METHODS IN ROBUST AND ADAPTIVE CONTROL

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The doctoral studies were conducted with Professor John B. Moore, as supervisor, and Professors Brian D.O. Anderson and Robert R. Bitmead, as advisors.

" I hereby declare that the results presented in this thesis, except as otherwise explicitly stated, are the results of original research and have not been submitted for any other degree to any other university or educational institution."

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It is very hard to express my appreciation for the support from my dear wife Huiqing Jin, I just say "I love you".

ABSTRACT

The thesis develops control system design methods and theory with an emphasis on achieving robust off-line and on-line controller designs. The topics studied cover four problem areas.

In the first problem area the objective is to improve controller loop robustness of an initial design. The first contribution we make here is the convenient parametrization of the class of model matching controllers, this being the first step to apply standard H^{∞}-optimization techniques to improve robustness. The second contribution in this topic is to extend the techniques of loop transfer recovery in linear quadratic Gaussian (LQG) designs to cope with nonminimum phase plants.

In the second problem area, the objective is to develop central tendency adaptive control methods so as to improve the transient performance of those standard adaptive schemes by taking the uncertainty of the plant parameter estimates into account in constructing the adaptive controller. Our contributions here are to the development of central tendency adaptive pole assignment and central tendency adaptive LQG control.

The third problem we tackle is avoidance of ill-conditioning which arises with the estimation and control of overparametrized systems. Here algorithms are proposed to cope with overparametrization in the signal model and achieve convergence to the unique model estimate which corresponds to the non-overparametrized model. Based on these estimation algorithms, a central tendency adaptive pole assignment control scheme is shown to be able to handle signal generating systems models which are possibly overparametrized. Furthermore, an identification scheme is proposed based on Kalman filter ideas to deal with possibly overparametrized systems with model order and plant parameter changes.

For the last problem area, motivated by the third problem area applications, we give some analysis results on standard estimation algorithms. One is to show that for identification of multidimensional linear regression models, the strictly positive real (SPR) conditions imposed on the plant noise can be side-stepped by introducing artificial noise into the regression vector to make the combined noise whiter. Convergence results are achieved without the SPR condition satisfied, and it is also shown that the whiter the noise environment, the more robust are the algorithms. Another result is to show how to side-step the strictly positive real condition in general ARMAX model identifications. This is achieved by a unique overparametrization so that the strictly positive real conditions can be relaxed at arbitrary degree. The last one is about tracking unknown randomly changed plant parameters, for linear stochastic system identification a using standard Kalman filter algorithm. Here we develop asymptotic properties of the algorithm and establish the tracking error bounds for the unknown randomly varying parameters.

PREFACE

The material included in this thesis is the results of original research co-operatively working with Prof. Moore (my supervisor) and /or Dr. Guo. The contributions on the joint research work with which I feel most closely identified with are the papers with myself as first author. More specifics of my contributions are now summarized.

As a first year graduate student, my contributions to the topics "Central tendency adaptive pole assignment" and "Loop recovery and robust state estimate feedback design" were to developing some of the technical results and to the simulation studies to demonstrate the effectiveness of the results. For the project "On improving control loop robustness of model matching controllers", some of my own insights and technical contributions were critical to the successful formulation and derivation of the results. The work on "Adaptive LQG controllers with central tendency properties" is primarily my own work, building on the ideas of the central tendency adaptive pole assignment study.

For the project "On adaptive estimation and pole assignment of overparametrized systems" the key theorems and proofs were developed in the first instance by myself with feedback from Prof. Gevers. Later these were enhanced to the stochastic case by working jointly with my supervisor. For the papers "Recursive identification of overparametrized systems" and "Adaptive estimation in the presence of order and parameter changes", I was able to work more independently.

In the paper "Robust recursive identification of multidimensional linear regression models", my contributions were to initially quantify the level of injected white noise to side-stepping the positive real condition on noise color. and to provide the insights crucial to a number of lemmas in the paper. In the paper "Tracking randomly varying parameters", I contributed to the development of some of the theorem proofs. In the paper "Computation of H^{∞}-norms of polynomials", a special case study in details turns out to be very helpful for the algorithm development. For the paper "Identification / prediction algorithms for ARMAX models with relaxed positive real conditions", I involved the development of almost all the theorem proofs and simulation studies.

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CHAPTER 1

INTRODUCTION

When one wants to control a plant, a starting point is the a priori knowledge about the plant and any assumptions concerning the plant. The resulting control scheme is very much dependent on such assumptions and a priori knowledge of the plant, as well as the control objectives. Certainly, with different assumptions, the control strategy varies, and the problems and difficulties in designing the controllers differ accordingly.

If one assumes that the plant is known exactly and is precisely described by a linear dynamical stochastic signal model which prescribes the variation with respect to time, the system order and the plant parameters, then a stabilizing fixed (non-adaptive) controller can be designed using text book methods. However, the true plant dynamics, as is inevitable in practice, may not perfectly match the assumed signal model. It is known that such a mismatch between the signal model and actual plant, may cause serious problems, namely cause instability of the closed loop system consisting of the actual plant and the controller designed for the nominal plant. This motivates for us the concept of a robust controller design. A robust controller is one which can cope with some mismatching between the signal model and the plant, achieving suitable stability margins when applied to the nominal plant. The robustness can be with respect to unmodelled dynamics, or to plant parameter estimation errors, or to both. Improving control loop robustness properties of a nominal design has attracted much attention, and there are many approaches proposed to do this. Loop transfer recovery, and H^∞-optimization are two such methods which appear to work in practice.

In many situations, the plant dynamics are only assumed to be known as one member of a specified class of models (say a linear system with known order but unknown parameters). Or they may drift very slowly inside of this class of models. To cope with such situations, the concept of adaptive control is relevant. Adaptive control normally consists of two steps, at least in what is known as indirect adaptive control. First, the plant parameters are estimated, then the controller is designed based on the information gained from the estimation of the plant. Thus the "learning" of the plant and the "controlling" of the plant take place in parallel, and as time goes on, it is the intention that as the plant parameters are identified, then the controller converges to the one which would have been used if the plant parameters were known. In the course of such adaption, the way to balance the effects of "learning" and "controlling" makes the complexity of the adaptive control schemes differ considerably. There are two extremes. One is called certainty equivalence principle, and the other is called dual control. In the certainty equivalence principle, the adaptive controller is designed using the estimates of the plant as if they were the true plant. This approach is easy to implement and simple to use as an on-line scheme. However, its disadvantage is that it can only be optimal asymptotically and the transient performance may be unnecessarily poor, even intolerable. On the other hand, the dual control objective is to give a "best" control in the presence of plant uncertainties. However schemes proposed under this heading are very formidable to design and are almost impossible to be implemented on-line. Thus to design an adaptive scheme which has a reasonable transient performance and also is easy to implement becomes a challenging task with the promise of a high payoff.

It is common, in many adaptive schemes, to assume that the plant to be controlled has a linear input-output signal model with known order but unknown parameters.

However, the choice of the signal model order, in some situation, is not an easy job. If the order is selected too low (it is called underparametrization), then the unmodelled dynamics can be destabilizing the whole system. this suggests that there is a tendency in practice to overparametrize the signal model to be on the "safe" side, that is to choose the order higher than it is. However, when an adaptive estimation algorithm (say least squares or Kalman filter) is employed to identify an overparametrized system, there is inevitably a lack of excitation in the regression vectors and normally there is no guaranteed convergence in this situation. In general, near pole zero cancellations in the estimated plant are expected, and they can be located anywhere being sensitively dependent on the initial conditions and noise sampling path. Furthermore, based on those estimates of the plant, some adaptive control schemes (typically pole assignment control) could ineluctably lead to excessive control signals (or so called ill-conditioned). Therefore the overparametrization on the signal model emerges as a significant problem.

In this thesis, we present our solutions to such robust and adaptive control problems as mentioned above. The solutions are only solutions under certain assumptions, and so are not claimed to be complete in any way. However, we believe they are an advance on current methods in the literature. The thesis consists of four parts based on the research papers published or submitted for publication in journals and international conferences, and is organized as follows.

In Part I, we develop and generalize some techniques for improving the controller loop robustness properties of an initial design. It is known that when a controller is designed for a nominal plant to satisfy some performance requirement, and achieves attractive closed loop transfer function properties, it does not always turn out to be a robust controller in the sense of tolerating plant changes form the nominal plant, or in other words the uncertainty of the plant parameters. Some approaches have been proposed to improve controller loop robustness while keeping the closed loop transfer function unchanged. One of the approaches is called "loop transfer recovery" (LTR) which mainly focuses on the linear quadratic Guassian (LQG) control design, but can be applied to any state estimate feedback based design. The idea of LTR is to represent the plant uncertainty by adding fictitious noise to the plant input while designing the LQG controller. For minimum phase plants, as the magnitude of the fictitious noise increases the control loop transfer functions approach to those for the state feedback design, which have attractive robustness properties. Then we say "loop recovery" occurs. Here as Chapter 2, we present our generalization on LTR, namely to handle non-minimum phase plant, which is based on the published paper [p1]. In Chapter 2, the loop recovery technique has been generalized for nonminimum phase plants in the following sense. The open loop properties of certain partial state feedback designs are recovered in a state estimation feedback controller design involving the addition of fictitious plant noise. The partial state is the state of a minimum phase factor in a minimum phase, all-pass factored form model. Of course, robust designs are expected only when these are achieved for the case of partial state feedback of only the minimum phase factor states. This may not always be possible. For the case on minimum phase plants, known designs and theory are recovered as a special case. The theory and designs of this chapter generalize to include frequency shaping of both the control objectives and the loop recovery.

Another approach of improving control loop robustness properties of an initial design, which is not limited to an LQG initial design, is to directly search for the most robust controllers among those which satisfy the performance requirement (say achieve a specific closed loop transfer function) or so called "model matching controller". Here in Chapter 3, we present the results on parametrization of the class of all model matching controllers, and a procedure to search for the most robust one, by performing an H^∞ optimization. These are based on the published paper [p2]. In this chapter, the class of all stabilizing controllers, for a two-degree-of-freedom control system which achieve a prescribed achievable transfer function, is first characterized. The characterization is in terms of an arbitrary proper stable transfer function. With this characterization, robust model matching is formulated as a standard H^∞-optimization problem. This means that standard controller designs for a nominal plant, such as LQG ones, can be enhanced to give improved robustness properties using H^∞-design techniques.

It is observed that in practice, to calculate the value of the H^∞ norm is usually done by a rather trivial method, i.e. plotting the absolute value of the function concerned on the unit circle. This involves some ad hoc selection of plotting intervals and interpolation to achieve an appropriate accuracy. Here in Chapter 4, we propose a recursive algorithm for the computation of H^∞ norm of polynomials or finite impulse response (FIR) transfer functions based on the published paper [p3]. The algorithm is shown to converge monotonically and the convergence rate is also established. Some examples are presented to illustrate the algorithm.

We report, in Part II, applications of so called central tendency control concept to adaptive pole assignment control and LQG control. As mentioned above, in designing indirect adaptive controls, there is a range of schemes to choose between the two extremes of applying the certainty equivalence principle and dual control.

When an adaptive scheme based on the certainty equivalence principle is employed, it can only be optimal asymptotically and there will be in general, circumstances where the transient performance is unnecessarily poor. And dual control normally is too formidable to be a practice proposition for on line schemes. In order to avoid excessive control signals due to inaccurate estimates of the plant during transient, and to achieve an on-line implementable scheme, the central tendency control was proposed. In the central tendency control, adaptive controllers are designed using measures of central tendency of the a posteriori probability function of the controller parameters. That is, given knowledge of plant uncertainty at each time instant, from estimation algorithms, and a controller design rule, the controller parameters is sought which maximized the likelihood of achieving the control objectives. Here we report the results on central tendency adaptive pole assignment control and LQG control respectively as Chapter 5, and Chapter 6 based on the papers [p4], [p5] presented in international conferences.

In Chapter 5, the concept of central tendency adaptive control is applied to adaptive pole assignment. At any iteration, given plant parameter estimates and their uncertainty, a controller is designed which is "most" likely to achieve the pole assignment objectives. Simulations show a factor of 100 improvement, in transient response in one example, over certainty equivalent adaptive pole assignment schemes at least for one example.

In Chapter 6, it is first observed that one particular standard certainty equivalence based version of an adaptive LQG controller in the literature tends to have a better transient performance than others of comparable complexity. Why is this so? Can we improve its transient performance even further? Can other adaptive LQG control schemes be modified to have improved transient performance? The main result of this chapter makes clear that improved transient performance tends to occur when the design rule is linearized so that the controller parameters are the most likely ones, according to the control design rule, given the plant uncertainty. In addition, since there is an option of different linearized design rules at each iteration, a particular one can be chosen to maximize a central tendency measure, thereby achieving central tendency adaptive LQG control. For central tendency adaptive LQG schemes, there is avoidance of excessive control action due to ill-conditioning associated with near unstable pole zero cancellation in plant estimates.

Part III is devoted to tackling the problems of overparametrization in adaptive schemes. For adaptive estimation / control scheme design, as mentioned early, it is common to assume a linear input -output signal model of specified order with unknown parameters. Since the underparametrization may cause destabilizing, there is a tendency in practice to overparametrize the signal model to be on the "safe" side. Thus overparametrization emerges as a significant problem in some applications. A specific situation is when the presence of some deterministic disturbances such as bias is assumed, when in fact any such disturbances are negligible. With overparametrization, there is a danger of ill-conditioning in adaptive estimation and in some adaptive control. When an adaptive estimation algorithm (say least squares, or Kalman filter) is employed to identify an overparametrized system, there is inevitably a lack of excitation in the regression vectors and normally there is no guaranteed convergence. Also insufficient excitation can lead to estimation with near pole-zero cancellations in the complex z-plane. Such pole-zero cancellation can occur anywhere sensitively dependent on the initial conditions and noise sampling path, and lead to excessive control signal in adaptive control schemes, such as pole assignment scheme. We propose an approach based on standard identification algorithms (namely least squares and extended least squares) to cope with adaptive estimation of overparametrized systems. The key idea in this approach is to introduce excitation signal into the regression vectors so as to enforce artificially the regression vector to have suitably excitation, even when the systems are overparametrized.

Here, as Chapter 7, our published paper [p6] is included. In this chapter, a first step is taken to avoid ill-conditioning in adaptive estimation and pole assignment schemes for the case when there is a signal model overparametrization. The methods proposed in this chapter are relatively simple compared with on-line order determination, being based on introducing suitable excitation in the "regression" vector of the parameter estimation algorithms to ensure parameter convergence. For the case when the model are non-unique in that pole zero cancellations can occur, the algorithms seek to estimate the unique model where the cancellations occur at the origin. Applying estimates of this (unique) model turns out to avoid ill-conditioning in central tendency adaptive pole assignment. For the case of one pole zero cancellation the convergence theory of the algorithm is complete.

Following on from our initial work, we present generalized results as Chapter 8 based on the published paper [p7]. In this chapter, a recursive identification algorithm based on extended least squares is proposed to deal with the contingency of overparametrization. The algorithm proposed here is relatively simple compared to those involving on-line order determination, being based on adaptively introducing suitable excitation into the algorithm to avoid ill-conditioning. In the case of extended least squares based adaptive estimation, then the regressors are appropriately stochastically perturbed. The algorithm is shown to converge to a unique defined signal model with any pole zero cancellations at the origin. Ill-conditioning is avoided.

In Chapter 9, based on the paper [p8], we report further research on tracking parameters changes in the presence of order changes. An approach to adaptive estimation and control is given when there are jump parameter changes which include order changes. Order changes can be viewed as the introduction of overparametrization, which in conventional algorithm causes ill-conditioning. Here, modified algorithms which involve the introduction of noise into the calculations are proposed and studied by theory and simulations.

Part IV consists of Chapters 10, 11 and 12, reporting some analysis results on adaptive estimation algorithms. One of them deals with linear regression model [p9]. It is known that stochastic adaptive estimation and control algorithms involving recursive prediction estimates have guaranteed convergence rates when the noise is not "too" colored, as when a positive real condition on the noise model is satisfied. Moreover, the whiter the noise environment the more robust are the algorithms. Chapter 10, which is based on the published paper [p9], shows that for linear regression signal models the suitable introduction of white noise into the estimation algorithm can make it more robust without compromising on convergence rates. Indeed, there is guaranteed attractive convergence rates independent of the process noise color. No positive real condition is imposed on the noise model.

However, the techniques used in Chapter 10 do not appliable to a general ARMAX signal model. Then in Chapter 11, which is based on the paper [p10], transformed extended least squares algorithms are proposed for ARMAX model identification with the objective of avoiding the positive real condition associated with standard equation error and output error algorithms. This is achieved by an overparametrization at the cost of additional richness requirements on excitation signals, but without introducing ill-conditioning or infinite dimensional calculations as in earlier methods. Results for the case of D-step-ahead prediction ELS algorithms for ARMAX models also explored in this chapter and some simulation studies are included to assess the relative performance characteristics of the proposed algorithms.

The other analysis result is [p11] on the standard algorithm namely Kalman filter. It is known that in linear stochastic system identification, when the unknown parameters are randomly time varying and can be represented by a Markov model, a natural estimation algorithm to use is the Kalman filter. In seeking an understanding of the properties of this algorithm, existing Kalman filter theory yields useful results only for the case where the noises are Guassian with covariances precisely known. In other cases, the stochastic and unbounded nature of the regression vector (which is regarded as the output gain matrix in state space terminology) precludes application of standard theory. In Chapter 12, based on the paper [p11], we develop asymptotic properties of the algorithm, in particular, we establish the tracking error bounds for the unknown randomly varying parameters.

A summary is drawn in Chapter 13 as the conclusions for the thesis and also some further research directions is pointed out there.

PUBLICATIONS

[p1] J.B. Moore and L. Xia, "Loop Recovery and Robust State Estimate Feedback Designs", **IEEE Transaction on Automatic Control** Vol.AC-32, pp. 512-517, 1987. Also see **Proc. of ACC**, pp. 1807-1812, Seattle 1985.

[p2] J.B. Moore, L. Xia and K. Glover, "On Improving Control-Loop Robustness of Model-Matching Controllers", **Systems and Control Letters** Vol. 7, pp. 83-87, 1986.

[p3] L. Guo, L. Xia and Y. Liu, "Recursive Algorithm for the Computation of H^{∞}-Norm of Polynomials", **IEEE Transaction on Automatic Control** Vol.AC-33, Oct. (to appear) 1988.

[p4] J.B. Moore, T. Ryall and L. Xia, "Central Tendency Adaptive Pole Assignment", **IEEE Transaction on Automatic Control** Vol.AC-34, Jan. (to appear) 1989. Also see "Central Tendency Pole Assignment", **Proc. of 25th CDC**, pp. 100-105, Athens, Greece. 1986.

[p5] L. Xia and J.B. Moore, "Adaptive LQG Controllers with Central Tendency Properties", to appear in **Proc. of IFAC Symposium on Identification and System Parameter Estimation**, Beijing, 1988.

[p6] L. Xia, J.B. Moore and M. Gevers, "On Adaptive Estimation and Pole Assignment of Overparametrized Systems", **Int. J. of Adaptive Control and Signal Processing**, Vol. 1 pp. 143-160. 1987. Also see **Proc. of 26th CDC**, pp.378-383, 1987

[p7] L. Xia and J.B. Moore, "Recursive Identification of Overparametrized Systems", **IEEE Transaction on Automatic Control** Vol. AC-33, Dec (to appear), 1988.

[p8] L. Xia, V. Krishnaumrthy and J.B. Moore, "Adaptive Estimation in the Presence of Order and Parameter Changes", submitted to **Int. J. of Adaptive Control and Signal Processing.**

[p9] L. Guo, L. Xia and J.B. Moore, "Robust Recursive Identification of Multidimensional Linear Regression Models", **Int. J. Control,** (to appear), 1988. Also see **Proc. of IFAC Workshop on Robust Adaptive Control.** 1988.

[10] J.B. Moore, M. Niedzwiecki and L. Xia, "Identification / Prediction Algorithms for ARMAX Models with Relaxed Positive Real Conditions", submitted to **Int. J. of Adaptive Control and Signal Processing.**

[p11] L. Guo, L. Xia and J.B. Moore, "Tracking Randomly Varying Parameters", to appear in **Proc of 27th CDC,** also submitted to **Mathematics of Control, Signals and System,** 1988.

PART I

On Improving Controller Robustness

Chapter 2

LOOP RECOVERY AND ROBUST STATE ESTIMATE FEEDBACK DESIGNS

1. INTRODUCTION

It is well known that plant input robustness properties of a state feedback (SF) design, such as measured by phase margins for example, can evaporate with a state estimate feedback (SEF) design [1]-[5]. An important class of SF design is linear quadratic (LQ) design, with the associated SEF design being then linear quadratic Gaussian (LQG) design.

A technique to improve robustness of SEF based designs, such as LQG designs, is to represent the plant uncertainty in the frequency band of interest by the addition to the plant input of fictitious noise in this band, [5,][6]. Such a technique for minimum phase plants leads to "loop recovery", because as the magnitude of the fictitious noise increases, the open loop transfer functions, and thus loop robustness properties, approach those for the SF design. The recovery of the SF robustness properties in the frequency band of the noise is at the expense of a reduced performance in the frequency band of the noise. The theory for the case of fictitious white noise is developed in [5], while the case of added colored noise is studied in [6].

In this chapter, the notion of loop recovery is extended to certain classes of SEF designs for nonminimum phase plants. The results are developed using an all-pass / minimum phase (i.e. inner/outer) factored form for the plants. The SEF controller designs can be based on applying known LQG and H^{∞} techniques [7],[8]. An initial state feedback design must constrain the feedback to feedback of only the minimum phase factor states, giving an outer state feedback (OSF) design (possibly dynamic). The addition of fictitious noise at the input to the minimum phase factor ensures that when state estimators are employed, there is loop recovery. Of course, certain nonminimum phase plants can never be "robustly" controlled, nor "robustly" controlled using state feedback of only the minimum phase factor states. In the latter case, when loop recovery is applied, it is not expected that the resulting design will be "robust".

In the case of two degree of freedom design, the technique of [8] building on those of [3] can be applied. A particularly convenient formulation arises using the results of [9].

In Section 2, a class of controllers is defined for which loop recovery properties are guaranteed. A design approach and an example design are included in Section 3 to add insight. Section 4 summarizes new results for frequency shaped state estimators by dualizing control results in [10]. Such results are useful for frequency shaped loop recovery. Conclusions are drawn in Section 5.

2. LOOP RECOVERY

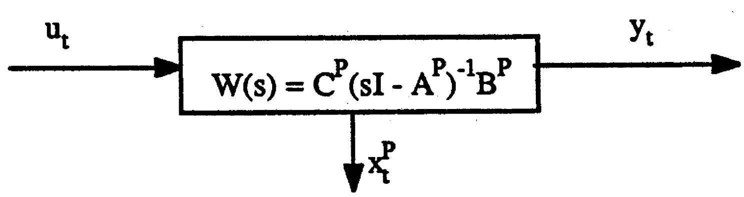
In this section, we consider in turn factored signal models, state estimation and controller constraints for loop recovery.

Factored Form Plant Model:

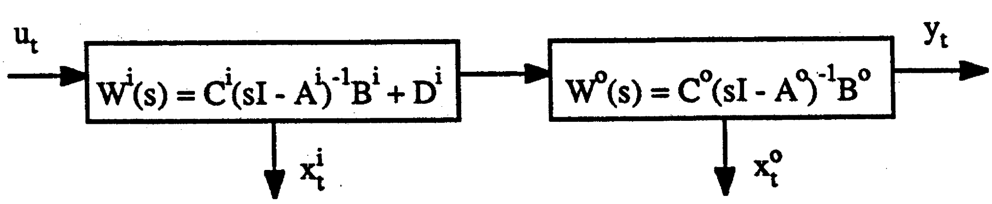
Let us consider a linear finite dimensional, time invariant plant with a transfer function matrix W(s). Consider also a standard inner/outer factorization of W(s) as, see [11]

$$W(s) = W^o(s)W^i(s) \tag{2.1}$$

Here $W^i(s)$ is asymptotically stable, and "inner", or equivalently "all-pass", in that $W^i(-s)^\tau W^i(s) = I$. Also, $W^o(s)$ is "outer" or "minimum phase" in that $W^o(s)$ has full rank for s in the right-half plane Re[s] > 0. Here also, $W^i(s)$ has its poles in Re[s] < 0, and the zeros of $W^o(s)$ are in Re[s] ≤ 0. Should W(s) be minimum phase, then trivially $W^i(S) = I$, $W^o(s) = W(s)$. Unstable modes of W(s) appear in $W^o(s)$. As guaranteed in the state space factorization approach of [11], there may exist cancellation of the poles $W^i(s)$ and the left half plane zeros of $W^o(s)$, and none between the zeros of $W^i(s)$ and the poles of $W^o(s)$. Actually, imaginary axis zeros of W(s) are not permitted for the algorithms of [11], but are permitted in much of the subsequent theory.



(a) Deterministic Model



(b) Factored (nonminimal) Model

Fig. 2.1. Plant Representations

Referring to Figure 2.1.a, let us consider a minimal state space realization $\{A^P,B^P,C^P\}$ for the plant with transfer function $W(s) = C^P(sI-A^P)^{-1}B^P$ and state x^P_t. Referring to Figure 2.1.b, let us consider a non-minimal representation of the plant factored into its all-pass and minimum phase factors. The all-pass system, $\{A^i,B^i,C^i,D^i\}$ assumed minimal, has states x^i_t and a transfer function $W^i(s) = [C^i\Phi^i(s)B^i+D^i]$ where $\Phi^i(s)$ denotes $(sI-A^i)^{-1}$. Also $D^{i\tau}D^i = I$. The minimum phase system $\{A^o,B^o,C^o\}$, assumed minimal, has states x^o_t and a transfer function $W^o(s) = C^o\Phi^o(s)B^o$, $\Phi^o(s) = (sI-A^o)^{-1}$.

Stochastic Model:

Consider the factored form model of Figure 2.1.b, but with process noise $v_t^P = N[0, Q^P]$ and measurement noise $w_t = N[0,R^P]$. These are assumed to be independent, zero mean and white, having covariance matrices $Q^P \geq 0$, $R^P > 0$ respectively. For simplicity, the process noise is assumed to be added at the plant input and Q^P is taken as $Q^P = q^P I$. In addition there is included in the model a fictitious process noise v^f_t, assumed to be inserted at the input to the minimum phase factor. Its purpose is to achieve loop recovery properties in its frequency band, It can be viewed as representing the effects of plant uncertainty in this frequency band. In the first instance, let us consider the case of white fictitious noise $v^f_t = N[0,qI]$. Denoting the state of the factored form plant model as x_t, then its state space equations are

$$x_t = \begin{bmatrix} x^o_t \\ x^i_t \end{bmatrix}, \quad v_t = \begin{bmatrix} v^f_t \\ v^P_t \end{bmatrix}, \quad \frac{dx_t}{dt} = Ax_t + Bu_t + \Gamma v_t, \quad y_t = Cx_t + w_t$$

$$A = \begin{bmatrix} A^o & B^oC^i \\ 0 & A^i \end{bmatrix}, \; B = \begin{bmatrix} B^oD^i \\ B^i \end{bmatrix}, \; \Gamma = \begin{bmatrix} B^o & B^oD^i \\ 0 & B^i \end{bmatrix}, \; C = [C^o \;\; 0] \qquad (2.2)$$

Should W(s) be minimum phase, then $x_t = x^P{}_t = x^o{}_t$, otherwise, x_t is a non-minimal state vector.

State Estimator:

Applying Kalman filter theory [1],[2] to the stochastic (non-minimal) plant model (2.2) yields the time invariant estimator

$$\hat{x}_t = \begin{bmatrix} \hat{x}^o{}_t \\ \hat{x}^i{}_t \end{bmatrix}, \quad K = \begin{bmatrix} K^o \\ K^i \end{bmatrix}, \quad Q = \Gamma \begin{bmatrix} qI & 0 \\ 0 & q^P I \end{bmatrix} \Gamma^\tau, \qquad R = R^P$$

$$\frac{d\hat{x}_t}{dt} = A\hat{x}_t + Bu_t + K(y_t - C\hat{x}_t), \quad \hat{x}_o = 0, \quad K = PC^\tau R^{-1},$$

$$PA^\tau + AP - PC^\tau R^{-1}CP + Q = 0, \; P \geq 0 \tag{2.3}$$

It is easy to verify that for $Q^P = q^P I$ the partitioned solution P has block $P_{12} = P_{21}{}^\tau = 0$. Also $K^i = 0$ (details to be seen in Lemma 1). Then for zero initial conditions, (2.3) gives

$$\hat{x}(s) = (sI - A + KC)^{-1}[Bu(s) + Ky(s)], \text{ or } \; \hat{x}^i(s) = \Phi^i(s)B^i u(s)\,,$$

$$\hat{x}^o(s) = \Phi^o(s)\{K^o[y(s) - C^o\hat{x}^o(s)] + B^o W^i(s)u(s)\}, \tag{2.4}$$

The estimation (2.3) is well defined when [A,C] is detectable, and in addition is asymptotically stable when $[A, Q^{\frac{1}{2}}]$ is stabilizable. Since the all-pass factor here is asymptotically stable, these conditions simplify as $[A^o, C^o]$ detectable, $[A^o, B^o(qI + q^P D^i D^{i\tau})^{\frac{1}{2}}]$ stabilizable. These are trivially satisfied for all finite $q > 0$, since $\{A^o, B^o, C^o\}$ is a minimal representation of $W^o(s)$.

Frequency Shaped Estimation:

For practical control law design using loop recovery techniques applied to minimum phase plants [6], the fictitious noise is usually frequency shaped so that robustness is achieved in the frequency band where otherwise the design is not robust. Thus fictitious noise is usually injected only at frequencies in the vicinity of the cross over (loop unity gain) point. The controller and performance is then unchanged outside such frequency bands. For practical designs, in the more general setting of nonminimum phase plants as here, the same thinking applies.

Frequency Shaped Fictitious Noise: Assume that the frequency shaped fictitious noise v^f_t has a power spectrum $qQ^f(s) = qQ^{f\tau}(-s)$ which is nonsingular a.e. The minimum phase stable spectral factor of $Q^f(s)$ is denoted by $[Q^f(s)]^{\frac{1}{2}}$. (2.5)

Frequency shaped estimation theory, see Section 4, now applies. Leaving aside any frequency shaped estimation of x^i_t since this is not required in subsequent loop recovery theory, the state estimation (2.4) generalizes as,

$$\hat{x}^o(s) = \Phi^o(s)\{K^o[y(s)-C^o\hat{x}^o(s)] + B^oW^i(s)u(s)\}, \tag{2.6}$$

where $K^o(s)$ is given from the spectral factorization

$$C^o\Phi^o(s)Q^o(s)\Phi^o(-s)^\tau C^{o\tau}+R = F^o(s)RF^o(-s)^\tau$$

$$C^o\Phi^o(s)K^o(s) = [F^o(s) - I], \tag{2.7}$$

Here $Q^o(s)$ is the spectrum of the noise signals affecting the states x^o_t and consists of the sum of two terms, $qB^oQ^f(s)B^{o\tau}$ due to v^f_t, and another term, $Q_v(s)$ due to v^P_t

prefiltered by $B^oW^i(s)$. We assume that there are no pole-zero cancellations in $C^o\Phi^o(s)Q^o(s)\Phi^o(-s)^\tau C^{o\tau}$. Here also, $F^o(s)$ is the unique minimum phase spectral factor with the same poles as $\Phi^o(s)$ and the stable poles of $Q^o(s)$. The theory of Section 4 tells us that with A^o, C^o observable, a stable proper $K^o(s)$ can be constructed satisfying (2.7) so that the optimal estimation (2.6) can be implemented. Indeed this property can also be verified using state space models for the shaping filter and augmenting those to the plant model and applying standard Riccati theory and spectral factorization concepts.

OSF Controller:

Consider the stabilizing outer state feedback control law, ignoring initial conditions, and associated open loop transfer function as

$$u^{OSF}(s) = L^{OSF}(s)x^o(s), \quad W_{OL}^{OSF}(s) = L^{OSF}(s)\Phi^o(s)B^oW^i(s) \qquad (2.8)$$

For the main result of this section, there is no concern about how $L^{OSF}(s)$ is designed. However, to use the result as a design tool we present subsequently one approach to the design of $L^{OSF}(s)$ based on LQ/H^∞ methods.

SEF Controller:

From (2.8), we see that the outer state feedback control u_t^{OSF} consists of dynamic feedback of the outer states x^o_t. The corresponding state estimate feedback controller using the estimator (2.3) to achieve estimates $\hat{x}^o_t$, replaces x^o_t with $\hat{x}^o_t$. Thus in obvious operator notation,

$$u_t^{OSF} = L^{OSF}x^o_t, \quad u_t^{SEF} = L^{OSF}\hat{x}^o_t \qquad (2.9)$$

Of course, ignoring initial conditions, $u^{OSF}(s) = u^{SEF}(s)$. However, the open loop transfer functions $W_{OL}^{OSF}(s)$, $W_{OL}^{SEF}(s)$ differ. The transfer function $W_{OL}^{SEF}(s)$ is depicted in Figure 2.2 and is (with $K^i = 0$, $L^{ISF} = 0$)

$$W_{OL}^{SEF}(s)=\{I-L^{OSF}(s)[I+\Phi^o(s)K^oC^o]^{-1}\Phi^o(s)B^o\}^{-1}\Phi^o(s)B^oW(s) \quad (2.10)$$

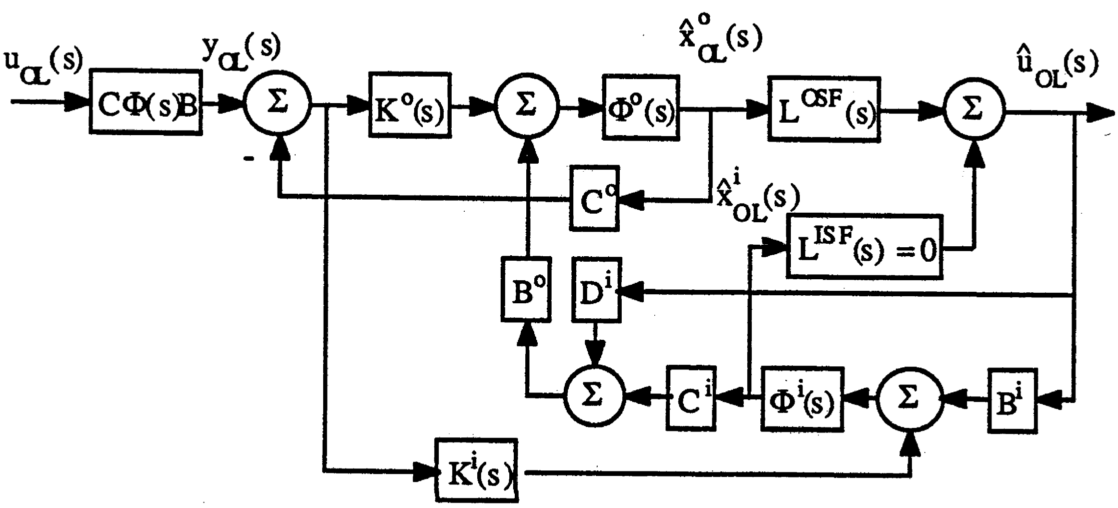


Fig. 2.2. Open-loop Transfer Function $W_{OL}^{SEF}(s)$

<u>Loop Recovery</u>

Loop recovery at the plant input is said to occur when for almost all s, the open loop state estimate feedback transfer function approaches the open loop state feedback transfer function as adjustments are made to the design of the state estimator. Here the adjustment is that $q \to \infty$ and we seek the loop recovery property

$$\lim_{q\to\infty} W_{OL}^{OSF}(s) = W_{OL}^{SEF}(s) \quad \text{a.e.} \qquad (2.11)$$

Lemma 1: Consider the factored plant (2.2) state estimator (2.3) and feedback control laws as in (2.9). Then for the case when $Q^P = q^P I$

$$P_{12} = P_{21}^{\tau} = 0, \quad K^i = 0, \quad K^o = P_{11} C^{o\tau} R^{-1}, \tag{2.12a}$$

$$P_{11} A^{o\tau} + A^o P_{11} - P_{11} C^{o\tau} R^{-1} C^o P_{11} + (q^P + q) B^o B^{o\tau} = 0, \; P_{11} \geq 0 \tag{2.12b}$$

Proof: The solution $P \geq 0$ of (2.3) is known to be unique under minimality of $[A^o, B^o, C^o]$ as in early discussions. We first show that taking $P_{12} = P_{21}^{\tau} = 0$ leads to a solution $P \geq 0$ to conclude that indeed $P_{12} = P_{21}^{\tau} = 0$. Taking $P_{12} = P_{21}^{\tau} = 0$ in (2.3) gives

$$P_{22} A^{i\tau} + A^i P_{22} + q^P B^i B^{i\tau} = 0, \qquad P_{22} \geq 0$$

which has a unique solution with $[A^i, B^i]$ stabilizable as here. From [12] Theorem 5.1, with a mild extension to include the non-square case we have

$$C^i P_{22} + q^P D^i B^{i\tau} = 0$$

Now (2.3) yields (2.12b) which has a solution with $[A^o, B^o, C^o]$ minimal as here. Consequently, $P = \mathrm{Diag}[P_{11} \; P_{22}] \geq 0$ is a solution to (2.3). Thus $P_{12} = P_{21}^{\tau} = 0$. Then (2.12) holds. ΔΔΔ

Remark The important aspect of this lemma is that the estimator gain is constructed in terms of only the outer factor parameters A^o, B^o, C^o and the covariances R, $q^P I$, qI. For the frequency shaped fictitious noise as in (2.5), the same results hold as established using the results of Section 4. Again $K^i = 0$ and $K^o(s)$ is constructed in terms of A^o, B^o, C^o, R, $q^P I$, $qQ^f(s)$.

Theorem 1 Consider the plant in factored form (2.1), (2.2) with $[A^o,B^o,C^o]$ minimal, and the optimal state estimation (2.6) under (2.5),(2.7). Consider also the SEF controller, as above, based on the OSF feedback control law (2.8). Then

$$\lim_{q\to\infty} C^o\Phi^o(s)[I + K^o(s)C^o\Phi^o(s)]^{-1}B^o[Q^f(s)]^{\frac{1}{2}} = 0, \text{ a.e.} \tag{2.13a}$$

$$\lim_{q\to\infty} q^{-\frac{1}{2}}K^o(s) = B^o[Q^f(s)]^{\frac{1}{2}}R^{-\frac{1}{2}}, \text{ a.e.} \tag{2.13b}$$

and the loop recovery property (2.11) holds.

Proof Rigorous proof techniques as in [2] are straightforward but tedious. Here a believable "proof" is given leaving out the technicalities. The spectral factorization (2.7) associated with the estimation of $x^o(s)$, can be written as

$$C^o\Phi^o(s)[B^oQ^f(s)B^{o\tau} + q^{-1}Q_v(s)]\Phi^o(-s)^\tau C^{o\tau} + q^{-1}R$$

$$= [I+C^o\Phi^o(s)K^o(s)](q^{-1}R)[I+C^o\Phi^o(-s)K^o(-s)]^\tau \tag{2.14}$$

In turn, as $q \to \infty$, since $B^oQ^f(s)B^{o\tau}$ and $q^{-1}Q_v(s)$ are non-negative definite a.e., then with $[A^o,B^o,C^o]$ minimal, (2.13a) and (2.13b) follow. Moreover, under (2.5)

$$\lim_{q\to\infty} [I + K^o(s)C^o\Phi^o(s)]^{-1}B^o = 0, \quad \text{a.e.} \tag{2.15}$$

With obvious notation, the open loop version of the expression for state and state estimates are (for zero initial conditions, refer to Fig.2.2)

$$\hat{x}^o_{OL} - x^o_{OL} = -\Phi^o[K^oC^o(\hat{x}^o_{OL} - x^o_{OL}) - B^oW^i(u_{OL} - \hat{u}_{OL})]$$

$$= \Phi^o[I + K^oC^o\Phi^o(s)]^{-1}B^oW^i(u_{OL} - \hat{u}_{OL}) \tag{2.16}$$

Applying (2.15), we have

$$\lim_{q \to \infty} \hat{x}^o{}_{OL}(s) = x^o{}_{OL}(s) \qquad \text{a.e.} \tag{2.17}$$

Then the loop recovery property (2.11) follows. ΔΔΔ

Remarks 1. If the plant is minimum phase, then outer state feedback control becomes full state feedback control, and standard "loop recovery" results are recovered as a special case. For the case when there are plant zeros on the imaginary axis, in the limit as $q \to \infty$, the state estimator loses its asymptotic stability property.

2. For the case when Q^P is more general than $q^P I$ so that $K^i(s) \neq 0$, then in (2.4) (2.6) (2.10), (2.13) - (2.16), $K^o(s)$ is replaced by $K^o(s)+B^oC^i\Phi^i(s)K^i(s)$. Also $\hat{x}^i(s) = \phi^i(s)\{B^iu(s) + K^i(s)[y(s) - C^o\hat{x}^o(s)]\}$.

3. A major observation of this chapter is that the above results can be extended using our analysis approach to achieve near loop recovery in frequency bands where $qQ^f(s)$ is "large" and $L^{ISF}(s)\Phi^i(s)B^i$ is "small" in some relative sense. It is readily shown that the transfer function $[W^{SEF}_{OL}(s) - W^{SE}_{OL}(s)]$ is then "small" in such a band. The requirement that $L^{ISF}(s)\Phi^i(s)B^i$ be "small" can be achieved for arbitrary control laws if $L^{ISF}(s)\Phi^i(s)B^i$ is "small" in the band of interest. Equivalently, this requirement is that the plant be "near minimum phase in the frequency band" of interest with $W^i(j\omega) \approx D^i$ in this band where $D^{i\tau}D^i = I$. Notice that if right half plane zeros of W(s) are in the far right half plane, then in the scalar case $W^i(s) \approx -1$, or if the zeros of W(s) are outside the frequency band but close to the $j\omega$ axis then $W^i(s) \approx +1$.

3. A Design Approach - Illustrative Example

An OSF controller design approach:

Consider the factored plant as (2.2) and associate with this plant a quadratic cost function,

$$V = \int_0^\infty (x_t^\tau Q^c x_t + u_t^\tau R^c u_t)dt \tag{3.1}$$

with $Q^c = Q^{c\tau} \geq 0$, $R^c = R^{c\tau} > 0$. The optimal control has the form

$$u_t = Lx_t = L^i x^i_t + L^o x^o_t, \quad [L^o \;\; L^i] = L \tag{3.2}$$

for some gain L found using standard techniques [1],[2]. For the more general case of frequency shaped LQ designs as in [10] in which Q^c, R^c generalize as $Q^c(s)=Q^{c\tau}(-s)$, $R^c(s) = R^{c\tau}(-s)$, then (3.2) generalizes as

$$u(s) = L(s)x(s) = L^i(s)x^i(s) + L^o x^o(s), [L^o(s)\; L^i(s)] = L(s) \tag{3.3}$$

Details are not developed here.

If we replace x^i_t by some estimate $\hat{x}^i_t$ obtained by any standard state estimator with stable dynamic, as in (2.3) or one with input u_t, and x^o_t, then the inner/outer state feedback control law (3.3) becomes an outer state feedback law. If in such an estimator $K^i = 0$, as in Lemma 1, ignoring initial conditions, $\hat{x}^i(s)=x^i(s)=\Phi^i(s)B^i u(s)$, and substitution into (3.3) yields an outer state feedback proper controller, with

$$u^{OSF}(s) = L^{OSF}(s)x^o(s),\; L^{OSF}(s) \triangleq [I - L^i(s)\Phi^i(s)B^i]^{-1}L^o(s) \tag{3.4}$$

The controller, denoted $L^{OSF}(s)$, being a state estimate feedback scheme in disguise, is stabilizing according to LQG theory, but may have poor loop robustness. When $K^i \neq 0$ the expression for $L^{OSF}(s)$ is more complicated, but the controller is still stabilizing. The robustness can perhaps be improved by optimizing an H^∞ robustness measure over the class of all controllers yielding the same closed loop transfer function. In our design approach, this class is the class of all optimal controllers in an LQ sense. Details on the optimization are not included here, see [7],[8]. Of course a "robust" design may not be achievable for certain non-minimum phase plants and quadratic indices. We do not address this fundamental problem here.

Example

A simple scalar example is studied to illustrate loop recovery properties and the robust design approach following from the theory of this chapter for general (non-minimum phase) plants. As an example, we work with an unstable non-minimum phase stochastic plant as in Section 2 given by transfer function

$$W(s) = (s - 6.1)(s^3 + s^2 + 23.25s + 50.5)^{-1}$$

Consider now the factored system $W(s) = W^o(s)W^i(s)$ where

$$W^o(s)=(s+6.1)(s^3+s^2+23.25s+50.5)^{-1}, \; W^i(s)=(s-6.1)(s+6.1)$$

with a realization in the notion of Section 2.

$$A^o = \begin{bmatrix} -1 & -23.25 & -50.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B^o = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; C^o = [0 \;\; 1 \;\; 6.1]$$

$$A^i = -6.1, \; B^i = 1, \; C^i = -12.2, \; D^i = 1.$$

The process and measurement noise covariances under such a realization are $Q_v = 1$, $R = 1$ and controller weightings for LQ design are $Q^c = C^{\tau}C$, $R^c = 1$.

For this (non-minimal) representation the LQ control law can be re-expressed in the form (3.2) with $L^o = [0.68\ -9.8\ -22.2]$ and $L^i = [-2.7]$. Also the outer state feedback controller as in (3.4) and open loop transfer function are

$$L^{OSF}(s) = (s + 6.1)(s + 8.8)^{-1}L^o$$

$$W_{OL}^{OSF}(s) = \frac{-0.7s^3 + 13.9s^2 - 37.7s - 135.4}{s^4 + 9.8s^3 + 32.1s^2 + 255.7s + 455.7}$$

The former is stable and the latter has two unstable poles. Figure 3.1 shows the Nyquist plot.

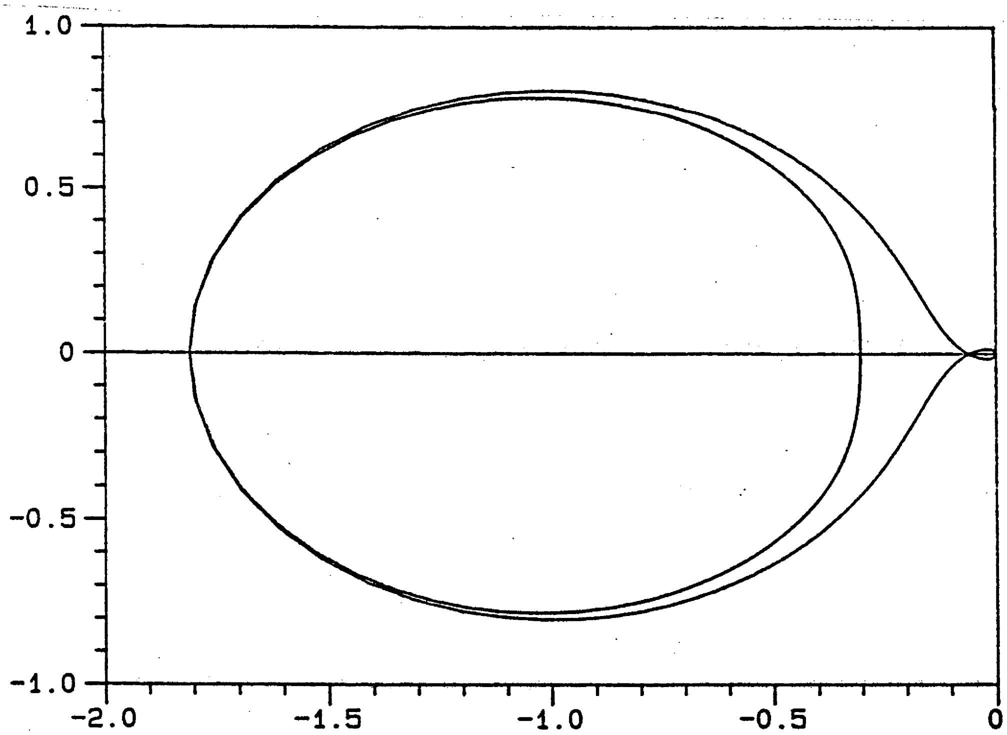


Fig. 3.1. Nyquist Plot for OSF

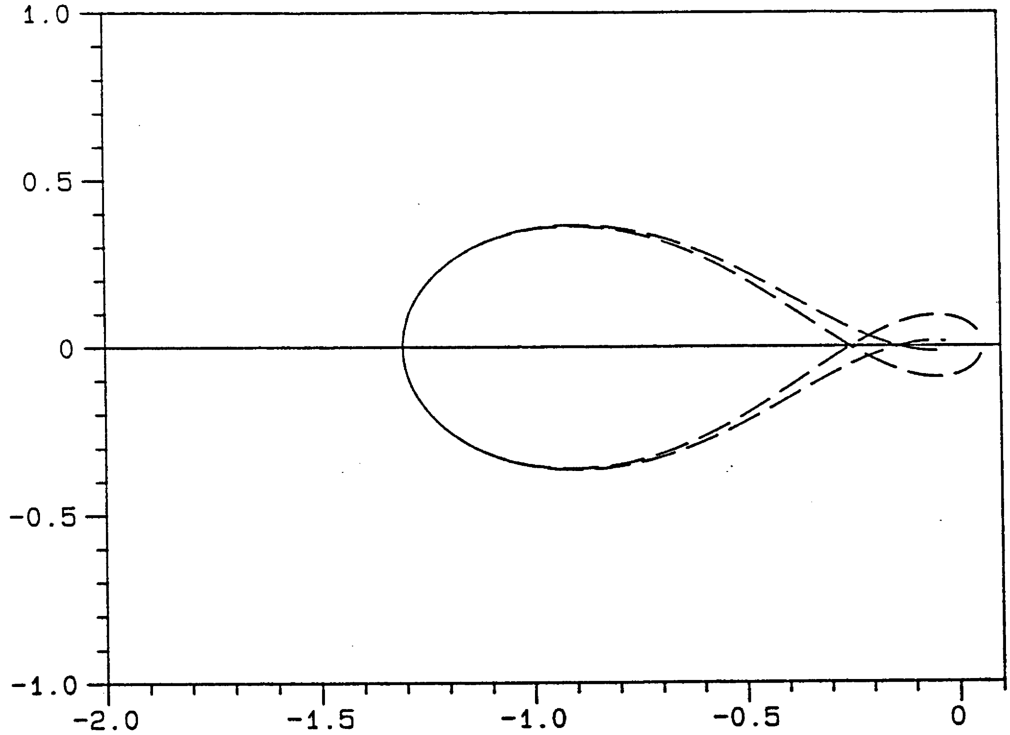


Fig. 3.2. Nyquist Plot for SEF

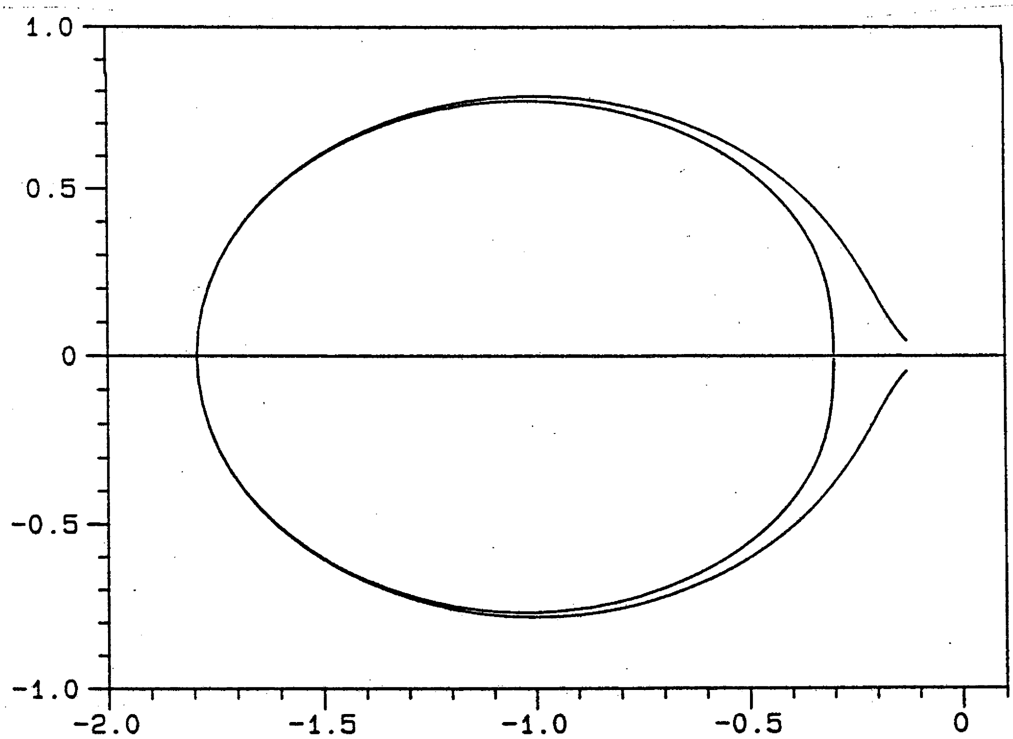


Fig. 3.3. Nyquist Plot for LTR

Now the state estimator is designed with $K^i = 0$, $K^o = [-3.6\ 0.9\ 0.2]^\tau$. Figure 3.2 shows the associated Nyquist plot with relatively poor robustness properties. Choose $v^f = N(0, 10000)$ to recover the outer state feedback loop robustness, now $K^i = 0$, $K^o = [68.1\ \ 11.7\ \ 0.8]^\tau$ and Figure 3.3 shows the loop recovery.

4. Frequency Shaped Estimation

When plant input noise is colored, rather than white as in the standard estimator theory, then it is straightforward to augment the plant model with the noise model and apply standard filter theory to the augmented plant. However, as the theory of [6] illustrates, the formulations and proof techniques are awkward. Here a direct approach in the frequency domain is taken so as to achieve the simplicity of derivation and formulation of the results of Section 2, which are essential for the frequency shaped loop recovery theory of Section 2.

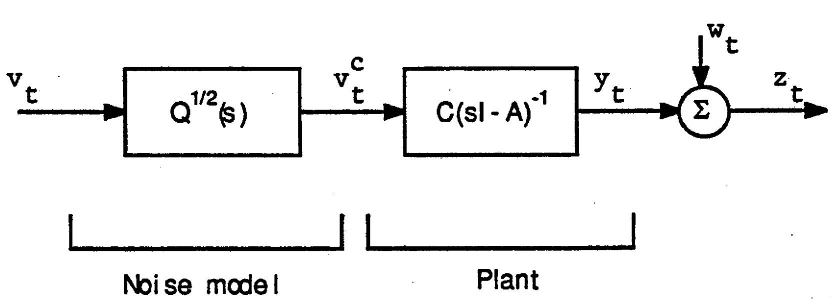


Fig. 4.1. Signal Model

Signal Model:

Let us consider a Wiener type signal model as in Figure 4.1 with v_t, w_t independent white noise sources with zero means and covariances I, $R = R^{\tau} > 0$ respectively. The plant input $v^c{}_t$ is colored noise with a spectrum Q(s) where $Q(s)^{\frac{1}{2}}$ is a stable minimum phase spectral factor of Q(s). The plant state model has a (strictly proper) transfer function $W(s)=C(sI-A)^{-1}$ for some (A,C) observable. The plant is possibly unstable. Its output is z_t. For subsequent results, Q(s) must be such that $W(s)Q(s)W(-s)^{\tau}$ has no hidden $j\omega$ axis modes.

Spectral Factorization:

Consider a spectral factorization of the spectrum of z_t as follows:

$$W(s)Q(s)W(-s)^{\tau}+R = F(s)F^{\tau}(-s)\ , \quad F(s) = [I+W_{OL}(s)]R^{\frac{1}{2}} \tag{4.1}$$

where F(s), $W_{OL}(s)$ have the same poles as $W(s)Q(s)^{\frac{1}{2}}$ and the spectral factor F(s) is minimum phase. In [10], this factorization is carried out for a dual control situation. First W is expressed as a matrix fraction decomposition $W = \tilde{M}\tilde{N}^{-1}$ with $\tilde{M}$, $\tilde{N} \in RH^{\infty}$ (rational proper asymptotically stable). Then $\tilde{N}(s)Q(s)\tilde{N}(-s)^{\tau} - \tilde{M}(s)R\tilde{M}(-s)^{\tau} = S(s)S(-s)^{\tau}$ is factorized to achieve S(s) (strictly) minimum phase and asymptotically stable under the assumption above. Next $W_{OL}(s)$ and F(s) are given from

$$W_{OL}(s) = \tilde{M}(s)^{-1}S(s)R^{-\frac{1}{2}} - I, \quad F(s) = \tilde{M}(s)^{-1}S(s)$$

As shown in [10]. $W_{OL}(s)$ is unique strictly proper (strictly) minimum phase, and has the following properties.

Lemma 2: With $W_{OL}(s)$ given from the spectral factorization (4.1), under the restrictions on W(s), Q(s), R above, with C^{-R} a right inverse of C there exists a $K(s) = (sI-A)C^{-R}W_{OL}(s) \in RH^{\infty}$ stabilizing W(s) such that

$$W(s)K(s) = W_{OL}(s)\,, \tag{4.2}$$

Moreover, there exists some rational P(s) such that [with $X_* \triangleq X(-s)^\tau$]

$$(I + WK)^{-1}WPC \in RH^\infty$$

$$(s^{-1}P)[(sI-A) + KC]_*[(sI-A) + KC](s^{-1}P)_* = KPK_* + Q$$

$$WK = W(s^{-1}P)C^\tau R^{-1}$$

$$W[(s^{-1}P)(sI-A)_*+(sI-A)(s^{-1}P)_*+(s^{-1}P)C^\tau R^{-1}C(s^{-1}P)_*-Q]W_*=0 \tag{4.3}$$

Optimal Estimation:

Consider the filter arrangement of Figure 4.2, which is asymptotically stable since under Lemma 2, K(s) stabilizes W(s) in feedback. Then it is immediate from the spectral factorization (4.1) that v_t is white zero mean and has a covariance R. It follows from the inverse problem of optimal filtering [13] that $\hat{y}_k$ is indeed the optimal filtered estimate of y_t in a least squares sense. Moreover, with [A,C] completely observable, a mild generalization of the argument in [13] gives that $\hat{x}_k$ is the optimal state estimate.

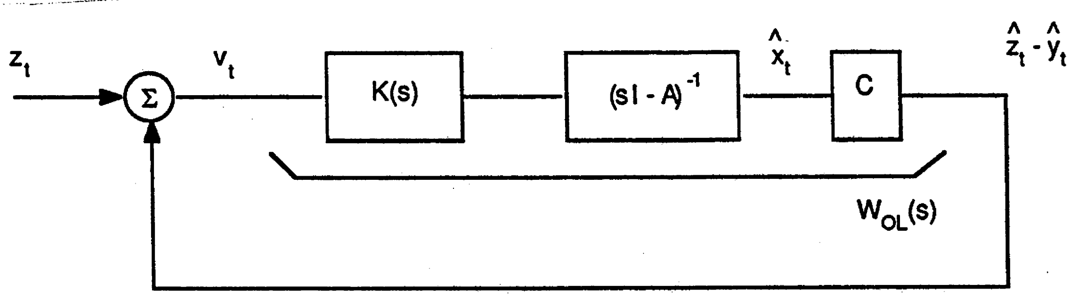


Fig. 4.2. Optimal Filter

5. CONCLUSIONS

The technique of loop recovery for improved robustness in state estimate feedback designs generalizes to cope with nonminimum phase plants. A generalization restricts the class of state feedback controllers to those which feedback only the state (or estimates) associated with the "minimum phase" states in an all-pass/minimum-phase factored signal model form. Of course, for some nonminimum phase plants, such controllers are not expected to achieve robust designs comparable to those for minimum phase plants. However, whatever loop robustness is achieved in such a partial state feedback design is recovered in a state estimate feedback design using the loop recovery techniques of this chapter. One specific method for a stabilizing partial state feedback design has been presented and a design example has been included to illustrate the approach. Also the results for frequency shaped state estimators have been developed.

REFERENCES

[1] B.D.O. Anderson and J.B. Moore, "**Linear Optimal Control**", Prentice Hall, 1971.

[2] H. Kwakernaak and R. Sivan, "**Linear Optimal Control Systems**", Wiley, New York, 1972.

[3] M.G. Safonov and M. Athans, "Gain Phase Margins for Multiloop LQG Regulators", **IEEE Trans. Auto. Control, Vol.AC-23**, No.2, pp173-179, April 1977.

[4] J.C. Doyle, "Guaranteed Margins for LQG Regulators", **IEEE Trans. Auto. Control, Vol.AC-23**, No.4, pp756-757, August 1977.

[5] J.C. Doyle and G. Stein, "Robustness with Observers", **IEEE Trans. Auto. Control, Vol.AC-24**, No.4, pp607-611, August 1979.

[6] J.B. Moore, D. Gangsaas, J.D. Blight, "Performance and Robustness Trades in LQG Regulation Design", **Proceedings of 20th Conference on Dec. and Control**, San Diego pp1191-1200. December 1981.

[7] B.A. Francis, J. Helton and G. Zames, "H^∞-Optimal feedback controller for linear mulitvariable systems", **IEEE Trans. Vol. AC-29** pp.888-900.

[8] J.B. Moore, L. Xia and K. Glover, "On improving Control Loop Robustness of Model Matching Controllers", **Systems and Control Letters** 7 (1986) pp. 83-87.

[9] J.B. Moore, K. Glover and A. Telford "On the Class of all Stabilizing Controllers", **Proc. of Conference on Dec. and Control**, Athens, Dec. 1986.

[10] J.B. Moore, "Robust Frequency Shaped LQ Control", submitted to Automatica.

[11] C. Chu and J. Doyle, "On Inner-outer and Spectral Factorizations", **Proceedings of 23rd Conference on Dec. and Control**, Los Vegas, Dec 1984., pp1764-1765

[12] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞}-error bounds", **Int.J. Control** Vol.39. No.6, pp 1115-1193.

[13] B.D.O. Anderson and J.B. Moore, "**Optimal Filtering**", Prentice Hall, 1979.

Chapter 3

On Improving Control-Loop Robustness of Model Matching Controllers

1 Introduction

In seeking robust controllers, H^∞-methods search over the (infinite) class of all stabilizing transfer function controllers for one that minimizes some L^∞-sensitivity measure. A key observation is that such problems can be reduced to solving a Nehari H^∞-optimization of an L^∞-norm, [1]-[5]. Appropriate generalizations of the fundamental results using more general indices [6] including frequency shaped indices [4], potentially lead to practical designs.

Here, a class of stabilizing two-degree-of-freedom controllers which achieve a specified (achievable) closed loop transfer function is conveniently characterized. Also, the search over this class of controllers for one which has optimum open loop robustness properties in an L^∞-sense is shown to reduce to solving a standard H^∞-optimization problem.

The a priori specification of a desired transfer function could arise, for example, from an optimal linear quadratic Gaussian (LQG) design, or some other standard method applied to the nominal plant. Referring to Figure 1, a preliminary design for a plant $P(s)$ could give proper controllers $K^o(s) = [K_1{}^o(s)\ K_2{}^o(s)]$ which achieve desired closed loop transfer function properties for the nominal plant but which have poor open loop robustness properties, as measured by the L^∞-norm $\| I + P(s)K_2{}^o(s) \|_\infty$, or some frequency shaped version of this. The results of this chapter then allow the definitions of a comprehensive class of controllers $K(Q,s)$ in terms of an arbitrary $Q(s) \in RH^{\infty}$ (proper stable transfer function), so that the closed loop transfer function is invariant of $Q(s)$. This allows the selection of a specific $Q(s)$, denoted $Q_{opt}(s)$, to minimize the index

$$\| I + P(s)K_2(Q,s) \|_{\infty}$$

or a related measure. The controller $K(Q_{opt},s)$ achieves the desired closed loop transfer function for the nominal plant $P(s)$ and improves open loop robustness.

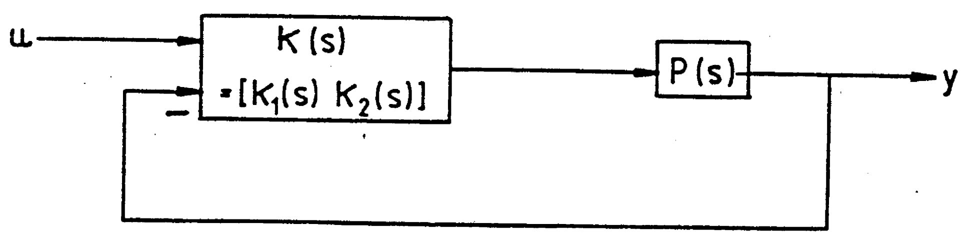


Figure 1. Control System with two-degree-freedom Controller

2. The Class of Stabilizing Controllers

Consider the class of two-degree-of-freedom controllers of Figure 1 where $P(s)$ is the plant and $K(s) = [K_1(s)\ K_2(s)] \in R_p$ (rational proper transfer function) is the controller. The controller is said to be stabilizing when all transfer functions between variables are stable. When $K_1(s) = I$, then the controller has one degree of freedom and is stabilizing if and only if

$$K_2(I + PK_2)^{-1} \in RH^{\infty}, \quad (I + PK_2)^{-1} \in RH^{\infty},$$

$$(I + PK_2)^{-1}P \in RH^\infty, \quad I - K_2(I + PK_2)^{-1}P \in RH^\infty \tag{2.1}$$

The class of such controllers is given in terms of an arbitrary stable Q(s) as in [7]:

$$K_2(Q) = (U - MQ)(V + NQ)^{-1}, \qquad Q \in RH^\infty$$

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \qquad \tilde{V}M + \tilde{U}N = I, \qquad \tilde{N}U + \tilde{M}V = I,$$

$$M, N, \tilde{M}, \tilde{N}, U, V, \tilde{U}, \tilde{V} \in RH^\infty \tag{2.2}$$

<u>Lemma 2.1</u> The class of two-degree-of-freedom controllers $K(s) = [K_1(s)\ K_2(s)]$ for the plant P(s), as in Figure 1, is stabilizing if and only if

$K_2(s)$ is stabilizing as one-degree-of-freedom controller for P(s) [namely (2.1) hold] (2.3a)

$$T_1 \triangleq (I + K_2P)^{-1}K_1 \in RH^\infty,$$

$$T_2 \triangleq P(I + K_2P)^{-1}K_1 \in RH^\infty \tag{2.3b}$$

<u>Proof</u>: All of the possible transfer functions between variables are given by those in (2.1) and (2.3) or are trivially related to those using the standard identity

$$(I + XY)^{-1}X = X(I + YX)^{-1}. \qquad \Delta\Delta\Delta$$

<u>Remarks</u> 1. Observe that if the controller is realized as two separate controllers, as is often the case in a classical servo design, then in addition, (2.3) would include the condition $K_1(s) \in RH^\infty$, details are omitted.

2. The class of all stabilizing two-degree-of-freedom controllers,

$K(s) = [K_1(s)\ K_2(s)]$ for the plant $P(s)$, can be characterized in terms of two arbitrary transfer functions, $Q_1(s)$, $Q(s) \in RH^\infty$ as follows, see also [8]. Characterize the matrix $K_2(s)$ in terms of arbitrary $Q(s) \in RH^\infty$ as in (2.2), and $K_1(s)$ in terms of arbitrary $Q_1(s) \in RH^\infty$ as

$$K_1 = (M + K_2N)Q_1, \quad Q_1 \in RH^\infty \tag{2.4}$$

A proof is as follows. From (2.4), and since $M, N \in RH^\infty$,

$$T_1 \triangleq (I + K_2P)^{-1}K_1 = MQ_1 \in RH^\infty$$

$$T_2 \triangleq P(I + K_2P)^{-1}K_1 = NQ_1 \in RH^\infty$$

and (2.3b) holds. From (2.2) then (2.3a) holds. Applying Lemma 2.1 then $K = [K_1\ K_2]$ defined from (2.2) and (2.4) is stabilizing. Also, given arbitrary stabilizing $K = [K_1\ K_2]$ for P, then K_2 is given from (2.2) in terms of arbitrary $Q \in RH^\infty$. Now define

$$Q_1 = M^{-1}(I + KP)^{-1}K_1$$

with M from (2.2). This has the property that $[M\ \ N]Q_1 \in RH^\infty$ with K_2 stabilizing and N also from (2.2). Since M, N are coprime, then $Q_1 \in RH^\infty$. Now (2.4) holds trivially.

3. A Class of Model Matching Controllers

To motivate the following results, let us consider that a two-degree-of-freedom stabilizing controller $K^* = [K_1^*\ \ K_2^*]$ is designed to meet performance requirements for the nominal plant P with

$$T_1^* \triangleq (I + K_2^*P)^{-1}K_1^* \in RH^\infty$$

$$T_2^* \triangleq P(I + K_2^*P)^{-1}K_1^* \in RH^\infty \tag{3.1}$$

Let us seek result which allow improvement of loop robustness of such a design while keeping the transfer functions $T_1^*(s)$, $T_2^*(s)$ invariant of any adjustments made to K_1, K_2.

The design technique to achieve $T_1^*(s)$, $T_2^*(s)$ is not important for our theory. However we could have in mind an optimal LQG design using a performance index with engineering significance. Often such designs results in poor robustness to plant uncertainty, in which case the following results could be useful.

Lemma 3.1 Consider the class of two-degree-of-freedom controllers

$$K(s) = [K_1(s)\ K_2(s)] \in R_p$$

for the plant P(s) as in Figure 1. Necessary and sufficient conditions for $K = [K_1\ K_2]$ to match the model transfer functions (3.1) and be stabilizing are that

$$K_1 = (I + K_2P)(I + K_2^*P)^{-1}K_1^* \equiv T_1^* + K_2T_2^* \tag{3.2a}$$

K_2 as one-degree-of-freedom controller for P(s) is stabilizing. (3.2b)

Moreover, the entire class of such stabilizing controllers can be characterized in terms of arbitrary stabilizing K_2 as a one-degree-of-freedom controller for P, as in Figure 2. In turn, this controller class can be characterized in terms of arbitrary $Q \in RH^\infty$ as in (2.2). Furthermore, closed loop transfer functions are affine in Q.

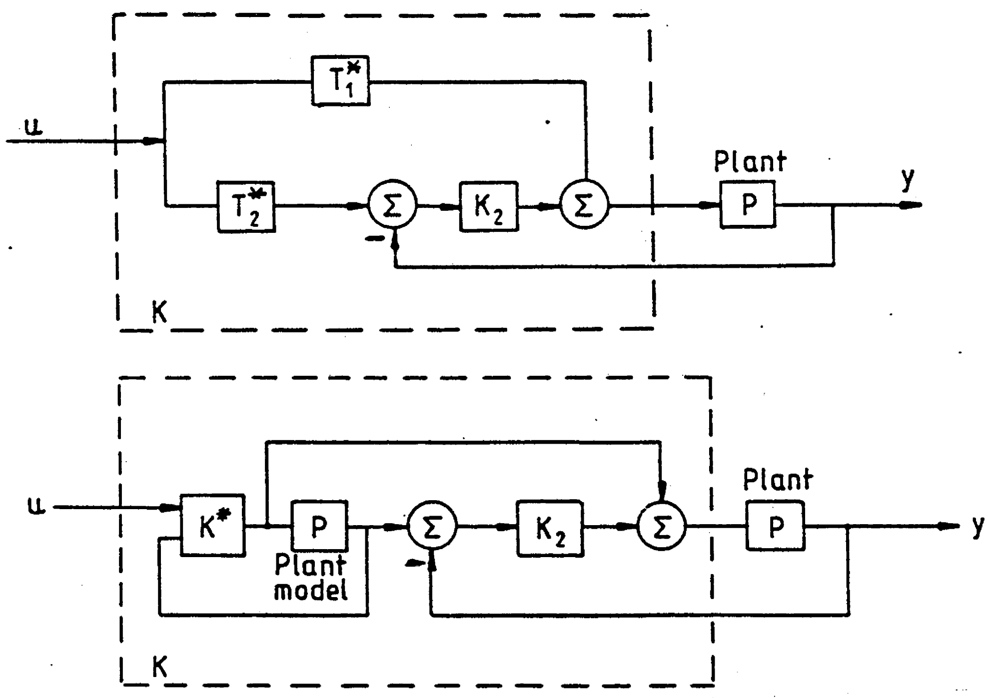


Fig. 2. Model Matching Controllers.

<u>Proof</u> A necessary condition for model matching is that

$$T_1 \equiv (I + K_2P)^{-1}K_1 = (I + K_2^*P)^{-1}K_1^* \equiv T_1^* \tag{3.3a}$$

which is equivalent to (3.2a). Now (3.3a) implies that

$$T_2 \equiv P(I + K_2P)^{-1}K_1 = P(I + K_2^*P)^{-1}K_1^* \equiv T_2^* \tag{3.3b}$$

and so (3.2a) is also sufficient for model matching. Clearly, (3.2b) is necessary for stability. Also, the property (3.2), together with (3.1) ensure that the model matching controller is stabilizing.

The structure of Figure 2 is verified since its transfer function is

$$P(I + K_2P)^{-1}T_1^* + PK_2(I + PK_2)^{-1}T_2^*$$

$$= P(I + K_2P)^{-1}(T_1^* + K_2T_2^*)$$

$$= P(I + K_2P)^{-1}K_1 = T_2^*$$

which is invariant of K_2 given that K_2 is stabilizing. Now substituting K_1 from (3.2a) into (2.3b), and applying (2.1), (2.2) the remaining results are obtained.

$\Delta\Delta\Delta$

<u>Remarks</u> 1. For the controllers of Figure 2, robustness properties are crucially dependent on K_2. Observe that for the nominal plant and zero initial conditions, the input to the block K_2 is zero since its transform is

$$T_2^*(s)u(s) - y(s)$$

Where $y(s) = T_2^*(s)u(s)$. In this case then the control is essentially feed-forward control and independent of K_2. Otherwise, the greater the gain K_2 at a particular frequency, the more significant is the feedback control via K_2 at that frequency.

2. In any realization of $K = [K_1\ K_2]$, it is important to avoid duplication of an unstable mode. For example, if $K_1(s)$, $K_2(s)$ were realized as separate controllers as in classical designs, then any unstable poles in K_2 that are also in K_1 would give closed loop instability. In the schemes of Figure 2, any instability is confined within a stable closed loop.

3. The schemes of Figure2 are in terms of an arbitrary stabilizing K_2 for the plant P, which in turn can be parametrized in terms of arbitrary $Q \in RH^\infty$ as in (2.1).

Now model matching is invariant of arbitrary $Q \in RH^\infty$ but control loop robustness properties are Q dependent. Consider the case when an original design is carried out based on an observer, such as an LQG design. Then Figure 2 can be redrawn as in Figure 3 for the case Q = 0. For the more general case of arbitrary $Q \in RH^\infty$, [9] shows that the controller is still stabilizing and represents the entire class of stabilizing proper controllers. Here we see, in addition, that model matching is invariant of $Q \in RH^\infty$, and the entire class of model matching controllers can be parametrized in terms of arbitrary $Q \in RH^\infty$ as in Figure 3. [An area for future research could be in (on line) adaptive Q selection to minimize residuals in the presence of varying disturbances to the plant.]

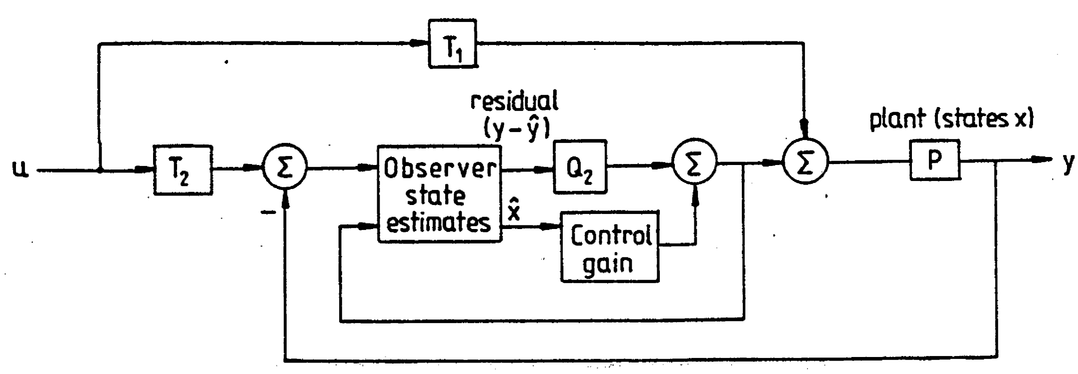


Fig. 3. Equivalent Observer-based Design.

4. Improving Robustness via H^∞-Optimization

Consider the controllers as in Figures 2, 3, where the designer is free to select $Q(s) \in RH^\infty$, and thus $K_2(Q)$ to optimize some robustness /performance measure. Consider the measure

$$J(Q) = \|W_1[I + PK_2(Q)]^{-1}W_2\|_\infty, \quad Q \in RH^\infty$$

where W_1, W_2 are square with $W_1(j\omega)$ full rank for all ω and W_1, W_2, $W_2^{-1} \in RH^\infty$. This is a standard H^∞-optimization problem for which a straightforward solution procedure exists [9]. More sophisticated cost functions can include performance measures but require search procedures [9].

Remarks 1. Should the controller $K(s) = [K_1(s)\ K_2(s)]$ be realized as two separate controllers $K_1(s)$, $K_2(s)$, then $K_1(s)$ must be asymptotically stable for a realization to be practical. Such an additional stability condition would render the optimization task a constrained one, in general, and not amenable to standard solution procedures. Details are omitted.

2. The more general task of searching for $Q_1(s)$, $Q(s) \in RH^\infty$ to achieve performance and robustness in terms of general indices is given in [8].

5. Conclusion Remarks

In this chapter, it is shown that model matching robust designs can be achieved by optimization over the class of all stabilizing two-degree-of-freedom controllers (realized without duplication of instability modes). The optimization task is shown to reduce to a standard one-degree-of-freedom optimization over the class of all stabilizing one-degree-of-freedom controllers. Such results are potentially the basis for robust controller designs to meet transfer function performance /robustness objectives, or to improve robustness of standard designs, such as LQG designs which focus primarily on performance for the nominal plant.

REFERENCE

[1] G. Zames and B. Francis, "Feedback, Minimax Sensitivity and Optimal Robustness", **IEEE Vol.AC-28**, pp 585-601, 1983

[2] B. Francis and G. Zames, "Design of H^∞-optimal Multivariable feedback Systems", **Proc. 22nd CDC**, pp , San Antonio, 1983

[3] B. Francis and G. Zames, "On H^∞-optimal Sensitivity Theory for SISO Feedback Systems", **IEEE Vol.AC-29**, pp 9-16, 1984

[4] B. Francis, "Notes on H^∞-optimal Linear Feedback Systems", **Lectures** presented in Linköping University, August, 1983

[5] B.A. Francis, J. Helton and G. Zames, "H^∞-optimal Feedback Controller for Linear Multivariable Systems", **IEEE AC-29**, pp 888-900, 1984

[6] H. Kwakernaak, "Minimax Frequency Domain Optimization of Multivariable Linear Feedback Systems", **IEEE AC-29**, pp 831-832, 1984

[7] D.C. Youla, H. Jabr and J.J. Bongiorno, "Modern Wiener-Hopf Design of Optimal Controllers: Part II, the Multivariable Case", **IEEE Vol.AC-21**, pp 319-338, 1976

[8] D.C. Youla and J.J. Bongiorno, "A Feedback Theory of Two-Degree-of-Freedom Optimal Wiener-Hopf Design", **IEEE Vol.AC-30**, pp 652-665, 1985

[9] J.C. Doyle, **Lecture notes for ONR /Honeywell Workshop on Advances in Multivariable Control**, Minneapolis, MN, 1984

Chapter 4

COMPUTATION OF H^∞-NORM OF POLYNOMIALS

1. INTRODUCTION

In recent years, H^∞-norm and its optimization are used more and more frequently in many areas of control theory and its applications. For example, H^∞-norm optimal controller synthesis approach [1][2], model/controller reduction and even some problems in system identification are closely related to H^∞-norms. The model/controller reduction is often best posed as a frequency weighted H^∞-optimal approximation problem [3]. For a given transfer function G(z), many approaches give a reduced order transfer function $G_r(z)$, normally, which is not optimal in H^∞ sense. Certainly, it is desirable to know the value of the H^∞-approximation error $\|G(z)-G_r(z)\|_\infty$. In system identification, if a monic polynomial C(z) is the moving average noise process transfer function in an ARMAX model, it is well known that for the convergence of the extended least square algorithm, a key condition is that $C(z)^{-1} - \frac{1}{2}I$ is strictly positive real (e.g.[4],[5]). It is easy to see that this condition is equivalent to the requirement $\|C(z)-1\|_\infty <1$. However, in practice, to calculate the value of the H^∞-norm is not a pleasant task. It is usually done by a rather trivial method, i.e., plotting the absolute value of the function concerned on the unit circle.

In this chapter, we propose a theoretical recursive algorithm for the computation of H^∞-norm of polynomials or FIR transfer functions (Section 2). We give in Section 3 the derivation of the algorithm and show that the guaranteed convergence rate of the algorithm is $O(\frac{\log n}{n})$. Simulation results of some examples are provided in

Section 4. Section 5 concludes the chapter with some remarks.

Before pursuing further, we need some concepts and definitions as following.

Let f(z) be a complex-valued function on the unit circle bounded almost everywhere, the set of all such functions is denoted by L^∞, with norm

$$\| f(z) \|_\infty = \operatorname*{ess\,sup}_{\theta \in [0\ 2\pi]} |f(e^{i\theta})| \tag{1}$$

The Hardy space H^∞ consists of all complex-valued functions which are analytic and of bounded modules on $|z| < 1$, with norm

$$\| f(z) \|_\infty = \sup_{|z|<1} | f(z) | \tag{2}$$

It is known that each f in H^∞ yields a unique L^∞ boundary function with the two norms equal. The set of such boundary functions is the subspace of L^∞-functions with Fourier coefficients zero for negative indices, and we can regarded H^∞ as a closed-subspace of space L^∞.

We also need the concept of space L^p, (p>0). It consists of all measurable complex functions f(z) defined on the unit circle $|z| = 1$ such that $|f(e^{i\theta})|^p$ is integrable with respect to Lebesgue measure, with norm

$$\| f(z) \|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}} \tag{3}$$

2. ALGORITHM DESCRIPTION AND MAIN RESULTS

Let C(z) be a polynomial with real coefficients and with degree r:

$$C(z) = C_0 + C_1 z + \cdots + C_r z^r, \quad C_0 C_r \neq 0, \tag{4}$$

Define a function f(z) as

$$f(z) = C(z)C(z^{-1}) \triangleq \gamma_0 + \sum_{j=1}^{r} \gamma_j (z^j + z^{-j}), \tag{5}$$

where

$$\gamma_j = \sum_{k=1}^{r} C_k C_{k+j}, \quad (C_k = 0, \text{ for } k > r) \tag{6}$$

To describe our algorithm, we need the following auxiliary variables:

$$\{ X_i(n), 1 \leq i \leq 2r, n \geq 1\} \text{ and } \{T(n), n \geq 1\}$$

which are recursively defined by (for $1 \leq k \leq r$)

$$X_{k+r}(n-1) = \frac{\sum_{j=1}^{r} (nj-k)\gamma_j X_{k-j}(n-1) - \sum_{j=1}^{r} (nj+k)\gamma_j X_{k+j}(n-1)}{(nr+k)\gamma_r} \tag{7}$$

$$X_k(n) = \frac{n \sum_{j=1}^{r} j\gamma_j [X_{k-j}(n-1) - X_{k+j}(n-1)]}{k\,[\gamma_0 + 2\sum_{j=1}^{r} \gamma_j X_j(n-1)]} \tag{8}$$

$$T(n) = \frac{n-1}{n} T(n-1) + \frac{1}{2n} \log[\gamma_0 + 2\sum_{j=1}^{r} \gamma_j X_j(n-1)], \tag{9}$$

where by definition

$$X_0(n) = 1 \quad \text{and } X_{-i}(n) = X_i(n), \quad 1 \leq i \leq 2r, \; n \geq 1$$

and where the initial conditions are

$$X_j(1) = \frac{\gamma_j}{\gamma_0},\ 1 \le j \le r;\quad T(1) = \frac{1}{2}\log(\gamma_0) \tag{10}$$

The n-th approximation for the norm $\|C(z)\|_\infty$ is defined by

$$J(n) = \exp\{T(n)\}, \qquad n \ge 1 \tag{11}$$

The asymptotic properties of the above algorithm are summarized in the following theorem.

Theorem 1: For any polynomial C(z) defined as in (4), the quantity J(n) given by (7)-(11) increases monotonically and converges to $\|C(z)\|_\infty$ as $n \to \infty$, with convergence rate

$$\|C(z)\|_\infty - J(n) \le (\|C(z)\|_\infty)\frac{\log n}{2n} + O(\frac{1}{n}) \tag{12}$$

3. Convergence Analysis

For the proof of Theorem 1, we first establish the following lemmas.

Lemma 1: For T(n) given by (9),

$$T(n) = \log(\|C(z)\|_{2n}),$$

holds for any $n \ge 1$. □□□

Proof: Define

$$M_k(n) = \frac{1}{2\pi}\int_0^{2\pi} f^n(e^{i\theta})e^{ki\theta}\, d\theta \tag{13}$$

for $n \geq 1$, $-2r \leq k \leq 2r$, where $f(e^{i\theta})$ is given by (5). It is easy to see that for any $n \geq 1$

$$M_{-k}(n) = M_k(n) \quad , \quad k=0,1,\dots,2r\,. \tag{14}$$

and

$$M_0(n) = \frac{1}{2\pi}\int_0^{2\pi} f^n(e^{i\theta})d\theta \; = (\| f^n(z) \|_n)^n = (\|C(z)\|_{2n})^{2n} \tag{15}$$

So for the proof of the lemma we need only to show that

$$T(n) = \frac{1}{2n}\log M_0(n) \tag{16}$$

We proceed as follows. By (5), (13) and (14) ,

$$M_0(n) = \frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{i\theta})f(e^{i\theta})d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{i\theta})[\gamma_0 + \sum_{k=1}^{r}\gamma_k(e^{ki\theta} + e^{-ki\theta})] \; d\theta$$

$$= \gamma_0 M_0(n-1) + 2\sum_{k=1}^{r}\gamma_k M_k(n-1)$$

$$= M_0(n-1)\left[\gamma_0 + 2\sum_{k=1}^{r}\frac{\gamma_k M_k(n-1)}{M_0(n-1)}\right] \tag{17}$$

consequently, we have

$$\frac{1}{2n}\log M_0(n) = \frac{n-1}{n}\left[\frac{\log M_0(n-1)}{2(n-1)}\right] + \frac{1}{2n}\log\left[\gamma_0 + 2\sum_{k=1}^{r}\frac{\gamma_k M_k(n-1)}{M_0(n-1)}\right]$$

Comparing this with (9), we see that for (16) it suffices to show that

$$X_j(n) = \frac{M_j(n)}{M_0(n)}, \quad 1 \le j \le r. \tag{18}$$

Now, by integral by parts from (13) and the fact that $f(z) = f(z^{-1})$ we have

$$M_k(n) = \frac{1}{2\pi}\int_0^{2\pi} f^n(e^{-i\theta})e^{ki\theta}\, d\theta$$

$$= \frac{1}{2\pi ki}\int_0^{2\pi} f^n(e^{-i\theta})\, de^{ki\theta}$$

$$= \frac{1}{2\pi i}\left\{ f^n(e^{-i\theta})e^{ki\theta}\Big|_0^{2\pi} - \int_0^{2\pi} e^{ki\theta}\, d[f^n(e^{-i\theta})]\right\}$$

$$= \frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})f'(e^{-i\theta})e^{(k-1)i\theta}\, d\theta \tag{19}$$

where

$$f'(e^{-i\theta}) = \frac{df(z)}{dz}\Big|_{z=e^{-i\theta}} = \sum_{j=1}^{r} j\gamma_j[e^{(1-j)i\theta} - e^{(1+j)i\theta}], \tag{20}$$

For (19), (20) we immediately have $(1 \le k \le r)$:

$$M_k(n) = \frac{n}{k}\sum_{j=1}^{r} j\gamma_j[M_{k-j}(n-1) - M_{k+j}(n-1)], \tag{21}$$

Multiplying $^1/M_0(n)$ on both sides of this equality and using (17), we know that the recursion (8) is true with $X_k(n)$ replaced by $M_k(n)/M_0(n)$.

To conclude (18), we still need to show that the recursion (7) also holds with $X_k(n)$ replaced by $M_k(n)/M_0(n)$. To this end, consider the following decomposition for $f'(e^{-i\theta})$:

$$f'(e^{-i\theta}) = g_1(e^{-i\theta}) - r\cdot f(e^{-i\theta})e^{i\theta}, \tag{22}$$

where

$$g_1(e^{-i\theta}) = r\gamma_0 e^{i\theta} + \sum_{j=1}^{r} \gamma_j[(r+j)e^{(1-j)i\theta} + (r-j)e^{(1+j)i\theta}]. \tag{23}$$

Substituting (22) into (19) we get

$$M_k(n) = \frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})[g_1(e^{-i\theta}) - r\cdot f(e^{-i\theta})e^{i\theta}]e^{(k-1)i\theta}\,d\theta$$

$$= \frac{-nr}{2\pi k}\int_0^{2\pi} f^n(e^{-i\theta})e^{ki\theta}\,d\theta + \frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})g_1(e^{-i\theta})\,e^{(k-1)i\theta}\,d\theta$$

$$= \frac{-nr}{k}M_k(n) + \frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta$$

By this identity we obtain for $1 \le k \le r$,

$$M_k(n) = \frac{n}{nr+k}\cdot\frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{-i\theta})g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta$$

$$= \frac{n}{nr+k}\left\{r\gamma_0 M_k(n-1) + \sum_{j=1}^{r}\gamma_j[(r+j)M_{k-j}(n-1) + (r-j)M_{k+j}(n-1)]\right\} \tag{24}$$

which in conjunction with (21) gives the recursive formula for $M_{k+r}(n-1)$:

$$M_{k+r}(n-1) = \frac{1}{(nr+k)\gamma_r}\left[\sum_{j=1}^{r}(nj-k)\gamma_j M_{k-j}(n-1) - \sum_{j=0}^{r-1}(nj+r)\gamma_j M_{k+j}(n-1)\right]$$

From here it is easy to see that (7) is true with $X_k(n-1)$ replaced by $M_k(n-1)/M_0(n-1)$. This proves the assertion (18) and hence the conclusion of the lemma. $\Delta\Delta\Delta$

<u>Lemma 2:</u> Let a complex function $f(z) \in L^{\infty}$, if $\frac{d}{d\theta}[\,|f(e^{i\theta})|^2] \in L^{\infty}$, then

$$0 \leq \| f(e^{i\theta}) \|_{\infty} - \| f(e^{i\theta}) \|_n \leq (\| f(e^{i\theta}) \|_{\infty})\frac{\log n}{n} + O(\frac{1}{n})$$

<u>Proof:</u> By (1) and (3) it is evident that for any $n \geq 1$,

$$\| f(e^{i\theta}) \|_n \leq \| f(e^{i\theta}) \|_{\infty}$$

Now denote

$$g(\theta) = | f(e^{i\theta}) |^2, \qquad \theta \in [0, 2\pi].$$

Since $g(\theta)$ is a continuous function of θ, there exists a $\theta_o \in [0,2\pi]$ such that

$$g(\theta_o) = \max_{\theta \in [0\ 2\pi]} g(\theta) = \| f(e^{i\theta}) \|_{\infty}^2$$

without loss of generality assume $\theta_o \in (0,2\pi)$.

By the Taylor's expansion we know that

$$g(\theta) = g(\theta_o) + g'(\xi)(\theta-\theta_o)$$

where ξ is some point between θ and θ_o.

From here we have for sufficiently large n,

$$\| f(e^{i\theta}) \|_n = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^n \, d\theta \right\}^{\frac{1}{n}}$$

$$= \left\{ \frac{1}{2\pi} \int_0^{2\pi} [g(\theta)]^{n/2} d\theta \right\}^{\frac{1}{n}}$$

$$= \left\{ \frac{1}{2\pi} \int_0^{2\pi} [g(\theta_o)]^{n/2} \, [1+\frac{g'(\xi)}{g(\theta_o)}(\theta-\theta_o)]^{n/2} d\theta \right\}^{\frac{1}{n}}$$

$$= [g(\theta_o)]^{1/2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} [1+\frac{g'(\xi)}{g(\theta_o)}(\theta-\theta_o)]^{n/2} d\theta \right\}^{\frac{1}{n}}$$

$$\geq \| f(z) \|_\infty \left\{ \frac{1}{2\pi} \int_{\theta_o-\frac{1}{n}}^{\theta_o+\frac{1}{n}} [1+\frac{g'(\xi)}{g(\theta_o)}(\theta-\theta_o)]^{n/2} d\theta \right\}^{\frac{1}{n}}$$

$$\geq \| f(z) \|_\infty \left\{ \frac{1}{2\pi} \int_{\theta_o-\frac{1}{n}}^{\theta_o+\frac{1}{n}} [1- \frac{\|g'(\theta)\|_\infty}{g(\theta_o)} \frac{1}{n}]^{n/2} d\theta \right\}^{\frac{1}{n}}$$

$$= \| f(z) \|_\infty \left[\frac{1}{n\pi}\right]^{\frac{1}{n}} \left[1- \frac{\|g'(\theta)\|_\infty}{g(\theta_o)} \cdot \frac{1}{n}\right]^{\frac{1}{2}}$$

$$= \| f(z) \|_\infty \cdot \exp(\frac{1}{n}\log\frac{1}{n\pi}) \cdot \left[1 + O(\frac{1}{n})\right]$$

$$= \| f(z) \|_\infty \cdot \left[1 - \frac{\log n\pi}{n} + O(\frac{\log^2 n}{n^2})\right]\left[1 + O(\frac{1}{n})\right]$$

$$= \| f(z) \|_\infty - (\| f(z) \|_\infty)\frac{\log n}{n} + O(\frac{1}{n})$$

This completes the proof of the lemma. ΔΔΔ

Proof of Theorem 1: By (11) and Lemma 1 we know that

$$J(n) = \|C(z)\|_{2n} \tag{25}$$

By the Hölder inequality it is easy to see that the L^p-norm $\|.\|_p$ is monotonically increasing in p and hence J(n) is monotonically increasing in n. The other results follow from (25) and Lemma 2. ΔΔΔ

4. Example Studies

To illustrate the algorithm works, two examples are studied. They are:

(i) $C(z) = 1 - z - z^2$

(ii) $C(z) = 1 + 2z + 3z^2$

It is easy to show in example (ii) that $\|C(z)\|_\infty = 6$. However, it is not straightforward to see in example (i) that $\|C(z)\|_\infty = \sqrt{5}$. After 1500 iterations, the H^∞-norm is approximated with relative error under 0.00154 in both cases, which are depicted in Figures 1 and 2, respectively.

5. Conclusions and Remarks

a) The proposed algorithm has itself theoretical interests as well as its application importance. Various algorithms for minimization (maximization) of functions exist

[6]-[8], but to the authors' knowledge, theoretical algorithms for computing the H^∞-norm, has not yet been studied elsewhere.

b) It is interesting to note that the principal part of the relative error of the algorithm is independent of the polynomial C(z), (i.e.,logn/2n). Furthermore, the error is monotonically decreasing to zero. So, for a given relative error, we can roughly decide the iteration step n to achieve the desired accuracy.

c) In this chapter, we have only considered the scalar polynomial case. Of course, for a given stable scalar rational function, one can first approximate it by a r-th order polynomial (with exponential decaying error $0(\lambda^r)$, $0<\lambda<1$) and then use the above method to approximate the H^∞-norm of the rational function. It is desirable to generate this results to general matrix transfer function case.

REFERENCES

[1] G. Zames and B.A. Francis, "Feedback, minimax sensitivity, and optimal robustness", **IEEE Trans. AC-28**, 1983, pp.585-601.

[2] B.A. Francis, "Notes on H -optimal linear feedback system", **Lecture notes**, Department of Electrical Engineering, Linko ping University, Sweden, 1983.

[3] B.D.O. Anderson and Yi Liu, "Controller reduction: Concepts and approaches", **Proc. American Control Conference**, 1987, Minnesota, U.S.A.,pp1-9.

[4] L. Ljung, "On positive real transfer function and the convergence of some recursive schemes", **IEEE**, **Vol.AC-22**, 1977, pp.539-551.

[5] H.F. Chen and L. Guo, "Convergence rate of least squares identification and adaptive control for stochastic systems", **Int. J. Control**, vol.44, no.5, 1986, pp.1459-1476.

[6] R.P. Brent, "**Algorithms for Minimization without Derivatives**". Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

[7] G.P.Szego,"**Minimization Algorithms: Mathematical Theories and Computer Results**", Academic Press, New York, 1972.

[8] M.R.Osborne,"**Finite Algorithms in Optimization and Data Analysis**", John Wiley & Sons, New York, 1985.

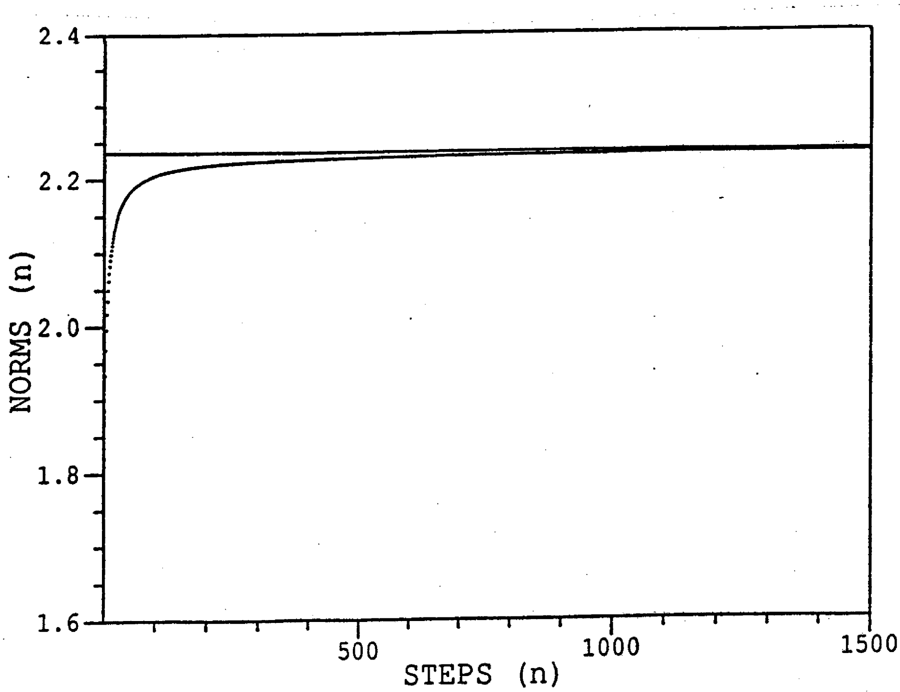


Fig. 1. Example 1

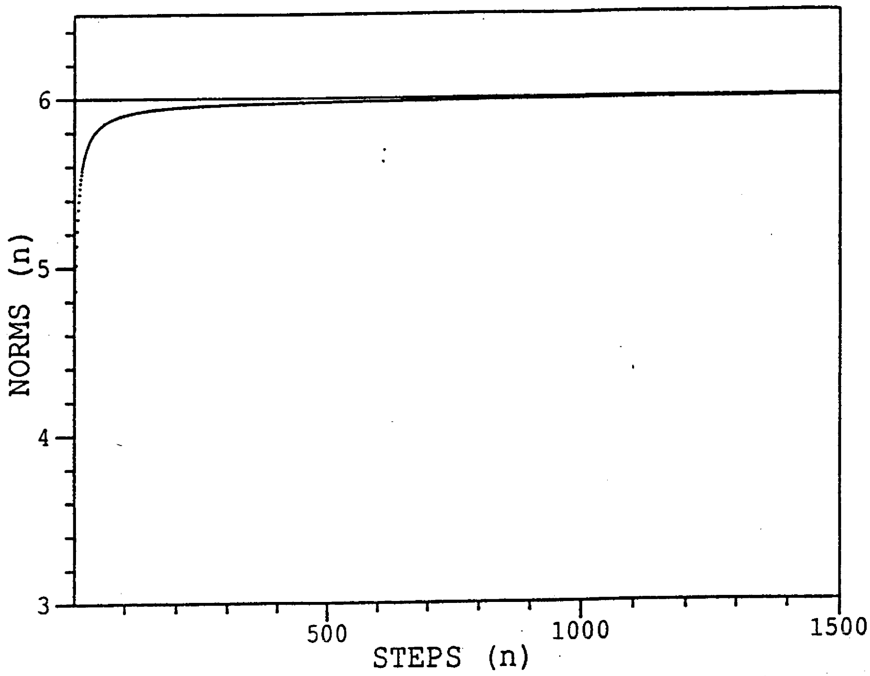


Fig. 2. Example 2

PART II

Central Tendency Adaptive Control

CHAPTER 5

CENTRAL TENDENCY ADAPTIVE POLE ASSIGNMENT

1. INTRODUCTION

In designing adaptive controllers, it is common to exploit the Certainty Equivalence Principle. That is, to use on-line parameter estimates in lieu of actual plant parameters in a controller design. When the parameter estimates are close to the actual plant parameters (assuming the plant is in the model set), then this approach makes some sense. However for adaptive schemes based on the certainty equivalence principle, where the plant model is unknown, there will be in general, circumstances where the transient performance is unnecessarily poor, perhaps intolerably so.

The notion of Dual Control, which gives a "best" control in the presence of plant uncertainties, is attractive until one gets down to implementing this concept. Indeed, most dual controllers, if they can be described, can not be implemented on line [1].

Consider a plant signal model where the plant is expressed in terms of the unit delay operator q^{-1} as

$$Ay_k = Bu_k + Cw_k \tag{1.1}$$

where

$$A(q^{-1}) = 1 + a_1q^{-1} + ... + a_nq^{-n}, \quad B(q^{-1}) = b_1q^{-1} + ... + b_mq^{-m},$$

$$C(q^{-1}) = 1 + c_1q^{-1} + ... + c_lq^{-l}.$$

u_k is the plant input, y_k its output, and w_k a zero mean "white" noise disturbance. The plant parameters are

$$\theta^{\tau} = [a_1 \; \dots \; a_n \; b_1 \; \dots \; b_m \; c_1 \; \dots \; c_l \,] \tag{1.2}$$

and least squares based estimates of θ are denoted $\hat{\theta}_k{}^{LS}$. With pole assignment as our objective, then a controller with parameters $\phi = \Phi(\theta)$ is calculated from a Bezout equation. In certainty equivalence adaptive control, the adaptive controller has parameters $\hat{\phi}_k = \Phi(\hat{\theta}_k)$.

In this chapter, we explore more sophisticated calculations for adaptive controller parameters $\hat{\phi}_k$ based on both parameter estimates $\hat{\theta}_k{}^{LS}$ and their uncertainty, as measured by an a posteriori error covariance estimate $\hat{P}_k{}^{LS}$. A cautious controller and central tendency controller are studied to achieve adaptive pole assignment objectives. The cautious controller adapts the same philosophical approach as the Astrom cautious control for minimum variance control [1]. The central tendency controller adapts the approach outlined in [2] for minimum variance control.

In this chapter, the notion of central tendency adaptive control is introduced with the same key objective as that of the cautious controller, namely to avoid ill conditioned calculations and improve transient performance. The term central tendency is taken from statistics, where a measure of central tendency associated with a probability density function is one, such as a mean, mode, or median, with a "high" probability, avoiding the "low" probability tails.

We recall here that the term central tendency controller defines a controller which loosely maximizes the probability of achieving the control objectives given plant parameter uncertainty information. The value of a random variable x which maximizes the probability of x is the mode, denoted mode[x]. However, in general for a nonlinear function of x, $l(x)$, mode$[l(x)] \neq l($mode$[x])$. It is therefore clear that with imprecise knowledge of a plant, a certainty equivalence approach to controller design may lead to a poor controller for the actual plant. A central tendency controller design which uses the most likely controller parameters based on the plant uncertainty should lead to designs which are less likely to be poor. Central tendency controllers should be more robust than certainty equivalence controllers.

The term central tendency adaptive control refers to indirect adaptive control schemes which attempt to calculate the most likely controller at each k, to achieve the control objectives, based on whatever knowledge is available of the plant parameter uncertainty. Let us denote the σ-algebra generated by all measurements of y_k and u_k up until time k as F_k. There are then conditioned a posteriori probability densities $f[\phi(\theta) \mid F_{k-1}]$, $f[\theta \mid F_{k-1}]$. Central tendency adaptive controllers work with measures of central tendency on $f[\phi \mid F_{k-1}]$ and thereby avoid the tails of $f[\phi \mid F_{k-1}]$, which are usually associated with ill-conditioned controllers. It also makes sense in some applications to work with the conditioned density on the controls directly that is on $f[u_k \mid F_{k-1}]$.

In Section 2, both Central Tendency Controllers and Cautious Controllers for adaptive pole assignment are proposed. Convergence properties are studied in Section 3, for schemes based on least squares identification, and relative merits are assessed. Simulation studies are given in Section 4. Conclusions are drawn in Section 5.

2. Central Tendency Adaptive Pole Assignment

In pole assignment, if the plant has a near pole zero cancellation then in general the control energy required is relatively large. Likewise in adaptive pole-assignment based on the Certainty Equivalence Principle, when the estimated plant has a near pole zero cancellation, the control energy could be excessive. One method to cope with this is to freeze the controller (suspend the application of the Certainty Equivalence principle) for the period of time when plant estimates have "near" pole zero cancellations. In practice, there could be difficulty in setting appropriate thresholds for "nearness".

The central tendency adaptive pole-assignment theory of this section leads to practical methods to assess whether or not there is a "near" pole zero cancellation and a method to select or construct a controller to use when there is such a cancellation. Also search procedures are noted for the more computationally intensive task of seeking the $\mathrm{Mode}[u_k^{PA} | F_{k-1}]$ where u_k^{PA} denotes a pole assignment control.

Pole-Assignment

For the plant (1.1) consider a pole assignment control scheme

$$Eu_k^{PA} = -Fy_k, \quad AE + BF = H \tag{2.1}$$

where

$$E(q^{-1}) = 1 + e_1 q^{-1} + \ldots + e_m q^{-m}, \; F(q^{-1}) = f_1 q^{-1} + \ldots + f_n q^{-n}$$

$$H(q^{-1}) = 1 + h_1 q^{-1} + \ldots + h_{n+m} q^{-n-m}$$

The zeros of $H(q^{-1})$ are the assigned poles. The specific form of the controller is such that its free parameters are equal in number to the plant parameters. Also, the controller has a built in delay for implementation purposes.

The polynomial equation (2.1b) can be rewritten as an algebraic equation in terms of the Sylvester matrix S_{AB} as

$$S_{AB}\begin{bmatrix}1\\ \phi\end{bmatrix} = h, \quad \phi^\tau = [\bar{e}^\tau \ \bar{f}^\tau], \quad h^\tau = [1 \ \bar{h}^\tau]$$

$$\bar{e}^\tau = [e_1 \ e_2 \ \ldots \ e_m], \quad \bar{f}^\tau = [f_1 \ f_2 \ \ldots \ f_n], \quad \bar{h}^\tau = [h_1 \ h_2 \ \ldots \ h_{n+m}],$$

$$S_{AB} = \begin{array}{c} \leftarrow m+1 \rightarrow \ \leftarrow n \rightarrow \\ \begin{bmatrix} 1 & & & 0 & & \\ a_1 & 1 & & 0 & 0 & \\ a_2 & a_1 & . & b_1 & 0 & . \\ . & . & . & b_2 & b_1 & . \\ & & . & . & & . \end{bmatrix} \end{array} \begin{array}{c} \uparrow \\ n+m+1 \\ \downarrow \end{array} \tag{2.2}$$

This relationship has the dual form

$$S_{EF}\begin{bmatrix}1\\ \theta\end{bmatrix} = h, \quad \theta^\tau = [\bar{a}^\tau \ \bar{b}^\tau], \tag{2.3}$$

A solution of (2.2) for ϕ involves S_{AB}^{-1}, which is known to exist if and only if the plant has no pole zero cancellations, or that [3,page 142]

$$q^nA(q^{-1}),\ q^mB(q^{-1}) \text{ are coprime} \tag{2.4}$$

Also, under (2.4) it is known that S_{EF}^{-1} exists, or equivalently that $q^nF(q^{-1})$, $q^mE(q^{-1})$ are coprime.

Persistence of Excitation

Should the plant disturbances (external or internal) include a white noise term w_k, then the controller selection (2.1) with $q^nF(q^{-1})$, $q^mE(q^{-1})$ coprime, ensures that the plant states $x_k^\tau = [y_{k-1} \ldots y_{k-n}\ u_{k-1} \ldots u_{k-m}]$ are persistently exciting, see [4,5]. The controller states x_k are identical to the plant states by design. [In pole assignment where the dimension of ϕ is less than that of θ, E has dimension (m-1) which is the minimum permissible, then in the absence of external excitation, one of the closed-loop system states u_{k-1} is a linear combination of the others, being constrained by the controller relationship, and excitation of plant states is not achieved.]

A Posteriori Densities

The evaluation of $f(u_k^{PA}| F_{k-1})$ to achieve Mode$[u_k^{PA}| F_{k-1}]$ appears to be too formidable for practical implementation. Consider now the density $f(\phi| F_{k-1})$. A differentiation of (2.2a) with respect to θ^τ leads after a number of steps (details are given in the appendix) to the Jacobian, under (2.4), and assuming $\det(S_{AB}) \neq 0$

$$J \triangleq \frac{\partial\phi}{\partial\theta^\tau} = -\bar{S}_{AB}^{-1}\bar{S}_{EF} \tag{2.5}$$

Where $\bar{S}_{AB}$ is obtained from S_{AB} by deleting the first row and column. Likewise for $\bar{S}_{EF}$. Now observe

$$|\det J| = \left|\frac{\det \bar{S}_{EF}}{\det \bar{S}_{AB}}\right| = \left|\frac{\det S_{EF}}{\det S_{AB}}\right| \tag{2.6}$$

Let us now assume that the a posteriori probability density $f(\theta | F_{k-1})$ is normal with mean θ_k^m and covariance P_k. Then using standard arguments

$$f[\phi(\theta) \mid F_{k-1}] = \kappa |\det J^{-1}(\theta)| \exp(-\frac{1}{2} \|\theta-\theta_k^m\|_{P_k^{-1}}) \qquad (2.7)$$

Here κ is some normalizing constant so that the integral of $f[\phi(\theta) \mid F_{k-1}]$ over all θ is unity. Also, in case det $J(\theta)$ is not properly defined, we set f=0.

Note that if a controller were chosen with the dimension of ϕ less than that of θ, then evaluation of $f(\phi \mid F_{k-1})$ could require integration out of an auxiliary variable. This would render the central tendency adaptive controllers of this section impractical for implementation.

Central Tendency Adaptive Pole Assignment:

The control u_k^{PA} of (2.1) can be re-expressed in terms of the controller states x_k^{CPA} as

$$u_k^{PA} = - \phi^\tau x_k^{CPA}, \quad x_k^{CPA\tau} = [\, u_{k-1}^{PA} \ldots u_{k-m}^{PA}\ y_{k-1} \ldots y_{k-n}] \qquad (2.8)$$

Thus, a central tendency control can be defined, using obvious notation, as

$$u_k^{CT} = - \text{Mode}[\, \phi \mid F_{k-1}]^\tau x_k^{CCT} \qquad (2.9a)$$

$$\text{Mode}[\, \phi \mid F_{k-1}] \text{ maximizes } f[\phi(\theta) \mid F_{k-1}] \qquad (2.9b)$$

However, there is practical difficulty in implementing (2.9b) over all θ, since for each θ investigated, the evaluation $\phi(\theta)$ from (2.2) is required. A compromise is to perform the maximization only over the set of θ for which $\phi(\theta)$ is of necessity evaluated, namely $\hat{\theta}_1$, $\hat{\theta}_2$, ... $\hat{\theta}_k$, or a subset of these e.g. $\hat{\theta}_k$, $\hat{\theta}_{k-1}$, $\hat{\theta}_{k-M}$ for some M. Let the optimizing ϕ be denoted ϕ_k^* and the associated $\hat{\theta}_i$ as $\hat{\theta}_k^*$. Now

$$u_k^{CT} = - \phi_k^{*\tau} x_k^{CCT} \qquad (2.10)$$

Properties.

In maximizing $f(\phi \mid F_{k-1})$, the definition (2.7) tells us that there is a "maximization" of $\det(S_{AB})$ balanced by a "minimization" of $\|\theta-\theta_k^m\|_{P_k^{-1}}$. Thus "near" pole-zero cancellations associated with $\hat{\theta}_k^*$ are avoided without departing "far" from θ_k^m.

A typical scenario for the application of the pseudo mode control law (2.10) for the case when $q^nA(q^{-1})$, $q^mB(q^{-1})$ are coprime is as follows. In the absence of a near pole-zero cancellation associated with $\hat{\theta}_k$ and with $\hat{\theta}_k$ converging to θ, then one expects that the controller at each k will be selected according to the Certainty Equivalence Principle. Should there be a "near" pole-zero cancellation associated with $\hat{\theta}_k$, then $\hat{\theta}_k$ will not maximize $f[\phi(\theta_i) \mid F_{k-1}]$ over $i=1,2,\ldots,k$, but rather an estimate $\hat{\theta}_i$ for $i<k$ will. One expects that on average u_k^{CT} will give an improved control signal, but not necessarily at all k. Should there be a "near" pole-zero cancellation associated with the plant, then if $\hat{\theta}_k$ is closer to θ than previous estimates, even though there is a "near" pole-zero cancellation associated with $\hat{\theta}_k$, then we would expect that the approximate pseudo mode control will be the Certainty Equivalence control. Precise convergence properties are given in the next section, and simulations are given in Section 4.

Cautious Adaptive Pole Assignment.

In implementing standard adaptive pole-assignment based on certainty equivalence, ill-conditioning occurs when $\det S_{\hat{A}_k\hat{B}_k}$ denoted Δ_k is small, since the control involves a factor Δ_k^{-1}. The Cautious Control methodology suggests that such a factor is replaced by $\Delta_k[\Delta_k^2 + \hat{V}_k]^{-1}$ where $\hat{V}_k$ is an estimate of the variance in uncertainty of Δ_k. Thus in obvious notion

$$u_k^{CC} = \Delta_k[\Delta_k^2 + \hat{V}_k]^{-1}u_k^{CE} \tag{2.11}$$

Here we take

$$\hat{V}_k = \frac{\partial \Delta_k}{\partial \hat{\theta}_k^{\tau}} P_k \frac{\partial \Delta_k}{\partial \hat{\theta}_k} \tag{2.12}$$

Where P_k is the covariance in uncertainty of parameter estimates $\hat{\theta}_k$, and

$$\frac{\partial \Delta_k}{\partial \hat{\theta}_k^{\tau}} = \Delta_k S_{\hat{A}_k\hat{B}_k}^{-\tau} \frac{\partial S_{\hat{A}_k\hat{B}_k}}{\partial \hat{\theta}_k} \tag{2.13}$$

which derives using standard expressions, for matrix M

$$\frac{\partial \ln M}{\partial M} = M^{-\tau}, \quad \frac{\partial \ln \det M}{\partial \det M} = (\det M)^{-1}$$

This scheme is simpler to implement than the central tendency schemes of the previous subsection, but does not appear to perform as well from simulations as in Section 4.

<u>A Pseudo-Mode Pole Assignment.</u>

Here the methodology of the pseudo-mode adaptive minimum variance controller of [2] is generalized using linearization for the pole assignment case. Following this approach leads to the pseudo-mode control, with * denoting pseudo-mode,

$$\mathrm{Mode}^*[u_k^{PA}| F_{k-1}] = \frac{2(\alpha_k^*\Delta_k - \bar{\phi}_k^{\tau}x_k^*)}{\Delta_k + \mathrm{sign}(\Delta_k)(\Delta_k^2 + 8\hat{V})^{1/2}} - \alpha_k^*$$

$$\alpha_k^* = \hat{V}_k^{-1}\hat{W}_k x_k^*, \quad \bar{\phi}_k = \Delta_k \phi_k \tag{2.14}$$

by analogy with (2.6), (2.8), (2.11), (2.12) of [2] where using linearization

$$\hat{W}_k = \frac{\partial \Delta_k}{\partial \hat{\theta}_k^\tau} P_k \frac{\partial \bar{\phi}_k}{\partial \hat{\theta}_k} \tag{2.15a}$$

$$\frac{\partial \bar{\phi}_k^\tau}{\partial \hat{\theta}_k} = \phi_k^\tau \frac{\partial \Delta_k}{\partial \hat{\theta}_k} + \Delta_k \frac{\partial \phi_k^\tau}{\partial \hat{\theta}_k} \tag{2.15b}$$

Here $\hat{V}_k$, $\frac{\partial \Delta_k}{\partial \hat{\theta}_k}$ and $\frac{\partial \phi_k^\tau}{\partial \hat{\theta}_k}$ can be calculated from expression (2.12) (2.13) and (2.5). This scheme is only mildly more complicated than the Cautious Adaptive scheme of the previous subsection.

3. LEAST SQUARES CONVERGENCE

Least Squares Algorithm.

Consider the signal model (1.1) with C=1 reorganized as

$$y_k = \theta^\tau x_k + w_k \tag{3.1}$$

where a priori $\theta = N[\theta_0, P_0]$,

$$E[w_k \mid F_{k-1}] = 0,\ E[w_k^2 \mid F_{k-1}] = \sigma^2, \quad w_k = N[0, \sigma^2]$$

$$\theta^\tau = [a_1 \ldots a_n\ b_1 \ldots b_m],\ x_k^\tau = [-y_{k-1} \ldots -y_{k-n}\ u_{k-1} \ldots u_{k-m}] \tag{3.2}$$

Consider also the Kalman filter identification, for $k \geq l$ where P_l^{KF} exists

$$\hat{\theta}_k^{KF} = \hat{\theta}_{k-1}^{KF} + \sigma^{-2} P_k^{KF} x_k \tilde{y}_{k|k-1}, \quad \tilde{y}_{k|k-1} = y_k - \hat{\theta}_k^{KF\tau} x_k$$

$$P_k^{KF} = P_{k-1}^{KF} - \frac{P_{k-1}^{KF} x_k x_k^\tau P_{k-1}^{KF}}{\sigma^2 + x_k^\tau P_{k-1}^{KF} x_k} = \left(\sum_{i=1}^{k} \sigma^{-2} x_i x_i^\tau + P_0^{-1} \right)^{-1} \qquad (3.3)$$

Readily established properties are

$$\tilde{y}_{k|k-1} = (1 - x_k^\tau \sigma^{-2} P_k^{KF} x_k)^{-1} \tilde{y}_{k|k}, \quad x_k^\tau \sigma^{-2} P_k^{KF} x_k \leq 1 \qquad (3.4)$$

$$\hat{\theta}_k^m \triangleq E[\hat{\theta} \mid F_{k-1}] = \hat{\theta}_k^{KF} = \left(\sum_{i=1}^{k} \sigma^{-2} x_i x_i^\tau + P_0^{-1} \right)^{-1} \left(\sum_{i=1}^{k} \sigma^{-2} x_i y_i + P_0^{-1} \theta_0 \right)$$

$$P_k \triangleq E[\tilde{\theta}_k \tilde{\theta}_k^\tau \mid F_{k-1}] = P_k^{KF}, \quad \tilde{\theta}_k = \theta - \hat{\theta}_k^{KF} \qquad (3.5)$$

The least squares equations are Kalman filter equations with "incorrect" initial condition [e.g. setting $\theta_0 = 0$, $P_0^{-1} = \varepsilon$, $\sigma^2 = I$ in (3.3)]. In this case the notation $\hat{\theta}_k^{LS}$, P_k^{LS} are employed. Estimates of P_k can be obtained from P_k^{LS} and estimates of σ^2 or as here one can simply take P_k^{LS} as an estimate of P_k. Other relationships which follow directly are

$$E[(\tilde{y}_{k|k} - w_k) \mid F_{k-1}] = 0, \quad E[(\tilde{y}_{k|k} - w_k)^2 \mid F_{k-1}] = \sigma^2 x_k^\tau P_k^{LS} x_k \leq \sigma^2$$

$$P_k = E[\theta\theta^\tau \mid F_{k-1}] - E[\theta \mid F_{k-1}] E[\theta^\tau \mid F_{k-1}] \qquad (3.6)$$

Convergence Properties

Lemma 3.1 For the signal model (3.1),(3.2) and Kalman filter scheme (3.3), or more generally the least squares variation, then for some θ^{LS}, P^{LS} (random variables), almost surely

$$\lim_{k \to \infty} P_k^{LS} x_k = 0, \qquad \lim_{k \to \infty} P_k^{LS} = P^{LS} \qquad (3.7)$$

$$\lim_{k \to \infty} \hat{\theta}_k^{LS} = \theta^{LS} \qquad (3.8)$$

$$P^{LS} = 0 \;\Rightarrow \theta^{LS} = \theta \tag{3.9}$$

Proof The proof is seen in [6].

Remark 1. The above asymptotic results do not presume signal model stability. Without such, there is of course ill-conditioning as $k \rightarrow \infty$.

2. The above lemma is invariant of any adaptive control generation of x_k based on $\hat{\theta}_1{}^{LS}$, ... $\hat{\theta}_{k-1}{}^{LS}$, as spelt out in [6]. We now apply the lemma in the adaptive (central tendency) pole assignment context.

Lemma 3.2 Consider the plant (1.1) satisfying the coprimeness condition (2.4) and with the alternative formulation (3.1). Consider also the approximate mode adaptive pole-assignment scheme of Section 2 based on the Kalman filter or least squares parameter estimation scheme (3.3), initialized by $\hat{\theta}_0(\hat{A}_0,\hat{B}_0)$ with $q^n\hat{A}_0(q^{-1})$, $q^m\hat{B}_0(q^{-1})$ coprime. Then $\phi_k{}^*$ exists for all k and for some ϕ^*,

$$\lim_{k\rightarrow\infty}\phi_k{}^* = \phi^* \quad \text{a.s.} \tag{3.10}$$

Moreover, for the associated polynomials $\hat{E}_k{}^*$, $\hat{F}_k{}^*$ and E^*, F^*, then

$$q^n\hat{F}_k{}^*(q^{-1}),\ q^m\hat{E}_k{}^*(q^{-1}) \quad \text{are coprime} \tag{3.11a}$$

$$q^nF^*(q^{-1}),\ q^mE^*(q^{-1}) \quad \text{are coprime} \tag{3.11b}$$

Furthermore, with the noise variance $\sigma^2 \neq 0$, then

$$P^{LS} = 0, \quad \theta^{LS} = \theta \tag{3.12}$$

and the adaptive pole assignment scheme is asymptotically optimal.

<u>Proof</u> First observe that ϕ_0 exists by the selection of $\hat{\theta}_0$ such that $q^n\hat{A}_0(q^{-1})$, $q^m\hat{B}_0(q^{-1})$ are coprime. Also, from the definition (2.7),

$$f(\phi \mid F_{k-1}, \phi=\phi_0) > 0 \quad \text{for all finite } k \tag{3.13}$$

$$\lim_{k\to\infty} f(\phi \mid F_{k-1}, \phi=\phi_0) = 0 \;\Leftrightarrow\; p^{LS} = 0,\ \theta^{LS} = 0,\ \hat{\theta}_0 \neq \theta \tag{3.14}$$

Furthermore, since $\det(S_{AB}) \neq 0$, from (2.7) observe that

$$p^{LS} = 0,\ \theta^{LS} = 0 \;\Rightarrow\; \lim_{k\to\infty} f(\phi \mid F_{k-1}, \phi=\phi_k) > 0 \tag{3.15}$$

Now let us consider ϕ_k^* of (2.10), and the associated $\theta_k^*(\hat{A}_k^*, \hat{B}_k^*)$ in obvious notation. Then $\det(S_{\hat{A}_k^*\hat{B}_k^*}) \neq 0$ and ϕ_k^* exists for all k, otherwise $f(\phi \mid F_{k-1}, \phi=\phi_1) \leq f(\phi \mid F_{k-1}, \phi=\phi_k^*) = 0$ contradicting (3.13). In considering the corresponding limiting properties, first observe that (3.11) holds, otherwise the property (3.5) that $\hat{\theta}_k$ converges is contradicted. Now under (3.11), $\lim_{k\to\infty} \det(S_{\hat{A}_k^*\hat{B}_k^*}) \neq 0$ and $\lim_{k\to\infty} \phi_k^*$ exists, otherwise

$$f(\phi \mid F_{k-1}, \phi=\phi_1 \text{ and } \phi_k) \leq f(\phi \mid F_{k-1}, \phi=\phi_k^*) = 0$$

leading to a contradiction in applying (3.13), (3.14), (3.15).

The result (3.12) follows from (3.10) and persistence of excitation results for reachable time-invariant linear systems with asymptotically time invariant linear system feedback in [4]. Details on this are omitted, save that here since $\sigma^2 > 0$, then y_k is sufficiently rich for x_k^c (the controller state) to be persistently exciting under (3.11b). <u>ΔΔΔ</u>

4. SIMULATION RESULTS

Consider now the application of the adaptive pole-assignment methods of Section 2 to the plant (1.1), taken from [7], with $A = 1-1.2q^{-1}$, $B = q^{-1}-3.1q^{-2}+2.2q^{-3}$, $C = 1$ and $\sigma^2=1$. There is one unstable pole at z=1.2, and nonminimum phase zeros at 1.1 and 2. The initial parameter estimates are $\hat{a}_1 = -1$, $\hat{b}_1 = 0$, $\hat{b}_2 = -2$, $\hat{b}_3 = 3$. We adaptively assign the poles of the characteristic polynomial of the closed loop system to the origin.

Figures 4.1 and 4.2 give the output y_k and control signal u_k of each algorithm respectively, and the nearness of the poles and zeros of the estimates of the plant given by each algorithm.

We can see clearly that during the adaptation, parameter estimates converge to the true parameters. Also, there are some points at which a pole and zero are very close to each other. Consequently, the control signal given by the certainty equivalence pole-assignment method is very large and the cautious control method gives a better result than the standard one, but still large. However, the control signal resulting form the approximate mode pole-assignment algorithm is reasonably small. The above example has been chosen, as in [7], because it illustrates very clearly the weakness of the standard pole-assignment scheme. Other examples where there is no "near" pole zero cancellation in the estimate of the plant, of course show negligible difference between the approximate mode and standard pole-assignment schemes (details are omitted). Also, it should be said that in seeking to give improved estimation of the conditional mode, on the example above, if anything the performance deteriorated over that of the approximate mode algorithm, illustrating that the implementations of this paper are not optimal.

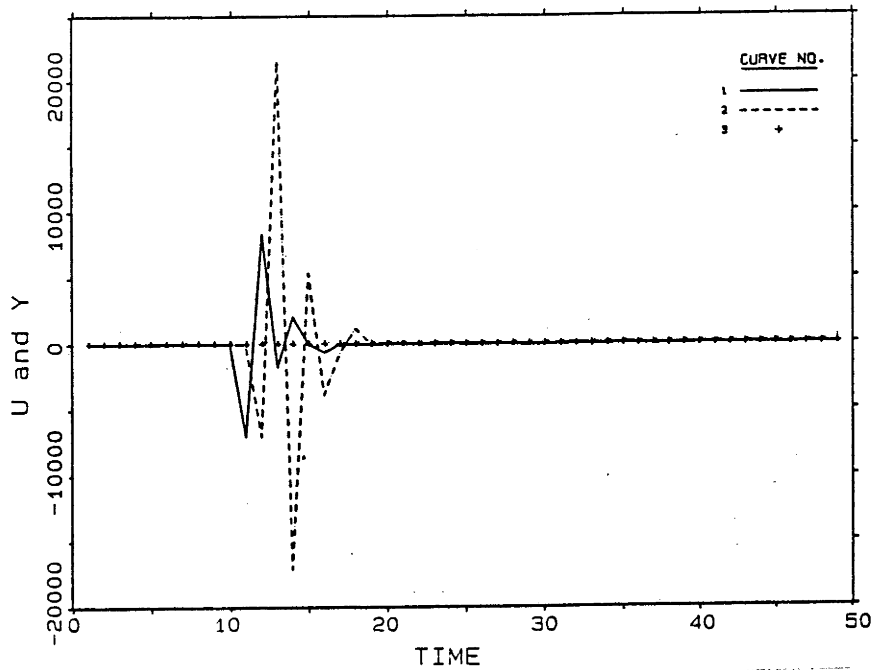


Fig. 4.1.a Certainty Equivalence Pole Assignment (y_k and u_k)

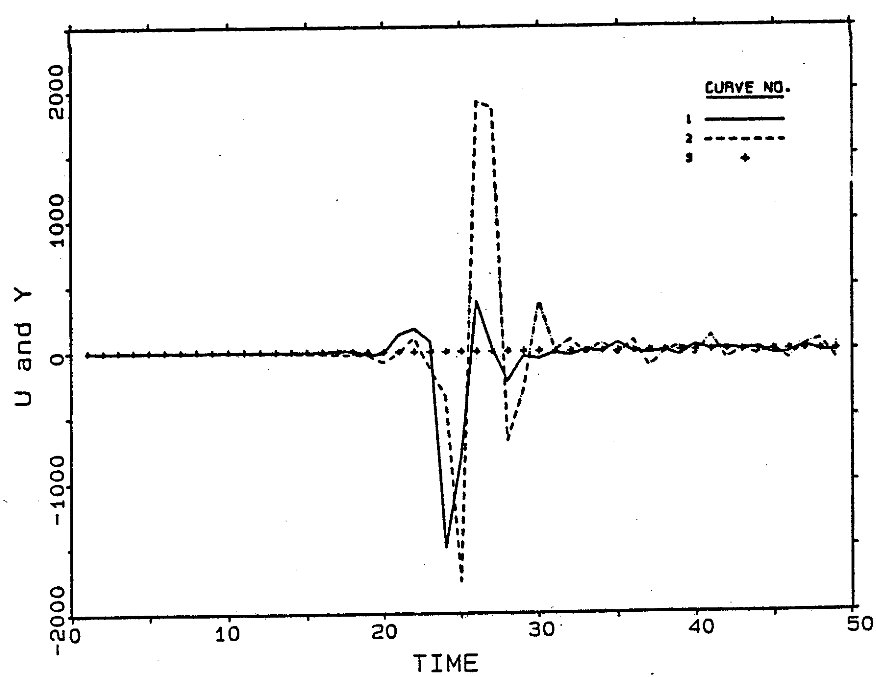


Fig. 4.1.b. Cautious Pole Assignment (y_k and u_k)

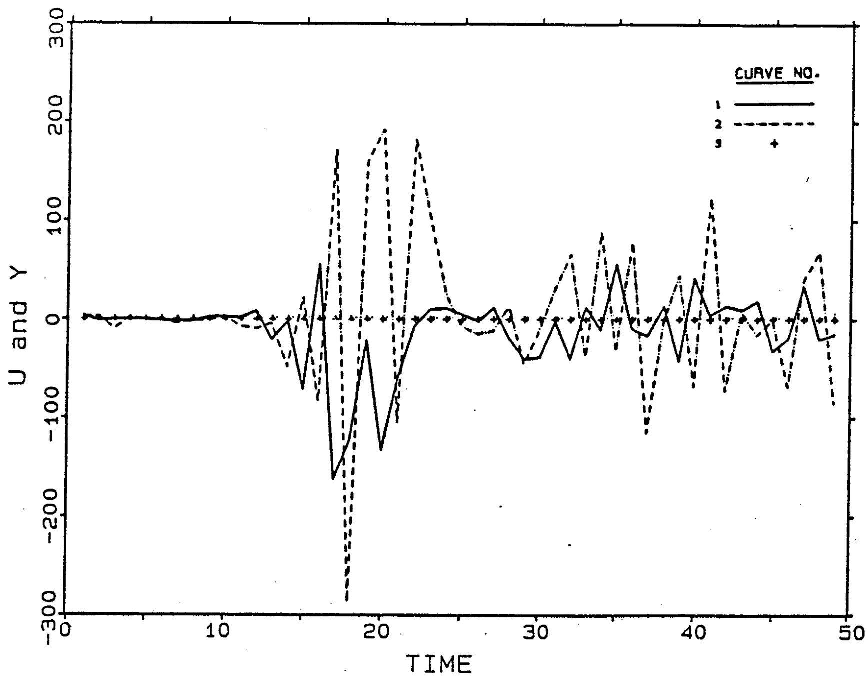


Fig. 4.1.c. Central Tendency Pole Assignment (y_k and u_k)

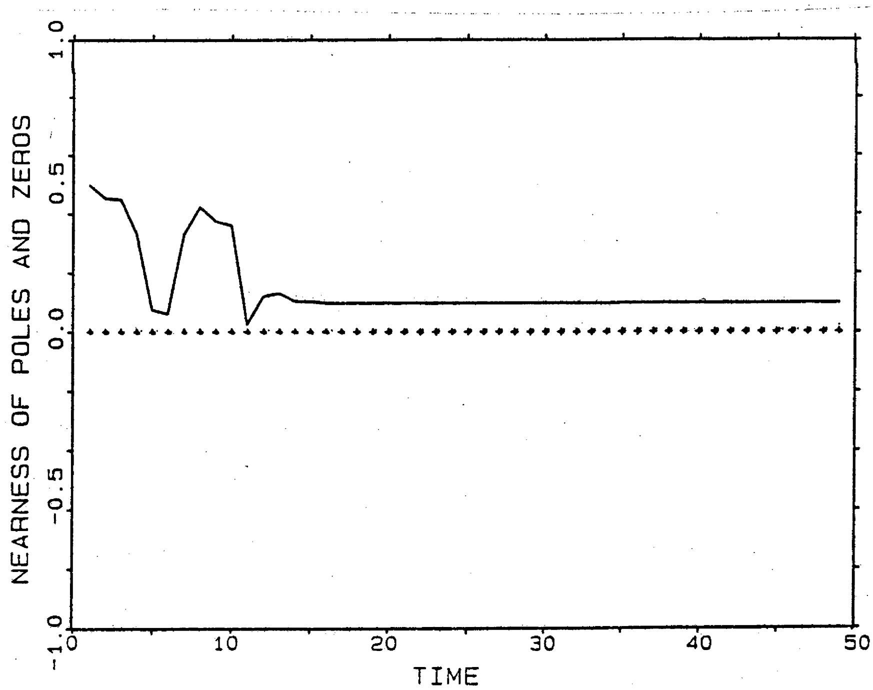


Fig. 4.2.a. Certainty Equivalence Pole Assignment

(Nearness of Poles and Zeros)

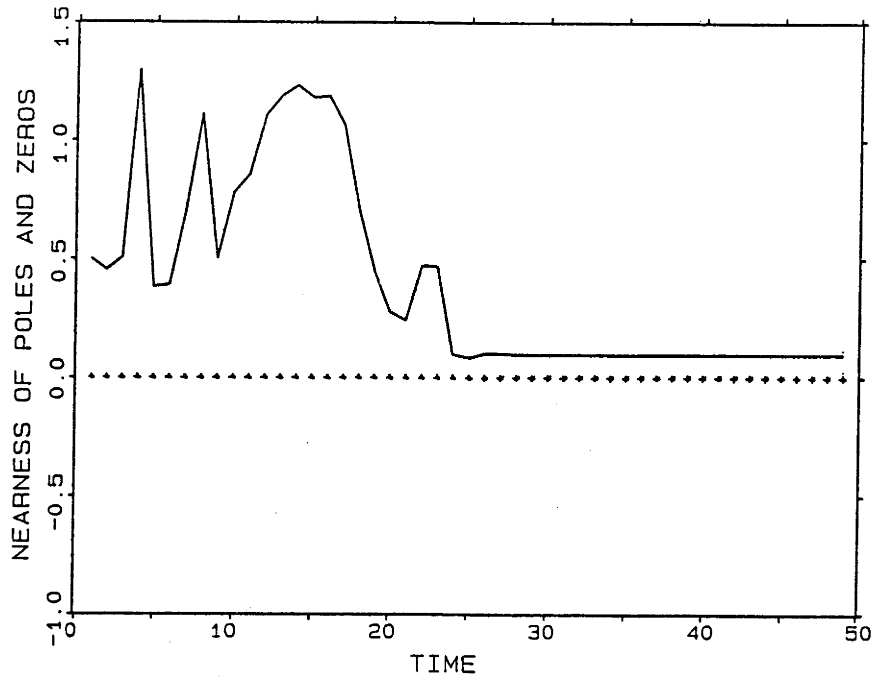


Fig. 4.2.b. Cautious Pole Assignment

(Nearness of Poles and Zeros)

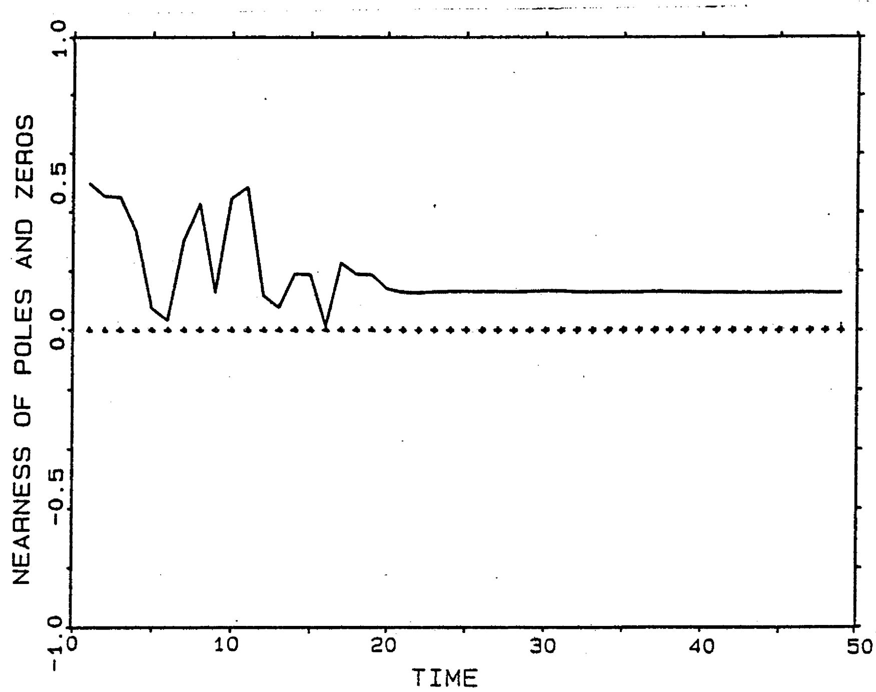


Fig. 4.2.c. Central Tendency Pole Assignment

(Nearness of Poles and Zeros)

5. CONCLUSIONS

This chapter has developed the concept of adaptive control designs based on central tendency measures. The specific case of Central Tendency Adaptive Pole Assignment using least squares parameter estimation has been studied in detail.

The convergence theory has shown how persistence of excitation and thus global convergence can be achieved by controller design, rather than by the addition of external signals.

Simulation studies have demonstrated the improved performance capabilities of the schemes of this chapter relative to those earlier proposed.

Areas for further research include the application of the ideas to adaptive schemes based on Extended Least Squares and Recursive Prediction Error Methods.

REFERENCES

[1] K.J. Astrom, "Theory and Applications of Adaptive Control - A Survey", **Automatica**, Vol.5, Sept. 1983, pp 471-486.

[2] T. Ryall, J.B. Moore "Central Tendency Minimum Variance Adaptive Control", Submitted for publication.

[3] T. Kailath, **Linear Systems**, Prentice Hall Inc, NY, (page 142) 1980.

[4] M. Green and J.B. Moore "Persistence of Excitation in Linear Systems", **Proc. of American Control Conference**, Boston, 1985.

[5] J.B. Moore "A Universality Advantage of Stochastic Excitation Signal for Adaptive Control", **Systems and Control Letters**, 9, pp55-58, 1987.

[6] H. Rootzen, J. Sternby, "Consistency in Least-squares Estimation: A Bayesian Approach", **Automatica** Vol. 20 pp 471_475, 1984.

[7] G.C. Goodwin and K.S, Sin, "Adaptive Control of Non Minimum Phase Plants", **IEEE Trans. Auto. Control**. April 1981, pp 478-483.

CHAPTER 6

ADAPTIVE LQG CONTROLLERS WITH CENTRAL TENDENCY PROPERTIES

1. INTRODUCTION

In designing adaptive controllers, it is common to exploit the certainty equivalence principle. That is, to use on-line parameter estimates in lieu of actual plant parameters in a controller design. When the parameter estimates are close to the actual plant parameters (assuming the plant is in the model set), then this approach makes some sense. However for adaptive schemes based on the certainty equivalence principle, where the plant model is unknown, there will, in general, be circumstances where the transient performance is unnecessarily poor. Perhaps intolerably so.

Central tendency adaptive schemes [1,2,3] tend to give improved transient performance over their corresponding simpler cousins, certainty equivalence based schemes. The simpler schemes use only parameter estimates, whereas the central tendency versions, in general, use parameter estimates and estimates of their uncertainty. The term central tendency is taken from statistics, where a measure of central tendency associated with a probability density function is one, such as a mean, mode, or median, with a "high" probability, avoiding the "low" probability tails. The term central tendency controller is introduced to define a controller [1,2,3] which loosely maximizes the probability of achieving the control objectives given plant parameter uncertainty information. Recall that the value of a random variable x which maximizes the probability of x is the mode, denoted mode[x]. Moreover, in general for a nonlinear function of x, $l(x)$, mode[$l(x)$] $\neq$ l(mode[x]). It is therefore clear that with imprecise knowledge of a plant, a certainty equivalence approach to controller design (using a nominal/best plant parameter estimate in the absence of knowledge of actual plant parameters), may lead to a poor controller for the actual plant. A central tendency controller design which uses the most likely controller parameters based on the plant uncertainty should lead to designs which are less likely to be poor.

For the case of adaptive minimum variance control [1], central tendency schemes are closed related to the simple cautious control schemes of [4]. They are less likely to be ill-conditioned leading to excessive control action than the cautious control schemes, and yet are only mildly more sophisticated.

For the case of adaptive pole assignment [2], central tendency schemes tend to avoid the ill-conditioning of the certainty equivalence schemes when plant estimates have near pole zero cancellations. They perform dramatically better in terms of transient performance than the corresponding certainty equivalence schemes in some situations, and appear to be no worse otherwise. However, there is additional complexity to appropriately incorporate the plant parameter uncertainty information into the controller. In one scheme a search is made of controller designs at previous iterations for the one which maximizes a readily calculated index based on the relationship between plant and controller parameters. What then of central tendency based adaptive linear quadratic Gaussian (LQG) schemes?

This chapter makes three observations which appear important. The first is that in terms of transient performance, one particular certainty equivalence based adaptive LQG controller in the literature [5] tends to have improved transient performance over others of comparable complexity. Why? The second observation is that this particular adaptive LQG scheme has a controller design rule which is linear in the parameter estimates and is thus itself inherently a primitive central tendency scheme. The third observation is that most certainty equivalence based adaptive LQG schemes can be upgraded in transient performance by linearization of the design rules, and further upgraded to fully fledged central tendency schemes by the addition of simple calculations which select from the present and previous controller designs the one that optimizes a measure of central tendency.

2. ADAPTIVE LQG CONTROLLERS VIA RICCATI RECURSIONS

In this section, we review certain adaptive LQG schemes for scalar stochastic input-output plant models, and make observation on their relative performance based on simulations.

Signal Model

Consider the auto-regression moving-average exogenous input (ARMAX) model

$$Ay_k = Bu_k + Cw_k \tag{2.1}$$

with input u_k, output y_k and zero mean white noise disturbance w_k. Here A,B,C are polynomial operators in terms of the unit delay q^{-1}. Thus

$$A(q^{-1}) = 1 + a_1q^{-1} + \dots + a_nq^{-n}, \quad B(q^{-1}) = b_1q^{-1} + \dots + b_mq^{-m},$$

$$C(q^{-1}) = 1 + c_1q^{-1} + \dots + c_lq^{-l}.$$

Without loss of generality $C(q^{-1})$ is assumed minimum phase.

Consider now a minimal state space representation for (2.1) as

$$x_{k+1}^M = \Phi^M x_k^M + \Gamma^M u_k + K^M w_k, \quad y_k = H^M x_k^M + w_k \tag{2.2a}$$

where

$$\Phi^M = \begin{bmatrix} -a_1 & 1 & \dots & 0 \\ -a_2 & & & \vdots \\ \vdots & & & 1 \\ -a_n & & & 0 \end{bmatrix}, \quad \Gamma^M = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad K^M = \begin{bmatrix} c_1 - a_1 \\ c_2 - a_2 \\ \vdots \\ c_n - a_n \end{bmatrix},$$

$$H^M = [1 \; 0 \; \dots \; 0] \tag{2.2b}$$

Consider also a non-minimal representation of (2.1). Thus

$$x_{k+1} = \Phi x_k + \Gamma u_k + K w_k \tag{2.3a}$$

where

$$\Phi^M = \begin{bmatrix} -a_1 & .. & -a_n & b_1 & .. & b_m & c_1 & .. & c_l \\ I_{n-1} & .. & 0 & 0 & .. & 0 & 0 & .. & 0 \\ 0 & .. & 0 & 0 & .. & 0 & 0 & .. & 0 \\ 0 & .. & 0 & I_{m-1} & .. & 0 & 0 & .. & 0 \\ 0 & .. & 0 & 0 & .. & 0 & 0 & .. & 0 \\ 0 & .. & 0 & 0 & .. & 0 & I_{l-1} & .. & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$H = [-a_1 \; \dots \; -a_n \; b_1 \; \dots \; b_m \; c_1 \; \dots \; c_l \,] = \theta \tag{2.3b}$$

Notice that

$$x_k^\tau = [y_{k-1} \dots y_{k-n} \; u_{k-1} \dots u_{k-m} \; w_{k-1} \dots w_{k-l}\,] \tag{2.4}$$

<u>Performance Index</u>

Let us associate with the above models performance indices as

$$I_k^M = \sum_{i=1}^{k}\left[x_i^{M\tau}Q^M x_i^M + R^M(u_i^M)^2\right],\ I_k = \sum_{i=1}^{k}(x_i^\tau Q x_i + Ru_i^2) \tag{2.5}$$

Extended Least Squares (ELS)

Considering the above signal model, a standard ELS algorithm for estimating the a_i, b_i, c_i based on the representation (2.4) and (2.5) is

$$\hat{\theta}_k = \hat{\theta}_{k-1} + P_k\bar{x}_k(y_k - \bar{x}_k^\tau\hat{\theta}_{k-1}) \tag{2.6a}$$

$$P_k = P_{k-1} - P_{k-1}\bar{x}_k(1 + \bar{x}_k^\tau P_{k-1}\bar{x}_k)^{-1}\bar{x}_k^\tau P_{k-1} \tag{2.6b}$$

$$\bar{x}_k^\tau = \left[y_{k-1} \ldots y_{k-n}\ u_{k-1}\ldots u_{k-m}\ \bar{w}_{k-1}\ldots\bar{w}_{k-l}\right] \tag{2.6c}$$

$$\bar{w}_k = y_k - \hat{\theta}_k^\tau\bar{x}_k \tag{2.6d}$$

for some $\hat{\theta}_0$ and $P_0 > 0$.

Explicit Adaptive LQG Controllers

Consider two adaptive LQG controllers based on the nominal representations (2.2) and (2.3) as follows (see also [5]). For the representation (2.3), we have

$$u_k = -L_k\hat{x}_k,\ \text{(Here } \hat{x}_k = \bar{x}_k) \tag{2.7a}$$

$$\hat{x}_{k+1} = \Phi_k\hat{x}_k + \Gamma u_k + K(y_k - H\hat{x}_k),\ \Phi_k \equiv \Phi(\theta = \hat{\theta}_k) \tag{2.7b}$$

$$L_k = \Omega_k\Gamma^\tau S_k\Phi_k,\quad \Omega_k = (\Gamma^\tau S_k\Gamma + R)^{-1} \tag{2.7c}$$

$$S_{k+1} = \Phi_k^\tau(S_k - S_k\Gamma\Omega_k\Gamma^\tau S_k)\Phi_k + Q,\quad S_0 = 0 \tag{2.7d}$$

Likewise, the second adaptive LQG scheme can be designed using x_k^M, Φ_k^M, Γ_k^M,

K_k^M, in lieu of x_k, Φ_k, Γ, K resulting in controllers

$$u_k^M = -L_k^M \hat{x}_k^M, \quad (\text{Here } \hat{x}_k^M \neq \hat{x}_k) \tag{2.8}$$

Observe that the equations for S_k, (likewise for S_k^M) are forward time-varying Riccati equations. Also observe that (2.2), (2.3) are innovation representations, so that the Kalman filter gains are K^M, K respectively. Moreover with L, L^M denoting the optimal LQG controller gains, then the minimality of (2.2) and standard Riccati /LQG theory tell us that

$$\lim_{k\to\infty} \hat{\theta}_k = \theta, \quad \Rightarrow \quad \lim_{k\to\infty} [L_k(\hat{\theta}_k)\ L_k^M(\hat{\theta}_k)] = [L\ L^M] \tag{2.9}$$

(Recall that with $\hat{\theta}_k$ converging to θ , minimality of { Γ^M, Φ^M, $(Q^M)^{1/2}$} tells us that { Γ^M, Φ^M, $(Q^M)^{1/2}$} is uniformly detectable and stabilizable, as then is { Γ_k, Φ_k, $Q^{1/2}$ } so that S_k^M, S_k converge to S^M, S the solutions of the algebraic Riccati equations associated with the optimal LQG controller and consequently, L_k^M, L_k converge to L_k^M, L as claimed.)

<u>Related Adaptive LQG Based Schemes</u>

The above schemes apply one recursion of the Riccati equation at each time instant using the latest estimates of the parameters $\hat{\theta}_k$. Variation allows 10 or so recursions of the Riccati equation at each iteration k, perhaps re-initializing at each iteration. For the case of an infinite number of recursions (assuming convergence), then the result would be equivalent to finding the relevant solution of an algebraic Riccati equation at each iteration, or equivalently, solving a spectral factorization and Bezout identity as in [6], see also references of [6]. In this latter case, the controls calculated from two representations above should be the same with matching initial conditions.

Preliminary Simulations

The simulation for the above two schemes (2.7),(2.8), have been done with the following plant, studied in [2],

$$y_k - 1.2y_{k-1} = u_{k-1} - 3.1\, u_{k-2} + 2.2u_{k-3} + w_k \tag{2.10}$$

with zero initial states. The initial estimates of the plant are $\hat{a}_1(0)=-1$, $\hat{b}_1(0)= 0$, $\hat{b}_2(0)=-2$, $\hat{b}_3(0)=3$. The associated performance indices are chosen as

$$I_k = \sum_{i=1}^{k}(y_i^2 + u_i^2)\,, \quad I_k^M = \sum_{i=1}^{k}\left[y_i^2 + (u_i^M)^2\right], \tag{2.11}$$

The controllers of the above schemes converge to the same (optimal) controller, but with different sample path dependent transient performances. For this particular plant, all stabilizing controllers are unstable. It appears that on average (but not in every sample path), the scheme (2.7) gives a better transient performance than the scheme (2.8). The plots of Figure 2.1 illustrate that for some sample paths, the scheme (2.7) is dramatically better than the scheme of (2.8). (Note the scale changes on the figures) These plots are typical of half the sample paths studied. We add that of the many sample paths studied, perhaps only one in five or six showed (2.8) significantly better than (2.7).

From the above simulation we are led to ask. Is there a reason for the significant difference in transient performance of the two adaptive LQG schemes? In terms of complexity both schemes are comparable, and in terms of philosophy of design both schemes are identical, what then is the crucial difference? We here conjecture

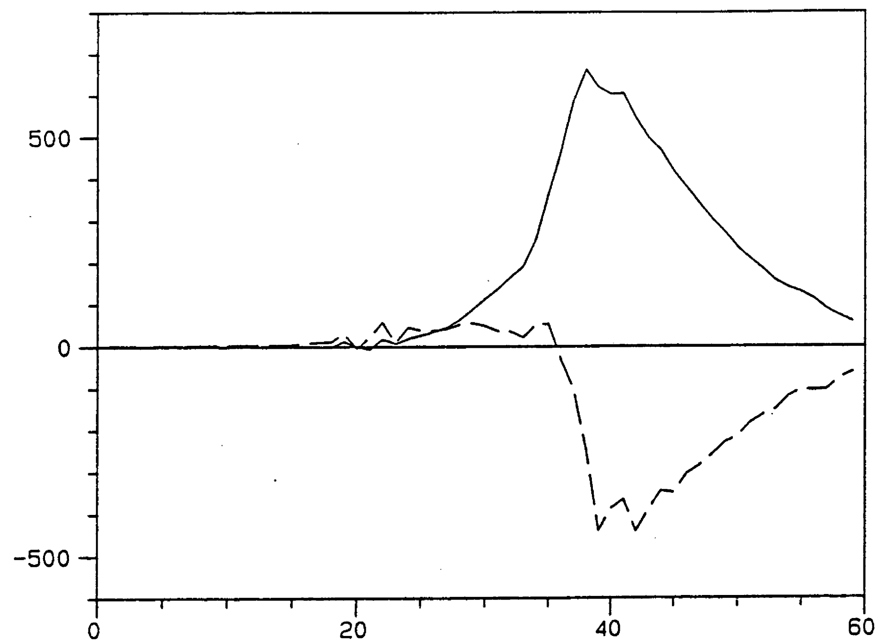


Fig. 2.1.a. Results of Scheme (2.8)

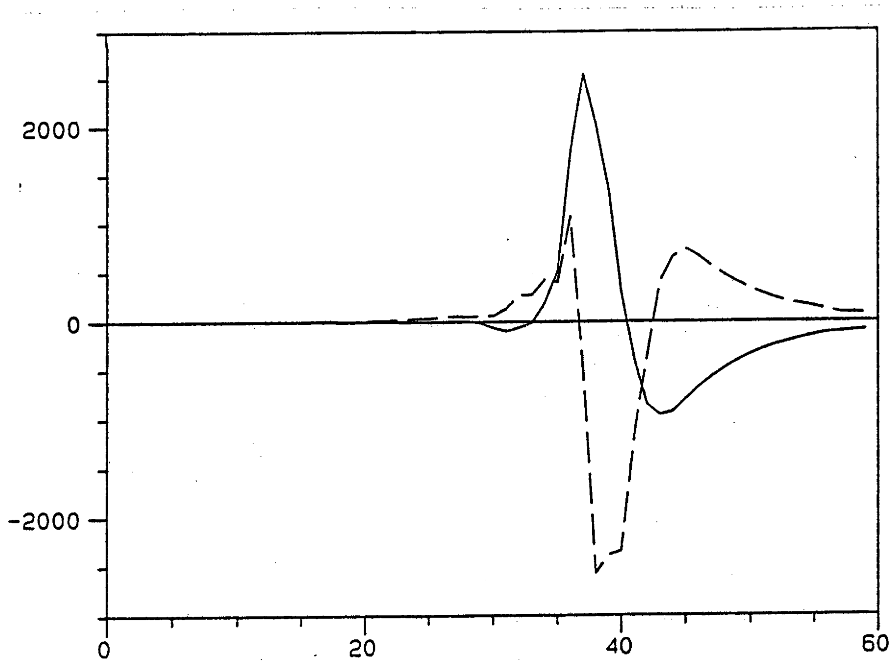


Fig. 2.1.b. Results of Scheme (2.7)

that the reason for the "improved" transient performance of the scheme (2.7) is that it has a linear relationship between the current controller parameters and the plant parameter estimates, and consequently has certain central tendency properties as defined in [1,2].

Linearity Property

Consider the adaptive LQG schemes (2.7) (2.8) where the controller parameters L_k, L_k^M are function of $\hat{\theta}_k$. Observe that

$$L_k(\hat{\theta}_k) \text{ is linear in } \hat{\theta}_k,\ L_k^M(\hat{\theta}_k) \text{ is non-linear in } \hat{\theta}_k \tag{2.12}$$

Consequently, with $\hat{\theta}_k$ the "best" estimate of θ given the measurements up to time k, given the controller design rule $L_k(.)$, then the certainty equivalence controller parameters $L_k(\hat{\theta}_k)$ is the corresponding "best" estimate of the controller parameters at time k. Given the design rule of $L_k^M(.)$, it is clear that $L_k^M(\hat{\theta}_k)$ is in general not the best estimate of the controller parameters. Thus (2.9) implies, according to the definition in [1,2], that

$$\begin{aligned} &L_k(\hat{\theta}_k) \text{ has central tendency properties,} \\ &L_k^M(\hat{\theta}_k) \text{ does not have central tendency properties.} \end{aligned} \tag{2.13}$$

The fact that $L_k^M(\hat{\theta}_k)$ is not in any sense a central tendency controller design rule means that in the presence of ill conditioning in the function $L_k^M(\hat{\theta}_k)$, then this control law design rule would lead to both "large" controller gains and "large" control signals. It is known that ill conditioning can occur when the plant model parametrized by $\hat{\theta}_k$ has near unstable pole zero cancellations. Certainly then the associated algebraic Riccati equation solution is ill conditioned, as is any iterative version of this. Observe that if in (2.7c), S_k is replaced by S_{k+1}, then $L_k(\hat{\theta}_k)$ would not be linear in $\hat{\theta}_k$.

3. Central Tendency Adaptive LQG Control

We have seen in the above section that the certainty equivalence adaptive LQG scheme (2.7) based on the non-minimal model has certain central tendency properties. How difficult then is it to modify the other versions of Section 2, or to strengthen the central tendency properties?

Linearized Riccati Based LQG Design Rules

It is fortuitous that the design rule (2.7) gives controller gains linear in $\hat{\theta}_k$. In this subsection, we mildly modify "all" Riccati based LQG design rules to have this property. First consider $L_k^M(\hat{\theta}_k)$ of (2.8). Here we propose a modification as

$$L_k^M = \Omega_k^M[\Gamma_k^{M\tau}S_k^M\Phi_{k-1}^M + \Gamma_{k-1}^{M\tau}S_k^M(\Phi_k^M - \Phi_{k-1}^M)] \tag{3.1}$$

with the properties

$$L_k^M(\hat{\theta}_k) \text{ linear in } \hat{\theta}_k, \tag{3.2}$$

$$\lim_{k\to\infty}\hat{\theta}_k = \theta \;\Rightarrow\; \lim_{k\to\infty} L_k^M(\hat{\theta}_k) = L^M(\theta) \tag{3.3}$$

To see (3.2), observe that the first term of (3.1) is linear in $\hat{b}_{i,k}$ for each i and independent of $\hat{a}_{i,k}$, while the second term is linear in $\hat{a}_{i,k}$ and independent of $\hat{b}_{i,k}$. Variations on (3.1) can also be devised to achieve the properties (3.2) (3.3).

Clearly the adaptive LQG design rules based on multiple iterations of the Riccati equation can be modified in the same way as suggested in (3.1), merely by upgrading S_k^M, Ω_k^M which depend only on $\hat{\theta}_{k-1}$, $\hat{\theta}_{k-2}$, ... and not on $\hat{\theta}_k$.

Simulations not reported in full detail here show that the linearized adaptive LQG law $L_k^M(\hat{\theta}_k)$ of (3.1), applied to (2.10) has on average improved transient performance over the non-modified scheme described in Section 2. In fact, now $L_k^M(\hat{\theta}_k)$ of (3.1) and $L_k(\hat{\theta}_k)$, for 50 or so noise sequences tested are on average comparable in performance, demonstrating again the power of a linearized controller design rule. Likewise for the adaptive LQG schemes based on algebraic Riccati equations.

Optimizing a Central Tendency Index

For design rules at time k with equal number of input variables (here elements of θ) and output variables (here elements of L), then the probability density function of L given θ, when $\theta = N[\hat{\theta}_k, P_k]$ is

$$f_k(L \mid \theta) = \kappa|\det J_k^{-1}(\theta)|\exp(-\frac{1}{2}\|\theta-\hat{\theta}_k\|_{P_k^{-1}}) \tag{3.4}$$

where $J_k(\theta) = \partial L_k(\theta)/\partial\theta$ and κ is a normalizing constant.

In [2], a central tendency design rule denotes a rule which avoids low probability designs [the tails of (3.4)] and seeks to maximize $f_k(L \mid \theta)$ in some way. A practical way to do this suggested in [2] is at time k to select a controller $L(\hat{\theta}_j)$ which optimizes the index, for some integer N,

$$\max_{j=0,1..N} f_k(L \mid \hat{\theta}_{k-j}) \tag{3.5}$$

This approach works well for adaptive pole assignment design when $J_k(\theta) = J(\theta)$ is time independent. In these schemes, when estimate of the plant has a near pole zero cancellation then $J_k^{-1} \approx 0$ which indicates severe ill-conditioning. Optimizing (3.5) avoids such ill-conditioning. Simulations show transient performance improvements from optimizing (3.5) even when there are no near pole zero cancellations in the estimate of the plant.

For adaptive LQG designs, the rules $L_k(\theta)$ Jacobians are time dependent. When the LQG design rule is nonlinear, then the calculation of $J_k(\hat{\theta}_{k-j})$ for j=0,1,..N is too tedious for practical implementation. When the rule is linearized, then $J_k(\hat{\theta}_{k-j})$ is invariant of j, so the index (3.5) is always optimized with $\theta = \theta_k$.

Here we propose a mild modification to the optimization task (3.5) as

$$\max_{j=0.1...N} f_{k-j}(L \mid \hat{\theta}_{k-j}) \tag{3.6}$$

Certainly this task is relatively straightforward to implement when the linearized controller design rules of Section 2 are employed. (Otherwise calculation can be simplified by neglecting terms which are tedious to calculate.).

The optimization task (3.6) shares the essential property of the central tendency approach, detailed in [2], namely that it avoids ill-conditioned calculations of $L(\hat{\theta}_k)$ leading to large controller gains and otherwise yields gains close to $L(\hat{\theta}_k)$. To see this, note that in the optimization task (3.6), the term $\exp(-\frac{1}{2} \|\theta - \hat{\theta}_k\|_{P_k^{-1}})$ is maximized when $\hat{\theta}_{k-j} = \hat{\theta}_k$, and the term $|\det J_{k-j}^{-1}(\hat{\theta}_{k-j})|$ is small when $L_{k-j}(\hat{\theta}_{k-j})$ is ill-conditioned. Adaptive LQG schemes based on solution of the algebraic Riccati equation are ill-conditioned when there are near unstable pole zero cancellations in the estimate of the plant. When only a few iterations of the Riccati equation are implemented, then ill-conditioning is likewise expected, but having less severity.

The gains from optimizing the central tendency measures here are expected to be greater for the schemes based on many iterations of the Riccati equation than for ones based on one or a few such iteration.

Simulations, not reported in full detail here, show that there is significant transient performance improvement for the example (2.10) in implementing the central tendency optimization (3.6). Taking N = 15 in (3.6), one noise sample function demonstrating dramatic improvement for a linearized adaptive LQG design rule based on the algebraic Riccati equation is presented in Figure 3.1. In these figures a comparison is made between the cases N = 0, 5, 15 where the case N = 0 can be interpreted as not taking any steps to optimize the measure (3.4).

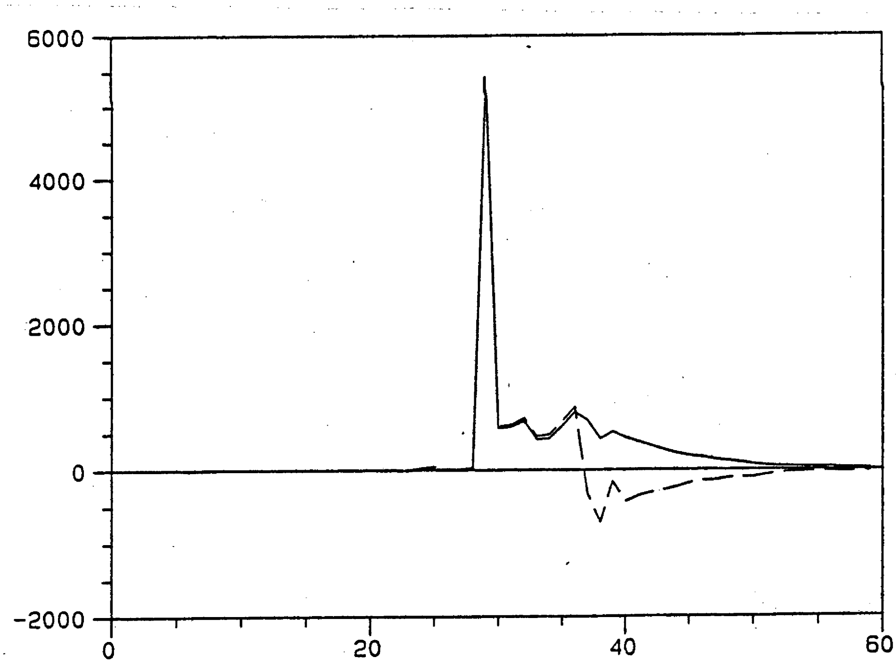


Fig. 3.1.a Results when N = 0

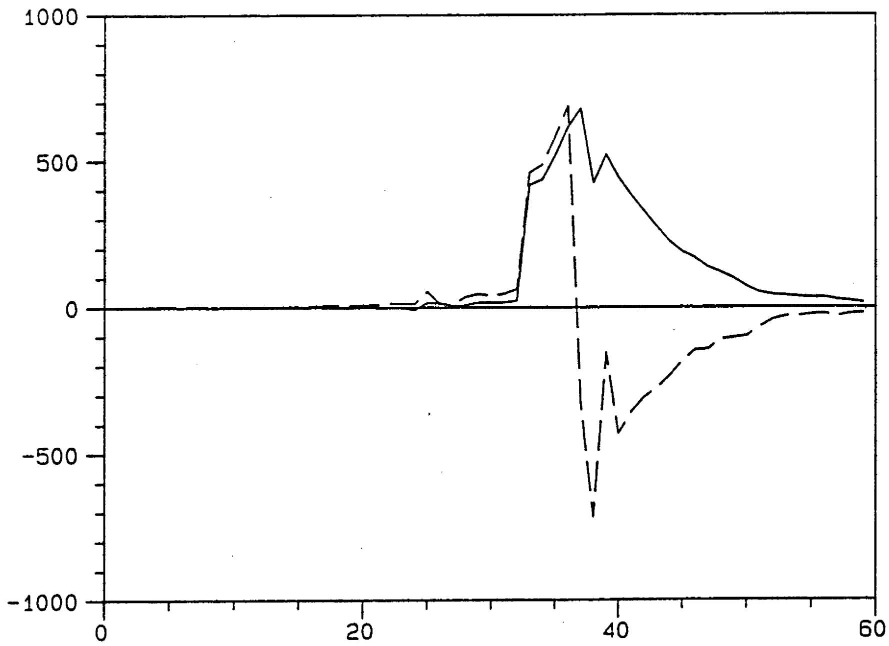


Fig. 3.1.b. Results when N = 5

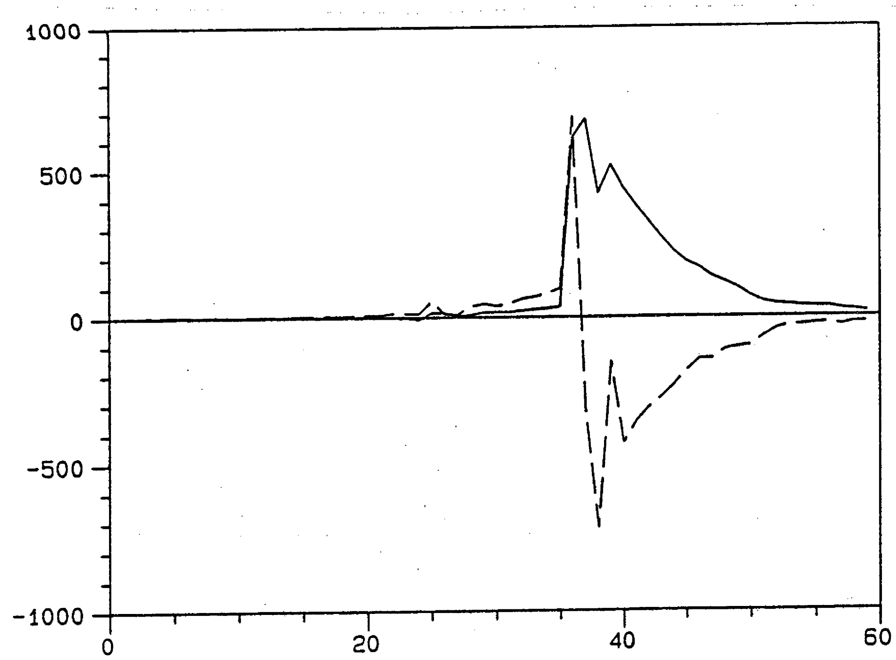


Fig. 3.1.c. Results when N = 15

4. CONCLUSIONS

Some very simple modifications to standard adaptive LQG schemes have been proposed to achieve optimization of central tendency measures, and thereby improved transient performance. The first proposal is to ensure a controller design rule at each iteration. The second is to select the best of previous and present controller designs to avoid ill-conditioning and thus unnecessarily large controller gains at each iteration.

Simulation studies have demonstrated the significance of the proposals on one "nasty" example prone to ill-conditioning in the controller design rule. Of course for less demanding controller designs which are well-conditioned, significant improvement in transient performance is not expected.

REFERENCES

1. T. Ryall, J.B. Moore and L. Xia. "Central Tendency Adaptive Control". **Proc. IEE Conference on Control.** pp116-121 July 1985.

2. J.B. Moore, T. Ryall, L. Xia. "Central Tendency Pole Assignment". **Proc. 25th CDC** pp100-105. Athens, Greece. Dec. 1986.

3. A. Chakravarty and J.B. Moore. "Flutter Suppression via Central Tendency Pole Assignment". **Proc. Control Engineering Conf.** pp78-80. Sydney, Australia. May 1986.

4. K. J. Astrom. "Theory and Applications of Adaptive Control- A Survey".

Automatica Val 19. pp471-486. 1983.

5. J.B. Moore " A Globally Convergent Recursive Adaptive LQG Regulator". **Proc. of IFAC Triennial World Congress** Budapest pp67-72 July 1984.

6. K.J. Astrom and Z.Y. Zhou. "A Linear Quadratic Gaussian Self-Tuner". **Workshop on Adaptive Control** pp1-20 Florence Italy. Oct. 1982

PART III

Adaptive Schemes for Overparametrized Systems

Chapter 7

ADAPTIVE ESTIMATION AND POLE ASSIGNMENT OF OVERPARAMETRIZED SYSTEMS

1. INTRODUCTION

In the practice of adaptive estimation and control there is a tendency to overparametrize signal models (plants) to be on the "safe" side. However, for overparametrized models there is a danger of ill-conditioning of both the adaptive estimation and the adaptive control algorithms applied to such plants. Of course, there is the twin danger of underparametrization, particularly in the absence of appropriate preprocessing of signals. The effects of underparametrization could be catastrophic and, since this is widely known, overparametrization emerges as a common problem. This paper shows that (after appropriate preprocessing) certain ill-conditioning associated with overparametrization can be avoided, without the need to perform on-line order determination with its associated significant increase in computational complexity.

For overparametrized signal models there can be a lack of excitation in regression vectors employed in parameter estimation and consequent ill-conditioning in the algorithms. Also, insufficient excitation can lead to identification of non-uniquely parametrized models which include pole-zero cancellations in the complex z-plane. When recursive estimates of the parameters of such non-uniquely parametrized models are applied for adaptive control, ill-conditioning leading to excessive controls can easily rise, particularly in adaptive pole assignment schemes.

Adaptive pole assignment schemes are perhaps the simplest schemes for adaptively stabilizing linear plants which are possibly nonminimum phase [1]. Also, they are the most natural form of adaptive scheme to use in some applications where it is required that the adaptive scheme behave as closely as possible to a nominal optimal design. However, a severe limitation for their application in practice has been their failure when the signal models are overparametrized. Adaptive pole assignment requires the solution of a linear algebraic equation which becomes ill-conditioned when estimates of the plant have near pole-zero cancellations. This is inevitable when the signal model is overparametrized. Some authors have proposed methods to cope with this difficulty using on-line estimation of plant order in some sense [2]. Such an approach increases the complexity of the adaptive scheme considerably.

The first contribution of this chapter, in Section 2, is to introduce excitation signals into the regression vectors for recursive (least squares based) parameter estimation in such a manner as to avoid ill-conditioning even when the model is overparametrized. For the special case when there is a potential non-uniqueness in the signal model owing to a pole-zero cancellation on the real axis, it is shown how the excitation can be designed so that the parameter estimates converge to those of a unique signal model, if one exists, otherwise to a model with a pole/zero cancellation at the origin. The introduced excitation does not excite the plant as in the case of added persistence of excitation signals. The estimation result of Section 2 is useful when applied in conjunction with the second contribution of the paper, in Section 3, which shows that when the parameter estimates converge so that the identified plant has a pole-zero cancellation at the origin, the associated central tendency adaptive pole assignment controller converges without ill-conditioning. Section 4 gives a novel property of Sylvester matrices required in the proof of the theory of Section 3, and Section 5 gives an illustrative simulation study. Conclusions are drawn in Section 6.

2. ALGORITHMS AND RESULTS - WHITE NOISE CASE

Signal Model.

Consider the following single-input single-output (SISO) input-output (stochastic) signal model class (plant) in terms of the unit delay operator q^{-1}, input u_k, output y_k and white noise disturbances w_k.

$$A(q^{-1})y_k = B(q^{-1})u_k + w_k$$

$$A(q^{-1}) = 1+a_1q^{-1} +\ldots +a_nq^{-n}, \quad B(q^{-1}) = b_1q^{-1} +\ldots+b_mq^{-m} \tag{2.1}$$

This can be rewritten as

$$y_k = \theta^\tau \bar{x}_k + w_k, \quad \theta^\tau = [a_1 \ a_2 \ldots a_n \ b_1 \ b_2 \ldots b_m]$$

$$\bar{x}^\tau = [-y_{k-1} \ldots -y_{k-n} \ u_{k-1} \ldots u_{k-m}] \qquad (2.1)'$$

The conditions on wk are more precisely:

The sequence $\{w_k\}$ is independent of u_k with $E[w_k \mid F_{k-1}] = 0$, $E[w_k^2 \mid F_{k-1}] \le \sigma_w^2 < \infty$, where F_k denotes the σ-algebra generated by $w_1, w_2 .. w_k$ (2.2)

Recursive Least Squares (RLS) Estimation

Consider that θ is estimated recursively minimizing a least squares criterion:

$$\bar{J}_k = \frac{1}{k}\left(\sum_{i=1}^{k} (y_i - \theta^\tau \bar{x}_i)^2 + (\theta - \bar{\theta}_0)^\tau \bar{B}_0(\theta - \bar{\theta}_0) \right)$$

via an RLS scheme as

$$\bar{\theta}_k = \bar{P}_k[\bar{B}_{k-1}\bar{\theta}_{k-1} + \bar{x}_k y_k], \quad \bar{B}_k = \bar{B}_{k-1} + \bar{x}_k\bar{x}_k^\tau$$

$$\bar{P}_k = \bar{P}_{k-1} - \bar{P}_{k-1}\bar{x}_k\bar{x}_k^\tau \bar{P}_{k-1}(1 + \bar{x}_k^\tau\bar{P}_{k-1}\bar{x}_k)^{-1} = \bar{B}_k^{-1} \tag{2.3}$$

for some initial conditions $\bar{\theta}_0$, $\bar{B}_0 > 0$.

Convergence Properties Review

To achieve a simple analysis making connection with Kalman filter theory as in [3], let us assume

> $\{w_k\}$ is normally distributed, and the a priori probability density associated with θ is $N[\bar{\theta}_0, \bar{P}_0]$ for some $\bar{P}_0 = \bar{B}_0^{-1} > 0$. (2.4)

This assumption is not needed for a more general theory based on stochastic Lyapunov functions $\tilde{\theta}_k^\tau\bar{B}_k\tilde{\theta}_k$ in [4,5]. where $\tilde{\theta}_k = \theta - \bar{\theta}_k$, but then the results are not quite as tidy.

For the model (2.1) (2.2), and the RLS scheme (2.3) under (2.4), Kalman filter theory tells us that, with $\tilde{\theta}_k = \theta - \bar{\theta}_k$,

$$\sigma^2\bar{P}_k = E[\tilde{\theta}_k\tilde{\theta}_k^\tau \mid F_{k-1}], \quad \bar{\theta}_k = E[\theta \mid F_{k-1}] \tag{2.5}$$

Moreover, from [3] there is almost sure convergence as

$$\lim_{k\to\infty} \bar{P}_k = \bar{P}^{LS}, \quad \lim_{k\to\infty} \bar{\theta}_k = \bar{\theta}^{LS}, \quad \text{a.s.} \tag{2.6}$$

for random variables $\bar{P}^{LS}$, $\bar{\theta}^{LS}$. With $\bar{x}_k$ sufficiently exciting in that $\bar{P}^{LS} = 0$, then [3] tells us that $\bar{\theta}^{LS} = \theta$. Also if $\bar{\theta}^{LS} = \theta$, then $\tilde{\theta}_k\tilde{\theta}_k{}^{\tau} \rightarrow 0$ as $k \rightarrow \infty$ and consequently, under (2.5), $\bar{P}_k \rightarrow 0$ as $k \rightarrow \infty$. Thus we have the following strong connection between sufficiency of excitation of $\bar{x}_k$ and parameter convergence.

<u>Lemma 2.1</u> For the RLS scheme (2.3) applied to the signal model (2.1) (2.2) under (2.4), then

$$\lim_{k\rightarrow\infty} \bar{P}_k = 0, \quad \Leftrightarrow \quad \lim_{k\rightarrow\infty} \bar{\theta}_k = \theta \tag{2.7}$$

<u>Proof</u> The proof is as above based on results in [3].

<u>On Sufficient Excitation</u>

In this subsection three specific excitation scenarios are studied using known results from [4,5,6]. These relate excitation of signal model inputs to outputs or states for reachable open loop time-invariant plants, with or without (possibly time-varying) feedback. The first two cases are a review of "known" results for the case when there is no overparametrization, while the third case deals with the case of overparametrized models.

Case (i): The simplest case to study is the non-overparametrized case when

$$q^nA(q^{-1}), q^mB(q^{-1}) \text{ are coprime,} \tag{2.8}$$

and u_k is suitably exciting in that

$$\lim_{k\rightarrow\infty} \left(\sum_{i=1}^{k} \bar{u}_i\bar{u}_i{}^{\tau} \right)^{-1} = 0, \quad \bar{u}_i{}^{\tau} = [u_{i-1}\ u_{i-2} \cdots u_{i-m-n}] \tag{2.9}$$

This latter condition is achieved when u_k includes at least (n+m)/2 distinct sinusoids decaying no faster than 1/k, as when uk is white noise or filtered white noise with a variance decaying no faster that 1/k.

Under (2.8), the model (2.1) is uniquely parametrized, $\bar{x}_k$ is reachable from u_k alone, and excitation of the inputs as in (2.9) implies excitation ot the states $\bar{x}_k$ [4,5]. Thus (2.8),(2.9) ensure that $\bar{P}_k \rightarrow 0$ as $k \rightarrow \infty$, and in turn $\bar{\theta}_k \rightarrow \theta$ as $k \rightarrow \infty$. for this case then, w_k need not be sufficiently exciting in any sense.

Case (ii): Another simple case to study is when (2.8) does not necessarily hold, but wk as well as uk is suitably exciting in that (2.9) is satisfied and

$$\left(\sum_{i=1}^{\infty} E[w_k^2 \mid F_{k-1}]\right)^{-1} = 0 \tag{2.10}$$

Under (2.9),(2.10), the model (2.1) is uniquely parametrized, $\bar{x}_k$ is reachable from u_k, w_k, and is sufficiently exciting to guarantee that $\bar{P}_k \rightarrow 0$ as $k \rightarrow \infty$, [4,5] and in turn that $\bar{\theta}_k \rightarrow \theta$ as $k \rightarrow \infty$. For this case then, the convergence as such is independent of whether or not the coprimeness condition (2.8) is satisfied.

Case (iii): The possibly overparametrized signal model situation of particular interest in this chapter, is when (2.8) possibly fails and there is no a priori guarantee of sufficient excitation of w_k as in (2.10). In this case the model (2.1) may not be uniquely parametrized, having one or more pole-zero cancellations in the z-plane. Also, $\bar{x}_k$ may not be sufficiently exciting to ensure that $\bar{P}_k \rightarrow 0$ as $k \rightarrow \infty$. Convergence can take place to a signal model with pole-zero cancellation anywhere in the complex z-plant. We seek to avoid such a situation and propose an RLS algorithm with additional excitation in the regression vector. It is derived using an alternative signal model formulation.

Alternative Signal Model Formulation

Consider (2.1) re-organized as

$$y_k = \theta^\tau x_k + (w_k - \theta^\tau v_k), \quad x_k = \bar{x}_k + v_k \tag{2.11}$$

where v_k is an excitation term to ensure that x_k is suitably exciting. Notice that v_k has no influence on y_k, u_k. Its selection in the next subsection is in accordence with a parameter estimation error measure, so that when parameter estimates are converging to their true values, v_k converges to zero.

RLS Estimation with Regression Vector Excitation

Consider that θ is estimated recursively minimising a least squares criterion:

$$J_k = \frac{1}{k}\left(\sum_{i=1}^{k}(y_i - \theta^\tau x_i)^2 + (\theta - \hat{\theta}_0)^\tau B_0(\theta - \hat{\theta}_0)\right) \tag{2.12}$$

via an RLS scheme as

$$\hat{\theta}_k = P_k[B_{k-1}\hat{\theta}_{k-1} + x_k y_k], \quad B_k = B_{k-1} + x_k x_k^\tau$$

$$P_k = P_{k-1} - P_{k-1}x_k x_k^\tau P_{k-1}(1 + x_k^\tau P_{k-1}x_k)^{-1} = B_k^{-1} \tag{2.13}$$

for some initial conditions $\hat{\theta}_0$, $B_0 > 0$.

Regression Vector Excitation Selection

Consider the signal model (2.1) formulated as (2.11). Let us assume that either there is no overparametrization in that (2.8) holds, or that there is the possibility of overparametrization which includes one pole-zero cancellation, leading to a non-uniquely parametrized model. In the latter case, we consider a unique parametrization with the properties (meaningful only when n>1, m>1)

$$a_n = b_m = 0, \quad q^{n-1}A(q^{-1}), \; q^{m-1}B(q^{-1}) \text{ are coprime} \tag{2.14}$$

For such a situation we propose a v_k selection as follows,

> The sequence $\{v_k\}$ is selected as an independent (Gaussian) zero mean white noise excitation term such that its covariance $\Sigma_v{}^2 = Dk^{-1}tr(P_{k-1})$ for $D = diag[0\ 0\ \ldots\ 0\ d_n{}^2\ \ 0\ \ldots 0\ d_{n+m}{}^2]$ where $d_n{}^2 > 0$, $d_{n+m}{}^2 > 0$. Denote the non-zero elements as $\sigma^2{}_{n,k}$, $\sigma^2{}_{n+m,k}$ (2.15)

The Gaussian assumption on v_k is to keep analysis simple and is not a necessary condition. In practice a more efficient excitation would be where elements had values in a bounded domain. Also, results are readily derived for the case when v_k is deterministic but containing a sufficient number of frequency components.

The following results are now a consequence of a straightforward application of results from [4,5].

> Lemma 2.2 Consider the linear signal model (2.1) (2.2) formulated as (2.11) with the v_k selection of (2.15).
>
> (i) Then under (2.8), x_k is the output of a linear time-invariant system reachable from u_k. Moreover, with u_k selected so that for some $\alpha > 0$

$$\lim_{k\to\infty} (\ln k)^{1+\alpha} \Big(\sum_{i=1}^{k} \bar{u}_i \bar{u}_i^{\tau} \Big)^{-1} = 0 \tag{2.16}$$

then

$$\lim_{k\to\infty} (\ln k)^{1+\alpha} P_k = 0, \quad \lim_{k\to\infty} (\ln k)^{1+\alpha} \bar{P}_k = 0, \quad \text{a.s.} \tag{2.17}$$

(ii) Also under (2.14), x_k is the output of a linear time-invariant system reachable from u_k and the elements $v_{n,k}$, $v_{n+m,k}$ of v_k. Then under (2.14) the excitation conditions on u_k of (2.4) and v_k, namely,

$$\lim_{k\to\infty} \Big(\sum_{i=1}^{k} \sigma^2_{n,i} \Big)^{-1} = \lim_{k\to\infty} \Big(\sum_{i=1}^{k} \sigma^2_{n+m,i} \Big)^{-1} = 0 \tag{2.18}$$

translate to excitation of x_k as

$$\lim_{k\to\infty} P_k = 0, \quad \text{a.s.} \tag{2.19}$$

<u>Proof</u> (i) Under the coprimeness condition (2.8), then from [5] it is immediate that x_k is the output of a linear time-invariant system driven by u_k, v_k, w_k and is reachable from u_k alone. Now should u_k be acting alone, then Lemma (3.2) of [4] applies to give that, for all k and some $K > 0$

$$\sum_{i=1}^{k+n+m} x_i x_i^{\tau} \geq K \sum_{i=1}^{k} \bar{u}_i \bar{u}_i^{\tau}$$

from which (for some $\alpha > 0$)

$$\lim_{k\to\infty} \Big(\sum_{i=1}^{k} \bar{u}_i \bar{u}_i^{\tau} \Big)^{-1} = 0, \Rightarrow \lim_{k\to\infty} \Big(\sum_{i=1}^{k} x_i x_i^{\tau} \Big)^{-1} = 0$$

$$\lim_{k\to\infty} (\ln k)^{1+\alpha} \Big(\sum_{i=1}^{k} \bar{u}_i \bar{u}_i^{\tau} \Big)^{-1} = 0 \Rightarrow \lim_{k\to\infty} (\ln k)^{1+\alpha} P_k = 0$$

Applying the results of Lemma (3.3) and its Remark 1 of [4], now tells us that when bounded variance white noise inputs v_k, w_k (independent of u_k) are also applied, the same implications hold, so that (2.16) implies (2.17a) as claimed. The result (2.17b) holds likewise.

(ii) Under (2.14), the elements $y_{k-1} \ldots y_{k-n-1}$, $u_{k-1} \ldots u_{k-m-1}$ of x_k are reachable from u_k alone [5], and the remaining elements $(y_{k-n} + v_{n,k})$, $(u_{k-m} + v_{n+m,k})$ of x_k are reachable from the non-zero elements of v_k, namely $v_{n,k}$ and $v_{n+m,k}$. Thus x_k is reachable from u_k, v_k under (2.14). Applying again the Lemma (3.3) and its Remark 1 of [4] gives directly that (2.4), (2.18) together imply (2.19). ΔΔΔ

<u>Main Results of Section</u>

<u>Theorem 2.1</u> Consider the signal model (2.1) (2.2) which is possibly overparametrized in that either (2.8) or (2.14) holding. Consider an RLS scheme (2.13) based on the alternative model formulation (2.11) with v_k selected as in (2.15) and (2.4) holding. Consider also that u_k is sufficiently exciting in that (2.16) holds. Then there is parameter convergence as

$$\lim_{k \to \infty} \hat{\theta}_k = \theta , \quad \text{a.s.} \tag{2.20}$$

where θ is the unique parameter associated with (2.1) under (2.8) or (2.14). Moreover (2.19) also holds.

<u>Proof</u> Part (i). In the case that (2.14) is satisfied, so that $a_n = b_m = 0$, then $\theta^\tau v_k = 0$. Now Lemma 2.1 applies with x_k replacing $\bar{x}_k$, so that (2.20) holds if and only if (2.19) holds. Assume that (2.19) does not hold, then since $P_k \le P_{k-1}$ for all k, $tr(P_k)$ converges to a non-zero element and from (2.15) the variances of v_k decay

at a rate k^{-1}. Thus (2.18) holds and in turn (2.19) holds under Lemma 2.2. This contradicts the assumption, so that (2.19) and (2.20) hold.

Part (ii) In the case that (2.8) is satisfied, Lemma 2.2 tells us that (2.16) implies (2.17). As a consequence from (2.15) then

$$\lim_{k\to\infty} \sum_{i=1}^{k} \sigma^2_{n,i} < \infty, \quad \lim_{k\to\infty} \sum_{i=1}^{k} \sigma^2_{n+m,i} < \infty$$

In turn we claim that

$$\lim_{k\to\infty} \sum_{i=1}^{k} (\theta^\tau v_i)^2 < \infty, \quad \text{a.s.} \tag{2.21}$$

This follows since, as is readily established under (2.15),

$$\sum_{i=1}^{k} (v^2_{n,i} - \sigma^2_{n,i}), \qquad \sum_{i=1}^{k} (v^2_{n+m,i} - \sigma^2_{n+m,i}),$$

are martigales bounded in L_2 and converge almost surely. Under (2.21) and (2.17b) we now claim

$$\lim_{k\to\infty} \bar{P}_k \sum_{i=1}^{k} w_i(\theta^\tau v_i) = 0, \quad \text{a.s.} \tag{2.22}$$

To see this observe that

$$E[M_k \mid F_{k-1}] = M_{k-1}, \qquad M_k = \sum_{i=1}^{k} w_i(\theta^\tau v_i)$$

$$E[M_k^2] = E\left[\sum_{i=1}^{k} \sum_{j=1}^{k} \theta^\tau v_i E[w_i w_j \mid F_{\min(i-1,\, j-1)}] v_j^\tau \theta \right] \leq E\left[\sum_{i=1}^{k} (\theta^\tau v_i)^2 \sigma_w^2 \right]$$

Thus M_k is a martigale on F_{k-1}, bounded in L_2 under (2.21), and so converges almost surely, so that (2.17b) implies (2.22).

Now under (2.17b) Lemma 2.1 can be applied to yield

$$\lim_{k\to\infty} \bar{P}_k \sum_{i=1}^{k} \bar{x}_i w_i = 0, \qquad \lim_{k\to\infty} \bar{P}_k \sum_{i=1}^{k} \bar{x}_i(\theta^\tau v_i) = 0, \quad \text{a.s.} \qquad (2.23)$$

The first rersult follows from (2.7) and the relationships

$$\bar{\theta}_k = \bar{P}_k \sum_{i=1}^{k} \bar{x}_i y_i, \qquad \tilde{\theta}_k = -\bar{P}_k \left(\sum_{i=1}^{k} \bar{x}_i w_i - B_0\right)$$

The second result follows from the first since $(v_i^\tau \theta)$ has the same essential properties as w_i in (2.2) (2.4).

The results (2.22),(2.23) lead in turn to the following convergence results

$$\bar{P}_k B_k = \bar{P}_k \sum_{i=1}^{k} (\bar{x}_i \bar{x}_i^\tau + \bar{x}_i v_i^\tau + v_i \bar{x}_i^\tau + v_i v_i^\tau) \to I, \ \text{a.s. as } k \to \infty$$

$$\hat{\theta}_k = P_k \sum_{i=1}^{k} x_i y_i = (\bar{P}_k B_k)^{-1} \bar{P}_k \sum_{i=1}^{k} (\bar{x}_i \bar{x}_i^\tau \theta + \theta^\tau v_i x_i^\tau \theta + \bar{x}_i w_i + \theta^\tau v_i w_i)$$

$$\to \theta, \quad \text{a.s.} \quad \text{as } k \to \infty$$

so that (2.20) holds as claimed. ΔΔΔ

Remarks 1. The specific v_k selection of the theorem is for the case when there is one possible pole-zero cancellation in the model. This is clearly one of the most important case, since in selecting a model order there is a tendency when in doubt to merely increase a likely order by one for safety.

2. If a bank of estimators is employed conditioned on different model orders, then the results above tell us that only odd (or even) orders need to be covered. Such a saving is a factor of two.

3. Rather than work with banks of estimators as in Remark 2 above, an ad hoc approach is to relex (2.15) and have $v_{1,k}$, $v_{2,k}$, .. $v_{n,k}$ each independent and suitably exciting with the variance of $v_{i,k}$ increasing with i. This would force pole-zero cancellations to occur near the origin, but would lead to biased estimates. We do not study this technique.

4. The algorithms and results of this section have been analysed for the simplest of stochastic signal models, namely, when $(y_k - \theta^\tau x_k)$ is white. For more general autoregressive moving average exogonous input (ARMAX) models, extended least squares (ELS) based algorithms can be employed. We claim that the technique of introducing v_k also can be made to extended the capabability of ELS based schemes. In particular, for those which are globally convergent under the coprimeness condition (2.9), [7] with the modifications they are globally convergent also when there is a possible overparametrization, as when (2.14) holds. Essentially the same theoretical approach applies, but the technical details are more tedious, so are not explored here.

3. ADAPTIVE POLE ASSIGNMENT

Pole Assignment

Let us seek an adaptive pole assignment scheme associated with the signal model

(2.1) so that there is asymptotic convergence to

$$H(q^{-1})y_k = KB(q^{-1})r_k \tag{3.1}$$

where $H(q^{-1}) = 1+h_1q^{-1} +...+ h_{n+m}\, q^{-n-m}$ is specified by the desired closed loop poles, K is a constant, and r_k is a reference input. This can be achieved by the following controller [2].

$$H(q^{-1})E(q^{-1})u_k = -H(q^{-1})F(q^{-1})y_k + Kr_k \tag{3.2}$$

where $E(q^{-1}) = 1 + e_1q^{-1} +...+e_mq^{-m}$, $F(q^{-1}) = f_1q^{-1} +...+f_nq^{-n}$ are given from the solution of the Bezout equation.

$$A(q^{-1})E(q^{-1}) + B(q^{-1})F(q^{-1}) = 1 \text{ , or } S(n,m)\,\phi = \alpha \tag{3.3}$$

where

$$\phi^\tau = [e_1\ e_2 \ldots\ e_m\ f_1\ f_2 \ldots f_n] \text{ , } \alpha^\tau = [-a_1\ -a_2\ \ldots -a_n\ 0 \ldots 0]$$

$$S(n,m) = \begin{array}{cc} \left[\begin{array}{cccccccccc} 1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a_1 & 1 & & & 0 & b_1 & & & & \cdot \\ \cdot & a_1 & \cdot & & \cdot & \cdot & b_1 & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & \cdot & & \cdot \\ a_n & & & & 1 & b_m & & & \cdot & \cdot \\ 0 & \cdot & & & \cdot & 0 & \cdot & & & b_1 \\ \cdot & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & a_n & 0 & \cdot & \cdot & & b_m \end{array}\right] & \begin{array}{c} \uparrow \\ n+m \\ \downarrow \end{array} \\ \longleftarrow m \longrightarrow \longleftarrow n \longrightarrow & \end{array} \tag{3.4}$$

It is known that solutions of (3.3) exist if and only if rank[S α] = rank[S]. Also from [8]

$$S(n,m) \text{ is non-singular} \Leftrightarrow q^nA(q^{-1}),\ q^mB(q^{-1}) \text{ are coprime} \qquad (3.5)$$

In certainty equivalence adaptive pole assignment, the estimates $\hat{\theta}_k$ are used in lieu of θ in (3.4) to compute on-line estimates $\hat{\phi}_k^{CE}$ of the controller parameters, so that in obvious notation

$$\hat{S}_k^{CE}(n,m)\hat{\phi}_k^{CE} = \hat{\alpha}_k^{CE} \qquad (3.6)$$

When $q^nA(q^{-1})$, $q^mB(q^{-1})$ are coprime, $\hat{S}_k(n,m)$ is non-singular and the solution of (3.6) exists. Otherwise it may not. Ill-conditioning in $\hat{S}_k^{-1}(n,m)$ can cause excessive values for $\hat{\phi}_k$. One modification to avoid large $\hat{\phi}_k$ is to select $\hat{\phi}_k = \hat{\phi}_{k-1}$ during ill-conditioning, but for unkown plants, it is not a priori clear how to quantify ill-conditioning to achieve a useful adaptive controller.

Central Tendency Control

In central tendency adaptive pole assignment [9], ill-conditioning in calculating the controller parameters is avoided without requiring prior information concerning the plant or controller. Suppose there is a Gaussian a posteriori probability density for the model parameters θ as $N[\hat{\theta}_k, \hat{\alpha}_k^2P_k]$ where $\hat{\alpha}_k^2$ is an estimate of σ^2, then there is an associated non-Gaussian probability density for the pole assignment controller parameters ϕ. A central tendency selection $\hat{\phi}_k^{CT}$ is one which maximizes this density, or at least avoids the tails of this density. Practical implementations are given in [9]. Associated with $\hat{\phi}_k^{CT}$ is some parameter estimate $\hat{\theta}_k^{CT}$ which is not in general $\hat{\theta}_k$. Thus, in obvious notation

$$\hat{S}_k^{CT}(n,m)\hat{\phi}_k^{CT} = \hat{\alpha}_k^{CT} \qquad (3.7)$$

The estimate $\hat{\theta}_k^{CT}$ has the property [9] that it is "close" to $\hat{\theta}_k$ but "far" from hypersurfaces for which $q^n\hat{A}_k(q^{-1})$, $q^m\hat{B}_k(q^{-1})$ are not coprime. As a consequence, the following property is claimed for central tendency adaptive pole assignment.

> The selections $\hat{\theta}_k^{CT} \rightarrow \theta$ are such that $[\hat{S}_k^{CE}(n,m)]^{-1}$ exists for all k, and if θ belongs the hypersurface (in Θ-space) defined by $q^nA(q^{-1})$, $q^mB(q^{-1})$ not coprime, then as $k \rightarrow \infty$, $\hat{\theta}_k^{CT}$ is contained in a cone centered at θ which excludes the tangent hyperplane at θ. (3.8)

Remarks 1. To give a geometric interpretation of (3.8), consider Figures 3.1 and 3.2. The heavy arc is the pole zero cancellation singular region for θ estimates

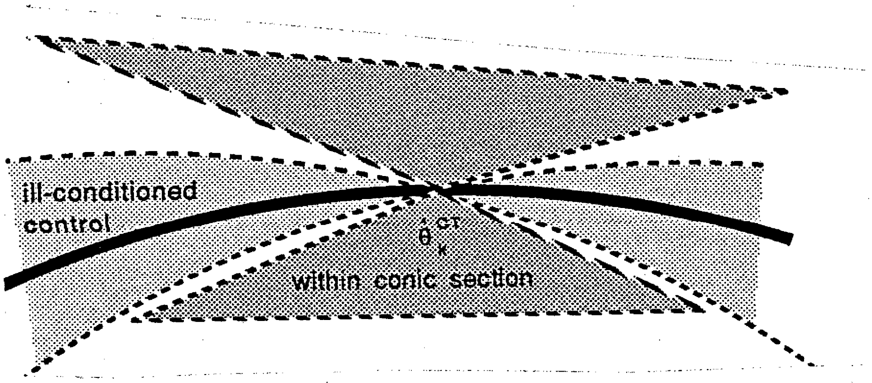


Fig. 3.1. Avoidance of Ill-conditioning

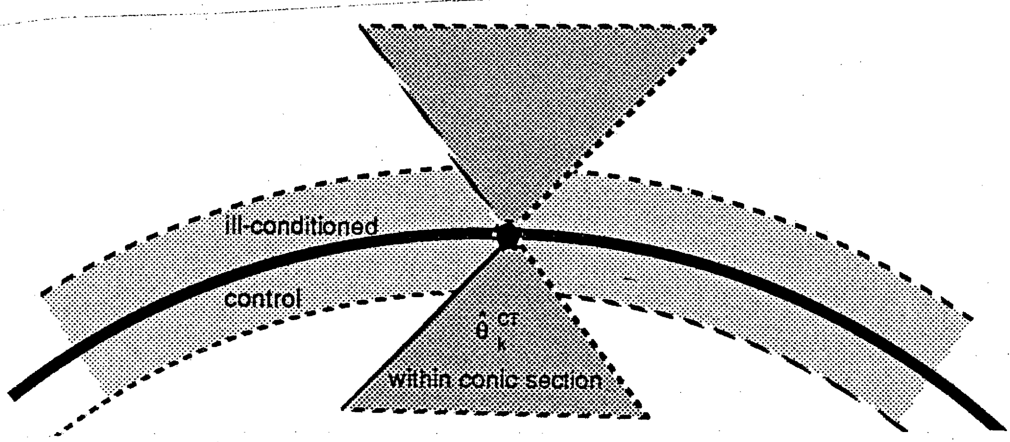


Fig. 3.2. Ill-conditioned Control

in Θ-space. The light shaded area is a zone of ill-conditioning control surrounding the singular arc. The heavy shaded cones are the conic regions of possible central tenency control estimates $\hat{\theta}_k^{CT}$ of (3.8). Figure 3.1 depicts the situation when the solution (2.14) is at the conic intersection and the cones avoid the ill-conditioned regions. Figure 3.2 depicts a solution not satisfying (2.14) when the central tendency estimates $\hat{\theta}_k^{CT}$ become ill-conditioned when converging.

2. The property (3.8) can be viewed as a vorollary of results rigorously proved in [9], although specific reference has not been made to (3.8) in [9]. The property is readily believable, but since it is not rigorously proved as such in [9] and is beyound the scope of this chapter, we here add the "qualification" to central tendency control that (3.8) be satisfied.

Two cases are studied now.

Case (i) Known Model Order

Theorem 3.1 Consider the signal model (2.1) for the case when it is not overparametrized, so that $q^nA(q^{-1})$, $q^mB(q^{-1})$ are coprime. Consider also RLS parameter estimates $\hat{\theta}_k$ from (2.13), and associated certainty equivalence [or central tendency] pole assignment with controller parameters $\hat{\phi}_k^{CE}$ [or $\hat{\phi}_k^{CT}$] and Sylvester matrices $\hat{S}_k^{CE}(n,m)$ [or $\hat{S}_k^{CT}(n,m)$]. Consider also that r_k is sufficiently exciting so that $\hat{\sigma}_k^2 P_k \to 0$ as $k \to \infty$ and there is parameter convergences with $\hat{\theta}_k \to \theta$ as $k \to \infty$. Then

$$\lim_{k\to\infty}[\hat{S}_k^{CE}(n,m)]^{-1} = S^{-1}(n,m), \text{ or } \lim_{k\to\infty}[\hat{S}_k^{CT}(n,m)]^{-1} = S^{-1}(n,m)$$

$$\lim_{k\to\infty}\hat{\phi}_k^{CE} = \phi, \text{ or } \lim_{k\to\infty}\hat{\phi}_k^{CT} = \phi \qquad (3.9)$$

Proof This is immediate from the coprimeness assumption and the property (3.5). Notice that there is no need of assumption (3.8).

Case (ii) Overparametrization

Theorem 3.2 Consider the signal model (2.1) with n>1, m>1, and (2.14) holding. Consider also RLS parameter estimates $\hat{\theta}_k$ of (2.13) and associated central tendency adaptive pole assignment controller parameters $\hat{\phi}_k^{CT}$ given from (3.7) with (3.8) satisfied. Then under RLS convergence of $\hat{\theta}_k^{CT}$ to θ as $k \to \infty$,

$$\lim_{k\to\infty} \det \hat{S}_k^{CT}(n,m) = 0, \quad \lim_{k\to\infty} \hat{\phi}_k^{CT} = \phi^* \tag{3.10}$$

where ϕ^* is a unique solution of $S(n,m)\phi^* = \alpha$ with zero elements as

$$\phi^* = [e_1^* \; e_2^* \ldots e_{m-1}^* \;\; 0 \;\; f_1^* \; f_2^* \ldots f_{n-1}^* \; 0\,]^\tau \tag{3.11}$$

Proof The proof is given in the Appendix, based on Sylvester matrix property studied in Section 4.

Remarks 1. It might be thought that the results can be more directly proved from properties of Diophantine equations. Although certain progress can be made along these lines, and indeed the results can be stated in such terms, it does not appear straghtforward to complete any proof without resort to Sylvester matrix properties as in Section 4.

2. This theorem result is dependent on the non-standard nature of the RLS algorithm with its internal perturbations v_k. In the presence of overparametrization, standard RLS estimation (when $v_k = 0$) will almost surely not converge to the

unique limits solution (2.14). This means that there is inevitable ill-conditioning. This situation applies even when a central tendency adaptive control law is implemented, as depicted in Figure 3.2.

3. The above theorem result is also facilitated by the central tendency property (3.8) of the adaptive controller. In the presence of overparametrization, even when the modified RLS algorithm of this chapter is implemented without (3.8) satisfied, there is a non-zero probability of ill-conditioning as suggested from Figure 3.1, at least during transients.

4. The theorem is developed in conjunction with the RLS estimation (2.4) which copes with possible overparametrization, by the a_n, b_m. Should some RLS based scheme cope with higher orders of overparametrization, the results of this theorem would still hold. The details of a more general proof are straightforward and are omitted here.

5. The result of the theorem also applies to the situation when (3.3) is replaced by

$$A(q^{-1})E(q^{-1}) + B(q^{-1})F(q^{-1}) = H(q^{-1}) \tag{3.12}$$

with $H(q^{-1})$ having degree no greater than n+m-2. This is the usual situation when $A(q^{-1})$ is of degree n-1, and $B(q^{-1})$ is of degree m-1. The proof details are a mild variation on that given here when (3.3) applies. The restriction on $H(q^{-1})$ implies that the associated α in the algebraic form of (3.12) corresponding to (3.3.b) has its last two entries zero.

6. The above result covers the cases when the plant has a delay of unity or greater. In these cases $B(q^{-1})$ specializes as having a factor q^{-N} where $N \geq 1$.

4. A PROPERTY OF SYLVESTER MATRICES

First recall (3.4) which associates with $\theta = [a_1 \ldots a_n \; b_1 \ldots b_m]^\tau$ a Sylvester matrix S(n,m). Let us denote the adjoint of S(n,m) as M(n,m) and determinent as D(n,m). Now consider a (linear) trajectory in Θ-space parametrized in terms of a scalar variable ξ, as

$$\theta(\xi) = [a_1 \ldots a_{n-1} \; (a\xi) \; b_1 \ldots b_{m-1} \; (b\xi)]^\tau, \quad |a| + |b| = 1 \tag{4.1}$$

Also denote the Sylvester matrix associated with $\theta(\xi)$ as $S_\xi(a_n,b_m)$, its adjoint matrix as $M_\xi(a_n,b_m)$ and its determinent as $D_\xi(a_n,b_m)$. Then we claim the following:

<u>Lemma 4.1</u> Consider $S_\xi^{-1}(a_n,b_m) = D_\xi^{-1}(a_n,b_m)M_\xi(a_n,b_m)$ for the case that

$$D(n-1,m-1) \neq 0, \quad ab_{m-1} - a_{n-1}b \neq 0 \tag{4.2}$$

Then the following limits exist as $\xi \to 0$ for all $j < n+m-1$

$$\lim_{\xi \to \infty} D_\xi^{-1}(a_n,b_m)[M_\xi(a_n,b_m)]_{i,j} = \begin{cases} [S^{-1}(n-1,m-1)]_{i,j} & \text{for } i < m \quad (4.3a) \\ [S^{-1}(n-1,m-1)]_{i-1,j} & m<i<m+n \quad (4.3b) \\ 0 & \text{for } i = m \text{ or } i = m+n \quad (4.3c) \end{cases}$$

<u>Proof</u> We consider in turn expressions for $D_\xi(a_n,b_m)$ and $M_\xi(a_n,b_m)$ in terms of order ξ and higher order terms denoted $O(\xi^2)$. Simple manipulations give an expression for $D_\xi(a_n,b_m)$ in terms of the elements of the last row of $S_\xi(a_n,b_m)$ and their minors, with the minors likewise expanded, as

$$D_\xi(a_n,b_m) = (-1)^n \xi (ab_{m-1} - a_{n-1}b) D(n-1,m-1) + O(\xi^2) \tag{4.4}$$

Noting that $[M_\xi(a_n,b_m)]_{i,j}$ is a (signed) determinent of a submatrix of $S_\xi(a_n,b_m)$, then again simple manipulations give an expression in terms of the elements of the last row of the submatrix of $S_\xi(a_n,b_m)$ and minors of this submatrix as

$$[M_\xi(a_n,b_m)]_{i,j}= (-1)^n(a\xi)[M_\xi(a_n,m-1)]_{i,j} + (b\xi)[M_\xi(a_n,b_m)]_{i,j} + O(\xi^2),$$

$$\text{for } i < m,\ j < n+m-1 \qquad (4.5)$$

Here $M_\xi(a_n,m-1)$ denotes the adjoint matrix of the lower dimensioned Sylvester matrix $S_\xi(a_n,m-1)$ which associates with the vector $[I_{n+m-1}\ 0]\theta(\xi) = [a_1...a_{n-1}\ (a\xi)\ b_1\ ...b_{m-1}]^\tau$. Using derivations similar to those giving (4.5), then

$$[M_\xi(a_n,m-1)]_{i,j} = b_{m-1}[M(n-1,m-1)]_{i,j} + O(\xi)$$

$$= b_{m-1}D(n-1,m-1)[S^{-1}(n-1,m-1)]_{i,j} + O(\xi) \text{ for } i < m,\ j < n+m-1 \qquad (4.6)$$

where $S^{-1}(n-1,m-1)$ exists under the assumption (4.2). The dual form of (4.6) is

$$[M_\xi(n-1,b_m)]_{i,j} = (-1)^{n-1} a_{n-1}D(n-1,m-1)[S^{-1}(n-1,m-1)]_{i,j} + O(\xi)$$

$$\text{for } i < m,\ j < n+m-1 \qquad (4.7)$$

Substitution of (4.6) (4.7) into (4.5) yields

$$[M_\xi(a_n,b_m)]_{i,j}= (-1)^n(ab_{m-1}-a_{n-1}b)\xi D(n-1,m-1)[S^{-1}(n-1,m-1)]_{i,j}+O(\xi^2)$$

$$\text{for } i < m,\ j < n+m-1 \qquad (4.8)$$

Dividing (4.8) by $D_\xi(a_n,b_m)$) from (4.4) and taking limits as $\xi \rightarrow 0$ under the assumption (4.2) leads to the result (4.3a). The result (4.3b) can be established along similar lines.

Similar arguments to the derivation of (4.5) (4.6) lead to

$$[M_\xi(a_n,b_m)]_{m,j} = (b\xi)[M_\xi(n-1,b_m)]_{m,j} + O(\xi^2)\ ,\ \text{for } j < n+m-1 \tag{4.9}$$

$$[M_\xi(n-1,b_m)]_{m,j} = (b\xi)O(\theta)\ ,\ \text{for } j < n+m-1 \tag{4.10}$$

where $O(\theta)$ denotes a quantity that is bounded in term of θ. From (4.9) (4.10),

$$[M_\xi(a_n,b_m)]_{m,j} = O(\xi^2)\ ,\ \text{for } j < n+m-1 \tag{4.11}$$

Similarly, we have

$$[M_\xi(a_n,b_m)]_{n+m,j} = O(\xi^2)\ ,\ \text{for } j < n+m-1 \tag{4.12}$$

Then dividing (4.11) and (4.12) by $D_\xi(a_n,b_m)$ from (4.4) and taking limits as $\xi \rightarrow 0$, under the assumption (4.2), we have (4.3c) as claimed. ΔΔΔ

Remarks 1. This result can be generalized to the case other a_i, b_i converge to zero using the same technique as above. Details are omitted.

2. The above result can be expressed in terms of an (n+m) x (n+m-2) matrix.

$$[S^+(\theta,\xi)]_{i,j} = D_\xi^{-1}(a_n,b_m)[M_\xi(a_n,b_m)]_{i,j} \qquad j < n+m+-1 \tag{4.13}$$

Thus under (4.2)

$$S^+(\theta,0) \triangleq \lim_{\xi\rightarrow\infty} S^+(\theta,\xi) = \begin{bmatrix} I_{m-1} & 0 \\ 0 & 0 \\ 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} S^{-1}(n-1,m-1) \tag{4.14}$$

Moreover, simple manipulations yield

$$S(n,m)\begin{bmatrix} I_{m-1} & 0 \\ 0 & 0 \\ 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S(n-1,m-1) \\ 0 \end{bmatrix}$$

so that

$$S(n,m)S^{+}(\theta,0) = \begin{bmatrix} I_{n+m-2} \\ 0 \end{bmatrix} \tag{4.15}$$

Corollary 4.2 Consider that $A(q^{-1},\xi)$, $B(q^{-1},\xi)$ are the polynomials associated with the $\theta(\xi)$ as in (4.1). Consider also that $E(q^{-1},\xi)$, $F(q^{-1},\xi)$ are the solution to

$$A(q^{-1},\xi)E(q^{-1},\xi) + B(q^{-1},\xi)F(q^{-1},\xi) = 1 \tag{4.16}$$

with the degree of $E(q^{-1},\xi)$ being m, and the degree of $F(q^{-1},\xi)$ being n. Then under the same conditions as in Lemma 4.1, the last coefficients of $E(q^{-1},\xi)$, $F(q^{-1},\xi)$ converge as

$$\lim_{\xi\to\infty} e_m(\xi) = 0, \quad \lim_{\xi\to\infty} f_n(\xi) = 0 \tag{4.17}$$

Proof The proof is straightforward from the result of Lemma 4.1, and in particular from (4.14). However, to prove it by just using the properties of polynomial equations appears too formidable without resort to the result on the Sylvester matrix described in Lemma 4.1.

Lemma 4.3 Consider the Bezout equation (3.3) under (2.14) with $m > 1$. Then a unique solution ϕ^* of (3.7) exists as

$$\phi^* = S^{+}(\theta,0)[I_{n+m-2}\ 0]\alpha \tag{4.18}$$

with the property that ϕ^* has n^{th} and $(n+m)^{th}$ elements which are zero.

Proof Under (2.14), of course $S^{-1}(n,m)$ does not exist, but (4.2) holds for a suitable selection of a,b. Also, the last two elements of α are zero. Now Lemma 4.1 holds under (4.2) so that (4.14),(4.15) apply. Thus with ϕ^* uniquely defined from (4.18)

$$S(n,m)\phi^* = \begin{bmatrix} I_{n+m-2} & 0 \\ 0 & 0 \end{bmatrix} \alpha = \alpha$$

and (3.3) is satisfied. Application of (4.14) under (4.16) gives guarentees that the n^{th} and $(n+m)^{th}$ elements of ϕ^* are zero. ΔΔΔ

5 SIMULATIONS

Consider now the application of the adaptive pole assignment scheme discussed in Sections 2 and 3, to the plant taken from [10] with

$$y_k - 1.2y_{k-1} = u_{k-1} - 3.1u_{k-2} + 2.2u_{k-3} + w_k, \quad K = 10, \quad H=1,$$

$$r_k = \begin{cases} 1 & k= 1..10;\ 21..30;\ 41..50 \\ -1 & k=11..20;\ 31..40;\ 51..60 \end{cases}$$

and variance of w_k decays as k^{-2}. (5.1)

Three cases studied are:

(i) The plant is overparametrized by one, i.e. plant is modelled as

$$y_k + a_1y_{k-1} + a_2y_{k-2} = b_1u_{k-1} + b_2u_{k-2} + b_3u_{k-3} + b_4u_{k-4} + w_k \tag{5.2}$$

and standard RLS estimation is used.

(ii) The plant is overparametrized by one too, but the RLS estimation discussed in Section 2 is employed.

(iii) The plant is not overparametrized, i.e. plant is modelled as

$$y_k + a_1 y_{k-1} = b_1 u_{k-1} + b_2 u_{k-2} + b_3 u_{k-3} + w_k \tag{5.3}$$

Figures 5.1, 5.2, 5.3 show the poor performance in the output, the estimates of the plant and the estimates of the controller in the case (i). And Figures 5.4, 5.5, 5.6 give the comparison of the outputs, the estimates of the plant and the estimates of the controller in cases (ii) and (iii).

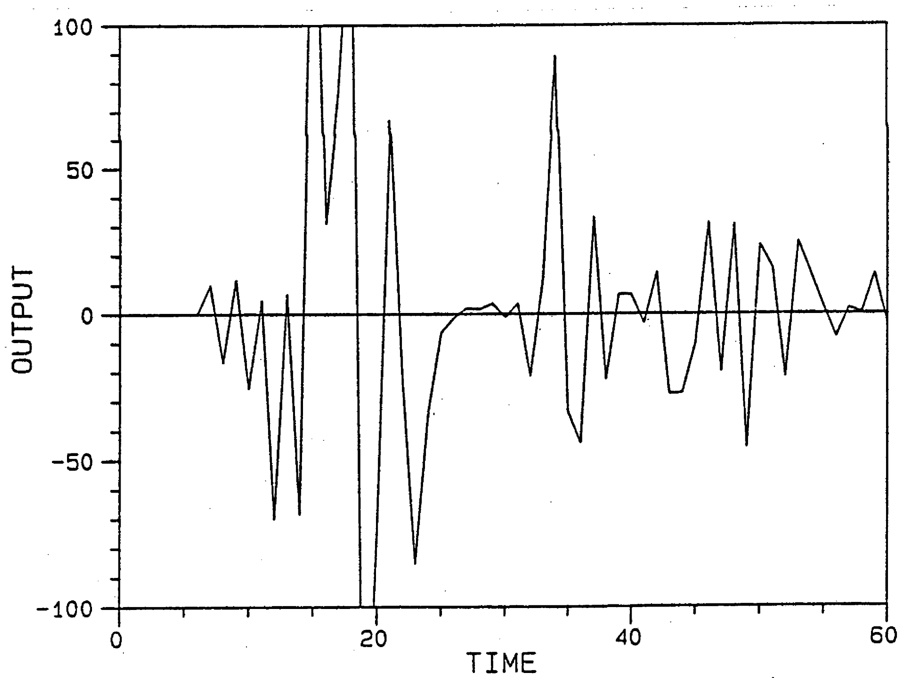


Fig. 5.1. Output (y_k) in Case (i)

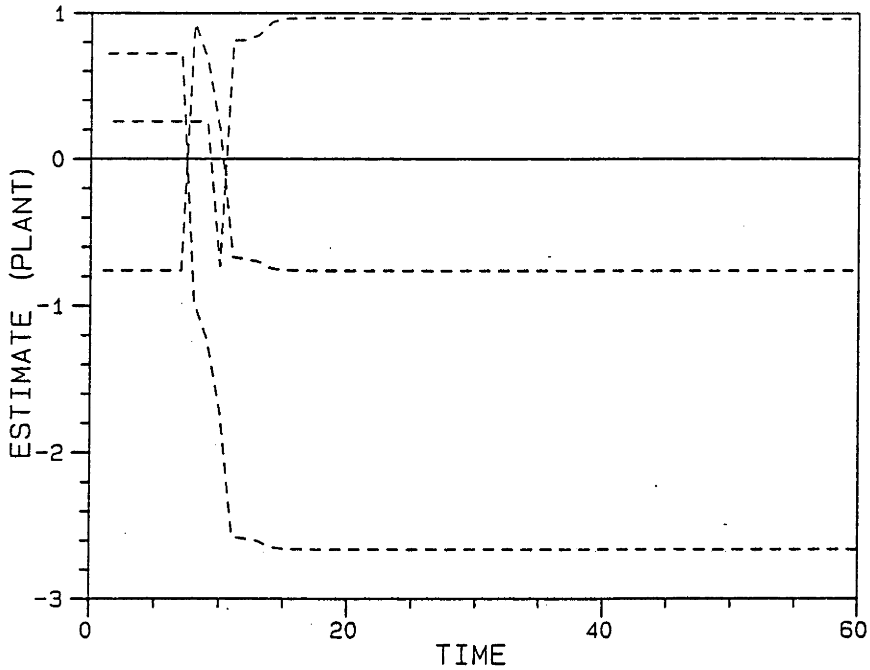


Fig. 5.2. Estimates of the Plant in Case (i)

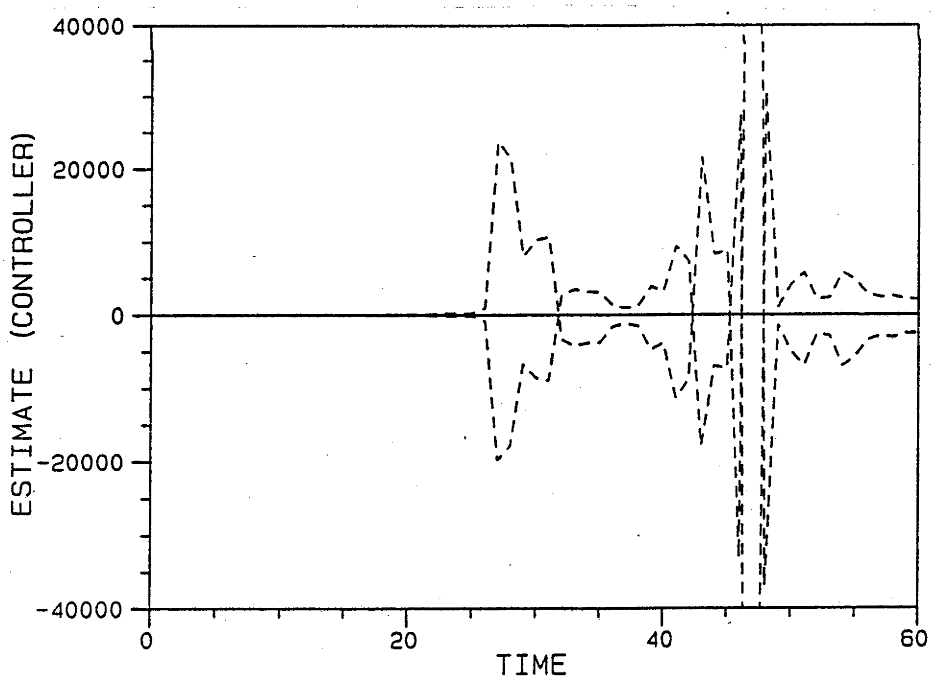


Fig. 5.3. Estimates of the Controller in Case (i)

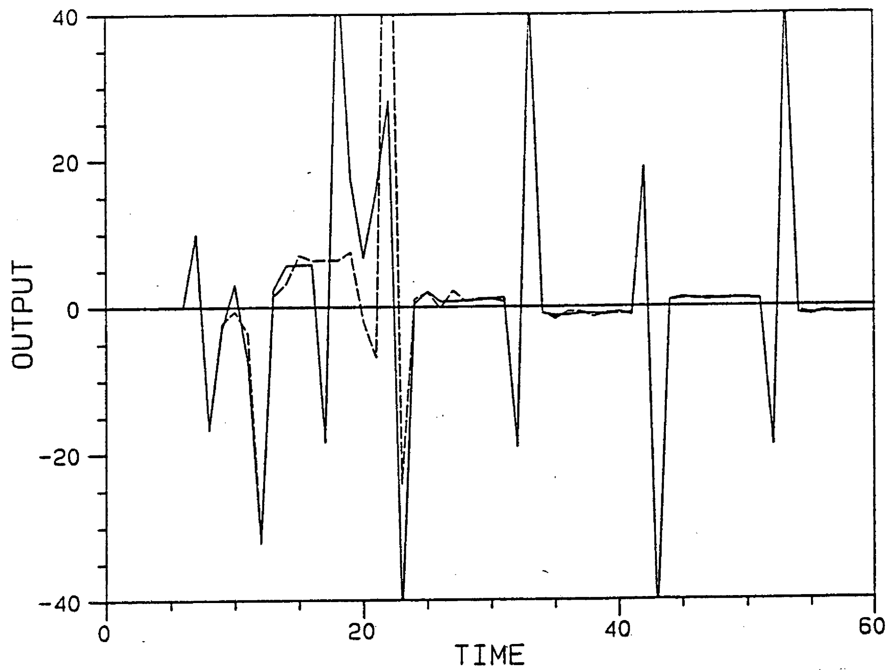


Fig. 5.4. Comparison of the outputs in Cases (ii) and (iii)

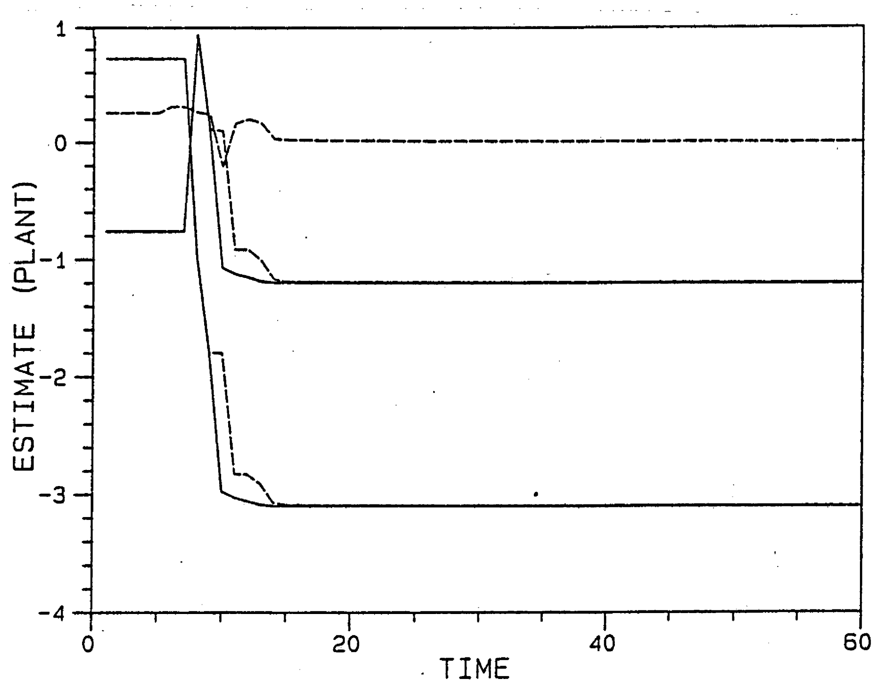


Fig. 5.5. Comparison of the Estimated Plants in Cases (ii) and (iii)

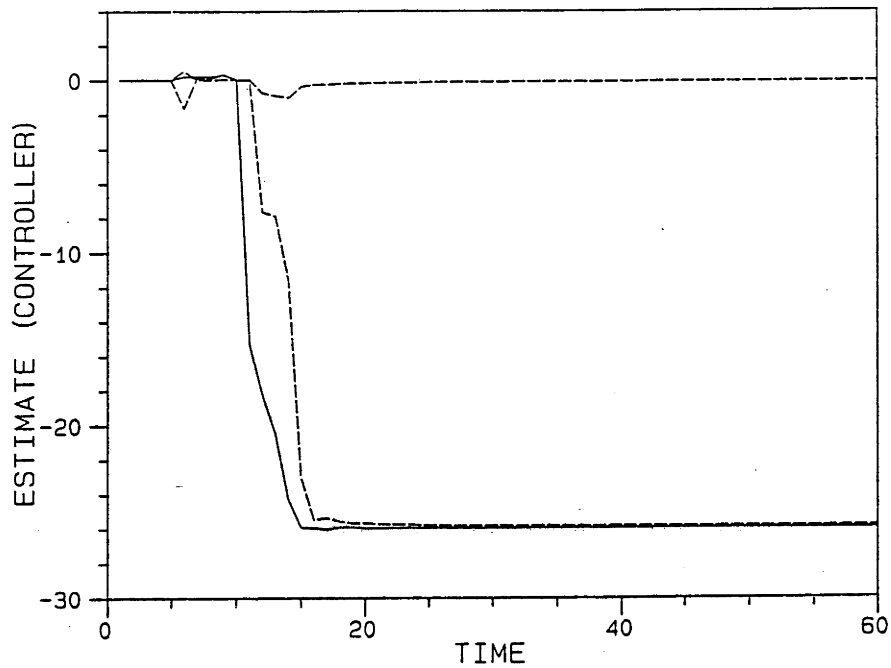


Fig. 5.6. Estimated Controllers in Cases (ii) and (iii)

From the figures, it is clearly shown that in the overparametrization presence, the adaptive pole assignment scheme based on the standard RLS estimation of the plant could yield quite poor performance; however, the scheme proposed in this chapter works considerably well. In fact, it converges to the optimal pole assignment controller as the standard adaptive pole assignment scheme does without overparametrization.

6 CONCLUSIONS

Non-standard adaptive estimation and control techniques have been proposed to avoid ill-conditioning which can arise when standard techniques are applied to signal models with overparametrization. The techniques avoid ill-conditioning and yield asymptotic optimality in the case when there is possibly one pole-zero cancellation in the assumed signal model. It seems reasonable to apply such techniques in conjunction with on-line model order estimation techniques since from finite data these possibly lead to overestimation of the order. The techniques have been studied for the case when exact pole zero cancellation occurs, but is known from simulations to avoid ill-conditioning when there are stable near pole zero cancellations. The results of this chapter are a starting point from which to cope with higher order pole-zero cancellations.

APPENDEX

Consider first the following lemma.

Lemma A.1 Under (2.13), (2.14), (3.8) (the conditions of Theorem 3.2)

$$\lim_{k\to\infty} S^{+}(\hat{\theta}_k^{CT},\xi_k) = S^{+}(\theta,0) \tag{A.1}$$

Proof With (3.8) satisfied and (2.13)

$$\lim_{k\to\infty} \hat{\theta}_k^{CT} = \theta$$

so that when (2.14) holds

$$\lim_{k\to\infty} \hat{a}_{n,k}^{CT} = \lim_{k\to\infty} a_k\xi_k = 0,$$

$$\lim_{k\to\infty} \hat{b}_{m,k}^{CT} = \lim_{k\to\infty} b_k\xi_k = 0,$$

Since $|a_k| + |b_k| = 1$, recalling (4.1), then

$$\lim_{k\to\infty} \xi_k = 0 \tag{A.2}$$

Now, under (3.8), (2.13), (2.14), we have

$$\lim_{\xi\to\infty} S^+(\hat{\theta}_k^{CT},\xi_k) = S^+(\hat{\theta}_k^{CT},0) = \begin{bmatrix} I_{m-1} & 0 \\ 0 & 0 \\ 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} [\hat{S}_k^{CT}(n-1,m-1)]^{-1} \tag{A.3}$$

Also under (3.8), (2.13), (2.14)

$$\lim_{k\to\infty} S^+(\hat{\theta}_k^{CT},0) = S^+(\theta,0) \tag{A.4}$$

From the continuity and (A.2), (A.1) is established. ΔΔΔ

Remark Without (3.8) satisfied, it can not be guaranteed that $S^+(\hat{\theta}_k^{CT},\xi_k)$ exists for all k. Also with $S^+(\hat{\theta}_k,\xi_k)$ large, then $\hat{\phi}_k$ is large, and ill-conditioning is said to occur.

Proof of Theorem 3.2

Consider the central tendency adaptive control with (3.8) satisfied and define for all k

$$[\hat{S}_k^{CT}(n,m)]^{-1} = [S^+(\hat{\theta}_k^{CT},\xi) \;\; *] \tag{A.5}$$

where * denotes terms not of interest. Thus for the signal model with m > 1, or in other words the last two entries of $\hat{\alpha}_k^{CT}$ zero,

$$\hat{\phi}_k^{CT} = [\hat{S}_k^{CT}(n,m)]^{-1}\,\hat{\alpha}_k^{CT} = [\hat{S}_k^{CT}(n,m)]^{-1}[I_{n+m-2}\;0]^{\tau}[I_{n+m-2}\;0]\,\hat{\alpha}_k^{CT}$$

$$= S^{+}(\hat{\theta}_k^{CT},\xi)[I_{n+m-2}\;0]\,\hat{\alpha}_k^{CT} \qquad \text{(A.6)}$$

Now applying (A.1) when taking limits as $k \to \infty$, we get

$$\lim_{k\to\infty}\hat{\phi}_k^{CT} = \lim_{k\to\infty} S^{+}(\hat{\theta}_k^{CT},\xi_k)[I_{n+m-2}\;0]\,\hat{\alpha}_k^{CT} = S^{+}(\theta,0)[I_{n+m-2}\;0]\alpha$$

The desired results (3.10), (3.11) follow from application of Lemma 4.3. ΔΔΔ

REFERENCES

[1] G.C. Goodwin and K.S. Sin, "**Adaptive Filtering Prediction and Control**", Prentice Hall, 1984

[2] S. Shinnaka and T. Suzuki, "Adaptive Pole Placement Applicable to Nonminimum-Phase Systems of Unknown Order and Unknown Dead-Time", **Denshi Tsushin Gakkai Ronbunshi**, Vol. 66-A, No.12, pp. 1228-1235, 1983.

[3] J. Sternby, "On Consistency for the Method of Least Squares Using Martingale Theory", **IEEE Trans. AC-22**, pp. 346, 1977

[4] J.B. Moore, "Persistence of Excitation in Extended Least Squares", **IEEE Trans. AC-28**, pp. 60-68. 1983.

[5] M. Green and J.B. Moore "Persistence of Excitation in Linear Systems" **Proc. of American Control Conference**, Boston, pp. 412-417. 1985.

[6] J.B. Moore, "A Universability Advantage of Stochastic Excitation Signals for Adaptive Control", **Systems and Control Letters**, Vol 9, pp55-58, 1987.

[7] J.B. Moore, "Stochastic Adaptive Control via Consistent Parameter Estimation", **Proc. IFAC Identification Conference**, pp. 611-616, York, 1985

[8] T. Kailath, "**Linear Systems**". Prentice Hall 1980

[9] J.B. Moore, T. Ryall and L. Xia, "Central Tendency Pole Assignment", **Proc. of 25th CDC**, pp. 100-105, Athens, Greece 1986.

[10] G.C. Goodwin and K.S. Sin, "Adaptive control of Non Minimum Phase Plants" **IEEE Trans. AC-26** pp. 478-483, 1981.

Chapter 8

RECURSIVE IDENTIFICATION OF OVERPARAMETRIZED SYSTEMS

1. INTRODUCTION

For adaptive estimation / control scheme design, it is common to assume a linear input -output signal model of specified order with unknown parameters. If the order of the signal model selected for design is too low (underparametrization), then the unmodelled dynamics can be destabilizing. This means that there is a tendency in practice to overparametrize the signal model to be on the "safe" side. Thus overparametrization emerges as a significant problem in some applications. A specific situation is when the presence of some deterministic disturbances such as bias is assumed, when in fact any such disturbances are negligible.

With overparametrization, there is a danger of ill-conditioning in adaptive estimation and in some adaptive control. When an adaptive estimation algorithm (say extended least squares) is employed to identify an overparametrized system, there is inevitably a lack of excitation in the regression vectors and normally there is no guaranteed convergence. Also insufficient excitation can lead to identification of near pole-zero cancellations in the complex z-plane, leading to excessive control signal in adaptive control schemes, such as pole assignment scheme [1]. Of course, an on-line order determination could be applied to get a suitable signal model [2,3]. However, such an approach increases the complexity considerably.

In an early work [4], it is suggested to inject an excitation signal into the regression vectors for recursive least squares parameter estimation. A design approach is presented in [4] which copes with a possible overparametrization by one for the recursive least squares estimation applied to systems with white noise perturbation which may or may not be persistently exciting. In this paper, we extend the approach in [4] to construct a regressor excitation signal which permits arbitrary degrees of overparametrization, and apply the approach to signal models with colored noise excitation. The formulations are presented in Section 2. Convergence proofs for recursive extended least squares estimation are given in Section 3. An example is reported in Section 4 to illustrate how the algorithm behaves. Conclusions are drawn in Section 5.

2. ALGORITHM

Signal Model The signal model considered is a time-invariant linear system described by an ARMAX model as

$$A(q^{-1})y_k = B(q^{-1})u_k + C(q^{-1})w_k \tag{2.1.a}$$

$$A(q^{-1}) = 1+\sum_{i=1}^{n} a_i q^{-i},\ B(q^{-1}) = \sum_{i=1}^{m} b_i q^{-i},\ C(q^{-1}) = 1+\sum_{i=1}^{l} c_i q^{-i} \tag{2.1.b}$$

Here u_k and y_k are the system input and output and w_k is the disturbance noise assumed being white noise zero mean. Or more precisely, the conditions on w_k are as follows.

The sequence $\{w_k\}$ is a martingale difference sequence with respect to F_{k-1}, with $E[w_k|F_{k-1}]=0$, $E[w_k^2|F_{k-1}]=\sigma_w^2 < \infty$, where F_k denotes the σ-algebra generated by $w_1 \cdots w_k$, $u_1 \cdots u_k$. (2.2.a)

We further assume that the noise $C(q^{-1})w_k$ is not "too colored" in that

$$C^{-1}(q^{-1}) - \frac{1}{2} \text{ is strictly positive real} \tag{2.2.b}$$

Also, let us assume that the plant inputs are persistently exciting in that

$$0 < \beta I \leq \frac{1}{k}\sum_{i=1}^{k} \bar{u}_i \bar{u}_i^{\tau} \leq \alpha I \quad \bar{u}_i^{\tau} = \left[u_i \cdots u_{i-n-m}\right] \tag{2.3}$$

for some α, β, k_0 and all $k \geq k_0$.

The signal model (2.1) can be rewritten as

$$y_k = \theta^{\tau}\phi_k + w_k \tag{2.4.a}$$

$$\theta^{\tau} = \left[a_1 \cdots a_n \; b_1 \cdots b_m \; c_1 \cdots c_l\right] \tag{2.4.b}$$

$$\phi_k^{\tau} = \left[-y_{k-1} \cdots -y_{k-n} \; u_{k-1} \cdots u_{k-m} \; w_{k-1} \cdots w_{k-l}\right] \tag{2.4.c}$$

<u>Perturbed Recursive Extended Least Squares</u> Consider the recursion

$$\theta_k = P_k\left[B_{k-1}\theta_{k-1} + \psi_k y_k\right], B_k = B_{k-1} + \psi_k\psi_k^{\tau}, P_k = B_k^{-1}, \tag{2.5}$$

where

$$\begin{aligned}
\bar{\phi}_k^{\tau} &= \left[-y_{k-1} \cdots -y_{k-n} \; u_{k-1} \cdots u_{k-m} \; \bar{w}_{k-1} \cdots \bar{w}_{k-l}\right], \\
\bar{w}_k &= y_k - \theta_k^{\tau}\psi_k, \quad \psi_k = \bar{\phi}_k + v_{k-1}, \quad \hat{w}_k = y_k - \theta_k^{\tau}\hat{\phi}_k \\
\hat{\phi}_k^{\tau} &= \left[-y_{k-1} \cdots -y_{k-n} \; u_{k-1} \cdots u_{k-m} \; \hat{w}_{k-1} \cdots \hat{w}_{k-l}\right]
\end{aligned} \tag{2.6}$$

initializing by

$$y_i = 0, u_i = 0, \bar{w}_i = 0, \hat{w}_i = 0, \text{ for } i \leq 0, \text{ and some } \theta_0, B_0 > 0.$$

For a system as in (2.1) with the possibility of overparametrization by up to $L < \min(n,m,l)$, the choice of v_k we propose is as follows

$$v_k \triangleq D_k \upsilon_k, \quad E\left[\upsilon_k \mid F_{k-1}\right] = 0, \quad E\left[\upsilon_k \upsilon_k^\tau \mid F_{k-1}\right] = \sigma_v^2 I,$$

$$\sup_k \left[\upsilon_k^\tau \upsilon_k\right] \leq \bar{V} < \infty,$$

$$D_k D_k \triangleq \mathrm{Diag}\left[\sigma_k^2(n) \cdots \sigma_k^2(1)\ \sigma_k^2(m) \cdots \sigma_k^2(1)\ \sigma_k^2(l) \cdots \sigma_k^2(1)\right]$$

$$\sigma_k(1) = \mathrm{tr}\left[\hat{B}_k^{-1}\right], \quad \hat{B}_k = \hat{B}_{k-1} + \hat{\phi}_k \hat{\phi}_k^\tau$$

$$\sigma_k(j) = \mathrm{tr}\left[\hat{B}_k^{-1}(j)\right], \quad \text{for } 1 < j < L, \text{ and } \quad \sigma_k(j) = 0 \quad \text{for } j > L \tag{2.7}$$

Here the σ-algebra F_k is extended to include that generated by $\{v_1, v_2 \cdots v_k\}$. Also $\hat{B}_k(j)$ is a matrix obtained from $\hat{B}_k$ by deleting rows and columns $(n+1-j)$ to n, $(n+m+1-j)$ to $(n+m)$ and $(n+m+l+1-j)$ to $(n+m+l)$. The $\hat{B}_k^{-1}(j)$ can be easily calculated from $\hat{B}_k^{-1}$ by inversion of a block matrix formula given in Appendix A.

This algorithm minimizes the index

$$J_k = \sum_{i=1}^{k} (y_i - \theta^\tau \psi_i)^2 + (\theta - \theta_0)^\tau B_0 (\theta - \theta_0) \tag{2.8}$$

3. CONVERGENCE

This section is devoted to proving a convergence result on the perturbed extended least squares algorithm proposed in the previous section. The proposed algorithm does not fit directly into the framework where known convergence analysis applies, yet our analysis specializes to known techniques. Let us consider the simplest situation of a stable signal model under persistently exciting input signals.

Theorem 3.1: Consider the signal model with assumed order (n,m,l) as in (2.1)-(2.4), possibly overparametrized by some $L < \min(n,m,l)$, and assumed stable. Consider the perturbed recursive extended least squares algorithm (2.6)-(2.8). Consider also that the input u_k is uniformly persistently exciting as in (2.3). Then there is parameter convergence as k goes to infinity,

$$\theta_k \rightarrow \theta \quad \text{a.s. at the rate of} \quad O([\ln(k)^{\mu}/k]^{\frac{1}{2}}) \tag{3.1}$$

where θ is the unique parameters associated with (2.1) with all possible pole / zero cancellations at the origin, and arbitrary constant $\mu > 1$. ***

As a step to prove the Theorem 3.1, let us first introduce the following lemma, which specializes to known results when D_k , $v_k = 0$ in the absence of overparametrization.

Lemma 3.2 Consider that the conditions of Theorem 3.1 apply and that θ is defined as in the theorem. Then

(1) $$\sum_{k=1}^{\infty} [\theta^{\tau} v_{k-1} v_{k-1}^{\tau} \theta] \leq K_v < \infty \tag{3.2}$$

(2) $$\lim_{k\rightarrow\infty} \sum_{i=1}^{k} [\psi_i^{\tau} P_i \psi_i / \mathrm{tr}(B_{i-1})] < \infty \tag{3.3}$$

(3) $$C(q^{-1}) z_k = h_k \tag{3.4}$$

$$z_k \triangleq y_k - \psi_k^\tau \theta_k - w_k,\ h_k \triangleq \psi_k^\tau \tilde{\theta}_k - \theta^\tau v_{k-1},\ \tilde{\theta}_k \triangleq \theta - \theta_k \tag{3.5}$$

(4) $$E[b_k w_k | F_{k-1}] = -\psi_k^\tau P_k \psi_k \sigma_w^2\ ,\ b_k \triangleq \psi_k^\tau \tilde{\theta}_k \tag{3.6}$$

(5) There are some constants $\rho_1>0$, $\rho_2>0$, and $K < \infty$ such that

$$S_k \triangleq \sum_{i=1}^{k}[2b_i z_i - b_i^2 - \rho_1 b_i^2 - \rho_2 z_i^2] + K \geq 0, \tag{3.7}$$

Proof:

(1). Suppose the system (2.1) is overparametrized by $0 \leq s \leq L$. Then with (2.3) and (2.2), it is well known that [5] $\hat{B}_k^{-1}(j) \to 0$ for $j > s$, at the rate of $O(1/k)$. Therefore $\sigma_k(j) \to 0$ at the rate of $O(1/k)$ for $j > s$, or

$$\sum_{k=1}^{\infty} \sigma_k^2(j) < \infty \tag{3.8}$$

Of course, for $j \leq s$, $\sigma_k(j)$ does not approach zero as $k \to \infty$, but terms involving $\sigma_k(j)$ for $j \leq s$ in $\theta^\tau D_k$ are zero, since the system is overparametrized by s. Combining this result with (3.8) and $\bar{V} < \infty$ in (2.7) leads to (3.2).

(2) Now $(\mathrm{tr}[B_i]I)^{-1} \leq P_i$ so that

$$\psi_i^\tau P_i \psi_i / \mathrm{tr}[B_{i-1}] \leq \psi_i^\tau P_i P_{i-1} \psi_i \tag{3.9}$$

Then from (2.5)

$$P_i = P_{i-1} - \frac{P_{i-1}\psi_i \psi_i^\tau P_{i-1}}{1+\psi_i^\tau P_{i-1}\psi_i}\ ,\qquad P_i \psi_i = \frac{P_{i-1}\psi_i}{1+\psi_i^\tau P_{i-1}\psi_i}$$

so that

$$\psi_i^\tau P_i P_{i-1} \psi_i = \mathrm{tr}\big[P_{i-1}\big] - \mathrm{tr}\big[P_i\big] \tag{3.10}$$

Now (3.3) follows from (3.9) by summing up on both sides of (3.10).

(3) Now with the definition for z_k of (3.5)

$$C(q^{-1})z_k = C(q^{-1})\Big[\, y_k - \psi_k^\tau \theta_k - w_k \Big]$$

$$= \Big[C(q^{-1}) - I\Big]\Big[y_k - \psi_k^\tau\theta_k\Big] + y_k - \psi_k^\tau\theta_k - C(q^{-1})w_k$$

But from (2.4) (2.6), manipulations yield

$$\bar{\phi}_k^\tau\theta = \Big[C(q^{-1}) - I\Big]\Big[y_k - \psi_k^\tau\theta_k\Big] + y_k - C(q^{-1})w_k$$

Thus

$$C(q^{-1})z_k = \bar{\phi}_k^\tau\theta - \psi_k^\tau\theta_k = \psi_k^\tau\tilde{\theta}_k - \theta^\tau v_{k-1}$$

where the second equality follows from (2.6) and definition of $\tilde{\theta}_k$ in (3.5). Applying the definition for h_k in (3.5) the result (3) is established.

(4) From the definition of b_k in (3.6),

$$b_k w_k = \psi_k^\tau \tilde{\theta}_k w_k \tag{3.11}$$

Also from (2.5),

$$B_k\theta_k = B_{k-1}\theta_{k-1} + \psi_k y_k = B_k\theta_{k-1} + \psi_k(y_k - \psi_k^\tau\theta_{k-1})$$

$$\theta_k = \theta_{k-1} + P_k\psi_k(\phi_k^\tau\theta + w_k - \psi_k^\tau\theta_{k-1})$$

$$\tilde{\theta}_k = \tilde{\theta}_{k-1} - P_k\psi_k(\phi_k^\tau\theta + w_k - \psi_k^\tau\theta_{k-1}) \tag{3.12}$$

Substituting (3.12) into (3.11) and taking conditional expectations on both sides gives the result (4). In this manipulation, recall that all quantities in the expression save w_k and v_k are F_{k-1} measurable.

(5) Proof is given in Appendix B. $\Delta\Delta\Delta$

Proof of Theorem 3.1: From (2.5), (3.5)

$$(B_{k-1} + \psi_k\psi_k^T)\theta_k = B_{k-1}\theta_{k-1} + \psi_k y_k$$

$$\theta_k = \theta_{k-1} + P_{k-1}\psi_k(y_k - \psi_k^T\theta_k)$$

$$\tilde{\theta}_k = \tilde{\theta}_{k-1} - P_{k-1}\psi_k(z_k + w_k) \tag{3.13}$$

Now defining $V_k \triangleq \tilde{\theta}_k^T B_k \tilde{\theta}_k$ and applying (3.13) , we have

$$V_k = \tilde{\theta}_k^T(B_{k-1} + \psi_k\psi_k^T)\tilde{\theta}_k$$

$$= V_{k-1} - 2\psi_k^T\tilde{\theta}_{k-1}(z_k+w_k) + \psi_k^T P_{k-1}\psi_k(z_k+w_k)^2 + \tilde{\theta}_k^T\psi_k\psi_k^T\tilde{\theta}_k$$

$$= V_{k-1} - 2\psi_k^T\tilde{\theta}_k(z_k+w_k) - \psi_k^T P_{k-1}\psi_k(z_k+w_k)^2 + \tilde{\theta}_k^T\psi_k\psi_k^T\tilde{\theta}_k$$

and with definition for b_k as in (3.6)

$$V_k = V_{k-1} + b_k^2 - 2b_k z_k - 2b_k w_k - \psi_k^T P_{k-1}\psi_k(w_k+z_k)^2 \tag{3.14}$$

Performing conditional expectation on both sides and applying (3.6) then

$$E\left[V_k \mid F_{k-1}\right] = V_{k-1} + E\left[b_k^2 - 2b_k z_k \mid F_{k-1}\right] + 2\psi_k^T P_k \psi_k \sigma_w^2 - \\ - E\left[\psi_k^T P_{k-1} \psi_k(w_k + z_k)^2 \mid F_{k-1}\right] \tag{3.15}$$

Now recall (3.7) for S_k and define X_k as follows.

$$X_k \triangleq \frac{V_k}{tr(B_k)} + \frac{S_k}{tr(B_{k-1})} + \rho_1 \sum_{i=1}^{k} \frac{b_i^2}{tr(B_{i-1})} + \rho_2 \sum_{i=1}^{k} \frac{z_i^2}{tr(B_{i-1})} +$$

$$+ \sum_{i=1}^{k} \frac{\psi_i^T P_{i-1} \psi_i}{tr(B_{i-1})} (z_i + w_i)^2 + \sum_{i=1}^{k} V_i \left[\frac{1}{tr(B_{i-1})} - \frac{1}{tr(B_i)} \right] \tag{3.16}$$

Then it is easy to verify that

$$E\left[X_k \mid F_{k-1}\right] \leq X_{k-1} + 2\frac{\psi_k^T P_k \psi_k \sigma_w^2}{tr(B_{k-1})} \tag{3.17}$$

Applying the martingale convergence theorem [2,p501], under (3.2) (3.3), we conclude that X_k converges almost surely as $k \to \infty$. This implies that,

$$\lim_{k\to\infty} \sum_{i=1}^{k} \frac{b_i^2}{tr(B_{i-1})} < \infty \quad \text{a.s.} \qquad \lim_{k\to\infty} \sum_{i=1}^{k} \frac{z_i^2}{tr(B_{i-1})} < \infty \quad \text{a.s.} \tag{3.18}$$

Also from the definitions of f_k , h_k and the Schwarz inequality,

$$\lim_{k\to\infty} \sum_{i=1}^{k} \frac{h_i^2}{tr(B_{i-1})} < \infty \quad \text{a.s.} \tag{3.19}$$

We now demonstrate that under the excitation and stability conditions of the theorem, for some δ and κ,

$$\lim_{k\to\infty} \sup\, (\lambda_{max} B_k / \lambda_{min} B_k) \leq \lim_{k\to\infty} \sup (tr B_k / \lambda_{min} B_k) \leq \delta < \infty \text{ a.s.} \tag{3.20}$$

$$\lim_{k\to\infty} \sup\, (\lambda_{max} B_k / k) \leq \kappa, \quad \text{a.s.} \tag{3.21}$$

Now consider that the system is overparametrized by $0 \leq s \leq L$, then the associated lack of excitation of certain modes reflects itself in the property that $\sigma_k(j) = \hat{B}_k^{-1}(j)$ converge to some nonzero random variable for $j \leq s$. Then from the result of [5] ψ_k is reachable from w_k, u_k and v_{k-1}, and since w_k, u_k, v_{k-1} is persistently exciting under (2.2), (2.3) and (2.7), ψ_k is persistently exciting. Now under the stability assumption, the technique of [2,p345] applies here to show that (3.20) holds.

With (3.20) holding, it is easy to show that

$$P_k \leq \delta I[\mathrm{tr}(B_k)]^{-1}$$

$$\sum_{i=1}^{k} \frac{\psi_i^T P_i \psi_i}{R_i} \leq \delta \sum_{i=1}^{k} \frac{\psi_i^T \psi_i}{\mathrm{tr}(B_i) R_i}, \qquad R_i \triangleq \left[\ln(\mathrm{tr}B_i)\right]^{\mu}, \mu > 1 \tag{3.22}$$

$$\sum_{i=1}^{k} \frac{\psi_i^T \psi_i}{\mathrm{tr}(B_i) R_i} = \sum_{i=1}^{k} \frac{\mathrm{tr}(B_i) - \mathrm{tr}(B_{i-1})}{\mathrm{tr}(B_i) R_i} = \sum_{i=1}^{k} \int_{\mathrm{tr}B_{i-1}}^{\mathrm{tr}B_i} \frac{dx}{\mathrm{tr}(B_i) R_i}$$

$$\leq \int_{\mathrm{tr}B_0}^{\mathrm{tr}B_k} \frac{dx}{x(\ln x)^{\mu}} = \left[\ln(\mathrm{tr}B_0)\right]^{1-\mu} - \left[\ln(\mathrm{tr}B_k)\right]^{1-\mu} < \infty \tag{3.23}$$

Now, define Q_k as follows,

$$Q_k \triangleq \frac{V_k}{R_k} + \frac{S_k}{R_k} + \rho_1 \sum_{i=1}^{k} \frac{b_i^2}{R_i} + \rho_2 \sum_{i=1}^{k} \frac{z_i^2}{R_i} +$$

$$+ \sum_{i=1}^{k} \frac{\psi_i^T P_{i-1} \psi_i}{R_i} (z_i + w_i)^2 + \sum_{i=1}^{k-1} V_i \left[\frac{1}{R_i} - \frac{1}{R_{i+1}}\right] \tag{3.24}$$

Then it is easy to verify that

$$E\left[Q_k \mid F_{k-1}\right] \leq Q_{k-1} + 2\frac{\psi_k^{\tau} P_k \psi_k \sigma_w^2}{R_k}$$

Applying the martingale convergence theorem [2,p501], under (3.2) (3.23), we conclude that Q_k converges almost surely as $k \to \infty$. This implies, recalling the definition of V_k

$$\lim_{k\to\infty} \sup \frac{\tilde{\theta}_k^{\tau} B_k \tilde{\theta}_k}{R_k} < \infty, \quad \text{a.s} \quad . \tag{3.25}$$

Thus $\tilde{\theta}_k^{\tau}\tilde{\theta}_k \to 0$, $\theta_k \to \theta$, from (3.21), at the rate of $O([\ln(k)^{\mu}/k]^{\frac{1}{2}})$ and the result (3.1) is established. Notice that here θ is such that (3.2) holds, and has the property that its elements a_i, b_j, c_k are zero for $i > n\text{-}s$, $j > m\text{-}s$, $k > l\text{-}s$. ΔΔΔ

Remark The extension of known techniques in the proof of the above theorem cope with the extra signals v_k in the signal model and thereby to cope with overparametrization. Clearly, the novel techniques also apply in related situation involving adaptive control (details are omitted here).

4. SIMULATIONS

To illustrate that the algorithm works, an artificial example is studied. The true plant has the input / output relations

$$y_k - y_{k-1} + 0.5y_{k-2} = -2.5u_{k-1} - 1.5u_{k-2} + w_k - 0.5w_{k-1}$$

The simulations show parameter estimates of a_1 and a_2 as in Figures 4.1, 4.2. In those figures, the curve 1 indicates the case when the signal model is assumed to be

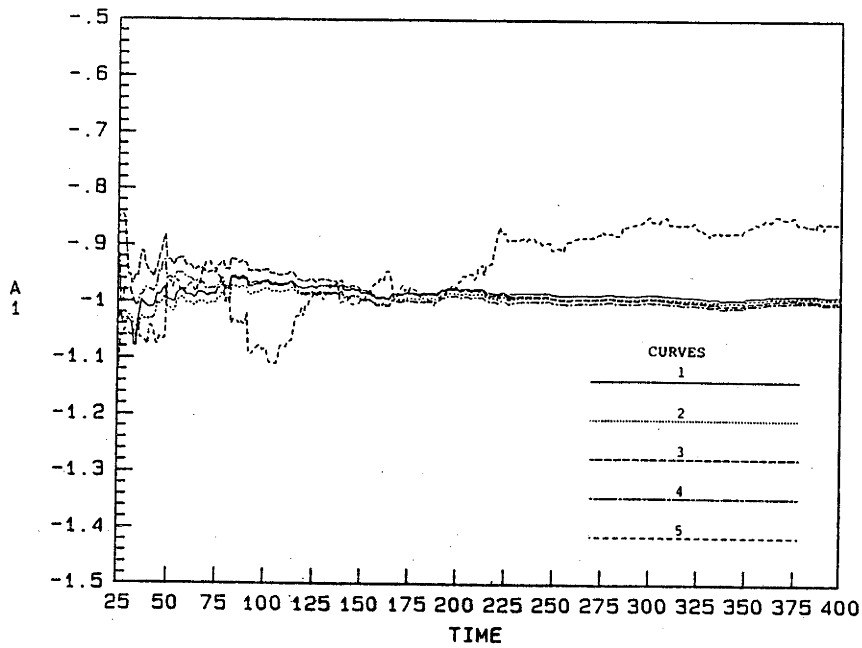


Fig. 4.1. Estimates of a_1

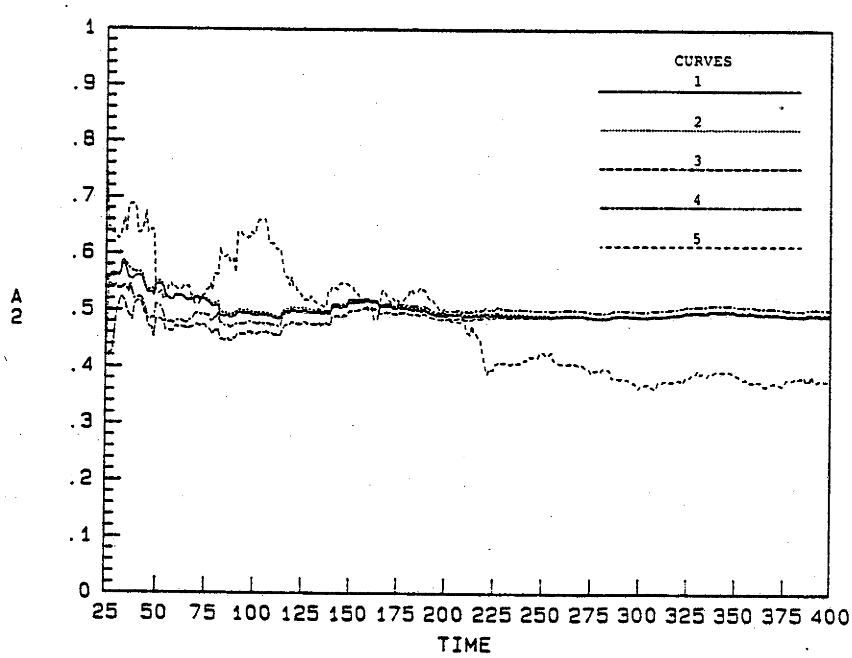


Fig. 4.2. Estimates of a_2

correctly ordered (here n=2, m=2, l=2), and normal extended least squares algorithm is used. The curves 2, 3, 4 depict the situations when the system is overparametrized by 1, 2, 3 (namely n=3,4,5; m=3,4,5; l=3,4,5) and our perturbed extended least squares algorithm proposed in Section 2 is employed with L=1,2,3 respectively. To make the comparison, we also give the estimates (as the curve 5) of the normal extended least squares algorithm working in the overparametrization environment.

From the figures, it is clear that the results of the perturbed least squares algorithm is essentially the same as that from normal extended least squares working on the correctly ordered signal model. On the other hand, normal extended least squares estimation applied with an overparametrized signal model gives biased estimates. These simulations are consistent with the result given in Section 3, and point to the usefulness of the techniques of the paper.

5. CONCLUSION

A perturbed least squares algorithm is proposed here to cope with the problem of identifying possibly overparametrized systems. It has been shown that such algorithms have guaranteed convergence rates the same as when the signal models are not overparametrized. Simulations show no deterioration of transient performance of the proposed algorithm in the presence of overparametrization.

APPENDIX A

For the inversion of a block matrix, we have the following result. Namely, consider a block matrix M and M^{-1} partitioned as follows,

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \tag{A.1}$$

then

$$A^{-1} = E - FH^{-1}G \tag{A.2}$$

To see this, from (A.1) we have AE + BG = I, AF + BH = 0 which implies that $B = -AFH^{-1}$ and $AE - AFH^{-1}G = I$, then (A.2) follows.

To apply this result in our calculation for $\hat{B}_k^{-1}(j)$ from $\hat{B}_k^{-1}(j-1)$, set $\hat{B}_k(j) = A$ and $\hat{B}_k^{-1}(j-1) = M^{-1}$, then $\hat{B}_k^{-1}(j)$ can be easily obtained from the partitioning of $\hat{B}_k^{-1}(j-1)$ and (A.2).

APPENDIX B (PROOF OF 3.7)

From the strict positive real condition (2.2b), and definitions (3.4) (3.5), we know [2] that there are some constants $\mu_1 > \mu_2 > 0$ and $K_1 < \infty$, such that for all $\{h_i\}$,

$$\sum_{i=1}^{k} [h_i(2z_i - h_i) - \mu_1 h_i^2 - \mu_2(2z_i - h_i)^2] + K_1 \geq 0, \qquad \forall k \tag{B.1}$$

which can be re-organized by selections

$$\mu_3 = \frac{\mu_1 - \mu_2}{1+2\mu_2} > 0, \ \mu_4 = \frac{4\mu_2}{1+2\mu_2} > 0, \ K_2 = \frac{K_1}{1+2\mu_2} < \infty,$$

as

$$\sum_{i=1}^{k} [h_i(2z_i - h_i) - \mu_3 h_i^2 - \mu_4 z_i^2] + K_2 \geq 0, \qquad \forall\, k \tag{B.2}$$

Now from definitions in (3.5) (3.6),

$$h_i = b_i - \theta^\tau v_{i-1} \tag{B.3}$$

Then for any given $\varepsilon_1 > 0$, manipulations on (B.3) yield

$$\begin{aligned}\sum_{i=1}^{k} h_i^2 &= (1-\varepsilon_1)\sum_{i=1}^{k} b_i^2 + \sum_{i=1}^{k}[\varepsilon_1 b_i^2 - 2b_i\theta^\tau v_{i-1} + (\theta^\tau v_{i-1})^2] \\ &= (1-\varepsilon_1)\sum_{i=1}^{k} b_i^2 + \sum_{i=1}^{k}[\varepsilon_1 b_i^2 - 2b_i\theta^\tau v_{i-1} + \frac{(\theta^\tau v_{i-1})^2}{\varepsilon_1}] - (\frac{1}{\varepsilon_1} - 1)\sum_{i=1}^{k}(\theta^\tau v_{i-1})^2\end{aligned}$$

Applying the inequality from (3.2), allows

$$\sum_{i=1}^{k} h_i^2 \geq (1-\varepsilon_2)\sum_{i=1}^{k} b_i^2 - (\frac{1}{\varepsilon_1} - 1)K_v \tag{B.4}$$

Also, by the Schwarz inequality and the inequality (3.2), for any given $\varepsilon_2 > 0$,

$$\begin{aligned}-2\sum_{i=1}^{k} \theta^\tau v_{i-1} z_i &\leq \varepsilon_2 \sum_{i=1}^{k} z_i^2 + \frac{1}{\varepsilon_2}\sum_{i=1}^{k}(\theta^\tau v_{i-1})^2 \\ &\leq \varepsilon_2 \sum_{i=1}^{k} z_i^2 + \frac{K_v}{\varepsilon_2}\end{aligned} \tag{B.5}$$

Moreover from (B.3), (B.5)

$$\begin{aligned}2\sum_{i=1}^{k} h_i z_i &= 2\sum_{i=1}^{k}(b_i - \theta^\tau v_{i-1}) z_i \\ &\leq 2\sum_{i=1}^{k} b_i z_i + \varepsilon_2 \sum_{i=1}^{k} z_i^2 + \frac{K_v}{\varepsilon_2}\end{aligned} \tag{B.6}$$

Substituting (B.4) (B.6) into (B.2) we have

$$0 \le \sum_{i=1}^{k} [h_i(2z_i - h_i) - \mu_3 h_i^2 - \mu_4 z_i^2] + K_2$$

$$\le \sum_{i=1}^{k} [2b_i z_i + \varepsilon_2 z_i^2 - (1+\mu_3)(1-\varepsilon_1)b_i^2 - \mu_4 z_i^2] + K_2 + \frac{K_v}{\varepsilon_2} + (\frac{1}{\varepsilon_1} - 1)K_v$$

Now by choosing $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, sufficiently small, we can define

$$\rho_1 = \mu_3 - \varepsilon_1 - \varepsilon_1\mu_3 > 0,\ \rho_2 = \mu_4 - \varepsilon_2 > 0, \text{ and } K = K_2 + (\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_1} - 1)K_v$$

and (3.7) is established. $\Delta\Delta\Delta$

REFERENCES

[1] J.B. Moore, T. Ryall, L. Xia "Central Tendency Pole Assignment". **Proc. of 25th CDC** pp100-105, Athens, Greece. 1986

[2] G.C. Goodwin, K. S. Sin **"Adaptive Filtering Prediction and Control"**. Prentice Hall 1984.

[3] H. F. Chen, L. Guo "Consistent Estimation of the Order of Stochastic Control Systems". **IEEE AC-32**, pp531-535. 1987

[4] L. Xia, J.B. Moore, M. Gevers "On Adaptive Estimation and Pole Assignment of Overparametrized Systems". **Int. J. of Adaptive Control and Signal Processing**. Vol 1, pp 143-160, 1987

[5] M. Green, J.B. Moore "Persistence of Excitation in Linear Systems". **Proc. of ACC** pp412-417 Boston 1985

Chapter 9

ADAPTIVE ESTIMATION IN THE PRESENCE OF ORDER AND PARAMETER CHANGES

1. INTRODUCTION:

Current adaptive estimation and control algorithms are designed to cope with plants of fixed order. However, many practical applications involve jump order changes for a plant. A typical example is the adaptive control of a robot arm when it grasps a flexible rod. There is a sudden 'jump' change in system order when the arm grasps the rod due to the additional flexure modes of the rod. Another significant application is the adaptive control of a structure in space under construction, or involved in a docking procedure. From our studies of the literature, the problem of dealing with jump order changes seems not to have been addressed seriously.

In this chapter, with the view to coping with time varying plants with jump order changes, we develop novel algorithms, building on earlier works [1,2]. These earlier studies propose least squares based adaptive estimation and control algorithms for unknown fixed plants, with only an upper bound on the plant order. The key contribution of [1,2] is the introduction of noise signals into standard estimation algorithms according to ill-conditioning measures. This ensures that identification of any pole-zero cancellation takes place at the origin and any overparametrization does not lead to ill-conditioning.

In this chapter, the approach of [1,2] is modified for the case when the plant may have time varying parameters by working with Kalman filter based estimation schemes, or recursive least squares with forgetting factor schemes. Again noise is introduced to the calculations according to ill-conditioning measures. However, the new measures are nontrivial generalizations of earlier ones, and a good deal simpler to implement. They are also able to detect and cope with jump order changes. For simplicity, the paper focuses on the special case of piecewise constant parameter values which permit jump order changes, but where parameter/order changes are relatively infrequent. Also attention is focussed on the adaptive estimation case, this being the basis for adaptive control.

The algorithm is detailed in Section 2, and some simulation results are given in Section 3. In Section 4, we draw a few lines as the conclusion.

2.PERTURBED KALMAN FILTER DETECTION/IDENTIFICATION SCHEME

Signal Model : In the first instance we consider a scalar, time-varying, linear system with changing order described by the DARMA model as

$$A_k(q^{-1})\, y_k = B_k(q^{-1})\, u_k \tag{2.1}$$

where

$$A_k(q^{-1}) = 1 + \sum_{i=1}^{n} a_k^{(i)} q^{-i}\,, \quad B_k(q^{-1}) = \sum_{i=1}^{m} b_k^{(i)} q^{-i}\,,$$

u_k and y_k are the system input and output; n and m are the upper bounds on the degree of the polynomials $A_k(q^{-1})$ and $B_k(q^{-1})$ respectively. We say that the system is overparametrized when the polynomials $q^n A_k(q^{-1})$ and $q^m B_k(q^{-1})$ have common zeroes. The degree of overparametrization is the number of common zeroes. Thus if there are no common zeroes, i.e. $q^n A_k(q^{-1})$ and $q^m B_k(q^{-1})$ are coprime, the degree of overparametrization is zero. A stochastic version of the signal model in (2.1) for estimation purposes can be rewritten as

$$\theta_{k+1} = \theta_k + w_k, \quad y_k = \theta_k^{\tau} \phi_k + v_k \tag{2.2}$$

$$\theta_k^{\tau} = [a_k^{(1)} \dots a_k^{(n)} \; b_k^{(1)} \dots b_k^{(m)}], \; \phi_k = [-y_{k-1} \dots -y_{k-n} \; u_{k-1} \dots u_{k-m}]$$

Here ϕ_k is termed a regression vector. The measurement noise term v_k is involved more for the purposes of algorithm design than to reflect any assumed persistently exciting measurement disturbance. The term w_k is the noise sequence which describes the changes in the parameter vector θ_k. For simplicity, we assume that θ_k is piecewise constant with infrequent changes, i.e. $w_k = 0$ for most k. An order change is merely a change in parameters such that there is a change in the degree of overparametrization in plant (2.1).

Excitation :

Assume that u_k is persistently exciting so that

$$0 < \alpha_2 I < \frac{1}{N} \sum_{i=k}^{k+N} \bar{u}_i \bar{u}_i^{\tau} < \alpha_1 I, \; \bar{u}_i^{\tau} = [u_i \dots u_{i-n-m}] \tag{2.3}$$

for some α_1, α_2, k_0, N and all $k > k_0$.

It is well known [3], that for intervals when $A_k(q^{-1})$ and $B_k(q^{-1})$ are constant and coprime, there is no pole-zero cancellation in the system model and the input excitation assumption (2.3) translated to an excitation condition on the regression vector ϕ_k as

$$0 < \beta_2 I < \frac{1}{N} \sum_{i=k}^{k+N} \phi_i \phi_i^\tau < \beta_1 I \qquad (2.4)$$

It is important that this excitation condition holds even in the absence of any actual persistently exciting measurement noise v_k. Of course, if v_k is persistently exciting, then ϕ_k is also persistently exciting and problems associated with overparametrization do not arise. To exclude this 'trivial' case, we assume that v_k may not be persistently exciting. If the degree of overparametrization is $L < \min(n,m)$, it can be proved that ϕ_k is persistently exciting only in the (n+m-L) dimensional subspace of the whole space $\Re^{n+m}$.

Standard Kalman Filter Identification:

We consider first a standard Kalman filter form for estimating the system parameters.

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \frac{P_k \phi_k e_k}{R + \phi_k^\tau P_k \phi_k}, \quad e_k = y_k - \phi_k^\tau \hat{\theta}_k \qquad (2.5a)$$

$$P_{k+1} = P_k - \frac{P_k \phi_k \phi_k^\tau P_k}{R + \phi_k^\tau P_k \phi_k} + Q, \quad P_0 > 0 \qquad (2.5b)$$

$R > 0$, $Q > 0$ are design parameters. (2.5c)

It has been proved [4] that if v_k, w_k have bounded variance, and ϕ_k is suitably exciting, the standard Kalman filter algorithm (2.5) gives an estimate $\hat{\theta}_k$ of θ_k with bounded tracking errors. Moreover [5], if w_k and v_k are zero mean and Gaussian with variances Q and R, respectively, then (2.5) will yield the conditional minimum variance (conditional mean) estimate, with conditional error covariance P_k. However, if ϕ_k is not suitably exciting, the estimate of θ_k using algorithm (2.5) does not necessarily have a bounded tracking error [4]. In fact, as will be shown later, P_k in (2.5) becomes unbounded when the plant is overparametrized, and consequently $\hat{\theta}_k$ becomes unbounded. To avoid any possible ill-conditioning associated with the lack of suitable excitation in the regressor, the approach in [1,2] of using a perturbed estimation algorithm will be used in the Kalman filter estimation. [Notice that the Kalman filter algorithm specializes to the least squares estimation scheme of [1,2] when Q = 0, R = 1.]

Perturbed Kalman Filter:

We propose the following perturbed Kalman filter identification algorithm:

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \frac{\bar{P}_k \psi_k e_k}{R + \psi_k^\tau \bar{P}_k \psi_k}, \quad e_k = y_k - \psi_k^\tau \hat{\theta}_k \tag{2.6a}$$

$$\bar{P}_{k+1} = \bar{P}_k - \frac{\bar{P}_k \psi_k \psi_k^\tau \bar{P}_k}{R + \psi_k^\tau \bar{P}_k \psi_k} + Q, \quad \bar{P}_0 > 0 \tag{2.6b}$$

$$\psi_k = \phi_k + \upsilon_k \tag{2.6c}$$

where υ_k is injected noise being white and zero mean to cope with the lack of persistent excitation in ϕ_k. With Kalman filter time-varying parameter identification, designing a suitable noise covariance is not a straightforward extension of that previously proposed in [1,2] when parameters are assumed constant. However, in common with the approach of [1,2], the selection is made based on the appropriate Riccati equation solution, here P_k. A crucial property of P_k is now studied.

Kalman Filter Error Covariance Property:

Theorem 2.1: Consider the signal model as in (2.2) with constant parameters (namely $\theta_k = \theta$) and Kalman filter algorithm (2.5) with $Q = qI$. Consider also that the input u_k is persistently exciting, i.e. (2.3) is satisfied.

(1) If there is no overparametrization, then for some bounds m_1, M_1

$$0 < m_1 < tr[P_k] < M_1 < \infty \tag{2.7}$$

(2) If there is overparametrization by $L < \min(n,m)$, i.e. the degree of overparametrization is L, then

$$0 < m_1 + qLk < tr[P_k] < M_1 + qLk \tag{2.8}$$

Proof : For part (1), from [4] and (2.4) we see that P_k is upper and lower bounded. Thus (2.7) is established. For part (2), since the system is overparametrized by L, ϕ_k in (2.2) is only exciting in the (n+m-L) dimensional subspace of the $\Re^{n+m}$ space of ϕ_k. Therefore, there exists a unitary matrix T such that

$$T\,T^\tau = I, \qquad T\,\phi_k = [\,\phi_k^{*\tau}\, 0 \ldots 0\,]^\tau = \Phi_k^*, \ \forall\, k \tag{2.9}$$

After the transformation in (2.9b), the last L rows of $T\,\phi_k$ are zero and ϕ_k^* is persistently exciting. Now recall Kalman filter algorithm (2.5), specialized here as,

$$P_{k+1} = P_k - \frac{P_k\, \phi_k\, \phi_k^\tau\ P_k}{R + \phi_k^\tau\, P_k\, \phi_k} + qI. \tag{2.10}$$

Premultiplying (2.10) by T and post multiplying by T^τ we have

$$T\, P_{k+1}\, T^\tau = T\, P_k\, T^\tau - \frac{TP_k\, \phi_k\, \phi_k^\tau\, P_k\, T^\tau}{R + \phi_k^\tau\, P_k\, \phi_k} + qTT^\tau. \tag{2.11}$$

Let $P_k^* = T P_k T^\tau$. Then using (2.9), we can rewrite (2.11) as

$$P_{k+1}^* = P_k^* - \frac{P_k^* \Phi_k^* \Phi_k^{*\tau} P_k^*}{R + \Phi_k^{*\tau} P_k^* \Phi_k^*} + qI \tag{2.12}$$

Then partitioning P_k^* and Φ_k^* as

$$P_k^* = \begin{bmatrix} P_k(1) & P_k(2) \\ P_k(2)^\tau & P_k(3) \end{bmatrix}, \quad \Phi_k^* = \begin{bmatrix} \phi_k^* \\ 0 \end{bmatrix} \tag{2.13}$$

(2.12) can be written as

$$P_{k+1}(1) = P_k(1) - \frac{P_k(1)\, \phi_k^*\, \phi_k^{*\tau} P_k(1)}{R + \phi_k^{*\tau} P_k(1)\, \phi_k^*} + qI_1 \tag{2.14}$$

$$P_{k+1}(2) = P_k(2) - \frac{P_k(1)\, \phi_k^*\, \phi_k^{*\tau} P_k(2)}{R + \phi_k^{*\tau} P_k(1)\, \phi_k^*} \tag{2.15}$$

$$P_{k+1}(3) = P_k(3) - \frac{P_k(2)\, \phi_k^*\, \phi_k^{*\tau} P_k(2)}{R + \phi_k^{*\tau} P_k(1)\, \phi_k^*} + qI2 \tag{2.16}$$

From (2.14) and also since ϕ_k^* is persistently exciting, it can be concluded that $P_k(1)$ is bounded from above and below. Also it can easily be proved [4] that since ϕ_k^* is persistently exciting, (2.15) implies that $P_k(2)$ converges exponentially to zero. Thus when k is large in (2.16), the following approximation holds:

$$P_k(3) \cong k\, qI2 \tag{2.17}$$

Now since $tr[P_k] = tr[P_k^*]$, we have

$$tr[P_k] = tr[P_k(1)] + tr[P_k(3)] \tag{2.18}$$

From the properties of $P_k(1)$, $P_k(3)$, the desired result (2.8) follows. ΔΔΔ

Corollary 2.1: Consider that the conditions for Theorem 2.1 hold. Consider that moving average on trace of P_k over period l as $P_m(k)$. Then there is a constant N_l such that for all $k > N_l$,

(1) if the system is not overparametrized, then the following inequality is true;

$$tr[P_k] - P_m(k-N_l) < N_l q \tag{2.19}$$

(2) if the system is overparametrized by $L_o > 0$, the following inequality is true

$$N_l qL_o < tr[P_k] - P_m(k-N_l) < N_l q(L_o+1) \tag{2.20}$$

Proof: The proof follows from application of (2.7) and (2.8) ΔΔΔ

Injected Noise Construction

Now based on Theorem 2.1 and the Corollary 2.1, we propose the following procedure for constructing the injected noise υ_k.

Step 1: Generate a zero mean unit covariance white noise sequence $\hat{\upsilon}_k$.

Step 2: Based on the standard Kalman filter algorithm (2.5), update P_k and perform moving average on $tr[P_k]$ over a period of l.

Step 3: Checking for some large constant N_l whether (2.19) holds, or if not, for what value of L_o, (2.20) holds. This latter constitutes a detection of the degree of the overparametrization L_o.

Step 4: If (2.19) holds, then let $\upsilon_k = 0$; if (2.20) holds for L_o then set

$$\upsilon_k = D_k \hat{\upsilon}_k, \quad D_k = \mathrm{Diag}[0 .. 0\ 1..1\ 0 .. 0\ 1..1] \tag{2.21}$$

where the pattern for the diagonal D is $(n-L_o)$ zeros followed by L_o ones and then $(m-L_o)$ zeros and L_o ones again. [If we set $L_o = 0$, for the case of no overparametrization as detected by (2.19) holding, then (2.21) is a comprehensive formula for υ_k design].

Properties of Perturbed Kalman Filter:

When the system is not overparametrized, then the perturbed Kalman filter specializes as the standard Kalman filter.

If the system is overparametrized, then the model parametrization is not unique, $\upsilon_k \neq 0$, and ψ_k (2.6c) is persistently exciting by virtue of the presence of υ_k. Because of the excitation of ψ_k, $\bar{P}_k$ is bounded above and below. Moreover, if the detection of the degree of overparametrization is correct, the addition of υ_k does not introduce any bias on the estimation. Furthermore, the injected noise υ_k forces on estimation of the unique plant parameters which give rise to a pole-zero cancellation at the origin. The reason for this is as follows. From (2.6) and denoting $\tilde{\theta}_k = \theta - \hat{\theta}_k$, we have (with v_k=0)

$$\phi_k^\tau\theta = \psi_k^\tau\theta, \quad \tilde{\theta}_{k+1} = \tilde{\theta}_k - \frac{\bar{P}_k\psi_k\psi_k^\tau\tilde{\theta}_k}{R+\psi_k^\tau\bar{P}_k\psi_k}, \tag{2.22}$$

Where θ is the unique plant parameter vector which forces a pole-zero cancellation at the origin for the interval in question. Then the argument as in [4] can be used to conclude that $\tilde{\theta}_k$ converges to zero, or in other words, the algorithm tries to estimate the "true" plant parameters.

Avoiding False Alarms:

It is noticed that only when N_l is suitably large, (2.19) or (2.20) hold. However, in order to adapt to a rapidly changing environment, N_l must be small. In this case, there might exist intervals where $tr[P_k]$ increases with a gradient more than predicted, given knowledge of the degree of overparametrization. This constitutes a false alarm since in this interval degree of system overparametrization is incorrectly detected. To avoid false alarms, we propose to consider the previous j detections in the calculations as follows. Consider a modified noise selection as

$$\upsilon_k = \left(\prod_{i=k-j}^{k} D_i \right) \hat{\upsilon}_k \tag{2.23}$$

The choice of the constant j is a trade-off between the sensitivity of system order change and false alarm. Another ad hoc approach is to make j changed based on the information from the estimation errors e_k (2.6). Details are omitted here.

3. SIMULATIONS

To give insights into the algorithm behavior, a number of simulation studies have been made. Here we report a typical one as follows. The signal model is assumed as a third order system with n = 3, m = 3 [recall (2.1)-(2.2)]. The plant parameter sets are

$$\theta_k = \begin{cases} [\,-0.8\ \ 0.8\ \ -0.2\ \ 1.5\ \ 0.5\ \ 0.4\,] & 0 < k \leq 200 \\ [\,-0.6\ \ 0.15\ \ 0\ \ 1.25\ \ -0.8\ \ 0\,] & 200 < k \leq 400 \\ [\,-0.8\ \ 0.8\ \ -0.2\ \ 1.5\ \ 0.5\ \ 0.4\,] & 400 < k \leq 600 \\ [\,-0.6\ \ 0\ \ 0\ \ 1.25\ \ 0\ \ 0\,] & 600 < k \leq 800 \end{cases}$$

A white noise sequence is used as the input signal. The design parameters are chosen as R=1, Q=0.1I, the interval for moving average is l = 18, and the interval for slope checking is N_l = 18. Figures 3.1 show the estimates of the plant

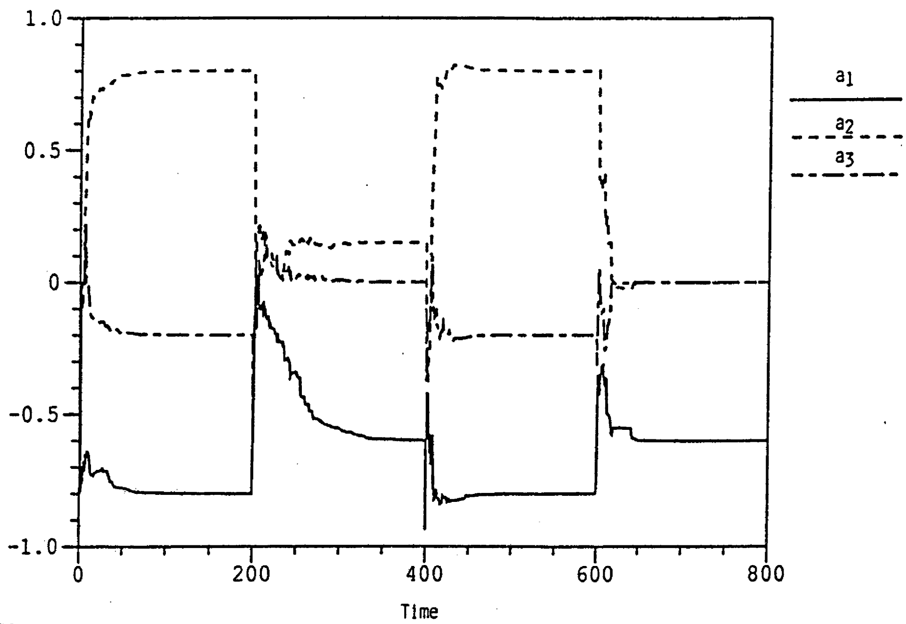


Fig. 3.1.a. Estimates of A Parameters (PKF)

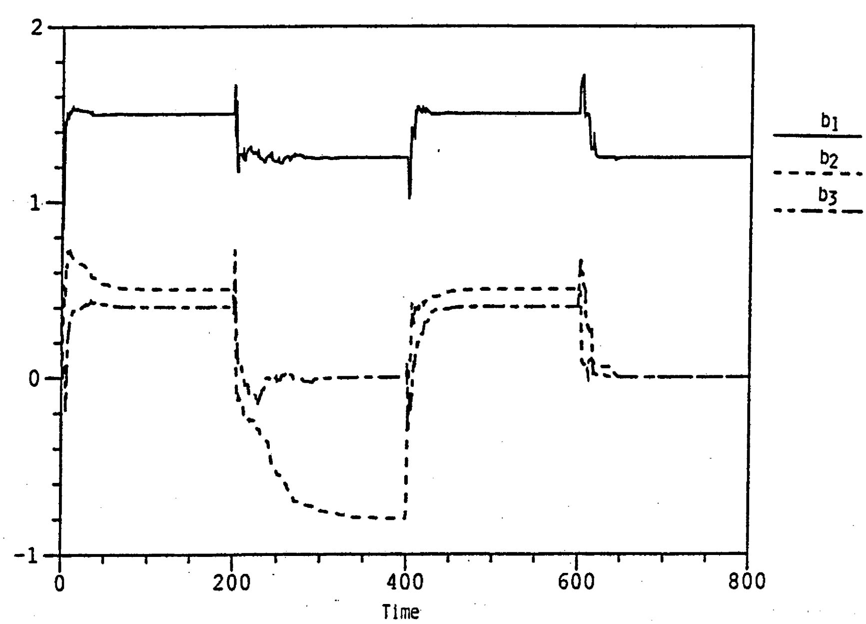


Fig. 3.1.b. Estimates of B Parameters (PKF)

parameters, and Figure 3.2 depicts the behavior of the trace of P_k . To make an easy comparison, Figures 3.3 give the estimates of the plant parameters using the standard Kalman filter algorithm.

From Figures 3.1 and 3.3, it is clear to see that for the time periods of 1 to 200 and 400 to 600, there is no overparametrization and the estimates of the plant parameters from both perturbed Kalman filter (PKF) algorithm and standard Kalman filter (SKF) algorithm are almost the same. For the periods of 200 to 400, and 600 to 800, there is overparametrization. The estimates from the PKF algorithm converge to the plant parameters which correspond to the pole-zero cancellation at the origin, or in other words, they converge to θ_k. However, the estimates from the SKF algorithm do not converge to the plant parameters θ_k ; pole-zero cancellations may occur anywhere depending on the initial conditions (when k_0=200, or k_0=600).

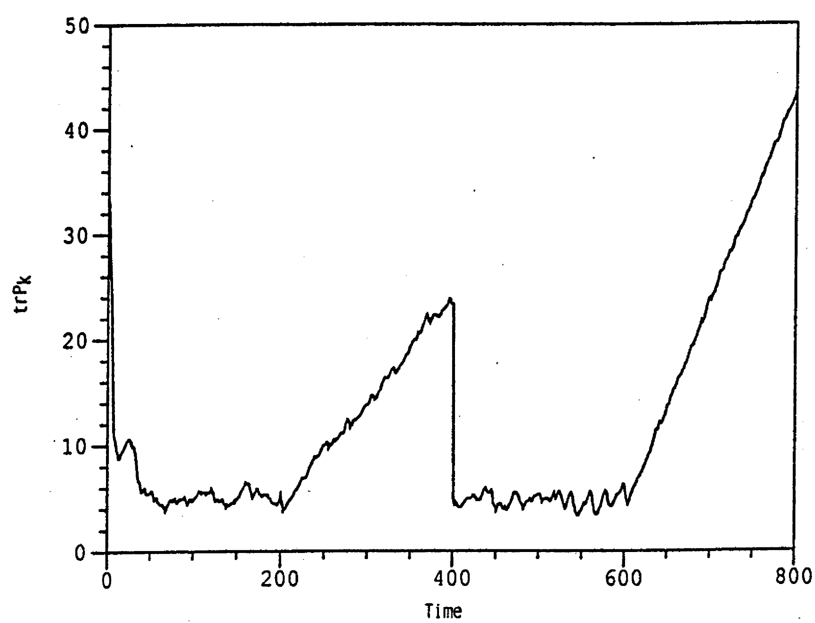


Fig. 3.2. Behavior of Tr(P_k)

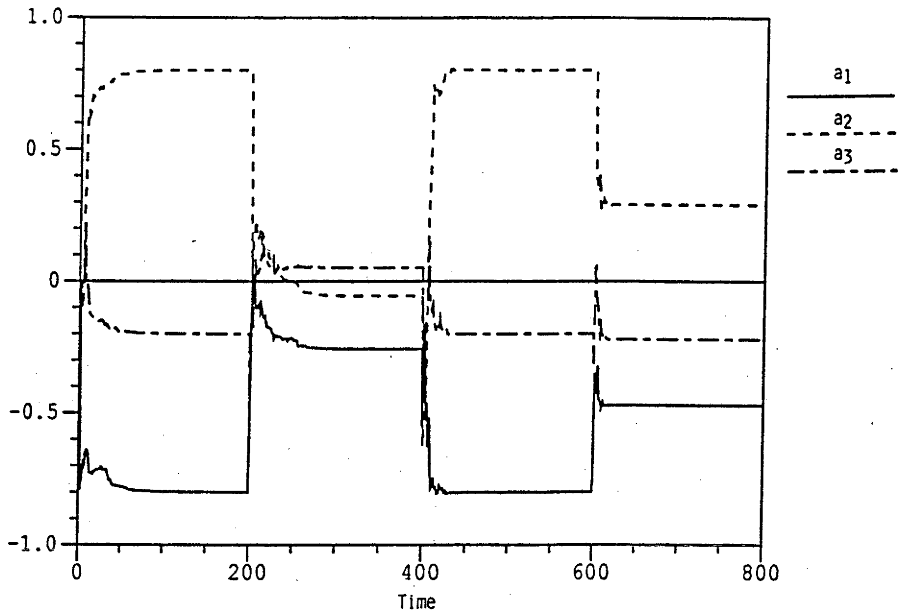


Fig. 3.3.a. Estimates of A Parameters (SKF)

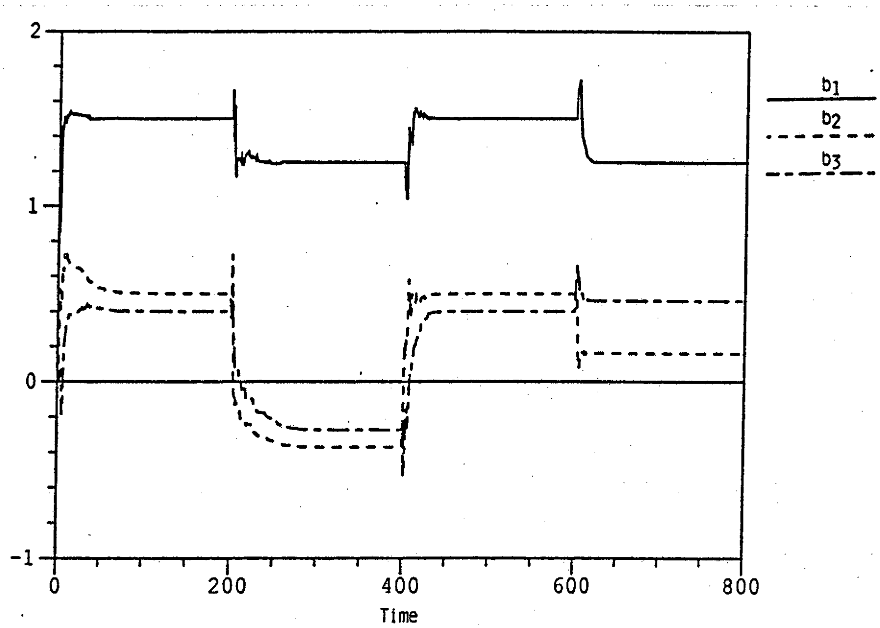


Fig. 3.3.b. Estimates of B Parameters (SKF)

From Figure 3.2, it is clear to see that when the signal model is not overparametrized as in periods 1 to 200 and 400 to 600, and also when the signal model is overparametrized as in periods 200 to 400 (L_0=1) and 600 to 800 (L_0=2), the trace of P_k has the gradient properties expected, being proportional to L_0. These and other simulations not reported here, confirm that the PKF algorithm performs virtually as well as if jump order changes do not occur. The PKF algorithm certainly performs as well as a SKF with order changes where appropriate, and cope better when there are false alarms.

4. CONCLUSIONS

An algorithm of adaptive estimation to cope with jump parameter changes and order jump changes is proposed. The key modification to standard algorithm is to introduce injected noise into the algorithm to handle the ill-conditioning due to lack of persistence of excitation caused by overparametrization. Theoretical analysis and simulations confirm that the algorithm has attractive properties, in that the algorithm performs as if the jump parameter changes did not include order changes.

REFERENCES

[1] L. Xia, J.B. Moore and M. Gevers. "On Adaptive Estimation and Pole Assignment of Overparametrized Systems", **Int. J. of Adaptive Control and Signal Processing.** Vol 1 pp 143-160

[2] L. Xia and J.B. Moore. "Recursive Identification of Overparametrized

Systems", submitted for publication

[3] M. Green and J.B. Moore. "Persistence of Excitation in Linear Systems", **Proc. of ACC** pp412-417 Boston 1985

[4] L. Guo, L. Xia and J.B. Moore. "Tracking Randomly Varying Parameters", submitted for publication

[5] D.Q. Mayne. "Optimal Non-stationary Estimation of the Parameters of a Linear System with Gaussian Inputs", **J. Electron. Control.** Vol. 14 p101, 1963.

PART IV

Some Analysis Results on Estimation Algorithms

Chapter 10

ROBUST RECURSIVE IDENTIFICATION OF MULTIDIMENSIONAL LINEAR REGRESSION MODELS

1. INTRODUCTION

Precise convergence rates are known for a number of stochastic adaptive schemes under a certain noise model positive real condition [1][2], first exposed as a convergence condition in [3][4]. Also robustness results are known[5]. The whiter the process noise, the more likely the positive real condition is satisfied, and the more robust are the algorithms.

In an earlier paper [6], a method is proposed to side-step the positive real condition for scalar variable noise models in stochastic adaptive estimation and control. The method has as a starting point the addition of white noise into the processing. Such additions ensure a whiter noise environment, which in turn ensure convergence and lend a certain robustness. The added noise can be seen as dominating unmodelled dynamics or unmodelled coloured noise. The method is made more powerful by additional processing involving on-line spectral factorization and parallel processing involving prewhitening filters. Simulations support the ideas of [6], although the theory in [6] is incomplete.

In this chapter, a number of the ideas of [6] are re-packaged in the context of a precise convergence analysis with the view to quantifying the extent of robustness enhancement and convergence rates. The techniques build on Kalman filtering theory, spectral factorization theory, and expand on those used for extended least squares convergence in [1,2]. Also, the work of [6] are nontrivially generalized to cope with multivariable signal models. Convergence rates are guaranteed without imposition of a positive real condition on the coloured noise model.

2. ALGORITHM DESCRIPTION AND MAIN RESULTS

Stochastic Model

Consider the following m-dimensional Linear Regression Model:

$$y(t) = \theta_o x(t) + \varepsilon(t), \qquad t \geq 0 \tag{2.1}$$

where y(t), x(t) and (t) are the m-, p- and m-dimensional observation vector, regression vector and modelling error respectively, and where θ_o is the mxp unknown parameter matrix.

Assume that the system noise $\varepsilon(t)$ is a moving average (MA) process:

$$\varepsilon(t) = w(t) + C_1 w(t-1) + \cdots + C_r w(t-r), \qquad t \geq 0 \tag{2.2}$$

with unknown matrix coefficients C_i, $1 \leq i \leq r$, where the driven noise $\{w(t)\}$ is assumed to be a Gaussian white noise sequence with

$$Ew(t) = 0, \quad Ew(t)w(t)^{\tau} = R_w > 0, \qquad t \geq 0. \tag{2.3}$$

Let us denote all the unknown parameters by

$$\theta = [\theta_o\ C_1 \cdots C_r]^{\tau} \tag{2.4}$$

Introduced Noise

To dominate unmodelled dynamics and/or noise which is highly coloured, consider the introduction of a Gaussian white noise sequence $\{v(t)\}$ which is independent of $\{w(t)\}$ with properties:

$$Ev(t) = 0, \quad Ev(t)v(t)^{\tau} = \sigma_v^2 I_m, \quad \sigma_v^2 > 0, \tag{2.5}$$

The "pre-whitening" idea proposed in [6] is to formulate the predictor in identification algorithm by using the following "pre-whitened" process

$$z(t) = y(t) + v(t), \qquad t \geq 0 \tag{2.6}$$

together with the output sequence $\{y(t)\}$.

Prediction Error Algorithm

Consider the prediction error algorithm processing (2.6):

$$\hat{\theta}(t+1) = \hat{\theta}(t) + P(t)\psi(t)[z(t+1)^{\tau} - \psi(t)^{\tau}\hat{\theta}(t)] \tag{2.7}$$

$$P(t) = P(t-1) - \frac{P(t-1)\psi(t)\psi(t)^{\tau}P(t-1)}{1 + \psi(t)^{\tau}P(t-1)\psi(t)}, \qquad P(0) > 0 \tag{2.8}$$

$$\psi(t) = [x(t)^{\tau} \;\; z(t)^{\tau}-\psi(t-1)^{\tau}\hat{\theta}(t) \cdots z(t-r+1)^{\tau}-\psi(t-r)\hat{\theta}(t-r+1)]^{\tau} \tag{2.9}$$

Estimates of the covariance of prediction errors are given by the following residual statistics:

$$\hat{R}_{\bar{w}}(t) = \frac{1}{t}\sum_{i=0}^{t-1}[z(i+1) - \hat{\theta}^{\tau}\psi(i)][z(i+1) - \hat{\theta}^{\tau}\psi(i)]^{\tau} \tag{2.10}$$

the terms of which have convenient recursive forms. Notice that when the introduced noise v(t) in (2.6) is set to zero, so that z(t) = y(t), then (2.9) reduces to the more 'standard' regression vector.

Theorems

Let us denote $\lambda_{min}(X)$ [$\lambda_{max}(X)$] as the minimum [maximum] eigenvalue of a matrix X and $\|X\| = \sqrt{\lambda_{max}(XX^*)}$ its norm, where X^* is the transpose complex conjugate of X. Let us also denote $\zeta(t) = \varepsilon(t) + v(t)$ and set

$$G_t^o = \sigma\{\zeta(i),\ i \le t\}, \qquad t \ge 0$$

Assume that the regression vector sequence $\{x(t), F_{t-1}\}$ is any adapted random sequence where

$$F_t = \sigma\{G_t^o \cup G_t^1\} \qquad t \ge 0$$

with $\{G_t^1\}$ being any family of non-decreasing σ-algebras such that G_t^1 is independent of G_{t+1}^o for any $t \ge 0$.

Theorem 2.1: For the system and algorithm described by (2.1)-(2.10), if in the pre-whitening of (2.5)-(2.6), σ_v^2 is chosen to satisfy

$$\sigma_v^2 > r\,\|R_w\|\,\|[C_1 \cdots C_r\|^2 - \lambda_{min}(R_w) \tag{2.11}$$

then the following convergence rates hold:

$$\text{(i)}\quad \|\hat{\theta}(t+1) - \bar{\theta}\| = O\left(\sqrt{\frac{\log\lambda_{max}(t)}{\lambda_{min}(t)}}\right), \quad \text{a.s.} \quad t \to \infty \tag{2.12}$$

$$\text{(ii)}\quad \|\hat{R}_{\bar{w}}(t) - R_{\bar{w}}\| = O\left(\sqrt{\frac{\log\lambda_{max}(t)}{t}}\right) + O\left(\frac{\log\lambda_{max}(t)}{t}\right) \quad \text{a.s. } t\to\infty \tag{2.13}$$

Here,

$$\bar{\theta} = [\theta_o \;\; D_1 \cdots D_r]^\tau \tag{2.14}$$

and $\{D_i,\ 1 \leq i \leq r,\ R_{\bar{w}}\}$ satisfies

$$D(z)R_{\bar{w}}D(z^{-1})^\tau = C(z)R_w C(z^{-1})^\tau + \sigma_v^2 I \tag{2.15}$$

with

$$C(z) \triangleq I + C_1 z + \cdots + C_r z^r \tag{2.16}$$

$$D(z) \triangleq I + D_1 z + \cdots + D_r z^r \tag{2.17}$$

Here also, $\lambda_{max}(t)$ $[\lambda_{min}(t)]$ denotes the maximum [minimum] eigenvalues of

$$\sum_{i=0}^{t} \psi(i)\psi(i)^\tau + \varepsilon I, \quad (\varepsilon > 0)$$ □□□

Theorem 2.2: Consider that the conditions of Theorem 2.1 apply and

$$\log\lambda_{max}{}^o(t) = o(\lambda_{min}{}^o(t)), \quad \text{a.s.} \quad t \to \infty \tag{2.18}$$

Then

$$\| \hat{\theta}(t+1) - \bar{\theta} \| = O\left(\sqrt{\frac{\log\lambda_{max}{}^o(t)}{\lambda_{min}{}^o(t)}}\right), \quad \text{a.s.} \quad t \to \infty \tag{2.19}$$

and

$$\| \hat{R}_{\bar{w}}(t) - R_{\bar{w}} \| = O\left(\sqrt{\frac{\log\log(t)}{t}}\right) + O\left(\frac{\log\lambda_{max}{}^o(t)}{t}\right), \quad \text{a.s. } t \to \infty \tag{2.20}$$

Here, $\lambda_{max}{}^o(t)$ $[\lambda_{min}{}^o(t)]$ denotes the maximum [minimum] eigenvalue of

$$\sum_{i=0}^{t} \psi^o(i)\psi^o(i)^\tau + \varepsilon I,$$

with

$$\psi^o(t) \triangleq [x^\tau(t) \;\; \bar{w}(t)^\tau \cdots \bar{w}(t-r+1)]^\tau \tag{2.21}$$

and $[\bar{w}(t), F_t\}$ is a Gaussian martingale difference sequence with

$$E\bar{w}(t)\bar{w}(t)^\tau \longrightarrow R_{\bar{w}}, \quad \text{as } t \to \infty$$

and satisfies, under (2.16)

$$C(z)w(t) + v(t) = D(z)\bar{w}(t) + O(e^{-\alpha t}) \qquad \text{for some } \alpha > 0. \qquad \square\square\square$$

<u>Remarks</u> 1. The classical linear regression model considered in mathematical statistics is a specialization of (2.1) with the so-called "design vector" x(t) deterministic and with the noise $\varepsilon(t)$ white. Obviously the stochastic model (2.1)-(2.2) considered in this paper is a more general one, namely, we allow that the regression vector x(t) to be a class of random vectors and the modelling error $\varepsilon(t)$ to be correlated. However, the restriction that $x(t) \in F_{t-1}$ essential to the convergence analysis excludes the specialization of (2.1),(2.2) to general ARMAX models - a crucial point not observed in [6].

2. For the case when the added noise v(t) in (2.6) is zero, then the condition (2.11) is usually replaced by a positive real condition on the noise model (even for the case where one is only interested in identifying θ_o). In particular it is required that

$$[C(z)^{-1} - \frac{1}{2}I] \text{ is strictly positive real} \tag{2.23}$$

(This condition is equivalent to $[C(z)-I]$ is strictly bounded real. To see the equivalence, recall that $X(z)$ is bounded real if and only if $Z(z) = [I-X(z)][I+X(z)]^{-1}$ is positive real.) It is the addition of sufficient noise into the algorithm which obviates the need for such a condition in the above Theorem. In identifying (2.1) with $C(z)$ unknown (2.23) can not be checked a priori. In contrast the condition (2.11) can be satisfied a priori with only a limited knowledge of the "unknown" process, namely some upper bound on $\|R_w\|$ and $\|[C_1 \cdots C_r]\|$. In the scalar variable case an upper bound on the term $\|[C_1 \cdots C_r]\|$ is numerically readily obtained since without loss of generality, $C(z)$ can be minimum phase. In this case, it is readily shown, see Appendix, that

$$\|[C_1 \cdots C_r]\| < \sqrt{(2r)!(r!)^{-2} - 1} \qquad (2.24)$$

3. Estimates $\hat{C}(t)$ and $\hat{R}_w(t)$ converging at the rates above to $C(z)$ and R_w can be determined from estimates $\hat{D}(t)$, $\hat{R}_{\overline{w}}(t)$ by an on-line spectral factorization corresponding to the off-line version (2.15) as in [7]. Of course, it is immediate that $C(z)$ and R_w can be uniquely determined from $D(z)$ and $R_{\overline{w}}$ via (2.15) to within an all-pass factor. Without loss of generality we can take $C(z)$ minimum phase. In this case $C(z)$ is uniquely determined from $D(z)$, $R_{\overline{w}}$.

4. The convergence rates of the estimates $\hat{\theta}(t)$ are virtually the same as that given in earlier theory for multivariable ARMAX models with $v(t)$ zero and (2.23) holding[1]. Of course, the covariance of the different error terms is inevitably higher because of the added noise, but this need not be the case with the additional processing proposed in [6] and studied in [8].

5. The requirement that $w(t),v(t)$ be Gaussian is a technical one required by the

particular martingale convergence theorem employed in subsequent analysis. It appears that by defining martingales in terms of orthogonal projections rather than in terms of conditional expectations could relax this requirement. Details are not explored here. Certainly simulations suggest that the Gaussian assumption is overly strong.

3. PRELIMINARY THEORY

Let $\{z(t),F_t\}$ and $\{\psi(t),F_t\}$ be two sequences of adapted random vectors [not necessarily defined by (2.6) and (2.9)]. Consider the following general prediction error algorithm based on a predictor $\hat{z}(t,\theta) \triangleq \hat{z}(t,\theta,\{z(0), \cdots z(t-1)\}) \in F_{t-1}$

$$\hat{\theta}(t+1) = \hat{\theta}(t) + P(t)\psi(t)[z(t+1)^{\tau} - \hat{z}(t+1,\hat{\theta}(t))^{\tau}] \tag{3.1a}$$

$$P(t) = P(t-1) - \frac{P(t-1)\psi(t)\psi(t)^{\tau}P(t-1)}{1 + \psi(t)^{\tau}P(t-1)\psi(t)}, \qquad P(0) > 0 \tag{3.1b}$$

Set

$$a(t) \triangleq [1 + \psi(t)^{\tau}P(t-1)\psi(t)]^{-1} \tag{3.2}$$

$$\xi(t) \triangleq a(t-1)[z(t)-\hat{z}(t,\hat{\theta}(t-1))] - \{z(t) - E[z(t) \mid F_{t-1}]\} \tag{3.3}$$

$$S_t(\theta,\alpha) \triangleq \sum_{i=1}^{t} [\xi(i+1)^{\tau}\tilde{\theta}(i+1)^{\tau}\psi(i) - \frac{1+\alpha}{2}\|\tilde{\theta}(i+1)^{\tau}\psi(i)\|^2], \ \alpha>0 \tag{3.4}$$

where $\tilde{\theta}(t) \triangleq \theta - \hat{\theta}(t)$ and θ is an arbitrary matrix of appropriate dimensions.

Lemma 3.1: Suppose that the adapted sequence $\{z(t),F_t\}$ satisfies:

$$\sup_{t\geq 0} E\Big[\, \| z(t)-E[z(t) \mid F_{t-1}] \|^{\beta} \mid F_{t-1} \Big] < \infty \quad \text{a.s.} \tag{3.5}$$

for some $\beta \geq 2$. Then for any θ and any $\alpha > 0$, the estimate $\hat{\theta}(t)$ given by (3.1) satisfies the following relation:

$$\| \hat{\theta}(t+1) - \theta \|^2 \leq \frac{-2S_t(\theta,\alpha)}{\lambda_{min}(t)} + O(\frac{\log^{1+\alpha\delta(\beta-2)}\lambda_{max}(t)}{\lambda_{min}(t)}) \text{ a.s.} \tag{3.6}$$

where $\lambda_{max}(t) = \lambda_{max}[P(t)^{-1}]$ and $S_t(\theta,\alpha)$ is defined in (3.4), and where $\delta(s) \triangleq 0$ for $x > 0$ and $\delta(s) \triangleq 1$ for $x = 0$.

Proof: See Appendix.

Remark: The proof techniques follow closely those of [1], but the result is in fact more general than that of [1]. Here (3.1) is a more general prediction error scheme than that of [1], which is an extended least squares scheme with $z(t+1,\hat{\theta}(t)) = \hat{\theta}(t)^{\tau}\psi(t)$.

Lemma 3.2: Consider that the conditions of Lemma 3.1 apply. Consider also that at some point , $E[z(t+1) \mid F_t]$ can be expressed by

$$E[z(t+1) \mid F_t] = \hat{z}(t+1,\hat{\theta}(t)) + \tilde{\theta}(t)^{\tau}\psi(t)+[H(z)-I]\tilde{\theta}(t+1)^{\tau}\psi(t)+\delta(t) \tag{3.7}$$

where $\tilde{\theta}(t) \triangleq \theta - \hat{\theta}(t)$, and $\delta(t)$ is an F_t-measurable random vector. Then, if the transfer matrix $H(z) - \frac{1+\alpha_o}{2} I$, ($\alpha_o > 0$) is positive real, then the following expansion holds:

$$\text{(i)}\quad \| \hat{\theta}(t+1) - \theta \|^2 \le O\left(\frac{\log^{1+\alpha\delta(\beta-2)}\lambda_{max}(t)}{\lambda_{min}(t)}\right) + \frac{2\sum_{i=1}^{t}\|\delta(i)\|^2}{(\alpha_o-\alpha)\lambda_{min}(t)} \qquad (3.8)$$

$$\text{(ii)}\quad \sum_{i=1}^{t}\|\tilde{\theta}(i+1)^\tau\psi(t)\|^2 \le O\left(\log^{1+\alpha\delta(\beta-2)}\lambda_{max}(t)\right)+\frac{4}{(\alpha_o-\alpha)^2}\sum_{i=1}^{t}\|\delta(i)\|^2 \qquad (3.9)$$

for any $\alpha \in (0, \alpha_o)$

Proof: By (A1) and (A4) in the Appendix and (3.7) we see that $\xi(t+1)$ defined by (3.3) can be rewritten as

$$\xi(t+1) = a(t)\{e(t+1) + E[z(t+1) \mid F_t] - \hat{z}(t+1,\hat{\theta}(t))\} - e(t+1)$$

$$= a(t)\{e(t+1) + \tilde{\theta}(t)^\tau\psi(t)+[H(z)-I]\tilde{\theta}(t+1)^\tau\psi(t)+\delta(t)\} - e(t+1)$$

$$= a(t)\{e(t+1) + \left[\tilde{\theta}(t+1)^\tau + P(t-1)\psi(t)[\xi(t+1)^\tau + e(t+1)^\tau]\right]\psi(t) +$$

$$+ [H(z)-I]\tilde{\theta}(t+1)^\tau\psi(t)+\delta(t)\} - e(t+1)$$

$$= a(t)\{(1 + \psi(t)^\tau P(t-1)\psi(t))e(t+1) + \xi(t+1)\psi(t)^\tau P(t-1)\psi(t) +$$

$$+ H(z)\tilde{\theta}(t+1)^\tau\psi(t)+\delta(t)\} - e(t+1)$$

From here we immediately get

$$a(t)\xi(t+1) = [1 - \psi(t)^\tau P(t-1)\psi(t)a(t)]\xi(t+1)$$

$$= a(t)[H(z)\tilde{\theta}(t+1)^\tau\psi(t)+\delta(t)]$$

and consequently

$$\xi(t+1) = H(z)\tilde{\theta}(t+1)^\tau\psi(t)+\delta(t) \qquad (3.10)$$

Since $H(z) - \frac{1+\alpha_o}{2} I$, is positive real, there exist constants K_o so that for $\forall\, \alpha \in (0, \alpha_o)$, from (3.4) and (3.10):

$$S_t(\theta,\alpha)=\sum_{i=1}^{t}\{[H(z)\tilde{\theta}(i+1)^\tau\psi(i)+\delta(i)]^\tau\tilde{\theta}(i+1)^\tau\psi(i) - \frac{1+\alpha_o}{2}\|\tilde{\theta}(i+1)^\tau\psi(i)\|^2\}$$

$$= \sum_{i=1}^{t}\{(H(z) - \frac{1+\alpha_o}{2} I)\tilde{\theta}(i+1)^\tau\psi(i)\}^\tau \cdot \tilde{\theta}(i+1)^\tau\psi(i) + K_o$$

$$+ \sum_{i=1}^{t}\delta(i)^\tau\tilde{\theta}(i+1)^\tau\psi(i) + \frac{\alpha_o-\alpha}{2}\sum_{i=1}^{t}\| \tilde{\theta}(i+1)^\tau\psi(i) \|^2 - K_o$$

$$\geq \sum_{i=1}^{t}\delta(i)^\tau\tilde{\theta}(i+1)^\tau\psi(i) + \frac{\alpha_o-\alpha}{2}\sum_{i=1}^{t}\| \tilde{\theta}(i+1)^\tau\psi(i) \|^2 - K_o \tag{3.11}$$

By the elementary inequality

$$\| a^\tau b \| \leq \frac{1}{2\varepsilon}\| a \|^2 + \frac{\varepsilon}{2}\| b \|^2 \quad (\forall \varepsilon > 0)$$

we see that for any $\alpha \in (0,\alpha_o)$

$$\sum_{i=1}^{t} | \delta(i)^\tau\tilde{\theta}(i+1)^\tau\psi(i) |$$

$$= \frac{1}{\alpha_o-\alpha}\sum_{i=1}^{t}\|\delta(i)\|^2 + \frac{\alpha_o-\alpha}{4}\sum_{i=1}^{t}\| \tilde{\theta}(i+1)^\tau\psi(i) \|^2 \tag{3.12}$$

Finally, by (3.11), (3.12) and Lemma 3.1, it follows that for any $\alpha \in (0,\alpha_o)$,

$$\| \hat{\theta}(t+1) - \theta \|^2 \leq \frac{-2\sum_{i=1}^{t}\delta(i)^\tau\tilde{\theta}(i+1)^\tau\psi(i)}{\lambda_{min}(t)} - \frac{(\alpha_o-\alpha)\sum_{i=1}^{t}\| \tilde{\theta}(i+1)^\tau\psi(i) \|^2}{\lambda_{min}(t)}$$

$$+ O(\frac{\log^{1+\alpha\delta(\beta-2)}\lambda_{max}(t)}{\lambda_{min}(t)})$$

$$\leq \frac{2}{\alpha_o-\alpha} \frac{\sum_{i=1}^{t} \|\delta(i)\|^2}{\lambda_{min}(t)} - \frac{(\alpha_o-\alpha)}{2} \frac{\sum_{i=1}^{t} \| \tilde{\theta}(i+1)^{\tau}\psi(i) \|^2}{\lambda_{min}(t)}$$

$$+ O\left(\frac{\log^{1+\alpha\delta(\beta-2)}\lambda_{max}(t)}{\lambda_{min}(t)}\right)$$

and then the conclusions of the lemma follow immediately.

Remark: In Lemmas 3.1, 3.2, no models are pre-postulated for $\{z(t), F_t\}$ and so the process $\{z(t)\}$ can be generated from an ARMAX model, since the restriction $x(t) \in F_{t-1}$ of Theorems 2.1, 2.2 is not imposed here. When $\{z(t),F_t\}$ with $F_t = \sigma\{z_i, i \leq t\}$ is generated from an ARMAX model and $\hat{\theta}(t)$ is given by the standard extended least squares algorithm, the process $\{\delta(t)\}$ appearing in Lemma 3.2 is zero. However, when there are unmodelled dynamics and time variations of the coefficients, then $\{\delta(t)\}$ is no longer zero [5]. Lemma 3.2 provides a unified approach to the convergence/robustness analysis of general prediction error algorithms such as psuedo-linear regression with or without filtering[9]. It is worth noting that the robustness properties of the algorithm are closely related to the passivity margin of the transfer function concerned (see (3.8)). Other applications of Lemma 3.2 will be noted elsewhere.

Lemma 3.3: Let C(z) be defined as in (2.16), and $\{w(t)\}$ and $\{v(t)\}$ be defined as in (2.3) and (2.5), then there exists a Gaussian martingale difference sequence $\{\bar{w}(t), G_t^o\}$ with

$$E\bar{w}(t)\bar{w}(t)^{\tau} \longrightarrow R_{\bar{w}}, \quad \text{(exponentially fast) as } t \rightarrow \infty$$

and a matrix polynomial D(z):

$$D(z) = I + D_1 z + \cdots + D_r z^r \tag{3.13}$$

such that

$$C(z)w(t) + v(t) = D(z)\bar{w}(t) + \eta(t) \tag{3.14}$$

where $\eta(t)$ is G_{t-1}^{o} -measurable and exponentially tending to zero as $t \to \infty$. Moreover, for any $\alpha_o \in [0, 1)$, if

$$\sigma_v^2 \geq r(\frac{1+\alpha_o}{1-\alpha_o})^2 \, \|R_w\| \cdot \| [C_1 \cdots C_r] \|^2 - \lambda_{min}(R_w) \tag{3.15}$$

then

$$D(z)^{-1} - \frac{1+\alpha_o}{2} I \qquad \text{is positive real} \tag{3.16}$$

Proof: Define $\zeta(t)$ as in Section 2:

$$\zeta(t) = C(z)w(t) + v(t)$$

and set,

$$A = \underbrace{\begin{bmatrix} 0 & I_m & \cdot & \cdot & 0 \\ & & \cdot & & \\ & & & \cdot & I_m \\ 0 & & & & 0 \end{bmatrix}}_{m(r+1)}, \; C = \begin{bmatrix} I_m \\ C_1 \\ \vdots \\ C_r \end{bmatrix}, \; H = [I_m \; 0 \; \cdots \; 0] \tag{3.18}$$

then $\zeta(t)$ can be expressed by

$$x^*(t+1) = Ax^*(t) + Cw(t+1), \quad \zeta(t) = Hx^*(t) + v(t)$$

According to the Kalman filtering theory $\zeta(t)$ can be generated by the following innovation model [10]

$$\hat{x}^*(t+1) = A\hat{x}^*(t) + K(t)\bar{w}(t), \quad \zeta(t) = H\hat{x}^*(t) + \bar{w}(t) \tag{3.19}$$

where $\hat{x}(t)$ is the estimator for x(t) and K(t) is the filter gain given by

$$K(t) = AP(t)H^\tau[HP(t)H^\tau + \sigma_v^2 I]^{-1} \tag{3.20}$$

$$P(t+1) = AP(t)A^\tau - AP(t)H^\tau[HP(t)H^\tau + \sigma_v^2 I]^{-1}HP(t)A^\tau + CR_wC^\tau \tag{3.21}$$

and where the innovation process $\{\bar{w}(t), G_t^o\}$ is a Gaussian martingale difference sequence with

$$E\bar{w}(t)\bar{w}(t)^\tau = HP(t)H^\tau + \sigma_v^2 I_m \tag{3.22}$$

By (3.17) and (3.19) we see that

$$C(z)w(t) + v(t) = H(I-Az)^{-1}K(t)\bar{w}(t-1) + \bar{w}(t) \tag{2.23}$$

Note that (3.19) is asymptotically stable, and hence [9,10]

$$P(t) \to P, \quad K(t) \to K, \quad \text{(exponentially fast)} \tag{3.24}$$

where P and K are defined by

$$P = APA^\tau - APH^\tau(HPH^\tau + \sigma_v^2 I)^{-1}HPA^\tau + CR_wC^\tau \tag{3.25}$$

$$K = APH^\tau(HPH^\tau + \sigma_v^2 I)^{-1} \tag{3.26}$$

Since $E\bar{w}(t)\bar{w}(t)^\tau \to HP(t)H^\tau + \sigma_v^2 I_m$, by the Borel-Cantelli Lemma it is easy to see that as $t \to \infty$,

$$\eta(t) \triangleq H(I - Az)^{-1}[K(t) - K]\bar{w}(t-1) \to 0, \text{ a.s. (exponentially)} \tag{3.27}$$

So by (3.25) and (3.27) we have

$$C(z)w(t) + v(t) = [H(I-Az)^{-1}Kz+I]\overline{w}(t) + \eta(t) \tag{3.28}$$

We write K as

$$K = [K_1^{\tau} \cdots \ K_{r+1}^{\tau}]^{\tau}$$

from (3.18) and (3.26) it can be seen that $K_{r+1} = 0$.

Set

$$D(z) = I + K_1 z + \cdots + K_r z^r \tag{3.29}$$

Then by (3.18) it can be verified that

$$D(z) = [H(I-Az)^{-1}Kz + I]$$

Therefore, by (3.28), we see that (3.14) is proved. We now proceed to prove (3.16).

By (3.14) it is easy to see that

$$\sum_{i=1}^{r} C_i R_w C_i^{\tau} + R_w + \sigma_v^2 I = \sum_{i=1}^{r} D_i R_{\overline{w}} D_i^{\tau} + R_{\overline{w}} \tag{3.30}$$

By (3.25) and (3.26) it is not difficult to see that

$$P = [A-KH]P[A-KH]^{\tau} + K\sigma_v^2 K^{\tau} + CR_w C^{\tau}$$

From this and (3.22), we immediately get

$$R_{\overline{w}} = HPH^{\tau} + \sigma_v^2 I_m \geq HCR_w C^{\tau} H^{\tau} + \sigma_v^2 I_m = R_w + \sigma_v^2 I_m \tag{3.31}$$

Consequently, by (3.30) and (3.31) we have

$$\sum_{i=1}^{r} C_i R_w C_i^\tau = \sum_{i=1}^{r} D_i R_{\overline{w}} D_i^\tau + R_{\overline{w}} - R_w - \sigma_v^2 I$$

$$\geq \sum_{i=1}^{r} D_i R_{\overline{w}} D_i^\tau \geq [\lambda_{min}(R_w) + \sigma_v^2] \sum_{i=1}^{r} D_i D_i^\tau$$

From here it follows that

$$\| [D_1 \cdots D_r] \|^2 = \lambda_{max}([D_1 \cdots D_r] \begin{bmatrix} D_1^\tau \\ \\ D_r^\tau \end{bmatrix})$$

$$\leq \frac{\lambda_{max}(R_w)}{\lambda_{min}(R_w) + \sigma_v^2} \lambda_{max}([C_1 \cdots C_r] \begin{bmatrix} C_1^\tau \\ \\ C_r^\tau \end{bmatrix})$$

$$= \frac{\lambda_{max}(R_w)}{\lambda_{min}(R_w) + \sigma_v^2} \| [C_1 \cdots C_r] \|^2 \tag{3.32}$$

and therefore, by (3.32) and (3.15) we see that

$$\| [D_1 \cdots D_r] \|^2 \leq \frac{1}{r} (\frac{1-\alpha_o}{1+\alpha_o})^2 \tag{3.33}$$

It is easy to see that

$$\| D_1 e^{i\theta} + D_2 e^{2i\theta} + \cdots + D_r e^{ir\theta} \|^2$$

$$= \lambda_{max}([D_1 \cdots D_r] \begin{bmatrix} e^{i\theta} I_m \\ \\ e^{ir\theta} I_m \end{bmatrix} [e^{-i\theta} I_m \cdots e^{-ir\theta} I_m] \begin{bmatrix} D_1^\tau \\ \\ D_r^\tau \end{bmatrix})$$

$$\leq \lambda_{max}([D_1 \cdots D_r] \begin{bmatrix} D_1^\tau \\ \\ D_r^\tau \end{bmatrix}) \cdot \lambda_{max}(\begin{bmatrix} e^{i\theta} I_m \\ \\ e^{ir\theta} I_m \end{bmatrix} [e^{-i\theta} I_m \cdots e^{-ir\theta} I_m])$$

$$= \| [D_1 \cdots D_r] \|^2 \cdot \lambda_{max}([e^{-i\theta} I_m \cdots e^{-ir\theta} I_m] \begin{bmatrix} e^{i\theta} I_m \\ \\ e^{ir\theta} I_m \end{bmatrix})$$

$$= \| [D_1 \cdots D_r] \|^2 \cdot \lambda_{max}(r \cdot I_m)$$

$$= \| [D_1 \cdots D_r] \|^2 \cdot r$$

From here and (3.33) we immediately have

$$\| D(e^{i\theta}) - I \| \leq \frac{1-\alpha_0}{1+\alpha_0} \tag{3.34}$$

By (3.34) it follows that for any $\theta \in [0,2\pi]$

$$\| \alpha_0[D(e^{-i\theta})-I] + \alpha_0[D(e^{i\theta})-I] + (1+\alpha_0)[D(e^{i\theta})-I][D(e^{-i\theta})-I] \|$$

$$\leq 2\alpha_0 \| D(e^{i\theta}) - I \| + (1+\alpha_0) \| D(e^{i\theta}) - I \|^2$$

$$\leq 2\alpha_0 \frac{1-\alpha_0}{1+\alpha_0} + \frac{(1-\alpha_0)^2}{1+\alpha_0} = 1 - \alpha_0$$

Consequently, for any $\theta \in [0,2\pi]$,

$$D(e^{-i\theta})^\tau + D(e^{i\theta}) - (1+\alpha_0)D(e^{i\theta})D(e^{-i\theta})^\tau$$

$$= (1-\alpha_0)I - \{\alpha_0[D(e^{-i\theta})-I] + \alpha_0[D(e^{i\theta})-I] + (1+\alpha_0)[D(e^{i\theta})-I][D(e^{-i\theta})-I]\}$$

$$\geq 0$$

Finally, we get

$$D(e^{i\theta})^{-1} + D(e^{-i\theta})^{-\tau} - (1+\alpha_0)I$$

$$= D(e^{i\theta})^{-1}[D(e^{-i\theta})^\tau + D(e^{i\theta}) - (1+\alpha_0)D(e^{i\theta})D(e^{-i\theta})^\tau]D(e^{-i\theta})^{-\tau}$$

$$\geq 0 \qquad \forall\, \theta \in [0,2\pi]$$

This proves the positive realness of $D(z)^{-1} - \frac{1+\alpha_0}{2}I$, and the proof of Lemma 2.3 is complete.

Remarks 1.Actually, the lower bound of σ_v^2 can be improved using the following result[10]. With initial condition P(0) = 0 in (3.21), P(t) increases monotonically (exponentially fast) to P. Thus the inequality

$$HPH^\tau \geq HP(1)H^\tau = HCR_wC^\tau H^\tau = R_w,$$

which is the essence of (3.31), can be strengthened as

$$HPH^\tau \geq HP(2)H^\tau = C_1R_wC_1^\tau - C_1R_w(R_w+\sigma_v^2I)^{-1} + R_w$$

$$= \sigma_v^2C_1\sqrt{R_w}\,(R_w+\sigma_v^2I)^{-1}\sqrt{R_w}\,C_1^\tau + R_w$$

and likewise as $HPH^\tau \geq HP(t)H^\tau$ for higher t. Since convergence of P(t) is exponentially fast to P, with a time constant linked to that of the Kalman filter, there are diminishing returns from taking t larger than (say) the dominant time constant of the Kalman filter. We do not explore this aspect of the results further here.

2. Of course, with appropriate initial conditions in the signal model and Kalman filter, $\eta(t)$ can be taken as zero. The term is left in our analysis to indicate a certain robustness in the noise modelling. The term $\eta(t)$ in (3.14) needs only to be square summable for the proofs of Theorems 2.1 and 2.2 to apply.

3. Actually, G_t^o defined in Section 2.4 can be expressed by

$$G_t^o = \sigma\{\bar{w}(i),\ i \leq t\}$$

because $\{\bar{w}(i)\}$ is the innovation sequence. Further since $\{\bar{w}(t), G_t^o\}$ is a Gaussian martingale difference sequence, it then follows that $\{\bar{w}(t)\}$ is an independent sequence. Also, since $\bar{w}(t+1) \in G_{t+1}^o$ and G_{t+1}^o is independent of G_t^1, it is clear that

$$E[\bar{w}(t+1) \mid F_t] = E[\bar{w}(t+1) \mid \sigma\{G_t^o \cup G_t^1\}]$$

$$= E[\bar{w}(t+1) \mid G_t^o]$$

$$= 0, \qquad \forall\, t \geq 0$$

this means that $\{\bar{w}(t), F_t\}$ is also a martingale difference sequence. All of these facts will be used in the sequel without explanations.

4. PROOF OF THEOREMS

Proof of Theorem 2.1, Part (i)

Here, we prove the first conclusion of Theorem 2.1, and give the proof for the second one in the Appendix. To prove (2.12), we need to verify the conditions of Lemma 3.2.

Note that in the present case

$$z(t) = y(t) + v(t), \quad \hat{z}(t+1, \hat{\theta}(t)) = \hat{\theta}(t)\psi(t) \tag{4.1}$$

so, by Lemma 3.3 we can rewrite (2.1) in the following form:

$$z(t+1) = y(t+1) + v(t+1)$$

$$= \theta_0 x(t+1) + C(z)w(t+1) + v(t+1)$$

$$= \theta_0 x(t+1) + D(z)\bar{w}(t+1) + \eta(t+1) \tag{4.2}$$

and so by (4.2) it follows that

$$\sup_t E[\, \|z(t) - E[z(t) \mid F_{t-1}]\, \|^3 \mid F_{t-1}]$$

$$= \sup_t E[\, \|\overline{w}(t)\|^3 \mid F_{t-1}] = \sup_t E[\, \|\overline{w}(t)\|^3 \mid G_{t-1}^{o}]$$

$$= \sup_t E[\, \|\overline{w}(t)\|^3] < \infty \tag{4.3}$$

since $\{\overline{w}(t)\}$ is Gaussian random sequence with uniformly bounded covariance (see (3.22)). By (4.2) we have the following expansion for $E[z(t+1) \mid F_t]$ at point $\overline{\theta}$:

$$E[z(t+1) \mid F_t] = \theta_o x(t+1) + [D(z)-I]\overline{w}(t+1) + \eta(t+1)$$

$$= \overline{\theta}^\tau \psi(t) + [D(z)-I][\overline{w}(t+1) - z(t+1) + \hat{\theta}(t+1)^\tau \psi(t)] + \eta(t+1)$$

$$= \overline{\theta}^\tau \psi(t) + [D(z)-I]D(z)^{-1}\{D(z)[\overline{w}(t+1) - z(t+1) + \hat{\theta}(t+1)^\tau \psi(t)]\} + \eta(t+1)$$

$$= \overline{\theta}^\tau \psi(t) + [I-D(z)^{-1}]\{z(t+1) - \theta_o x(t+1) - \eta(t+1) - D(z)[z(t+1) - \hat{\theta}(t+1)^\tau \psi(t)]\} + \\ + \eta(t+1)$$

$$= \overline{\theta}^\tau \psi(t) + [I-D(z)^{-1}]\{-\overline{\theta}^\tau \psi(t) + z(t+1) - \eta(t+1) - z(t+1) + \\ + \hat{\theta}(t+1)^\tau \psi(t)\} + \eta(t+1)$$

$$= \hat{\theta}(t)^\tau \psi(t) + \tilde{\theta}(t)^\tau \psi(t) + [D(z)^{-1} - I][\tilde{\theta}(t+1)^\tau \psi(t)] + D(z)^{-1}\eta(t+1) \tag{4.4}$$

where $\tilde{\theta}(t) = \overline{\theta} - \hat{\theta}(t)$. Now, since

$$\sigma_v^2 > r\, \|R_w\| \cdot \|\, [C_1 \cdots C_r]\, \|^2 - \lambda_{min}(R_w)$$

there exists $\alpha_o > 0$ such that

$$\sigma_v^2 \geq r\left(\frac{1+\alpha_o}{1-\alpha_o}\right)^2 \|R_w\| \cdot \|\, [C_1 \cdots C_r]\, \|^2 - \lambda_{min}(R_w)$$

and hence by Lemma 3.3 we know that

$$D(z)^{-1} - \frac{1+\alpha_o}{2} I \qquad \text{is positive real} \tag{4.5}$$

so by (4.3), (4.4), (4.5) we know that Lemma 3.2 is applicable, and we then have for $0 < \alpha < \alpha_o$

$$\| \hat{\theta}(t+1) - \bar{\theta} \|^2 \leq O\left(\frac{\log\lambda_{max}(t)}{\lambda_{min}(t)}\right) + \frac{(\alpha_o-\alpha)}{2} \frac{\sum_{i=1}^{t} \| D(z)^{-1}(z)\eta(i+1) \|^2}{\lambda_{min}(t)}$$

$$= O\left(\frac{\log\lambda_{max}(t)}{\lambda_{min}(t)}\right)$$

Since $\eta(t) \to 0$ exponentially fast. This proves the first conclusion of Theorem 2.1.

<u>Proof of Theorem 2.2:</u>

Set

$$\xi(t) = z(t) - \hat{\theta}(t)^{\tau}\psi(t-1) - \bar{w}(t) \tag{4.6}$$

$$\psi^{\xi}(t) = [0 \;\; \xi(t)^{\tau} \cdots \; \xi(t-r+1)^{\tau}]^{\tau} \tag{4.7}$$

By a similar argument as used in the proof of (4.4) we know that

$$D(z)\xi(t) = \tilde{\theta}(t)^{\tau}\psi(t-1) + \eta(t) \tag{4.8}$$

Since D(z) is strictly positive real it must be stable, by (4.4), (4.7), (4.8) and Lemma 3.2, we have

$$\sum_{i=1}^{t} \| \psi^{\xi}(i+1) \|^2 = O\left(\sum_{i=1}^{t} \|\tilde{\theta}(i+1)\psi(i)\|^2 \right) + O(1)$$

$$= O(\log\lambda_{max}(t)) \tag{4.9}$$

Note that

$$\psi(t) = \psi^o(t) + \psi^\xi(t) \tag{4.10}$$

and hence by use of (4.9) and (4.10) similar to the proof of Theorem 2 in [1,p1465-6] we know that

$$\lambda_{max}(t) = O(\lambda_{max}{}^o(t)\,), \quad \lambda_{min}{}^o(t) = O(\lambda_{min}(t)\,) \quad \text{a.s.}$$

This result together with Theorem 2.1 yield the desired results immediately, and Theorem 2.2 is now established.

5. CONCLUSIONS

This chapter has shown that modifying standard ELS algorithms for Linear Regression model identification can obviate the need for a positive real condition on the colored noise model. Estimates of the regression part parameter matrix θ_o in (2.1) and those of the modified noise model D_i are achieved without any compromise on convergence rates. The recovery of the original noise model parameters C_i by an on-line spectral factorization has been studied in [7]. Also, a method to remove an estimator error variance increase by additional processing is currently under study. The methods and theory of the chapter fall short of giving precise results for avoiding the positive real condition for general ARMAX models.

APPENDIX

Proof of Bound (2.24)

With

$$C(z) = 1 + \sum_{i=1}^{r} C_i z^i \triangleq \prod_{i=1}^{r} (1+\alpha_i z)$$

in the scalar case, the minimum phase condition is that $|\alpha_i| < 1$ for all i. Denoting the binomial coefficients $\binom{r}{i}$ as

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}, \quad 0 \le i \le r$$

comparing the coefficients of z^i in the above identity and noting $|\alpha_i| < 1$, we know that

$$|C_i| < \binom{r}{i}, \quad 1 \le i \le r$$

Consequently, it follows that

$$1 + \sum_{i=1}^{r} C_i^2 < \sum_{i=1}^{r} \binom{r}{i}^2$$

But, by comparing the coefficients of z^r in the follwoing identity

$$(1+z)^r(1+z)^r = (1+z)^{2r}$$

it is easy to know

$$\sum_{i=1}^{r} \binom{r}{i}^2 = \binom{2r}{r}$$

and hence,

$$1 + \sum_{i=1}^{r} C_i^2 < (2r)!(r!)^{-2}$$

which is tantamount to (2.24).

Remarks: 1. The bound (2.24) is sharp for all r.

2. A similar bound is not available for the multidimensional case, unless extra conditions in addition to the minimum phase assumption on C(z) are imposed.

Proof of Lemma 3.1

Set

$$e(t) \triangleq z(t) - E[z(t) \mid F_{t-1}] \tag{A1}$$

we see that $\{e(t), F_t\}$ is a martingale difference sequence, and satisfies

$$\sup_{t\geq 0} E\left[\|e(t)\|^{\beta} \mid F_{t-1} \right] < \infty \quad \text{a.s. } \beta \geq 2 \tag{A2}$$

By (3.1b) and (3.2) it is easy to see that

$$P(t)\psi(t) = [P(t-1) - a(t)P(t-1)\psi(t)\psi(t)^{\tau}P(t-1)]\psi(t)$$

$$= a(t)P(t-1)\psi(t) \tag{A3}$$

and then by (3.3) and (A1) we know

$$P(t)\psi(t)[z(t+1)^{\tau} - \hat{z}(t+1,\hat{\theta}(t))^{\tau}] = P(t-1)\psi(t)[\xi(t+1)^{\tau} + e(t+1)^{\tau}]$$

So we can rewrite (3.1a) as

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - P(t-1)\psi(t)[\xi(t+1)^{\tau} + e(t+1)^{\tau}] \tag{A4}$$

with $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$, for any θ. From (3.1b) it is known that

$$P(t) = [\sum_{i=1}^{t}\psi(i)\psi(i)^{\tau} + P(0)^{-1}]^{-1}, \quad t \geq 0 \tag{A5}$$

We now estimate the last term on the right-hand side of Theorem 1 in [1]. By (A4) and (A5) a similar treatment used as in the proof of (19) in [1, p.1462] leads to

$$\begin{aligned}
&\mathrm{tr}\, \tilde{\theta}(t+1)^{\tau}P(t)^{-1}\tilde{\theta}(t+1) \\
&\leq \mathrm{tr}\, \tilde{\theta}(t)^{\tau}P(t-1)^{-1}\tilde{\theta}(t) + \|\tilde{\theta}(t+1)\psi(t)\|^{2} - 2\xi(t+1)^{\tau}\tilde{\theta}(t)^{\tau}\psi(t) \\
&\qquad - 2e(t+1)^{\tau}\tilde{\theta}(t+1)^{\tau}\psi(t), \qquad\qquad \forall\, \theta \\
&= \mathrm{tr}\, \tilde{\theta}(t)^{\tau}P(t-1)^{-1}\tilde{\theta}(t) - 2\left[\xi(t+1)^{\tau}\tilde{\theta}(t)^{\tau}\psi(t) \supseteq - \frac{1+\alpha_{0}}{2}\|\tilde{\theta}(i+1)^{\tau}\psi(i)\|^{2}\right] - \\
&\qquad - \alpha\, \|\tilde{\theta}(t+1)^{\tau}\psi(t)\|^{2} - 2e(t+1)^{\tau}\tilde{\theta}(t+1)^{\tau}\psi(t), \quad \forall\theta, \forall\, \alpha > 0
\end{aligned} \tag{A6}$$

Summing both sides of (A6) and using (3.4) we get

$$\begin{aligned}
&\mathrm{tr}\, \tilde{\theta}(t+1)^{\tau}P(t)^{-1}\tilde{\theta}(t+1) \\
&\leq \mathrm{tr}\, \tilde{\theta}(1)^{\tau}P(0)^{-1}\tilde{\theta}(1) - 2S_{t}(\theta,\alpha) - \alpha \sum_{i=1}^{t} \|\, \tilde{\theta}(i+1)^{\tau}\psi(i)\, \|^{2} - \\
&\qquad - 2 \sum_{i=1}^{t} e(i+1)^{\tau}\tilde{\theta}(i+1)^{\tau}\psi(i), \quad \forall\theta, \quad \forall\, \alpha>0
\end{aligned} \tag{A7}$$

We now estimate the last term on the right-hand side of (A7). Since $\{e(t), F_t\}$ is a martingale difference sequence and satisfies (A2), by Lemma 2 in [11] we know that for any F_t-measurable matrix $M(t)$,

$$\sum_{i=0}^{t} M(i)e(i+1) = O\left([\, \sum_{i=0}^{t} \|M(i)\|^{2}]^{\eta+\frac{1}{2}}\right), \quad \text{a.s.} \quad \forall\, \eta > 0 \tag{A8}$$

Set

$$\eta(t) = E[z(t+1) \mid F_t] - \hat{z}(t+1,\hat{\theta}(t)) \tag{A9}$$

Obviously, $\eta(t)$ is F_t-measurable, and by (A1), (A9) it follows from (3.1) that

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - P(t)\psi(t)[\eta(t)^\tau + e(t+1)^\tau], \quad \forall\theta \tag{A10}$$

Then by using (A8) and (A10) similar to the proof of (22) in [1, pp.1463] we have

$$\left| \sum_{i=1}^{t} e(i+1)^\tau\tilde{\theta}(i+1)^\tau\psi(i) \right| = O\left([\sum_{i=0}^{t} \|\tilde{\theta}(i+1)^\tau\psi(i)\|^2]^{\eta+\frac{1}{2}} \right) + \\ +O\left(\sum_{i=0}^{t} \psi(i)^\tau P(i)\psi(i)\|e(i+1)\|^2 \right), \quad \forall\ \eta>0. \tag{A11}$$

But, by (A3) and both (29) and (30) in [1, pp.1465] we know that

$$\sum_{i=0}^{t} \psi(i)^\tau P(i)\psi(i)\|e(i+1)\|^2 = O(\log^{1+\alpha\delta(\beta-2)}\lambda_{max}(t)) \tag{A12}$$

Finally, putting (A11) and (A12) into (A7) and taking $\eta < \frac{1}{2}$, we see that for any $\alpha > 0$ and any θ:

$$\mathrm{tr}\ \tilde{\theta}(t+1)^\tau P(t)^{-1}\tilde{\theta}(t+1) \leq O(1) - S_t(\theta,\alpha) + O(\log^{1+\alpha\delta(\beta-2)}\lambda_{max}(t))$$

and the desired result follows from here immediately and therefore the proof of Lemma 3.1 is completed.

Proof of Theorem 2.1(ii)

We now prove the second result (2.15) of Theorem 2.1.

Multiplying $P(t)^{-1}$ on both sides of (2.9) and using (A5) we have

$$P(t)^{-1}\hat{\theta}(t+1) = [P(t)^{-1} - \psi(t)\psi(t)^{\tau}]\hat{\theta}(t) + \psi(t)z(t+1)^{\tau}$$

$$= P(t-1)^{-1}\hat{\theta}(t) + \psi(t)z(t+1)^{\tau}$$

Consequently, from this and (4.2) and (4.10) we know

$$P(t)^{-1}\hat{\theta}(t+1) = P(0)^{-1}\hat{\theta}(1) + \sum_{i=1}^{t} \psi(i)z(i+1)^{\tau}$$

$$= P(0)^{-1}\hat{\theta}(1) + \sum_{i=1}^{t} \psi(i)[(\psi(i)^{\tau}-\psi^{\xi}(i)^{\tau})\bar{\theta} + \bar{w}(i+1)^{\tau} + \eta(i+1)^{\tau}]$$

$$= P(0)^{-1}\hat{\theta}(1) + \sum_{i=1}^{t} \psi(i)\psi(i)^{\tau}\bar{\theta} - \sum_{i=1}^{t} \psi(i)[\psi^{\xi}(i)^{\tau}\bar{\theta} - \eta(i+1)^{\tau}] +$$
$$+ \sum_{i=1}^{t} \psi(i)\bar{w}(i+1)^{\tau}$$

and hence

$$\tilde{\theta}(t+1) = [\sum_{i=1}^{t}\psi(i)\psi(i)^{\tau} + P(0)^{-1}]^{-1} \{P(0)^{-1}\tilde{\theta}(1) -$$
$$- \sum_{i=1}^{t} \psi(i)\bar{w}(i+1)^{\tau} + \sum_{i=1}^{t} \psi(i)[\psi^{\xi}(i)^{\tau}\bar{\theta} - \eta(i+1)^{\tau}] \} \qquad \text{(A13)}$$

Again by (4.2) and (4.10) from (2.12) we know

$$t\,\hat{R}_{\bar{w}}(t) = \sum_{i=0}^{t-1} [\bar{\theta}^{\tau}\psi^{o}(i) + \bar{w}(i+1) + \eta(i+1) - \hat{\theta}(t)^{\tau}\psi(i)]\cdot[\bar{\theta}^{\tau}\psi^{o}(i) +$$
$$+ \bar{w}(i+1) + \eta(i+1) - \hat{\theta}(t)^{\tau}\psi(i)]^{\tau}$$

$$= \sum_{i=0}^{t-1} [\tilde{\theta}(t)^{\tau}\psi(i) - \bar{\theta}^{\tau}\psi^{\xi}(i) + \bar{w}(i+1) + \eta(i+1)]\cdot[\tilde{\theta}(t)^{\tau}\psi(i) -$$
$$-\bar{\theta}^{\tau}\psi^{\xi}(i) + \bar{w}(i+1) + \eta(i+1)]^{\tau}$$

$$=\tilde{\theta}(t)^\tau \sum_{i=0}^{t-1} \psi(i)\psi(i)^\tau \tilde{\theta}(t) + \sum_{i=0}^{t-1} [-\bar{\theta}^\tau\psi^\xi(i) + \bar{w}(i+1) + \eta(i+1)]\psi(i)^\tau\tilde{\theta}(t) +$$

$$+ \tilde{\theta}(t)^\tau \sum_{i=0}^{t-1} \psi(i)[-\bar{\theta}^\tau\psi^\xi(i) + \bar{w}(i+1) + \eta(i+1)]^\tau +$$

$$+ \sum_{i=0}^{t-1} [\bar{\theta}^\tau\psi^\xi(i) - \bar{w}(i+1) - \eta(i+1)]\cdot[\bar{\theta}^\tau\psi^\xi(i) - \bar{w}(i+1) - \eta(i+1)]^\tau$$

$$\triangleq S_1(t) + S_2(t) + S_2(t)^\tau + S_3(t) \tag{A14}$$

But, by the Schwarz inequality and (4.9) we get

$$\| [\sum_{j=0}^{t-1}\psi(j)\psi(j)^\tau + P(0)^{-1}]^{-1/2} \sum_{i=0}^{t-1}\psi(i)\psi^\xi(i)^\tau \|^2$$

$$\leq \sum_{i=0}^{t-1} \| [\sum_{j=0}^{t-1}\psi(j)\psi(j)^\tau + P(0)^{-1}]^{-1/2}\psi(i) \|^2 \sum_{i=0}^{t-1} \|\psi^\xi(i)\|^2$$

$$= \mathrm{tr}\ \{ \sum_{i=0}^{t-1} \psi(i)^\tau[\sum_{j=0}^{t-1}\psi(j)\psi(j)^\tau + P(0)^{-1}]^{-1}\psi(i)\ \} \sum_{i=0}^{t-1} \|\psi^\xi(i)\|^2$$

$$= O(1) \cdot \sum_{i=0}^{t-1} \|\psi^\xi(i)\|^2$$

$$= O\ (\log\lambda_{max}(t-1))$$

and hence

$$\| [\sum_{j=0}^{t-1}\psi(j)\psi(j)^\tau + P(0)^{-1}]^{-1/2} \sum_{i=0}^{t-1}\psi(i)\psi^\xi(i)^\tau \| = O(\sqrt{\log\lambda_{max}(t)}\,) \tag{A15}$$

We need the following estimates for weighted sum of martingale difference sequences [12]. Let $\{X_t,F_t\}$ be an adapted vector sequence and $\{e_t,F_t\}$ be martingale difference sequence with

$$\sup_t E\Big[\ \| e_t \|^{2+\delta} \mid F_{t-1}\ \Big] < \infty \quad (\delta > 0)$$

Then, for $\varepsilon > 0$,

$$\|(\sum_{i=0}^{t} X_i X_i^{\tau}+\varepsilon I)^{1/2} \sum_{i=0}^{t} X_i e_{i+1}^{\tau}\| = O\left([\log\lambda_{max}(\sum_{i=0}^{t} X_i X_i^{\tau} + \varepsilon I)]^{1/2}\right) \quad \text{(A16)}$$

Applying (A15) and (A16) to $S_1(t)$ and $S_2(t)$ defined in (A14) and noting (A13) it is not difficult to show that

$$S_1(t) = O(\log\lambda_{max}(t)), \quad \text{a.s.} \quad \text{(A17)}$$

$$S_2(t) = O(\log\lambda_{max}(t)), \quad \text{a.s.} \quad \text{(A18)}$$

By (4.9) and the inequality (A8) we see that the last term in (A14) can be estimated by

$$S_3(t) = \sum_{i=0}^{t-1} \bar{w}(i+1)\bar{w}(i+1)^{\tau} + O(\log\lambda_{max}(t)), \quad \text{(A19)}$$

By Lemma 3.3, $\{\bar{w}(t), F_t\}$ is a Gaussian martingale difference sequence and $E\bar{w}(t)\bar{w}(t)^{\tau} \to R_{\bar{w}}$ (exponentially fast), and hence by the laws of the iterated logarithm[13], it is not difficult to convince oneself that

$$\frac{1}{t} \sum_{i=0}^{t-1} [\bar{w}(i+1)\bar{w}(i+1)^{\tau} - R_{\bar{w}}] = O\left(\sqrt{\frac{\log\log t}{t}}\right), \quad \text{a.s.} \quad \text{(A20)}$$

Finally, putting (A17)-(A20) into (A14), the result, equation (2.13), follows.

REFERENCES

[1] H.F. Chen and L.Guo, "Convergence rate of least squares identification and adpative control for stochastic systems", **Int.J.Control**, Vol.44,

No.5, 1986, pp.1459-1476.

[2] T.L. Lai and C.Z. Wei, "Extended least squares and their application to adaptive control and prediction in linear systems", **IEEE Trans.Auto.Control**, Vol.AC-31, 1986, pp.898-906.

[3] G. Ledwich and J.B. Moore, "Multivariable self-tuning filters", **Differential Games and Control Theory, II**, Lecture Notes in Pure and Applied Mathematics, Dekker, 1977, pp.345-376.

[4] L. Ljung, "On positive real transfer function and the convergence of some recursive schemes", **IEEE Trans.Auto.Control**, Vol.AC-22, 1977, pp.539-551.

[5] H.F. Chen and L. Guo, "Robustness analysis of identification and adaptive control for stochastic systems", **Systems and Control Letters**, Vol.9, No.2, 1987.

[6] J.B. Moore, "Side-stepping the positive real restriction for stochastic adaptive schemes", **Richerche Di Automatica**, vol.8, No.1, October 1982, pp.501-523.

[7] L. Guo and J.B. Moore, "Adaptive spectral factorization with applications to system identification", 1987, submitted for publication.

[8] J.B. Moore and L. Guo, "On asymptotically efficient linear regression model identification", in preparation.

[9] G.C. Goodwin and K.S. Sin, **Adaptive Filtering, Prediction and**

Control, Prentice- Hall, Inc., Englewood Cliffs, New Jersey, 1984.

[10] B.D.O. Anderson and J.B. Moore, **Optimal Filtering**, Prentice-Hall Inc., Englewood Cliffs, NJ, 1979.

[11] H.F. Chen and L. Guo, "Optimal adaptive control and consistent parameter estimates for ARMAX model with quadratic cost", **SIAM J. Control and Optimization,** Vol.25, No.4, 1987.

[12] T.L. Lai and C.Z. Wei, "On the concept of excitation in least squares identification and adaptive control", **Stochastics**, Vol.16, 1986, pp227-254.

[13] W.F. Stout, **Almost Sure Convergence**, Academic Press, New York, 1974,.

Chapter 11

IDENTIFICATION / PREDICTION ALGORITHMS FOR ARMAX MODELS WITH RELAXED POSITIVE REAL CONDITIONS

1. INTRODUCTION AND BACKGROUND

A classic problem in on-line identification, signal processing and adaptive control is recursive estimation associated with auto-regressive, moving-average exogenous input (ARMAX) models. For simplicity here we restrict affection to scalar variable models

$$A(q^{-1})\, y_k = B(q^{-1})\, u_k + C(q^{-1})\, w_k \tag{1.1}$$

where q^{-1} is the unit delay operator, $D \geq 1$,

$$A(q^{-1}) = \sum_{i=0}^{n} a_i q^{-i},\ \ B(q^{-1}) = \sum_{i=D}^{m} b_i q^{-i},\ C(q^{-1}) = \sum_{i=0}^{l} c_i q^{-i},\ a_0 = c_0 = 1$$

and w_k is zero mean, bounded-variance "white" noise. Note that D is a pure delay (dead-time) of the system and that, for the sake of notational convenience, it was incorporated in the corresponding polynomial. Consequently, there are n and l unknown coefficients in polynomials A and C, respectively, but only m-D+1 coefficients in the B polynomial. For convergence analysis studies based on martingale theory, it is usual to assume

$$E[w_k|F_{k-1}] = 0,\ E[w_k^2|F_{k-1}] \leq \sigma_w^2,\ E[\|w_k\|^\beta\, |F_{k-1}] < \infty, \text{ for some } \beta > 2 \tag{1.2}$$

where F_k is the σ-algebra generated by $w_0, w_1, \ldots w_k, u_0, \ldots u_k$. A special case when (1.1) holds is when there is a stable deterministic system $A(q^{-1})z_k = B(q^{-1})u_k$ with measurements y_k of z_k contaminated with added noise w_k. Thus $y_k = z_k + w_k$. Then (1.1) holds with

$$A(q^{-1}) = C(q^{-1}), \;\; z_k \triangleq y_k - w_k, A(q^{-1})z_k = B(q^{-1})u_k, \tag{1.3}$$

For extended least squares (ELS) based recursive identification schemes applied to models (1.1), a crucial sufficient condition for almost sure noise estimation convergence results is that

$$C^{-1}(z^{-1}) - \frac{1}{2} \quad \text{is Strictly Positive Real (SPR), or equivalently,}$$

$$C(z) - 1 \qquad \text{is Strictly Bounded Real (SBR).} \tag{1.4}$$

This condition first pointed out in [1,2], see also [3,4], also appears to be a necessary one, at least for some noise sample paths [4], and is the focus of attention in this chapter. The condition (1.4) requires that $C^{-1}(z^{-1})$ is asymptotically stable, and in the case (1.3), that $A^{-1}(z^{-1})$ is asymptotically stable. Actually it is known that without loss of generality we can take $C^{-1}(z^{-1})$ stable, unless (1.3) constrains $C(z^{-1}) = A(z^{-1})$.

We consider the Equation Error ELS scheme for the general model (1.1), (1.2) and Output Error ELS scheme for the special case when a (1.1) is constrained as in (1.3). The regression vector for the Equation Error ELS algorithm associated with (1.1) is

$$\phi_k = [y_{k-1} \ldots y_{k-n} \; u_{k-D} \ldots u_{k-m} \; w_{k-1} \ldots w_{k-l}]' \quad \text{(Eq. Err.)} \tag{1.5a}$$

denoted also ϕ_k^{EE}. For the Output Error ELS algorithm under (1.1)-(1.3), the regression vector is

$$\phi_k = [(y_{k-1}-w_{k-1}) \ldots (y_{k-n}-w_{k-n})\ u_{k-D} \ldots u_{k-m}]' \qquad \text{(Out. Err.)} \qquad (1.5b)$$

also denoted ϕ_k^{OE}. It is known [3] that there is almost sure parameter convergence, if in addition to (1.4) the following persistence of excitation (PE) condition holds for ϕ_k^{EE} or ϕ_k^{OE}, respectively,

$$\lim_{k\to\infty} \frac{\log \lambda_{max} P_k^{-1}}{\lambda_{min} P_k^{-1}} = 0, \ \text{a.s.} \qquad (1.6)$$

where $P_k^{-1} = (\sum_{i=1}^{k} \phi_i\phi_i' + P_0^{-1})$ for some $P_0 > 0$. In order to translate this condition on regressions ϕ_k to conditions on external excitation signals u_k, w_k, it is necessary and sufficient [5] that ϕ_k be reachable from u_k, w_k, or equivalently that for the cases (1.5a) and (1.5b), respectively

$$\mathcal{A}(z),\ \mathcal{B}(z),\ \mathcal{C}(z) \text{ are coprime} \qquad (1.7a)$$

$$\mathcal{A}(z),\ \mathcal{B}(z) \text{ are coprime,} \qquad (1.7b)$$

where $\mathcal{A}(z) \triangleq z^nA(z^{-1})$, $\mathcal{B}(z) \triangleq z^mB(z^{-1})$, $\mathcal{C}(z) \triangleq z^lC(z^{-1})$.

The ad hoc basis behind extended least squares is that state (regression) estimates $\hat{\phi}_k$ are used in lieu of states (regression) ϕ_k to obtain recursive least squares estimates $\hat{\theta}_k$ of parameters θ, and $\hat{\theta}_k$ is used in lieu of θ to obtain minimum variance estimates $\hat{\phi}_k$ of ϕ_k. Thus the formidable task of simultaneously obtaining a (nonlinear) optimal estimate of both θ, and ϕ_k is abandoned in favour of a tractable algorithm. It is not surprising then, that the suboptimal extended least squares scheme fails if the state estimation is too poor, as when the noise is highly colored. The SPR (SBR) condition limits the noise color. It should not be surprising that attempts to relax the SPR (SBR) condition will result in high order calculations, even approaching infinite dimensional calculations as for the optimal (nonlinear) estimation of θ, ϕ_k. Several proposals have been made to relax this condition.

In [6], the overparametrized signal model

$$F_N(q^{-1})A(q^{-1})y_k = F_N(q^{-1})B(q^{-1})u_k + F_N(q^{-1})C(q^{-1})w_k \tag{1.8}$$

is studied where the common factor $F_N(q^{-1}) = \sum_{k=0}^{N-1} f_k q^{-k}$, is not uniquely defined. It turns out that for certain convergence properties to be established, there must exist $F_N(z^{-1})$ such that

$$C^{-1}(z^{-1})F_N^{-1}(z^{-1}) - \frac{1}{2} \quad \text{is SPR, or equivalently,}$$

$$C(z^{-1})F_N(z^{-1}) - 1 \qquad \text{is SBR.} \tag{1.9}$$

It is shown in [6], that for each stable $C^{-1}(z^{-1})$, there exists some $F_N(q^{-1})$ with N suitable large, such that (1.9) holds. The common factor $F_N(q^{-1})$ in (1.8) means that the regression vector

$$\phi_k = [y_{k-1} \ldots y_{k-n-N+1}\ u_{k-D} \ldots u_{k-m-N+1}\ w_{k-1} \ldots w_{k-l-N+1}]' \quad \text{(Ov. Par.)} \tag{1.10}$$

also denoted ϕ_k^{OP}, is not reachable form u_k, w_k, and thus can not be persistently exciting, in general. Moreover, the estimates $\hat{\phi}_k$ can not be persistently exciting so that the calculations are fundamentally ill-conditioned and impractical for application. The convergence theory does give stable estimation error convergence under (1.4), in the same sense as for the nonoverparametrized case under (1.9), although there is never consistent parameter estimation for $N > 1$.

In [7], by taking $N = \infty$ and $F_\infty(q^{-1}) = C^{-1}(q^{-1})$, then the overparametrized model (1.8) is now uniquely parametrized as

$$F_\infty(q^{-1})A(q^{-1})y_k = F_\infty(q^{-1})B(q^{-1})u_k + w_k$$

which is amenable to least squares estimation, albeit infinite dimensional. Now (1.9) with $F_\infty(q^{-1})$ as above is trivially satisfied. Convergence analysis is studied in [7]. In translating excitation requirements on the infinite regressions ϕ_k to the inputs u_k, w_k, for parameter convergence,it turns out that the input excitations must be suitably rich as when containing an infinite set of frequencies. A variation with N a function of k requiring $N \rightarrow \infty$ as $k \rightarrow \infty$ is studied in [8]. Neither of these schemes are practical to implement, but their study is important to give results for the case of a sequence of models $C_i(q^{-1})$ with one or more zeros z_i approaching the unit circle as $i \rightarrow \infty$.

A technique that avoids the SPR condition by dominating the plant colored noise by white noise, although applying to related schemes [9], does not appear to apply to the ARMAX case as claimed in [10].

In this chapter, for equation error identification a uniquely parametrized over-parametrization approach is used which forces the unique selection $F_N(q^{-1}) = F_c(q^{-1})$ where

$$1 = C(q^{-1})F_c(q^{-1}) + q^{-N}G_c(q^{-1}) \qquad (1.11)$$

Here $F_c(q^{-1}) = \sum_{i=0}^{N-1} f_i q^{-i}$ with $f_0 = 1$ is the unique (N-1)th degree truncation of $C^{-1}(q^{-1})$ and $G_c(q^{-1})$ is the unique remainder term. The signal model (1.1) can now be transformed to the unique overparametrized model

$$F_c(q^{-1})A(q^{-1})y_k = F_c(q^{-1})B(q^{-1})u_k - G_c(q^{-1})w_{k-N} + w_k \tag{1.12}$$

Notice that $G_c(q^{-1})$ is of degree l - 1with l coefficients $g_0, g_1, .. g_{l-1}$, so that the degree of overparametrization is less than, by N, that for the signal models (1.8). In other words, here $F_N(q^{-1})$ is constrained so that the coefficients of $w_{k-1}, w_{k-2} ... w_{k-N+1}$ in (1.8) are zero. There is no need then to estimate these known zero coefficients. The associated Transformed Equation Error ELS scheme has a regression vector

$$\phi_k = [y_{k-1}..y_{k-n-N+1}\ u_{k-D}..u_{k-m-N+1}\ w_{k-N}..w_{k-l-N+1}]' \quad \text{(Tr.Eq.Err.)} \tag{1.13}$$

also denoted ϕ_k^{TEE}.

A crucial advantage of working with the signal model form (1.12), is that as shown in the next section there is reachability from u_k, w_k of the regression vector ϕ_k^{TEE} of (1.13) associated with (1.12) under A(z), B(z), C(z) coprime as for ϕ_k^{EE}, see (1.7a). As a consequence, with suitable excitation on u_k, w_k, then the ill-conditioning inherent in the ELS algorithm based on the overparametrized model (1.8) is avoided. Indeed, with appropriate input excitation, there can be consistent estimation of the unique overparametrized model (1.12) parameters, although not directly the parameters of (1.1). However, the parameters of (1.1) can be recovered in an on-line parallel least squares estimation exercise using $\hat{\phi}_k$ from the Transformed Equation Error ELS algorithm associated with (1.12).

The SPR (SBR) condition (1.9) in the case $F_N(z^{-1}) = F_c(z^{-1})$ can be re-organized in terms of the remainder term $G_c(z^{-1})$ as

$$G_c(z^{-1}) \text{ is SBR} \tag{1.14}$$

Not surprisingly, and as shown in a later section, for $C^{-1}(z^{-1})$ asymptotically stable and N sufficiently large, then (1.14) is satisfied. With the zeros of $C(q^{-1})$ approaching the unit circle, then N approaches infinity. Thus apart from this borderline situation, there is inherently an implementation advantage of the finite dimensional methods proposed here relative to the infinite dimensional methods of [7,8].

For output error identification, under (1.3), the relevant Transformed Output Error model studied in this chapter is

$$y_k = G_c(q^{-1})(y_{k-N} - w_{k-N}) + F_c(q^{-1})B(q^{-1})u_k + w_k \tag{1.15}$$

with regressions

$$\phi_k = [(y_{k-N}-w_{k-N}) \dots (y_{k-n-N+1}-w_{k-n-N+1})\ u_{k-D} \dots u_{k-m-N+1}]' \qquad \text{(Tr. Out. Err.)} \tag{1.16}$$

also denoted ϕ_k^{TOE}. There are corresponding advantages of working with this model as with (1.12) for equation error algorithms. The strictly positive real condition (1.9), or equivalently the strictly bounded real condition (1.14) applies. Also a necessary and sufficient condition for reachability of ϕ_k^{TOE} (and $\hat{\phi}_k^{TOE}$) is that A(z) [= C(z)], B(z) are coprime as for ϕ_k^{OE} in (1.7b).

The transformed signal models (1.12), (1.15) proposed for ELS estimation, are intermediate in some sense to the transformed signal model used for recursive D-step-ahead prediction of [11] when D = N, and indeed D-step-ahead prediction schemes also require the convergence condition (1.14) when D = N. This chapter proposes D-step-ahead prediction algorithm under (1.14) for the more general situation when $N \geq D$, and addresses the issue of possible ill-conditioning with such schemes. An approach is also suggested to avoid such ill-conditioning using the key ideas of this chapter.

The modified estimation algorithms and associated theorems are presented in the next section, and the case of D-step-ahead prediction is studied in Section 3. Some extensions to multivariable ARMAX models are given in Section 4. Example studies are presented in Section 5, and conclusions drawn in Section 6.

2. Transformed ELS Algorithms and Theorems

Signal Models: Consider the ARMAX models (1.1), (1.2) [or (1.1) - (1.3)], re-organized as those uniquely parametrized, but overparametrized, models (1.12) [or (1.15)], under (1.10). The measurements are linear in the parameters θ and the relevant regressions ϕ_k of (1.13) [or (1.16)] as

$$y_k = \phi_k'\theta + w_k \tag{2.1}$$

where θ is the vector of parameters associated with the coefficients, suitably arranged, of

$$F_c(q^{-1})A(q^{-1}),\ F_c(q^{-1})B(q^{-1}),\ G_c(q^{-1}) \qquad \text{(Tr. Eq. Err) (2.2a)}$$

$$G_c(q^{-1}),\ F_c(q^{-1})B(q^{-1}) \qquad \text{(Tr. Out. Err) (2.2b)}$$

A posteriori ELS Algorithm : The a posteriori noise estimates $\hat{w}_k$ are given in terms of the most recent parameter and state estimates $\hat{\theta}_k$, $\hat{\phi}_k$ respectively as

$$\hat{w}_k = y_k - \hat{\phi}_k'\hat{\theta}_k \tag{2.3}$$

The state estimates $\hat{\phi}_k$ are obtained by replacing w_{k-i} in ϕ_k by $\hat{w}_{k-i}$, and the parameter estimates $\hat{\theta}_k$ are given by a least squares scheme with $\hat{\phi}_k$ replacing ϕ_k as

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \hat{P}_k\hat{\phi}_k(y_k - \hat{\phi}_k'\hat{\theta}_{k-1}) \tag{2.4}$$

$$\hat{P}_k = \hat{P}_{k-1} - \frac{\hat{P}_{k-1}\hat{\phi}_k\hat{\phi}_k'\hat{P}_{k-1}}{1 + \hat{\phi}_k'\hat{P}_{k-1}\hat{\phi}_k} = (\sum_{i=1}^{k} \hat{\phi}_i\hat{\phi}_i' + \hat{P}_0^{-1})^{-1} , \quad \hat{P}_0 > 0 \tag{2.5}$$

<u>SPR Condition:</u> Using the passivity - martingale convergence theory as in [1] and [12], a convergence condition is that a certain subsystem W is required to be strictly passive. The system W has input $(\tilde{\theta}_k'\phi_k)$ and output $(\frac{1}{2}\tilde{\theta}_k\phi_k + \theta'\tilde{\phi}_k)$ where $\tilde{\theta}_k = \theta - \hat{\theta}_k$, $\tilde{\phi}_k = \phi_k - \hat{\phi}_k$. To exploit this result here we introduce the following lemma.

<u>Lemma 2.1</u> (i) Consider the Transformed Equation Error ELS, or the Transformed Output Error ELS algorithm, and associated subsystem W with input $\tilde{\theta}_k'\hat{\phi}_k$ and output $(\frac{1}{2}\tilde{\theta}_k\hat{\phi}_k + \theta'\tilde{\phi}_k)$. Then the system W is a time invariant linear system with a transfer function dependant only on the noise model parameters $C(q^{-1})$ as $[C(q^{-1})F_c(q^{-1})]^{-1} - \frac{1}{2}$.

(ii) Strict passivity of W is equivalent to (1.9), or equivalently (1.14).

<u>Proof</u> (i) For the Transformed Equation Error ELS, from the definitions of $\tilde{\theta}_k$ and $\tilde{\phi}_k$, and (1.11), (1.12), (2.3)

$$\theta'\tilde{\phi}_k = [C(q^{-1})F_c(q^{-1}) - 1]\,(w_k - \hat{w}_k)$$

$$\hat{w}_k - w_k = y_k - \hat{\theta}_k'\hat{\phi}_k - w_k = \tilde{\theta}_k'\hat{\phi}_k + \theta'\tilde{\phi}_k$$

Thus

$$\theta'\tilde{\phi}_k = [C(q^{-1})F_c(q^{-1})]^{-1}[1 - C(q^{-1})F_c(q^{-1})]\ \tilde{\theta}_k'\hat{\phi}_k$$

and the transfer function of W is

$$[C(q^{-1})F_c(q^{-1})]^{-1}[1 - C(q^{-1})F_c(q^{-1})] + \frac{1}{2} = [C(q^{-1})F_c(q^{-1})]^{-1} - \frac{1}{2}$$

For the Transformed Output Error ELS, a similar argument applies. Note also that W is linear and time invariant, so that part (ii) follows immediately. ΔΔΔ

The following lemma tells us that for N sufficiently large the SPR (SBR) condition is satisfied.

Lemma 2.2 Consider any polynomial

$$C(z^{-1}) = \sum_{i=0}^{l} c_i z^{-i} = \prod_{i=1}^{l} (1 - z_i z^{-1}), \quad \text{with } c_0 = 1 \tag{2.6}$$

such that $|z_i| \leq R < 1$ for all i. Consider also for any N, a polynomial pair $\{F_c(z^{-1}), G_c(z^{-1})\}$ with degrees N-1 and l-1 respectively, defined uniquely by the long division (1.11). Then there exists $N_0(R)$ such that for all $N \geq N_0(R)$

$$\sum_{i=0}^{l-1} |g_i| < 1 \tag{2.7}$$

and the SBR condition (1.14) is satisfied.

Proof: From (1.7), $F_c(z^{-1}) = \sum_{i=0}^{N-1} f_i z^{-i}$ is the truncation of $C^{-1}(z^{-1}) \triangleq \sum_{i=0}^{\infty} f_i z^{-i}$. (Here $c_0 = f_0 = 1$). Also,

$$z^{-N} \sum_{i=0}^{l-1} g_i z^{-i} = C(z^{-1}) \sum_{i=N}^{\infty} f_i z^{-i}$$

or equivalently,

$$g_i = \sum_{j=0}^{i} f_{N+j}\, c_{i-j}\,, \quad i = 1,2, \dots l\text{-}1, \quad g_0 = f_N \tag{2.8}$$

Since (1.14) is implied by (2.7) , we shall look for upper bounds on $|\, f_i \,|$ and $|\, c_i \,|$ to achieve (2.7). Recalling (2.6), and denoting

$$\bar{C}(z^{-1}) \triangleq (1 - Rz^{-1})^l = \sum_{i=0}^{l} \bar{c}_i z^{-i}, \quad \bar{c}_0 = 1, \quad \bar{C}^{-1}(z^{-1}) \triangleq \sum_{i=0}^{\infty} \bar{f}_i z^{-i} \tag{2.9}$$

then, in loose but obvious notation, for all i,

$$|\, c_i \,| = |\, \Sigma\, \Pi(\text{terms } z_j) \,| \le \Sigma\, \Pi(\text{terms } |z_j|) \le \Sigma\, \Pi(\text{terms } R) = |\, \bar{c}_i \,|$$

Likewise $|\, f_i \,| \le |\, \bar{f}_i \,|$ for all i. Now

$$\bar{f}_i = \frac{R^i (i+l)!}{i!} > 0, \quad \sum_{j=0}^{l} |\, \bar{c}_j \,| \le (1 + R)^l < 2^l \tag{2.10}$$

and denoting

$$\bar{g}_i \triangleq \sum_{j=0}^{i} f_{N+j}\, |\, \bar{c}_{i-j} \,|\,, \quad \bar{f}_{max} \triangleq \max\, \{f_{N+j}\} \text{ for } 0 \le j \le 1\text{-}1$$

then

$$\sum_{i=0}^{l-1} \bar{g}_i = \sum_{i=0}^{l-1} \sum_{j=0}^{i} \bar{f}_{N+j}\, |\, \bar{c}_{i-j} \,| \le \bar{f}_{max} \sum_{i=0}^{l-1} \sum_{j=0}^{i} |\, \bar{c}_{i-j} \,| < \bar{f}_{max}\, 2^l \tag{2.11}$$

where the last inequality follows from (2.10). Also from (2.10)

$$\bar{f}_{max} \le R^N (N+2l - 1)! \,/\, (N+l\text{-}1)! < R^N (N+2l - 1)^l \tag{2.12}$$

Thus, for any N such that

$$2R^{\frac{N}{l}} (N+2l - 1) < 1 \tag{2.13}$$

(1.14) is satisfied. Note that $N_0(R)$ can be defined as the smallest value of N such that (2.13) holds. $\Delta\Delta\Delta$

Persistence of Excitation: We are interested in having suitably excited regression vectors ϕ_k and their estimates $\hat{\phi}_k$ for two reasons. First, if $\hat{\phi}_k$ is not suitably excited, then the estimation algorithm, and in particular the calculation of P_k will suffer from numerical ill-conditioning. Second, persistence of excitation of ϕ_k as in [3] is needed to assure the strong consistency of the identification scheme. Here we study excitation of the regression vectors ϕ_k^{TEE}, ϕ_k^{TOE} of (1.13) (1.16) associated with signal models (1.12) and (1.15).

Lemma 2.3 A necessary and sufficient condition for the regression vector ϕ_k^{TEE} of (1.13) associated with signal model (1.12) to be reachable from inputs u_k, w_k is that the coprimeness condition (1.7a) holds. For ϕ_k^{TOE} of (1.16) (1.15), a necessary and sufficient condition for reachability from u_k is that (1.7b) hold.

Proof (i) Here we apply techniques developed in [5]. First let us rewrite the Z-transform of (1.1) as

$$\begin{aligned} y(z) &= \mathcal{A}^{-1}(z)\mathcal{B}(z)z^{n-m}u(z) + \mathcal{A}^{-1}(z)\mathcal{C}(z)z^{n-l}w(z) \\ &= T_1(z)u(z) + T_2(z)w(z) = T(z)v(z) \end{aligned}$$

where $v(z) = [u(z)\ w(z)]'$, $T(z) = [T_1(z)\ T_2(z)]$, and $\mathcal{A}(z) \triangleq z^n A(z^{-1})$, $\mathcal{B}(z) \triangleq z^m B(z^{-1})$, $\mathcal{C}(z) \triangleq z^l C(z^{-1})$. Observe that

$$\begin{bmatrix} y(z) \\ u(z) \\ w(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ e_1 \\ e_2 \end{bmatrix} v(z)$$

where $e_1 = [1 \quad 0]$, $e_2 = [0 \quad 1]$. Consider the regression vector ϕ_k^{TEE} and note that

$$\phi^{TEE}(z) = T^{TEE}(z)\, v(z)$$

$$T^{TEE}(z) = \Big[[z^{-1}T(z)]' \cdot\cdot [z^{-n-N+1}T(z)]' \; [z^{-D}e_1'] \cdot\cdot [z^{-m-N+1}e_1]' \; [z^{-N}e_2]' \cdot\cdot [z^{-l-N+1}e_2]'\Big]'$$

We know [5] that $\phi^{TEE}(\cdot)$ is reachable from $v(\cdot)$ iff $T^{TEE}(z)$ has full row rank over R, for all z. Furthermore manipulations show that for any $\alpha = [\alpha_1' \; \alpha_2' \; \alpha_3']'$, $\alpha_1 = [\alpha_{1,1} \cdot\cdot \alpha_{1,n+N-1}]'$, $\alpha_2 = [\alpha_{2,1} \cdot\cdot \alpha_{1,m+N-D}]'$, $\alpha_3 = [\alpha_{3,1} \cdot\cdot \alpha_{1,l-1}]'$, the condition $\alpha' T^{TEE}(z) = 0$ is equivalent to, with $p = \min\{ n,m,l\}$.

$$\alpha'(z)H(z) \triangleq [\alpha_1(z) \;\; \alpha_2(z) \;\; \alpha_3(z)] \begin{bmatrix} z^{p-n}T_1(z) & z^{p-n}T_2(z) \\ z^{p-m} & 0 \\ 0 & z^{p-l} \end{bmatrix} = 0 \quad (2.14)$$

where $\alpha_1(z) = \alpha_{1,1}z^{n+N-2} + \cdots + \alpha_{1,n+N-1}$, $\alpha_2(z) = \alpha_{2,1}z^{m+N-2} + \cdots + \alpha_{2,m+N-D}$, $\alpha_3(z) = \alpha_{3,1}z^{l-1} + \cdots + \alpha_{3,l}$.

Let N(H) denote the left nullspace of H(z), i.e. the set of all polynomial vectors $\alpha(z)$ obeying (2.14). Define also

$$P(k_1, k_2, k_3) = \{\text{polynomials } [\alpha_1(z) \;\; \alpha_2(z) \;\; \alpha_3(z)]' \text{ with } \deg\alpha_i(z) < k_i\}$$

Then it is immediate from (2.14) that $T^{TEE}(z)$ will be full row rank iff

$$P(n+N-1,\, m+N-D,\, l) \cap N(H) = 0 \qquad (2.15)$$

We will show that a necessary and sufficient condition for (2.15) to hold is that (1.7a) holds.

Necessity: Straightforward calculation shows that

$$[\, -\mathcal{A}(z) \;\; \mathcal{B}(z) \;\; \mathcal{C}(z)\,]H(z) = 0, \text{ i.e. } [\, -\mathcal{A}(z) \;\; \mathcal{B}(z) \;\; \mathcal{C}(z)\,] \in N(H)$$

We have also,

$$[\,-\mathcal{A}(z)\ \ \mathcal{B}(z)\ \ \mathcal{C}(z)\,] \in P(n+1,\, m\text{-}D+1,\, l+1)$$

Thus $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ coprime is a necessary condition for (2.15) to hold.

Sufficiency: Since for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ coprime the vector $[\,-\mathcal{A}(z)\ \ \mathcal{B}(z)\ \ \mathcal{C}(z)\,]'$ forms a basis for N(H), any other element of N(H) can be obtained as

$$[\,\mathcal{A}_1(z)\ \ \mathcal{B}_1(z)\ \ \mathcal{C}_1(z)\,] = \beta(z)[\,-\mathcal{A}(z)\ \ \mathcal{B}(z)\ \ \mathcal{C}(z)\,]$$

where $\deg \beta(z) \geq 1$. However, since $\deg \mathcal{C}_1(z) \geq \deg \mathcal{C}(z) = l+1$, coprimeness of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ implies (2.15).

(ii) Now consider the regression vector ϕ_k^{TOE}. Here following the approach of (i) above.

$$[y(z) - w(z)] = \mathcal{A}^{-1}(z)\mathcal{B}(z)z^{n-m}u(z) = T(z)u(z), \quad \phi^{TOE}(z) = T^{TOE}(z)u(z)$$

where $T^{TOE}(z) = \left[z^{-N}T(z) \cdots z^{-n-N+1}T(z)\ z^{-D} \cdots z^{-m-N+1}\right]'$, Thus

$$\alpha' T^{TOE}(z) = 0 \iff \alpha'(z)H(z) \triangleq [\alpha_1(z)\ \ \alpha_2(z)]\begin{bmatrix} z^{p-n}T(z) \\ z^{p-m} \end{bmatrix} = 0$$

where $\alpha_1(z) = \alpha_{1,1}z^{n-1} + \cdots + \alpha_{1,n}$, $\alpha_2(z) = \alpha_{2,1}z^{m+N-D-1} + \cdots + \alpha_{2,m+N-D}$. Thus $T^{TOE}(z)$ will be full row rank iff $P(n, m+N\text{-}D) \cap N(H) = 0$, which in turn holds iff (1.7b) holds since $[\,-\mathcal{A}(z)\ \ \mathcal{B}(z)]'H(z) = 0$, using arguments as in (i).

$\Delta\Delta\Delta$

<u>Convergence of the Transformed ELS Algorithms</u> Standard techniques (see e.g. [3] and [13]) apply to achieve convergence properties of the Transformed ELS

algorithms. We summarize as the following lemma.

Lemma 2.4 Consider the Transformed Equation Error ELS [or Transformed Output Error ELS] algorithm (2.3) - (2.5) associated with the signal model (1.12) [or with the signal model (1.15)]. If the N is chosen sufficiently large such the SBR condition (1.14) is satisfied, then, as $k \to \infty$,

$$\| \theta - \hat{\theta}_k \|^2 = O(\frac{\log \lambda_{max}\hat{P}_k^{-1}}{\lambda_{min}\hat{P}_k^{-1}}) \quad \text{a.s.} \tag{2.16a}$$

$$\sum_{i=1}^{k} \| w_i - \hat{w}_i \|^2 = O(\log \lambda_{max}\hat{P}_k^{-1}) \quad \text{a.s.} \tag{2.16b}$$

Moreover, under (1.6), with $P_k^{-1} = \sum_{k=1}^{r} \phi_i\phi_i' + P_0^{-1}$ defined in terms of ϕ^{TEE} [or ϕ^{TOE}], then as $k \to \infty$,

$$\| \theta - \hat{\theta}_k \|^2 = O(\frac{\log \lambda_{max}P_k^{-1}}{\lambda_{min}P_k^{-1}}) \to 0 \quad \text{a.s.} \tag{2.16c}$$

$$\sum_{i=1}^{k} \| w_i - \hat{w}_i \|^2 = O(\log \lambda_{max}P_k^{-1}) \quad \text{a.s.} \tag{2.16d}$$

Furthermore, under (1.7a) [or (1.7b)] ϕ_k is reachable from inputs u_k, w_k [or u_k] and for suitably rich bounded variance inputs u_k, w_k [or u_k] then as $k \to \infty$, $\lim\inf \lambda_{min}P_k^{-1}/k > 0$ and for stable signal models with bounded inputs.

$$\| \theta - \hat{\theta}_k \|^2 = O[k^{-1}\log k] \quad \text{a.s.} \tag{2.16e}$$

Proof The results (2.16a)-(2.16d) are implicitly established in the proof of the theorems of [3]. Of course, the focus of [3] is a signal model with different interpretations for θ, ϕ_k than here (namely the signal specialization of the model used here when N=1), and thus a different transfer function required to be SPR, but in fact the proofs are invariant of such interpretation as long as the subsystem with input $(\tilde{\theta}_k'\hat{\phi}_k)$ and output $(\theta'\tilde{\phi}_k + \frac{1}{2}\tilde{\theta}_k'\hat{\phi}_k)$ be strictly passive. The second part follows immediately via persistence of excitation results in [5].

Least Squares Parameter Recovery So far, we have succeeded in showing that consistent estimates of parameters of the transformed signal models (1.12) or (1.15) can be obtained under the relaxed SBR condition (1.14) and PE condition (1.6). This may be helpful in some situations, e.g. the self-tuning minimum variance regulator for (1.1) can be based on the overparametrized model (1.12). In the next step we will show that identification of the original signal model parameters A, B and C can be also accomplished under the same condition if the following LS algorithm, operating in parallel to the ELS algorithm, is utilized.

$$\bar{\theta}_k = \bar{\theta}_{k-1} + \bar{P}_k\bar{\phi}_k(y_k - \bar{\phi}_k'\bar{\theta}_{k-1}) \tag{2.17a}$$

$$\bar{P}_k = \bar{P}_{k-1} - \frac{\bar{P}_{k-1}\bar{\phi}_k\bar{\phi}_k'\bar{P}_{k-1}}{1 + \bar{\phi}_k'\bar{P}_{k-1}\bar{\phi}_k} = (\sum_{i=1}^{k} \bar{\phi}_i\bar{\phi}_i' + \bar{P}_0^{-1})^{-1}, \quad \hat{P}_0 > 0 \tag{2.17b}$$

where the regression vector $\bar{\phi}_k$ is given in terms of y_k, u_k, and the noise estimate $\hat{w}_k$ from the relevant Transformed ELS scheme, as

$$\bar{\phi}_k \triangleq [\, y_{k-1} \dots y_{k-n} \;\; u_{k-D} \dots u_{k-m} \;\; \hat{w}_{k-1} \dots \hat{w}_{k-l} \,]' \qquad \text{(Eq. Err.)} \tag{2.18a}$$

$$\bar{\phi}_k = [(y_{k-1}-\hat{w}_{k-1}) \dots (y_{k-n}-\hat{w}_{k-n}) \;\; u_{k-D} \dots u_{k-m}]', \qquad \text{(Out. Err.)} \tag{2.18b}$$

Also $\bar{\theta}$ are the parameters suitably arranged from the coefficients of the polynomials

$A(z^{-1}),\ B(z^{-1}),\ C(z^{-1})$ (Eq. Err.)

$A(z^{-1}),\ B(z^{-1})$ (Out. Err.)

Note that the noise terms $\hat{w}_k$ in the regression vector (2.18) are regarded as measurable. For this reason the algorithm (2.17), despite its similarity to

(2.4),(2.5), has an almost standard least squares form. The only nonstandard feature of the proposed scheme is the fact that the regression vector (2.18) differs from the "true" one as in (1.5). We will show that as long as the ELS algorithm converges the discrepancy between $\bar{\phi}_k$ and ϕ_k as in (1.5) is asymptotically negligible, i.e. it does not affect either consistency or the asymptotic rate of convergence of the LS scheme.

<u>Lemma 2.5</u> Consider the least squares algorithm (2.17) with the signal model (1.1) [or (1.3)] under the relaxed SPR condition (1.14), where $\hat{w}_k$ is generated from the ELS algorithm (2.3)-(2.5). Then as $k \to \infty$,

$$\| \bar{\theta} - \bar{\theta}_k \|^2 = O\left(\frac{\log \lambda_{max} \bar{P}_k^{-1}}{\lambda_{min} \bar{P}_k^{-1}}\right) \quad \text{a.s.} \tag{2.19}$$

Moreover, under (1.6) with P_k in terms of ϕ^{TEE} [or ϕ^{TOE}], then as $k \to \infty$,

$$\| \bar{\theta} - \bar{\theta}_k \|^2 \leq O\left(\frac{\log \lambda_{max} P_k^{-1}}{\lambda_{min} P_k^{-1}}\right) \quad \text{a.s.} \tag{2.20}$$

Furthermore, for suitably rich bounded variance inputs u_k, w_k [or u_k] then as $k \to \infty$, $\lim\inf \lambda_{min} P_k^{-1}/k > 0$ and for stable signal models with bounded inputs.

$$\| \bar{\theta} - \bar{\theta}_k \|^2 \leq O[k^{-1} \log k] \quad \text{a.s.} \tag{2.21}$$

<u>Proof</u> Let us rewrite the signal model (1.1) [or (1.3)] as

$$y_k = \phi_k' \bar{\theta} + w_k = \bar{\phi}_k' \bar{\theta} + \tilde{\phi}_k' \bar{\theta} + w_k, \qquad \tilde{\phi}_k \triangleq \phi_k - \bar{\phi}_k \tag{2.22}$$

Then from (2.17), (2.22)

$$(\bar{\theta} - \bar{\theta}_k) = \bar{P}_k [\sum_{i=1}^{k} \bar{\phi}_i (\bar{\phi}_i' \bar{\theta} - y_i) + \bar{P}_0^{-1} (\bar{\theta} - \bar{\theta}_0)]$$

$$= - \bar{P}_k[\sum_{i=1}^{k} \bar{\phi}_i w_i - \bar{P}_0^{-1}(\bar{\theta} - \bar{\theta}_0)] + \bar{P}_k \sum_{i=1}^{k} \bar{\phi}_i \tilde{\phi}_i' \bar{\theta} \tag{2.23}$$

The first right hand side term above is the least squares parameter estimation error associated with the model $y_k = \bar{\phi}_k'\bar{\theta} + w_k$, so that by known results [14, 3] (in fact an appropriate specialization of Lemma 2.4), then

$$\| \bar{P}_k[\sum_{i=1}^{k} \bar{\phi}_i w_i - \bar{P}_0^{-1}(\bar{\theta} - \bar{\theta}_0)] \|^2 = O(\frac{\log \lambda_{max} \bar{P}_k^{-1}}{\lambda_{min} \bar{P}_k^{-1}}) \tag{2.24}$$

The second right hand side term of (2.23) is bounded as follows,

$$\|\bar{P}_k \sum_{i=1}^{k} \bar{\phi}_i \tilde{\phi}_i' \bar{\theta} \|^2 = \|\bar{P}_k^{\frac{1}{2}} \sum_{i=1}^{k} \bar{P}_k^{\frac{1}{2}} \bar{\phi}_i \tilde{\phi}_i' \bar{\theta} \|^2 \leq \lambda_{max}(\bar{P}_k) \| \sum_{i=1}^{k} \bar{P}_k^{\frac{1}{2}} \bar{\phi}_i \tilde{\phi}_i' \bar{\theta} \|^2 \tag{2.25}$$

Now from the Schwarz inequality,

$$\| \sum_{i=1}^{k} \bar{P}_k^{\frac{1}{2}} \bar{\phi}_i \tilde{\phi}_i' \bar{\theta} \|^2 \leq \sum_{i=1}^{k} \|\bar{P}_k^{\frac{1}{2}} \bar{\phi}_i \|^2 \cdot \sum_{i=1}^{k} \| \tilde{\phi}_i' \bar{\theta} \|^2 \tag{2.26}$$

where

$$\sum_{i=1}^{k} \|\bar{P}_k^{\frac{1}{2}} \bar{\phi}_i \|^2 = \sum_{i=1}^{k} \bar{\phi}_i' \bar{P}_k \bar{\phi}_i = \mathrm{tr}\, [\bar{P}_k \sum_{i=1}^{k} \bar{\phi}_i \bar{\phi}_i'] < \dim(\bar{\phi}) \tag{2.27}$$

$$\sum_{i=1}^{k} \| \tilde{\phi}_i' \bar{\theta} \|^2 = O(\log \lambda_{max} \hat{P}_k^{-1}) \qquad \text{[via (2.16b)]} \tag{2.28}$$

Thus (2.25)-(2.28) imply that

$$\|\bar{P}_k \sum_{i=1}^{k} \bar{\phi}_i \tilde{\phi}_i' \bar{\theta} \|^2 = O(\frac{\log \lambda_{max} \hat{P}_k^{-1}}{\lambda_{min} \bar{P}_k^{-1}}) \tag{2.29}$$

Now

$$\lambda_{max}\, \hat{P}_k^{-1} = O(\sum_{i=1}^{k} y_i^2 + \sum_{i=1}^{k} u_i^2 + \sum_{i=1}^{k} \hat{w}_k^2) = O(\lambda_{max}\, \bar{P}_k^{-1}) \tag{2.30}$$

so that substitution (2.24), (2.29) and (2.30) into (2.23) gives the desired result (2.19). Moreover, under (1.6) with P_k in terms of ϕ^{TEE} [or ϕ^{TOE}], we have [3]

$$\frac{\log \lambda_{max}\hat{P}_k^{-1}}{\lambda_{min}\hat{P}_k^{-1}} = O(\frac{\log \lambda_{max}P_k^{-1}}{\lambda_{min}P_k^{-1}}) \rightarrow 0\,, \quad \text{as } k \rightarrow \infty \tag{2.31}$$

Furthermore, recalling the definition for $\hat{\phi}_k$ and $\bar{\phi}_k$, we know that $\bar{\phi}_{k-N+1}$ is a subvector of $\hat{\phi}_k$. Thus $\bar{P}_{k-N+1}$ is obtained by deleting appropriate rows and columns of $\hat{P}_k$ and thus (since $\lambda_{min}(X) = \min \alpha'X\alpha/\alpha'\alpha$, over α)

$$\lambda_{min}\, \hat{P}_k^{-1} \leq \lambda_{min}\, \bar{P}_{k-N+1}^{-1} + O(1) \leq \lambda_{min}\, \bar{P}_k^{-1} + O(1) \tag{2.32}$$

where O(1) accounts for bounded terms due to initial conditions. Thus, we have from (2.31) (2.32),

$$\frac{\log \lambda_{max}\bar{P}_k^{-1}}{\lambda_{min}\bar{P}_k^{-1}} \leq O(\frac{\log \lambda_{max}\hat{P}_k^{-1}}{\lambda_{min}\hat{P}_k^{-1}}) = O(\frac{\log \lambda_{max}P_k^{-1}}{\lambda_{min}P_k^{-1}}) \rightarrow 0, \text{ as } k \rightarrow \infty$$

and the result (2.20) follows. To establish the result (2.21) the same argument as in the proof of Lemma 2.4 applies. $\Delta\Delta\Delta$

3. D-Step-Ahead Prediction

Signal Model: Let us consider again the signal model (1.1), re-organized via (1.11) as (1.12). Consider also the long division of $C(q^{-1})$ by $A(q^{-1})$ as

$$C(q^{-1}) = A(q^{-1})F_a(q^{-1}) + q^{-D}G_a(q^{-1}) \tag{3.1}$$

where $F_a(q^{-1})$ is of degree (D-1) and $G_a(q^{-1})$ is of degree (r-1) with $r = \max\{l-D+1,n\}$. Also define

$$d_k = F_a(q^{-1})w_k \tag{3.2}$$

Now post multiplication of (1.12) by $F_a(q^{-1})$, and re-organization leads to the D-step-ahead Equation Error ELS model,

$$y_k=G_c(q^{-1})(y_{k-N}-d_{k-N})+F_a(q^{-1})F_c(q^{-1})B(q^{-1})u_k+G_a(q^{-1})F_c(q^{-1})y_{k-D}+d_k \tag{3.3}$$

Note that although the error term d_k in (3.3) is not white any more (as in the case of one-step-ahead prediction) it is the moving average process of order D-1. Consequently, for $N \geq D$, d_k is orthogonal to all other terms appearing on the right hand side of (3.3). Due to this fundamental property the model (3.3) can be used for adaptive D-step-ahead prediction. The corresponding D-step-ahead Output Error ELS model is

$$y_k = G_c(q^{-1})(y_{k-N} - w_{k-N}) + F_c(q^{-1})B(q^{-1})u_k + w_k \tag{3.4}$$

The associated regressors are,

$$\phi_k=[(y_{k-N}-d_{k-N})..(y_{k-N-l+1}-d_{k-N-l+1})\ u_{k-D}..u_{k-m-N-D+2}\ y_{k-D}..y_{k-r-N-D+2}]'$$

(D-step Eq. Err.) (3.5a)

$$\phi_k = [(y_{k-N} - w_{k-N}) .. (y_{k-n-N+1} - w_{k-n-N+1})\ u_{k-D} .. u_{k-m-N+1}]'$$

(N-step Out. Err.) (3.5b)

also denoted ϕ_k^{DEE}, ϕ_k^{DOE}, respectively.

The signal models can then be expressed in the form (2.1) where θ is the vector of parameters associated with the coefficients, suitably arranged, of

$$G_c(q^{-1}),\ F_a(q^{-1})F_c(q^{-1})B(q^{-1}),\ G_a(q^{-1})F_c(q^{-1}) \quad \text{(D-step Eq. Err.)} \quad (3.6a)$$

$$G_c(q^{-1}),\ F_c(q^{-1})B(q^{-1}). \quad \text{(D-step Out. Err.)} \quad (3.6b)$$

Of course when D=1, the signal models of previous sections are recovered. Also with N=D, the N-step-ahead prediction signal model of [11] is recovered.

<u>SPR Condition</u> Applying the ELS algorithm to the above D-step ahead models, the subsystem with input $(\tilde{\theta}_k'\hat{\phi}_k)$ and output $(\frac{1}{2}\tilde{\theta}_k\hat{\phi}_k + \theta'\tilde{\phi}_k)$ is readily calculated, as in the proof of Lemma 2.1, via the key intermediate results $\tilde{\theta}_k\hat{\phi}_k + \theta'\tilde{\phi}_k = \hat{d}_k - d_k$, $\theta'\tilde{\phi}_k = G_c(q^{-1})q^{-N}(\hat{d}_k - d_k)$, to be $W = [C(z^{-1})F_c(z^{-1})]^{-1} - \frac{1}{2}$. The key convergence condition for the D-step-ahead ELS algorithm is that this system W be SPR, or equivalent (1.14) hold, as for the 1-step-ahead ELS algorithms of the previous sections. Lemma 2.2 applies to this case also.

<u>Persistence of Excitation</u> We consider first the D-step Equation Error ELS scheme.

<u>Lemma 3.1</u> (i) The regression vector ϕ_k^{DEE} given by (3.5a) is reachable from u_k, w_k iff the following conditions are satisfied (for $N \geq D$)

$$\mathcal{A}(z),\ \mathcal{B}(z),\ \mathcal{C}(z) \quad \text{are coprime} \quad (3.8a)$$

$$\mathcal{F}_a(z),\ \mathcal{C}(z) \quad \text{are coprime} \quad (3.8b)$$

(ii) The regression vector ϕ_k^{DOE} given by (3.5b) is reachable from u_k iff

$$\mathcal{A}(z),\ \mathcal{B}(z) \quad \text{are coprime} \quad (3.9)$$

<u>Proof</u> (i) From the signal model (1.1), and the relations (3.1), (3.2), after a

few manipulations, we have

$$y(z) = z^{n-m}\frac{\mathcal{B}(z)}{\mathcal{A}(z)}u(z) + z^{n-l}\frac{\mathcal{C}(z)}{\mathcal{A}(z)}w(z) \triangleq T_1(z)u(z) + T_2(z)w(z)$$

$$y(z) - d(z) = z^{n-m}\frac{\mathcal{B}(z)}{\mathcal{A}(z)}u(z) + z^{-D+n-r+1}\frac{\mathcal{G}_a(z)}{\mathcal{A}(z)}w(z)$$

$$\triangleq T_1(z)u(z) + T_3(z)w(z), \quad d(z) = z^{-D+1}\mathcal{F}_a(z)w(z) \tag{3.10}$$

where $\mathcal{A}(z) = z^{n-m}A(z^{-1})$, $\mathcal{B}(z) = z^mB(z^{-1})$, $\mathcal{C}(z) = z^lC(z^{-1})$, $\mathcal{F}_a(z) = z^{-D+1}F_a(z^{-1})$, $\mathcal{G}_a(z) = z^{r-1}G_a(z^{-1})$. Then for ϕ_k in (3.5a)

$$\phi(z) = \begin{bmatrix} z^{-N}T_1(z) & z^{-N}T_3(z) \\ \vdots & \vdots \\ z^{-l+1}z^{-N}T_1(z) & z^{-l+1}z^{-N}T_3(z) \\ z^{-D} & 0 \\ \vdots & \vdots \\ z^{-m-N+2}z^{-D} & 0 \\ z^{-D}T_1(z) & z^{-D}T_2(z) \\ \vdots & \vdots \\ z^{-r-N+2}z^{-D}T_1(z) & z^{-r-N+2}z^{-D}T_2(z) \end{bmatrix} \begin{bmatrix} u(z) \\ w(z) \end{bmatrix} \tag{3.11}$$

and a necessary and sufficient condition for $\phi(z)$ to be reachable from u(z), w(z) is

$$P(l,\ m+N-1,\ r+N-1) \cap N(H) = 0 \tag{3.12}$$

where the definitions for $P(\cdot)$ and N(H) are the same as in the proof of Lemma 2.3, and the transfer function matrix H(z) is given, with $p = \min\{l, m+N-1, r+N-1\}$ by

$$H(z) = \begin{bmatrix} z^{p-l}z^{-N}T_1(z) & z^{p-l}z^{-N}T_3(z) \\ z^{p-m-N+1} & 0 \\ z^{p-r-N+1}z^{-D}T_1(z) & z^{p-r-N+1}z^{-D}T_2(z) \end{bmatrix} \tag{3.13}$$

Now we see that, from (3.1), (3.9),

$$\left[\mathcal{C}(z) \quad -\mathcal{F}_a(z)\mathcal{B}(z) \quad -\mathcal{G}_a(z)\right] H(z) \tag{3.14}$$

$$=\begin{bmatrix} z^{p-l-N}T_1(z)\mathcal{C}(z) - z^{p-m-N+1-D}\mathcal{F}_a(z)\mathcal{B}(z) - z^{p-r-N+1-D}T_1(z)\mathcal{G}_a(z) \\ z^{p-l-N}T_3(z)\mathcal{C}(z) - z^{p-r-N+1-D}T_2(z)\mathcal{G}_a(z) \end{bmatrix}' = 0$$

and $\begin{bmatrix}\mathcal{C}(z) & -\mathcal{F}_a(z)\mathcal{B}(z) & -\mathcal{G}_a(z)\end{bmatrix} \in P(l+1,\ m-1,\ r)$. Therefore, following the same argument as in the proof of Lemma 2.3, (3.12) holds iff

$$\mathcal{C}(z),\ -\mathcal{F}_a(z)\mathcal{B}(z),\ -\mathcal{G}_a(z) \text{ are coprime} \tag{3.15}$$

Moreover, (3.15) is equivalent to

$\mathcal{A}(z),\ \mathcal{B}(z),\ \mathcal{C}(z)$ are coprime

$\mathcal{F}_a(z),\ \mathcal{C}(z)$ are coprime

and the result is established.

(ii) To prove the result (3.9) the similar outline can be followed. Here we do not give the details but only the main equations as follows.

$$\phi(z)=\begin{bmatrix} z^{-N}T(z) \\ \vdots \\ z^{-n-N+1}T(z) \\ z^{-D} \\ \vdots \\ z^{-m-N+1} \end{bmatrix} u(z),\ T(z)=z^{n-m}\frac{\mathcal{A}(z)}{\mathcal{B}(z)} \quad \text{[corresponding to (3.11)]}$$

$$P(n,\ N+m-D) \cap N(H) = 0 \quad \text{[corresponding to (3.12)]}$$

$$H(z) = \begin{bmatrix} z^{p-n-N}T(z) \\ z^{p-m-N} \end{bmatrix},\ p = \min\{n,\ N+m-D\} \quad \text{[corresponding to (3.13)]}$$

$$[\ \mathcal{A}(z) \quad -\mathcal{B}(z)]\ H(z) = 0 \quad \text{[corresponding to (3.14)]}$$

Then the (3.9) follows. ΔΔΔ

<u>Remarks</u> 1. The reachability condition (3.8a) is of course identical to (1.7a), the

reachability condition for the other Equation Error ELS methods of Sections 1, 2. The condition (3.8b) is additional, and fails on hypersurfaces in the coefficient space of $A(q^{-1})\, C(q^{-1})$ coefficients. There is then a potential for ill-conditioning in addition to that for the 1-step-ahead Equation Error ELS scheme of Sections 1, 2. The following simple example shows that coprimeness of $\mathcal{F}_a(z)$ and $\mathcal{C}(z)$, is not in general, guaranteed by coprimeness of $\mathcal{A}(z)$ and $\mathcal{C}(z)$.

Example: Consider the case where $C(z^{-1}) = 1 + cz^{-1}$ and $A(z^{-1}) = 1 + az^{-2}$. Observe that

$$1 + cz^{-1} = (1 + cz^{-1})(1 + az^{-2}) - z^{-2}a(1 + cz^{-1})$$

Hence $F_a(z^{-1}) = C(z^{-1})$ for $D = 2$ and all values of a and c. Observe also that the coefficient of z^{-1} in $A(z^{-1})$ given above is zero, hence the coprimeness condition fails on a hypersurface in the coefficient space.

2 The Transformed Equation Error ELS scheme of the previous sections allows identification of $G_c(q^{-1})$, via a converging estimate $\hat{G}_c(q^{-1})$. This knowledge can be used in the D-step Equation Error ELS algorithm to simplify the algorithm —— just replace $G_c(z^{-1})$ by its estimate $\hat{G}_c(z^{-1})$. The regression vector associated with such a model is now

$$\phi_k = [u_{k-D} \dots u_{k-m-N-D+2}\ y_{k-D} \dots y_{k-r-N-D+2}]'$$

which is readily shown to be reachable from u_k, w_k. In this way the ill-conditioning associated with failure of (3.8b) can be obviated.

4. Extension to Multivariable ARMAX Model

Consider a multivariable version of the ARMAX model. In particular, consider the scalar model (1.1) with the scalar output y, input u and noise w variables replaced by the corresponding k_1-, k_2-, and k_1- vectors, respectively and the polynomials A, B, C defined as follows

$$A(q^{-1}) = \sum_{i=0}^{n} A_i q^{-i}, \quad B(q^{-1}) = \sum_{i=D}^{m} B_i q^{-i}, \quad C(q^{-1}) = \sum_{i=0}^{l} C_i q^{-i}, \quad A_0 = C_0 = I$$

Here A_i, B_i, and C_i are $k_1 \times k_1$, $k_1 \times k_2$ and $k_1 \times k_1$ matrices of coefficients, respectively. There is a natural extension of the results obtained so far to this multivariate case.

Let us first review some standard results on identification of multivariate ARMAX models in standard (i.e. non-transformed) form. Define the $k_1 \times (k_1 n + k_2 m + k_1 l)$ coefficient matrix θ and the $(k_1^2 n + k_1 k_2 m + k_1^2 l) \times 1$ regression vector ϕ_k as follows

$$\theta = [A_1 \cdots A_n \; B_D \cdots B_m \; C_1 \cdots C_l],$$

$$\phi_k = [y_{k-1}' \cdots y_{k-n}' \; u_{k-D}' \cdots u_{k-m}' \; w_{k-1}' \cdots w_{k-l}']'$$

and consider the matrix version of the ELS algorithm

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \hat{P}_k \hat{\phi}_k (y_k - \hat{\phi}_k' \hat{\theta}_{k-1}) \tag{4.1}$$

$$\hat{P}_k = (\sum_{i=1}^{k} \hat{\phi}_i \hat{\phi}_i' + \hat{P}_0^{-1})^{-1}, \quad \hat{P}_0 > 0 \tag{4.2}$$

Reachability of ϕ_k is guaranteed by the coprimeness conditions

$$\mathcal{A}(z),\ \mathcal{B}(z),\ \mathcal{C}(z) \text{ are left coprime} \tag{4.3}$$

where left coprimeness is defined as being full row rank for all z, or equivalently where there is no unimodular left coprime factor (a more standard coprimeness definition) and $[A_n\ B_m\ C_l]$ is full row rank, as in [5], [16]. Let us stress the fact that there exist uniquely parametrized multivariate ARMAX models which do not admit representations satisfying (4.3) [16]. complete, although usually less parsimonious, representations (canonical forms) can be obtained by imposing certain structural constraints upon the transfer matrices $\mathcal{A}(z)$, $\mathcal{B}(z)$ and $\mathcal{C}(z)$ (such as the requirement that $\mathcal{A}(z)$ or $\mathcal{C}(z)$ should be diagonal) - see [17] for a more thorough discussion of the identifiability problem for multivariate systems. The convergence of the ELS algorithm (4.1), (4.2) under (4.3) and A_n full row rank condition is studied in [3].

As in the univariate vase, a strict positive real condition, here $[C^{-1}(z^{-1}) - \frac{1}{2}]$ is SPR, is crucial to guarantee almost sure convergence of the algorithm. This condition can be relaxed in exactly the same way by means of overparametrization (all basis conditions and results can be redefined in terms of matrix polynomials). In particular, the reachability conditions for transformed models have exactly the same form as in the scalar case (provided that the word "coprime" is replaced by the "left coprime" which includes the full row rank conditions at z=0). The use of special canonical forms opens another avenue for analysis of multivariate models. For example, by forcing $\mathcal{C}(z)$ to be diagonal one can rewrite the vector ARMAX equation as k_1 separate scalar equations, i.e. the equations for $y_k^{(i)}$, involves only the noise variables $w_{k-n}^{(i)}$ for various n but not the noise variables $w^{(j)}$, $j \neq i$. This allows us to carry out identification of coefficients in each equation separately (by ignoring correlations between different noise inputs). This may be a great advantage from the practical point of view even though it decreases statistical efficiency of the estimates.

5. EXAMPLE STUDIES

First, simulations on a system from [4] are performed. The system certainly fails to satisfy the associated SPR condition, namely (1.4), with the ARMAX representation

$$y_k + 0.9y_{k-1} + 0.95y_{k-2} = u_{k-1} + w_k + 1.5w_{k-1} + 0.75w_{k-2} \qquad (5.1)$$

However, when reformulated as in (1.11) (1.12) for N=6, the associated SPR condition (1.9) is satisfied. Figure 5.1.a shows the estimates of the system parameters via the two stage scheme proposed in this chapter for the case N=6. It is clear that the parameter estimates converge to the correct values, in contrast to the case when N=1, shown in Figure 5.1.b. With N=4, it turns out that the SPR condition is not satisfied, but as indicated in the Figure 5.1.c, there is a performance improvement over the case when N=1, at least in terms of "bias".

Another example is studied, in which the system with ARMAX model as

$$y_k - 0.9y_{k-1} + 0.2y_{k-2} = 1.5u_{k-1} + w_k - 0.6w_{k-1} + 0.1w_{k-2} \qquad (5.2)$$

satisfies the SPR condition. The transformed algorithm is used to identify the system with N=4, and the estimates of the system parameters are given in Figure 5.2.a For comparison, a standard ELS algorithm is also run, with the estimates shown in Figure 5.2.b. From Figure 5.2, we see that when the two stage algorithm of this chapter for N=4 is used, even in the case that the SPR condition (1.4) is satisfied, the transient performance does not deteriorate relative to that for the standard ELS scheme.

To study the effect on relaxation of the SPR condition when N is increasing, we plot, as in Figures 5.3, the region for 2nd order C polynomials in which the SPR condition is satisfied. These show the benefits of working with $N \geq 1$ as far as the SPR condition is concerned.

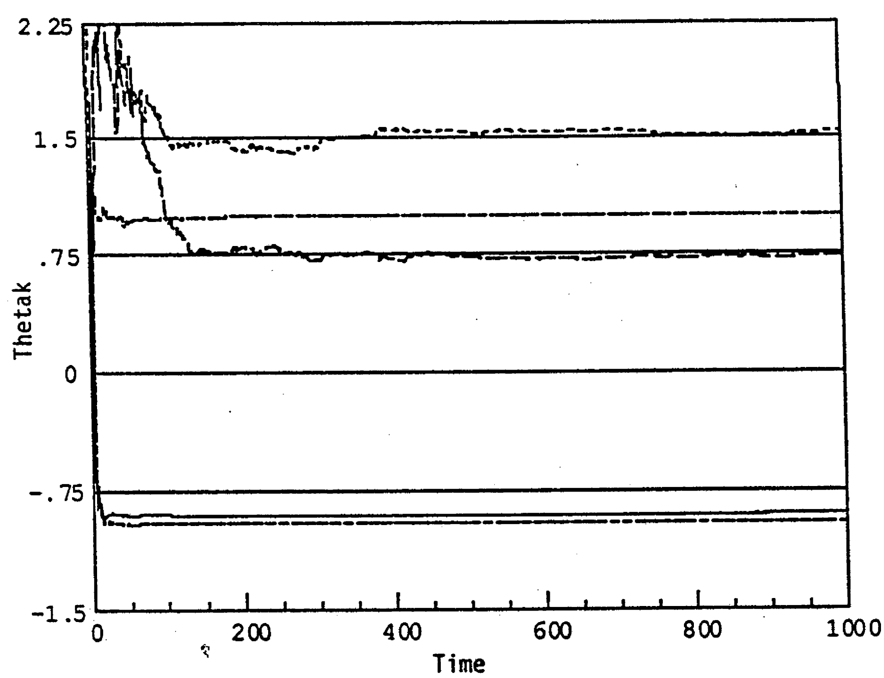


Fig 5.1.a. Comparison on estimates of system (5.1)

(a) transformed algorithm (N=6)

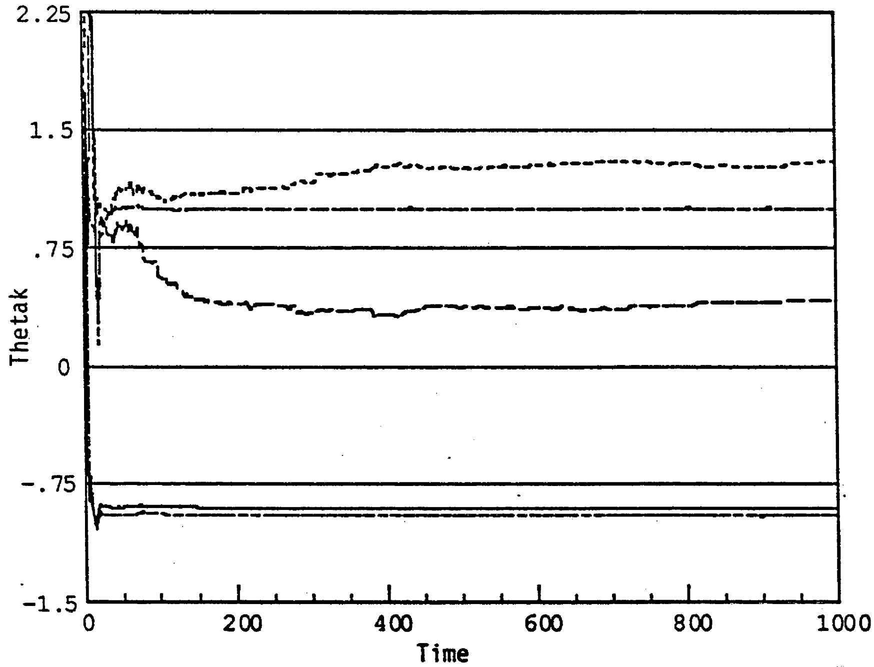


Fig 5.1.b. Comparison on estimates of system (5.1)

(b) standard algorithm (N=1)

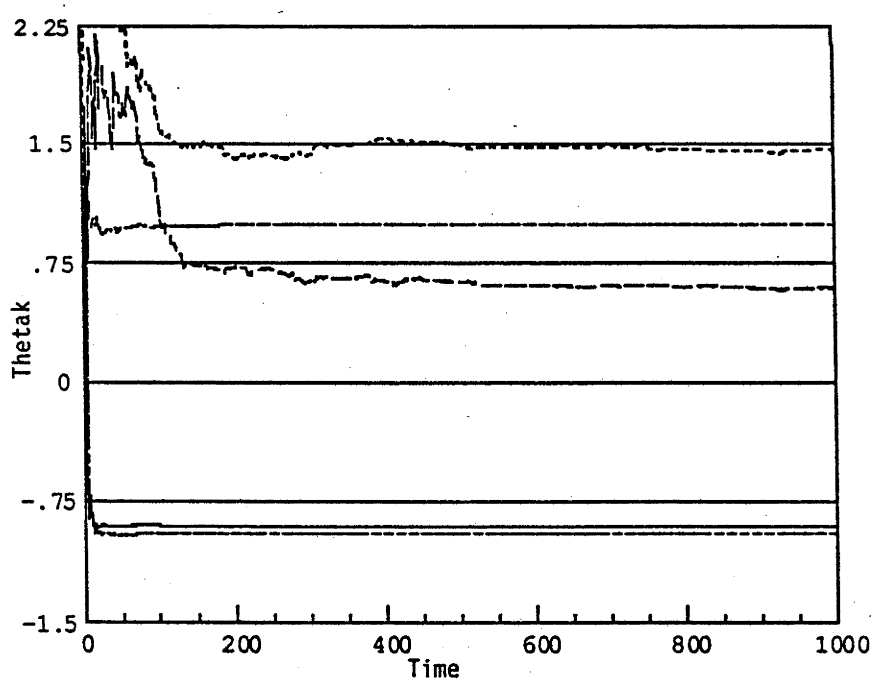


Fig 5.1.c. Comparison on estimates of system (5.1)

(c) transformed algorithm (N=4)

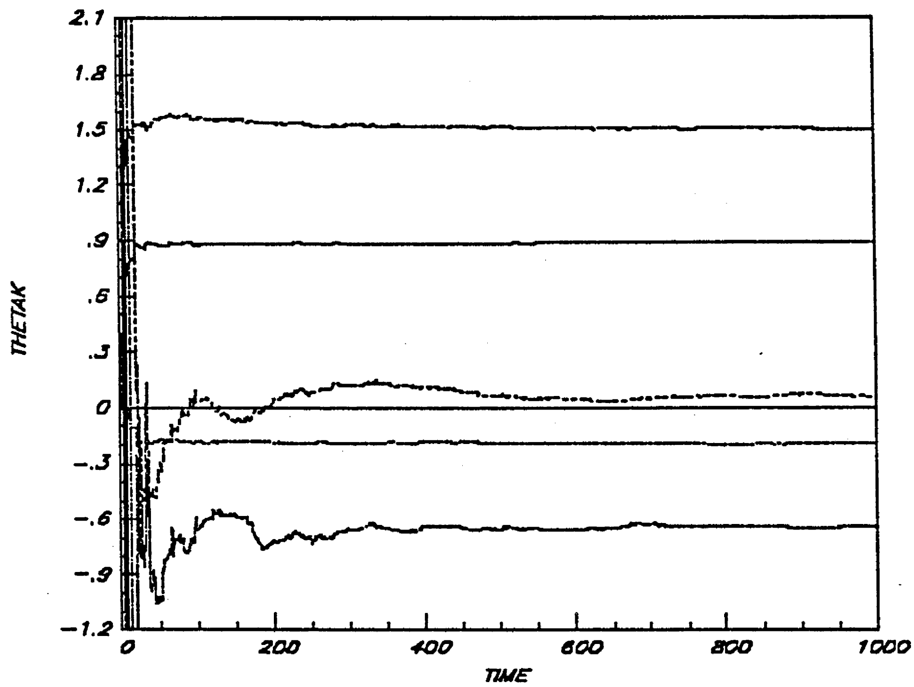


Fig 5.2.a. Comparison on estimates of system (5.2)

(a) transformed algorithm (N=4)

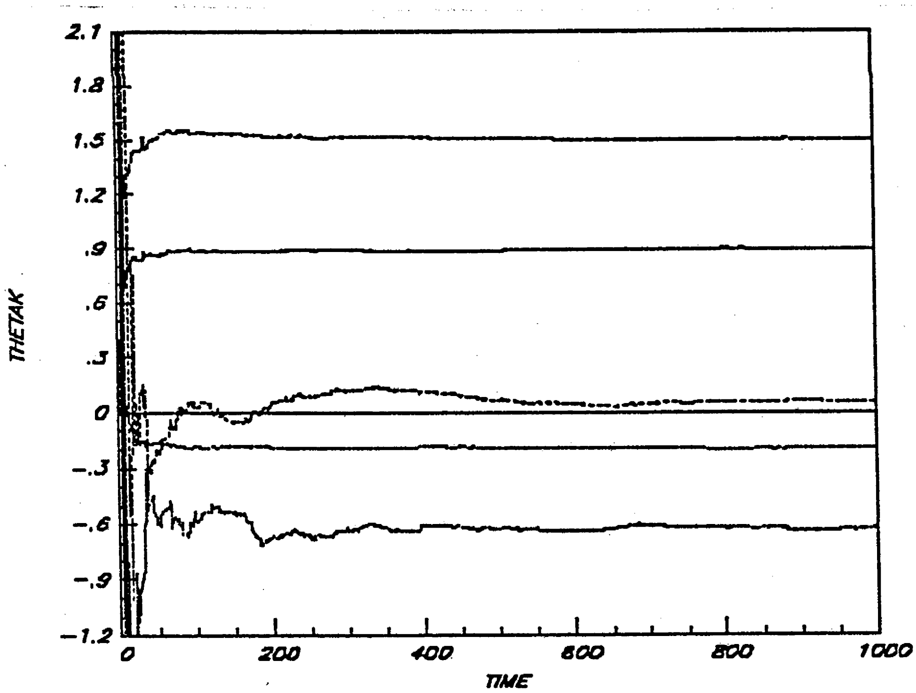


Fig 5.2.b. Comparison on estimates of system (5.2)

(b) standard algorithm (N=1)

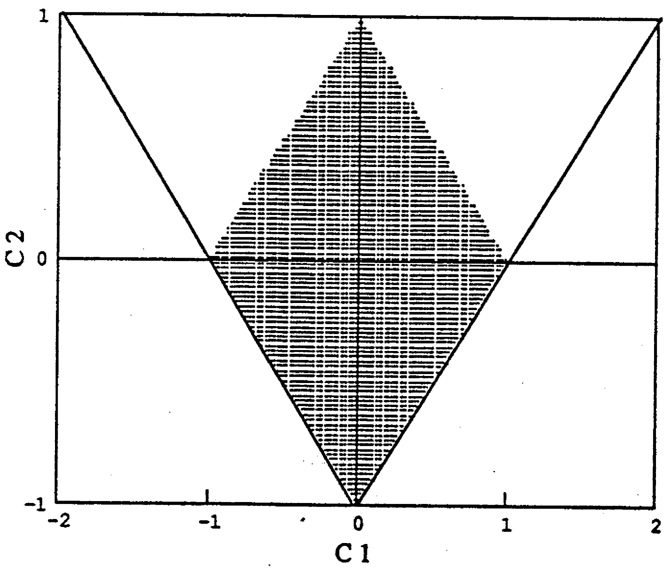


Fig 5.3.a. The SPR regions for 2nd order polynomials with transformed algorithms (N=1)

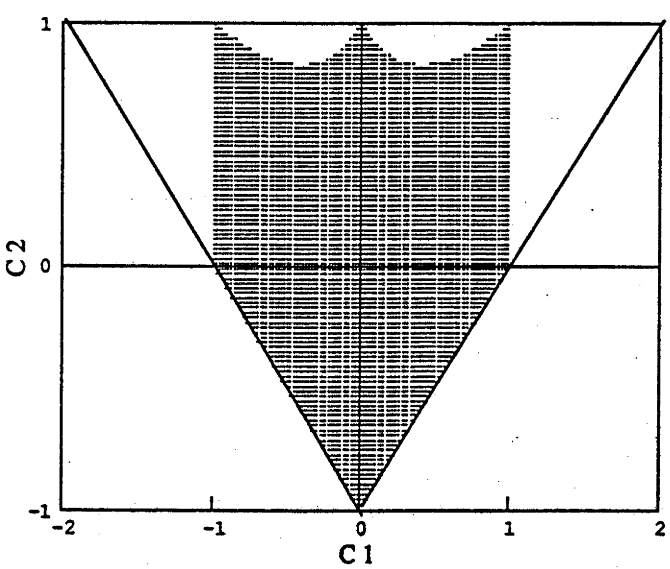


Fig 5.3.b. The SPR regions for 2nd order polynomials with transformed algorithms (N=2)

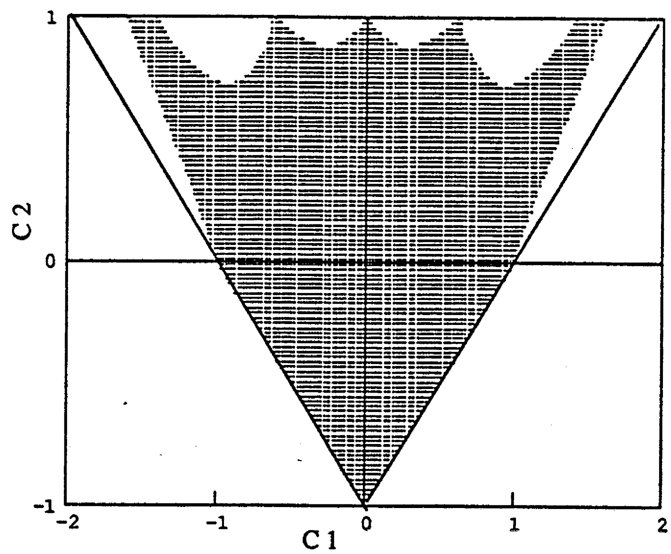


Fig 5.3.c. The SPR regions for 2^{nd} order polynomials with transformed algorithms (N=4)

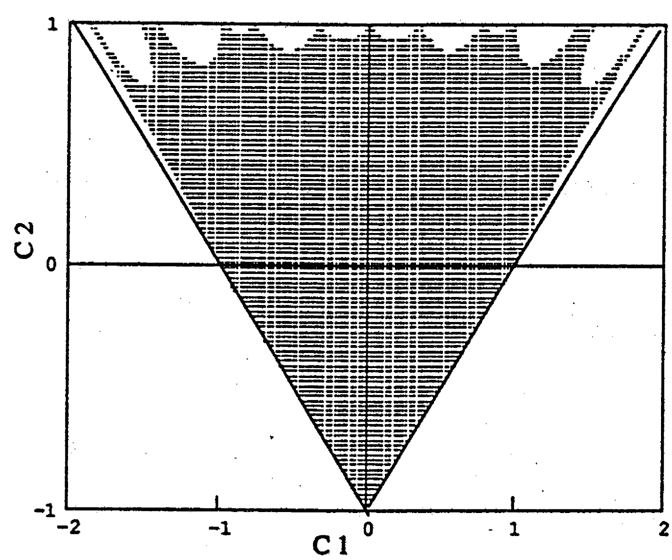


Fig 5.3.d. The SPR regions for 2^{nd} order polynomials with transformed algorithms (N=8)

6. Conclusions

A new method of ELS-based identification and D-step-ahead prediction for ARMAX models is suggested. The proposed algorithms are suitable for both the equation error and output error identification and they converge under relaxed (to an arbitrary degree) positive real conditions associated with standard ELS algorithms. Side-stepping of positive real conditions is achieved by means of transforming the model into an equivalent (uniquely) overparametrized form and additional (optional) LS processing. Maintaining suitable excitation of the regression vectors, necessary for consistent parameter estimation, is achieved under reasonable conditions. Thus numerical ill-conditioning is avoided in the generic case. Computer simulation studies illustrate the comparative attractive performance properties of the algorithm and illustrate the effect of N on the satisfaction of the SPR condition.

References

[1] G. Ledwich and J.B. Moore, "Multivariable self-tuning filters", **Differential Games and Control Theory, II.** Lecture Notes in Pure and Applied Mathematics, Dekker 1977, pp 345-376

[2] L. Ljung, "On positive real functions and the convergence of some recursive schemes", **IEEE Trans. Auto. Control, Vol. AC-22,** 1977, pp 539-551

[3] H. Chen and L. Guo, "Convergence rate of least squares identification and

adaptive control for stochastic systems", **Int. J. Control. Vol. 44**, pp 1459-1476, 1986

[4] L. Ljung, T. Soderstrom and I. Gustavsson, "Counterexamples to general convergence of a commonly used recursive identification method", **IEEE Trans. Auto. Control, Vol. AC-20**, 1975, pp 643-652

[5] M. Green and J.B. Moore, "Persistence of excitation in linear systems", **Systems and Control Letters Vol. 7**, 1986, pp 351-360

[6] S. Shan and G.F. Franklin, "On satisfying strict positive real condition for convergence by overparametrization", **IEEE Trans. Auto. Control, Vol. AC-27**, pp 715-716

[7] L. Guo, D.W. Huang and E.J. Hannan, "On ARX(∞) approximation", (submitted for publication) 1988

[8] L. Guo and D. Huang, "Least squares identification for ARMAX models without the positive real condition", (submitted for publication) 1988

[9] L. Guo, L. Xia and J.B. Moore, "Robust recursive identification of multidimensional linear regression models", **Int. J. Control.** (to appear) 1988

[10] J.B. Moore, "Side-stepping the positive real restriction for stochastic adaptive schemes", **Richerche Di Automatica, Vol.8**, 1982, pp 501-523

[11] R. Kumar and J.B. Moore, "Convergence of adaptive minimum variance

algorithms via weighting coefficient selection", **IEEE Trans. Auto. Control, Vol. AC-27,** 1982, pp 146-153.

[12] J.B. Moore and G. Ledwich, "Multivariable adaptive parameter and states estimators with convergence analysis", **J. Austr. Math. Soc. Vol. 21**, part 2, pp 176-197

[13] G.C. Goodwin and K.S. Sin, "**Adaptive filtering, prediction and control**", Prentice Hall, 1984

[14] T.L. Lai and C.Z. Wei, "Least squares estimation in stochastic regression models with application to identification and control of dynamic systems", **The Annals of Statistics, Vol. 10**, pp 154-166

[15] K.L. Hitz and B.D.O. Anderson, "Discrete positive real functions and their applications in system stability", **Proc. IEE Vol. 116,** pp 153-155, 1969

[16] E. J. Hannan, "The identification problem for multiple equation systems with moving average errors", **Econonmetrica. Vol. 39**, pp 751-765, 1971

[17] R. L. Kashyap and A. R. Rao, "**Dynamic stochastic models from empirical data**", Academic Press 1976.

Chapter 12

TRACKING RANDOMLY VARYING PARAMETERS

1. INTRODUCTION

Let us first define the signal model class and estimation algorithm.

Signal Model: Consider the following linear regression model

$$y_k = \varphi_k^\tau \theta_k + v_k, \quad k \geq 0, \tag{1.1a}$$

$$\theta_{k+1} = F\theta_k + w_{k+1}, \quad E\,\|\theta_o\|^2 < \infty \tag{1.1b}$$

with θ_k viewed as time varying unknown parameters with a Markov model representation. The noise sources $\{w_k\}$ and $\{v_k\}$ are mutually independent and also independent themselves, with zero mean and covariances

$$E[w_{k+1}w_{k+1}^\tau] = Q_w \geq 0, \quad E[v_{k+1}v_{k+1}^\tau] = R_v > 0 \tag{1.2}$$

(Generalization of our theory to the case of time varying covariances is straightforward). The measurement y_k is assumed scalar, and the regression vector φ_k is stochastic and belongs to F_{k-1}— the σ- algebra generated by $\{y_0, y_1, \cdots y_{k-1}\}$.

Much of the work done in stochastic system identification has been concerned with identifying the parameters θ_k in (1.1) for the case when $\theta_k = \theta_0$ is constant, that is, when $F = I$ and the covariances of w_k is zero. Typically, φ_k is viewed as the regression vector of an ARMAX model and least squares identification of θ_0 is applied. When θ_k is time varying, one natural approach to use is to model θ_k as in

(1.1b) where all eigenvalues of F lie in or on the unit circle, i.e., $|\lambda_i(F)| \leq 1$, for all i. In such cases, the natural performance criterion is tracking error bounds.

Estimation Algorithm (Kalman Filter):

Consider the following estimation algorithm associated with (1.1) as

$$\hat{\theta}_{k+1} = F\hat{\theta}_k + \frac{FP_k\varphi_k}{R+\varphi_k^\tau P_k\varphi_k}(y_k - \varphi_k^\tau\hat{\theta}_k) \tag{1.3a}$$

$$P_{k+1} = FP_kF^\tau - \frac{FP_k\varphi_k\varphi_k^\tau P_kF^\tau}{R+\varphi_k^\tau P_k\varphi_k} + Q, \tag{1.3b}$$

where $P_o \geq 0$, $Q > 0$ and $R > 0$ as well as $\hat{\theta}_o$ are deterministic and can be arbitrarily chosen [here Q and R may be regarded as a priori estimates for Q_w and R_v respectively. We stress that even if Q_w is singular, here Q must be chosen as nonsingular to achieve a short memory algorithm. That is the adaptation gain in (1.3a) does not diminish to zero.].

It is known that if the noise source $\{ w_k^\tau, v_k \}$ is a Gaussian white noise sequence, then $\hat{\theta}_k$ generated by (1.3) is the best estimate for θ_k, and P_k is the estimation error covariance, i.e.

$$\hat{\theta}_k = E[\theta_k \mid \mathcal{F}_{k-1}], \quad P_k = E[\tilde{\theta}_k\tilde{\theta}_k^\tau \mid \mathcal{F}_{k-1}],$$

provided that $Q = Q_w$, $R = R_v$, $\hat{\theta}_o = E[\theta_o]$ and $P_o = E[\tilde{\theta}_o\tilde{\theta}_o^\tau]$, where $\tilde{\theta}_k$ is the estimation error:

$$\tilde{\theta}_k = \theta_k - \hat{\theta}_k \tag{1.4}$$

This remarkable result was first observed by Mayne [1] and expanded on by various authors e.g. Astrom and Wittenmark [2], Kitagawa and Gersch [3].

In the nongaussian case, however, the properties of (1.3) applied to (1.1) have not been well studied. The reasons for this may be explained as follows: a). In the time varying case, there is no almost sure parameter convergence. Also, the successful stochastic Lyapunov function technique, as well as the martingale limit approach used in least squares (LS) convergence analysis (e.g.,Ljung [4], Moore [5], Lai and Wei [6], and Chen and Guo [7]), fail in the present case. This is so even though (1.3) is the standard LS algorithm when $F = I$, $Q = 0$ and $R = 1$. Similar observations are also made by Meyn and Caines [8]; b). The algorithm (1.3) is a Kalman filter when $Q = Q_w$ and $R = R_v$. It is optimal in a linear minimum variance sense when φ_k is deterministic (e.g. Anderson and Moore [9]), and not stochastic as here. Thus, the stochastic nature of the regressors precludes applicability of the useful properties of the Kalman filter, even when Q_w and R_v are precisely known; c).The existing theory for time varying linear systems usually requires that the system output gain matrix (i.e., φ_k, in the present case) is bounded in k (e.g. Anderson and Moore [10]). This requirement turns out to be unrealistic in applying the theory to general adaptive control and identification problems. This is especially so in the stochastic case, because φ_k may contain the past system inputs and outputs, and the system noise may be unbounded. Hence, the unbounded nature of the regressors $\{\varphi_k\}$ also precludes the direct application of the standard theory.

In this chapter, we establish tracking error bounds for the case of randomly varying parameters. The main concern is with the following three cases:

(i). Parameters generated from a stable linear model, i.e., (1.1b) with

$|\lambda_i(F)| < 1$, for any i;

(ii).Drifting parameters, i.e., (1.1b) with $F = I$; and

(iii). Disturbed parameters ,i.e., $\theta_k = \theta_0 + w_k$.

2. Tracking Error Bound

In the sequel, we denote $\lambda_{max}(A)$ and $\lambda_{min}(A)$ the maximum and minimum eigenvalues of a matrix A respectively, and $\|A\| = \{\lambda_{max}(AA^\tau)\}^{1/2}$ its norm, so that $\|A\| = \lambda_{max}(A)$ when A is symmetric and nonnegative definite.

Let us first denote

$$K_k = FP_k\varphi_k(R + \varphi_k^\tau P_k\varphi_k)^{-1} \tag{2.1}$$

and rewrite (1.3) as

$$\hat{\theta}_{k+1} = F\hat{\theta}_k + K_k(y_k - \varphi_k^\tau\hat{\theta}_k), \tag{2.2a}$$

$$P_{k+1} = (F - K_k\varphi_k^\tau)P_k(F - K_k\varphi_k^\tau)^\tau + K_kRK_k^\tau + Q, \tag{2.2b}$$

The lower bounds to the tracking error is relatively straightforward by combining (1.1b) and (2.2a), indeed, we have

Theorem 2.1. Consider the signal model (1.1) and algorithm (1.3), if $\sup_k E\|w_k\|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$, then

$$\inf_k E\|\tilde{\theta}_k\|^2 \geq tr(Q_w), \tag{2.3}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|\tilde{\theta}_i\|^2 \geq \operatorname{tr}(Q_w), \quad \text{a.s}, \tag{2.4}$$

where Q_w and $\tilde{\theta}_k$ are defined by (1.2) and (1.4) respectively.

Proof: By (1.1) and (2.2a), the error equation is

$$\tilde{\theta}_{k+1} = (F - K_k \varphi_k^\tau)\tilde{\theta}_k - K_k v_k + w_{k+1} \tag{2.5}$$

Set

$$f_k = (F - K_k \varphi_k^\tau)\tilde{\theta}_k - K_k v_k$$

then $\{f_k^\tau w_{k+1}\}$ is a martingale difference sequence with respect to the σ-algebra generated by $\{v_{i-1}, w_i, i \leq k+1\}$, so the first assertion (2.3) follows from (2.5) and the orthogonality of f_k and w_{k+1} immediately.

Now, by an estimation for the weighted sum of martingale difference sequences (e.g. Chen and Guo [11], pp. 848), we know that

$$\sum_{i=1}^{n} f_i^\tau w_{i+1} = O(\{\sum_{i=1}^{n} \|f_i\|^2\}^{(1/2)+\eta}), \quad \text{a.s.}$$

for any $\eta > 0$. Consequently, by taking $\eta < \frac{1}{2}$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\|f_i\|^2 + 2f_i^\tau w_{i+1})$$

$$= \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|f_i\|^2 \{1 + O([\sum_{i=1}^{n} \|f_i\|^2]^{\eta - (1/2)})\} \geq 0.$$

From this inequality and (2.5) it follows that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \| \tilde{\theta}_i \|^2 = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \{ \|f_i\|^2 + 2f_i^\tau w_{i+1} + \|w_{i+1}\|^2 \}$$

$$\geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \|w_{i+1}\|^2 \geq \operatorname{tr}(Q_w), \text{ a.s.,}$$

which is the second assertion (2.4). Hence the proof is complete. ΔΔΔ

The upper bounds for the tracking error depend on the stability of the equation

$$\xi_{k+1} = (F - K_k \varphi_k^\tau) \xi_k, \quad k \geq 0, \tag{2.6}$$

as can be seen from (2.5), which we will show depend on the bounds of $\{P_k\}$.

A lower bound to P_k is easy to get, since from (2.2b) :

$$P_k \geq Q > 0, \quad \text{for any } k \geq 1. \tag{2.7}$$

However, upper bounds for $\{P_k\}$ are far from obvious for general F and $\{\varphi_k\}$. Let us first see the role played by the upper bound of $\{P_k\}$ in the stability of the equation (2.6).

Lemma 2.1. Assume that there exists a random constant b such that

$$\sup_{k \geq 0} \| P_k \| \leq b < \infty, \text{ a.s.,} \tag{2.8}$$

then for K_k defined by (2.1),

$$\left\| \prod_{k=i}^{j-1} (F - K_k \varphi_k^\tau) \right\| \leq \beta \alpha^{j-i}, \text{ a.s., for any } j > i \geq 0, \tag{2.9}$$

where α and β are defined by

$$\alpha = \frac{\|F\|(a+b)b^{1/2}}{[a^3 + \|F\|^2(a+b)^2 b]^{1/2}}, \tag{2.10}$$

$$\beta = [b/a]^{1/2}, \quad a = \lambda_{min}(Q). \tag{2.11}$$

The proof is given in Appendix A. The precise expressions of α and β in (2.10) and (2.11) lead directly to the following important observation.

Remark 2.1. If b, the upper bound of P_k, is a deterministic constant, then the exponential bounds claimed in (2.9) are also deterministic.

This fact is very crucial in establishing the upper bound for the tracking errors in terms of mathematical expectations in the sequel.

Let us now proceed to establish the upper bound for the tracking errors by considering different parameter models separately.

A. Parameters Generated from a Stable Model.

In this case, $|\lambda_i(F)| < 1$ for all i, then by (1.3b),

$$\begin{aligned} P_{k+1} &\leq FP_kF^\tau + Q \\ &\leq F^2P_{k-1}(F^\tau)^2 + FQF^\tau + Q \leq \\ &\leq \sum_{i=0}^{k} F^iQ(F^\tau)^i + F^{k+1}P_0(F^\tau)^{k+1}, \quad \text{for any } k \geq 0, \end{aligned} \tag{2.12}$$

and hence

$$b \triangleq \| \sum_{i=0}^{\infty} F^iQ(F^\tau)^i \| + \sup_{k\geq 0} \| F^kP_0(F^\tau)^k\|, \tag{2.13}$$

can serve as a finite deterministic upper bound for $\{P_k\}$ since P_0 is deterministic.

This enables us to establish the following results.

Theorem 2.2. Consider the signal model (1.1) with $|\lambda_i(F)| < 1$, for any i, and the estimation algorithm (1.3). Then

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \| \tilde{\theta}_i \|^p \le \left(\frac{\beta}{1-\alpha}\right)^p [L_p(w) + \|F\|(b/R)^{1/2} L_p(v)]^p, \tag{2.14}$$

and

$$\limsup_{n\to\infty} E \| \tilde{\theta}_n \|^p \le \left(\frac{\beta}{1-\alpha}\right)^p [M_p(w) + \|F\|(b/R)^{1/2} M_p(v)]^p, \tag{2.15}$$

here $\tilde{\theta}_n = \theta_n - \hat{\theta}_n$, b, α and β are given by (2.13), (2.10) and (2.11) respectively, and $p > 1$ is any real number such that

$$L_p(v) \triangleq \limsup_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \| v_i \|^p \right\}^{1/p} < \infty, \quad \text{a.s.} \tag{2.16}$$

$$L_p(w) \triangleq \limsup_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \| w_i \|^p \right\}^{1/p} < \infty, \quad \text{a.s.} \tag{2.17}$$

$$M_p(v) \triangleq \sup_i \{ E \| v_i \|^p \}^{1/p} < \infty, \tag{2.18a}$$

$$M_p(w) \triangleq \sup_i \{ E \| w_i \|^p \}^{1/p} < \infty, \tag{2.18b}$$

and $E \| \theta_0 \|^p < \infty$.

Proof. By (2.5) we have

$$\tilde{\theta}_{k+1} = \prod_{j=0}^{k} (F - K_j \varphi_j^\tau) \tilde{\theta}_0 + \sum_{i=0}^{k} \left[\prod_{j=i+1}^{k} (F - K_j \varphi_j^\tau) \right] (-K_i v_i + w_{i+1})$$

Applying Lemma 2.1 we see that

$$\| \tilde{\theta}_{k+1} \| \leq \beta \, \alpha^{k+1} \| \tilde{\theta}_0 \| + \beta \sum_{i=0}^{n} \alpha^{k-i} \left(\| K_i v_i \| + \| w_{i+1} \| \right), \qquad (2.19)$$

then applying the Minkowski inequality gives

$$\left(\sum_{i=0}^{n} \| \tilde{\theta}_{i+1} \|^p \right)^{1/p} \leq \beta \| \tilde{\theta}_0 \| \left(\sum_{k=0}^{n} \alpha^{p(k+1)} \right)^{1/p} + \beta \left\{ \sum_{k=0}^{n} \left(\sum_{i=0}^{k} \alpha^{k-i} \| K_i v_i \| \right)^p \right\}^{1/p}$$

$$+ \beta \left\{ \sum_{k=0}^{n} \left(\sum_{i=0}^{k} \alpha^{k-i} \| w_{i+1} \| \right)^p \right\}^{1/p}, \qquad (2.20)$$

now, by the Holder inequality it follows that ($1/p + 1/q = 1$):

$$\left(\sum_{i=0}^{k} \alpha^{k-i} \| K_i v_i \| \right)^p = \left\{ \sum_{i=0}^{k} \alpha^{(k-i)/q} \left[\alpha^{(k-i)/p} \| K_i v_i \| \right] \right\}^p$$

$$\leq \left(\sum_{i=0}^{k} \alpha^{k-i} \right)^{p/q} \left(\sum_{i=0}^{k} \alpha^{k-i} \| K_i v_i \|^p \right) \leq \left(\frac{1}{1-\alpha} \right)^{p/q} \sum_{i=0}^{k} \alpha^{k-i} \| K_i v_i \|^p$$

and then

$$\sum_{k=0}^{n} \left(\sum_{i=0}^{k} \alpha^{k-i} \| K_i v_i \| \right)^p \leq \left(\frac{1}{1-\alpha} \right)^{p/q} \sum_{i=0}^{n} \sum_{k=i}^{n} \alpha^{k-i} \| K_i v_i \|^p$$

$$\leq \left(\frac{1}{1-\alpha} \right)^{(p/q)+1} \sum_{i=0}^{n} \| K_i v_i \|^p = (1-\alpha)^{-p} \sum_{i=0}^{n} \| K_i v_i \|^p \qquad (2.21)$$

Let us now consider the upper bound for K_i. Since b is an upper bound for $\|P_k\|$, then by (2.1),

$$\| K_k \|^2 \leq \|F\|^2 \frac{\varphi_k^\tau P_k^2 \varphi_k}{(R + \varphi_k^\tau P_k \varphi_k)^2} \leq \|F\|^2 b \frac{\varphi_k^\tau P_k \varphi_k}{(R + \varphi_k^\tau P_k \varphi_k)^2}$$

$$\leq \|F\|^2 b / R, \quad \text{for any } k \geq 0, \qquad (2.22)$$

which together with (2.16) and (2.21) yields

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \Big(\sum_{i=0}^{k} \alpha^{k-i} \| K_i v_i \| \Big)^p \le \Big[\frac{\|F\|(b/R)^{1/2}}{1-\alpha} \Big]^p \, [L_p(v)]^p \, . \quad (2.23)$$

Similarly,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \Big(\sum_{i=0}^{k} \alpha^{k-i} \|w_{i+1}\| \Big)^p \le \Big(\frac{1}{1-\alpha} \Big)^p [L_p(w)]^p \qquad (2.24)$$

Finally, the first result (2.14) follows from (2.20), (2.23) and (2.24).

Let us now consider (2.15). The inequality corresponding to (2.20) can also be derived by the Minkowski inequality and takes the form :

$$(E\|\tilde{\theta}_{k+1}\|^p)^{1/p} \le \beta\, \alpha^{k+1} (E \| \tilde{\theta}_0 \|^p)^{1/p} + \beta \, \Big\{ E\Big(\sum_{i=0}^{k} \alpha^{k-i} \| K_i v_i \| \Big)^p \Big\}^{1/p}$$
$$+ \beta \, \Big\{ E\Big(\sum_{i=0}^{k} \alpha^{k-i} \| w_{i+1} \| \Big)^p \Big\}^{1/p} .$$

From this, a similar argument as used in the proof of (2.14) leads to (2.15) because in this case the constants b, α and β are all deterministic. This completes the proof. $\Delta\Delta\Delta$

Remark 2.2. From the proof of Theorem 2.2 we see that the independence assumptions made on the noise sequences $\{w_k\}$ and $\{v_k\}$ are not really used, indeed, Theorem 2.2 holds for any random sequences $\{w_k\}$ and $\{v_k\}$ satisfying (2.16)-(2.18). In particular, w_k, which appeared in the parameter model (1.1b) may have non-zero mean.

Remark 2.3. We have recently applied the property (2.15) with $p = 4 + \delta$ for some $\delta > 0$, to adaptive control problems (Guo and Meyn [12]), and it appears that the non-trivial stochastic adaptive control problem considered by Meyn and Caines [8] can be generalized to the case where the noises are nongaussian with unknown covariances.

Remark 2.4. Observe that there is no excitation requirement to achieve the bounds of the theorem. Of course, from (1.3b) and the matrix inversion lemma,

$$P_{k+1} = F\,[(P_k)^{-1} + \varphi_k R^{-1} \varphi_k^\tau\,]^{-1} F^\tau + Q$$

and it is clear that the greater the excitation of φ_k, the smaller is P_{k+1} in norm and the lower are the tracking error bounds (α,β are smaller).

B. Drifting Parameters.

In this case, $F = I$, and similar arguments as used in (2.12) for the boundedness proof of $\{P_k\}$ fail. Moreover, it turns out that it is impossible to establish the upper bounds for P_k without further assumptions on the regressors $\{\varphi_k\}$. To see this, let us take $\varphi_k = 0$, for all $k \geq 0$, then by (1.3b),

$$P_{k+1} = P_k + Q = P_0 + (k+1)Q \xrightarrow[k \to \infty]{} \infty \qquad (2.25)$$

Nevertheless, we have the following results.

Lemma 2.2. Consider that there exists a strictly increasing sequence of random integers $\{t_n\}$ with $t_0 = 0$, $d \stackrel{\Delta}{=} \sup_k (t_n - t_{n-1}) < \infty$, a.s., and random constants $\delta > 0$, $M < \infty$ such that for any $k \geq 1$,

$$\lambda_{min}(k) \geq \delta \text{ , a.s.} \qquad (2.26)$$

and

$$\lambda_{max}(k) / \lambda_{min}(k) \;\le\; M, \quad a.s, \tag{2.27}$$

where $\lambda_{max}(k)$ and $\lambda_{min}(k)$ denote the maximum and minimum eigenvalues of the matrix

$$\sum_{i=t_{k-1}+1}^{t_k} \varphi_i \varphi_i^{\tau} \tag{2.28}$$

respectively. Then $\{P_k\}$ defined by (1.3b) with $F = I$, has the following upper bound:

$$\sup_k \|P_k\| \le \|P_0\| + R/\delta + [\,1 + (1+M)d]\,\|Q\| < \infty, \text{ a.s.} \tag{2.29}$$

The proof of this lemma is given in Appendix B. The conditions (2.26)-(2.27) can be regarded as certain kinds of excitations, thus, the divergence phenomena as in (2.25) may be explained as lack of excitation of $\{\varphi_k\}$.

It is interesting to compare the conditions (2.26)-(2.27) with the standard persistence of excitation condition used in the analysis of short memory adaptive control algorithms in the literature (e.g. Anderson et al,[13]). That is, there exist constants $0 < \delta_1 \le \delta_2 < \infty$, and $N < \infty$ such that

$$\delta_1 I \le \sum_{i=k}^{k+N} \varphi_i \varphi_i^{\tau} \le \delta_2 I, \text{ for any } k \ge 0. \tag{2.30}$$

This implies that $\{\varphi_k\}$ is a bounded sequence. Clearly, Condition (2.26)-(2.27) is weaker than (2.30), and it means that $\lambda_{max}(k)$ and $\lambda_{min}(k)$ may grow at the same rate, and does not necessarily mean that $\{\varphi_k\}$ is bounded. As an example, let us take φ_k as a scalar slope function : $\varphi_k = c\,k$, $c \ne 0$, then, clearly

(2.30) fails, while (2.26)-(2.27) still holds because $\lambda_{max}(k)$ and $\lambda_{min}(k)$ coincide in this case. A related but different excitation condition to (2.26)-(2.27) has been introduced and studied in (Chen and Guo,[14]) for the analysis of short memory gradient algorithms when the regressors $\{\varphi_k\}$ are possibly unbounded.

Remark 2.5. Lemma 2.2 can be generalized to the case where $F \neq I$, and a similar bound as in (2.29) is also achieved. Similar results as in the following Theorem 2.3 are also available. However, in this case, the matrix given by (2.28), which are used in defining $\lambda_{min}(k)$ and $\lambda_{max}(k)$, will involve the matrix F in general.

Theorem 2.3. Consider the signal model (1.1) with F = I, and the estimation algorithm (1.3), consider also that the conditions in Lemma 2.2 apply. Then

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|\tilde{\theta}_i\|^p \leq \left(\frac{\beta}{1-\alpha}\right)^p \left[L_p(w) + (b/R)^{1/2} L_p(v)\right]^p \tag{2.31}$$

Here $\tilde{\theta}_k = \theta_k - \hat{\theta}_k$, α and β are defined by (2.10) and (2.11) with F = I and with the upper bound b for $\{P_k\}$ given in (2.29). Also, $L_p(w)$, $L_p(v)$ and $p > 1$ are defined in (2.16)-(2.17).

Proof. The proof is actually the same as that for (2.14). Note that the result (2.15) is also achieved in the present case provided that the quantity on the R.H.S. of (2.29) is deterministic. $\Delta\Delta\Delta$

As an example, let us now consider the i.i.d. noise case, and without loss of generality assume that θ_k is one dimensional. More precisely, let $\{w_k\}$ be i.i.d. random variables with mean zero and variance $\sigma^2 > 0$. Putting F = I in (1.1b) we get

$$\theta_n = \theta_{n-1} + w_n = \theta_0 + S_n, \quad S_n = \sum_{i=1}^{n} w_i \tag{2.32}$$

Consequently, by Strassen's invariance principle (Strassen, [15]),

$$\limsup_{n \to \infty} \left(\sum_{i=1}^{n} S_i^2\right) / (n^2 \log\log n) = 8\sigma^2/\pi^2 \cdot \text{a.s.},$$

On the other hand, by a result of Donsker and Varadhan [16, pp.751],

$$\liminf_{n \to \infty} \left(\sum_{i=1}^{n} S_i^2\right) (\log\log n) / n^2 = \sigma^2/4 \cdot \text{a.s.}$$

Hence with probability 1, the averaged value of parameters

$$\frac{1}{n} \sum_{i=1}^{n} (\theta_i)^2 \sim \frac{1}{n} \left(\sum_{i=1}^{n} S_i^2\right), \qquad n \to \infty$$

fluctuate in the interval

$$\left[\left(\frac{1}{4} + o(1)\right) \sigma^2 n / \log\log n, \quad (8\pi^{-2} + o(1))\, \sigma^2 n \log\log n \right]$$

as $n \to \infty$. Thus from this and the result (2.31) we see that the estimation algorithm (1.3) can indeed perform the non-trivial task of tracking rapidly varying parameters in the long run average sense.

Let us consider another situation.

C. Disturbed Parameters.

By disturbed parameters we mean that the parameters can be modeled by

$$\theta_k = \theta_0 + w_k \tag{2.33}$$

with unknown θ_0 and noise $\{w_k\}$. This case, is not a specialization of (1.1b), but can still be studied by use of the theory developed.

> Theorem 2.4. Consider the signal model (1.1a) with parameters described by (2.33), and the algorithm (1.3) with F = I. Consider also that conditions of lemma 2.2 apply. Then
>
> $$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \| \tilde{\theta}_i \|^p \le \left(\frac{\beta}{1-\alpha}\right)^p [(b/a)L_p(w) + (b/R)^{1/2} L_p(v)]^p \tag{2.34}$$
>
> where $\tilde{\theta}_k = \theta_0 - \hat{\theta}_k$, $a = \lambda_{min}(Q)$, and the constants α, β, b, $L_p(w)$ and $L_p(v)$ are all the same as those in Theorem 2.3.

Proof. With $\tilde{\theta}_k = \theta_0 - \hat{\theta}_k$ and F = I, the error equation (2.5) is now changed to

$$\tilde{\theta}_{k+1} = (I - K_k\varphi_k^\tau)\tilde{\theta}_k - K_k v_k + K_k\varphi_k^\tau w_{k+1}$$

Note that by (A1) in Appendix A, $K_k\varphi_k^\tau$ is bounded by

$$\|K_k\varphi_k^\tau\| \le b/a$$

Hence, a similar argument as used in the proof of (2.14) leads to the desired result (2.34). The details are not repeated here. ΔΔΔ

3 CONCLUSIONS

When the Kalman filter is applied to estimation of randomly varying parameters, our results show that it has quite reasonable tracking properties ---- even in the nongaussian case when it is not an optimal filter.

If the parameters are generated from a stable model, we have seen that there is no restriction on the regressors to achieve tracking error bounds. The bounds obtained have application for adaptive controller analysis.

If the parameters are drifting, as when the parameter model is unstable, the theory of the chapter shows that the regressors must be suitably exciting to achieve tracking error bounds. For the case of parameters disturbed by noise, there is again an excitation requirement to achieve tracking error bounds.

APPENDIX A

Proof of Lemma 2.1:

We first establish the upper bound for $K_k\varphi_k^\tau$ as follows (note that φ_k may be unbounded),

$$\| K_k\varphi_k^\tau \| \leq \| F \| \|P_k\| \| \varphi_k \|^2 \;/\; (R + \varphi_k^\tau P_k \varphi_k)$$

$$\leq \| F \| \; b \; \| \varphi_k \|^2 \;/\; [\lambda_{min}(Q) \; \| \varphi_k \|^2], \quad \text{(by (2.7))},$$

$$\leq \| F \| \frac{b}{a} \tag{A1}$$

Let us then denote for simplicity $F_k = F - K_k\varphi_k^\tau$. An upper bound for F_k is

$$\| F_k \| \leq \| F \| \left(1 + \frac{b}{a} \right) \tag{A2}$$

Now consider the following inequalities. By (A2), (2.2b) and the matrix inversion Lemma,

$$P_k^{-1} - F_k^\tau P_{k+1}^{-1} F_k$$

$$= P_k^{-1} - F_k^\tau \left[F_k P_k F_k^\tau + K_k R K_k^\tau + Q \right]^{-1} F_k$$

$$\geq P_k^{-1} - F_k^\tau \left[F_k P_k F_k^\tau + Q \right]^{-1} F_k$$

$$= \left[P_k + P_k F_k^\tau Q^{-1} F_k P_k \right]^{-1}$$

$$\geq \left[P_k + (P_k)^{1/2} \| (P_k)^{1/2} F_k^\tau Q^{-1} F_k (P_k)^{1/2} \| (P_k)^{1/2} \right]^{-1}$$

$$\geq \left[P_k + \| F \|^2 \left(1 + \frac{b}{a} \right)^2 \frac{b}{a} P_k \right]^{-1}$$

$$= \left[1 + \| F \|^2 \left(1 + \frac{b}{a} \right)^2 \frac{b}{a} \right]^{-1} P_k^{-1}$$

Consequently, by the definition (2.10) for α:

$$F_k^\tau P_{k+1}^{-1} F_k \leq \alpha^2 P_k^{-1}, \quad \text{for any } k \geq 0.$$

Thus, noting (2.7) and (2.8), and repeatedly using this inequality, we get

$$\| \prod_{k=i}^{j-1} F_k \|^2 \leq b \| \left(\prod_{k=i}^{j-1} F_k \right)^\tau P_j^{-1} \left(\prod_{k=i}^{j-1} F_k \right) \|$$

$$\leq b\, \alpha^{2(j-i)} \| P_i^{-1} \| \leq \left(\frac{b}{a} \right) \alpha^{2(j-i)}, \quad \text{for any } j > i \geq 0. \qquad \Delta\Delta\Delta$$

APPENDIX B

Proof of Lemma 2.2.

Clearly, if the result holds for any deterministic sequences $\{\varphi_k\}$ and $\{t_k\}$ and deterministic constants δ and M, then the stochastic case can be proved by applying the result for each sample path. So, without lose of generality, we can assume that all the quantities appearing in the lemma are deterministic in the following proof.

Let us first establish the upper bound for the subsequence $\{ P_{t_n+1} \}$.

To this end, we introduce an auxiliary stochastic system

$$x_{k+1} = x_k + Q^{1/2}\eta_k^1 \tag{A3}$$

$$z_k = \varphi_k^\tau x_k + (R)^{1/2}\eta_k^2 \tag{A4}$$

where $\{ \eta_k^{1\tau}, \ \eta_k^2 \}$ is an i.i.d. Gaussian random sequence with zero mean and unity covariance. Assume further that $var(x_o) = P_0$ and x_o is independent of $\{\eta_k^{1\tau}, \eta_k^2\}$.

Denote $\hat{x}_k$ the estimation for x_k based on $\{z_o, ..., z_k\}$ which is given by the Kalman filter, then it is well known that (e.g. Anderson and Moore [9]) P_k defined by (1.3b) (or (2.2b) with F = I) can be represented by

$$P_{k+1} = \Sigma_k + Q, \quad \text{for any } k \geq 0. \tag{A5}$$

where

$$\Sigma_k = E(x_k - \hat{x}_k)(x_k - \hat{x}_k)^\tau \quad .$$

Let us consider another linear estimate $\hat{x}^*_n$ for x_n at time $n = t_k$ as follows

$$\hat{x}^*_{t_k} = w^{-1}(k) \sum_{i=t_{k-1}+1}^{t_k} \varphi_i z_i \,, \; W(k) \triangleq \sum_{i=t_{k-1}+1}^{t_k} \varphi_i \varphi_i^{\tau} \,.$$

Note that by (A3) and (A4),

$$x_{t_k} - \hat{x}^*_{t_k} = W^{-1}(k) \{W(k) x_{t_k} - \sum_{i=t_{k-1}+1}^{t_k} \varphi_i z_i \}$$

$$= W^{-1}(k) \{ \sum_{i=t_{k-1}+1}^{t_k} \varphi_i \varphi_i^{\tau} \sum_{j=i+1}^{t_k} Q^{1/2} \eta^1_{j-1} - \sum_{i=t_{k-1}+1}^{t_k} \varphi_i (R)^{1/2} \eta^2_i \}$$

$$= I_1(k) + I_2(k) \tag{A6}$$

We now proceed to estimate the covariances of $I_1(k)$ and $I_2(k)$ as follows.

Denote

$$S_i \triangleq \sum_{j=t_{k-1}+1}^{i} \varphi_j \varphi_j^{\tau}, \; S_{t_{k-1}} \triangleq 0, \; T_i \triangleq \sum_{j=i}^{t_k} Q^{1/2} \eta^1_{j-1} \,, \; T_{t_k+1} \triangleq 0 \,.$$

By summation by parts we have

$$\sum_{i=t_{k-1}+1}^{t_k} \varphi_i \varphi_i^{\tau} \sum_{j=i+1}^{t_k} Q^{1/2} \eta^1_{j-1} = \sum_{i=t_{k-1}+1}^{t_k} (S_i - S_{i-1}) T_{i+1}$$

$$= \sum_{i=t_{k-1}+1}^{t_k - 1} S_i (T_{i+1} - T_{i+2}) + S_{t_k} T_{t_k+1} - S_{t_{k-1}} T_{t_{k-1}+2}$$

$$= \sum_{i=t_{k-1}+1}^{t_k - 1} S_i Q^{1/2} \eta^1_i \,.$$

Then by orthogonality of $\{\eta_i^1\}$ and monotonicity of $\{S_i\}$:

$$\|E\, I_1(k)\, I_1^\tau(k)\| \le \| W^{-1}(k) \sum_{i=t_{k-1}+1}^{t_k-1} S_i Q S_i\, W^{-1}(k) \|$$

$$\le \| W^{-1}(k) \sum_{i=t_{k-1}+1}^{t_k-1} (S_i)^{1/2} S_i (S_i)^{1/2}\ W^{-1}(k) \|\, \|Q\|$$

$$\le \| W^{-1}(k) \sum_{i=t_{k-1}+1}^{t_k-1} S_i\, W^{-1}(k) \|\ \lambda_{max}(S_{t_k})\, \|Q\|$$

$$\le \| W^{-1}(k) \sum_{i=t_{k-1}+1}^{t_k-1} S_{t_k} W^{-1}(k) \|\ \lambda_{max}(k)\, \|Q\|$$

$$\le (t_k - t_{k-1})\, \| W^{-1}(k)\|\ \lambda_{max}(k)\, \|Q\|$$

$$\le d\, \|Q\|\ \lambda_{max}(k) / \lambda_{min}(k) \le d\, M\, \|Q\| , \qquad \text{(A7)}$$

while for $I_2(k)$ we have

$$\|E\, I_2(k)\, I_2^\tau(k)\| = \| W^{-1}(k) \sum_{i=t_{k-1}+1}^{t_k} \varphi_i R \varphi_i^\tau\, W^{-1}(k)\|$$

$$= R\ \|W^{-1}(k)\| \le R\, \delta^{-1} \qquad \text{(A8)}$$

Thus by the orthogonality of $I_1(k)$ and $I_2(k)$ from (A6) - (A8) we get

$$\| E\, (x_{t_k} - \hat{x}^*_{t_k})\ (x_{t_k} - \hat{x}^*_{t_k})^\tau\| \le R\delta^{-1} + dM\|Q\|.$$

From this and the optimality of the Kalman filter

$$\Sigma_{t_k} = E(x_{t_k} - \hat{x}_{t_k})\ (x_{t_k} - \hat{x}_{t_k})^\tau \le \quad E\, (x_{t_k} - \hat{x}^*_{t_k})\ (x_{t_k} - \hat{x}^*_{t_k})^\tau$$

the following upper bound for P_{t_k+1} follows by noting (A5) :

$$\| P_{t_k+1} \| \leq R/\delta + (1 + dM) \|Q\| , \text{ for all } k \geq 1.$$

To complete the proof, we have to establish the upper bound for $\{P_n\}$.

Since $\{t_k\}$ is a sequence of strictly increasing integers, $t_k \dashrightarrow \infty$, as $k \dashrightarrow \infty$, then for any integer $n \geq t_1 + 1$, there exists a integer $k \geq 1$ such that

$$t_k+1 \leq n \leq t_{k+1} \quad .$$

From this and the following inequality (by(1.3b) with F = I) :

$$P_{k+1} \leq P_k + Q , \quad \text{for any } k \geq 0 \tag{A9}$$

we obtain

$$\|P_n\| \leq \|P_{t_k+1}\| + (n - t_k) \|Q\|$$

$$\leq R/\delta + (1+dM)\|Q\| + (t_{k+1} - t_k) \|Q\|$$

$$\leq R/\delta + [1 + d(M+1)] \|Q\| , n \geq t_1+1 , \tag{A10}$$

while for the case where $n \leq t_1$, by (A9),

$$\|P_n\| \leq \|P_0 + t_1 Q\| = \| P_0 + (t_1 - t_0) Q\|$$

$$\leq \|P_0\| + d \|Q\| \tag{A11}$$

Finally, the desired result follows by combining (A10) and (A11). $\Delta\Delta\Delta$

REFERENCES

[1] D.Q. Mayne, "Optimal Non-stationary Estimation of the Parameters of a Linear System With Gaussian Inputs", **J. Electron. Contr.** Vol. 14, p101, 1963.

[2] K.J. Astrom and B. Wittenmark "Problems of Identification and Control", **J. Math. Anal. Applic.** 34, pp 90-113, 1971.

[3] G. Kitagawa and W. Gersh, " A Smoothness Priors Time Varying AR Coefficient Modelling of Non-stationary Covariance Time Series" **IEEE Trans. AC** Vol. AC-30, pp48-56, 1985.

[4] L. Ljung, "Consistency of the Least - Squares Identification Method" **IEEE Trans. AC** Vol. AC-21, pp 779-781, 1976.

[5] J.B. Moore, "On Strong Consistency of Least Squares Identification Algorithm", **Automatica** ,Vol. 14 pp 505-509. 1978.

[6] T. L. Lai and C. Z. Wei "Least Squares Estimation in Stochastic Regression Model with Applications to Identification and Control of Dynamic Systems", **Ann. Statist.**, 10, pp 154-166, 1982.

[7] H.F. Chen and L. Guo, "Convergence Rate of Least - Squares Identification and Adaptive Control for Stochastic Systems", **Int. J. Control** Vol.44, No. 5, pp 1459-1476, 1986.

[8] S.P. Meyn and P. E. Caines, "A New Approach to Stochastic Adaptive Control", **IEEE Trans. AC.** Vol. AC-32, pp 220-226, 1986

[9] B.D.O. Anderson and J.B. Moore, **"Optimal Filtering"**, Prentice-Hall, Englewood Cliffs, NJ. 1979.

[10] B.D.O. Anderson and J.B. Moore,"Detectability and Stability of Discrete-time Linear Systems", **SIAM J. on Control and Optimization** ,Vol. 19, No. 1, pp 20-32, 1981.

[11] H.F. Chen and L. Guo, "Optimal Adaptive Control and Consistent Parameter Estimates for ARMAX Model with Quadratic Cost", **SIAM J. on Control and Optimization** Vol. 25, No. 4, pp 845-867, 1987.

[12] L. Guo and S.P. Meyn, "Short Memory Stochastic Adaptive Control", Submitted for publication.

[13] B.D.O. Anderson et al, **"Stability of Adaptive Systems: Passivity and Averaging Analysis"** Boston, Mass:MIT Press, 1986.

[14] H. F. Chen and L. Guo, "Adaptive Control via Consistent Estimation for Deterministic Systems", **Int. J. Control** ,Vol. 45, No. 6, pp2183-2202, 1987.

[15] V. Strassen, "An Invariance Principle for the Law of Iterated Logarithm", **Z. Wahrsch. Gebiete** 3, pp 211-226, 1964.

[16] M. D. Donsker and S. R. S. Varadhan, "On Laws of the Iterated Logarithm for Local Times" **Comm. Pure. Appl. Math** 30, pp 707-753, 1977.

CHAPTER 13

CONCLUSIONS OF THE THESIS

This thesis addresses some robust and adaptive control problems. The research work has resulted in new off-line controller design methods, new on-line adaptive controller algorithms, and related theory giving various characterizations, convergence properties and performance bounds. Where theory is inadequate to fully describe behavior, we have used simulation studies to demonstrate the advantages of the new techniques. Areas for further refinements are apparent from the work.

New Design Methods

To enhance robustness properties of initial designs, we have generalized the idea of loop transfer recovery to cope with non-minimum phase plants by working on an inner / outer factored representation of the plants. Also a method to seek a robust controller with transfer function constrain is given based on a convenient parametrization of the class of all model matching stabilizing controller and two-degree-freedom stabilizing controller.

New Algorithms

In order to improve the transient performance in adaptive schemes we have developed the design approaches for central tendency adaptive pole assignment and central tendency adaptive LQG. It appears from our studies that these are successful schemes sometimes giving dramatic performance improvement over standard techniques, sure they are design to avoid ill-conditioning in the controller update calculation. In seeking to avoid ill-conditioning due to overparametrization of signal models we have proposed a perturbed versions of standard least squares algorithm, a perturbed extended least squares algorithm, and a perturbed Kalman filter detection / identification algorithm. There is also proposed a recursive algorithm to calculate H^{∞}-norms of polynomials.

New Theory

Convergence results on the adaptive controller algorithms proposed are developed for stochastic environments. These are non-trivial applications of standard martingale convergence methods applied in such a way that the theory guides the design of the algorithm modifications, so as to avoid the ill-conditioning associated with standard methods. Besides, it has been proved that for linear regression signal models, the suitable introduction of whiter noise into the estimation algorithm can make it more robust without compromising on convergence rates. The whiter the noise environment the more robust are the algorithm, and the noise color conditions imposed on plant noise model can be side-stepped. Moreover, side-stepping the colored noise restrictions for general ARMAX model identification has been fulfilled by means of introducing overparametrization and working with a transformed signal model. Asymptotic properties of the Kalman filter have also been developed when it is employed for tracking unknown randomly time varying parameters in linear stochastic system identification, and the tracking error bounds are given with reasonable excitation assumptions.

Confirmation via Simulations

Simulations have been employed to illustrate the effectiveness of the proposed techniques not completely described by theory. Thus in the loop recovery methods, the relationship between fictitious noise added and the loop recovery achieved is illustrated. For H^∞-norm calculations, the number of iterations required to achieve reasonable accuracy is illustrated. In assessing the effectiveness of the method for avoiding ill-conditioning in adaptive estimation and control, it is natural to check transient performance characteristics by simulations as in the thesis.

Further Research

Although we have reported here some solutions to robust and adaptive control problems, there is scope for further researches in this area. Here we point out some directions for further research as following.

* It would be interesting to combine the loop transfer recovery approach for robustness enhancement and the H^∞-optimization together to achieve improved trade-offs between performance and robustness.

* For calculation of H^∞-norm, our theory is currently limited to polynomials. A generalization worths of study is to cope with rational transfer functions and also multidimensional versions. A more challenging task is the optimization on H^∞-norm of polynomials or rational transfer functions subject to some constraints.

* There is a need to develop central tendency adaptive control schemes based on other estimation and / or control methods.

* Developing adaptive schemes to control linear stochastic systems with randomly time-varying parameters rather than just to perform on-line identification is an important challenging task for the future.