SCALAR-TENSOR THEORIES OF GRAVITATION

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CONTENTS

Chapter One

INTRODUCTION

The problem the writer wishes to consider here, is essentially one related to the classical field description of Nature.

The framework of General Relativity provides a theory for the geometry of the four dimensional space-time manifold and at the same time gives a description of the gravitational field in terms of the metric tensor, while the electromagnetic field can be interpreted in terms of a particular second rank, skew-symmetric tensor —— the covariant curl of a vector field defined on the manifold. However the scalar field, the simplest geometric object that could be defined on the manifold, does not seem to be experimentally evident when it is interpreted as a third classical long range field. In spite of this lack of experimental evidence and as there appears to be no theoretical objection to the existence of such a long range field, the problem is to introduce the scalar field into the classical scheme of things and to construct a viable theory containing all three long range fields.

It is interesting to compare the physical descriptions involved with these fields. Both the gravitational and the electromagnetic fields have gauge-like degrees of freedom and before a situation could be physically relevant these degrees of freedom must be fixed —— for the gravitational field by imposing coordinate conditions while for the electromagnetic field, after coordinate conditions have been imposed the gauge of the field potential must be chosen. As a consequence of these gauge freedoms, in order that the fields couple consistently with matter sources, the energy momentum tensor of the source must be covariantly conserved and the electromagnetic current density of the source must be conserved. The scalar field on the other hand has no gauge-like degree of freedom and consequently has no conserved "charge" as a source. Thus for example, in contrast to the other two fields, no constraints exist by which the scalar field could be separated into a source "bound" part and a free "wave" part.

In recent years the problem of incorporating the scalar field into the description of gravitation has led to the investigation of a special class of gravitation theories — the scalar-tensor gravitation theories.

With the previously mentioned problem in mind, the goals of this thesis are

(i) to review work that has been done on these theories and

(ii) to discuss them in a way that compares them to the theory of gravitation given in General Relativity.

Chapter Two basically gives an historical background and introduces more specific motives for considering the scalar field as a fundamental physical field.

Chapter Three considers the important class of scalar-tensor gravitation theories based on a Riemann space-time and Chapter Four continues this theme by looking at the "most developed" and perhaps simplest member - the Brans-Dicke theory.

For completeness the "massive Brans-Dicke" theories and some special scalar-tensor theories are looked at briefly in Chapter Five.

Chapter Six returns to the scalar-tensor model of gravitation developed in Chapter Three and looks at the implications for the model in more general space-times.

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Chapter Two

BACKGROUND

The electromagnetic and gravitational fields were described in the introduction as classical long range fields. These fields are responsible for forces that fall off inversely proportional to the square of the distance apart of the interacting bodies (sources); in contrast to short range forces which show an exponential behaviour. Einstein (1916) (1), attributed to the space-time manifold a Riemann structure and gave "meaning" to the gravitational field in terms of curvature through his gravitational field equations

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \quad , \qquad 2.1$$

where G is Newton's gravitational constant.

The notation established here is used in most sections. Units of length and time are chosen such that $c = 1$, although with this understanding some formulae may still contain c . Greek indices range over the values $\{0,1,2,3\}$, the coordinates x^0 and x^i , $(i = 1,2,3)$, are assumed time-like and space-like respectively and the signature of the space-time metric, $g_{\mu\nu}$ is $- + + +$. The Riemann and Ricci tensors have the respective forms

$$R^{\lambda}{}_{\mu\nu\rho} = \begin{Bmatrix}\lambda\\ \mu\rho\end{Bmatrix}_{,\nu} - \begin{Bmatrix}\lambda\\ \nu\mu\end{Bmatrix}_{,\rho} + \begin{Bmatrix}\eta\\ \mu\rho\end{Bmatrix}\begin{Bmatrix}\lambda\\ \nu\eta\end{Bmatrix} - \begin{Bmatrix}\eta\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\lambda\\ \rho\eta\end{Bmatrix}$$

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} = \begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}_{,\lambda} - \begin{Bmatrix}\lambda\\ \mu\lambda\end{Bmatrix}_{,\nu} + \begin{Bmatrix}\eta\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\lambda\\ \lambda\eta\end{Bmatrix} - \begin{Bmatrix}\eta\\ \mu\lambda\end{Bmatrix}\begin{Bmatrix}\lambda\\ \nu\eta\end{Bmatrix}$$

where $\begin{Bmatrix}\sigma\\ \mu\nu\end{Bmatrix} = \tfrac{1}{2}g^{\lambda\sigma}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda})$ and partial differentiation is denoted by a comma.

The close relation between the Riemann curvature tensor and gravitational effects is further illustrated, for example, in the equations of geodesic deviation, (2)

$$\frac{D^2}{D\tau^2}\,\delta x^{\alpha} = R^{\alpha}{}_{\mu\nu\rho}\,\delta x^{\nu}\,\frac{dx^{\mu}}{d\tau}\,\frac{dx^{\rho}}{d\tau} \quad , \qquad 2.2$$

where $x^{\alpha}(\tau)$, $x^{\alpha}(\tau) + \delta x^{\alpha}(\tau)$ define the paths of a pair of neighbouring, freely falling particles and $\frac{D}{D\tau}$ denotes the absolute derivative along the curve $x^{\alpha}(\tau)$. A freely falling particle is at rest in a coordinate frame falling with it, whereas a pair of neighbouring freely falling particles will show a relative acceleration given by eq. 2.2. To an observer travelling with the frame this motion will indicate the presence of a gravitational field.

The electromagnetic field on the other hand, appears in this picture as a field "embedded" in space-time, the geometry of which is determined by gravitation. A resolution of this difference in the roles of the two long range fields was proposed by Weyl (1918), (3, 4). However, along with other attempts at unification it was generally considered to be physically unsatisfactory, and so except for some special references, the electromagnetic field is included in the source side (i.e. the right hand side of eq. 2.1) of Einstein's field equations or of these equations in any subsequently modified form.

Basic to Weyl's approach was a generalisation of Riemann space-time - the Weyl space-time, for short. This space-time has been revived quite recently by some authors (e.g. Ross, Lord, and Omote , (5, 6, 7)) as a framework for scalar-tensor gravitation theories and for this reason it deserves a few comments about its historical origins, in addition to the treatment given in Chapter Six.

Curvature in Riemann space-time can be related to the idea of the parallel displacement of a vector - the transport of a vector by parallel displacement around a closed curve resulting in the final direction of the vector being different from its initial direction. Weyl supposed that the transported vector has a different length as well as a different direction and so for Weyl space-time, unless two points are infinitesmally close together lengths at these points can only be compared with respect to a path joining them. Because a determination of length at one point leads to only a first order approximation to a determination of length at neighbouring points one must set up, arbitrarily, a standard of length at each point and with lengths referred to this local standard a definite number can be given for the length of a vector at a point. If a vector which has length, l, at a point with coordinates x^{α} is parallelly displaced to the point with coordinates $x^{\alpha} + \delta x^{\alpha}$ then its change of length Weyl gave to be

$$\delta l^2 = - l^2 \phi_{\mu} \delta x^{\mu}, \qquad 2.3$$

where ϕ_{μ} are the components of a vector field.

For parallel displacement around a small closed curve the total change of length of the transported vector turns out to be

$$\Delta l^2 = - l^2 f_{\mu\lambda} \delta a^{\mu\lambda} \quad , \qquad 2.4$$

where $\delta a^{\mu\lambda}$ describes the element of area enclosed by the curve, and

$$f_{\mu\lambda} = \phi_{\mu,\lambda} - \phi_{\lambda,\mu} \quad . \qquad 2.5$$

Weyl set ϕ_{μ} proportional to the electromagnetic potential A_{μ} and so $f_{\mu\lambda}$ is made proportional to the electromagnetic field tensor, $F_{\mu\lambda}$. Thus the electromagnetic potential determines by eq. 2.3 the behaviour of length on parallel displacement and the electric and magnetic fields find expression in the derived tensor, $f_{\mu\lambda}$. This tensor can be shown to be independent of the initial choice of length standard, which is a necessary condition if it is to be physically meaningful.

A difficulty of the theory was an apparent conflict between eq. 2.3 and the interpretation given above, with the idea that atomic standards of length and time appear absolute and independent of space-time position.* If the coefficient of proportionality between ϕ_μ and A_μ is assumed real and put equal to $\frac{2C}{e}$ (e is the **charge** of an electron and C is dimensionless) then a recent experiment, (8), places an upper bound on C of 10^{-47}. Such a figure, however, does not exclude the geometric interpretation of the electromagnetic field in terms of the Weyl space-time if, for example, Weyl's original idea of equivalent initial length standards is modified to give special status to atomic standards.

Aside from introducing the Weyl space-time these comments emphasize a feature of length standards in Riemann space-time, where once they are defined in terms of atomic standards at a point, parallel displacement allows the comparison of lengths taken at separated points. Without getting involved in problems of measurement we shall just assume that on this basis, the physical descriptions of atomic systems are independent of space-time position and that by using these systems the space-time interval measured between neighbouring events is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \qquad 2.6$$

where $g_{\mu\nu}$ is identified with the gravitational field variable appearing in eq. 2.1.

With this brief and rather indirect look at some of the ideas involved in the Riemann space-time we return to look at Einstein's field equations and his description of gravitation in order to provide a background before introducing the scalar field.

Because the dynamical variable of the gravitational field in General Relativity is the metric tensor, it plays an important geometry-determining role in space-time and the field equations can be understood to couple the geometry of space-time to matter. Thus the only physical constraint imposed by these equations

* so under parallel displacement a vector maintains its length with respect to atomic standards.

on the nature of matter is that its energy-momentum tensor has zero divergence.

Implicit in all of the discussion so far has been the division between the gravitational field and matter. In his discussion of the action principle formulation of his field equations, Einstein (9), stated an assumption to ensure that this distinction carried over to the action principle — that is, the Lagrangian density could be divided into two parts, one of which refers to the gravitational field and contains only the metric tensor and its derivatives. The appropriate density for this part is the Riemann scalar density and apart from a cosmological term the resulting action for the free gravitational field is unique in giving field equations which are linear in the second derivatives of the metric and which in the weak-field limit give the Newtonian case.

In spite of this success in giving empty space-time field equations that are unique modulo a cosmological term, the action principle without further assumptions, does not offer much insight into the nature of the energy-momentum tensor. So the action principle remains an important method for constructing field equations.

In order to make progress later on, much use is made of the <u>Principle of Minimal Coupling</u>, (e.g. 10), which Anderson notes is not an essential part of General Relativity. If a material system is considered in Special Relativity as a set, χ of matter fields then its Euler-Lagrange equations of motion, in some inertial frame, will follow from an action principle

$$\delta\int L_M d^4x = 0 \quad ,$$

for suitable variations of the variables χ. The action ($\delta\int L_M d^4x$) will depend on the Lorentz metric, $\eta_{\mu\nu}$, and with $\eta_{\mu\nu}$ replaced by $g_{\mu\nu}$ this action when added to the free gravitation field action gives the required action[1] for the gravitational and matter field equations. If $\mathcal{L}_{NG}$ denotes the matter or nongravitational part of the full Lagrangian density (i.e. with the principle assumed, $\mathcal{L}_{NG}$ is the minimally coupled L_M) then the energy-momentum tensor of the material system is defined in terms of the system's "response" to the metric field by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \left[\frac{\partial \mathcal{L}_{NG}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^{\alpha}} \frac{\partial}{\partial \left(\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right)} \mathcal{L}_{NG} + \dots \right] \qquad 2.8$$

This definition still holds without appeal to the Principle of Minimal Coupling but in this case the connection described between $\mathcal{L}_{NG}$ and L_M could not be supposed to hold.

An aspect of the field equations noted here then, is that the nature of the energy-momentum tensor is determined by criteria outside of General Relativity. To work within the framework of General Relativity leads to an extreme position such as suggested by McCrea (11), that the Einstein tensor is to be interpreted or identified as an energy-momentum tensor and the central question is then which geometric constraints imposed by the field equations are physically meaningful. This rather formal approach has been developed a little by Harrison, (12), in a way, to suggest that scalar-tensor gravitation theories are in fact derivative from Einstein's theory by suitable interpretations of the energy-momentum tensor. However this view is a bit unorthodox and we shall return to it later.

It was mentioned in the introduction that the gravitational field, as described by Einstein, has a gauge-like degree of freedom. This of course corresponds to

[1] this minimalcoupling prescription is sometimes, when needed, supplemented with the rule that :-

partial derivatives ⟶ covariant derivatives.

the coordinate transformations of the metric tensor and one can always introduce at a point a local coordinate system, sometimes characterised as a locally freely falling system, in which for a sufficiently small neighbourhood of the point the metric is the Lorentz metric. By describing physics in this neighbourhood in terms of such a coordinate system the effects of the gravitational field are transformed away. Essentially it is this feature of General Relativity - that gravitational or cosmological effects can be made to vanish in the small, which has been questioned and led to the scalar-tensor gravitation theories.

In 1937 Dirac, (13), (and 1938 (14)), suggested that an expanding model of the Universe not only provided a cosmic time scale but also allowed the possibility that the gravitational constant may vary with this time. By taking the ratio of the age of the Universe to a unit of time fixed by atomic constants (e.g. $\frac{e^2}{mc^3}$ or $\frac{h}{mc^2}$) one obtains a number, t, of the order 10^{40} , and by taking the ratio of the gravitational force to the electric force between typically charged particles one obtains a dimensionless expression, $\frac{Gm^2}{e^2}$, of the order 10^{-40} . So for this epoch

$$\frac{Gm^2}{e^2} \sim t^{-1} , \qquad 2.9$$

and with m and e supposed constant this relation becomes

$$G \sim t^{-1} , \qquad 2.10$$

which Dirac's hypothesis implied, held for all epochs. In more general terms Dirac's hypothesis, (14), stated that "any two of the very large dimensionless numbers occurring in Nature are connected by a simple mathematical relation in which the coefficients are of the order of magnitude unity" and as a consequence if a number varies with epoch then other dimensionless numbers may be required to vary with epoch in order to keep the relations between them.

A feature of the cosmology which Dirac was led to, is that fundamental significance need not be given to these numbers. For example, from eq. 2.9, the ratio of gravitational to electric forces is small because the Universe is old. Although Dirac's cosmology could almost be ruled out by present observations, the idea remains that fundamental constants and in particular, the gravitational constant, may not in fact be constant. Some observable effects of a variable gravitational constant have been discussed by Jordan, (15), and by Dicke, (16), but because these effects are geophysical or cosmological, the systems involved are complex and the numerical data available is insufficient as evidence for variation of the gravitational constant. Recent results by Shapiro, (17), using planetary radar systems, and atomic clocks put an experimental limit on the fractional time variation of the gravitational constant as 4×10^{-10}/year and so the idea of a variable gravitational constant is still a conjecture which has not been established by direct observation.

Einstein's equations appear at present to describe local gravitational effects quite adequately and one could expect these equations to hold for the Universe as a whole. But, since the equations require the gravitational constant, when measured in units defined by atomic standards, to be constant they need to be modified if Dirac's hypothesis is assumed to be valid. A simple way to introduce a variable gravitational constant into the field equations is to make the gravitational constant a new local scalar field variable depending on position in space-time.

Historically, Jordan (1948) was the first to use this approach to incorporate a variable gravitational constant in a field theory of gravitation. He originally used the five-dimensional representation of General Relativity developed by Kaluza (1921) and Klein (1926) and later (1955), (18), he and others developed the theory as a four-dimensional scalar-tensor formalism. These earlier references to Jordan's theory are given more completely by Pauli, (19), and a comprehensive review of Jordan's theory is given in an article by Brill, (20). The most widely known theory of gravitation which includes a variable gravitational constant is the Brans-Dicke theory (1961) which is formally, very closely related to Jordan's theory. From 1961 onwards, the existence of such a long-range scalar field seemed feasible (but perhaps experimentally doubtful) and in the writer's opinion the most interesting developments to come from the Brans-Dicke theory relate to the problem of constructing dynamical laws involving the gravitational field variable, the scalar field variable and matter field variables. Finally, one notes that, to introduce the scalar field as a long range cosmological field for the purpose of obtaining a variable gravitational constant is by no means the only way of giving expression to Dirac's hypothesis.

In a pattern similar to that described above, other authors have postulated a scalar field and introduced scalar field terms into Einstein's field equations in order to deduce from these equations preferable models of the Universe. Hoyle's equations (1948) (21) implied that matter was not conserved and gave a steady-state model of the Universe. Here the scalar field was related to the creation of matter, in contrast to the scalar field postulated by Rosen (1969) (22) which had no interaction with matter. Rosen's equations gave an oscillating model of the Universe.

The Machian idea of a connection between local physical laws and properties of the Universe as a whole has already been partly met, with Dirac's hypothesis. In an effort to explain inertia, the Brans-Dicke theory was based more on Mach's Principle than on Dirac's hypothesis. Some further references to these ideas are given in Chapter Four while passing mention is made here to Caloi and Firmani's (1970) (23) modified Brans-Dicke theory in which radiation is given a more Machian property in determining along with matter, the inertia of a body. However their theory is restrictive and applies only to a homogeneous, isotropic space-time in which the matter content can be represented as a perfect fluid. Gursey's (24) theory is Machian motivated in a different kind of way and this theory is discussed in Chapter Five.

It is apparent that the scalar field has been introduced into the General Relativistic framework to incorporate many quite different physical features which have been thought desirable and found not to follow from the usual interpretations of Einstein's field equations. The field equations of the scalar-tensor gravitation theories that have been devised, possess cosmological solutions describing a variety of models of the Universe. So with these rather general comments summarizing (and substituting for) what could have been a lengthy look at the individual theories, the relation between the scalar-tensor and Einstein's descriptions of gravitation is taken up with reference to Harrison's papers (12, 25).

Harrison, (12), states that the scalar-tensor field equations "constitute in fact a limited and particular class of equations that derive from General Relativity and are of lesser generality". He arrives at this view after showing that the forms of the action principles of different scalar-tensor theories can be transformed into each other and into the form of the action principle for General Relativity, by recalibrations (i.e. conformal or scaling transformations) of the field variables. Thus, together with the observation noted earlier that the physical nature of the energy-momentum tensor lies outside of the scope of the theory of General Relativity, a scalar-tensor gravitation theory seems to be, (25), "a specialised application of the theory of General Relativity". In this way the scalar-tensor and

Einstein's theories of gravitation do not have the same status as gravitation theories. General Relativity becomes in a sense a generic theory where one works in a Riemann space-time and postulates field equations based on assumptions about the content of the energy-momentum tensor in Einstein's field equations. A classic example of this procedure is given by McCrea (1951), (26), who found that Hoyle's results (1948) could be derived from Einstein's field equations if negative stress was allowed in the energy-momentum tensor of the Universe. Another example is implied by remarks of Dirac (1938) that, assuming the gravitational constant was variable with respect to atomic standards of measurement, Einstein's equations should hold for units which vary appropriately with respect to the atomic standards.

Perhaps this view emphasizes the geometrization of gravitation achieved by General Relativity and the special importance placed on the interpretation of the Einstein tensor.

In contrast, the assumption of the following chapters is that one wants the scalar field to be an integral part of the description of gravitation — for the purpose of giving position dependence to the gravitational constant, inertial mass or just to offer new models of the Universe — and therefore the scalar-tensor and Einstein's theories of gravitation are to be on equal footing as gravitation theories.