Performance Monitoring and Assessment of Optimality under a Linear Quadratic Criterion

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Statement

The content of this thesis is the result of original research, and has not been submitted for a higher degree at any other university or institution.

These doctoral studies were conducted under supervision of Prof. Robert R. Bitmead and Dr. Peter L. Bartlett. Nearly all work in this thesis has been carried out in collaboration with them, however the majority of the work is my own.

Leonardo César Kammer

Canberra, 9 July 1998.

List of Publications

A number of papers leading to this thesis have been published or submitted to conferences and journals as listed below.

Journal Papers

[J1] Kammer, L. C., R. R. Bitmead and P. L. Bartlett (1998). Optimal controller properties from closed-loop experiments. *Automatica* **34**(1), 83–91.

[J2] Kammer, L. C., R. R. Bitmead and P. L. Bartlett (1998). Direct iterative tuning via spectral analysis. Submitted to *Automatica.*

Conference Papers

[C1] Kammer, L. C., R. R. Bitmead and P. L. Bartlett (1996). Tracking adaptive identification: approach via model falsification. In: *Proc. of the 13th IFAC World Congress.* San Francisco, U.S.A.. pp. 261–266.

[C2] Kammer, L. C., R. R. Bitmead and P. L. Bartlett (1996). Signal-based testing of LQ-optimality of controllers. In: *Proc. of the 35th IEEE CDC.* Kobe, Japan. pp. 3620–3624.

[C3] Kammer, L. C., R. R. Bitmead and P. L. Bartlett (1998). Direct controller adjustment by on-line performance assessment. To be presented at the *IFAC Workshop on Adaptive Control and Signal Processing.* Glasgow, U.K.

[C4] Kammer, L. C., R. R. Bitmead and P. L. Bartlett (1998). Direct iterative tuning via spectral analysis. Submitted to the *37th IEEE CDC.*

[C5] De Bruyne, F. and L. C. Kammer (1998). Iterative feedback tuning with guaranteed stability. In preparation.

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Abstract

The central theme of this thesis is the testing of closed-loop performance optimality under a linear quadratic (LQ) criterion. Since performance optimality is an attribute of the closed-loop system necessarily comprising the real plant and the optimal controller, the mechanisms for testing this attribute must look at the loop as an undivided entity. That is, these mechanisms ought not to resort to a parametric model of the plant, a characteristic of the so called model-free methods.

Previous results on performance monitoring and assessment under a simpler criterion are strong motivations for the development of the research in this thesis. Unfortunately the complexity of model-free mechanisms rapidly increases as the performance criterion becomes more generic. In order to test for LQ performance optimality, the reference signal is used for exciting the closed-loop system, thus making the measurable signals more informative. The actual assessment of optimality is based on frequency flatness of a power-spectral density constructed from the signals measured during excitation.

Further manipulation of the excited closed-loop signals results in a power-spectral density containing the characteristic polynomial of the optimally controlled system. This polynomial can then be estimated via phase reconstruction of the spectral density. The relevance of this result lies in the estimation of the closed-loop poles that would be obtained under optimal performance, from signals collected under any stabilising controller. Linear quadratic performance monitoring and direct adaptive pole-placement are examples of applications benefiting from these estimates of the optimal close-loop poles.

The analysis of the cost function as a functional on the space of controller parameters brings new insight into assessment of optimality. This perspective reveals geometric properties associated with the condition of local optimality, inclusive of systems under reduced complexity controllers. The main features of interest are the first and second derivatives of the cost function with respect to the controller parameters. In the specific case of a linear quadratic cost function these derivatives are obtained in both frequency and time domains.

Apart from testing local optimality, the derivatives of the cost function are used for performing gradient-based controller tuning. During experiments with this sort of tuning the author has observed examples of sudden instability of the closed-loop system. This motivated the development of a mechanism for guaranteeing stability of the closed-loop system, which is applicable to any iterative control scheme.

The developments in this thesis are initial steps towards a model-free mechanism for assessment of optimal performance under a linear quadratic criterion. It seems that the estimate of the optimal characteristic polynomial is the most likely seed of such an achievement.

Contents

Chapter 1

Introduction

1.1 Thesis Motivation and Significance

Consider the control engineering problem of anticipating the potential benefits of redesigning or tuning a controller. In an industrial context, the expenditure of large sums on the redesign of a control system will be most powerfully argued if there is a quantified estimate of the likely benefit. Equally, if a loop is suspected of under-performance, assessment of its distance from optimality can help decide, in this order, among: controller tuning, if the optimal performance with the current controller structure is acceptable; controller redesign, if the optimal performance with a different controller structure would be significantly better; and, in the last case, redesign of the sensor/plant/actuator system to improve the best achievable performance. Thus an important issue in the applicability of advanced (or other) control is to provide some assessment of its prospective benefits and so to reduce some of the risks associated with this expenditure decision.

In the absence of an open-loop process model, a comparison between currently achieved performance and optimality (according to some criterion) is problematic. Using closed-loop signals one might proceed by identification of an open-loop model, followed by controller design and comparison between expected and actual behaviours. This introduces issues of modelling accuracy in addition to control design. If there is to be any prospect of developing generic control loop performance monitors applicable to, say, nonlinear systems, then predicating the method on the availability of a model could introduce harsh requirements of model identification prior to performance analysis.

Furthermore, performance optimality is an attribute of the closed-loop system necessarily comprising the *real plant* and the optimal controller. In contrast to parametric modelling of the process, non-parametric characterisations of the closed-loop system, usually easy to obtain, are allowable in the so called 'model-free' methods.

The most significant result available on model-free assessment of optimal performance is limited to a certain class of plants and to a criterion that only penalises the variance of the process output. This means that the control effort necessary to achieve optimal performance might become too large to be acceptable in practice. Nevertheless, knowledge of the minimum achievable output variance is important as a lower limit against which the current performance can be compared.

The regulation problem where the minimum variance criterion applies is shown in Figure 1.1. The process output, y_t, is intended to remain close to a *constant* (here zero) reference value in spite of the influence of the additive disturbances, v_t. Moreover, the class of disturbances under analysis is restricted to stochastic processes generated by filtered white noise. Therefore, deterministic disturbances and the tracking problem are not addressed here.

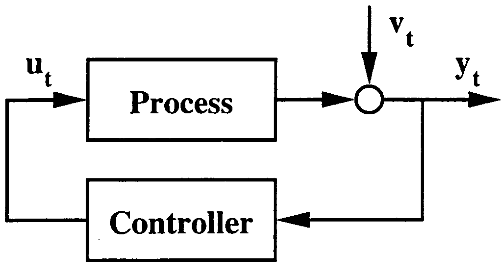

Figure 1.1: Regulation problem

The primary motivation behind the results obtained throughout this thesis is the development of a model-free mechanism for assessment of optimal performance under a linear quadratic criterion. Such a criterion establishes a compromise between the variance of the process output, y_t, and the variance of the control action, u_t, in a regulation problem and ought not to be artificially limited to too small a class of plants.

Unfortunately the complexity of model-free mechanisms rapidly increases as the performance criterion becomes more generic. Actually, the very existence of such a mechanism is not guaranteed for all criteria. In order to complete the set of information required for performance assessment, it is possible to make the measurable signals more informative by exciting the loop. If this is not enough then the missing information must be estimated or supplied by the user. But every piece of information that has to be introduced *a priori* corresponds to a part of the system that is not analysed by the performance assessment mechanism. These comments raise the issue of determining the capacity of a method to analyse the real system, which could range from a powerful model-free mechanism up to a blind acceptance of a model-based controller design.

The need for some knowledge about the plant and the use of external excitation are obstacles for the widespread use of these mechanisms in industrial plants. A more realistic scenario is their application to some individual loops that deserve the costly effort to achieve optimality. In fact, if it is unclear how to assess the financial benefit of achieving optimality in a particular loop then it is very likely that the problem is best solved with a robust controller. Also if the loop is relatively important, but not economically crucial in the production line, then the optimal design might be followed by a detuning of the controller in order to gain robustness at the expense of performance.

The contributions introduced by this thesis are initial steps towards a model-free mechanism for assessment of optimal linear quadratic performance. Despite the ultimate goal being the analysis of general systems, the scope of investigation in this thesis is restricted to scalar, discrete-time, linear, time-invariant systems in order to assess the problem complexity and feasibility. Among those contributions are the mechanisms for testing and monitoring linear quadratic optimality. Moreover, the development of these mechanisms has generated by-products with immediate application in direct adaptive control, controller tuning and assessment of local optimality for reduced order controllers, as is usually the case of industrial plants under PID control.

Further appraisal of this thesis' significance can only be made in the future, based on how many ideas it will trigger in other researchers' minds; and the author would be profoundly disappointed if that sums to zero.

1.2 Synopsis of the Thesis

Chapter 2: Performance Monitoring and Assessment in Control
The terms 'performance monitoring' and 'performance assessment' are part of the technical jargon but might assume different meanings for different authors. This chapter provides definitions for these terms followed by a brief review of the most significant model-free mechanisms for performance monitoring and assessment currently available in the literature. The 'model-free' status is given to these mechanisms for their use of non-parametric characterisations of the closed-loop system: statistical charts, autocorrelation functions and impulse responses.

Chapter 3: Assessment of Linear Quadratic Optimality
Spectral analysis of the process input and output signals, observed during perturbed closed-loop operation, provides the elements for testing performance optimality under a linear quadratic criterion. This mechanism is model free when all states of the process are measurable or reconstructed with a Kalman filter. Similar results are obtained for output feedback control but, in this case, part of the noise dynamics must be known.

Chapter 4: Linear Quadratic Performance Monitoring
The mechanism for monitoring linear quadratic performance resorts to an estimate of the optimal closed-loop characteristic polynomial. This polynomial is obtained via signal reconstruction of special power-spectral densities containing the optimal closed-loop poles. Estimates of the optimal closed-loop characteristic polynomial can also be used for other purposes, like direct adaptive pole-placement.

Chapter 5: Assessment of Local Optimality
The analysis of a linear quadratic cost function as a functional on the space of controller parameters brings new insight into assessment of optimality. This perspective reveals geometric properties associated with the condition of local optimality, inclusive of systems under reduced complexity controllers. The main features of interest are the first and second derivatives of the cost function with respect to the controller parameters, which are derived in both frequency and time domains.

Chapter 6: Controller Tuning
An additional use for the estimates of first and second derivatives of the cost

function is in gradient-based controller tuning. This chapter presents one possible way of computing appropriate directions for controller tuning. The scheme includes a mechanism for guaranteeing stability of the closed-loop system and a criterion for stopping the iterative tuning.

Chapter 7: Directions for Future Research

The thesis concludes with a summary of achievements followed by directions for near and long-range future research.

Appendix - Tracking Adaptive Identification: Approach via Model Falsification

The material in this appendix was presented at the IFAC World Congress in July, 1996. Despite not sharing the same primary motivation as the other contributions in the thesis, this paper was important as the author's introductory study of linear quadratic control. The main contribution of this paper is the improvement of a certainty equivalence estimator, adapted from results on the problem of learning from experts, by translating it into the parameter bounding context and by adding adaptivity.

1.3 Summary of Original Contributions

A number of original contributions have been made during the course of research and development for this doctoral dissertation. A brief description of these contributions follows.

Chapter 3: Assessment of Linear Quadratic Optimality

Development of a model-free test of linear quadratic optimality for systems under state feedback. This test requires that all states of the plant should be directly measurable or reconstructed with a Kalman filter. The mechanism for assessment of optimality is initially developed for systems with no noises or disturbances, and then it is extended to systems under influence of stochastic noises. This test basically states the equivalence between optimality and whiteness of a certain power-spectral density comprised of signals collected during loop excitation. Actually, this is the dual of a well known Kalman filtering result.

Development of a test of linear quadratic optimality for systems under output feedback. Under this situation part of the noise dynamics must be known. The test states that optimality implies whiteness of a certain power-spectral density, but the converse depends on specific plant characteristics.

Chapter 4: Linear Quadratic Performance Monitoring

Estimation of the optimal closed-loop characteristic polynomial. This contribution begins by composing a couple of power-spectral densities containing the optimal closed-loop poles. Given that power-spectral densities contain only the magnitude information, the phase contents of these noisy signals are reconstructed via an error-reduction algorithm with parametric model fitting. Each estimated transfer function has the optimal closed-loop characteristic polynomial as denominator.

Formulation of a mechanism for monitoring linear quadratic performance. This performance monitor is based on a whitening filter for the process output signal. The construction of such a filter uses the denominator of the feedback control law and an estimate of the optimal closed-loop characteristic polynomial. If the closed-loop performance is optimal then the filtered signal is a spectrally flat process.

Analysis of minimum variance control of nonminimum-phase plants. In view of the previous developments, minimum variance control is revisited for the problematic case of nonminimum-phase plants. The conclusion is that just a minor simplification can be obtained in comparison to the general linear quadratic setup.

Chapter 5: Assessment of Local Optimality

Estimation of first and second derivatives of a linear quadratic cost function with respect to the controller parameters. Three ways of obtaining these derivatives are developed: with frequency-domain non-parametric models of the closed-loop system, with time-domain non-parametric models of the closed-loop system and from signals filtered through the loop. For this latter development only the estimation of the second derivative is an original contribution of this thesis; the estimation of first derivative from signals filtered through the loop was originally presented by Hjalmarsson *et al.* (1994*a*).

Formulation of a mechanism for assessment of local optimality. An operating point has locally optimal performance if the gradient vector is null and the Hessian matrix is positive definite. This simple fact generates a fairly robust method for detecting local optimality based on comparison of the current linear quadratic cost with the minimum of the convex approximation generated by the estimates of first and second derivatives of the cost function.

Chapter 6: Controller Tuning

Computation of appropriate tuning directions. The availability of unbiased estimates of first and second derivatives of the cost function greatly improves the quality of the approximation of the closed-loop behaviour around its operating point. This improvement is explored by a meticulous selection of tuning direction according to local conditions.

Establishment of guaranteed closed-loop stability. The insertion of a generalised stability margin limits drastic changes in the controller parameters and guarantees stability of the closed-loop system. The stability margin is computed from cross-spectral densities of the closed-loop system under external excitation.

Computation of tuning directions from a non-parametric model of the process. Originally the closed-loop system has to be excited for each new computation of tuning direction. Using a non-parametric model of the process, it is possible to compute new tuning directions, without re-exciting the loop, after a minimum has been reached along the previous line search.

Appendix - Tracking Adaptive Identification: Approach via Model Falsification

Improvement of a confidence-function parameter estimator. The certainty-equivalence estimator presented in (Kumar 1995) computes a density function that assigns a confidence level to each possible parameter vector. The original contribution presented in this appendix improves that estimator by translating it into the parameter bounding context, via dead-zone action, and by adding adaptivity.

Chapter 2

Performance Monitoring and Assessment in Control

2.1 Introduction

It is surprisingly difficult to find definitions of the terms 'performance monitoring' and 'performance assessment', in the control systems literature. Apparently these terms assume different meanings depending on the sub-area where they are applied. The definitions adopted in this thesis are given in the sequel.

> **Performance monitoring** is the act of comparing the current performance measurement with a set of pre-specified criteria, using normal operating data.

In spite of the diversity of mechanisms suggested for the task of monitoring the performance of a closed-loop system, the use of *normal operating data* is always a common denominator among them. This clearly means that whenever one has to resort to special interferences on the loop, e.g. to inject a pseudo-random signal or to open the loop, in order to acquire the desired result, then the mechanism employed should not be classified as a performance monitor. Another common factor is the action of *comparing* current performance measurement with a set of pre-specified criteria, resulting in a binary conclusion about the achieved system performance, e.g. 'is the current closed-loop performance optimal according to a given criterion?'.

Performance assessment, on the other hand, has two different meanings in the literature: assessment of current performance and assessment of achievable performance.

> **Assessment of (performance) optimality** is the act of testing whether the current performance is optimal or not, according to a given criterion.

This is similar to performance monitoring, except that some sort of interference is performed on the loop. A recent example of this kind of performance assessment is found in (Kendra and Çinar 1997).

Assessment of optimal performance is the act of establishing the best achievable closed-loop performance for a process under a given criterion.

This meaning is more common in the literature and refers to establishing *limits of performance* of the closed-loop system. Mechanisms for assessment of optimal performance range from model-based approaches, e.g. (Boyd and Barratt 1991), to completely model-free ones, e.g. (Lynch and Dumont 1996).

The effectiveness of the mechanisms for performance monitoring and assessment is tightly related to the amount, and quality, of *a priori* information required to apply them. This idea is described by Åström (1991) in terms of levels of knowledge about the plant. Such a detailed definition goes beyond the purpose of this thesis, but the message conveyed is important for qualitative evaluation of those mechanisms. Model-based approaches are extremely flexible to handle, but the requirement that a (good) model of the plant be available becomes its greatest weakness. At the other extreme are the so called model-free methods, which are usually specific to a single performance criterion but do not need to rely on much *a priori* information about the plant.

The aim of this thesis is to develop mechanisms for performance monitoring and assessment under linear quadratic criteria. Such mechanisms can be trivially conceived based on a model of the plant, therefore the challenge is to elaborate methods that use the minimum amount of information about the plant, i.e. as close as possible to model-free methods. The term 'model free' characterises the absence of a *parametric* model of the *process* under control. Statistical models and frequency-domain plots are examples of *non-parametric* characterisations of the *closed-loop system* and therefore are allowable in model-free methods.

This chapter presents a review of some existing mechanisms available for performance monitoring in control that are oriented towards model-free approaches. The presentation is kept brief and simple, leaving the formalisms to later chapters where new results are developed. An important method for assessment of absolute performance under a minimum variance criterion is also described in this chapter.

2.2 Statistical Process Control

Probably the most basic form of performance monitoring is provided by Statistical Process (quality) Control (SPC). The requirements of system performance are specified by statistical properties, which are then constantly monitored with appropriate charts. Initially designed and applied for performing quality control in the *parts* industry (batch processing), SPC charts have also been used to monitor the performance of controlled systems in the *process* industry (continuous processing) (Box and Kramer 1992, Vander Wiel *et al.* 1992).

Statistical process control monitors system performance via time-domain plots of process output deviation from set point, such as Shewhart charts, cumulative sums (CUSUM) charts and exponentially weighted moving average (EWMA) charts. An example of Shewhart chart is shown in Figure 2.1, where the central horizontal line is the set point; the dashed lines, at two standard deviations, are warning levels; and at three standard deviations are the specification levels.

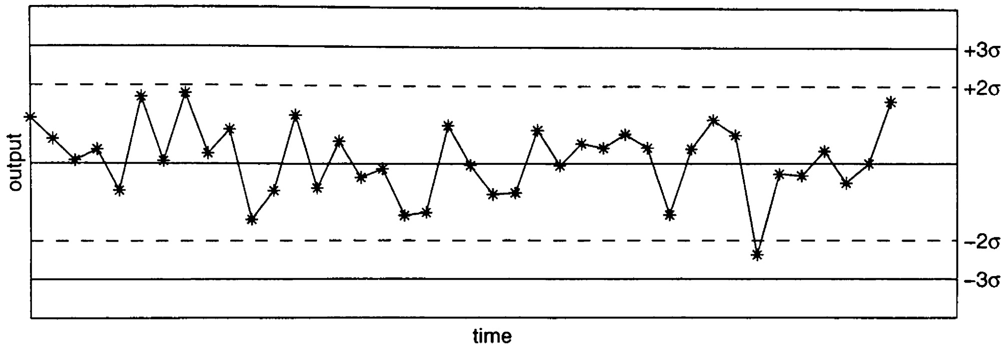

Figure 2.1: Shewhart chart

Shewhart charts are designed to signal excessive deviations of the process output from a given set point. However, when the focus of concern is that the mean value of the output be kept at the set point level, CUSUM charts are the tools to be used instead. As its name implies, a CUSUM chart plots the cumulative sum of output deviation from set point, thus detecting shifts in the process mean. Additionally, EWMA charts provide means for monitoring certain moving average behaviour of the process output.

These types of charts are individually directed towards monitoring particular aspects of the process output that are given by epistemic statistical model; they are not directly designed to the task of monitoring best achievable performance like minimum output variance, for instance. In spite of that, the following set of decision rules, known as the Western Electric zone rules, suggests a way to detect nonrandom patterns in Shewhart charts (Montgomery 1991). The process is out of control if either

- One point plots outside the 3σ control limits;
- Two out of three consecutive points plot beyond the 2σ warning limits;
- Four out of five consecutive points plot at a distance of 1σ or beyond from the centre line;
- Eight consecutive points plot on one side of the centre line.

These heuristic rules can be interpreted as a fast mechanism for testing whiteness of a stochastic process, and signal whiteness is frequently associated with optimality of the systems presented in the next sections.

2.3 Kalman Filtering

The theory of control design based on state-space models is very well established, but it is unrealistic to assume that all states of the system can be measured. The usual approach to this issue is to design an estimator to reconstruct unmeasurable states using past and present values of the inputs and outputs of the process. Under a stochastic framework where the output measurements are affected by Gaussian disturbances, it is possible to determine the optimal estimator, frequently called the Kalman filter, which minimises the variance of the estimation error (Åström and Wittenmark 1990).

Kalman filters exhibit the property that the difference between measured and estimated outputs, known as the innovation sequence, is a Gaussian white-noise sequence (Kailath 1968). This property is not only a sufficient condition but also a necessary one for the optimality of a Kalman filter, therefore simple whiteness tests suffice to monitor the optimal performance of estimators (Mehra 1970).

It turns out that the Kalman filtering problem is dual to the linear quadratic control problem, the object of study in this thesis. Actually, the equivalence of filtering optimality with whiteness of the innovation sequence is used in the next chapter for proving a result on assessment of linear quadratic control optimality.

2.4 Minimum Variance Control

For the regulation problem depicted in Figure 2.2, minimum variance control (MVC) is one of the possible design techniques available. Its theoretical significance is mainly due to computational simplicity and intuitive reasoning. The MVC design of the controller, C, minimises the variance of the measured process output, y_t, which is an effect of the Gaussian white noise at e_t.

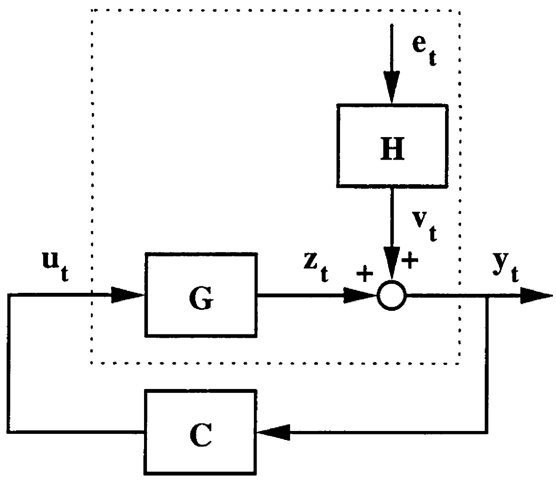

Figure 2.2: Regulation setup

As a starting point for industrial regulators, MVC is a very common control objective since it focuses all effort on rejecting output variation. Minimum variance control represents an archetypical ideal control which, if necessary, is detuned to balance output variation with unacceptable control magnitude.

One way to describe the reasoning behind MVC design is based on the minimisation of the influence at the measured output, y_t, of an impulse at the noise source, e_t. The behaviour of relevant signals is shown in Figure 2.3. All initial conditions of the system are at zero when the impulse occurs at time $t = 0$. The disturbance signal, v_t, becomes the impulse response of the stable and stably invertible filter

H, and this signal appears immediately at the measured output, y_t. The controller can also effect immediate action at the plant input, u_t, but the plant dynamics, G, are assumed to have an internal delay of d sampling times, therefore the signal z_t can only start counteracting the disturbances at time $t = d$ samples. Given that the impulse response of H and the dynamics of G are both known, the minimum variance control law computes the signal u_t such that $z_t = -v_{t|t-d}$ for all $t \geq d$ samples.

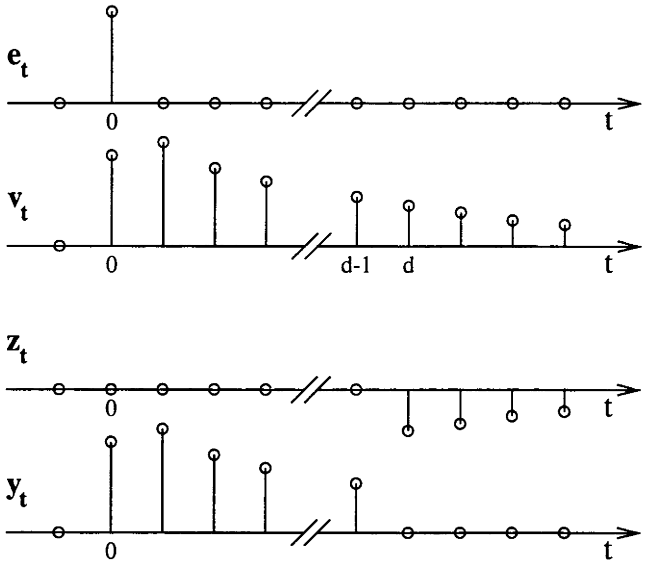

Figure 2.3: MVC impulse response

In a stochastic setup e_t is a sequence of independent and identically distributed random values, and v_t becomes a convolution of impulse responses. Nonetheless the control law that minimises the variance of y_t is the one described above. A complete development of MVC design can be found in (Åström and Wittenmark 1990), (Box and Jenkins 1976) and other textbooks on stochastic control.

Whenever e_t is a sequence of independent random variables and the dynamics of G and H are linear, the minimum variance control law is also linear. If the process has unstable zeroes, the control effort required to achieve minimum variance becomes unbounded and the design needs reformulating. Under this circumstance, the MVC with bounded control signal is still a linear control law but the output signal does not exhibit the behaviour shown in Figure 2.3.

2.4.1 Monitoring of MVC Performance

Returning to Figure 2.3, it can be observed that the influence at y_t of an impulse at e_t is completely extinguished after d samples, that is, $y_t \cdot y_{t+\tau} = 0$, for any $t \in \mathbb{R}$ and any $\tau \geq d$. An equivalent conclusion is reached when e_t is a sequence of independent, stationary, zero mean random variables:

$$r(\tau) \triangleq \mathrm{E}[y_t \cdot y_{t+\tau}] = 0, \qquad \forall \tau \geq d, \tag{2.1}$$

if and only if the controller is the optimal one in terms of minimum variance (Åström 1970). The operator 'E' denotes expectation over the noise distribution.

An equivalent way of describing (2.1) is that under MVC the process output is a moving average of order $d-1$. This property leads to a straightforward monitoring mechanism for MVC optimality, as long as the delay of the process is known and all zeroes of G are stable. When the process has unit delay the monitoring becomes a whiteness test, which is an appropriate situation for using a Shewhart chart.

2.4.2 Assessment of Optimal MVC Performance

Another look over the impulse responses of the system reveals that whichever control action is applied, z_t does not change until $t = d$ samples and, as a consequence, the first 'd' elements of the response of y_t to an impulse at e_t are invariant to the control action. With any controller under operation it is possible to assess the impulse response of y_t under minimum variance control, provided the delay of G is known.

In a practical situation e_t is a noise source to which the user does not have access. Nonetheless, it is possible, although not straightforward, to use the time series y_t to fit an auto-regressive moving average (ARMA) model to the closed-loop transfer function

$$T = \frac{H}{1+G\tilde{C}}, \tag{2.2}$$

where $\tilde{C}$ is any stabilising controller. The first 'd' elements of the impulse response of T determine what would be the response of y_t to an impulse at e_t if $\tilde{C}$ were the minimum variance controller. Moreover, the minimum achievable output variance for that process is the sum of the squares of the initial 'd' elements of the impulse response of T (Harris 1989).

The significance of the result stated above is that one can assess the minimum possible variance of the output, from normal operating data and without any knowledge about the plant apart from the delay. The only requirement for the controller is to stabilise the loop.

Performance assessment of operating control loops is important in determining the potential benefits of controller redesign or tuning, and several practical implementations of this result have been suggested, with successful application to industrial processes (Lynch and Dumont 1996, Desborough and Harris 1992, Stanfelj *et al.* 1991, Eriksson and Isaksson 1994). Two recent papers extend the original scheme into multivariable systems: (Harris *et al.* 1996) and (Huang *et al.* 1997).

The problem of assessing the minimum variance performance of plants with unstable zeroes is still open. The article (Tyler and Morari 1995) mistakenly claimed to have solved this problem and that of assessing the performance of unstable plants (see Section 4.3). The latter problem is not actually a real one since the presence of unstable poles in the process does not affect the design of the minimum variance controller, and the methodologies for monitoring and assessing performance remain exactly the same as described above. The real problem is the presence of unstable zeroes in the process, which can be intrinsic to the physics of the plant or due to fast sampling of continuous-time systems (Åström *et al.* 1984). This point will be further developed in Section 4.3, where it is shown that knowledge of the delay and the position of the unstable zeroes of the plant is not enough to assess the performance of the loop.

2.5 Conclusion

Simple mechanisms for performance monitoring of closed-loop systems are currently available in the literature. This chapter reviews some of the mechanisms that are close to being model free, culminating with the assessment of absolute performance of plants with no unstable zeroes under the minimum variance criterion.

The main criticism of using the minimum variance criterion is that the control action is not penalised and its variance might become too large. In practical terms a criterion that penalises the control action, like the linear quadratic criterion, is often more appropriate. The drawback is that the complexity of the criterion implies extra complexity in the performance monitoring and assessment methods, as seen in the next chapters.

Chapter 3

Assessment of Linear Quadratic Optimality

3.1 Introduction

A natural extension of the minimum variance criterion is to add a term that penalises the variance of the control action. Such a criterion is called linear quadratic (LQ) control and is expressed as

$$J^{lq} = \lim_{N_t \to \infty} \frac{1}{N_t} \sum_{t=0}^{N_t - 1} (y_t^2 + \lambda\, u_t^2), \tag{3.1}$$

where λ is a non-negative scalar that weights the penalty on the variance of the control action. The choice of λ should be based on financial assessment of the relative merits of each variance; unfortunately this is rarely feasible in real situations. The practical approach is to start with some (small) λ in the design of the control law that minimises J^{lq}, followed by an analysis of achieved variances and robustness. In case they are out of specification another value of λ should be tried, maybe combined with fine tunings of the controller parameters.

The optimal LQ controller can be designed quite easily, provided *a model* of the plant is available. The issue then becomes the quality assessment of that model for describing the dynamics of the real plant at the operating point. Given that the model becomes just an instrument for designing the control law, the fundamental question to be answered by an LQ test of optimality follows:

Is the actual system closed-loop performance optimal according to J^{lq}?

The term 'test (or assessment) of LQ optimality' is different from 'LQ performance estimation'. The latter concept means a straightforward computation of (3.1) from a finite amount of data, and this does not provide any information about performance optimality.

As seen before, it is simple to test for optimality of estimators: the Kalman filters are uniquely characterised by the whiteness of the innovations sequence (Mehra 1970). Despite the duality between Kalman filters and LQ control, a signal analogous to the innovations sequence is not directly available for measurement in the LQ counterpart. Nor is whiteness appropriate since linear quadratic control requires no stochastic framework.

The test for minimum variance performance optimality is another step towards the desired result on linear quadratic control. As seen in the last chapter, it is relatively easy to monitor minimum variance optimality since the only information needed about the plant is its delay (Åström 1970). This result is valid only for plants with no unstable zeroes; in the general case a lot more has to be known about those plants.

In order to monitor linear quadratic optimality without any constraint on the process characteristics, except for its linearity and time invariance, the amount of information required about the plant—at our current stage of research—is so large that an identification procedure would be necessary. The solution adopted in this thesis is to make the measurable signals more informative by inserting an exogenous excitation into the closed-loop system, therefore moving from performance monitoring into assessment of optimality. This procedure leads to a truly model-free test for LQ optimality when state feedback is used to control the process (Section 3.2). For the output-feedback case (Section 3.3), part of the noise dynamics must be known; apart from that, no sort of parametric modelling is performed.

3.2 State Feedback

The first results on assessment of optimality under an LQ criterion are specific to plants that have all internal states measurable. The process is required to be controllable and observable, but its model is not known. Under these circumstances a model-free mechanism for assessment of optimality is developed, initially for deterministic systems and later extended to cope with stochastic disturbances.

3.2.1 Noise-free System

The process to be controlled is given in state-space form, with no noises or disturbances acting on the system:

$$x_{t+1} = \Phi x_t + \Gamma u_t \tag{3.2a}$$

$$y_t = \Lambda x_t, \tag{3.2b}$$

where x_t is the n-dimensional state vector, and Φ, Γ and Λ are unknown matrices of dimension $n \times n$, $n \times 1$ and $1 \times n$, respectively. Due to this deterministic setup the LQ criterion of (3.1) is modified to

$$J_{det}^{lq} = \lim_{N_t \to \infty} \sum_{t=0}^{N_t - 1} (y_t^2 + \lambda\, u_t^2), \tag{3.3}$$

meaning that the control action should take the states of (3.2) from their initial values towards the origin in such a way that (3.3) is minimised.

The control law that minimises J_{det}^{lq} is obtained indirectly via an Algebraic Riccati Equation (ARE):

$$S = \Phi^T S \Phi - \Phi^T S \Gamma \left(\Gamma^T S \Gamma + \lambda\right)^{-1} \Gamma^T S \Phi + \Lambda^T \Lambda,$$

resulting in an n-dimensional state-feedback vector K_λ^{lq}:

$$\begin{aligned} u_t &= -\left(\Gamma^T S \Gamma + \lambda\right)^{-1} \Gamma^T S \Phi x_t \\ &= K_\lambda^{lq} x_t. \end{aligned}$$

The notation 'K_λ^{lq}' is used to emphasise the dependence on λ of the optimal control law.

The mechanism for assessment of optimality consists of adding a non-degenerate —that is, $\phi_r(\omega) \neq 0, \forall \omega$ —stationary random process to the control action and observing the power-spectral density of the plant input and output signals. The control action then becomes

$$u_t = K x_t + r_t \tag{3.4}$$

where $\{r_t\}$ has zero mean and power-spectral density $\phi_r(\omega)$. The n-dimensional vector K is a candidate for K_λ^{lq}. The power-spectral densities of u_t and y_t are given by

$$\begin{aligned} \phi_u(\omega) &= \left|1 + K(\mathrm{e}^{i\omega} I - \Phi - \Gamma K)^{-1}\Gamma\right|^2 \phi_r(\omega), \\ \phi_y(\omega) &= \left|\Lambda(\mathrm{e}^{i\omega} I - \Phi)^{-1}\Gamma\right|^2 \phi_u(\omega) = \left|\Lambda(\mathrm{e}^{i\omega} I - \Phi - \Gamma K)^{-1}\Gamma\right|^2 \phi_r(\omega). \end{aligned}$$

Theorem 3.1. *Given the process model described by* (3.2), *and the control action in* (3.4) *such that the closed-loop system is stable, then*

$$\frac{\phi_y(\omega)}{\phi_r(\omega)} + \lambda \frac{\phi_u(\omega)}{\phi_r(\omega)} = \beta, \tag{3.5}$$

for all $\omega \in [0, 2\pi)$ *and some* $\beta \in \mathbb{R}^+$*, if and only if the control law defined by* K *is the optimal* K_λ^{lq}*. The exact value of the constant* β *is* $\Gamma^T S \Gamma + \lambda$*.*

Proof. To prove sufficiency, K is taken as the optimal K_λ^{lq} then the *Return Difference Equality,*

$$\begin{aligned} &\lambda + \Gamma^T(z^{-1}I - \Phi)^{-T}\Lambda^T\Lambda(zI - \Phi)^{-1}\Gamma \\ &\qquad = [1 - K(z^{-1}I - \Phi)^{-1}\Gamma]^T(\Gamma^T S \Gamma + \lambda)[1 - K(zI - \Phi)^{-1}\Gamma], \end{aligned} \tag{3.6}$$

holds (Åström and Wittenmark 1990, Section 11.2). Therefore

$$\begin{aligned} &\left[1 - K(z^{-1}I - \Phi)^{-1}\Gamma\right]^{-T}\left[\lambda + |\Lambda(zI - \Phi)^{-1}\Gamma|^2\right]\left[1 - K(zI - \Phi)^{-1}\Gamma\right]^{-1} \\ &\qquad = \Gamma^T S \Gamma + \lambda, \end{aligned}$$

$$\left[\lambda + |\Lambda(zI - \Phi)^{-1}\Gamma|^2\right]|1 + K(zI - \Phi - \Gamma K)^{-1}\Gamma|^2 = \Gamma^T S \Gamma + \lambda,$$

and taking $z = \mathrm{e}^{i\omega}$, one obtains

$$\lambda \frac{\phi_u(\omega)}{\phi_r(\omega)} + \frac{\phi_y(\omega)}{\phi_r(\omega)} = \Gamma^T S \Gamma + \lambda.$$

Since the value of $\Gamma^T S \Gamma + \lambda$ is constant for all ω, the result is obtained.

To prove necessity, the initial statement, (3.5), is expanded into

$$\begin{aligned}\Gamma^T(z^{-1}I - \Phi - \Gamma K)^{-T}Q_c(zI - \Phi - \Gamma K)^{-1}\Gamma \\ + \left[1 + K(z^{-1}I - \Phi - \Gamma K)^{-1}\Gamma\right]^T \lambda \left[1 + K(zI - \Phi - \Gamma K)^{-1}\Gamma\right] = \beta, \quad (3.7)\end{aligned}$$

where Q_c ($= \Lambda^T\Lambda$ for J^{lq}_{det} given by (3.3)) is the state-weighting matrix in a state-space quadratic cost criterion.

The original LQ-control problem on (3.2) is dual to the problem of state estimation for the system defined by

$$\bar{x}_{t+1} = \Phi^T \bar{x}_t + \Lambda^T \bar{u}_t + v_t \quad (3.8a)$$

$$\bar{y}_t = \Gamma^T \bar{x}_t + e_t, \quad (3.8b)$$

where

$$\begin{aligned}\mathrm{E}[v_t v_t^T] &= R_1 \\ \mathrm{E}[e_t e_t^T] &= R_2 \\ \mathrm{E}[e_{t+i} v_{t+j}^T] &= 0, \quad \forall i, j.\end{aligned}$$

The state estimates satisfy

$$\hat{\bar{x}}_{t+1|t} = \Phi^T \hat{\bar{x}}_{t|t-1} + \Lambda^T \bar{u}_t + M(\bar{y}_t - \Gamma^T \hat{\bar{x}}_{t|t-1}).$$

The vector M is sought to minimise the criterion $\mathrm{E}[(\bar{x}_t - \hat{\bar{x}}_{t|t-1})^T(\bar{x}_t - \hat{\bar{x}}_{t|t-1})]$, and the output prediction error is

$$\begin{aligned}\tilde{\bar{y}}_t &= \bar{y}_t - \Gamma^T \hat{\bar{x}}_{t|t-1} \\ &= \Gamma^T(zI - \Phi^T + M\Gamma^T)^{-1} v_t + [1 - \Gamma^T(zI - \Phi^T + M\Gamma^T)^{-1}M]e_t.\end{aligned}$$

Invoking this duality, (3.7) becomes

$$\begin{aligned}\Gamma^T(zI - \Phi^T + M\Gamma^T)^{-1}R_1(z^{-1}I - \Phi^T + M\Gamma^T)^{-T}\Gamma \\ + \left[1 - \Gamma^T(zI - \Phi^T + M\Gamma^T)^{-1}M\right] R_2 \left[1 - \Gamma^T(z^{-1}I - \Phi^T + M\Gamma^T)^{-1}M\right]^T \\ = \gamma, \quad \gamma \in \mathbb{R}^+, \quad (3.9)\end{aligned}$$

therefore

$$\mathrm{E}[\tilde{\bar{y}}_t \tilde{\bar{y}}_t^T] = \gamma. \quad (3.10)$$

Given that the system is linear and the sequences $\{v_t\}$ and $\{e_t\}$ are mutually uncorrelated zero-mean Gaussian white noises, then (3.10) implies that $\{\tilde{\bar{y}}_t\}$ (the innovations sequence) is zero mean and white; yet, according to Theorem 6.1 in (Anderson and Moore 1979, Section 5.6), the state estimator is optimal—usually known as the Kalman predictor. Re-invoking the duality, the original controller is proved to be optimal. □

Unfortunately, while the innovations sequence in the Kalman predictor can be measured, the equivalent sequence for optimal LQ control cannot. Therefore correlation methods used in adaptive filtering (Mehra 1970) are not directly applicable to LQ control.

Example 3.1. A linear plant, described by

$$x_{t+1} = \begin{bmatrix} 1.2 & 1 & 0 \\ 0 & -0.5 & 1 \\ 0.2 & 0.1 & -0.3 \end{bmatrix} x_t + \begin{bmatrix} 0.2 \\ 0 \\ 0.1 \end{bmatrix} u_t$$
$$y_t = \begin{bmatrix} 1 & 0.01 & 0.2 \end{bmatrix} x_t,$$

is controlled according to

$$u_t = \begin{bmatrix} -2.851 & -1.805 & -0.8001 \end{bmatrix} x_t,$$

which minimises J^{lq}_{det} for $\lambda = 0.2$. An example of the autonomous response to an initial state is given in Figure 3.1, for $x_0 = \begin{bmatrix} 1 & -1 & 0.5 \end{bmatrix}^T$.

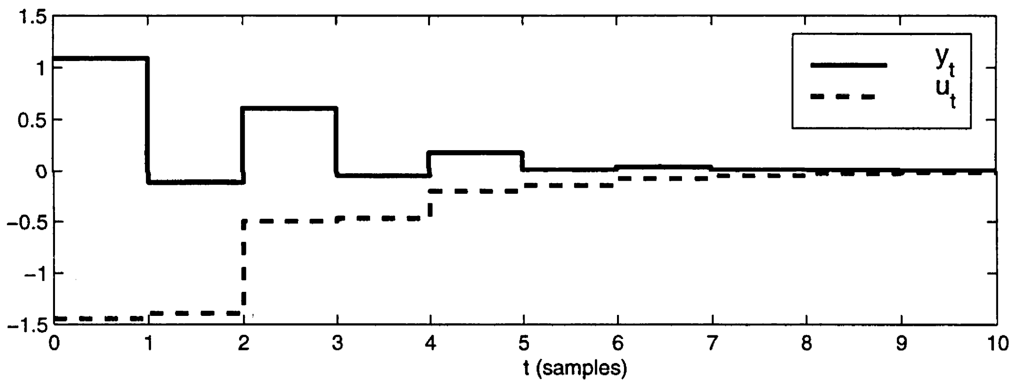

Figure 3.1: Response of the autonomous system

In order to assess optimality of the control law, a white-noise signal of zero mean and $\phi_r(\omega) = \sigma^2 = 1$ is added to u_t, producing the power-spectral densities shown in Figure 3.2. The fact that $\phi_y(\omega) + \lambda\phi_u(\omega)$ is constant over all frequencies indicates that the plant is optimally controlled.

3.2.2 System with Noise

In the presence of stochastic additive plant noises, the state-space description of the process becomes

$$x_{t+1} = \Phi x_t + \Gamma u_t + v_t \quad (3.11a)$$
$$y_t = \Lambda x_t + e_t, \quad (3.11b)$$

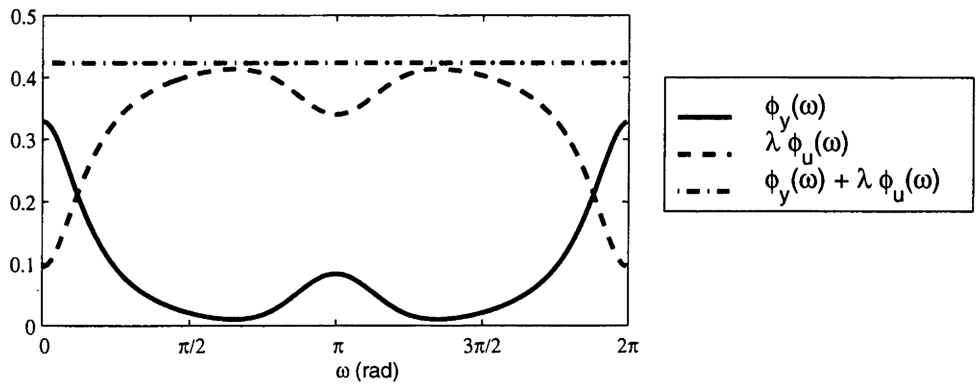

Figure 3.2: Power-spectral densities

where $\{v_t\}$ and $\{e_t\}$ are white-noise processes with zero mean and covariance given by

$$\begin{aligned} \mathrm{E}[v_t v_t^T] &= R_1 \\ \mathrm{E}[v_t e_t^T] &= R_{12} \\ \mathrm{E}[e_t v_t^T] &= R_{21} = R_{12}^T \\ \mathrm{E}[e_t e_t^T] &= R_2. \end{aligned}$$

Within this stochastic framework the linear quadratic cost function in (3.1) is well defined. The optimal control law is independent of R_1, R_{12} and R_2, and is actually the same as if the system were noise free.

Any stabilising controller, K, given as an n-dimensional vector, can be tested for LQ optimality via an injection of a non-degenerate random process at r_t, as given by (3.4). The cross-spectral densities between u_t and r_t and between y_t and r_t are

$$\begin{aligned} \phi_{ur}(\omega) &= \left[1 + K(\mathrm{e}^{i\omega}I - \Phi - \Gamma K)^{-1}\Gamma\right] \phi_r(\omega), \\ \phi_{yr}(\omega) &= \Lambda(\mathrm{e}^{i\omega}I - \Phi)^{-1}\Gamma \ \phi_{ur}(\omega). \end{aligned}$$

In this case, the following result holds, by means of Theorem 3.1 and straightforward calculations.

Theorem 3.2. *Given the process described by* (3.11), *and the control action in* (3.4) *such that the closed-loop system is stable, then*

$$\left|\frac{\phi_{yr}(\omega)}{\phi_r(\omega)}\right|^2 + \lambda \left|\frac{\phi_{ur}(\omega)}{\phi_r(\omega)}\right|^2 = \beta,$$

for all $\omega \in [0, 2\pi)$ *and some* $\beta \in \mathbb{R}^+$, *if and only if* K *is the optimal state feedback* K_λ^{lq}. *The exact value of the constant* β *is* $\Gamma^T S \Gamma + \lambda$.

Remark. The cross-spectral densities $\phi_{yr}(\omega)$ and $\phi_{ur}(\omega)$ can be obtained from closed-loop data via spectral analysis given that $\{r_t\}$ is independent of $\{v_t\}$ and $\{e_t\}$ (Ljung 1987).

Due to the 'Separation Principle' of linear optimal control, this result is still valid when non-measurable states are reconstructed with a Kalman filter. The implementation of this filter, however, implies certain knowledge of the plant, which is a drawback. In the extreme case only the output of the plant is measurable, and the reconstruction of all states requires full knowledge of the plant. The Kalman filter dynamics are then identical to the part of the noise dynamics needed for the output feedback mechanism of performance assessment, as is seen in the next section.

3.3 Output Feedback

When none of the states of the plant are measurable, the design of the optimal control law is affected and the mechanisms presented in the last section are not directly applicable. Our analysis follows with an input/output auto-regressive moving-average exogenous input (ARMAX) model of the process:

$$A(q)y_t = B(q)u_t + C(q)e_t, \tag{3.12}$$

where $\{e_t\}$ is a zero-mean white-noise process. The polynomials $A(q)$ and $C(q)$ are monic, $\deg C = \deg A > \deg B$, and all the zeroes of $C(q)$ are strictly inside the unit disc.

The control law that minimises J^{lq} is obtained indirectly via spectral factorisation:

$$\eta P(z)P(z^{-1}) = \lambda A(z)A(z^{-1}) + B(z)B(z^{-1}), \quad \eta \in \mathbb{R}^+, \tag{3.13}$$

where the monic polynomial $P(z)$ is the characteristic polynomial of the closed-loop system. $P(z)$ is unique given that it is monic, $\deg P = \deg A$ and all the zeroes of $P(z)$ are inside the unit disc or on the unit circle. The optimal control law is given by

$$u_t = -\frac{S^{lq}(q)}{R^{lq}(q)} y_t,$$

where $S^{lq}(z)$ and $R^{lq}(z)$ are solutions to the pole-positioning Diophantine equation

$$A(z)R^{lq}(z) + B(z)S^{lq}(z) = P(z)C(z). \tag{3.14}$$

Some constraints are imposed on $R^{lq}(z)$ and $S^{lq}(z)$ in order to guarantee uniqueness of the solution: $R^{lq}(z)$ is monic, $\deg S^{lq} \leq \deg R^{lq} = \deg A$ and $S^{lq}(0) = 0$. Still, if $A(0) = 0$ or the polynomials $A(z)$ and $B(z)$ have common factors, these conditions are not enough to guarantee uniqueness of the solution, thus requiring additional equations to be used instead of (3.14) (Åström and Wittenmark 1990, Section 12.5).

If a control law of the kind

$$u_t = -\frac{S(q)}{R(q)} y_t,$$

is optimal, it has to satisfy the constraints described above. This implies that the order of the process ($\deg A$) has to be known *a priori*. The mechanism for assessment of LQ optimality uses the control action

$$u_t = -\frac{S(q)}{R(q)} y_t + \frac{C(q)}{R(q)} r_t, \tag{3.15}$$

where $\{r_t\}$ is a non-degenerate random process of zero mean and power-spectral density $\phi_r(\omega)$. Observe that knowledge of $C(z)$, part of the noise dynamics, is also needed for the realisation of the test.

The injection of $\{r_t\}$ into the loop results in the cross-spectral densities

$$\phi_{ur}(\omega) = \frac{A(\mathrm{e}^{i\omega})C(\mathrm{e}^{i\omega})}{A(\mathrm{e}^{i\omega})R(\mathrm{e}^{i\omega}) + B(\mathrm{e}^{i\omega})S(\mathrm{e}^{i\omega})} \phi_r(\omega)$$

and

$$\phi_{yr}(\omega) = \frac{B(\mathrm{e}^{i\omega})C(\mathrm{e}^{i\omega})}{A(\mathrm{e}^{i\omega})R(\mathrm{e}^{i\omega}) + B(\mathrm{e}^{i\omega})S(\mathrm{e}^{i\omega})} \phi_r(\omega).$$

Theorem 3.3. *Take the process described by* (3.12) *and the control action in* (3.15) *such that the closed-loop system is stable. If $R(z)$ and $S(z)$ are the polynomials $R^{lq}(z)$ and $S^{lq}(z)$ comprising the optimal control law, then*

$$\left|\frac{\phi_{yr}(\omega)}{\phi_r(\omega)}\right|^2 + \lambda \left|\frac{\phi_{ur}(\omega)}{\phi_r(\omega)}\right|^2 = \eta, \tag{3.16}$$

for all $\omega \in [0, 2\pi)$ and some $\eta \in \mathbb{R}^+$.

Furthermore, if the process has $A(z)$ and $B(z)$ coprime and $A(0) \neq 0$, then the condition (3.16) *implies optimality of $R(z)$ and $S(z)$.*

The exact value of the constant η can be obtained from (3.13).

Proof. For sufficiency, straightforward calculations lead to the result:

$$\begin{aligned}\left|\frac{\phi_{yr}(\omega)}{\phi_r(\omega)}\right|^2 + \lambda \left|\frac{\phi_{ur}(\omega)}{\phi_r(\omega)}\right|^2 &= \frac{|B(\mathrm{e}^{i\omega})|^2 + \lambda |A(\mathrm{e}^{i\omega})|^2}{|A(\mathrm{e}^{i\omega})R^{lq}(\mathrm{e}^{i\omega}) + B(\mathrm{e}^{i\omega})S^{lq}(\mathrm{e}^{i\omega})|^2} |C(\mathrm{e}^{i\omega})|^2 \\ &= \frac{\eta |P(\mathrm{e}^{i\omega})|^2}{|P(\mathrm{e}^{i\omega})C(\mathrm{e}^{i\omega})|^2} |C(\mathrm{e}^{i\omega})|^2 = \eta.\end{aligned}$$

To prove necessity, (3.16) is expanded as

$$\frac{|B(z)|^2 + \lambda |A(z)|^2}{|A(z)R(z) + B(z)S(z)|^2} |C(z)|^2 = \eta,$$

which leads to

$$|P(z)C(z)|^2 = |A(z)R(z) + B(z)S(z)|^2.$$

Since the loop is stable, the closed-loop poles are positioned at their optimal locations, according to (3.13). Under the constraints that $A(z)$ and $B(z)$ are coprime and $A(0) \neq 0$, the control law under test provides the unique solution to this pole-placement problem, namely the optimal control law. □

Remark. The cross-spectral densities $\phi_{yr}(\omega)$ and $\phi_{ur}(\omega)$ can be obtained from closed-loop data via spectral analysis given that $\{r_t\}$ is independent of $\{e_t\}$.

This result can be compared with Theorem 3.2, which leads to the conclusion that the Kalman filter brings to the feedback controller the necessary information about the noise dynamics. Åström and Wittenmark (1990, Section 12.5) presented the pole-positioning aspects of the linear quadratic Gaussian (LQG) design: the poles of the optimal observer (Kalman filter) are positioned at the roots of $C(z)$, while the state feedback is responsible for positioning the poles of the closed-loop system at the roots of $P(z)$. That means, if $C(z)$ is known then the test for optimality can be performed on the polynomial control law, otherwise an observer should be implemented and adjusted to have a white innovations sequence (necessary and sufficient condition for optimality), followed by LQ performance assessment on the estimated state feedback.

Notice that the test of LQ optimality is performed on the reference-to-output path, which excludes the additive noise dynamics from influencing the results. This explains the need for knowledge about the noise dynamics and its use in the excitation filter, but this also means that the quality of our information about the noise is not under test.

Example 3.2. Consider the open-loop unstable and nonminimum-phase process described by (3.12), with

$$\begin{aligned} A(z) &= (z-1.70)(z-0.50 \pm 0.40\mathrm{i}) \\ B(z) &= -0.80(z-1.25)(z-0.85) \\ C(z) &= z(z-0.80)(z-0.30), \end{aligned}$$

and the variance of $\{e_t\}$ being $\sigma_e^2 = 0.2$.

Under minimum variance control the variances of $\{y_t\}$ and $\{u_t\}$ are $\sigma_y^2 = J^{mv} = 0.986$ and $\sigma_u^2 = 17.8$, respectively. In order to reduce the control action, the cost function of (3.1), with $\lambda = 0.1$, is adopted. Under LQ control the variances become $\sigma_y^2 = 1.20$ and $\sigma_u^2 = 8.44$.

Since the process contains an unstable zero, the mechanisms for minimum variance performance monitoring and assessment, presented in Section 2.4, are not applicable. With an appropriate external signal, the test of linear quadratic optimality can be performed either when the polynomial $C(z)$ is known, or when the non-measurable states of the plant are reconstructed by Kalman filters. For this example the polynomial $C(z)$ is assumed known so the test can be performed according to Theorem 3.3.

The excitation is performed with a white-noise signal whose variance trades-off amplitude (signal-to-noise ratio) and duration of excitation in order to improve the quality of the results. Under this particular choice of excitation, the spectral density in the test of optimality can be rewritten as

$$|\phi_{yr}(\omega)|^2 + \lambda |\phi_{ur}(\omega)|^2 = \eta\, \sigma^4.$$

In this example a variance $\sigma^2 = 4.00$ results in the following increase in the output variance: $\Delta\sigma_y^2 = 4.25$ for the LQ controller and $\Delta\sigma_y^2 = 4.00$ for the minimum variance controller.

The system is run for some time (256 steps) in order to present a stationary behaviour and after that 4096 pairs (u_t, y_t) of excited signals are collected for spectral analysis. Figure 3.3 shows the spectral densities obtained during the test of optimality when the system is operating under LQ-optimal control. Figure 3.4 shows equivalent spectral densities for the minimum variance controller. Both figures show the true expected values of the spectral densities as thin lines.

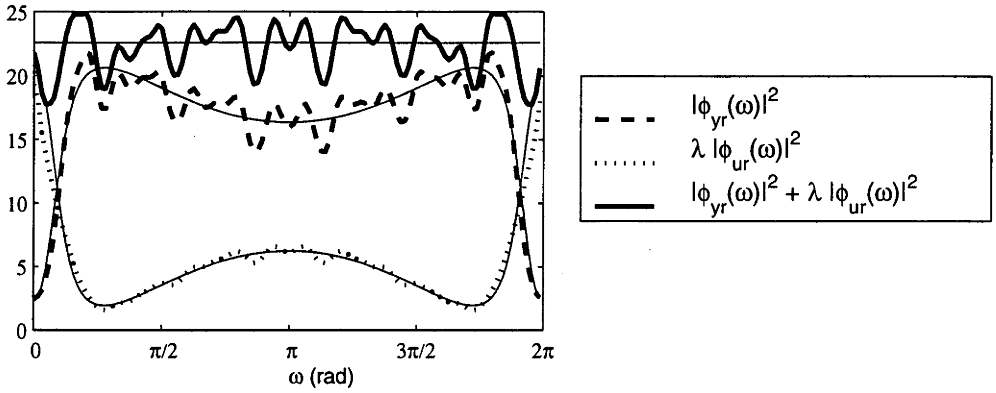

Figure 3.3: Test of LQ optimality with the optimal controller

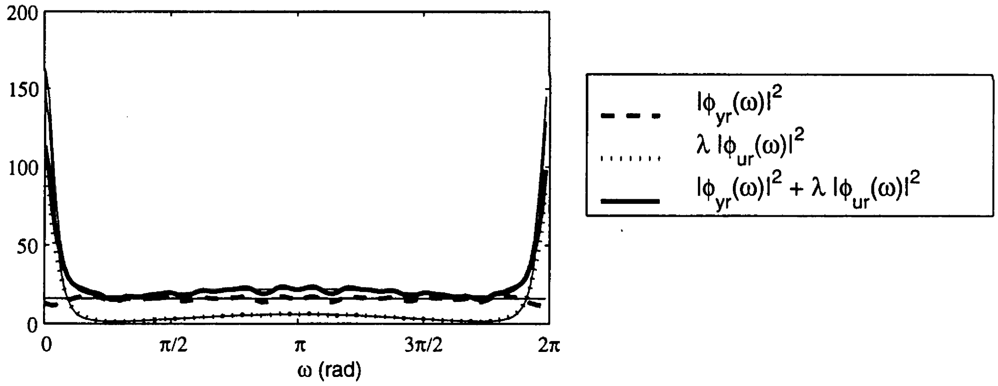

Figure 3.4: Test of LQ optimality with the MV controller

As anticipated, Figure 3.3 shows that $|\phi_{yr}(\omega)|^2 + \lambda|\phi_{ur}(\omega)|^2$ has constant amplitude over the frequency range. This does not happen in Figure 3.4 (MV controller) for the criterion J^{lq} with $\lambda = 0.1$. Observe that in this case $|\phi_{yr}(\omega)|^2$ is flat, corresponding to the optimal performance under an LQ criterion with $\lambda = 0$.

3.4 Conclusion

This chapter has introduced mechanisms for assessment of linear quadratic optimality based on spectral analysis of the process input and output signals observed during perturbed closed-loop operation. The simplest form of LQ performance assessment is feasible when all states of the plant are measurable, resulting in a model-free mechanism. If some of the states of the plant are not measurable, they have to be optimally reconstructed with an observer, but that requires a state-space model of the plant. Another approach is to use the mechanism for performance assessment of systems under output feedback. The drawback is that the noise dynamics in the ARMAX model, $C(z)$, has to be known; hence, the test is not completely model free. The extension of these results into multivariable systems is believed to be possible, although messy.

The state-space test of LQ optimality uses the duality between Kalman filtering and LQ control. Indeed, as the Kalman filter has the whiteness of the innovations sequence, the LQ-control dual to this property is the spectral flatness of a signal equivalent to the innovations sequence. Unfortunately this signal, in the LQ control, is not directly measurable in the time domain. Moreover, the difficulties arising in the output-feedback control are due to the combined solution of the state estimation and the LQ control. The information required for the realisation of the test, $C(z)$, contains the optimal locations for the estimator poles. That is, this information would be indirectly provided to the system if the Kalman filter were actually implemented.

In a retrospective analysis, it is noticeable that the complexity of the mechanisms for testing performance optimality increases as the criterion becomes more generic. The initial whiteness test provided by statistical process control can detect optimality only of unit-delay minimum-phase plants under a minimum variance criterion. Correlation analysis extends the test to plants with any delay, as long as this delay is known, but the plant is still constrained to have all zeroes stable and the criterion is limited to the minimum variance control. The mechanisms presented in this chapter extend the test of optimality into the linear quadratic criterion with no constraints on the plant characteristics, but at the cost of exciting the loop and, in the output-feedback case, knowing the noise dynamics.

The need for some knowledge about the plant (in the output-feedback case) and an external excitation are obstacles for the widespread use of this test in industrial plants. A more realistic scenario is its application to some individual loops that deserve the costly effort to achieve optimality, as well as a design validation step for the optimal controller of the model of the plant.

The response to an impulse at the noise source is a key element in minimum variance performance monitoring, but it does not seem to present any special property related to linear quadratic performance or minimum variance of plants with unstable zeroes. These two cases have in common the property that at optimality the closed-loop system contains at least one pole away from the origin, therefore the impulse response of the system is infinite in its extent. The next chapter focuses on this characteristic polynomial of the optimally performing closed-loop system.

Chapter 4

Linear Quadratic Performance Monitoring

4.1 Introduction

Assessment of optimality of operating control loops is important in determining which are the loops that need redesigning or tuning. Even more important are mechanisms for performance monitoring because they do not disturb the system's normal operation. Ideally, normal operating data should provide all necessary information about the loop performance optimality. Under the minimum variance criterion this ideal situation is not too distant, provided the plant has no unstable zeroes. However, under a linear quadratic criterion,

$$J^{lq} = \lim_{N_t \to \infty} \frac{1}{N_t} \sum_{t=0}^{N_t - 1} (y_t^2 + \lambda\, u_t^2), \tag{4.1}$$

even assessment of optimality is problematic because the noise dynamics of the ARMAX model

$$A(q)y_t = B(q)u_t + C(q)e_t \tag{4.2}$$

must be known.

Using excited closed-loop signals one might proceed by identification of an open-loop model, applying say the methods of (Van den Hof and Schrama 1993), followed by controller design and comparison between expected and actual behaviours. This introduces issues of modelling accuracy in addition to control design. Independent of this approach, one can use the excited closed-loop signals to estimate the closed-loop poles that would have been obtained under optimal LQ performance. This chapter shows how to perform such an estimation and how this result can be used towards LQ performance monitoring. Moreover, the particular case of MV performance monitoring of nonminimum-phase plants shows that the presence of closed-loop poles away from the origin significantly influences the amount of information required to monitor or assess performance optimality.

4.2 LQ Optimal Characteristics

Among the terms resulting from system identification, the noise model often has the lowest accuracy or reliability because the identification procedures attempt to capture all unmodelled effects as part of this noise model. This can include drifts, offsets, quantisation error, signal variabilities, under-modelling as well as the process disturbance. This problem motivates the development of a more complex mechanism for LQ performance assessment and monitoring that does not require any knowledge about $C(z)$.

4.2.1 Spectral Densities with Optimality Attributes

The feedback control law used for regulating the ARMAX plant, described by (4.2), is as follows:

$$u_t = -\frac{S(q)}{R(q)} y_t + r_t, \tag{4.3}$$

where $r_t \equiv 0$ during normal operation. Extra information about the system is obtained with temporary external excitation at r_t, which for simplicity is chosen here as a white-noise process of zero mean and variance σ^2. The resultant cross-spectral densities are

$$\phi_{ur}(\omega) = \frac{A(\mathrm{e}^{i\omega})R(\mathrm{e}^{i\omega})}{A(\mathrm{e}^{i\omega})R(\mathrm{e}^{i\omega}) + B(\mathrm{e}^{i\omega})S(\mathrm{e}^{i\omega})}\sigma^2$$

and

$$\phi_{yr}(\omega) = \frac{B(\mathrm{e}^{i\omega})R(\mathrm{e}^{i\omega})}{A(\mathrm{e}^{i\omega})R(\mathrm{e}^{i\omega}) + B(\mathrm{e}^{i\omega})S(\mathrm{e}^{i\omega})}\sigma^2.$$

If the model of the plant were available, the characteristic polynomial of the optimal closed-loop system, $P(z)$ would be computed via the spectral factorisation

$$\eta P(z)P(z^{-1}) = \lambda A(z)A(z^{-1}) + B(z)B(z^{-1}), \tag{4.4}$$

with $\eta \in \mathbb{R}^+$. This expression is used for obtaining

$$\frac{\eta^{-1}|B(z)|^2}{|P(z)|^2} = \frac{|B(z)|^2}{|B(z)|^2 + \lambda|A(z)|^2},$$

which can have both numerator and denominator of its right-hand side fraction multiplied by

$$\frac{|R(z)|^2}{|A(z)R(z) + B(z)S(z)|^2}\sigma^4,$$

leading to

$$\frac{\eta^{-1}|B(\mathrm{e}^{i\omega})|^2}{|P(\mathrm{e}^{i\omega})|^2} = \frac{|\phi_{yr}(\omega)|^2}{|\phi_{yr}(\omega)|^2 + \lambda|\phi_{ur}(\omega)|^2}. \tag{4.5}$$

Similar developments are employed for establishing

$$\frac{\eta^{-1}|A(e^{i\omega})|^2}{|P(e^{i\omega})|^2} = \frac{|\phi_{ur}(\omega)|^2}{|\phi_{yr}(\omega)|^2 + \lambda|\phi_{ur}(\omega)|^2}. \tag{4.6}$$

Results (4.5) and (4.6) are special because: i) their right-hand sides are built from cross-spectral densities acquired with any stabilising controller in the loop; ii) their left-hand sides are equivalent to power-spectral densities of filtered white noise, whose filters' poles are the optimal ones.

From those power-spectral densities one can think of estimating their filter transfer functions to obtain the stable polynomial $P(z)$. This idea, developed in the next section, results in a method for phase retrieval and estimation from frequency-domain data. The zeroes of $A(z)$ and $B(z)$ can be estimated from that procedure only if the process is known to be stable and minimum-phase, otherwise one cannot distinguish between a certain estimated zero and its inverse because their contributions to the power-spectral density are identical.

4.2.2 Estimation of $P(z)$

The problem to be solved in this section is:

Problem 4.1. *Given a noisy measurement, $|\hat{\phi}(\omega)|^2$, of the squared magnitude of some spectrum*

$$\phi(\omega) = \mathcal{F}\left[\frac{L(z)}{P(z)}\right],$$

find the best estimate, $\hat{P}(z)$, of the monic denominator polynomial $P(z)$, whose zeroes lie strictly inside the unit circle.

Direct methods for identification from frequency-response data need complete spectral information (magnitude and phase) as input. If one takes the square root of $|\hat{\phi}(\omega)|^2$, the phase of the spectrum is lost and some form of signal reconstruction has to be used.

Generic signal-reconstruction problems can be solved by an iterative algorithm known as the error-reduction algorithm. This algorithm bounces back and forth between the object domain, where the object-domain constraints are applied, and the Fourier domain, where the respective constraints are applied (Dainty and Fienup 1987).

The error-reduction algorithm can be applied to solve the phase-retrieval problem that arises when only the magnitude of a signal is known, as found in Problem 4.1. For this instance, the object-domain constraints are given by the order of the polynomials $L(z)$ and $P(z)$, and the frequency-domain constraint is the magnitude of the noisy measurement. A block diagram of the algorithm is shown in Figure 4.1. It consists of the following basic steps:

1. Form an initial estimate for $\hat{\phi}(\omega)$;

2. Construct a long finite impulse response (FIR) filter by inverse discrete Fourier transform (IDFT): $g(z) = \mathcal{F}^{-1}[\hat{\phi}(\omega)]$;

3. Fit an n-th order auto-regressive moving-average (ARMA) model to the FIR filter, $g(z)$, by model reduction: $\hat{L}(z)/\hat{P}(z)$;

4. Compute the discrete Fourier transform (DFT) of the ARMA model: $\phi_m(\omega) = \mathcal{F}[\hat{L}(z)/\hat{P}(z)]$;

5. Form a new spectrum estimate, $\hat{\phi}(\omega) = |\hat{\phi}(\omega)|\, e^{i \arg[\phi_m(\omega)]}$, and return to Step 2.

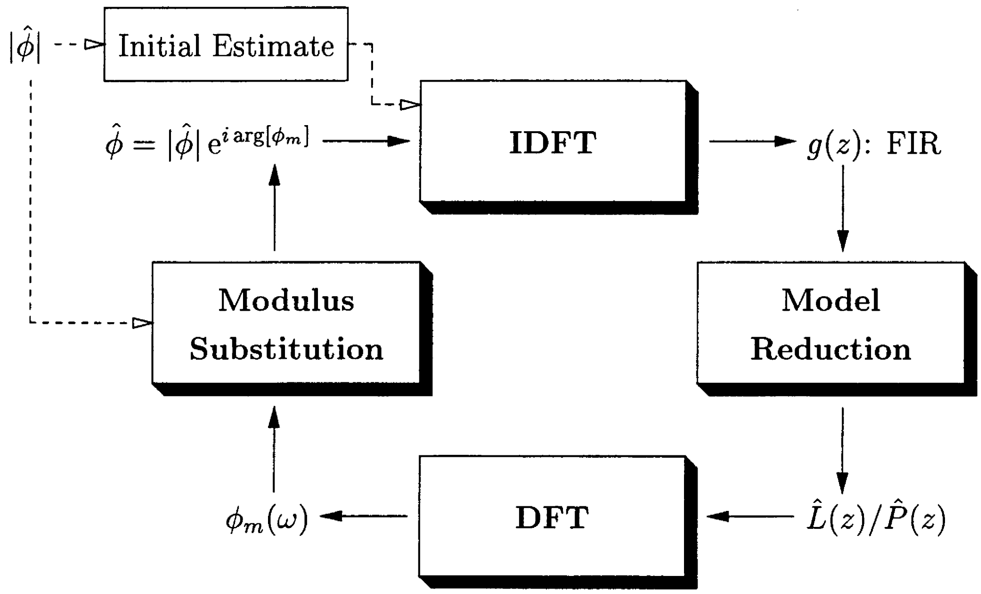

Figure 4.1: Block diagram of the algorithm for phase retrieval

There are two open issues in the algorithm as stated so far: the initial estimate creation and the model reduction. According to the author's experience, a reasonably good initial estimate can be obtained through the solution of the following least-squares problem:

$$\min_{\hat{L},\hat{P}} \sum_{\omega} \left[|\hat{\phi}(\omega)|^2 \hat{P}(e^{i\omega})\hat{P}(e^{-i\omega}) - \hat{L}(e^{i\omega})\hat{L}(e^{-i\omega}) \right]^2 .$$

The Fourier transform of $\hat{L}(z)/\hat{P}(z)$ generates the initial estimate of the phase.

The model-reduction step is implemented with a Steiglitz-McBride iteration (Steiglitz and McBride 1965). This method attempts to minimise the squared error between the impulse response $\hat{g}(z)$ of $\hat{L}(z)/\hat{P}(z)$ and the FIR filter $g(z)$ by iteratively pre-filtering $g(z)$ and applying a least-squares method (Ljung 1987, page 297).

A convergence analysis of the error-reduction algorithm is presented in (Dainty and Fienup 1987). It is shown that the algorithm 'converges' in the weak sense that the squared error cannot increase with an increasing number of iterations. For our particular case, the convergence of the whole procedure is dependent on the convergence of the model-reduction method, the Steiglitz-McBride iteration, which is analysed in (Stoica and Söderström 1981).

If the plant order is known and the spectrum measurement is exact, the initial $\hat{P}(z)$ is also exact and this is not changed by the iterative procedure. When the spectrum measurement includes noisy components, approximate values are obtained for $\hat{P}(z)$.

Example 4.1. Consider the same plant used in Example 3.2, an open-loop unstable and nonminimum-phase process described by (4.2), with

$$\begin{aligned} A(z) &= (z-1.70)(z-0.50 \pm 0.40\mathrm{i}) \\ B(z) &= -0.80(z-1.25)(z-0.85) \\ C(z) &= z(z-0.80)(z-0.30), \end{aligned}$$

and the variance of $\{e_t\}$ being $\sigma_e^2 = 0.2$.

This time the external excitation is performed according to (4.3), therefore it is not required that the polynomial $C(z)$ be known. Two different controllers are used during the experiments: the LQ optimal for $\lambda = 0.1$ and the MV controller. With a white-noise excitation of variance $\sigma^2 = 1$, the increases in the output variance are $\Delta\sigma_y^2 = 6.18$ for the LQ controller and $\Delta\sigma_y^2 = 4.25$ for the minimum variance controller.

The cross-spectral densities obtained with each of the controllers are used for building estimates of the power-spectral densities $\frac{\eta^{-1}|B(e^{i\omega})|^2}{|P(e^{i\omega})|^2}$ and $\lambda\frac{\eta^{-1}|A(e^{i\omega})|^2}{|P(e^{i\omega})|^2}$, where $P(z)$ is the optimal characteristic polynomial associated with this example's plant and J^{lq} with $\lambda = 0.1$. These power-spectral densities, obtained with 4096 samples collected from the loop, are presented in Figures 4.2 and 4.3, which as expected are invariant to the control law. The term $\phi_J(\omega)$, used in the legend of these figures, is defined as $|\phi_{yr}(\omega)|^2 + \lambda|\phi_{ur}(\omega)|^2$.

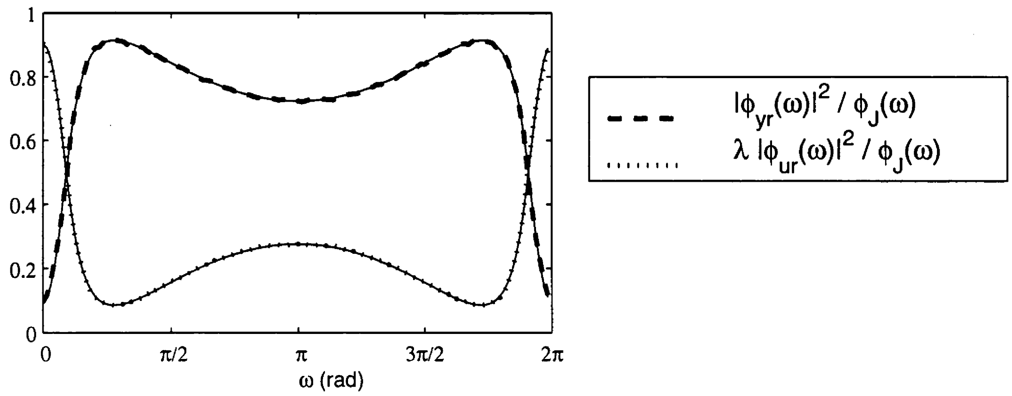

Figure 4.2: Spectral densities derived from the LQ controller

The next step is to use these power-spectral densities for estimating the optimal characteristic polynomial. This estimation is performed with the error-reduction algorithm presented in the current section, and the poles of the resultant polynomials are shown in Table 4.1. The actual optimal poles are $0.7346 \pm 0.1336\mathrm{i}$ and 0.0886; the accuracy of estimation of these poles is remarkably good, given that no explicit plant model is used.

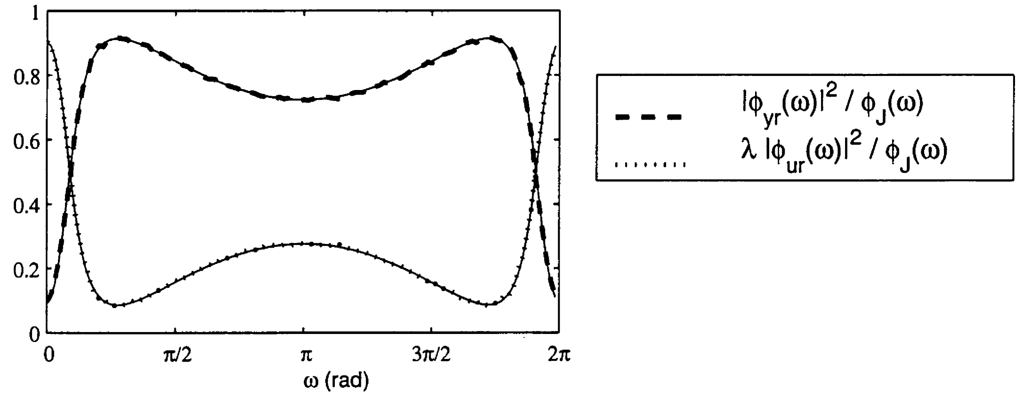

Figure 4.3: Spectral densities derived from the MV controller

Table 4.1: Estimated closed-loop poles

	From $\lvert\phi_{yr}(\omega)\rvert^2/\phi_J(\omega)$	From $\lambda\lvert\phi_{ur}(\omega)\rvert^2/\phi_J(\omega)$
LQ controller	$0.7389 \pm 0.1188\mathrm{i}, 0.1564$	$0.7462 \pm 0.1410\mathrm{i}, 0.0874$
MV controller	$0.7183 \pm 0.1389\mathrm{i}, 0.0996$	$0.7378 \pm 0.1390\mathrm{i}, 0.0957$

4.2.3 LQ Performance Monitoring

Undoubtedly the estimates of the optimal closed-loop poles do not have intrinsic relevance, although they can become very useful for other techniques. One example is to use these estimates for guiding direct adaptive pole-placement algorithms (e.g. Elliot 1982, Åström and Wittenmark 1989). Another example of the use of the optimal closed-loop poles is in monitoring linear quadratic performance.

The mechanism for LQ performance monitoring uses the signal

$$w_t \triangleq \frac{P(q)}{R_+(q)R_-{}^*(q)} y_t, \tag{4.7}$$

where the factorisation of the controller denominator $R(q) = R_+(q)R_-(q)$ is such that all zeroes of $R_+(q)$ are inside the unit disc and all zeroes of $R_-(q)$ are outside the unit disc or on the unit circle. The reciprocal polynomial $R_-{}^*(q)$ is chosen to be monic, therefore guaranteeing uniqueness of that factorisation.

Lemma 4.2. *Given the process model described by* (4.2) *under control of the LQ optimal output feedback*

$$u_t = -\frac{S^{lq}(q)}{R^{lq}(q)} y_t, \tag{4.8}$$

then the signal $\{w_t\}$*, defined in* (4.7)*, is a white-noise process.*

Proof. Under optimal control law the system output is given by

$$\begin{aligned} y_t &= \frac{C(q)R^{lq}(q)}{A(q)R^{lq}(q)+B(q)S^{lq}(q)}e_t \\ &= \frac{R^{lq}(q)}{P(q)}e_t. \end{aligned}$$

Substitute this expression into (4.7) to obtain

$$w_t = \frac{R_-^{lq}(q)}{R_-^{lq^*}(q)}e_t.$$

Given that $\{e_t\}$ is a white-noise process and

$$\left| \frac{R_-^{lq}(\mathrm{e}^{i\omega})}{R_-^{lq^*}(\mathrm{e}^{i\omega})} \right| = 1,$$

then the covariance function of w_t is

$$\begin{aligned} r_w(\tau) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{i\tau\omega}\,\phi_w(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{i\tau\omega}\,\phi_e(\omega)\,d\omega \\ &= r_e(\tau) = \begin{cases} \sigma^2, & \tau = 0 \\ 0, & \tau \neq 0, \end{cases} \end{aligned}$$

implying that $\{w_t\}$ is a white-noise process. □

Unfortunately the optimal control law is not the only one that whitens $\{w_t\}$. The general expression of the control law that does that is

$$\begin{aligned} S(z) &= S^{lq}(z) - Q(z)A(z) \\ R(z) &= R^{lq}(z) + Q(z)B(z) \end{aligned}$$

where $Q(z)$ is any polynomial. Taking into consideration that the order of the plant is known—it is needed for the estimation of $P(z)$ —the order of the controller is constrained, reducing $Q(z)$ to the set of scalars. Another consideration about the optimal control law is that it has $S(0) = 0$, hence if $A(0) \neq 0$ the unique choice for $Q(z)$ is to equal zero (Åström and Wittenmark 1990).

The performance monitoring mechanism works according to the following rules:

IF $\{w_t\}$ is NOT a white-noise process
THEN the controller is NOT optimal.

IF $\{w_t\}$ is a white-noise process AND $A(0)$ is known to be $\neq 0$
THEN the controller is optimal.

If $\{w_t\}$ is a white-noise process and $A(0) = 0$, or $A(0)$ is not known to be $\neq 0$, then nothing can be concluded about the optimality of the controller.

Example 4.2. Consider the system in Example 4.1 where estimates of $P(z)$ were computed from signals of the loop under excitation. The aim now is to monitor the loop performance, under normal operating data, by checking the covariance function of the signal w_t.

Initially the minimum variance controller is used, with $P(z)$ estimated from $\lambda|\phi_{ur}(\omega)|^2/\phi_J(\omega)$:

$$\hat{P}(z) = (z - 0.7378 + 0.1390i)(z - 0.7378 - 0.1390i)(z - 0.0957).$$

The signal $\{w_t\}$ is formed according to (4.7), with $\{y_t\}$ being a sequence of 4096 samples collected from normal operating data. The initial terms of the covariance function of $\{w_t\}$ are presented in Figure 4.4. Observe that for several values of τ the covariance function is not within the 95% confidence limits. Consequently, $\{w_t\}$ is not a white-noise process implying that the controller is not LQ optimal.

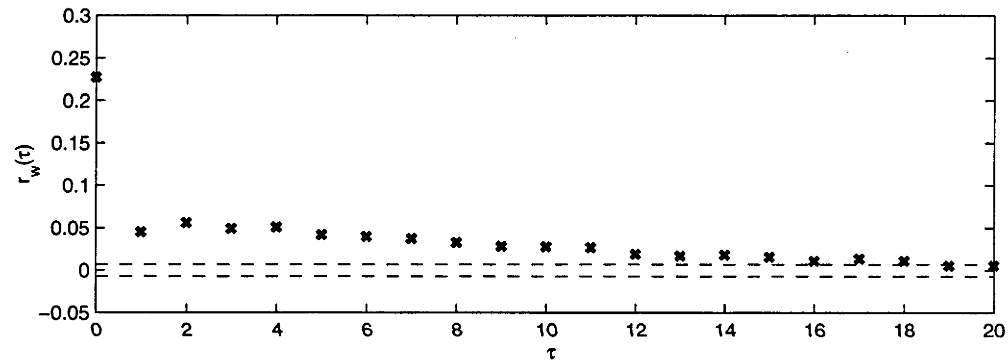

Figure 4.4: Covariance function of w_t under the MV controller

The same procedure is repeated with the LQ controller and

$$\hat{P}(z) = (z - 0.7462 + 0.1410i)(z - 0.7462 - 0.1410i)(z - 0.0874).$$

The initial terms of the covariance function of $\{w_t\}$ are presented in Figure 4.5. As expected, the covariance function shows that $\{w_t\}$ is a white-noise process.

4.3 On MV of Nonminimum-phase Plants

The problem of assessing the minimum variance performance of plants with unstable zeroes was tackled by Tyler and Morari (1995), and their conclusion was that MV performance assessment of this kind of plant depends on knowing only the plant delay and the position of the unstable zeroes of the plant. Unfortunately this conclusion is incorrect since almost all the plant model has to be known. The current section shows the details of this latter conclusion.

For plants described by the ARMAX model (4.2) with no unstable zeroes, the

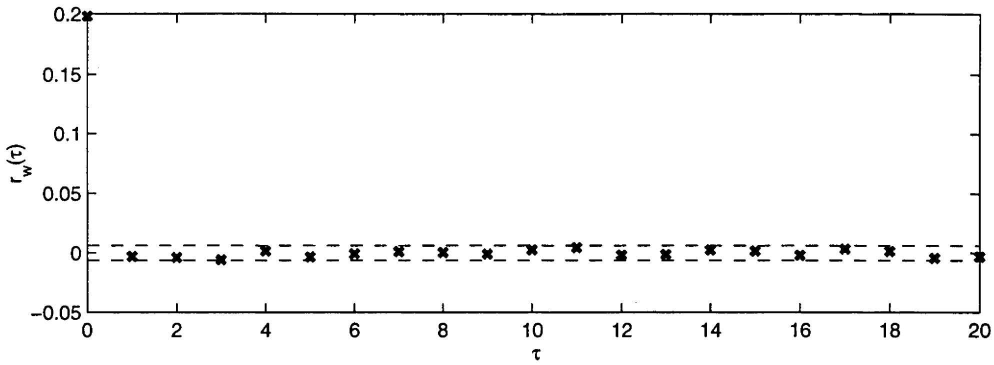

Figure 4.5: Covariance function of w_t under the LQ controller

MV control law is

$$\begin{aligned} u_t &= -\frac{M^*(q^{-1})}{B^*(q^{-1})F^*(q^{-1})}y_t \\ &= -\frac{M(q)}{B(q)F(q)}y_t, \end{aligned}$$

where $F(q)$ and $M(q)$ are polynomials obtained from

$$q^{d-1}C(q) = A(q)F(q) + M(q)$$

with $d = \deg A - \deg B > 0$, $\deg F = d - 1$ and $\deg M < \deg A = n$. Under this control action the output of the plant becomes

$$\begin{aligned} y_t &= F^*(q^{-1})e_t \\ &= \frac{F(q)}{q^{d-1}}e_t. \end{aligned} \tag{4.9}$$

That is, $\{y_t\}$ is a moving-average process of order $d-1$. It is important to emphasise that all d elements of the MV-system impulse response are actually invariant to the control law.

These powerful features, relevant for MV performance monitoring and assessment, are absent when the plant contains at least one unstable zero. Minimum variance closed-loop systems with this kind of plant have pole(s) away from the origin, therefore their closed-loop impulse responses are infinite in their extent—as they also are under LQ optimality.

The MV control law of plants with unstable zeroes is

$$\begin{aligned} u_t &= -\frac{D^*(q^{-1})}{B_+{}^*(q^{-1})E^*(q^{-1})}y_t \\ &= -\frac{D(q)}{B_+(q)E(q)}y_t, \end{aligned}$$

where $E(q)$ and $D(q)$ are obtained from

$$q^{d-1}C(q)B_-{}^*(q) = A(q)E(q) + B_-(q)D(q)$$

with $\deg E = d + \deg B_- - 1 = \deg F + \deg B_-$ and $\deg D < \deg A = n$ (Åström and Wittenmark 1990). The output of the plant then becomes

$$\begin{aligned} y_t &= \frac{E^*(q^{-1})}{B_-(q^{-1})} e_t \\ &= \frac{E(q)}{q^{d-1} B_-{}^*(q)} e_t. \end{aligned} \tag{4.10}$$

The part of the impulse response of (4.10) that is invariant to the control law can be isolated:

$$\begin{aligned} y_t &= \frac{C(q)}{A(q)} e_t + \frac{B_-(q)D(q)}{A(q)} e_t \\ &= \frac{F(q)}{q^{d-1}} e_t + \frac{B_-{}^*(q)M(q) - B_-(q)D(q)}{q^{d-1}A(q)B_-{}^*(q)} e_t. \end{aligned} \tag{4.11}$$

From the analysis of

$$\frac{E(q)}{B_-{}^*(q)} = F(q) + \frac{B_-{}^*(q)M(q) - B_-(q)D(q)}{A(q)B_-{}^*(q)}$$

it can be concluded that

$$B_-{}^*(q)M(q) - B_-(q)D(q) = L(q)A(q), \tag{4.12}$$

where

$$L(q) = E(q) - F(q)B_-{}^*(q)$$

with $\deg L = \deg B_-$. The substitution of (4.12) into (4.11) results

$$y_t = \frac{F(q)}{q^{d-1}} e_t + \frac{L(q)}{q^{d-1}B_-{}^*(q)} e_t. \tag{4.13}$$

This last expression highlights three features of the impulse response of the system:

- The initial d terms are invariant to the control law.
- The terms from $d + 1$ up to $d + \deg B_-$ depend on $L(q)$ and $B_-(q)$, where $L(q)$ depends on $A(q)$ and $C(q)$, among others.
- The following terms are determined from $B_-(q)$ and previous terms of the impulse response.

If it were not for the terms from $d + 1$ to $d + \deg B_-$, MV performance monitoring and assessment of nonminimum-phase plants would just require the extra knowledge of $B_-(q)$, as it was mistakenly claimed in (Tyler and Morari 1995). Unfortunately the only information about the model that is not needed for MV performance assessment of nonminimum-phase plants is $B_+(q)$, i.e. the stable zeroes of the plant.

One can think of implementing a performance monitoring mechanism for nonminimum-phase plants akin to that presented in Section 4.2.3. This is feasible, but the extra knowledge of $B_-(q)$ is not enough for that because $P(z) = B_-{}^*(z)B_+(z)$. Unless all zeroes of the plant are known, the mechanism for MV performance monitoring of nonminimum-phase plants has to be preceded by an experiment for estimation of $P(z)$.

4.4 Conclusion

The weakest aspect of the mechanisms introduced in Chapter 3 is the requirement that the noise dynamics, in the output feedback case, are known. This weakness is not present within the mechanism presented in this chapter, but it comes at the cost of increasing the scheme complexity. From excited closed-loop operation one builds spectral densities containing the optimal closed-loop poles, which are then estimated via algorithms of signal reconstruction.

The complexity of this procedure rivals that of identification and LQ(G) design. The significance of the model-free approach is that its results might extend to yield qualitatively valuable extensions of SPC methods in far wider circumstances where the efficiency of system identification is unknown.

Estimates of the optimal closed-loop poles can be used for several purposes, including adaptive pole-placement algorithms. These algorithms have the limitation that there is no explicit trade-off between control effort and bandwidth of the closed-loop system (Trulsson and Ljung 1985). In particular, the mechanisms for direct adaptive pole-placement need a sensible target to aim for, but there is no estimation of a plant model on which to base the computation of a 'desired' loop performance, in terms of closed-loop poles. The LQ optimal characteristic polynomial provides a sensible choice of poles, with an explicit trade-off between process output and control variances, and guaranteed robustness.

It is expected that the estimates of the optimal closed-loop poles can lead to the development of a mechanism of absolute performance assessment under an LQ criterion, but in this chapter the focus is on performance monitoring. After the initial experiment is performed and the estimates are obtained, there is no need for further disturbances to the normal operation of the loop. This is especially important for monitoring loop performance during controller tuning. In the long term, though, those estimates tend to lose accuracy as changes in the plant dynamics occur with time.

The amount of information required for performance monitoring is also analysed in the particular case of minimum variance control of nonminimum-phase plants. The conclusion is that it is not significantly lower than in the generic LQ problem.

All statements made so far are with respect to the optimal performance under any controller structure, which might not always be the desired analysis since the vast majority of industrial controllers have a fixed structure and few parameters. The next chapter addresses the problem of assessment of local performance optimality, where the focus turns to the computation of gradients of the cost function with respect to the parameters of the controller.

Chapter 5

Assessment of Local Optimality

5.1 Introduction

Instead of proceeding with the investigation of properties associated with absolute optimality of the closed-loop performance, let us now look at the linear quadratic cost function as a functional on the space of controller parameters. This new perspective reveals geometric properties that are intrinsically associated with the condition of local optimality, namely null gradient vector and positive definite Hessian matrix. Moreover, these properties are also valid for assessment of local optimality under reduced complexity controllers.

Another benefit of using functional analysis is that frequency-weights can be introduced into the linear quadratic cost function without increasing the complexity of the development. Hence, the cost function becomes

$$J^{lq}(\rho) = \lim_{N_t \to \infty} \frac{1}{N_t} \sum_{k=1}^{N_t} \left\{ [F_y(q)\, y_k(\rho)]^2 + \lambda\, [F_u(q)\, u_k(\rho)]^2 \right\}, \tag{5.1}$$

where the linear filters $F_y(q)$ and $F_u(q)$ introduce frequency-dependent weights on the variance of $\{y_t\}$ and $\{u_t\}$, respectively. The vector ρ contains the parameters of the controller, therefore the signals $\{y_t(\rho)\}$ and $\{u_t(\rho)\}$ are obtained with the loop closed under the particular controller parametrised by

$$\rho = \begin{bmatrix} \rho_1 & \rho_2 & \dots & \rho_p \end{bmatrix}.$$

Derivatives of (5.1) with respect to the controller parameters can be obtained in both time and frequency domains using special closed-loop signals and full knowledge of the control law, hence there is no explicit need for modelling the plant. Nevertheless, in order to reach these mathematical results the following linear model of the plant is used:

$$y_t = G(q)\, u_t + H(q)\, e_t, \tag{5.2}$$

where $H(q)$ is a stable and stably invertible filter and $\{e_t\}$ is a zero-mean white-noise process of variance σ_e^2.

The structure of the controller is assumed to have two degrees of freedom, despite only the regulation problem being addressed in this thesis. The control action is given by

$$u_t = -C_{fb}(q,\rho)\, y_t + C_{ff}(q,\rho)\, r_t. \tag{5.3}$$

This generic structure also encompasses usual implementations of the widely used PID controllers.

Within this basic framework, the following sections present the mechanisms for estimation of the derivatives of $J^{lq}(\rho)$ with respect to the parameters of the controller, initially in the frequency domain and later in the time domain. These derivatives are the core elements for assessing the condition of local optimality of the closed-loop performance.

5.2 Frequency-Domain Analysis

By means of Parseval's Theorem the cost function can be expressed in the frequency domain, thus allowing the computation of its derivatives with respect to the controller parameters also in the frequency domain. The cost function is transformed into

$$J^{lq}(\rho) = \lim_{N_\omega \to \infty} \frac{1}{N_\omega} \sum_{k=0}^{N_\omega - 1} \left[|F_y(\mathrm{e}^{i\omega_k})|^2 \phi_y(\omega_k,\rho) + \lambda |F_u(\mathrm{e}^{i\omega_k})|^2 \phi_u(\omega_k,\rho) \right], \tag{5.4}$$

where $\omega_k = 2\pi\, k/N_\omega$; $\phi_y(\omega,\rho)$ and $\phi_u(\omega,\rho)$ are the power spectral densities of the signals $\{y_t\}$ and $\{u_t\}$, respectively, obtained with the loop closed under the particular controller $\{C_{fb}(q,\rho), C_{ff}(q,\rho)\}$.

The reference signal, r_t, is assumed to be kept identically zero during normal operation, therefore spectral analysis of the output signal results in

$$\phi_y(\omega,\rho) = \left| \frac{H(\mathrm{e}^{i\omega})}{1 + G(\mathrm{e}^{i\omega})\, C_{fb}(\mathrm{e}^{i\omega},\rho)} \right|^2 \sigma_e^2, \tag{5.5}$$

and of the plant input results in

$$\phi_u(\omega,\rho) = \left| \frac{H(\mathrm{e}^{i\omega})\, C_{fb}(\mathrm{e}^{i\omega},\rho)}{1 + G(\mathrm{e}^{i\omega})\, C_{fb}(\mathrm{e}^{i\omega},\rho)} \right|^2 \sigma_e^2, \tag{5.6}$$

provided the closed-loop system is stable.

The frequency-domain characteristics of the system under stationary excitation through the reference signal are also needed. The cross-spectral density between $\{y_t\}$ and $\{r_t\}$ is given by

$$\phi_{yr}(\omega,\rho) = \frac{G(\mathrm{e}^{i\omega})\, C_{ff}(\mathrm{e}^{i\omega},\rho)}{1 + G(\mathrm{e}^{i\omega})\, C_{fb}(\mathrm{e}^{i\omega},\rho)} \phi_r(\omega), \tag{5.7}$$

and the cross-spectral density between $\{u_t\}$ and $\{r_t\}$ is

$$\phi_{ur}(\omega,\rho) = \frac{C_{ff}(\mathrm{e}^{i\omega},\rho)}{1 + G(\mathrm{e}^{i\omega})\, C_{fb}(\mathrm{e}^{i\omega},\rho)} \phi_r(\omega). \tag{5.8}$$

With this set of equations it is possible to proceed with the computation of the derivatives of $J^{lq}(\rho)$ with respect to the controller parameters. Equally, it would be possible to proceed directly to the identification of a plant model, but this concept is proscribed here for reasons outlined earlier.

5.2.1 Derivatives of the Cost Function

For the sake of clarity, the cost function is expressed in terms of auxiliary functions.

$$J^{lq}(\rho) = \lim_{N_\omega \to \infty} \frac{1}{N_\omega} \sum_\omega \left[g(\omega,\rho)\, h(\omega,\rho) \right]$$

$$g(\omega,\rho) = |F_y(\mathrm{e}^{i\omega})|^2 + \lambda\, |F_u(\mathrm{e}^{i\omega})|^2\, |C_{fb}(\mathrm{e}^{i\omega},\rho)|^2$$

$$h(\omega,\rho) = \frac{|H(\mathrm{e}^{i\omega})|^2}{|1+G(\mathrm{e}^{i\omega})\, C_{fb}(\mathrm{e}^{i\omega},\rho)|^2} \sigma_e^2 \qquad \Big(= \phi_y(\omega,\rho) \Big).$$

The function arguments were presented above so as to emphasise variable dependencies. In the sequel, the notation is shortened by dropping these arguments.

The derivative of $J^{lq}(\rho)$ with respect to a parameter of the controller is

$$\frac{\partial J^{lq}}{\partial \rho_l} = \lim_{N_\omega \to \infty} \frac{1}{N_\omega} \sum_\omega \left[\frac{\partial g}{\partial \rho_l} h + g \frac{\partial h}{\partial \rho_l} \right]$$

$$\frac{\partial g}{\partial \rho_l} = \lambda\, |F_u|^2 \left(C_{fb} \frac{\partial C_{fb}^*}{\partial \rho_l} + \frac{\partial C_{fb}}{\partial \rho_l} C_{fb}^* \right) = 2\,\lambda\, |F_u|^2\, \mathfrak{Re}\left(C_{fb}^* \frac{\partial C_{fb}}{\partial \rho_l} \right)$$

$$\frac{\partial h}{\partial \rho_l} = -2\,\phi_y\, \mathfrak{Re}\left(\frac{G}{1+G\, C_{fb}} \frac{\partial C_{fb}}{\partial \rho_l} \right),$$

where C_{fb}^* is the complex conjugate of C_{fb} and $\mathfrak{Re}(\cdot)$ is the function that returns the real part of its argument. Also the second derivative of the cost function can be computed from the initial signals.

$$\frac{\partial^2 J^{lq}}{\partial \rho_l\, \partial \rho_k} = \lim_{N_\omega \to \infty} \frac{1}{N_\omega} \sum_\omega \left[\frac{\partial^2 g}{\partial \rho_l\, \partial \rho_k} h + \frac{\partial g}{\partial \rho_l} \frac{\partial h}{\partial \rho_k} + \frac{\partial g}{\partial \rho_k} \frac{\partial h}{\partial \rho_l} + g \frac{\partial^2 h}{\partial \rho_l\, \partial \rho_k} \right]$$

$$\frac{\partial^2 g}{\partial \rho_l\, \partial \rho_k} = 2\,\lambda\, |F_u|^2\, \mathfrak{Re}\left(\frac{\partial C_{fb}}{\partial \rho_l} \frac{\partial C_{fb}^*}{\partial \rho_k} + C_{fb}^* \frac{\partial^2 C_{fb}}{\partial \rho_l\, \partial \rho_k} \right)$$

$$\frac{\partial^2 h}{\partial \rho_l\, \partial \rho_k} = 2\,\phi_y\, \mathfrak{Re}\left[\frac{G}{1+G\, C_{fb}} \left(2 \frac{G}{1+G\, C_{fb}} \frac{\partial C_{fb}}{\partial \rho_l} \frac{\partial C_{fb}}{\partial \rho_k} - \frac{\partial^2 C_{fb}}{\partial \rho_l\, \partial \rho_k} \right) + \left| \frac{G}{1+G\, C_{fb}} \right|^2 \frac{\partial C_{fb}}{\partial \rho_l} \frac{\partial C_{fb}^*}{\partial \rho_k} \right].$$

By observing the expressions for first and second derivatives of the cost function, one can verify that almost all terms are known or obtainable from normal operating data. The only term remaining can be obtained via an intrusive experiment performed on the loop by injecting a stationary non-degenerate signal $\{r_t\}$. Actually, from (5.7) it is straightforward to compute

$$\frac{G(e^{i\omega})}{1+G(e^{i\omega})\,C_{fb}(e^{i\omega},\rho)} = \frac{\phi_{yr}(\omega,\rho)}{C_{ff}(e^{i\omega},\rho)\,\phi_r(\omega)}. \tag{5.9}$$

These first and second derivatives of $J^{lq}(\rho)$ compose its gradient (*row*) vector,

$$\nabla J = \begin{bmatrix} \frac{\partial J^{lq}}{\partial \rho_1} & \frac{\partial J^{lq}}{\partial \rho_2} & \cdots & \frac{\partial J^{lq}}{\partial \rho_p} \end{bmatrix}, \tag{5.10}$$

and its Hessian matrix,

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 J^{lq}}{\partial \rho_1 \partial \rho_1} & \frac{\partial^2 J^{lq}}{\partial \rho_1 \partial \rho_2} & \cdots & \frac{\partial^2 J^{lq}}{\partial \rho_1 \partial \rho_p} \\ \frac{\partial^2 J^{lq}}{\partial \rho_2 \partial \rho_1} & \frac{\partial^2 J^{lq}}{\partial \rho_2 \partial \rho_2} & \cdots & \frac{\partial^2 J^{lq}}{\partial \rho_2 \partial \rho_p} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 J^{lq}}{\partial \rho_p \partial \rho_1} & \frac{\partial^2 J^{lq}}{\partial \rho_p \partial \rho_2} & \cdots & \frac{\partial^2 J^{lq}}{\partial \rho_p \partial \rho_p} \end{bmatrix}, \tag{5.11}$$

where p is the number of parameters in the controller.

5.2.2 Detection of Local Optimality

The quantities ∇J and $\mathbf{H}$ are the basis of the mechanism for assessment of local optimality: an operating point given by ρ has locally optimal performance, according to (5.1), if $\nabla J = 0$ and $\mathbf{H}$ is positive definite. The converse statement is conditioned on the absence of pathological characteristics in $J^{lq}(\rho)$, one of which occurs when the complexity of the controller is higher than necessary.

A finite set of data is used in practice to estimate the index $J^{lq}(\rho)$ given by (5.1). The same is true about the use of (5.4) and its derivatives; that is, a finite set of frequencies composes the estimated spectral functions. Obviously the overall quality of the mechanism for assessment of local optimality depends on the quality of the estimated spectral functions. In practical terms, the verification of ∇J being small becomes relative to the 'magnitude' of $\mathbf{H}$. Fairly robust results are obtained through comparison of the current LQ cost with the minimum of the convex approximation generated by ∇J and $\mathbf{H}$, as in

$$\Delta J^{lq}{}_{\%} = \frac{0.5\,\nabla J\,\mathbf{H}^{-1}\,\nabla J}{J^{lq}(\rho)} \cdot 100\%, \tag{5.12}$$

provided $\mathbf{H}$ is positive definite.

Example 5.1. Consider the process in Examples 4.1 and 4.2:

$$y_t = \frac{-0.8q^2 + 1.68q - 0.85}{q^3 - 2.7q^2 + 2.11q - 0.697}\,u_t + \frac{q^3 - 1.1q^2 + 0.24q}{q^3 - 2.7q^2 + 2.11q - 0.697}\,e_t,$$

with $\sigma_e^2 = 0.2$. In those examples two different controllers were tested for loop optimality: the minimum variance controller and the linear quadratic controller for $\lambda = 0.1$. Given that the criterion used for testing these controllers was J^{lq} with $\lambda = 0.1$, the MV controller was detected not to be optimal, but the optimality of the LQ controller could not be established without the knowledge that $A(0) \neq 0$.

Here the same signals from Example 4.1 are used for estimating the derivatives of the cost function with respect to the parameters of each controller. Moreover, these derivatives build the mechanism for detection of local optimality according to (5.12).

Under MV control law the estimated Hessian matrix is positive definite and the estimated gradient vector is

$$\nabla J = \begin{bmatrix} -7.819 & -6.524 & -5.394 & 44.70 & 37.22 & 30.52 \end{bmatrix}.$$

The quantity $\Delta J^{lq}\%$ predicts that the current estimated cost function, $J^{lq} = 2.68$, can be reduced by approximately 21.9%.

With the LQ-optimal controller a positive definite estimate of the Hessian matrix is also obtained , as well as the following estimated gradient vector:

$$\nabla J = \begin{bmatrix} -0.0561 & -0.1106 & -0.1046 & -0.9532 & -1.074 & -1.257 \end{bmatrix}.$$

In this case the quantity $\Delta J^{lq}\%$ predicts that the current estimated cost function, $J^{lq} = 1.99$, can be reduced by approximately 0.42%. This controller is definitely optimal, at least locally, according to the accuracy provided by the number of samples collected during the experiment.

5.3 Time-Domain Analysis

From the initial descriptions of the plant and the controller, respectively (5.2) and (5.3), one can derive expressions for the closed-loop signals under normal operating conditions, i.e. $r_t \equiv 0$. These expressions are

$$y_t(\rho) = \frac{H(q)}{1 + G(q)\, C_{fb}(q,\rho)}\, e_t \tag{5.13}$$

and

$$u_t(\rho) = -\frac{H(q)\, C_{fb}(q,\rho)}{1 + G(q)\, C_{fb}(q,\rho)}\, e_t. \tag{5.14}$$

From this point onwards the function arguments are dropped, with the exception of a few cases in which variable dependencies need to be emphasised. Furthermore, the computation of the average over time of these ergodic signals is replaced by the shorter notation of the 'average' operator:

$$\mathcal{AV}[\cdot] = \lim_{N_t \to \infty} \frac{1}{N_t} \sum_t [\cdot].$$

5.3.1 Derivatives of the Cost Function

Straightforward calculations lead to the derivatives of $J^{lq}(\rho)$, in (5.1), with respect to a parameter of the controller.

$$\frac{\partial J^{lq}}{\partial \rho_l} = 2\,\mathcal{AV}\left\{[F_y\, y_t]\left[F_y \frac{\partial y_t}{\partial \rho_l}\right] + \lambda\,[F_u\, u_t]\left[F_u \frac{\partial u_t}{\partial \rho_l}\right]\right\} \tag{5.15a}$$

$$\begin{aligned}\frac{\partial y_t}{\partial \rho_l} &= -\frac{H}{1+G\,C_{fb}}\,\frac{G}{1+G\,C_{fb}}\,\frac{\partial C_{fb}}{\partial \rho_l}\,e_t\\ &= -\frac{G}{1+G\,C_{fb}}\,\frac{\partial C_{fb}}{\partial \rho_l}\,y_t\end{aligned} \tag{5.15b}$$

$$\begin{aligned}\frac{\partial u_t}{\partial \rho_l} &= \left[-\frac{H}{1+G\,C_{fb}} + \frac{H\,C_{fb}}{1+G\,C_{fb}}\,\frac{G}{1+G\,C_{fb}}\right]\frac{\partial C_{fb}}{\partial \rho_l}\,e_t\\ &= -\frac{H}{1+G\,C_{fb}}\,\frac{1}{1+G\,C_{fb}}\,\frac{\partial C_{fb}}{\partial \rho_l}\,e_t\\ &= -\frac{1}{1+G\,C_{fb}}\,\frac{\partial C_{fb}}{\partial \rho_l}\,y_t.\end{aligned} \tag{5.15c}$$

Moreover, the second derivatives are obtainable from similar calculations.

$$\begin{aligned}\frac{\partial^2 J^{lq}}{\partial \rho_l\,\partial \rho_k} = 2\,\mathcal{AV}\Bigg\{&\left[F_y \frac{\partial y_t}{\partial \rho_l}\right]\left[F_y \frac{\partial y_t}{\partial \rho_k}\right] + [F_y\, y_t]\left[F_y \frac{\partial^2 y_t}{\partial \rho_l\,\partial \rho_k}\right]\\ &+ \lambda\left[F_u \frac{\partial u_t}{\partial \rho_l}\right]\left[F_u \frac{\partial u_t}{\partial \rho_k}\right] + \lambda\,[F_u\, u_t]\left[F_u \frac{\partial^2 u_t}{\partial \rho_l\,\partial \rho_k}\right]\Bigg\}\end{aligned} \tag{5.16a}$$

$$\frac{\partial^2 y_t}{\partial \rho_l\,\partial \rho_k} = \frac{G}{1+G\,C_{fb}}\left(2\,\frac{G}{1+G\,C_{fb}}\,\frac{\partial C_{fb}}{\partial \rho_l}\,\frac{\partial C_{fb}}{\partial \rho_k} - \frac{\partial^2 C_{fb}}{\partial \rho_l\,\partial \rho_k}\right) y_t \tag{5.16b}$$

$$\frac{\partial^2 u_t}{\partial \rho_l\,\partial \rho_k} = \frac{1}{1+G\,C_{fb}}\left(2\,\frac{G}{1+G\,C_{fb}}\,\frac{\partial C_{fb}}{\partial \rho_l}\,\frac{\partial C_{fb}}{\partial \rho_k} - \frac{\partial^2 C_{fb}}{\partial \rho_l\,\partial \rho_k}\right) y_t. \tag{5.16c}$$

Observing the expressions of first and second derivatives of the cost function, one can verify that the only terms that are neither known nor obtainable from normal operation are

$$T(q,\rho) \triangleq \frac{G(q)}{1+G(q)\,C_{fb}(q,\rho)} \tag{5.17}$$

and

$$S(q,\rho) \triangleq \frac{1}{1+G(q)\,C_{fb}(q,\rho)}. \tag{5.18}$$

Thus, the problem becomes how to obtain the required information about these two terms that also appear in the expressions of the closed-loop system under excitation:

$$y_t = \frac{H}{1+G\,C_{fb}}\,e_t + C_{ff}\,T\,r_t$$

and

$$u_t = -\frac{H\,C_{fb}}{1 + G\,C_{fb}}\,e_t + C_{ff}\,S\,r_t.$$

A few years ago, Hjalmarsson *et al.* (1994*a*) introduced the idea of using the real system for filtering appropriate signals and obtaining an unbiased estimate of the first derivatives of $J^{lq}(\rho)$. Given that their main goal was to tune the controller parameters, there was no need for obtaining unbiased estimates of the second derivatives. This contrasts with the mechanism for assessment of local optimality, which requires an extension of that method, called iterative feedback tuning (IFT), with two additional signal filterings through the real system. This is further developed in Section 5.3.3.

Another possible solution is to perform a single experiment on the closed loop to fit a high order model into $T(q,\rho)$ and $S(q,\rho)$, as suggested by De Bruyne and Carrette (1997). Based on this latter idea and the results on frequency-domain analysis, it follows that non-parametric models—impulse responses—of the stable filters $T(q,\rho)$ and $S(q,\rho)$ suffice to achieve the desired results. These impulse responses are obtained via correlation analysis of excited signals.

5.3.2 Stable Filterings

There exists the possibility that some filters in the derivative expressions are not stable, for instance $\frac{\partial C_{fb}}{\partial \rho_l}$. If that happens, each unstable filter has to be factorised as

$$F(q) = \frac{N(q)}{D_+(q)\,D_-(q)},$$

and the frequency weights $F_y(q)$ and $F_u(q)$ must include every single filter

$$F_a(q) \triangleq \frac{D_-(q)}{D_-{}^*(q)}.$$

This procedure does not affect the cost function because $F_a(q)$ is an all-pass filter (Hjalmarsson *et al.* 1994*b*). The signal filtering then occurs through the stable transfer function

$$F_a(q)\,F(q) = \frac{N(q)}{D_+(q)\,D_-{}^*(q)}.$$

With the first and second derivatives of $J^{lq}(\rho)$, the gradient vector and the Hessian matrix are formed according to (5.10) and (5.11), respectively. The mechanism for assessment of local optimality remains identical to that presented in Section 5.2.2. That is, an operating point has locally optimal performance if $\nabla J = 0$ and $\mathbf{H}$ is positive definite. A robust way to perform such an analysis is through the computation of $\Delta J^{lq}_{\%}$ introduced in (5.12).

Example 5.2. Consider the system in Example 5.1: an unstable and nonminimum-phase process under MV and LQ control being tested for local optimality. In that example the computation of the derivatives of the cost function were obtained via spectral analysis of closed-loop signals. Here those same signals generate estimates of the closed-loop impulse responses, with each controller, via correlation analysis. The computation of the derivatives of the cost function proceeds according to (5.15) and (5.16).

The closed-loop impulse responses obtained under MV control are shown in Figure 5.1. Under this control law the estimated Hessian matrix is positive definite and the estimated gradient vector is

$$\nabla J = \begin{bmatrix} -6.890 & -6.063 & -5.383 & 38.21 & 34.29 & 30.69 \end{bmatrix}.$$

The quantity $\Delta J^{lq}\%$ predicts that the current estimated cost function, $J^{lq} = 2.68$, can be reduced by approximately 18.8%.

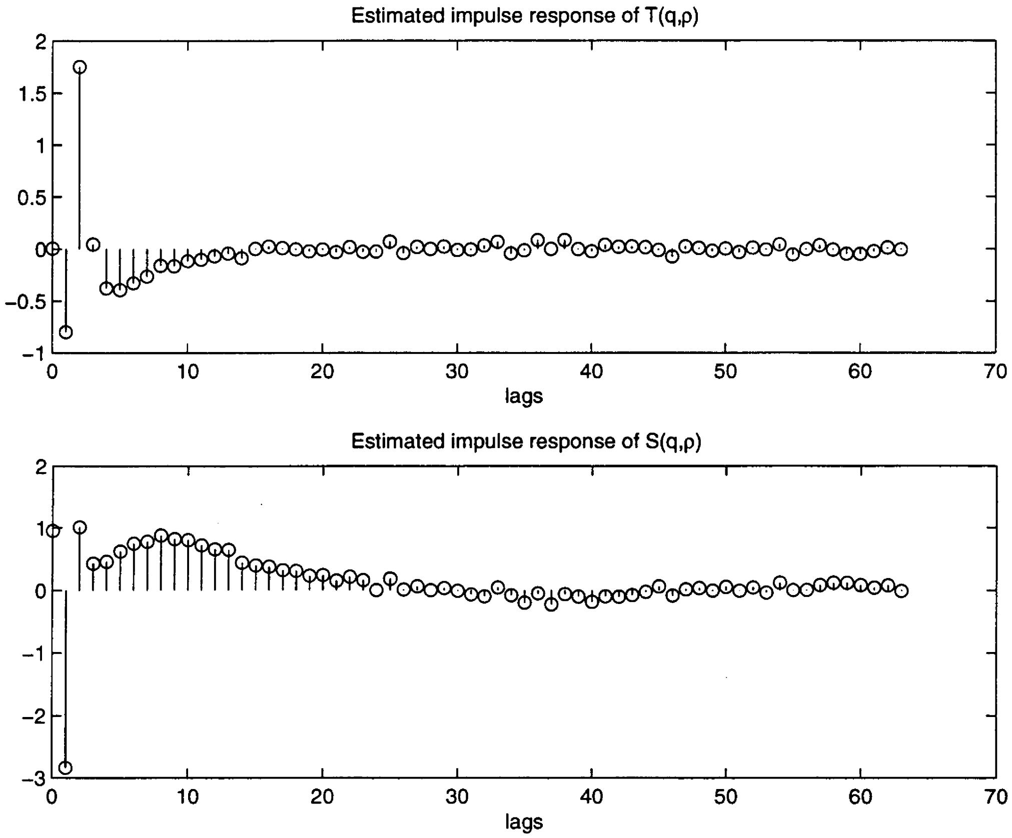

Figure 5.1: Impulse response estimates under the MV controller

The closed-loop impulse responses obtained under the LQ-optimal controller are shown in Figure 5.2. With these impulse responses a positive definite estimate of the Hessian matrix is also obtained , as well as the following estimated gradient vector:

$$\nabla J = \begin{bmatrix} -0.0316 & -0.1058 & -0.1477 & 0.4531 & 0.5830 & 0.6762 \end{bmatrix}.$$

In this case the quantity $\Delta J^{lq}\%$ predicts that the current estimated cost function, $J^{lq} = 1.99$, can be reduced by approximately 0.04%.

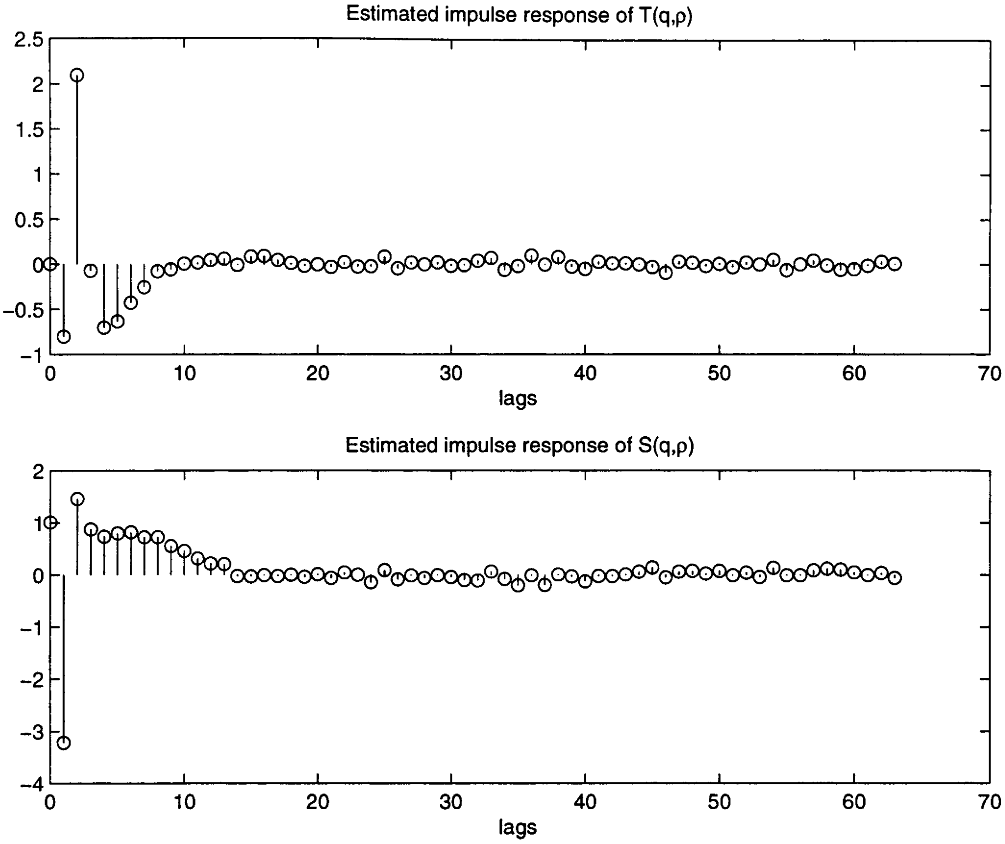

Figure 5.2: Impulse response estimates under the LQ controller

The results in this example are very similar to those obtained from frequency-domain analysis. In this case, ∇J of the LQ-optimal controller is even closer to a null vector.

5.3.3 Extended Iterative Feedback Tuning

The mechanism presented in (Hjalmarsson *et al.* 1994*a*) uses a finite amount of plant-output data, collected during normal operation, as the external excitation in a separate stage. The resultant signals are

$$\begin{aligned} y_t^{(1)} &= \frac{H}{1 + G\,C_{fb}}\, e_t^{(1)} \\ u_t^{(1)} &= -\frac{H\,C_{fb}}{1 + G\,C_{fb}}\, e_t^{(1)} \end{aligned}$$

and

$$\begin{aligned} y_t^{(2)} &= \frac{H}{1 + G\,C_{fb}}\, e_t^{(2)} + C_{ff}\,T\,y_t^{(1)} \\ u_t^{(2)} &= -\frac{H\,C_{fb}}{1 + G\,C_{fb}}\, e_t^{(2)} + C_{ff}\,S\,y_t^{(1)}. \end{aligned}$$

Given that $\{e_t{}^{(1)}\}$ and $\{e_t{}^{(2)}\}$ are uncorrelated, the following expressions provide the reasoning behind the computation of unbiased estimates for the first derivatives of $J^{lq}(\rho)$:

$$\mathcal{AV}\left\{[F_y\, y_t]\left[F_y\frac{\partial y_t}{\partial \rho_l}\right]\right\} = \mathcal{AV}\left\{[F_y\, y_t{}^{(1)}]\left[-F_y\frac{1}{C_{ff}}\frac{\partial C_{fb}}{\partial \rho_l}\, y_t{}^{(2)}\right]\right\} \tag{5.19a}$$

and

$$\mathcal{AV}\left\{[F_u\, u_t]\left[F_u\frac{\partial u_t}{\partial \rho_l}\right]\right\} = \mathcal{AV}\left\{[F_u\, u_t{}^{(1)}]\left[-F_u\frac{1}{C_{ff}}\frac{\partial C_{fb}}{\partial \rho_l}\, u_t{}^{(2)}\right]\right\}. \tag{5.19b}$$

In order to obtain unbiased estimates of the second derivatives of $J^{lq}(\rho)$, another two stages of excitation have to be performed on the real system:

$$\begin{aligned} y_t{}^{(3)} &= \frac{H}{1+G\,C_{fb}}\, e_t{}^{(3)} + C_{ff}\,T\,y_t{}^{(1)} \\ u_t{}^{(3)} &= -\frac{H\,C_{fb}}{1+G\,C_{fb}}\, e_t{}^{(3)} + C_{ff}\,S\,y_t{}^{(1)}, \end{aligned}$$

and

$$\begin{aligned} y_t{}^{(4)} &= \frac{H}{1+G\,C_{fb}}\, e_t{}^{(4)} + C_{ff}\,T\,y_t{}^{(2)} \\ u_t{}^{(4)} &= -\frac{H\,C_{fb}}{1+G\,C_{fb}}\, e_t{}^{(4)} + C_{ff}\,S\,y_t{}^{(2)}. \end{aligned}$$

Given that $\{e_t{}^{(1)}\}$, $\{e_t{}^{(2)}\}$, $\{e_t{}^{(3)}\}$ and $\{e_t{}^{(4)}\}$ are mutually uncorrelated, the following expressions outline the computation of the second-derivative estimates:

$$\mathcal{AV}\left\{\left[F_y\frac{\partial y_t}{\partial \rho_l}\right]\left[F_y\frac{\partial y_t}{\partial \rho_k}\right]\right\} = \mathcal{AV}\left\{\left[F_y\frac{1}{C_{ff}}\frac{\partial C_{fb}}{\partial \rho_l}\, y_t{}^{(2)}\right]\left[F_y\frac{1}{C_{ff}}\frac{\partial C_{fb}}{\partial \rho_k}\, y_t{}^{(3)}\right]\right\}, \tag{5.20a}$$

$$\begin{aligned} &\mathcal{AV}\left\{[F_y\, y_t]\left[F_y\frac{\partial^2 y_t}{\partial \rho_l\,\partial \rho_k}\right]\right\} \\ &= \mathcal{AV}\left\{[F_y\, y_t{}^{(1)}]\left[F_y\left(\frac{2}{C_{ff}^2}\frac{\partial C_{fb}}{\partial \rho_l}\frac{\partial C_{fb}}{\partial \rho_k}\, y_t{}^{(4)} - \frac{1}{C_{ff}}\frac{\partial^2 C_{fb}}{\partial \rho_l\,\partial \rho_k}\, y_t{}^{(3)}\right)\right]\right\}, \end{aligned} \tag{5.20b}$$

$$\mathcal{AV}\left\{\left[F_u\frac{\partial u_t}{\partial \rho_l}\right]\left[F_u\frac{\partial u_t}{\partial \rho_k}\right]\right\} = \mathcal{AV}\left\{\left[F_u\frac{1}{C_{ff}}\frac{\partial C_{fb}}{\partial \rho_l}\, u_t{}^{(2)}\right]\left[F_u\frac{1}{C_{ff}}\frac{\partial C_{fb}}{\partial \rho_k}\, u_t{}^{(3)}\right]\right\} \tag{5.20c}$$

and

$$\begin{aligned} &\mathcal{AV}\left\{[F_u\, u_t]\left[F_u\frac{\partial^2 u_t}{\partial \rho_l\,\partial \rho_k}\right]\right\} \\ &= \mathcal{AV}\left\{[F_u\, u_t{}^{(1)}]\left[F_u\left(\frac{2}{C_{ff}^2}\frac{\partial C_{fb}}{\partial \rho_l}\frac{\partial C_{fb}}{\partial \rho_k}\, u_t{}^{(4)} - \frac{1}{C_{ff}}\frac{\partial^2 C_{fb}}{\partial \rho_l\,\partial \rho_k}\, u_t{}^{(3)}\right)\right]\right\}. \end{aligned} \tag{5.20d}$$

This involved mechanism for obtaining unbiased estimates of the second derivatives is the one preferred by model-free purists. Still there is the need to stabilise the signal filterings via the mechanisms in Section 5.3.2, finally followed by the test of local optimality.

Example 5.3. Consider the process

$$y_t = \frac{1}{q-0.7}\,u_t + \frac{q-0.1}{q-0.9}\,e_t,$$

with $\sigma_e^2 = 0.5$, under the feedback control action

$$u_t = -\frac{1.5q^2 - 0.3q}{q^2 - 0.5q + 0.1}\,y_t.$$

The stable closed-loop system is intended to operate with good performance according to a linear quadratic criterion with $\lambda = 0.8$ and $F_y \equiv F_u \equiv 1$.

In order to apply the test of local optimality, the vector of controller parameters is chosen as

$$\rho = \begin{bmatrix} 1.5 & -0.3 & -0.5 & 0.1 \end{bmatrix},$$

and the feedforward controller of (5.3) is simply a unitary gain.

A simulation of the loop under normal operation ($r_t \equiv 0$), for $N_t = 4096$ time samples, results in $J^{lq}(\rho) = 1.87$. The loop output signal, $\{y_t\}$, of this simulation becomes the reference signal, $\{r_t\}$, in a second and third simulation stages, where each stage lasts 4096 time samples. Finally, the loop output signal of the second stage is used as the reference signal of a fourth stage. With these signals and full knowledge of the control law, one can proceed to the estimation of the gradient vector and the Hessian matrix according to (5.15a), (5.19), (5.16a) and (5.20).

The eigenvalues of the estimated Hessian matrix are $\{20.4, 8.75, 1.59, -0.109\}$, hence this matrix is not positive definite and the feedback control law is not optimal for the specified criterion. Moreover, the second order approximation of the functional $J^{lq}(\rho)$, at the current operating point, contains a saddle point with anticipated cost $J^{lq} = 1.59$. This indicates that the minimal cost is very likely to be less than 1.59. As a matter of fact, the optimal control law results in $J^{lq} = 0.726$.

Example 5.4. Consider the same system and criterion from the previous example. Now, only the two initial stages of simulation are performed on the loop, therefore an unbiased estimate of the gradient vector can be directly computed from these signals, according to (5.15a) and (5.19).

In order to proceed with the computation of the Hessian matrix, according to (5.16), the signals of the second stage are used for fitting impulse responses into $T(q,\rho)$ and $S(q,\rho)$, given by (5.17) and (5.18), respectively. These estimated impulse responses, shown in Figure 5.3, each contain $\sqrt{N_t} = 64$ lags.

The eigenvalues of the estimated Hessian matrix are $\{21.7, 9.07, 1.69, -0.0824\}$, which are fairly close to the true values $\{20.3, 9.46, 1.35, -0.0964\}$. Given that the estimated Hessian is not positive definite, the feedback control law is not optimal for the specified criterion.

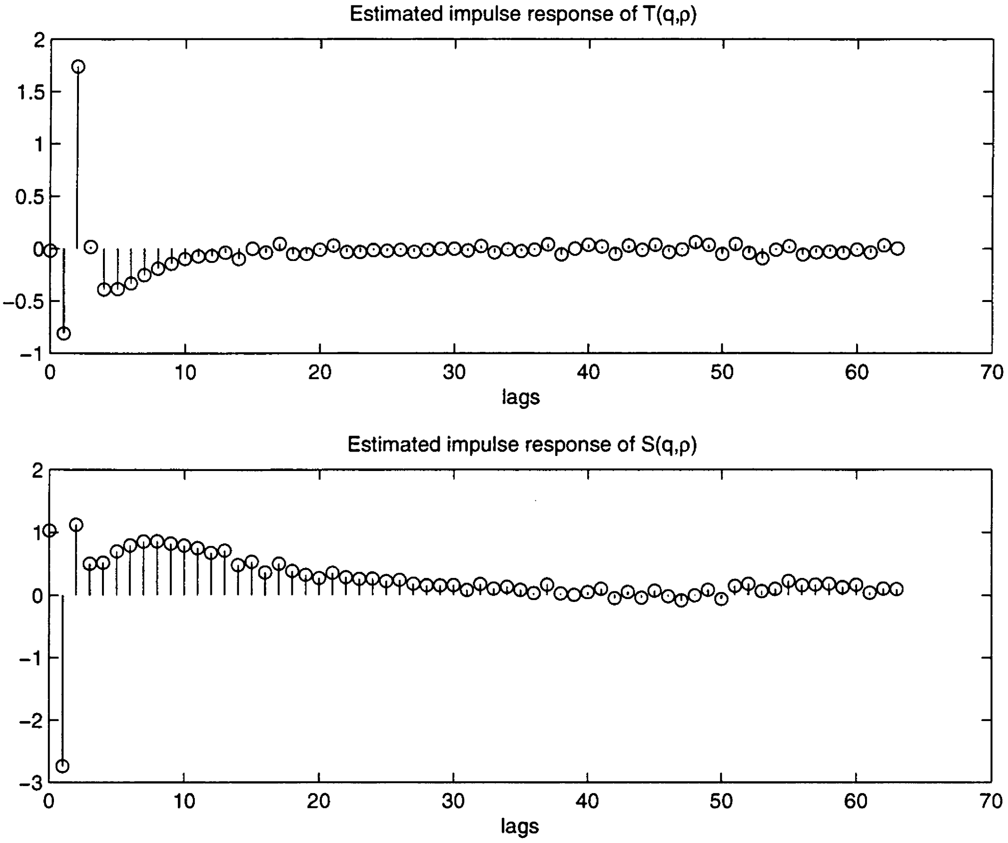

Figure 5.3: Impulse response estimates

5.4 Conclusion

Throughout the current chapter the linear quadratic cost function is viewed as a functional on the space of controller parameters. This perspective reveals geometric properties associated with the condition of local optimality, inclusive of systems under reduced complexity controllers. The main features of interest are the first and second derivatives of the cost function with respect to the controller parameters, which are derived in both frequency and time domains.

The first and second derivatives of the cost function are then used for constructing the gradient vector and the Hessian matrix, respectively. Under locally optimal performance the gradient vector is null and the Hessian matrix is positive definite, but more robust assessment can be obtained through comparison of the current LQ cost with the minimum of the (convex) second order approximation generated by the gradient vector and the Hessian matrix.

An attempt to extend the frequency-domain analysis into multivariable systems:

$$
\begin{aligned}
y_{1_t} &= G_{11}\, u_{1_t} + G_{12}\, u_{2_t} + H_1\, e_{1_t} \\
y_{2_t} &= G_{21}\, u_{1_t} + G_{22}\, u_{2_t} + H_2\, e_{2_t}
\end{aligned}
$$

under the control law

$$u_{1_t} = -C_{11}\, y_{1_t} - C_{12}\, y_{2_t} + r_{1_t}$$
$$u_{2_t} = -C_{21}\, y_{1_t} - C_{22}\, y_{2_t} + r_{2_t},$$

failed to deliver a simple mechanism for estimation of derivatives. Some expressions for the first derivative are of the form

$$\frac{\partial \phi_{y_2}}{\partial \rho_{11}} = -2\,|\phi_{y_2e_1}|^2 \sigma_{e_1}^2\, \Re\mathfrak{e}\left(\frac{\phi_{y_1e_1}}{\phi_{y_2e_1}} \frac{\phi_{y_2r_1}}{\phi_{r_1}} \frac{\partial C_{11}}{\partial \rho_{11}} \right) - 2\,|\phi_{y_2e_2}|^2 \sigma_{e_2}^2\, \Re\mathfrak{e}\left(\frac{\phi_{y_1e_2}}{\phi_{y_2e_2}} \frac{\phi_{y_2r_1}}{\phi_{r_1}} \frac{\partial C_{11}}{\partial \rho_{11}} \right),$$

which would require a great deal of information about the open-loop noise dynamics. The reason for this problem is the existence of cross-couplings in the observable signals. Particular plants with a single source of noise (e.g. $e_{2_t} \equiv 0$) lead to a more tractable problem, but in the generic case it is difficult to justify all the complexity associated with obtaining these derivatives directly from data, which might be even higher than the complexity of identifying the plant from closed-loop signals.

Ideally the mechanisms for performance assessment or monitoring should also provide means for improving the system performance. In Chapter 4 the optimal closed-loop poles used for monitoring LQ performance were found to be applicable in algorithms for direct adaptive pole-placement. Similarly, the first and second derivatives of the cost function can be applied in the computation of appropriate directions of controller tuning. The next chapter deals with this topic and additional features of a mechanism for controller tuning with guaranteed stability.

Chapter 6

Controller Tuning

6.1 Introduction

Performance improvement of, initially stable, closed-loop systems is one of the main tasks entrusted to control engineers. Although adaptive controllers could be used to perform this task, they introduce extraneous nonlinearities into the system's behaviour (Mareels and Bitmead 1986). These nonlinear dynamics are problematic because the control law varies at each sampling time depending on recent signals. A safer way to deal with the task of tuning controllers for time-invariant processes is to apply techniques of iterative control, which are like block-wise slow adaptive control techniques. The basic idea is to observe the closed-loop signals, under a fixed controller, for a longer period of time (after the subsidence of initial conditions) and then to decide on a new control law. This guarantees block-wise time-invariance of the closed-loop dynamics.

All iterative methods currently available for the accomplishment of the tuning task start with an experiment performed on the loop. An external excitation is injected into the closed-loop system and two possible ways might be followed: modelling of the plant followed by control design, or direct computation of performance derivatives followed by controller adjustment. This is similar to the distinction between 'indirect' and 'direct' adaptive control. The underlying reasoning for choosing one approach or another includes the degree of confidence the user has in the information obtained from the experiment.

The option of modelling the plant is usually followed by the design of a control law, as in (Zang *et al.* 1991, Schrama 1992, Lee *et al.* 1993), and this reflects a high degree of confidence in that model. In most of these approaches the structure of the controller is not assumed to be constrained, and the solution might be of high order. When the complexity of the control law is restricted, as for example in the widely used PID controllers, methods based on derivatives of the performance with respect to the controller parameters become more appropriate, thus avoiding plant or controller model reduction. An important characteristic of this concept is that the plant model is used to describe the system's behaviour around the current operating point, and this model is constantly put under test on the real system as the controller parameters are changed. The idea of modelling the plant to compute derivatives and tuning directions has been explored in the context of adaptive control (Trulsson and Ljung 1985), but, to the author's knowledge, it has only been considered briefly in iterative control (Hjalmarsson *et al.* 1994*b*).

There are, however, serious difficulties associated with performing parametric plant modelling: structure selection, parameter estimation, model validation, etc. Additionally, with plant modelling followed by computation of performance derivatives with respect to controller parameters, the complexity of these expressions could be quite high due to the dimension of the plant model, and could be very sensitively dependent on plant and controller parametrisations.

As presented in Chapter 5, the derivatives of the cost function are obtainable directly from filtered data and non-parametric models of the closed-loop system. These mechanisms are advantageous alternatives to the computation of a parametric model of the plant. The basic idea is to use those derivatives to compute a direction for adjusting the controller parameters; once the best controller is found in that direction, another experiment has to be performed in order to obtain a new direction of descending cost. This reflects a complete non-reliance on explicit plant modelling.

The current chapter presents a method for computing tuning directions based on unbiased estimates of first and second derivatives of the following frequency-weighted linear quadratic cost function:

$$J^{lq}(\rho) = \lim_{N_t \to \infty} \frac{1}{N_t} \sum_{k=1}^{N_t} \left\{ [F_y(q)\, y_k(\rho)]^2 + \lambda\, [F_u(q)\, u_k(\rho)]^2 \right\}. \tag{6.1}$$

This tuning procedure guarantees closed-loop stability due to the insertion of a stability margin that limits drastic changes in the controller parameters.

The computation of the stability margin requires estimates of the closed-loop dynamics, hence the emphasis of this chapter is on those mechanisms for estimation of derivatives based on non-parametric models of the closed-loop system. These non-parametric models can be impulse responses or cross-spectral densities, but since they convey the same information, the latter ones are adopted throughout this chapter under a generic notation:

$$T(\rho) \triangleq \frac{G}{1 + G\, C_{fb}(\rho)} \tag{6.2}$$

and

$$S(\rho) \triangleq \frac{1}{1 + G\, C_{fb}(\rho)}. \tag{6.3}$$

Notice that from $T(\rho)$ and $S(\rho)$ it is straightforward to estimate a non-parametric model for the open-loop plant dynamics. This non-parametric model of the plant allows immediate computation of a new tuning direction after the previous search has reached a minimum. This idea is further explored in Section 6.4.

6.2 Tuning Directions

Unlike adaptive control, where new directions have to be computed in between adjacent samples, the computation of tuning directions in iterative control is performed block-wise and, therefore, is allowed to be very elaborate. Usually these computations are not performed frequently and they are not seriously limited by time constraints. These circumstances favour the use of Newton's method, which is accurate but complicated due to a matrix inversion.

Newton's method uses the gradient vector, $\boldsymbol{\nabla} J$, and the Hessian matrix, $\mathbf{H}$, to approximate the local behaviour of the cost function around the current operating point. According to this approximation, the following change in the controller parameters would lead to an operating point with all first derivatives of the cost function identically zero:

$$\Delta\rho = -\mathbf{H}^{-1}\,\boldsymbol{\nabla} J^{T}. \tag{6.4a}$$

That operating point, in the approximation, can be a minimum, a saddle or a maximum, depending on whether the Hessian matrix is positive definite, indefinite or negative definite, respectively. In fact, only for the first of these situations is $\Delta\rho$ an appropriate direction of tuning.

A good alternative to Newton's method, when it fails to deliver a direction that leads towards minimum cost, is the combination of the method of Steepest Descent and a *negative curvature* descent direction (Fletcher 1987, page 49). In its original formulation, the method of Steepest Descent results in the gradient vector being the direction of parameter tuning. This method does not provide an estimate of the order of magnitude of the change in the controller parameters to find a point of minimum cost. Given that the Hessian matrix is available, the following improvement to the original method results in a $\Delta\rho$ that corresponds to the minimum approximated cost function in the direction of the gradient vector:

$$\Delta\rho = -\gamma\,\boldsymbol{\nabla} J^{T}, \tag{6.4b}$$
$$\gamma = \frac{\boldsymbol{\nabla} J\,\boldsymbol{\nabla} J^{T}}{\boldsymbol{\nabla} J\,\mathbf{H}\,\boldsymbol{\nabla} J^{T}},$$

as long as γ is positive. This improvement has its basis in the analysis of the original method on a purely-quadratic problem.

Under some circumstances, the Steepest Descent direction alone makes the tuning scheme extremely slow in terms of performance improvement. In such cases it is worth alternating those directions with directions of negative curvature. The author has experienced very good results by using the direction of the eigenvector associated with the smallest (most negative) eigenvalue of the Hessian matrix. This is shown in Figure 6.1: the crosses on the surface correspond to an initial tuning along the Steepest Descent direction, followed by the circles in the direction of the eigenvector associated with the negative eigenvalue. The directions of the eigenvectors of $\mathbf{H}$ are plotted as thick dashed lines on the contour plot.

In the unfortunate situation of γ being negative (indicating a total lack of local convexity), the original method of Steepest Descent is used instead. That is,

$$\Delta\rho = -\boldsymbol{\nabla} J^{T}. \tag{6.4c}$$

In summary, the direction of controller tuning is obtained by one of (6.4a-c), where the choice depends on local conditions.

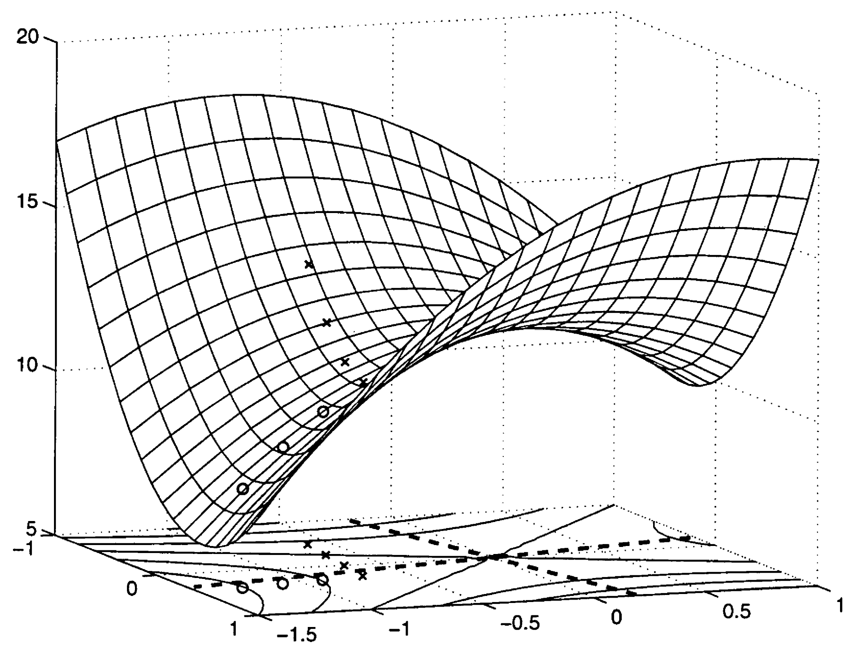

Figure 6.1: Saddle surface

Alternatively one could apply the customary approach of using a positive definite matrix in (6.4a), instead of the Hessian matrix. According to the author's experience, the best choice of positive definite matrix is provided by the following transformation of **H**:

$$\mathbf{H}^{+} = \sum_{k=1}^{n} |\gamma_k| \xi_k \xi_k^T,$$

where γ_k is the k-th eigenvalue of **H** and ξ_k is its respective eigenvector (Greenstadt 1967). If all eigenvalues of the Hessian matrix are non-negative then $\mathbf{H}^{+} \equiv \mathbf{H}$. This approach is simpler but more prone to problems than the combination of Newton's method and Steepest Descent introduced before (see Examples 6.1 and 6.2).

Whichever method is chosen for the computation of the tuning directions, it is not likely that the real system conforms entirely to the quadratic approximation, especially when the operating point is far from a local minimum. In practice the parameters of the controller are changed along the line

$$\rho = \rho_0 + \alpha \, \Delta\rho \tag{6.5}$$

in the search for a point of minimum cost. The vector ρ_0 contains the initial parameters of the controller, and α is varied along $\mathbb{R}^+$.

An initial value for α and its subsequent increments can be chosen on-line, according to several factors:

- The minimum cost function along (6.5) is expected to be at $\alpha \simeq 1$, except when the original method of Steepest Descent is used;

- The anticipated behaviour of the cost function is given by

$$J^{lq}(\rho) \simeq J^{lq}(\rho_0) + \alpha\, \nabla J\, \Delta\rho + \frac{\alpha^2}{2}\, \Delta\rho^T\, \mathbf{H}\, \Delta\rho; \tag{6.6}$$

- Stability of the closed-loop system has to be maintained (see Section 6.3).

Some basic characteristics of the tuning algorithm provide clues for deciding whether the control law is nearly optimal or not. One example is the anticipated reduction of the cost function given by (6.6), at $\alpha = 1$. Actually, this leads to

$$\Delta J^{lq}{}_{\%} = \frac{0.5\, \nabla J\, \mathbf{H}^{-1}\, \nabla J}{J^{lq}(\rho)} \cdot 100\%,$$

the expression adopted in the last chapter for assessing local optimality. Such knowledge is important for stopping the tuning process, thus avoiding unnecessary iterations.

The content of this section is independent of the mechanism employed to obtain the estimates of the gradient vector and Hessian matrix. Therefore the following examples use the true values of these quantities, in a comparison of the proposed method with the alternative transformation of the Hessian matrix into a positive definite matrix.

Example 6.1. Consider the process

$$y_t = \frac{1}{q-0.7}\, u_t + \frac{q-0.1}{q-0.9}\, e_t,$$

with $\sigma_e^2 = 0.5$, under the feedback control action

$$u_t = -\frac{1.5q^2 - 0.3q}{q^2 - 0.5q + 0.1}\, y_t.$$

This stable closed-loop system, also used in Example 5.3, is now tuned with respect to a linear quadratic criterion with $\lambda = 0.8$ and $F_y \equiv F_u \equiv 1$. The initial controller parameters are given by

$$\rho = \begin{bmatrix} 1.5 & -0.3 & -0.5 & 0.1 \end{bmatrix}.$$

The methodology for computation of tuning directions presented in this section is compared with the use of the positive definite matrix $\mathbf{H}^+$ instead of the Hessian matrix. Both methodologies achieve the goal of reducing the cost function to its minimum. Figure 6.2 shows the evolution of the cost function along the tuning iterations. A close look at the path followed by the controller parameters, in Figure 6.3, reveals marked intermediate divergence of some of these parameters in the $\mathbf{H}^+$ methodology. This divergence delays the reaching of the locally convex region surrounding the optimal performance, which occurs at iteration 5.

According to the more elaborate methodology presented in this section, the three initial tuning directions are taken along the Steepest Descent. Due to a very low reduction of the cost function from iteration 2 to iteration 3, the tuning proceeds along the direction of the eigenvector associated with the negative eigenvalue

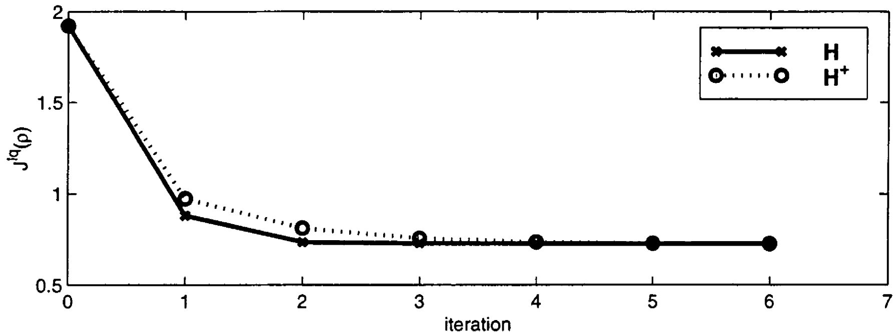

Figure 6.2: Reduction of the cost function

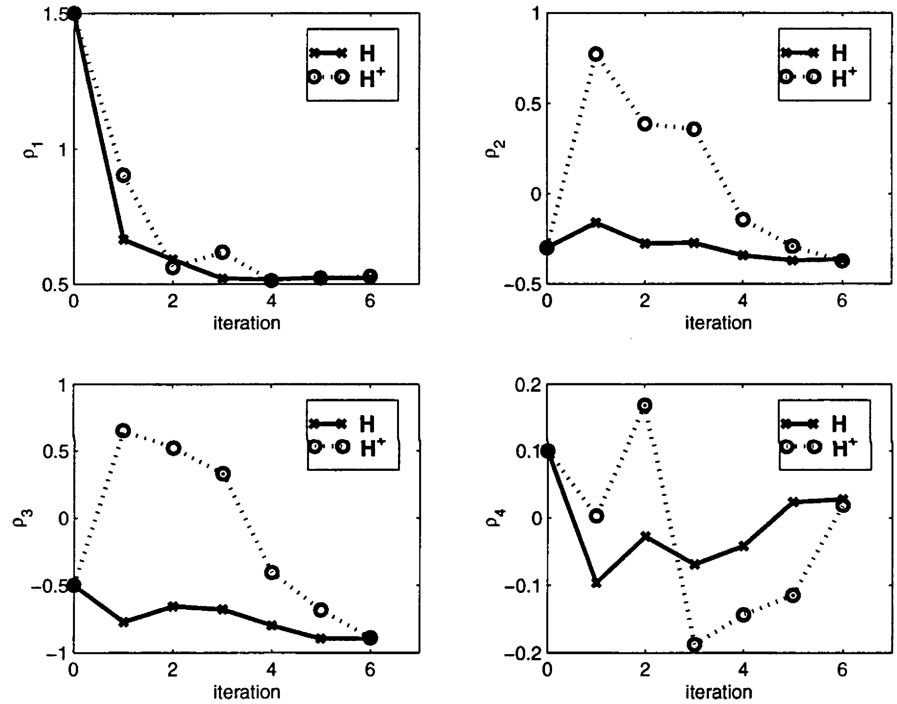

Figure 6.3: Evolution of the controller parameters

of **H**. This leads the system performance into the locally convex region, evidenced by a positive definite Hessian matrix at iteration 4. It is worth emphasising that the direction taken at iteration 3 is obtained from the estimates computed for iteration 2, therefore it does not require new estimates for the derivatives of the cost function.

The next example highlights a problem of closed-loop instability as the controller is tuned. Both methodologies for computation of tuning directions might lead to such a problem, but throughout a significant number of experiments the $\mathbf{H}^+$ approach proved to be remarkably more likely to provide problematic tuning directions than the methodology that combines Newton's method and Steepest Descent. In fact, the author has never found an example where the problem occurs with the combined methodology.

Example 6.2. Consider the process

$$y_t = \frac{0.1(q-0.2)}{(q-0.7)(q-0.5)} u_t + \frac{(q-0.6)(q-0.3)}{(q-0.7)(q-0.5)} e_t,$$

with $\sigma_e^2 = 1$, under the feedback control action

$$u_t = -\frac{0.51q^2 - 0.32q}{q^2 - 1.1q + 0.15} y_t.$$

This stable closed-loop system is tuned according to a linear quadratic criterion with $\lambda = 0.1$ and $F_y \equiv F_u \equiv 1$.

With the true values of ∇J, $\mathbf{H}$ and $J^{lq}(\rho)$, and the tuning direction computed with the $\mathbf{H}^+$ methodology, the line search occurs as depicted in Figure 6.4. The cost function is given by the solid line, while the second order approximation of this function is the dotted line. Observe that if α is taken at intervals of 0.02 units, the closed-loop system suddenly becomes unstable at $\alpha = 0.18$.

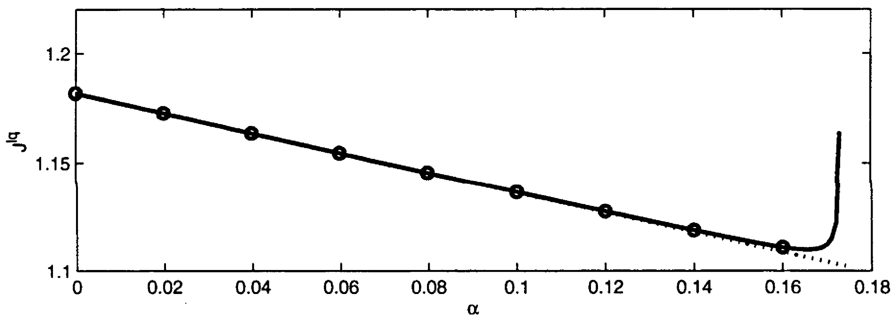

Figure 6.4: Tuning towards instability

The details of the controller parameters along the tuning direction are given in Table 6.1. Despite the complexity of the control law being the same as the absolute optimal one, a pole-zero pair tends to cancel near the unit disc, causing the instability. The absolute optimal performance has $J^{lq} = 1.1094$, but under the control law

$$u_t = -\frac{0.4140q}{q - 0.1685} y_t,$$

$J^{lq}(\rho) = 1.1097$, leaving almost no improvements to be obtained by the complementary pole-zero pair.

Such a problem is accentuated as the complexity of the controller increases above the necessary one. The extra pole(s) and zero(es) might tend to cancel each other near the unit disc, causing the closed-loop system to be dangerously close to instability. This effect had been observed previously by Deistler *et al.* (1978), but in a different context: estimation of ARMA models.

Table 6.1: Controller parameters

α	ρ	Controller poles	Controller zero
0.00	[0.5100 − 0.3200 − 1.100 0.1500]	0.1595, 0.9405	0.6275
⋮	⋮	⋮	⋮
0.16	[0.4140 − 0.4049 − 1.160 0.1671]	0.1685, 0.9917	0.9781
0.18	[0.4020 − 0.4156 − 1.168 0.1692]	0.1695, 0.9982	1.0337

6.3 Stability Margins

The previous example shows that maintenance of closed-loop stability is a strong constraint that should be imposed on the task of tuning. Although it is always possible to step back to the previous controller parameters, occurrences of instability on the way to the optimal control law tend to reduce confidence in the method being used. It is intuitive to think that the cost function increases as the closed-loop system approaches regions of instability, but that is not always the case, at least not to the degree of tuning discretisation one would expect in practical situations.

Given the non-reliance on a plant model, the only information available about the degree of system stability is provided by estimates of the closed-loop dynamics. Faced with the inaccuracies associated with these estimates, the author has opted for a rather conservative strategy to guarantee stability of the closed-loop system. The following stability assertions were originally presented in (Vinnicombe 1993), but with the plant and the feedback controller interchanged.

For a stable closed-loop system, the generalised stability margin is defined as

$$b_{G,C_{fb}} \triangleq \left\| \begin{matrix} \frac{1}{1+GC_{fb}} & \frac{C_{fb}}{1+GC_{fb}} \\ \frac{G}{1+GC_{fb}} & \frac{GC_{fb}}{1+GC_{fb}} \end{matrix} \right\|_{\infty}^{-1}, \tag{6.7}$$

and as 0 for unstable closed-loop systems. Moreover, the *Vinnicombe distance* between two feedback controllers, C_0 $(= C_{fb}(\rho_0))$ and C_1 $(= C_{fb}(\rho_1))$, is defined as

$$\delta_\nu(C_0, C_1) \triangleq \left\| (1 + C_1 C_1^*)^{-1/2} (C_1 - C_0) (1 + C_0 C_0^*)^{-1/2} \right\|_{\infty}, \tag{6.8}$$

provided the winding number condition, given by (6.9), is satisfied, and $\delta_\nu = 1$ otherwise.

$$\delta_\nu(C_0, C_1) < 1 \quad \Leftrightarrow \quad \begin{cases} \|1 + C_1^* C_0\| \neq 0 \quad \forall \omega, \quad \text{and} \\ \text{wno}(1 + C_1^* C_0) + \eta(C_0) - \eta(C_1) = 0 \end{cases}, \tag{6.9}$$

where wno$(\cdot)$ denotes the winding number, or number of counterclockwise encirclements of the origin of the Nyquist plot, and $\eta(\cdot)$ denotes the number of poles outside the unit circle.

Vinnicombe (1993) has shown the following:

Given a plant G and a feedback controller C_0 such that the closed-loop system is stable, then the loop remains stable under any feedback controller, C_1, satisfying $\delta_\nu(C_0, C_1) < b_{G,C_0}$.

As mentioned before, this result is rather conservative because it gives a sufficient, but not necessary, condition for the stability of the closed-loop system.

The insertion of Vinnicombe's result into the gradient tuning scheme depends on the availability of a model of the closed-loop system. Although this could be a parametric model, it is simpler to use non-parametric characterisations of the closed-loop dynamics, like the cross-spectral densities $T(e^{i\omega}, \rho)$ and $S(e^{i\omega}, \rho)$. The generalised stability margin is computed as

$$b_{G,C_{fb}(\rho_0)} = \left\| \begin{matrix} S(\rho_0) & S(\rho_0)\, C_{fb}(\rho_0) \\ T(\rho_0) & T(\rho_0)\, C_{fb}(\rho_0) \end{matrix} \right\|_\infty^{-1} . \tag{6.10}$$

For each new set of controller parameters, given by (6.5), the winding number condition is tested, followed by the inequality

$$\delta_\nu(C_{fb}(\rho_0), C_{fb}(\rho)) < b_{G,C_{fb}(\rho_0)}.$$

If any of these tests fails, the controller $C_{fb}(\rho)$ is not guaranteed to stabilise the closed-loop system. Then the user has the options of finding, and applying, the largest α in (6.5) that satisfies the tests; or performing a new experiment with the current controller, for which a new direction of search and a new stability margin are obtained.

Example 6.3. Consider the same setup of Example 6.2, but now let us augment the tuning scheme with the generalised stability margin. The initial control law results in $b_{G,C_{fb}} = 0.680$. Figure 6.5 shows the Vinnicombe distance from the current feedback control law to the initial one, as α increases.

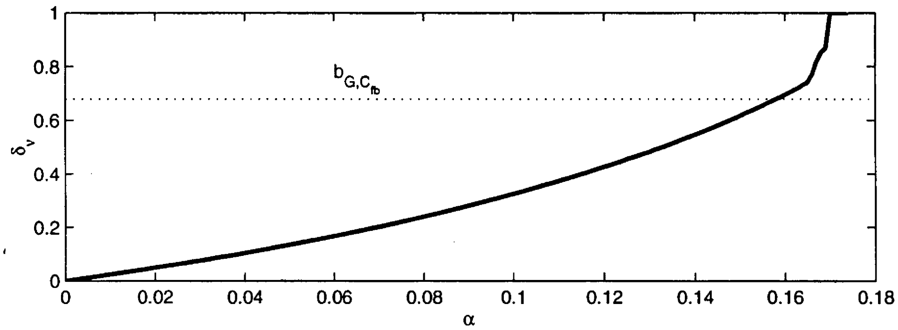

Figure 6.5: Distance between feedback controllers

The line search should stop at $\alpha \approx 0.158$ because further controllers are not guaranteed to stabilise the closed-loop system, according to Vinnicombe's criterion. At that point the tuning proceeds with the computation of a new direction and its corresponding stability margin.

6.4 Indirect Tuning

The reason for applying external excitations into the loop is to obtain the closed-loop non-parametric terms $T(\rho)$ and $S(\rho)$ needed in the computation of both the derivatives of the cost function and the generalised stability margin. After an initial experiment, with the controller $\{C_{fb}(\rho_0), C_{ff}(\rho_0)\}$, it is possible to recompute $T(\rho_j)$ and $S(\rho_j)$, for any $C_{fb}(\rho_j)$, based on $T(\rho_0)$ and $S(\rho_0)$:

$$T(\rho_j) = \frac{T(\rho_0)}{S(\rho_0) + T(\rho_0)\, C_{fb}(\rho_j)}, \tag{6.11}$$

$$S(\rho_j) = \frac{S(\rho_0)}{S(\rho_0) + T(\rho_0)\, C_{fb}(\rho_j)}. \tag{6.12}$$

This procedure resorts to an implicit non-parametric modelling of the plant, therefore accuracy tends to reduce as the mismatch between $C_{fb}(\rho_j)$ and $C_{fb}(\rho_0)$ increases.

The main point of this procedure is to compute new directions of controller tuning, without re-exciting the loop, after a minimum has been reached along the initial line search. This modified scheme becomes 'indirect' by resorting to a model of the plant. Despite the seeming philosophical shift, the plant model is only used for computing tuning directions while the sequence of line searches, performed on the real system, constantly tests the quality of that model.

On the other hand, the generalised stability margin, $b_{G,C_{fb}(\rho_0)}$, must only be computed with the signals obtained from excitation experiments, and never estimated from (6.11) and (6.12). The distance between feedback controllers, $\delta_\nu(C_0, C_1)$, has to be taken from the current controller to the one used during excitation, irrespective of intermediate line searches.

A new loop excitation is only required when the current plant model fails to deliver a direction of decreasing cost or when the stability condition is violated.

Example 6.4. Consider the plant

$$y_t = \frac{0.1}{q\,(q-0.76)^2}\, u_t + \frac{q-0.2}{q-0.98}\, e_t,$$

with $\sigma_e^2 = 0.2$, under control of a PID controller. The controller has the three usual parameters to be adjusted: K_p, T_i and T_d for the proportional gain, integral time and derivative time, respectively. Its structure is given by

$$u_t = K_p\,\frac{q-b_1}{q-1}\, r_t - K_p\,\frac{c_1\, q^2 + c_2\, q + c_3}{q^2 - (1+a_1)\, q + a_1}\, y_t,$$

with

$$a_1 = \frac{T_d}{G_d\, t_s + T_d} \qquad b_1 = 1 - \frac{t_s}{T_i}$$

$$c_1 = 1 + G_d\, a_1 \qquad c_2 = \frac{t_s}{T_i} - 1 - a_1 - 2\, G_d\, a_1 \qquad c_3 = a_1\left(1 + G_d - \frac{t_s}{T_i}\right),$$

where t_s, the sampling time, is chosen to be 1 time unit, therefore T_i and T_d are directly given in time units. The variable G_d, known as the derivative filtering parameter, can easily be included in the parameter set, but for this example it will be fixed at $G_d = 20$. Also the derivative action is fixed at $T_d = 0$ in this example, for graphical reasons that will become evident later.

The tuning criterion is

$$\widehat{J^{lq}} = \frac{1}{N_t} \sum_{k=1}^{N_t} \left[(F_y\, y_k)^2 + \lambda\, (F_u\, u_k)^2 \right],$$

with $F_y = 1$; $F_u = \frac{q-1}{q}$ due to the integrator in the control law; $\lambda = 0.1$ and $N_t = 8192$ samples collected during normal operation, for each parameter set. The number of samples taken in the computation of $\widehat{J^{lq}}$ represents a compromise between tuning speed and accuracy, and it can be made variable, with increasing number of samples as the controller approaches optimal performance.

The initial controller parameters are: $\{K_p : 0.4; T_i : 45\}$. A loop excitation with a white-noise signal of zero mean and $\sigma_r^2 = 6$ results in a signal to noise ratio of less than 2 in the variances of both $\{y_t\}$ and $\{u_t\}$. Still, reasonable estimates for $T(\rho)$ and $S(\rho)$ could be obtained, as shown in Figure 6.6. The true values of magnitude and phase for each estimate are given as thin lines.

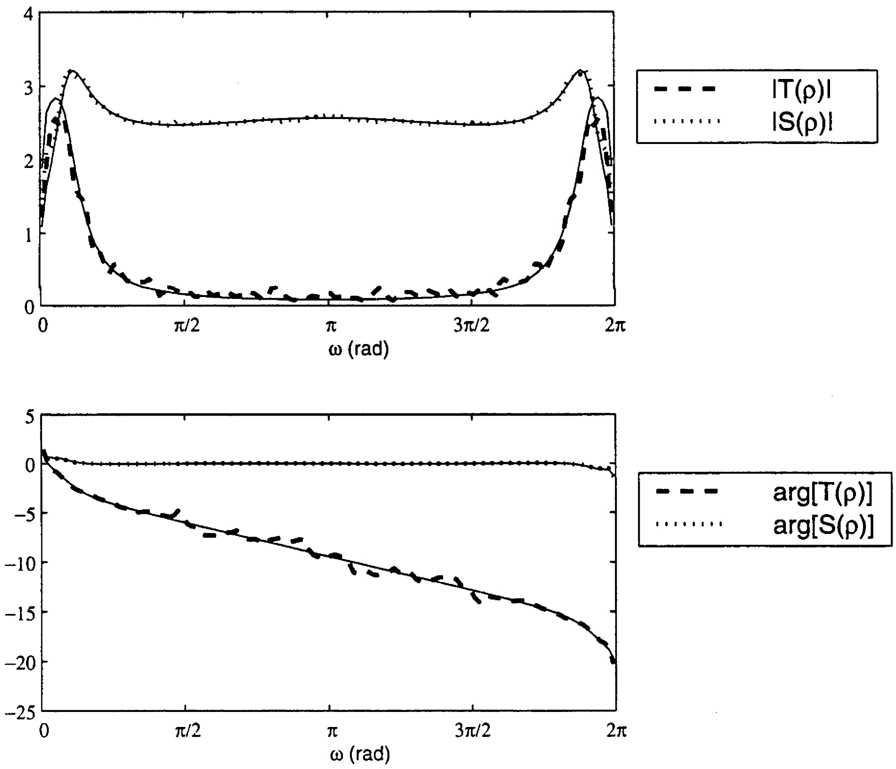

Figure 6.6: Cross-spectral densities of the closed-loop system

Measurements of the cost function along the tuning directions are presented in Figure 6.7 as solid lines, as well as the anticipated cost function (6.6) as dotted lines. The measured cost function is highly affected by the noise realisation, hence the establishment of each point of minimum is improved by fitting a low order polynomial to the line-search curve; this example uses a fourth order polynomial for this purpose.

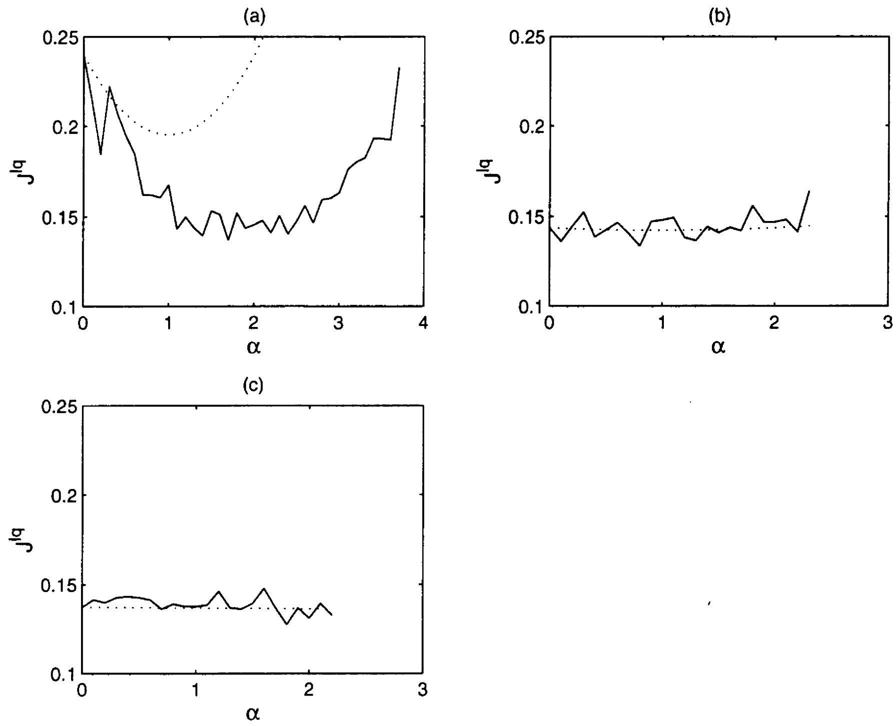

Figure 6.7: Cost function along the tuning directions

The initial Hessian matrix contains one negative eigenvalue, therefore the tuning begins in the direction of steepest descent (Figure 6.7.a). The new parameter set becomes $\{K_p : 1.14; T_i : 45\}$. The second tuning direction comes from the closed-loop estimates and the current controller. Again, the Hessian matrix contains one negative eigenvalue. Figure 6.7.b shows that almost no improvements are obtained along the direction of steepest descent ($\{K_p : 1.02; T_i : 45\}$), therefore the new direction is given by the eigenvector associated with the negative eigenvalue of the Hessian matrix. The tuning stops at $\{K_p : 1.01; T_i : 18.6\}$ because the stability margin is reached (Figure 6.7.c).

The contour plot of $J^{lq}(\rho)$, Figure 6.8, provides a different perspective of these tuning directions. The line searches are shown as thick solid lines, with their chosen minima marked with crosses. At the initial point, marked with a circle, the external excitation provides the loop stability margin $b_{G,C_{fb}(\rho_0)} = 0.6619$. This margin constrains the controller parameters to remain inside the region delimited by the thick dashed line.

After reaching the stability boundary, a new loop excitation has to be performed, generating a new stability margin. The Hessian matrix is now positive definite and the line search generates Figure 6.9.a. Any possible improvement on J^{lq} is entirely hidden under the inaccuracies associated with its estimate $\widehat{J^{lq}}$. Another search is performed from the same initial controller, but with $N_t = 65536$. The result is shown in Figure 6.9.b and a faithful choice of $\alpha = 1$ results in $\{K_p : 1.01; T_i : 16.3\}$, the point marked with a pentagram in Figure 6.8.

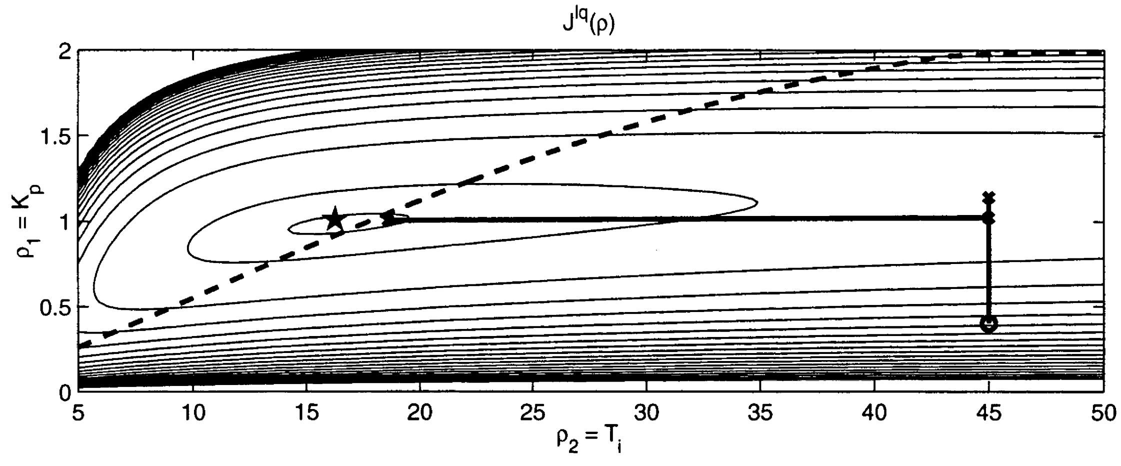

Figure 6.8: Overview of the tuning directions

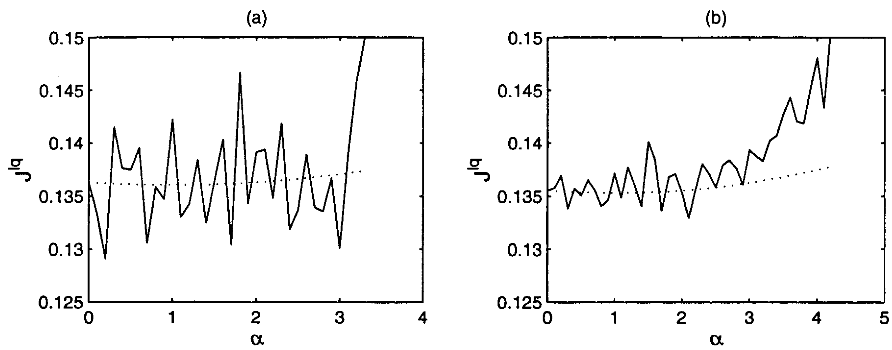

Figure 6.9: Final tuning

6.5 Conclusion

Beyond detection of local optimality, estimates of first and second derivatives of the cost function provide the information for computing appropriate directions of controller tuning. Once the best controller is found in this direction, new estimates are obtained for computing tuning directions. This iterative procedure continues until the performance is locally optimal.

Like other gradient-based schemes, this approach might present sensitivity problems when the controller is over-parametrised. The insertion of Vinnicombe's result into the tuning scheme guarantees closed-loop stability for a certain limit of controller variation. Although the computation of this limit requires estimates of the closed-loop dynamics, non-parametric estimates of these quantities are readily available from the procedure for estimation of the derivatives of the cost function.

The non-parametric estimates of the closed-loop system immediately provide a non-parametric model for the open-loop plant dynamics. One can use this model to compute new tuning directions without the need to re-excite the loop. This new concept is 'indirect' since it resorts to a model of the plant.

In comparison with the iterative feedback tuning (IFT) scheme, presented in (Hjalmarsson *et al.* 1994*a*), the scheme presented in this chapter is more complex due to the unbiased estimates of the second derivatives and to the generalised stability margin. Each of these sources of complexity brings a special feature to the tuning scheme that is not available in the IFT: detection of local optimality and guarantee of stability.

Chapter 7

Directions for Future Research

7.1 Thesis Achievements

This thesis describes the development of mechanisms for assessing and monitoring optimality of the loop performance under a linear quadratic criterion. Since performance optimality relates to the control of the real plant, those mechanisms are designed to look at the closed-loop system as a whole, without modelling the plant.

The initial results obtained here refer to absolute performance optimality, i.e. the optimal performance under the appropriate controller structure. This characteristic establishes those results as initial steps towards a model-free mechanism for assessment of absolute performance under a linear quadratic criterion. It is the author's belief that the estimate of the optimal characteristic polynomial has strong potential application in the development of such a mechanism.

A second set of results in this thesis refers to local performance optimality. These results on performance assessment and controller tuning are based on derivatives of the cost function with respect to the controller parameters, therefore they can easily accommodate controllers of reduced complexity as well as different performance criteria. Examples of sudden instability of the closed-loop system, during controller tuning, motivated the development of the mechanism for guaranteeing loop stability. Also, in order to reduce the amount of external excitations, a non-parametric model of the plant is used for computing new tuning directions after a point of minimum has been reached in the line search, even though the quality of that model is expected to reduce as the controller develops further.

The mechanisms introduced in this thesis contribute to the solution of the practical problem of evaluating the economic benefits of controller tuning or redesign. And, in an industrial context, there are large sums of money to be saved by improving the performance of some important control loops.

7.2 Future Research

The following topics are logical extensions of the work presented in this thesis. Future research into these topics could lead to improved mechanisms for performance assessment and monitoring, and for controller tuning. Some of these topics are already under investigation, with a high likelihood of success.

- Undoubtedly, the search for a model-free mechanism for assessment of optimal performance is still in place as there are some non-parametric methods yet to be tried on the closed-loop system. At the least, it should be possible to reach a definitive statement about the solvability of this problem.

- Given the difficulties associated with the linear quadratic criterion, it would be worthwhile pursuing simpler methods for performance monitoring and assessment under a different criterion containing some sort of penalty on the control action. Perhaps a frequency-weighted minimum variance criterion is enough to constrain large variations in the control action, but it is very likely that an appropriate choice for the frequency weighting depends on the characteristics of the open-loop plant.

- The mechanisms for monitoring and assessment of linear quadratic optimality are limited to the linear case, especially because spectral analysis is constrained to linear systems. Associated methods for testing optimality of appropriate classes of nonlinear systems could be pursued.

- This thesis only addresses the regulation problem of (weakly) stationary stochastic disturbances, mainly because in this case the measurable signals are ergodic processes. That is, a time average over any finite set of data provides unbiased estimates of the cost function. The conversion of the results in this thesis into a purely deterministic setup might be straightforward, but a combination of deterministic and stochastic signals deserves careful consideration, especially because the expected value of the cost function usually depends on the number of samples collected from the loop.

- Both assessment of local optimality and computation of directions for controller tuning are generic to any criterion whose first and second derivatives are computable and also obtainable from closed-loop estimations; some criteria commonly used in industrial environments might satisfy these restrictions.

- The techniques of gradient adjustment of loop performance might be extensible to the improvement of observer (or estimator) performance. This can be an alternative to the adaptive estimation mechanisms of (Mehra 1970) and (Anderson and Moore 1979).

- The noisy collection of cost functions along the line searches deserves a statistical treatment. The use of confidence levels can lead to well-defined stopping criteria for these line searches. This topic is of practical interest because the linear quadratic cost function tends to be very flat in an extensive region around the optimal operating point.

Appendix

TRACKING ADAPTIVE IDENTIFICATION: APPROACH VIA MODEL FALSIFICATION

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Abstract. Identification schemes that are capable of taking account of model uncertainties and also tracking the time-varying dynamics of the process offer great power to adaptive control systems. This paper introduces a new approach to parameter identification that has these characteristics. The parameter estimator uses a normalised confidence distribution function to compute bounds on the process parameters. This distribution function adds some characteristics of certainty equivalence schemes to the "traditional" parameter bounding estimator, and straightforwardly solves the problems of adaptive tracking and detection of outliers.

Keywords. Identification; parameter estimation; bounded noise; time-varying systems; adaptive control; experiment design.

1. INTRODUCTION

There is a clear distinction between parameter bounding and certainty equivalence estimators, the two classes of estimator schemes most frequently used in system identification. Parameter bounding estimation is performed by the computation of the set of process model parameters that satisfy prescribed constraints on model-output errors. These estimators intrinsically take account of model uncertainties since the estimation produces a set of feasible process parameter vectors. Certainty equivalence estimation may be realised in many different ways, but their common characteristic is to produce a single parameter vector which is used by the control design as if it was the actual process description. These estimators do not take account of model uncertainties but are easily modified to track time-varying process

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dynamics, especially those estimators based on recursive algorithms with some sort of forgetting factor [8].

Reference [6] presents a new certainty equivalence estimator and controller "adapted" from recent results on the problem of "learning from experts." The heart of that estimator is an "unnormalised" density function ($q(t, \theta)$) that maps from a $2n$ dimensional parameter (θ) into a $(0, 1]$ range. This function is updated at each step and its value for some θ corresponds to the confidence that θ°, the actual process parameter vector, is θ. Thus $q(t, \theta)$ measures the non-falsification of the parameter θ by the data until time t.

This paper improves Kumar's estimator, described above, by translating it into the parameter bounding context and adding adaptivity. The first step in this translation is the addition of a dead-zone action in the computation of the confidence distribution function. With the new distribution function we can easily pick up the set of models that satisfies the constraints on model-output errors: the set of θs with maximum confidence value. The advantage of this scheme over "traditional" parameter bounding ones is that no value of θ is completely discarded, and hence this estimator is able to track time-varying process dynamics. The characteristics of the adaptive tracking are tightly related to the excitation and process time variation characteristics. It might happen that after a stepwise change in the process parameters, the estimation result jumps from the previous region to the new actual one instead of sliding continuously over Θ, the space of feasible models.

The system description is given in Section 2, where the controller has an indirect Self-Tuning Regulator structure, i.e. the process parameters are estimated and afterwards used in the control action design. Section 3 presents the necessary modifications to the identification scheme in order to permit fast adaptive tracking of time-varying dynamics. The subsequent section discusses the problem of excitation design in a parameter bounding context. In Section 5 some properties of the system convergence are analysed and in the last section we present the conclusions. Simulation results of a first order system are shown throughout this paper in order to illustrate some system behaviour.

2. CONTROL DESCRIPTION

2.1 *The Process*

The process to be controlled is given by

$$y(t) = \phi^T(t-1)\theta^\circ + w(t), \quad |w(t)| \leq \eta \tag{1}$$

where

$$\phi(t-1) := (y(t-1), \dots, y(t-n), u(t-1), \dots, u(t-n))^T,$$

and

$$\theta^\circ := (a_1, \dots, a_n, b_1, \dots, b_n)^T.$$

As usual $u(t)$ and $y(t)$ are, respectively, the input and the output to the process, while $w(t)$ encompasses the whole collection of bounded errors related to the modelling of the system.

It is assumed, for our purposes here, that θ° is in the interior of Θ, a closed $2n$ dimensional region such that each process represented by $\theta \in \Theta$ is open-loop stable.

2.2 *Identification*

The identification is based on that proposed by [6] in which, instead of selecting a single point parameter estimate in Θ, all parameters are feasible and we adjust a confidence "density" function $q(t,\theta)$ at each point $\theta \in \Theta$. The confidence function at a particular θ reflects the confidence of the parameter value with the available data up to that time.

We commence with all parameter values being feasible and unfalsified by data by taking

$$q(0,\theta) \equiv 1, \quad \forall\theta \in \Theta.$$

With new data the q values are relaxed towards zero according to the prediction error performance using a multiplier $F(t,\theta)$:

$$q(t,\theta) = F(t,\theta)q(t-1,\theta), \tag{2}$$

with

$$F(t,\theta) = 1 - \mu\frac{\max\left\{0, \left|y(t) - \phi^T(t-1)\theta\right| - \varepsilon\right\}}{n(t-1)}, \tag{3}$$

where

$$n(t-1) = \max\{1,\eta\} + 2\max\left\{1, \sup_{\Theta} ||\theta||\right\} ||\phi(t-1)||,$$

and

$$0 < \mu < 1, \quad \varepsilon \geq 0.$$

For θ values yielding prediction error less then ε, F is equal to one. Poorer prediction error performance yields F less than one and a diminution of confidence, q, for the corresponding value of θ.

The value of ε defines the dead zone level and should be properly chosen in order to avoid bursting phenomena or poor signal-to-noise ratio performance.

Proposition 1. Define

$$\hat{\Theta}(t) := \{\theta : q(t,\theta) = 1\},$$

then, whenever $\eta \leq \varepsilon$, θ° is guaranteed to be in the region $\hat{\Theta}(t)$.

This region may be equivalently posed in a parameter bounding fashion:

$$\hat{\Theta}(t) = \bigcap_{j=1}^{t} \{\theta : |y(j) - \phi^T(j-1)\theta| \leq \varepsilon\}.$$

The result of the identification procedure, $\hat{\Theta}(t)$, is subsequently used in the computation of a robust control action.

The main distinction between our approach and that of [6] is the use of a region of unfalsified models, as in [5], rather than single parameter selection as the centre of mass of posterior mean. This region leads us to robust control design as opposed to the certainty equivalence design of Kumar.

2.3 *Control Action*

There are many possible approaches to the robust control design for $\hat{\Theta}(t)$ as in [7,10]. In this work we implement a simple strategy for the design of the control action: we compute an optimal control for a certainty equivalent parameter obtained from the region $\hat{\Theta}(t)$, and then relax this control action, if necessary, in order to keep the entire region stable (recall the open-loop stability assumption.)

The certainty equivalent estimation of θ° at time t, defined as $\hat{\theta}(t)$, will be given by the centre of mass of the region $\hat{\Theta}(t)$:

$$\hat{\theta}(t) := \frac{\int_{\hat{\Theta}(t)} \theta \, \mathrm{d}\theta}{\int_{\hat{\Theta}(t)} \mathrm{d}\theta}. \tag{4}$$

For $\hat{\theta}(t)$ we compute the control law that minimises the following performance criterion

$$J = \lim_{t \to \infty} \frac{1}{t} \sum_{j=1}^{t} \{[y_r - y(j)]^2 + \lambda [u_r - u(j)]^2\},$$

where $\lambda \geq 0$, y_r is a constant reference value, and u_r is the control action value that in steady-state keeps $y(t) = y_r$ when $w(t) \equiv 0$. This is a stationary infinite horizon LQ tracking problem whose solution is obtained via Algebraic Riccati Equation (ARE) as shown in [2]. In order to apply this ARE, the process characterised by $\hat{\theta}(t)$ is described in a state space form

$$\hat{X}(t+1) = A\,\hat{X}(t) + B\,u(t)$$

$$y(t) = C\,\hat{X}(t).$$

With this procedure we obtain a control action in the form

$$u(t) = f\,y_r + G\,\hat{X}(t), \tag{5}$$

where f effects a null steady state error when $w(t) \equiv 0$, and G is the gain vector obtained via ARE. The stability of the closed loop system is determined by the eigenvalues of the matrix $(A + B\,G)$. If at least one (A, B) in $\hat{\Theta}(t)$ does not satisfy the condition of stability for the given G, we calculate another gain vector

$$\tilde{G} = \alpha\, G,$$

where α is the supremum value in $[0, 1)$ such that the matrix $(A + B\,\tilde{G})$ has all eigenvalues inside the unit disc, for all (A, B) in $\hat{\Theta}(t)$.

If we relax the condition of open-loop stability for all $\theta \in \Theta$, the strategy adopted for the robust control law might not be valid any more and another approach should be used. In the worst case there might be no control law that stabilises the whole set $\hat{\Theta}(t)$, and another criteria might be chosen in the control action design.

3. ADAPTATION PROPERTIES

The use of parameter bounding identification in tracking adaptive control systems is still embryonic despite some attempts such as [11]. The main reason for this unused combination is that parameter bounding cannot naturally cope with time-varying processes. Some considerations must be made for the possibility of process variation since $\hat{\Theta}(t)$ might become an empty set. One way to avoid this is to reset the system identification, but this is a drastic proceeding since a sudden pulse disturbance might cause an outlier. In [11] the ideas of ages and generations are introduced, but again a sudden pulse disturbance might cause drastic effects.

The solution adopted in this work is to re-normalise the density function. Another improvement adopted here is to fix a minimum value to the density function in order to speed up adaptive tracking. With these modifications Equation (2) changes to

$$\hat{q}(t, \theta) = F(t, \theta) q(t - 1, \theta),$$

$$q(t, \theta) = \max \left\{ q_{min}, \frac{\hat{q}(t, \theta)}{\max_\theta \{\hat{q}(t, \theta)\}} \right\}.$$

The procedure described above ensures $\hat{\Theta}(t)$ contains at least one element. Observe that no value of θ is completely discarded, even after being falsified by data for several times, hence any value of θ might be "defalsified" by data. In order to accelerate the adaptation, we assign a minimum value for the function q. Instead of having to rise from almost zero to one, the values of $q(t, \theta)$ will rise much faster from q_{min} to one. The value of q_{min} is determined by a compromise between adaptation speed and robustness to outliers.

Figure 1 shows one possible behaviour of the identification scheme for a first order system. The system is constantly under external excitation in the form of stepwise changes in the reference (y_r). After a while all values of θ such that $q(t, \theta) > q_{min}$ are concentrated close to θ° (Fig. 1.i). After a stepwise change in the process parameters there will not necessarily occur an immediate drift towards the new parameters, as in the least-squares methods, but the confidence levels of

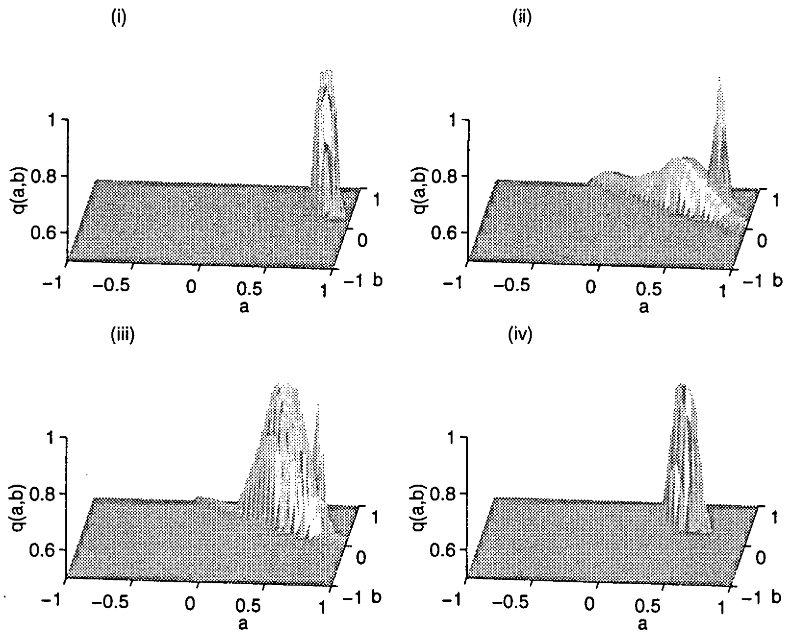

Figure 1: Jumpwise Adaptation

θ values close to the new θ° will rise relative to the old ones (Fig. 1.ii) and after some time the region $\hat{\Theta}(t)$ might jump to a new position in Θ (Fig. 1.iii). Some time later all values of θ such that $q(t,\theta) > q_{min}$ are again concentrated near θ° (Fig. 1.iv). The adaptation speed depends on the excitation and on the old and new θ°.

4. EXCITATION DESIGN

The control action proposed so far thoroughly emphasises system performance and ignores excitation issues. Instead of dealing with both problems together, for instance adopting some minimax criteria, this scheme separates them and restricts the action of excitation signal to special situations.

The optimal design of an excitation signal might become very difficult if the control law is allowed to change at every step because the control action could cancel this excitation or the control and output signals could grow unboundedly [3]. The solution adopted in [3] and references therein is block processing, i.e. the control law is frozen during the period of excitation (p steps).

The main goal of the excitation is to reduce the set $\hat{\Theta}(t+p)$ such that the following criterion is minimised

$$J_{max} = \max_{\theta \in \hat{\Theta}(t+p)} \{J(\theta)\},$$

where $J(\theta)$ is a performance measure defined according to the system specifications. In order to simplify the excitation design, the goal may be relaxed to the minimisation of the diameter or the volume of $\hat{\Theta}(t+p)$, as in [10,12]. With no *a priori* knowledge of $\hat{\Theta}(t)$, $\{\phi(t), \ldots, \phi(t+p)\}$ must span the space of Θ for the observations to yield a bounded set $\hat{\Theta}(t+p)$ [9]. Reference [1] gives a sufficient condition for this spanning problem, in terms of inputs alone.

In the context of this work the excitation design problem is posed as follows: given the set of unfalsified process parameters $\hat{\Theta}(t)$ and a fixed control law

$$u(t) = F(q)\, y_r + G(q)\, y(t) + r(t),$$

where $r(t)$ is a bounded excitation signal, which sequence $r(t)$, $r(t+1)$, … ,$r(t+p-1)$ minimises the volume of $\hat{\Theta}(t+p)$?

Despite the complexity of the answer, which depends on several factors, the amplitude of $\|\phi\|$ is the major element for minimisation of the volume of $\hat{\Theta}$ [9]. Observe that the best excitation for some θs in $\hat{\Theta}(t)$, in terms of magnitude of $\|\phi\|$, may be extremely poor for some other θs also in $\hat{\Theta}(t)$.

The generic order excitation design problem will not be addressed in this paper. A simplified version over a first order process is given in the following example.

Example 2. Consider a first order process and a controller as described in Section 2. For our analysis we take y_r to be zero and $w(t)$ zero. In such conditions

$$y(t) = a^\circ\, y(t-1) + b^\circ\, u(t-1),$$

$$u(t) = g\, y(t) + r(t), \quad |r(t)| \leq A,$$

and the closed loop response is described by

$$y(t+1) = (a^\circ + b^\circ\, g) y(t) + b^\circ\, r(t).$$

As the closed system is stable, $y(t)$ is bounded with its maximum amplitude given by

$$y_m = \frac{|b^\circ|(1 + |a^\circ + b^\circ\, g|)}{1 - (a^\circ + b^\circ\, g)^2} > 0.$$

The set $\hat{\Theta}(t)$ may be described as the intersection of the regions given by

$$|a^*\, y(j) + b^*\, u(j)| \leq \varepsilon, \;\; j = 0, 1, \ldots, t-1, \tag{6}$$

where $a^* = a - a^\circ$, and $b^* = b - b^\circ$. The effect of each new data can be viewed as a "cut" over the region Θ. Given two distinct states of the system: (y_1, u_1) and (y_2, u_2), the area of their regions' intersection is

$$S = \frac{4\varepsilon^2}{|u_1\, y_2 - u_2\, y_1|} = \frac{4\varepsilon^2}{|r_1\, y_2 - r_2\, y_1|}. \tag{7}$$

This equation establishes that the area of the intersection is inversely proportional to the amplitude of $|r_1\, y_2 - r_2\, y_1|$. This means that there are two different possibilities of choices for minimisation of S:

- $r_1 = r_2 = A, \;\; y_1 = -y_2 = y_m;$

- $r_1 = -r_2 = A, \;\; y_1 = y_2 = y_m.$

Given the characteristics of Equation (6), both solutions are identical. Another property is that since no other state can diminish the area S, the minimisation of the maximum diameter of $\hat{\Theta}(t)$ is also obtained.

The main conclusion is that in the presence of a dead zone, the amplitude of the signals plays an important role in the minimisation of uncertainties.

Starting from any $y(t)$, the best sequence of excitation within a p-step boundary is given by

$$
\begin{aligned}
r(t) &= A\,\mathrm{sgn}[(a^\circ + b^\circ g)y(t)],\\
r(t+1) &= A\,\mathrm{sgn}[y(t)],\\
r(t+2) &= r(t),\\
r(t+3) &= r(t+1),\\
&\vdots\\
r(t+p-2) &= A\,\mathrm{sgn}[(a^\circ + b^\circ g)^{p-1}y(t)],\\
r(t+p-1) &= -A\,\mathrm{sgn}[(a^\circ + b^\circ g)^{p}y(t)].
\end{aligned}
$$

In the first $p-1$ steps the excitation will raise the absolute value of the process output, and in the last step it will do the opposite. Observe that in this design the only necessary information about the actual process is the sign of $a^\circ + b^\circ g$. If we have additional information about the noise characteristics this scheme can be altered to avoid dependence of a single "cut" in one direction, but the compromise with the signal amplitude is evident.

Figure 2 shows the result of an experiment with excitation signal designed for a ten step period. Parts i, ii and iii of this figure are contour plots of the confidence distribution functions where the region $\hat{\Theta}(t)$ is delimited by thick lines. Observe that the identification scheme cannot learn much about the process after the first ten steps due to the small amplitude of the initial conditions (Fig. 2.i). Parts ii and iii of Figure 2 show the distribution function before and after the last "cut", respectively.

There are two situations where the excitation signal is specially important: in the beginning of operation and after the process dynamics have varied. The former situation is characterised by a big volume of $\hat{\Theta}(t)$ while the other situation might be detected by a significant diminution in the volume of $\hat{\Theta}(t)$. These symptoms can be useful in the trigger design of the excitation periods.

5. ANALYSIS

The estimator described in Sections 2 and 3 acts as any other parameter bounding estimator whenever there is no variation on the process dynamics. Some convergence analyses of this sort of estimator can be found in References [4], [9] and others, but the developments in parameter bounding have a strong intuitive appeal. In this section we analyse the transient behaviour of our estimator after a stepwise process variation.

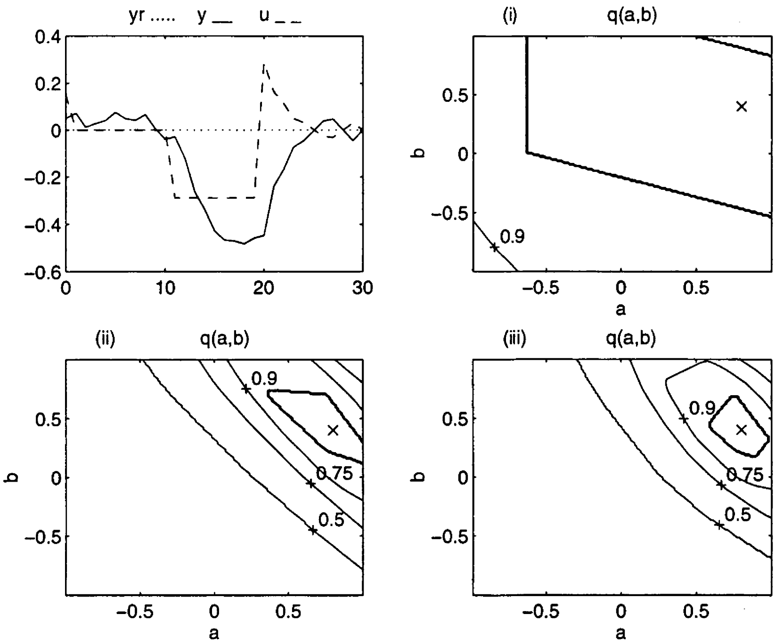

Figure 2: Excitation Effects

Suppose the process description changes significantly at time t_0 from θ_1° to θ_2°. At that time the estimator has properly identified the process such that $q(t_0, \theta_2^\circ) = q_{min}$ and the set

$$\Theta_1 := \{\theta_1 : q(t_0, \theta_1) > q_{min}, \theta_1 \in \Theta\}$$

surrounds θ_1°. Our aim is to find out when the set $\hat{\Theta}(t_0 + \Delta t)$ will contain the new process description (θ_2°) and how to speed up this convergence.

Proposition 3. Given that $F(t, \theta_2^\circ) = 1$, $\forall t > t_0$, the necessary condition for θ_2° to be in $\hat{\Theta}(t_0 + \Delta t)$ is straightforwardly given by

$$q(t_0, \theta_1) \cdot \prod_{i=t_0+1}^{t_0+\Delta t} \left[1 - \mu \frac{\max\left\{0, \left|\phi^T(i-1)[\theta_2^\circ - \theta_1] + w(i)\right| - \varepsilon\right\}}{n(i-1)} \right] \leq q_{min},$$

$$\forall \theta_1 \in \Theta_1. \quad (8)$$

Remark 4. As long as $w(t) \leq \varepsilon$ the identification never diverges, even though small values of $\phi^T(t-1)[\theta_2^\circ - \theta_1]$ might not contribute to convergence. The sufficient condition for some data $\phi(t-1)$ to contribute to identification convergence is

$$\left|\phi^T(t-1)[\theta_2^\circ - \theta_1]\right| > 2\varepsilon,$$

for some $\theta_1 \in \{\theta : q(t-1, \theta) > q_{min}, \theta \in \Theta_1\}$. Despite the importance of the amplitude of $\phi(t-1)$, its direction relative to the vector $[\theta_2^\circ - \theta_1]$ is crucial. Hence optimal excitation design would require knowledge of θ_2°.

From (8) we can conclude that convergence speed might be increased by:

- increasing μ;
- increasing the product $\phi^T(t)[\theta_2^\circ - \theta_1]$, that depends partially on $||\phi(t)||$;
- decreasing ε (exclude from ε the modelling of small drifts on process dynamics);
- reducing Θ such that $\sup_\Theta ||\theta||$ decreases (reduce the *a priori* uncertainty);
- increasing q_{min}.

The appropriateness of each item described above depends on specific characteristics of the (set of) process(es) to be controlled.

6. CONCLUSIONS

A new approach to parameter bounding identification has been presented. This approach is capable of tracking time-varying processes since none of the process parameters are entirely discarded, even after being falsified by data.

An application of this identification scheme in an adaptive controller has been suggested. The robust control law chosen to this controller separates performance assessment from excitation. Actually the excitation issue is only addressed during special periods triggered by some indications that are characteristic of parameter bounding estimators.

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