]\Psi)[\alpha] \\
&= \int \mathrm{d}^2\sigma\, \mathrm{d}s f(\gamma_1(\sigma,s)) \cdot \hat{T}^1[\gamma_1](s)\Psi[\eta \cup \alpha] - \int \mathrm{d}^2\sigma\, \mathrm{d}s f_a(\gamma_1(\sigma,s))\hat{T}[\eta] \cdot \\
&\qquad\qquad \hbar \sum_\epsilon (-1)^{n(\epsilon)} \Delta^a[\gamma_1, \alpha](s)\Psi[(\gamma_{1,\sigma} * \alpha)^{n(\epsilon)}] \\
&= \int \mathrm{d}^2\sigma\, \mathrm{d}s f_a(\gamma_1(\sigma,s))\hbar \sum_\epsilon (-1)^{n(\epsilon)} \Delta^a[\gamma_1, \eta \cup \alpha](s)\Psi[(\gamma_{1,\sigma} * (\alpha \cup \eta)^\epsilon] - \\
&\quad \int \mathrm{d}^2\sigma\, \mathrm{d}s f_a(\gamma_1(\sigma,s))\hbar \sum_\epsilon (-1)^{n(\epsilon)} \Delta^a[\gamma_1, \alpha](s)\Psi[\eta \cup (\gamma_{1,\sigma} * \alpha)^\epsilon] \\
&= \int \mathrm{d}^2\sigma\, \mathrm{d}s f_a(\gamma_1(\sigma,s))\hbar \sum_\epsilon (-1)^{n(\epsilon)} \{\Delta^a[\gamma_1, \alpha](s)\Psi[(\gamma_{1,\sigma} * \alpha)^\epsilon \cup \eta] + \\
&\qquad\qquad \Delta^a[\gamma_1, \eta](s)\Psi[(\gamma_{1,\sigma} * \eta)^\epsilon \cup \alpha]\} - \\
&\quad \int \mathrm{d}^2\sigma\, \mathrm{d}s f_a(\gamma_1(\sigma,s))\hbar \sum_\epsilon (-1)^{n(\epsilon)} \Delta^a[\gamma_1, \alpha](s)\Psi[(\gamma_{1,\sigma} * \alpha)^\epsilon \cup \eta] \\
&= \int \mathrm{d}^2\sigma\, \mathrm{d}s f_a(\gamma_1(\sigma,s))\hbar \sum_\epsilon (-1)^{n(\epsilon)} \Delta^a[\gamma_1, \eta](s)\Psi[\alpha \cup (\gamma_{1,\sigma} * \eta)^\epsilon] \\
&= \int \mathrm{d}^2\sigma\, \mathrm{d}s\, \mathrm{d}u f_a(\gamma_1(\sigma,s))\hbar \sum_\epsilon (-1)^{n(\epsilon)} \dot{\eta}^a(u)\delta^3(\eta(u), \gamma_1(\sigma,s))\Psi[(\gamma_{1,\sigma} * \eta)^\epsilon \cup \alpha] \\
&= \hbar \sum_\epsilon (-1)^{n(\epsilon)} \int \mathrm{d}u f_a(\eta(u))\dot{\eta}^a(u)\Psi[(\gamma_1(\sigma(u)), \cdot\,) * \eta)^\epsilon \cup \alpha] \\
&= \hbar \sum_\epsilon (-1)^{n(\epsilon)} \oint_\eta f \cdot \hat{T}^1[(\gamma_{1,\sigma(u)} * \eta)^\epsilon]\Psi[\alpha],
\end{aligned}
$$

where $\sigma = \sigma(u)$ is now regarded as a function of u, $\sigma : I \to I \times I$. Whence,

$$[\hat{T}^1[\gamma_1](f), \hat{T}[\eta]] = \hbar \sum_\epsilon (-1)^{n(\epsilon)} \oint_\eta f \cdot \hat{T}^1[(\gamma_{1,\sigma(u)} * \eta)^{n(\epsilon)}]. \quad \square$$

In general, $\hat{T}^{a_1 \cdots a_n}[\gamma](s_1, \ldots, s_n)$ may be regarded as a distribution belonging to the space $L'_n(\Sigma,\gamma) \otimes L(\hat{\mathcal{M}}^*)$, where $L_n(\Sigma,\gamma)$ (equipped with a suitable topology) is the space of test functions φ which regularise $\hat{T}^{a_1 \cdots a_n}[\gamma](s_1, \ldots, s_n)$, $L'_n(\Sigma,\gamma)$ is the topological dual of $L_n(\Sigma,\gamma)$ and $L(\hat{\mathcal{M}}^*)$ is the space of linear transformations on $\hat{\mathcal{M}}^*$. That is, $\hat{T}^n[\gamma_n](\varphi) : \hat{\mathcal{M}}^* \to \hat{\mathcal{M}}^*$ is a linear transformation.

The phrase "loop representation" is meant in the following sense. Let $\mathfrak{T}_n$ denote the space of the T^n-observables and define the T-algebra $\mathfrak{T}$ to be the graded sum of $\mathfrak{T}_n$'s: $\mathfrak{T} \stackrel{\text{def}}{=} \bigoplus_{n \in \mathbb{N}} \mathfrak{T}_{n-1}$. Finally, let $\lambda_n : \mathfrak{T}_n \to \hat{\mathcal{M}}^*$ be given by $\lambda_n(T^{a_1\ldots a_n}[\gamma,A])\Psi \stackrel{\text{def}}{=} \hat{T}^{a_1\ldots a_n}[\gamma,A]\Psi$. Strictly, $\lambda_n(T^{a_1\ldots a_n}[\gamma,A])$ has to be regularised; but in order to keep the discussion simple here, that bit of technicality will be skipped. Then, the *loop representation* is the map $\lambda : \mathfrak{T} \to \hat{\mathcal{M}}^*$ is defined by its restriction to $\mathfrak{T}_n$: $\lambda|\mathfrak{T}_n \stackrel{\text{def}}{=} \lambda_n$.

7.3. Reality Conditions

The work explored in this section is purely speculative and should only be considered as such. Here, an attempt is made to seek for an explicit expression of the reality condition $A^i_a + (A^i_a)^\dagger = 2\Gamma^i_a$ in the loop representation.[2] The motivation can be found in the quantum T^1-operator. First, observe that the following expansion

$$U[\gamma,A] = 1 + \sum_{n=1}^{\infty} \int_0^1 \mathrm{d}t_n \cdots \int_0^{t_2} \mathrm{d}t_1 A_{a_n}(\gamma(t_n))\dot{\gamma}^{a_n}(t_n)\ldots A_{a_1}(\gamma(t_1))\dot{\gamma}^{a_1}(t_1)$$

of the complexified SU(2) holonomy implies heuristically at least that the quantum operator $\hat{T}[\gamma,A]$, where $T[\gamma,A] = \operatorname{tr} U[\gamma,A]$, can be expressed as an infinite sum $\hat{T}[\gamma,A] = \sum_{n=0}^{\infty} \hat{t}^n_\gamma[A]$, where

$$t^n_\gamma[A] \stackrel{\text{def}}{=} \operatorname{tr}\left(\int_0^1 \mathrm{d}t_n \cdots \int_0^{t_2} \mathrm{d}t_1 A_{a_n}(\gamma(t_n))\dot{\gamma}^{a_n}(t_n)\ldots A_{a_1}(\gamma(t_1))\dot{\gamma}^{a_1}(t_1)\right).$$

This property together with the definition of the action of $\hat{T}^0$ on the multi-loop functionals—*cf.* Eqn (3.5.1) in §3.5—act to motivate the following ansatz:

$$\hat{t}^n_\gamma[A]\Psi_m \stackrel{\text{def}}{=} \begin{cases} \Psi_m[\gamma,\cdot\,] & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

where Ψ_m is an m-loop functional. For the case when $n = 0$, set $\hat{1} \stackrel{\text{def}}{=} \hat{t}^0_\gamma[A]$ and define it as follows:

$$\hat{1}\Psi \stackrel{\text{def}}{=} \begin{cases} \Psi & \text{if } \Psi \in \mathbb{C}, \\ 0 & \text{if } \Psi \in \hat{\mathcal{M}}^*_n \text{ for } n > 0, \end{cases}$$

where the space $\hat{\mathcal{M}}^*_0$ of constant multi-loop functionals is identified with $\mathbb{C}$.

Next, observe that $\hat{t}^1_\gamma[A]$—and more generally, the $\hat{t}^n_\gamma[A]$ operators—may be written potentially as either

$$\hat{t}^1_\gamma[A] = \operatorname{tr}\int_0^1 \mathrm{d}t\hat{A}_a(\gamma(t))\hat{\dot{\gamma}}^a(t) \quad \text{or} \quad \hat{t}^1_\gamma[A] = \operatorname{tr}\int_0^1 \mathrm{d}t\dot{\gamma}^a(t)\hat{A}_a(\gamma(t)). \tag{7.3.1}$$

[2] As far as the author is aware, implementing the reality conditions explicitly in the loop representation has not appeared in any literature.

Because the former involves the product of two operators at the same space point $\gamma(t)$, there is the question of operator ordering—$\hat{\gamma}^a \hat{A}_a$ or $\hat{A}_a \hat{\gamma}^a$—to be resolved. And indeed, as neither $\hat{A}_a \Psi$ nor $\hat{\gamma}^a \Psi$ are known *a priori*, the question is moot. Hence, only the latter case will be considered in this analysis. However, the point to note about (7.3.1) is that if the explicit manner in which the operator $\dot{\gamma}^a(t)\hat{A}_a(\gamma(t))$ acts on Ψ is known, then in principle, the implementation of the reality conditions in the loop representation would be an easy task to accomplish.

Consider a loop functional $\psi \in \hat{\mathcal{M}}_1^*$ and the action $\hat{t}^1_\gamma[A]$ on it: $(\hat{t}^1_\gamma[A]\psi)$ is a constant loop functional on $\mathcal{M}_1$ with the value $\psi[\eta]$. That is,

$$\left(\int_0^1 \mathrm{d}t\, \dot{\gamma}^a(t)\hat{A}_a(\gamma(t))\psi\right)[\eta] = \psi[\eta] \quad \forall \eta \in \mathcal{M}_1. \tag{7.3.2}$$

It thus remains to come up with a suitable definition for the action of $\hat{A}_a(\gamma(t))$ on ψ so that (7.3.2) holds. Then, one can proceed to solve explicitly how $\hat{t}^n_\gamma[A]$ acts on Ψ_n, an n-loop functional, to yield the $(n-1)$-loop functional $\Psi_n[\gamma,\cdot]$. However, even assuming that the action of $\hat{A}$ can be solved, implementing the reality conditions might even then prove to be somewhat obscure.

So, returning to (7.3.1), another alternative is to proceed along the following lines. Consider the following ansatz:

$$\hat{t}^1_\gamma[A]\psi \stackrel{\text{def}}{=} \int_0^1 \mathrm{d}t A_a(\gamma(t))\hat{\dot{\gamma}}^a(t)\psi.$$

This is quite reasonable since being in the loop representation, it is only natural to expect that the loop takes on the active rôle as an operator rather than the connection 1-form. Furthermore, in this form, it is an easy matter to implement the reality condition. Thus, one has only to emerge with a suitable definition for the quantum loop operator $\hat{\dot{\gamma}}^a(t)$ to act on ψ so as to recover the equality given in (7.3.2). Unfortunately, this problem is yet to be solved.

There is, however, a less appealing aspect of the T^0-operator that could bring the entire analysis to a halt: to wit, the fact that $\hat{T}[\gamma, A]\Psi = \Psi[\gamma,\cdot]\ \forall A \in \mathcal{A}$ strongly suggests that the operator $\hat{T}[\gamma, A]$ is independent of A! This impasse would of course prove to be somewhat of an embarrassment.[3] Work is currently in progress along all possible avenues of thought.

[3] A somewhat rash—if not desperate, and hopefully not too incongruous—quick fix to this problem might be to consider the former case of (7.3.1) and set $\hat{A}_a$ to be the identity operator 1. Then, the reality condition $A_a + A_a^\dagger = 2\Gamma_a$ would imply that $\hat{\Gamma}_a = 1 = \hat{A}_a{}^\dagger$. Perhaps in the light of the action of the T^n-operators on Ψ for $n \in \mathbb{N}_0$, this suggestion might not be as inane as it seems since the operators do not appear to have any dependence on A whatsoever.

7.4. Conclusions

In this thesis, a brief historical perspective of quantum gravity was traced out. Arguments were put forward in favour of a non-perturbative treatment of quantum gravity, and the pros and cons of quantising gravity in the loop representation outlined. Ashtekar's Hamiltonian formulation of general relativity was given in some detail in Chapter 2 and the self-dual representation as well as the loop representation theory of quantum gravity summarised therein.

In Chapter 3, the topological structure of the multi-loop space $\mathcal{M}$ was analysed and it was found to be a second countable metrizable space consisting of the disjoint union of the n-loop spaces. Moreover, a subset of the space of continuous multi-loop functionals was also constructed. In Chapter 4, a precise relationship between a subset of $\aleph_0$-loops and the 3-geometries of Σ was established without the introduction of a lattice spacing, and a non-trivial diffeomorphism-invariant, outer regular Borel multi-loop measure was constructed in Chapter 5. Furthermore, the Hermitian conjugates of the T^n-operators were also evaluated explicitly in that chapter.

An SU(2) gauge-invariant promeasure was constructed on the space of Ashtekar connection 1-forms in Chapter 6, and some basic properties of $\mathcal{A}$ was established. In particular, $\mathcal{A}$ was shown to be an infinite-dimensional manifold. Some speculations concerning the possible construction of a diffeomorphism-invariant, SU(2) gauge-invariant promeasure was also described. Finally, in this chapter, the formulation of an explicit expression for the reality condition in the loop variables was essayed although the programme was not completed.

All in all, formulating a theory of quantum gravity in the loop representation proved to be rather rewarding even though there are problems still associated with it. The loop representation has given researchers a much deeper and richer insight into the convoluted state of gravity at the quantum level. Yet the ultimate question still remains: is it possible to have a consistent theory of quantum gravity or is general relativity intrinsically incompatible with quantum theory? Certainly the loop representation holds a great promise towards solving this time-honoured conundrum, unless belike, there lies a more profound mystery behind the scene that nature is loth to yield! Or as Shakespeare's Hamlet [7, 167–168, p. 805] so aptly puts:

> *"There are more things in heaven and earth, Horatio,*
> *Than are dreamt of in your philosophy."*
>
> —Hamlet, Act I, Scene V.

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APPENDIX

A. The Compact C$^\infty$-Topology

The definition of a *compact* C^∞*-topology* will be reviewed [5, pp. 32–33, §§4.1–4.3]. Let $J^n[\Sigma]$ be the space of C^n-jets from Σ into Σ and denote an element in $J^n[\Sigma]$ by either $j^n(f(x))$ or $[f,x]_n$ (which ever proves more convenient). Fix an atlas $\mathfrak{A}_\Sigma = \{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ on Σ and set $\mathfrak{A}_\Sigma(U_\alpha) = \{\, U \subset U_\alpha \mid U \text{ open} \,\}$. Then, $\mathfrak{B}_\Sigma = \bigcup_\alpha \mathfrak{A}_\Sigma(U_\alpha)$ forms a base for Σ. Let $J^0[\Sigma] = \Sigma \times \Sigma$ and let $\pi^0_1 : J^1[\Sigma] \to J^0[\Sigma]$ by $j^1\phi(p) \mapsto (p, \phi(p))$. Set $U^1_{\alpha\alpha'} \equiv (\pi^0_1)^{-1}(U_\alpha \times U_{\alpha'})$ and define $p^1_\pm : J^1[\Sigma] \to \Sigma$ by $p^1_+ : j^1\phi(p) \mapsto \phi(p)$ and $p^1_- : j^1\phi(p) \mapsto p$. Then, it is clear that $U^1_{\alpha\alpha'} = (p^1_-)^{-1}(U_\alpha) \cap (p^1_+)^{-1}(U_{\alpha'})$. Finally, let $\mathfrak{A}^1_{\alpha\alpha'} = \{\,(\pi^0_1)^{-1}(U \times U') \mid U \times U' \subset U_\alpha \times U_{\alpha'} \text{ open} \,\}$. Then, $\mathfrak{B}^1 = \bigcup_{\alpha,\alpha'} \mathfrak{A}^1_{\alpha\alpha'}$ forms a base for $J^1[\Sigma]$. Following [6, Definition 4.1.5, p. 94], define $\Psi^1_{\alpha\alpha'} : U^1_{\alpha\alpha'} \cong {}^3B_{\varepsilon_\alpha}(x_\alpha) \times {}^3B_{\varepsilon_{\alpha'}}(x_{\alpha'}) \times {}^{N_1}B_{\varepsilon_1}(x_1)$ by

$$[\phi, p]_1 \mapsto (\psi_\alpha(p), \psi_{\alpha'}(\phi(p)), D_\alpha(j^1\phi(p))),$$

where ${}^nB_\varepsilon(x)$ is an open ε-ball in $\mathbb{R}^n$ and $D_\alpha j^1\phi(p) \stackrel{\text{def}}{=} \left\{ \frac{\partial}{\partial x^i_\alpha} \phi_{\alpha\alpha'}(\psi_\alpha(p)) \right\}_i$ for some $N_1 \in \mathbb{N}$ such that $D_{\alpha\alpha'} : U^1_{\alpha\alpha'} \cong {}^{N_1}B_{\varepsilon_1}(x_1)$ and $\phi_{\alpha\alpha'} \stackrel{\text{def}}{=} \psi_{\alpha'} \circ \phi \circ \psi_\alpha^{-1}$. The pair $(U^1_{\alpha\alpha'}, \Psi^1_{\alpha\alpha'})$ defines a chart on $J^1[\Sigma]$. Denote $\Psi^1_{\alpha\alpha'}$ symbolically by $\psi_\alpha \times \psi_{\alpha'} \times D_\alpha$.

Now, define $p^2_\pm : J^2[\Sigma] \to \Sigma$ by $p^2_- : j^2\phi(p) \mapsto p$ and $p^2_+ : j^2\phi(p) \mapsto \phi(p)$. Furthermore, define $\pi^0_2 : J^2[\Sigma] \to J^0[\Sigma]$ by $\pi^0_2(j^2\phi(p)) = (p, \phi(p))$ and let $U^2_{\alpha\alpha'} \stackrel{\text{def}}{=} (\pi^0_2)^{-1}(U_\alpha \times U_{\alpha'})$. Then, $U^2_{\alpha\alpha'} \equiv (p^2_-)^{-1}(U_\alpha) \cap (p^2_+)^{-1}(U_{\alpha'})$. And as with the case for $J^1[\Sigma]$, the pair $(U^2_{\alpha\alpha'}, \Psi^2_{\alpha\alpha'})$ defines a chart in $J^2[\Sigma]$, where $\Psi^2_{\alpha\alpha'} \stackrel{\text{def}}{=} \psi_\alpha \times \psi_{\alpha'} \times D_\alpha \times D^2_\alpha$ and $D^2_\alpha : U^2_{\alpha\alpha'} \cong {}^{N_2}B_{\varepsilon_2}(x_2)$, for some $\varepsilon_2 > 0$ and some $N_2 \in \mathbb{N}$, is defined by $D^2_\alpha(j^2\phi(p)) \stackrel{\text{def}}{=} \left\{ \frac{\partial^2}{\partial x^{i_1}_\alpha \partial x^{i_2}_\alpha} \phi_{\alpha\alpha'}(\psi_\alpha(p)) \right\}_{i_1 \leqq i_2}$. Also, define $\pi^1_2 : J^2[\Sigma] \to J^1[\Sigma]$ by $[\phi, p]_2 \mapsto [\phi, p]_1$. Then, by definition, $\pi^0_2 = \pi^0_1 \circ \pi^1_2$ and $\pi^1_2(U^2_{\alpha\alpha'}) = U^1_{\alpha\alpha'}$.[1] Finally, let $\mathfrak{A}^2_{\alpha\alpha'} = \{\,(\pi^0_2)^{-1}(U \times U') \mid U \times U' \subset U_\alpha \times U_{\alpha'} \text{ open} \,\}$; then, $\mathfrak{B}^2 = \bigcup_{\alpha,\alpha'} \mathfrak{A}^2_{\alpha\alpha'}$ forms a base for $J^2[\Sigma]$.

---

[1] For $j^1\phi(p) \in U^1_{\alpha\alpha'} \Rightarrow \pi^0_1(j^1\phi(p)) = (p, \phi(p)) \in U_\alpha \times U_{\alpha'} \Rightarrow j^2\phi(p) \in U^2_{\alpha\alpha'}$ and so, $U^1_{\alpha\alpha'} \subseteq \pi^1_2(U^2_{\alpha\alpha'})$. Conversely, $j^1\phi'(p') \in \pi^1_2(U^2_{\alpha\alpha'}) \Rightarrow j^2\phi'(p') \in U^2_{\alpha\alpha'} \Rightarrow (p', \phi'(p')) \in U_\alpha \times U_{\alpha'} \Rightarrow$ the converse set-inequality, as required.

By induction, given $J^n[\Sigma]$, $(\pi_n^0)^{-1}(U_\alpha \times U_{\alpha'}) = (p_-^n)^{-1}(U_\alpha) \cap (p_+^n)^{-1}(U_{\alpha'})$ and $\pi_n^{n-1}(U_{\alpha\alpha'}^n) = U_{\alpha\alpha'}^{n-1}$. Furthermore, the pair $(U_{\alpha\alpha'}^n, \Psi_{\alpha\alpha'}^n)$ forms a chart on $J^n[\Sigma]$ as follows: $\Psi_{\alpha\alpha'}^n \stackrel{\text{def}}{=} \psi_\alpha \times \psi_{\alpha'} \times \prod_{i=1}^n D_\alpha^i$, where

$$D_\alpha^\ell : [\phi, p]_\ell \mapsto \left\{ \frac{\partial^\ell \phi_{\alpha\alpha'} \circ \psi_\alpha(p)}{\partial x_\alpha^{i_1} \dots \partial x_\alpha^{i_\ell}} \right\}_{i_1 \leqq \cdots \leqq i_\ell} \in \mathbb{R}^{N_\ell}$$

with some $N_\ell \in \mathbb{N}$ such that $D_\alpha^\ell(U_{\alpha\alpha'}^\ell) = {}^{N_\ell}B_{\varepsilon_\ell}(x_\ell)$. Tersely, $\Psi_{\alpha\alpha'}^n : U_{\alpha\alpha'}^n \cong {}^3B_{\varepsilon_\alpha}(x_\alpha) \times {}^3B_{\varepsilon_{\alpha'}}(x_{\alpha'}) \times \prod_{i=1}^n {}^{N_i}B_{\varepsilon_i}(x_i)$. The topology on $J^n[\Sigma]$ is generated by the base $\mathfrak{B}^n = \bigcup_{\alpha\alpha'} \mathfrak{A}_{\alpha\alpha'}^n$, where $\mathfrak{A}_{\alpha\alpha'}^n = \{ (\pi_n^0)^{-1}(U \times U') \mid U \times U' \subset U_\alpha \times U_{\alpha'} \text{ open} \}$.

It follows from the construction that $\{J^n[\Sigma], \pi_n^{n-1}, \mathbb{N}\}$ forms an inverse sequence. Let $J^\infty[\Sigma] \stackrel{\text{def}}{=} \varprojlim J^n[\Sigma]$ denote the limit of the inverse sequence. Then, $\mathfrak{B}^\infty = \{ (\pi^n)^{-1}(U) \mid U \in \mathfrak{B}^n \ \forall n \}$ defines a base of $J^\infty[\Sigma]$, where $\pi^n \stackrel{\text{def}}{=} p^n | J^\infty[\Sigma]$ and $p^n : \prod_{i \in \mathbb{N}} J^i[\Sigma] \to J^n[\Sigma]$ is the nth projection. Observe from [3, Proposition 2.5.1, p. 98] that $J^\infty[\Sigma]$ is closed in the Cartesian product $\prod_{i \in \mathbb{N}} J^i[\Sigma]$.

The *compact* (or *weak*) C^∞*-topology* on $C^\infty(\Sigma, \Sigma)$ is the topology induced by the map $j^\infty : C^\infty(\Sigma, \Sigma) \to C(\Sigma, J^\infty[\Sigma])$ defined by $f \mapsto j^\infty f \stackrel{\text{def}}{=} [f, \cdot]_\infty$ such that it is a topological imbedding. Let $\text{Diff}(\Sigma) \subset C^\infty(\Sigma, \Sigma)$ denote the set of C^∞-diffeomorphisms on Σ. The composition mapping $\circ : \text{Diff}(\Sigma) \times \text{Diff}(\Sigma) \to \text{Diff}(\Sigma)$ given by $(f, g) \mapsto f \circ g$ defines a group structure on $\text{Diff}(\Sigma)$. Indeed, the group structure is compatible with the compact C^∞-topology on $\text{Diff}(\Sigma)$ [4, Ex. 9, p. 64]. Lastly, observe from [4, Theorem 1.6, p. 38] that $\text{Diff}(\Sigma)$ is open in $C^\infty(\Sigma, \Sigma)$ (as Σ is compact implies that the weak and strong C^∞-topology coincide).

This section will conclude with a brief sketch of the compact C^∞-topology on the space Γ_2^+ of (admissible) Riemannian metrics on Σ. Let $p_\Sigma : S_2^+\Sigma \to \Sigma$ be the symmetric covariant 2-tensor bundle over Σ and $p_{\Sigma n} : J^n[p_\Sigma] \to \Sigma$ be the C^n-jet bundle of the cross-sections of $S_2^+\Sigma$. Then, defining $\pi_{\Sigma 1}^0 : J^1[p_\Sigma] \to \Sigma \times S_2^+\Sigma$ as above by $j^1 q(x) \mapsto (x, q(x))$ and $\pi_{\Sigma n}^m : J^n[p_\Sigma] \to J^m[p_\Sigma]$ by $j^n q(x) \mapsto j^m q(x)$ whenever $m \leqq n$, one again obtains an inverse sequence $\{J^n[p_\Sigma], \pi_{\Sigma n}^{n-1}, \mathbb{N}\}$, where $J^0[p_\Sigma] \stackrel{\text{def}}{=} \Sigma \times S_2^+\Sigma$. Finally, let $J^\infty[p_\Sigma]$ denote the inverse limit of the sequence and set $\pi_\Sigma^n \stackrel{\text{def}}{=} p_\Sigma^n | J^\infty[p_\Sigma]$, where $p_\Sigma^n : \prod_{i \in \mathbb{N}} J^i[p_\Sigma] \to J^n[p_\sigma]$ is the nth projection. The topology of Γ_2^+ is then defined by the (topological) imbedding $j^\infty : \Gamma_+^2 \hookrightarrow C(\Sigma, J^\infty[p_\Sigma])$.

B. Differential Calculus on $\mathcal{M}_n$

Here, the concept of derivatives will be formulated on locally path connected, metrizable spaces. The motivation originated from the attempt to express the loop constraints as derivatives of loop functionals. However, this attempt failed to eventuate. The construction sketched here extends the concept of differential calculus defined on linear spaces to non-linear spaces that do not admit manifold structures modelled on a locally convex linear space.

Now, consider a locally path connected metric space (X, d) and let $\{C_\beta\}_{\beta \in B}$ be the set of (path) components of X that partitions it: that is, $\bigcup_{\beta \in B} C_\beta = X$. Fix an $x_\beta \in C_\beta$ for each $\beta \in B$, and regard C_β as the pointed subspace (C_β, x_β) in X in all that follows. From here on, x_β will be denoted by 0_β.

Given $\varepsilon \in \mathbb{R}$, if $H_\varepsilon : X \cong X$ is a homeomorphism such that $d(H_\varepsilon(x), x) = |\varepsilon| \ \forall x \in X$, call H_ε an *ε-isometry* on X. Let $H = \{H_\varepsilon\}_{\varepsilon \in \mathbb{R}}$ be a 1-parameter group of isometries on X such that each $H_\varepsilon | C_\beta : C_\beta \cong C_\beta \ \forall \beta \in B$. Next, consider 1-parameter groups $\{H\}$ of isometries such that each H generates a set of curves in C_β that foliates it for each $\beta \in B$. That is, on each C_β, there exists a set $\Lambda(\beta) \subset C_\beta$ such that the family $h_\beta = \{h_{\beta,x}\}_{x \in \Lambda(\beta)}$ of curves $h_{\beta,x} : \mathbb{R} \to C_\beta$ in C_β defined by $h_{\beta,x}(t) \stackrel{\text{def}}{=} H_t(x)$, where $h_{\beta,x}(0) = x$, satisfies:

(i) $C_\beta = \bigcup_{x \in \Lambda(\beta)} h_{\beta,x}(\mathbb{R})$,

(ii) $h_{\beta,x}(\mathbb{R}) \cap h_{\beta,x'}(\mathbb{R}) = \varnothing, \ \forall x \neq x'$,

(iii) for each $x \in \Lambda(\beta)$, $h_{\beta,x}(t) = h_{\beta,x}(t') \Rightarrow t \equiv t'$,

This 1-parameter group H is called a *d-translate* on X and $H_\varepsilon \in H$ is called an *ε d-translation* on X. Let $\mathcal{T}(X, d)$ denote the set of all d-translates on X. The family h_β of curves induced by $H \in \mathcal{T}(X, d)$ is called an *H-flow* on X.

Let $\mathcal{T}_0(X, d) \subset \mathcal{T}(X, d)$ be the set of all d-translates such that (a) for each $\beta \in B$ and $x \in C_\beta$, there exists a unique $H \in \mathcal{T}_0(X, d)$ with $h_{\beta,0_\beta}(\varepsilon) = x$ for some $\varepsilon \in \mathbb{R}$, where h_β is an H-flow on X, (b) given a pair $\gamma, \eta \in C_\beta$, $\exists\, r \in \mathbb{R}$, $\delta_\gamma \eta \in C_\beta$ and some $\tilde{H} \in \mathcal{T}_0(X, d_X)$ such that $\tilde{h}_{\beta,0_\beta}(r) = \delta_\gamma \eta$ and $\tilde{h}_{\beta,\gamma}(\tilde{r}) = \eta$ for some $r, \tilde{r} \in \mathbb{R}$, and (c) given an H-flow $\{h_{\beta,x}\}_{x \in \Lambda(\beta)}$ and an H'-flow $\{h'_{\beta,x}\}_{x \in \Lambda(\beta)}$ with $H, H' \in \mathcal{T}_0(X, d_X)$, there exists a continuous map $R_\beta : I \times C_\beta \to C_\beta$ such that $R_{\beta 1} \circ h_{\beta,x} = h'_{\beta,x}$ for each $x \in \Lambda(\beta)$, where $R_{\beta t} \stackrel{\text{def}}{=} R_\beta(t, \cdot)$. Denote this unique d-translate by $H[x]$.[2] In other words, $H_\varepsilon[x](0_\beta) = x$ for some $\varepsilon \in \mathbb{R}$ and condition (b) translates to $\tilde{H}_{\tilde{r}}[\delta_\gamma \eta](\gamma) = \eta$. Finally, in all that follows, if (X, d_X) and (Y, d_Y) are locally path connected metric spaces, let

$$\varepsilon^+(x) \stackrel{\text{def}}{=} \begin{cases} d_X(0_\beta, x) & \text{if } x \in C_\beta[X], \\ d_Y(0'_\beta, x) & \text{if } x \in C_\beta[Y], \end{cases}$$

[2]Clearly, $H[x] = H[y]$ *iff* x and y both lie on the same "orbit"; that is, $\exists\, \delta_x, \delta \in \mathbb{R}$ such that $H_{\delta_x}[x](0_\beta) = x$ and $H_{\delta_x + \delta}[x](0_\beta) = y$.

where $C_\beta[X]$ denotes a path component of X *etc.*, for each β, with $C_\beta[Y]$ regarded as the pointed space $(C_\beta[Y], 0'_\beta)$, and set

$$\varepsilon(x) \stackrel{\text{def}}{=} \begin{cases} \varepsilon^+(x) & \text{if } H_{\varepsilon^+(x)}[x](0_\beta) = x, \\ -\varepsilon^+(x) & \text{if } H_{-\varepsilon^+(x)}[x](0_\beta) = x. \end{cases}$$

B.1. Definition. Given a continuous function $f : X \to \mathbb{R}$, a fixed point $x \in C_\beta$ and any point $y \in C_\beta$, define

$$Df(x;y) \stackrel{\text{def}}{=} \begin{cases} \lim\limits_{t\to 0} \dfrac{f(H_t[y](x)) - f(x)}{t} & \text{if } y \neq x_\beta, \\ 0 & \text{if } y = x_\beta, \end{cases}$$

where the limit is defined with respect to the d-topology. If the limit $Df(x;y)$ exists, then f is said to be *d-differentiable* at x along $H[y]$, and $Df(x;y)$ is the *d-derivative* of f at x along $H[y]$. The notations $Df(x;y)$, $Df(x)(y)$ and $f'(x;y)$ will be used interchangeably.

The following properties of D are easy to verify:

(1) D is a *derivation*; that is, $D(f(x) + cg(x)) = Df(x) + cDg(x)$, and $D(f(x)g(x)) = f(x)D(g(x)) + g(x)D(f(x))$ $\forall\, d$-differentiable $f, g \in C(X, \mathbb{R})$ and $c \in \mathbb{R}$;

(2) $Df(x) = 0\ \forall\, x \in C_\beta \Leftrightarrow f$ is constant on C_β.

The second derivative $D^2 f(x)$ at x is defined to be

$$D^2 f(x; u, v) = \lim_{t\to 0} \frac{Df(H_t[v](x); u) - Df(x;u)}{t}.$$

Unfortunately, $D^2 f(x) : C_\beta \times C_\beta \to \mathbb{R}$ is not a symmetric mapping in general. However, if need be, the symmetrised derivative operator can always be defined as

$$D^2_{\mathrm{s}} f(x; u, v) \stackrel{\text{def}}{=} \tfrac{1}{2}(D^2 f(x; u, v) + D^2 f(x; v, u)).$$

More generally, f is said to be *n-times d-differentiable* at x if f is d-differentiable and Df is $(n-1)$-times d-differentiable at x. Lastly, $Df(x)$ is called the *differential* of f at x, and it will also be denoted by $f'(x)$.

It is clear that if d and d' are two equivalent metrics on X, then a real function f on X is d-differentiable *iff* it is d'-differentiable. The above definition can trivially be extended to complex functionals on X. Moreover, it is also clear from Definition B.1 that continuity is a necessary condition for d-differentiability. The concept of "topological" differentiation can be generalised to mappings between two locally path connected metric spaces. This is done in the following way.

Let (X, d_X) and (Y, d_Y) be any pair of locally path connected metric spaces, $f : (X, d_X) \to (Y, d_Y)$ be a continuous map, and suppose that $f(C_\beta[X]) \subseteq C_{\beta'}[Y]$ for each β. Recall also that for any fixed point $x_0 \in C_\beta[X]$, $H[x_0] \in \mathcal{T}_0(X, d_X)$ is defined to be the unique d_X-translate that joins 0_β to x_0; that is, $H_{\varepsilon(x_0)}[x_0](0_\beta) = x_0$.

B.2. Definition. Given a continuous mapping $f : X \to Y$ with $f(C_\beta[X]) \subseteq C_{\beta'}[Y]$, fix a point $x_0 \in C_\beta[X]$ and consider an arbitrary point $x \in C_\beta[X]$. Then, f is said to be (d_X, d_Y)-*differentiable* (or *metrically differentiable* for brevity) at x_0 along $H[x] \in \mathcal{T}_0(X, d_X)$ should the following limit

$$Df(x_0; x) \stackrel{\text{def}}{=} \lim_{t \to 0} H_{\varepsilon(x)}[H_{-\varepsilon(f(x_0))}[f(x_0)](f(H_t[y](x_0)))](0'_{\beta'})$$

exist. Again, all three symbols $Df(x_0; x)$, $Df(x_0)(x)$ and $f'(x_0; x)$ will be used interchangeably.

It is intuitively clear from the construction that $Df(x_0) : C_\beta[X] \to Y$ is continuous. This will be verified below. First, for each fixed $t \in \mathbb{R}$, let $\mathcal{T}_0(X, d_X; t) \stackrel{\text{def}}{=} \{ H_t \in H \mid H \in \mathcal{T}_0(X, d_X) \}$ and set $\mathcal{T}_0(X, d_X; \mathbb{R}) = \bigcup_{t \in \mathbb{R}} \mathcal{T}_0(X, d_X; t)$. Endow $\mathcal{T}_0(X, d_X; \mathbb{R})$ with the topology of pointwise convergence. Recall that this topology is generated by all basic subsets $N(f; x_1, \ldots, x_n, U_1, \ldots, U_n)$ defined by

$$\{ g \in \mathcal{T}_0(X, d_X; \mathbb{R}) \mid g(x_i) \in U_i, \forall i = 1, \ldots, n < \infty \},$$

for each $f \in \mathcal{T}_0(X, d_X; \mathbb{R})$ and finite collection of points $\{x_1, \ldots, x_n\} \in X$ together with open subsets $U_i \subset C_{\beta'}[Y]$ of $f(x_i)$. If $U_i = B_{\varepsilon_i}(f(x_i))$ for each i, denote $N(f; x_1, \ldots, x_n, U_1, \ldots, U_n)$ by $N(f; x_1, \ldots, x_n, \varepsilon_1, \ldots, \varepsilon_n)$ for simplicity.

B.3. Lemma. *Let* $\psi_\beta : \mathbb{R} \times (C_\beta[X] - \{0_\beta\}) \to \mathcal{T}_0(X, d_X; \mathbb{R})$ *be defined by* $(t, x) \mapsto H_t[x]$. *Then,* ψ_β *is a continuous surjection.*

Proof. By definition, ψ_β is surjective. Hence, it only remains to establish that it is continuous. Fix $(t_0, x_0) \in \mathbb{R} \times (C_\beta[X] - \{0_\beta\})$ and consider a neighbourhood $N(\psi_\beta(t_0, x_0)) \stackrel{\text{def}}{=} N(H_{t_0}[x_0]; x_1, \ldots, x_n, \varepsilon_1, \ldots, \varepsilon_n)$ about $\psi_\beta(t_0, x_0)$ in $\mathcal{T}_0(X, d_X; \mathbb{R})$. Let $\delta_i^+ = d_X(x_0, x_i)$,

$$\delta_i = \begin{cases} \delta_i^+ & \text{if } H_{\delta_i^+}[x_0](x_0) = x_i, \\ -\delta_i^+ & \text{if } H_{-\delta_i^+}[x_0](x_0) = x_i, \end{cases}$$

and set $T(B_\delta(x_i)) = H_{-\delta_i}[x_0](B_\delta(x_i))$, where $\delta = \frac{1}{2}\min\{\varepsilon_1, \dots, \varepsilon_n\}$. Then, by definition, $T(B_\delta(x_i))$ is open and $x_0 \in T(B_\delta(x_i))$ for each i. Consequently, $D_{x_0} \stackrel{\text{def}}{=} \bigcap_{i=1}^n T(B_\delta(x_i))$ is a neighbourhood of x_0 in $C_\beta[X]$. Let $\varepsilon > 0$ be any radius such that $B_\varepsilon(x_0) \subset D_{x_0}$ and set $\hat{\varepsilon} = \frac{1}{2}\varepsilon$. Furthermore, let $\hat{\delta} = \frac{1}{2}\delta$ and define $X_i \subset C_\beta$ to be the subset

$$X_i \stackrel{\text{def}}{=} \{ H_t(0) \mid d_X(H_t(x_i), H_{t_0}[x_0](x_i)) \in (\varepsilon_i - \tilde{\delta}, \varepsilon_i),\ t \in (t_0 - \hat{\delta}, t_0 + \hat{\delta}) \},$$

where $\tilde{\delta} \in (0, \frac{1}{2}\delta)$ is any fixed small positive number. Finally, let $r = \frac{1}{2}\inf\{ d_X(x, x_0) : x \in X_i,\ i = 1, \dots, n \}$ and $\tilde{\varepsilon} = \min\{\hat{\varepsilon}, r\}$. Then, from the construction, given any $(t, x) \in (t_0 - \hat{\delta}, t_0 + \hat{\delta}) \times B_{\tilde{\varepsilon}}(x_0)$, $H_t[x](x_i) \in B_{\varepsilon_i}(H_{t_0}[x_0](x_i))$ for each i, where $H_t[x] = \psi(t, x)$, and continuity thus follows. □

B.4. Corollary. *Let $f : X \to Y$ be continuous and set $\hat{f}^x_{x_0}(t) = H_{-\varepsilon(f(x_0))}[f(x_0)](f(H_t[x](0_\beta)))$. Then, for each fixed t, $\hat{f}^x_{x_0}(t)$ is a continuous mapping with respect to $x \in C_\beta[X]$.*

Proof. Set $\hat{f}_{x_0}(t)(x) = \hat{f}^x_{x_0}(t)$. Then, $\hat{f}_{x_0}(t) = H_{-\varepsilon(f(x_0))}[f(x_0)] \circ f \circ H_t[\,\cdot\,](0_\beta)$ is a composition of continuous functions. □

B.5. Proposition. *$Df(x_0) : C_\beta[X] \to C_{\beta'}[Y]$ is continuous.*

Proof. Recall that for any $x \in C_\beta[X]$, $Df(x_0; x) = \lim\limits_{t\to 0} H_{\varepsilon(x)}[\hat{f}^x_{x_0}(t)](0'_{\beta'})$. Since $\lim\limits_{t\to 0} H_{\varepsilon(\cdot)}[\,\cdot\,](0'_{\beta'})$ is continuous in x by Lemma B.3 and $Df(x_0) = \lim\limits_{t\to 0} h_{\varepsilon(\cdot)}[\,\cdot\,](0_{\beta'}) \circ \hat{f}_{x_0}(t)(\cdot)$, the continuity of $Df(x_0)$ follows at once from Corollary B.4. □

It is easy to see that if f is not continuous, then it is not metrically differentiable. Note also that if d'_X and d'_Y are two metrics equivalent to d_X and d_Y respectively, then f is (d_X, d_Y)-differentiable *iff* f is (d'_X, d'_Y)-differentiable. Finally, it follows from the definition that $D\,\mathrm{id}_X(x_0) = \mathrm{id}_X |C_\beta[X]$ for each β.

B.6. Theorem. *Let $f : X \to Y$ and $g : Y \to Z$ be metrically differentiable on X and Y respectively. Fix $x_0 \in C_\beta[X]$ and set $y_0 = f(x_0) \in C_{\beta'}[Y]$. Then, $D(g \circ f)(x_0) = Dg(y_0) \circ Df(x_0)$. In particular, this holds for $Z = \mathbb{R}$.*

Proof. Let $x \in C_\beta[X]$ be any point. Then,

$$D(g \circ f)(x_0; x) = \lim_{t\to 0} H_{\varepsilon(x)}[H_{-\varepsilon(g\circ f(x_0))}[g \circ f(x_0)](g \circ f(H_t[x](x_0)))](0'_{\beta'}).$$

Moreover, observe from the definition of $f'(x_0; x)$ that $\forall\, \varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ such that

$$d_Y(f(H_t[x](x_0)), H_t[f'(x_0; x)](f(x_0))) < \varepsilon\ \forall\, x \in C_\beta[X]$$

whenever $|t| < \delta_\varepsilon$. In particular, $\forall\, x \in C_\beta[X]$,

$$\lim_{t\to 0} d_Y(f(H_t[x](x_0)), H_t[f'(x_0;x)](f(x_0))) = 0.$$

Hence, this together with the equality $\varepsilon(f'(x_0;x)) \overset{\text{def}}{=} d_Y(0'_{\beta'}, f'(x_0;x)) = \varepsilon(x)$ imply at once that

$$\begin{aligned}
&\lim_{t\to 0} H_{\varepsilon(x)}[H_{-\varepsilon(g\circ f(x_0))}[g\circ f(x_0)](g\circ f(H_t[x](x_0)))](0'_{\beta'})\\
&= \lim_{t\to 0} H_{\varepsilon(f'(x_0;x))}[H_{-\varepsilon(g(y_0))}[g(y_0)]\circ g(f(H_t[f'(x_0;x)](y_0)](0'_{\beta'})\\
&= \lim_{t\to 0} H_{\varepsilon(\cdot)}[H_{-\varepsilon(g(y_0))}[g(y_0)](g(H_t[\,\cdot\,](y_0)))](0'_{\beta'})\circ f'(x_0;x).
\end{aligned}$$

Whence, the arbitrariness of $x \in C_\beta[X]$ implies that $(g\circ f)'(x_0) = g'(y_0)\circ f'(x_0)$, as required. To establish the last assertion, it will suffice to note that for any $v \in C_\beta[X]$,

$$\begin{aligned}
(g\circ f)'(x_0;v) &= \lim_{t\to 0}\frac{g\circ f(H_t[v](x_0)) - g\circ f(x_0)}{t}\\
&= \lim_{t\to 0}\frac{g(f(H_t[v](x_0))) - g(f(x_0))}{t}\\
&= \lim_{t\to 0}\frac{g(H_t[f'(x_0;v)](f(x_0))) - g(f(x_0))}{t}\\
&= \lim_{t\to 0}\frac{g(H_t[\,\cdot\,](f(x_0))) - g(f(x_0))}{t}\circ f'(x_0;v). \qquad \square
\end{aligned}$$

The definition for higher order derivatives can also be defined. Suppose that $f : X \to Y$ is of class C^1; that is, $Df : C_\beta[X]\times C_\beta[X] \to C_{\beta'}[Y]$ is continuous for each β. Fix a pair $(x_0, x) \in C_\beta[X]\times C_\beta[X]$ and consider $u \in C_\beta[X]$. Then, the second derivative of f at x_0 is defined as

$$f''(x_0;x)(u) \overset{\text{def}}{=} \lim_{t\to 0} H_{\varepsilon(u)}[H_{-\varepsilon(f'(x_0;x))}[f'(x_0;x)](f'(H_t[u](x_0));x)](0'_{\beta'})$$

provided that the limit exists. $f''(x_0;x)(u)$ will also be denoted by $f''(x_0;x,u)$ or $D^2f(x_0)(x,u)$, *etc.* In general, f is *n-times differentiable* if f is differentiable and $D^{n-1}f$ is also differentiable. It is again rather unfortunate that for each $n > 1$, $Df^n(x_0) : C^n_\beta[X] \to Y$ is not, in general, a totally symmetric mapping, unlike differentials defined on locally convex linear spaces (which are symmetric mappings).

B.7. Theorem. *$\tilde{\mathcal{L}}_\Sigma$ is locally path connected.*

Proof. Fix a $\gamma \in \tilde{\mathcal{L}}_\Sigma$ and define $C(\gamma)$ to be the set of all points in $\tilde{\mathcal{L}}_\Sigma$ which can be joined to γ by a curve. Explicitly, it consists of all $\eta \in \tilde{\mathcal{L}}_\Sigma$ such that $\exists\, p : I\times I \to \Sigma$

continuous with $p(0,\cdot) = \gamma$ and $p(1,\cdot) = \eta$. First, observe that as Σ is path connected, $C(\gamma)$ is never trivial; that is, $C(\gamma) \neq \{\gamma\}$. Second, note trivially that if $\gamma_1, \gamma_2 \in C(\gamma)$, let $h_i : I \times I \to \Sigma$ be a homotopy from γ to γ_i, $i = 1, 2$. Then, h_{12} given by

$$h_{12}(s,t) \stackrel{\text{def}}{=} \begin{cases} h_1(1-2s,t) & \text{for } 0 \leqq s \leqq \frac{1}{2}, \\ h_2(2s-1,t) & \text{for } \frac{1}{2} \leqq s \leqq 1, \end{cases}$$

defines a homotopy from γ_1 to γ_2. Consequently, it is enough to show that $C(\gamma)$ is closed in $\tilde{\mathcal{L}}_\Sigma$.

Let $\{\gamma_n\}_n$ be a sequence in $C(\gamma)$ which converges to $\hat{\gamma} \in \tilde{\mathcal{L}}_\Sigma$. Recall that $\gamma_n \to \hat{\gamma}$ in the ρ-topology of $\tilde{\mathcal{L}}_\Sigma$ means that given any $\varepsilon > 0$, $\exists\, N > 0$ such that

$$(\dagger) \quad \sup\{d(\gamma_n(t), \hat{\gamma}(t)) : t \in I\} + \operatorname{ess\ sup}\{\|D^\ell \gamma_n(t) - D^\ell \hat{\gamma}(t)\| : t \in I,\, \ell \geqq 1\} < \varepsilon$$

whenever $n > N$. Hence, it is evident from ($\dagger$) that by taking $N > 0$ sufficiently large (*i.e.*, by making $\varepsilon > 0$ sufficiently small), there exists a curve from γ_n to $\hat{\gamma}$. To show this, set $c_n^i = \gamma_n^i - \hat{\gamma}^i$, $i = 1, 2, 3$ (working with local coordinates now). Then, by ($\dagger$), $c_n \in \tilde{\mathcal{L}}_\Sigma$ and $\Gamma^i(t) = \gamma_n^i - t c_n^i$ defines a curve joining γ_n to $\hat{\gamma}$: $\Gamma(0) = \gamma_n$ and $\Gamma(1) = \hat{\gamma}$, where n is any fixed integer satisfying $n > N$. So, $\hat{\gamma} \in C(\gamma)$ and the assertion is established. □

Since local path connectedness is invariant under quotient mappings and is also preserved under finite Cartesian products [3, 6.3.10(b), (c), p. 376], the following corollary is evident.

B.8. Corollary. *$\mathcal{M}_n$ is locally path connected for* $1 \leqq n < \infty$. □

Hence, the metrizability of $\mathcal{M}_n$ for each $n < \infty$ means that differentiation can be defined on it. It is easy to see that Σ is simply connected *iff* $\tilde{\mathcal{L}}_\Sigma$ is path connected (and hence *iff* $\mathcal{M}_n$ is, for each n). However, since no restrictions regarding the connectivity of Σ were made *a priori* (aside from being connected), it is unclear off-hand that $\tilde{\mathcal{L}}_\Sigma$ is path connected. And indeed, it is not expected of a Cauchy spatial 3-slice of a general space-time that it be simply connected. The ultimate aim here is to attempt to obtain alternative—but hopefully illuminating—forms of the quantum loop constraints. Unfortunately, the programme appears to reach an impasse! To conclude this appendix, it will suffice to remark that alternative (but equivalent) expressions for the quantum loop constraints already exist [1, 2].

References

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List of Notations

For the sake of being concise, let X denote a topological space, $U \subseteq X$ and S a non-empty set.

1_X, id_X – the identity map of X.

2^X – the power set of X; *i.e.*, the set of all subsets of X.

$\aleph_0$ – the cardinality of $\mathbb{N}$.

$\mathcal{A}$ – the space of Ashtekar connection 1-forms.

$B_\varepsilon(x)$ – a ball of radius ε centred about x.

$\bar{\psi}$ – the complex conjugation of ψ.

$\mathrm{Diff}^+(\Sigma)$ – the group of orientation-preserving smooth diffeomorphisms on Σ.

Γ_2^+ – the space of Riemannian 3-metrics on Σ.

e – exponential map.

I – the unit interval $[0,1]$.

$\ln$ – natural logarithm.

$\mathcal{M}$ – the multi-loop space.

$\mathcal{M}_n$ – the n-loop space.

$\mathbb{N}_0$ – $\mathbb{N} \cup \{0\}$.

$|S|$ – the cardinality of S.

$\overline{\mathbb{R}}_+$ – the closed semi-infinite interval $[0,+\infty]$.

Σ – a closed, connected, compact Riemannian 3-manifold.

X' – the topological dual of X.

X^n – the Cartesian product of n copies of X.

X^∞ – the countably infinite Cartesian product of X.

$\overline{U}$ – the closure of U.

U° – the interior of U.

U^c – the complement of U: $X - U$.