ACTA
ACADEMIAE PAEDAGOGICAE AGRIENSIS
NOVA SERIES TOM. XXIV.

AZ ESZTERHÁZY KÁROLY TANÁRKÉPZŐ FŐISKOLA
TUDOMÁNYOS KÖZLEMÉNYEI

REDIGIT—SZERKESZTI
PÓCS TAMÁS, V. RAISZ RÓZSA

SECTIO MATEMATICAE

TANULMÁNYOK A MATEMATIKAI TUDOMÁNYOK KÖRÉBŐL

REDIGIT—SZERKESZTI
KISS PÉTER, RIMÁN JÁNOS

EGER, 1997

ACTA
ACADEMIAE PAEDAGOGICAE AGRIENSIS

NOVA SERIES TOM. XXIV.

AZ ESZTERHÁZY KÁROLY TANÁRKÉPZŐ FŐISKOLA
TUDOMÁNYOS KÖZLEMÉNYEI

REDIGIT—SZERKESZTI
PÓCS TAMÁS, V. RAISZ RÓZSA

SECTIO MATEMATICAE

TANULMÁNYOK A MATEMATIKAI TUDOMÁNYOK KÖRÉBŐL

REDIGIT—SZERKESZTI
KISS PÉTER, RIMÁN JÁNOS

EGER, 1997

EMTEX—JATEX

EKTF
LÍCEUM KIADÓ

A kiadásért felelős:
az Eszterházy Károly Tanárképző Főiskola főigazgatója
Megjelent az EKTF Líceum Kiadó műszaki gondozásában
Kiadóvezető: Csőke Lajos
Felelős szerkesztő: Rimán János
Műszaki szerkesztő: Szabó Tünde
Megjelent: 1997 Példányszám: 50

Készült: Molnár és Társa '2001' Kft. nyomdája, Eger
Ügyvezető igazgató: Molnár György

A characterization of the identity function

BUI MINH PHONG*

Abstract. We prove that if a multiplicative function f satisfies the equation $f(n^2+m^2+3)=f(n^2+1)+f(m^2+2)$ for all positive integers n and m, then either $f(n)$ is the identity function or $f(n^2+m^2+3)=f(n^2+1)=f(m^2+2)=0$ for all positive integers.

Throughout this paper $\mathbf{N}$ denotes the set of positive integers and let $\mathcal{M}$ be the set of complex valued multiplicative functions f such that $f(1)=1$.

In 1992, C. Spiro [3] showed that if $f \in \mathcal{M}$ is a function such that $f(p+q)=f(p)+f(q)$ for all primes p and q, then $f(n)=n$ for all $n \in \mathbf{N}$. Recently, in the paper [2] written jointly with J. M. de Koninck and I. Kátai we proved that if $f \in \mathcal{M}$, $f(p+n^2)=f(p)+f(n^2)$ holds for all primes p and $n \in \mathbf{N}$, then $f(n)$ is the identity function. It follows from results of [1] that a completely multiplicative function f satisfies the equation $f(n^2+m^2)=f(n^2)+f(m^2)$ for all $n,m \in \mathbf{N}$ if and only if $f(2)=2$, $f(p)=p$ for all primes $p \equiv 1 \pmod 4$ and $f(q)=q$ or $f(q)=-q$ for all primes $p \equiv 3$ (mod 4).

The purpose of this note is to prove the following

Theorem. *Assume that $f \in \mathcal{M}$ satisfies the condition*

$$f\left(n^2+m^2+3\right)=f(n^2+1)+f(m^2+2) \tag{1}$$

for all $n,m \in \mathbf{N}$. Then either

$$f(n^2+1)=f(m^2+2)=f(n^2+m^2+3)=0 \quad \text{for all} \quad n,m \in \mathbf{N}, \tag{2}$$

or $f(n)=n$ for all $n \in \mathbf{N}$.

Corollary. *If $f \in \mathcal{M}$ satisfies the condition (1) and $f(n_0^2+1) \neq 0$ for some $n_0 \in \mathbf{N}$, then $f(n)$ is the identity function.*

First we prove the following lemma.

Lemma. *Assume that the conditions of Theorem 1 are satisfied. Then either (2) is satisfied for all $n \in \mathbf{N}$ or the conditions*

$$\begin{gathered} f(n^2+1)=n^2+1, \quad f(m^2+2)=m^2+2 \quad \text{and} \\ f(n^2+m^2+3)=n^2+m^2+3 \end{gathered} \tag{3}$$

* Research (partially) supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA 2153 and T 020295.

simultaneously hold for all $n, m \in \mathbf{N}$.

Proof. From (1), we have

$$f(n^2+1)+f(m^2+2)=f(m^2+1)+f(n^2+2)$$

for all $n, m \in \mathbf{N}$, and so

$$f(n^2+2)-f(n^2+1)=f(3)-f(2):=D \quad \text{for all} \quad n \in \mathbf{N}. \tag{4}$$

Thus, the last relation together with (1) implies that

$$f(n^2+m^2+3)=f(n^2+1)+f(m^2+1)+D \tag{5}$$

holds for all $n, m \in \mathbf{N}$. Let $S_j := f(j^2+1)$. It follows from (5) that if k, l, u and $v \in \mathbf{N}$ satisfy the condition

$$k^2+l^2=u^2+v^2,$$

then

$$f(k^2+1)+f(l^2+1)+D=f(u^2+1)+f(v^2+1)+D,$$

which shows that

$$k^2+l^2=u^2+v^2 \quad \text{implies} \quad S_k+S_l=S_u+S_v. \tag{6}$$

We shall prove that

$$S_{n+12}=S_{n+9}+S_{n+8}+S_{n+7}-S_{n+5}-S_{n+4}-S_{n+3}+S_n \tag{7}$$

holds for all $n \in \mathbf{N}$.

Since

$$(2j+1)^2+(j-2)^2=(2j-1)^2+(j+2)^2$$

and

$$(2j+1)^2+(j-7)^2=(2j-5)^2+(j+5)^2,$$

we get from (6) that

$$S_{2j+1}+S_{j-2}=S_{2j-1}+S_{j+2} \tag{8}$$

and

$$S_{2j+1}+S_{j-7}=S_{2j-5}+S_{j+5}.$$

These with (8) imply that

$$\begin{aligned} S_{j+5} - S_{j+2} + S_{j-2} - S_{j-7} &= S_{2j-1} - S_{2j-5} \\ &= S_{j+1} - S_{j-3} + S_{2j-3} - S_{2j-5} = S_{j+1} - S_{j-3} + S_j - S_{j-4}, \end{aligned}$$

which proves (7) with $n = j - 7$.

By (8), we have

$$S_7 = S_{2\cdot 3+1} = 2S_5 - S_1,$$

$$S_9 = S_{2\cdot 4+1} = S_7 + S_6 - S_2 = S_6 + 2S_5 - S_2 - S_1$$

and

$$S_{11} = S_{2\cdot 5+1} = S_9 + S_7 - S_3 = S_6 + 4S_5 - S_3 - S_2 - 2S_1.$$

Finally, by using (6) and the facts

$$8^2 + 1^2 = 7^2 + 4^2, \quad 10^2 + 5^2 = 11^2 + 2^2 \quad \text{and} \quad 12^2 + 1^2 = 9^2 + 8^2,$$

we have

$$S_8 = S_7 + S_4 - S_1 = 2S_5 + S_4 - 2S_1,$$

$$S_{10} = S_{11} + S_2 - S_5 = S_6 + 3S_5 - S_3 - 2S_1$$

and

$$S_{12} = S_9 + S_8 - S_1 = S_6 + 4S_5 + S_4 - S_2 - 4S_1.$$

Thus, to complete the proof of the lemma , by using (1), (4), (5) and (7), it is enough to prove that either $S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = 0$ or

(9) $$S_j = j^2 + 1 \quad \text{for} \quad j = 1, 2, 3, 4, 5, 6.$$

Repeated use of (1), using the multiplicativity of f, gives $S_1 = f(1^2 + 1) = f(2)$,

(10) $$S_2 = f(2^2 + 1) = f(5) = f(1^2 + 1^2 + 3) = f(2) + f(3),$$

(11) $$S_3 = f(3^2 + 1) = f(10) = f(2)f(5) = f(2)^2 + f(2)f(3).$$

and thus

$$f(11) = f(2^2 + 2^2 + 3) = f(5) + f(6) = f(2) + f(3) + f(2)f(3).$$

On the other hand, it follows from (4) that

$$\begin{aligned} f(11) &= f(3^2+2) = f(10) + D = f(2)f(5) + D \\ &= f(2)^2 + f(2)f(3) + f(3) - f(2), \end{aligned}$$

which, together with the last relation, implies

$$f(2)^2 = 2f(2), \tag{12}$$

and

$$f(13) = f(1^2+3^2+3) = f(2) + f(11) = 2f(2) + f(2)f(3) + f(3).$$

Finally, the relation (10) together with the fact

$$f(8) = f(1^2+2^2+3) = f(2) + f(6) = f(5) + f(3)$$

show that

$$f(2)f(3) = 2f(3). \tag{13}$$

Moreover

$$S_5 = f(5^2+1) = f(26) = f(2)f(13) = 4f(2) + 6f(3), \tag{14}$$

$$\begin{aligned} S_6 &= f(6^2+1) = f(37) = f(3^2+5^2+3) \\ &= f(11) + f(26) = 5f(2) + 9f(3), \end{aligned} \tag{15}$$

$$2f(17) = f(4^2+4^2+3) - D = f(35) - D = f(5)f(7) - D, \tag{16}$$

and

$$f(3)f(7) = f(21) = f(3^2+3^2+3) = 2f(10) + D = 3f(2) + 5f(3). \tag{17}$$

The equation (12) shows that either $f(2) = 0$ or $f(2) = 2$. Assume that $f(2) = 0$. Then (13) implies that $f(3) = 0$ and so, by using (10)–(17) we have

$$S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = 0,$$

from which follows that (2) is true.

Assume now that $f(2) = 2$. In this case we have $f(5) = 2 + f(3)$, $f(8) = 2 + 2f(3)$. We shall prove that $f(3) = 3$. It follows from (15) and using the fact

$$f(37) = f(1^2 + 6^2 + 3) - f(3) = f(5)f(8) - f(3) = 2f(3)^2 + 5f(3) + 4$$

that

$$2f(3)^2 - 4f(3) - 6 = 0. \tag{17}$$

On the other hand, from (4) we infer that

$$f(6)f(11) - f(3)f(13) = f(66) - f(65) = f(3) - f(2),$$

consequently

$$3f(3)^2 - 7f(3) - 6 = 0.$$

This together with (17) proves that $f(3) = 3$, and so (10)–(17) imply that

$$S_j = j^2 + 1 \quad (j = 1, 2, 3, 4, 5, 6).$$

This completes the proof of (9) and so the lemma is proved.

Proof of the theorem

In the proof of the theorem, using the lemma, we can assume that (3) is satisfied, that is

$$\begin{aligned} &f(n^2 + 1) = n^2 + 1, \quad f(m^2 + 2) = m^2 + 2 \quad \text{and} \\ &f(n^2 + m^2 + 3) = n^2 + m^2 + 3. \end{aligned} \tag{18}$$

It is clear from (18) that $f(n) = n$ for all $n \leq 7$.

Assume that $f(n) = n$ for all $n < T$, where $T > 7$. We shall prove that $f(T) = T$. It is clear that T must be a prime power, that is $T = q^\alpha$ with $\alpha \in \mathbf{N}$ and some prime q .

It is easily seen that if $\alpha = 1$, then $q > 7$ and there are positive integers $n, m \leq \frac{q-1}{2}$ such that $n^2 + m^2 + 3 = qN$, $(q, N) = 1$ and $N < q$. Thus, we have $f(q) = q$.

Assume now that $\alpha \geq 2$ and $q > 3$. We consider the congruence

$$n^2 + m^2 + 3 \equiv 0 \pmod{q^\alpha}.$$

Let

$$\mathcal{A}_q(3) := \left\{1 \leq m \leq q - 1 : \left(\frac{-m^2 - 3}{q}\right) = 1\right\}.$$

Then we have

$$\sharp\mathcal{A}_q(3) = \sum_{\substack{m=1\\(m^2+3,q)=1}}^{q-1} \frac{1}{2}\left(1+\left(\frac{-m^2-3}{q}\right)\right) = \sum_{m=0}^{q-1}\frac{1}{2}\left(1+\left(\frac{-m^2-3}{q}\right)\right)$$

$$-\sum_{\substack{m=0\\ q\mid m^2+3}}^{q-1} \frac{1}{2}\left(1+\left(\frac{-m^2-3}{q}\right)\right) - \frac{1}{2}\left(1+\left(\frac{-3}{q}\right)\right)$$

$$= \frac{1}{2}\left(q-\left(\frac{-1}{q}\right)-2-2\left(\frac{-3}{q}\right)\right).$$

By our assumption, the last relation implies that $\sharp\mathcal{A}_q(3) \geq 1$. Thus, there are integers $m \in \{1,\dots,q-1\}$, $1 \leq n_1 \leq q^\alpha - 1$, $(n_1,q) = 1$ and $1 \leq n_2 := q^\alpha - n_1 \leq q^\alpha - 1$ such that

$$n_i^2 + m^2 + 3 = q^\alpha N_i \quad (i = 1,2).$$

It follows from the above relations that

$$q^\alpha(N_2 - N_1) = (q^\alpha - n_1)^2 - n_1^2 = q^{2\alpha} - 2q^\alpha n_1,$$

that is

$$N_2 - N_1 = q^\alpha - 2n_1.$$

Since $(n_1,q) = 1$, we obtain that at least one of N_1 or N_2 is coprime to q. Let $n \in \{n_1,n_2\}$ and $N \in \{N_1,N_2\}$ such that $n^2 + m^2 + 3 = q^\alpha N$, $(N,q) = 1$. Then $\alpha \geq 2$ implies that

$$N \leq \frac{1}{q^\alpha}\left[(q^\alpha-1)^2 + (q-1)^2 + 3\right] < q^\alpha.$$

Thus,

$$Nf(q^\alpha) = f(N)f(q^\alpha) = f(Nq^\alpha) = f(n^2+m^2+3) = n^2+m^2+3 = Nq^\alpha,$$

which shows that $f(q^\alpha) = q^\alpha$ as we wanted to establish.

To complete the proof of the theorem, it remains to consider the cases $q = 2$ and $q = 3$. Let $q = 2$ and $T = 2^\alpha$, where $\alpha \geq 3$.

Since $-7 \equiv 1 \pmod 8$, we have -7 is a quadratic residue modulo 2^α and therefore there exists $n_\alpha \in [0, 2^{\alpha-1}-1]$ such that $n_\alpha^2+7 = n_\alpha^2+2^2+3 \equiv 0 \pmod{2^\alpha}$, and consequently, $[n_\alpha + 2^{\alpha-1}]^2 + 7 \equiv 0 \pmod{2^\alpha}$. Define N_1, and N_2 by $n_\alpha^2 + 7 = 2^\alpha N_1$ and $[n_\alpha + 2^{\alpha-1}]^2 + 7 = 2^\alpha N_2$. We easily deduce from these two equation and the fact $7 < T = 2^\alpha$ that

$$N_1 < 2^\alpha, \quad N_2 < 2^\alpha \quad \text{and} \quad N_2 - N_1 = n_\alpha + 2^{\alpha-2}.$$

It follows from the last relation and the fact 2 does not divide n_α that one of N_1 or N_2 is odd, and so $f(2^\alpha) = 2^\alpha$.

Finally, let $q = 3$ and $T = 3^\alpha$, where $\alpha > 1$. We consider the congruence

$$n^2 + 2 \equiv 0 \pmod{3^\alpha}.$$

Similarly as above, one can deduce that there are positive integers $n, N \in \mathbf{N}$ such that $n^2 + 2 = 3^\alpha N$, $(N, 3) = 1$ and $N < 3^\alpha$. Thus these together with (18) implies that $f(3^\alpha) = 3^\alpha$.

The theorem is proved.

References

[1] P. V. Chung, Multiplicative functions satisfying the equation $f(m^2 + n^2) = f(m^2) + f(n^2)$, *Math. Slovaca*, **46** (1996), No. 2–3, 165–171.

[2] J.-M. De Koninck, I. Kátai and B. M. Phong, A new charateristic of the identity function, *J. Number Theory* (to appear).

[3] C. Spiro, Additive uniqueness set for arithmetic functions, *J. Number Theory* **42** (1992), 232–246.

Bui Minh Phong
Department of Computer Algebra
Eötvös Loránd University
Múzeum krt. 6-8
H-1088 Budapest,
Hungary

The generalization of Pascal's triangle from algebraic point of view

GÁBOR KALLÓS*

Abstract. In this paper we generalize Pascal's Triangle and examine the connections between the generalized triangles and powering integers and polynomials respectively.

The interesting and really romantic Pascal's Triangle is a favourite research field of mathematicians for a very long time. The table of binomial coefficients has been named after Blaise Pascal, a French scientist, but was known already by the ancient Chinese and others before Pascal (Edwards [1]).

Among the elements of the triangle a lot of interesting connections exist. One of them is that from the n-th row of the triangle with positional addition we get the n-th power of 11 (Figure 1.), where n is a non-negativ integer, and the indices in the rows and columns run from 0.

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \\ &&&&&\dots&&&&& \end{array}$$

$$1 = 11^0,\ 11 = 11^1,\ 121 = 11^2,\ 1331 = 11^3,\ 14641 = 11^4,\ 161051 = 11^5, \dots$$

Figure 1: The powers of 11 in Pascal's triangle

This comes immediately from the binomial equality

$$\binom{n}{0}10^n + \binom{n}{1}10^{n-1} + \binom{n}{2}10^{n-2} + \dots + \binom{n}{n-1}10^1 + \binom{n}{n}10^0 = 11^n$$

An interesting way of generalizing is if we construct triangles in which the powers of other numbers appear. To achieve this, let us consider Pascal's Triangle as the 11-based triangle, and take the following.

---

* This paper was completed during the stay of the author at Paderborn University, Germany in summer 1996.

Definition. Let a and b integers, with $0 \leq a, b \leq 9$. Then we can get the k-th element in the n-th row of the ab-based triangle if we add the $k-1$-th elemetn in the $n-1$-th row b-times to the k-th element in the $n-1$-th row a-times. If $k-1 < 0$ or $k > n-1$ (id est the element in the $n-1$-th row does not exist according to the traditional implementation) then we consider this element to be 0 (Figure 2.). The indices in the rows and columns of the ttiangle run from 0.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 4 & & 7 & & \\
 & 16 & & 56 & & 49 & \\
64 & & 336 & & 588 & & 343 \\
 & & & \cdots & & &
\end{array}
$$

Figure: 2: The 47-based triangle

Example. In the third row of the 47-based triangle $64 = 7 \cdot 0 + 4 \cdot 16$ and $336 = 7 \cdot 16 + 4 \cdot 56$.

Proposition 1. *By positional addition from the n-th row of the ab-based triangle we get the n-th power of $ab(10a+b)$.*

Proof. From the expansion of $(10a+b)^n$ we get

$$(10a+b)^n = \binom{n}{0} a^n 10^n + \binom{n}{1} a^{n-1} b 10^{n-1} + \cdots + \binom{n}{n-1} a b^{n-1} 10 + \binom{n}{n} b^n.$$

This is exactly the number we get after positional addition from the n-th row.

The structure of the ab-based triangle is relatively simple. We have the following.

Proposition 2. *The k-th element in the n-th row of the ab-based triangle is $a^{n-k} b^k C_n^k$, where C_n^k (the number of combinations of n things taken k at a times) is the k the element in the n-th row of Pascal's Triangle.*

Proof. We prove by induction. In the first row we have $a = a^1 \cdot 1$ and $b = b^1 \cdot 1$. Let us now assume, that the $k-1$-th element in the $n-1$-th row is $a^{n-k} b^{k-1} C_{n-1}^{k-1}$ and the k-th element in the $n-1$-th row is $a^{n-k-1} b^k C_{n-1}^k$. Then the k-th element in the n-th row by definition is

$$
\begin{aligned}
b a^{n-k} b^{k-1} C_{n-1}^{k-1} + a a^{n-k-1} b^k C_{n-1}^k &= a^{n-k} b^k C_{n-1}^{k-1} + a^{n-k} b^k C_{n-1}^k \\
&= a^{n-k} b^k \left(C_{n-1}^{k-1} + C_{n-1}^k \right) = a^{n-k} b^k C_n^k.
\end{aligned}
$$

Proposition 3. Connection with the binomial theorem.

The elements in the n-th row of the ab-based triangle are the coefficients of the polynomials $(ax + by)^n$.

Proof. If we substitute ax with $10a$ and by with b in the Proof of Proposition 1, and use that the k-th element in the n-th row of the ab-based triangle is $a^{n-k}b^kC_n^k$ we get the statement.

Example. From the 47-based triangle $(4x + 7y)^3 = 64x^3 + 336x^2y + 588xy^2 + 343y^3$.

The base-number of the triangle can consist of not only 2, but arbitrarily many digits.

Definition. Let $0 \leq a_0, a_1, a_2, \ldots, a_{m-2}, a_{m-1} \leq 9$ be integers. Then we can get the k-th element in the n-th row of the $a_0a_1a_2 \ldots a_{m-2}a_{m-1}$-based triangle if we multiply the $k-m$-th element in the $n-1$-th row by a_{m-1}, the $k-m+1$-th element in the $n-1$-th row by a_0, and add the products. If for some i we have $k-m+i < 0$ or $k-m+i > n-1$ (id est some element in the $n-1$-th row does not exists according to the traditional implementation) then we consider this element to be 0. The indices in the rows and columns of the triangle run from 0 (Figure 3.).

1

4 3 5

16 24 49 30 25

64 144 348 387 435 225 125

...

Figure 3: The 435-based triangle

Remarks. In the above definition we can allow for the base-number not only $0 \leq a_0, a_1, a_2, \ldots, a_{m-2}, a_{m-1} \leq 9$ digits, but arbitrary integers, rational and irrational numbers. Thus for example we can build triangles with base of root expressions (Figure 4.).

1

$\sqrt{2}$ $\sqrt{3}$ $\sqrt{5}$

2 $2\sqrt{6}$ $2\sqrt{10}+3$ $2\sqrt{15}$ 5

$2\sqrt{2}$ $6\sqrt{3}$ $6\sqrt{5}+9\sqrt{30}$ $6\sqrt{30}+3\sqrt{3}$ $9\sqrt{5}+15\sqrt{2}$ $15\sqrt{3}$ $5\sqrt{5}$

...

Figure 4: The $\sqrt{2}$ $\sqrt{3}$ $\sqrt{5}$-based triangle

In the combinatorical literature the 11...1-based triangles (k pieces of 1 digits) with name order k (Vilenkin [2]) or k-th Pascal's triangle (Gerőcs [3]) can be found. The authors gave this different definition because of different approach.

Theorem 1. *From the n-th row of the $a_0a_1a_2\dots a_{m-2}a_{m-1}$-based triangle after positional addition we get the n-th power of the base-number $a_0a_1a_2\dots a_{m-2}a_{m-1}$ is obviously in the first row of the triangle.*

Let us now assume, that in the $n-1$-th row ($n > 1$) we have the elements $b_0, b_1, b_2, \dots, b_{p-1}, b_p$ (where p equals to $m+(m-1)(n-2)-1 = mn-m-n+1$, because in the first row there are m pieces of elements and in every new row there are $m-1$ pieces more), and from these elements with positional addition we get the $n-1$-th power of the number $a_0a_1a_2\dots a_{m-2}a_{m-1}$. Then we can write out $(a_0 10^{m-1} + a_1 10^{m-2} + \cdots + a_{m-2}10 + a_{m-1})^n$ as

$$\begin{aligned}(b_0 10^p + b_1 10^p + b_1 10^{p-1} + \cdots + b_{p-1}10 + b_p)a_0 10^{m-1}\\ + (b_0 10^p + b_1 10^{p-1} + \cdots + b_{p-1}10 + b_p)a_1 10^{m-2}\\ \dots\\ + (b_0 10^p + b_1 10^{p-1} + \cdots + b_{p-1}10 + b_p)a_{m-2}10\\ + (b_0 10^p + b_1 10^{p-1} + \cdots + b_{p-1}10 + b_p)a_{m-1}.\end{aligned}$$

By adding these expressions (using that $p = mn - m - n + 1$) we get

$$\begin{aligned}&a_0b_0 10^{mn-n} + (a_0b_1 + a_1b_0)10^{mn-n-1} + (a_0b_2 + a_1b_1 + a_2b_0)10^{mn-n-2}\\ &+ \cdots + (a_{m-1}b_0 + a_{m-2}b_1 + a_{m-3}b_2 \cdots + a_1b_{m-2} + a_0b_{m-1})10^{mn-m-n-1}\\ &+ \cdots + (a_{m-2}b_p + a_{m-1}b_{p-1})10 + a_{m-1}b_p.\end{aligned}$$

And this is exactly the number we get after positional addition from the n-th row of the triangle.

Consideration of effectivity. This method is easy to algorithmize, it is enough to store the proceeding row and the base-number to determine one row of the triangle. In the row we work with relatively small numbers (compare with the final result), and we have to multiply only with digits. However, to reach the n-th row we need to determine $(2 + (n-1)(m-1))\frac{n}{2} = O(n^2m)$ elements. So obviously, if we need only the $n$the power of the base-number some other methods are more effective (Knuth [4]). However, if we need all the (non-negative integer) powers up to n of the base-number this method is competitive. It is especially interesting that with this method the first some powers of a base number of a few digits can even be determined by heart. It is similar to some methods of by heart calculate artists (Surányi [5]).

In Proposition 3 we have seen a connection of the ab-based triangle with the binomial theorem. Thus, we expect for the $a_0a_1a_2\dots a_{m-2}a_{m-1}$-based triangle a relation with the polynomial theorem. However, the structure of the latter triangle is much more complicated. See for example the triangle with abc base-number (Figure 5.) The elements in the n-th row are some sums of the coefficients of the polynomials $(ax+by+cz)^n$.

$$1$$
$$a \qquad b \qquad c$$
$$a^2 \qquad 2ab \qquad 2ac+b^2 \qquad 2bc \qquad c^2$$
$$a^3 \qquad 2a^2b \qquad 3a^2c+3ab^2 \qquad 6abc+b^3 \qquad 3ac^2+3b^2c \qquad 3bc^2 \qquad c^3$$
$$\dots$$

Figure 5: The abc-based triangle

To discover the connection of the general triangle with the polynomial theorem we need the following.

Definition. For the digits of the base-number let he weight of a digit be its distance from the centerline. So $w(a_0) = -w(a_{m-1})$, $w(a_1) = -w(a_{m-2})$, etc. If the base number is odd, then $w(a_{(m-1)/2}) = 0$. Let the unit of the weights be the distance of two neighbouring elements in the triangle, id est $w(a_i) = w(a_{i+1}) = 1$.

Example. In the abc-based triangle $w(a) = -1$, $w(b) = 0$ and $w(c) = 1$, in the $abcd$-based triangle $w(a) = -1.5$, $w(b) = -0.5$, $w(c) = 0.5$ and $w(d) = 1.5$.

We would like to extend this idea to the elements of the other rows. Because the elements of the triangle are sums, consider first the parts of them. For such an expression let the weight of the part be the sum of the weights of its digits. If a digit is on the i-th power then we count its weight i-times.

Example. One part of the third element in the third row of the abc-based triangle is $3a^2c$ (Figure 5.). For this expression we have $q(3a^2c) = 2w(a) + w(c) = -1$.

Lemma 1. *In an element of the general triangle the weights of the parts are identical, and this weight is the distance of the element from the centerline.*

Proof. We get this result by induction immediately from the construction of the triangle.

Lemma 2. *Let us consider an expression* $a_0^{i_0} a_1^{i_1} \cdots a_{m-2}^{i_{m-2}} a_{m-1}^{i_{m-1}}$, *for which* $i_0 + i_1 + \cdots + i_{m-2} + i_{m-1} = n$. *Then we can find this expression with some coefficient as a part of the element with the same weight in the* n*-th row of the general triangle.*

Proof. Let us assume indirectly that this expression does not exist in the n-th row of the general triangle as a part of the element with corresponding weight. We should get this expression from parts of elements of the previous row

$$\Big(a_0^{i_0-1} a_1^{i_1} \cdots a_{m-2}^{i_{m-2}} a_{m-1}^{i_{m-1}}, a_0^{i_0} a_1^{i_1-1} \cdots a_{m-2}^{i_{m-2}} a_{m-1}^{i_{m-1}},$$

$$a_0^{i_0} a_1^{i_1} \cdots a_{m-2}^{i_{m-2}-1} a_{m-1}^{i_{m-1}}, a_0^{i_0} a_1^{i_1} \cdots a_{m-2}^{i_{m-2}} a_{m-1}^{i_{m-1}-1}\Big)$$

with multiplication (by $a_0, a_1, \ldots, a_{m-2}, a_{m-1}$). Thus, these parts of the elements can't exist in the previous row. Proceeding backwards with this method we conclude that in the first line some digits of the base-number do not exist, and this is a contradiction.

Lemma 3. *For the coefficient* e *of the expression* $ea_0^{i_0} a_1^{i_1} \cdots a_{m-1}^{i_{m-1}}$ *with* $i_0 + i_1 + \cdots + i_{m-1} = n$ *in the* n*-th row of the general triangle we have* $e = \frac{n!}{i_0! i_1! \cdots i_{m-1}!}$.

Proof. We prove with induction. In the first row the statement is true. Let us now assume that in the $n-1$-th row there are the following expressions as parts of the elements:

$$e_0 a_0^{i_0-1} a_1^{i_1} \cdots a_{m-1}^{i_{m-1}}, \ e_1 a_0^{i_0} a_1^{i_1-1} \cdots a_{m-1}^{i_{m-1}}, \ldots, \ e_{m-1} a_0^{i_0} a_1^{i_1} \cdots a_{m-1}^{i_{m-1}-1},$$

with coefficients

$$e_0 = \frac{(n-1)!}{(i_0-1)! i_1! \cdots i_{m-1}!}, \quad e_1 = \frac{(n-1)!}{i_0!(i_1-1)! \cdots i_{m-1}!}, \ldots,$$

$$e_{m-1} = \frac{(n-1)!}{i_0! i_1! \cdots (i_{m-1}-1)!}$$

(by the induction assumption). Thus, the coefficient e of the expression $ea_0^{i_0} a_1^{i_1} \cdots a_{m-1}^{i_{m-1}}$ is the sum of the coefficients e_i $(0 \le i \le m-1)$

$$e = e_0 + e_1 + \cdots + e_{m-1} = \frac{(n-1)!}{(i_0-1)!(i_1-1)! \cdots (i_{m-1}-1)!}$$

$$\left(\frac{1}{i_1 i_2 \cdots i_{m-1}} + \frac{1}{i_0 i_2 \cdots i_{m-3} i_{m-1}} + \cdots + \frac{1}{i_0 \cdots i_{m-3} i_{m-2}}\right) =$$

$$\frac{(n-1)!}{(i_0-1)!(i_1-1)!\cdots(i_{m-1}-1)!}\left(\frac{i_0+i_1+\cdots+i_{m-1}}{i_0 i_1 \cdots i_{m-1}}\right) = \frac{n!}{i_0! i_1! \cdots i_{m-1}!}.$$

By these three Lemmas we have proved the following.

Theorem 2. *The elements in the n-th row of the $a_0 a_1 a_2 \cdots a_{m-2} a_{m-1}$-based triangle are exactly the sums of the coefficients of the polynomial $(a_0 x_0 + a_1 x_1 + a_2 x_2 + \cdots + a_{m-2} x_{m-2} + a_{m-1} x_{m-1})^n$, in which the weights of the parts are identical.*

Like among the binomial coefficients in Pascal's triangle (for example Edwards [1] and Vilenkin [2]), in the general triangle there are also interesting connecitons among the elements. One of them comes immediately from the second Theorem.

Corollary. *In the n-th row of the $a_0 a_1 a_2 \ldots a_{m-2} a_{m-1}$-based triangle the sum of the elements (with normal addition) is $(a_0 + a_1 + a_2 + \cdots + a_{m-2} + a_{m-1})^n$.*

Proof. If we set in the polynomial

$$(a_0 x_0 + a_1 x_1 + a_2 x_2 + \cdots + a_{m-2} x_{m-2} + a_{m-1} x_{m-1})^n,$$

$1 = x_0 = x_1 = x_2 = \cdots = x_{m-2} = x_{m-1}$, then from Theorem 2 in the n-th row of the triangle there are the coefficients of the "polynomial" $(a_0 + a_1 + a_2 + \cdots + a_{m-2} + a_{m-1})^2$.

Remark. In Pascal's triangle from this Corollary we get the well known combinatorical equality

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Another possibility to power polynomials is that we extend the property for the general triangle, that the elements in the n-th row of Pascal's Triangle are the coefficients of the binomial $1 + x$.

Proposition 4. *The elements in the n-th row of the general triangle are exactly the coefficients of the polynomials $(a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-2} x^{m-2} + a_{m-1} x^{m-1})^n$, the k-th element is the coefficient of x^k.*

Proof. We prove by induction. In the first row the statement is true. Let us now assume, that in the $n-1$-th row there are the coefficients of the polynomial $(a_0 + a_1x + a_2x^2 + \cdots + a_{m-2}x^{m-2} + a_{m-1}x^{m-1})^{n-1}$, the k-th element is the coefficient of x^k. If we multiply this polynomial by $a_0 + a_1x + a_2x^2 + \cdots + a_{m-2}x^{m-2} + a_{m-1}x^{m-1}$, and add up the results (similarly as in the proof of Theorem 1), we get the n-th power of the basic polynomial. But according to the forming rules of the triangle, the coefficients of this polynomial are exactly the elements of the n-th row.

Example. From the third row of the 435-based triangle (Figure 3.)

$$(4 + 3x + 5x^2)^3 = 64 + 144x + 348x^2 + 387x^3 + 435x^4 + 225x^5 + 125x^6.$$

Consideration of effectivity. The powering of polynomials is considerably more complex operation as powering of (integer) numbers. However, the consideration above applies here, too. So if we need only the n-th power of the base-polynomial some other methods are more effective (Knuth [4], Geddes [6].) However, if we need all the (non-negativ integer) powers up to n then this method is competitive.

References

[1] EDWARDS, A. W. F. Pascal's Arithmetical Triangle, Charles Griffin and Company Ltd, Oxford University Press, 1987.

[2] VILENKIN, N. J.: Kombinatorika. Tankönyvkiadó, Budapest, 1987.

[3] GERŐCS L.: A Fibonacci-sorozat általánosítása. Tankönyvkiadó, Budapest, 1988.

[4] KNUTH, D. E., The Art of Computer Programming, Vol. 2. Seminumerical Algoritms, Addison–Wesley, 1981.

[5] SURÁNYI, J.: Számoljunk ügyesen. KÖMAL XXV. kötet, 1962.

[6] GEDDES, K. O., CZAPOR, S. R., LABAHN, G., Algorithms for Computer Algebra, Kluwer Academic Publishers, 1991.

SZÉCHENYI ISTVÁN COLLEGE
HÉDERVÁRI U. 3.
H-9026 GYŐR, HUNGARY
E-mail: kallos@rs1.szif.hu

On some connections between Legendre symbols and continued fractions

ALEKSANDER GRYTCZUK

Abstract. In this note we give a complement of some results of Friesen given in [2] about some connections between Legendre symbols and continued fractions.

1. Introduction

In the paper [1] P. Chowla and S. Chowla gave several conjectures concerning continued fractions and Legendre symbols. Let $d = pq$, where p, q are primes such that $p \equiv 3 \pmod 4$, $q \equiv 5 \pmod 8$ and let $\sqrt{d} = [q_0; \overline{q_1, \ldots, q_s}]$ be the representation of $\sqrt{d}$ as a simple continued fraction. Denote by $S = \sum_{i=1}^{s} (-1)^{s-i} q_i$. Then P. Chowla and S. Chowla conjectured the following relationship: $\left(\frac{p}{q}\right) = (-1)^s$, where $\left(\frac{p}{q}\right)$ is the Legendre's symbol. This conjecture has been proved by A. Schinzel in [3]. Further interesting results for $d = pq \equiv 1 \pmod 4$ and for $d = 2pq$ was given by C. Friesen in [2]. From his results summarized in the Table 1 on page 365 of [2] it follows that in the following cases: $p \equiv 3 \pmod 8, q \equiv 1 \pmod 8$ or $p \equiv 7 \pmod 8$, $q \equiv 1 \pmod 8$ or $p \equiv 1 \pmod 8$, $q \equiv 3 \pmod 8$ or $p \equiv 1 \pmod 8$, $q \equiv 7 \pmod 8$ are not known a connection between Legendre's symbol and the representation of $\sqrt{pq}$ as a simple continued fraction. In this connection we prove the following Theorem:

Theorem. *Let $d = pq \equiv 3 \pmod 4$ and $\sqrt{pq} = [q_0; \overline{q_1, \ldots, q_s}]$, then $s = 2m$; $c_m = 2, p, q$; and*

$$\left(\frac{p}{q}\right) = (-1)^{m\frac{q-1}{2}}, \qquad \text{if } c_m = p$$

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{s+q-1}{2}}, \qquad \text{if } c_m = q$$

$$\left(\frac{2}{p}\right)\left(\frac{2}{q}\right) = (-1)^m, \qquad \text{if } c_m = 2$$

where c_m is defined by the following recurrent formulas:

$$q_m = \left[\frac{q_0 + b_m}{c_m}\right], \quad b_m + b_{m+1} = c_m q_m, \quad d = pq = b_{m+1}^2 + c_m c_{m+1}.$$

2. Proof of the Theorem

In the proof of the Theorem we use the following lemmas:

Lemma 1. *Let $\sqrt{d} = [q_0; \overline{q_1, \ldots, q_s}]$ be the representation of $\sqrt{d}$ as a simple continued fraction. Then*

(1) $q_n \left[\frac{q_0+b_n}{c_n}\right]$, $b_n + b_{n+1} = c_n q_n$, $d = b_{n+1}^2 + c_n c_{n+1}$, *for any integer* $n \geq 0$

(2) *if* $s = 2r+1$ *then minimal number* k*, for which* $c_k = c_{k+1}$ *is* $k = \frac{s-1}{2}$

(3) *if* $s = 2r$ *then minimal number* k*, for which* $b_k = b_{k+1}$ *is* $k = \frac{s}{2}$

(4) $1 < c_n < 2\sqrt{d}$, *for* $1 \leq n \leq s-1$

(5) $P_{n-1}^2 - dQ_{n-1}^2 = (-1)^n c_n$, *where* P_n/Q_n *is* n*-th convergent of* $\sqrt{d}$.

This Lemma is a collection of the well-known results of the theory of continued fractions.

Lemma 2. Let $\sqrt{d} = [q_0; \overline{q_1, \ldots, q_s}]$. The equation $x^2 - dy^2 = -1$ is solvable if and only if the period s is odd. Moreover, if $p \equiv 3 \pmod 4$ and p is a divisor of d then this equation is unsolvable.

This Lemma is well-known result given by Legendre in 1785.

For the proof of the Theorem we remark that by the condition $d = pq \equiv 3 \pmod 4$ it follows that $p \equiv 3 \pmod 4$ or $q \equiv 3 \pmod 4$ and consequently from Lemma 2 we obtain that the period $s = 2m$. From (5) of Lemma 1 we get

$$P_{m-1}^2 - pqQ_{m-1}^2 = (-1)^m c_m. \tag{6}$$

On the other hand by (1) and (3) of Lemma 1 it follows that

$$2b_{m+1} = q_m c_m, \quad d = pq = b_{m+1}^2 + c_m c_{m+1}. \tag{7}$$

From (7) we obtain

$$4pq = c_m(q_m^2 c_m + 4c_{m+1}). \tag{8}$$

By (8) it follows that $c_m = 1, 2, 4, p, q, pq, 2pq, 4pq$. Using (4) of Lemma 1 we get that $c_m = 1, 2, 4, p, q$. If $c_m = 1$ then it is easy to see that (6) is impossible. If $c_m = 4$ then from (6) we obtain

$$P_{m-1}^2 - pqQ_{m-1}^2 = (-1)^m 4. \tag{9}$$

Since $(P_{m-1}, Q_{m-1}) = 1$ then by (9) it follows that P_{m-1} and Q_{m-1} are odd and consequently we obtain $P_{m-1}^2 \equiv Q_{m-1}^2 \equiv 1 \pmod 4$. Since $pq \equiv 3 \pmod 4$ then by (9) it follows that $1 \equiv P_{m-1}^2 = pqQ_{m-1}^2 + (-1)^m 4 \equiv 3 \pmod 4$ and we get a contradiction. Therefore, we have $c_m = p, q, 2$. Let $c_m = p$ then from (6) we obtain

$$pX^2 - qQ_{m-1}^2 = (-1)^m, \quad \text{where} \quad P_{m-1} = pX. \tag{10}$$

From (10) and the well-known properties of Legendre's symbol we obtain

$$\left(\frac{p}{q}\right) = \left(\frac{(-1)^m}{q}\right) = \left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}m}. \tag{11}$$

In similar way, for the case $c_m = q$ we get

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}m}. \tag{12}$$

By (12) and the reciprocity law of Gauss we obtain

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{s+q-1}{2}}.$$

If $c_m = 2$ then by (6) it follows that $\left(\frac{2(-1)^m}{p}\right) = \left(\frac{2(-1)^m}{q}\right) = 1$. Hence, in virtue of $pq \equiv 3 \pmod 4$ we obtain $\left(\frac{2}{p}\right)\left(\frac{2}{q}\right) = (-1)^m$ and the proof is complete.

References

[1] P. Chowla and S. Chowla, Problems on periodic simple continued fractions, *Proc. Nat. Acad. Sci. USA* **69** (1972), 37–45.

[2] C. Friesen, Legendre symbols and continued fractions, *Acta Arith.* **59** 4. (1991), 365–379.

[3] A. Schinzel, On two conjectures of P. Chowla and S. Chowla concerning continued fractions, *Ann. Math. Pure Appl.* **98** (1974), 111–117.

Intitute of Mathematics
Department of Algebra and Number Theory
T. Kotarbiński Pedagogical University
Pl. Słowiański 9, 65-069 Zielona Góra
Poland

Remark on Ankeny, Artin and Chowla conjecture

ALEKSANDER GRYTCZUK

Abstract. In this paper we give two new criteria connected with well-known and still open conjecture of Ankeny, Artin and Chowla.

Introduction

In the paper [2] Ankeny, Artin and Chowla conjectured that, if $p \equiv 1 \pmod 4$ is a prime and $\varepsilon = 1/2(T+U\sqrt{p}) > 1$ is the fundamental unit of the quadratic number field $K = Q(\sqrt{p})$ then $p \nmid U$. It was shown by Mordell [5] in the case $p \equiv 5 \pmod 8$ and by Ankeny and Chowla [3] for the remaining primes $p \equiv 1 \pmod 4$ that $p \mid U$ if and only if $p \mid B_{\frac{p-1}{2}}$, where B_{2n} is $2n$-th Bernoulli number. Another criterion has been given by T. Agoh in [1]. Beach, Williams and Zarnke [4] verified the conjecture of Ankeny, Artin and Chowla for all primes $p < 6270713$. Sheingorn [6], [7] gave interesting connections between the fundamental solution $\langle x_0, y_0 \rangle$ of the non-Pellian equation

$$(1) \qquad x^2 - py^2 = -1, \quad p \equiv 1 \pmod 4, \quad p \text{ is a prime}$$

and the manner of the reflection lines on the modular surface and also of the $\sqrt{p}$ Riemann surface. We prove the following two theorems:

Theorem 1. *Let $p \equiv 1 \pmod 4$ be a prime and $p = b^2 + c^2$. Moreover, let $\sqrt{p} = [q_0; \overline{q_1, q_2, \ldots, q_s}]$ be the representation of $\sqrt{p}$ as a simple continued fraction and let $\langle x_0, y_0 \rangle$ be the fundamental solution of (1). Then $p \mid y_0$ if and only if $p \mid cQ_r + bQ_{r-1}$ and $p \mid Q_r - cQ_{r-1}$, where $r = \frac{s-1}{2}$ and P_n/Q_n is n-th convergent of $\sqrt{p}$.*

Theorem 2. *Assume that the assumptions of the Theorem 1 are satisfied. Then $p \mid y_0$ if and only if $p \mid 4bQ_rQ_{r-1} - (-1)^{r+1}$, where $r = \frac{s-1}{2}$ and P_n/Q_n is n-th convergent of $\sqrt{p}$.*

Basic Lemmas

Lemma 1. *Let $\sqrt{d} = [q_0; \overline{q_1, \ldots, q_s}]$ be the representation of $\sqrt{d}$ as a simple continued fraction. Then*

$$(2) \qquad q_n = \left[\frac{q_0 + b_n}{c_n}\right], \quad b_n + b_{n+1} = c_n q_n, \quad d = b_{n+1}^2 + c_n c_{n+1}$$

(3) *if* $s = 2r+1$ *then minimal number* k, *for which* $c_{k+1} = c_k$ *is* $k = \frac{s-1}{2}$,

$$dQ_{n-1} = b_n P_{n-1} + c_n P_{n-2}, \tag{4}$$
$$P_{n-1} = b_n Q_{n-1} + c_n Q_{n-2}, \tag{6}$$
$$P_{n-1}^2 - dQ_{n-1}^2 = (-1)^n c_n, \tag{7}$$

where P_n/Q_n *is the* n*-th convergent of* $\sqrt{d}$.

This Lemma is a collection of well-known results of the theory of continued fractions.

From Lemma 1 we can deduce for the case $d = p \equiv 1 \pmod 4$ and $r = \frac{s-1}{2}$ the following:

Lemma 2. *Let* $p \equiv 1 \pmod 4$ *be a prime and let* $\sqrt{p} = [q_0; \overline{q_1, \dots, q_s}]$, *where* $s = 2r+1$ *then*

$$p = b_{r+1}^2 + c_r^2 = b^2 + c^2; \quad b_{r+1} = b, \quad c_r = c \tag{8}$$
$$pQ_r = bP_r + cP_{r-1} \tag{9}$$
$$P_r = bQ_r + cQ_{r-1} \tag{10}$$
$$P_{r-1} = cQ_r - bQ_{r-1} \tag{11}$$
$$P_r Q_{r-1} - Q_r P_{r-1} = (-1)^{r+1} \tag{12}$$
$$P_r^2 - pQ_r^2 = (-1)^{r+1} c \tag{13}$$
$$P_{r-1}^2 - pQ_{r-1}^2 = (-1)^r c \tag{14}$$
$$P_{r-1}^2 + P_r^2 = p(Q_{r-1}^2 + Q_r^2). \tag{15}$$

Lemma 3. *Let* $\sqrt{d} = [q_0; \overline{q_1, \dots, q_s}]$ *and* $s = 2r+1$, *then* $Q_{s-1} = Q_{\frac{s-1}{2}-1}^2 + Q_{\frac{s-1}{2}}^2$ *and*

$$P_{s-1} = P_r Q_r + P_{r-1} Q_{r-1}.$$

Proof. First we prove that for $k = 1, 2, \dots, \frac{s-1}{2}$ we have

$$Q_{s-1} = Q_k Q_{s-(k+1)} + Q_{k-1} Q_{s-(k+2)}. \tag{16}$$

Really, since $q_{s-1} = q_1$, $Q_1 = q_1$, $Q_0 = 1$ then we obtain $Q_{s-1} = q_{s-1} Q_{s-2} + Q_{s-3} = Q_1 Q_{s-2} + Q_0 Q_{s-3}$ and (16) is true for $k = 1$. Suppose that (16) is true for $k = m$, i.e.

$$Q_{s-1} = Q_m Q_{s-(m+1)} + Q_{m-1} Q_{s-(m+2)}. \tag{17}$$

Then, for $k = m + 1$ in virtue of $Q_{s-(m+1)} = q_{s-(m+1)} Q_{s-m-2} + Q_{s-m-3}$ and $q_{s-(m+1)} = q_{m+1}$ we get $Q_{s-(m+1)} = q_{m+1} Q_{s-m-2} + Q_{s-m-3}$. By (17) and the last equality it follows that $Q_{s-1} = Q_{m+1} Q_{s-m-2} + Q_m Q_{s-m-3}$ and inductive proof of (16) is finished. Putting $k = \frac{s-1}{2}$ and ovbserving that $s - k - 1 = \frac{s-1}{2}$, $s - k - 2 = \frac{s-1}{2} - 1$, we obatin $Q_{s-1} = Q^2_{\frac{s-1}{2}-1} + Q^2_{\frac{s-1}{2}}$. In similar way we obtain that $P_{s-1} = P_r Q_r + P_{r-1} Q_{r-1}$ and the proof of Lemma 3 is complete.

Proof of Theorems

Proof of Theorem 1. Suppose that $p \mid y_0$. Then by (13) of Lemma 2 we have

$$c = (-1)^{r+1}(P_r^2 - pQ_r^2). \tag{18}$$

From Lemma 2 we also obtain

$$b = (-1)^{r+1}(pQ_r Q_{r-1} - P_r P_{r-1}). \tag{19}$$

Let $L = cQ_r + bQ_{r-1}$. Then by (18) and (19) it follows that

$$L = (-1)^{r+1}\left(P_r(P_r Q_r - P_{r-1}Q_{r-1}) - pQ_r(Q_r^2 - Q_{r-1}^2)\right). \tag{20}$$

On the other hand from Lemma 2 we have

$$P_r Q_r - P_{r-1} Q_{r-1} = b(Q_r^2 + Q_{r-1}^2). \tag{21}$$

Substituting (21) to (20) we obtain

$$L = (-1)^{r+1}\left(bP_r(Q_r^2 + Q_{r-1}^2) - pQ_r(Q_r^2 - Q_{r-1}^2)\right). \tag{22}$$

By Lemma 3 it follows that $y_0 = Q_{s-1} = Q_r^2 + Q_{r-1}^2$ and therefore from (22) we get $p \mid L$. From (10) and (11) of Lemma 2 we have

$$P_r^2 + P_{r-1}^2 = (bQ_r + cQ_{r-1})^2 + (cQ_r - bQ_{r-1})^2. \tag{23}$$

On the other hand it is well-known the following indentity:

$$(bQ_r + cQ_{r-1})^2 + (cQ_r - bQ_{r-1})^2 = (cQ_r + bQ_{r-1})^2 + (bQ_r - cQ_{r-1})^2. \tag{24}$$

From (23) and (24) we obtain

$$P_r^2 + P_{r-1}^2 = (cQ_r + bQ_{r-1})^2 + (bQ_r - cQ_{r-1})^2. \tag{25}$$

From (15) of Lemma 2 and the assumption that $p \mid y_0$ we obtain

$$p^2 \mid P_r^2 + P_{r-1}^2. \tag{26}$$

By (25), (26) and the fact that $p \mid L, L = cQ_r + bQ_{r-1}$ it follows that $p \mid bQ_r - cQ_{r-1}$. Now, we can prove the converse of the theorem. Assume that

$$p \mid cQ_r + bQ_{r-1}, \qquad p \mid bQ_r - cQ_{r-1}. \tag{27}$$

From (15) of Lemma 2 and Lemma 3 we obtain

$$P_r^2 + P_{r-1}^2 = p(Q_r^2 + Q_{r-1}^2) = pQ_{s-1} = py_0. \tag{28}$$

By (27) and (25) it follows that $p^2 \mid P_r^2 + P_{r-1}^2$ and therefore from (28) we get $p \mid y_0$. The proof of the Theorem 1 is complete.

Proof of the Theorem 2. From Lemma 3 we have $P_{s-1} = P_rQ_r + P_{r-1}Q_{r-1}$. Substituting (10) and (11) of Lemma 2 to this equality we obtain

$$P_{s-1} = b(Q_r^2 - Q_{r-1}^2) + 2cQ_rQ_{r-1}. \tag{29}$$

By (29) easily follows that

$$P_{s-1}^2 + 1 = b^2(Q_r^2 - Q_{r-1}^2)^2 + 4bcQ_rQ_{r-1}(Q_r^2 - Q_{r-1}^2) + 4c^2Q_r^2Q_{r-1}^2 + 1. \tag{30}$$

On the other hand from Lemma 2 we can deduce that

$$c(Q_r^2 - Q_{r-1}^2) + (-1)^{r+1} = 2bQ_rQ_{r-1}. \tag{31}$$

From (30) and (31) we obtain

$$c^2(P_{s-1}^2 + 1) = (b^2 + c^2)\left(4(b^2 + c^2)Q_r^2Q_{r-1}^2 - 4b(-1)^{r+1}Q_rQ_{r-1} + 1\right). \tag{32}$$

Since $\langle x_0, y_0\rangle = \langle P_{s-1}, Q_{s-1}\rangle$ then $P_{s-1}^2 + 1 = pQ_{q-1}^2$. Suppose that $p \mid y_0$. Then we have

$$p^3 \mid P_{s-1}^2 + 1. \tag{33}$$

By (33) and (32) it follows that

$$p \mid 4bQ_rQ_{r-1} - (-1)^{r+1}, \tag{34}$$

because $p = b^2 + c^2$. Now, we can assume that the relation (34) is satisfied. Using (32) we obtain

$$p^2 \mid c^2(P_{s-1}^2 + 1). \tag{35}$$

Since $p = b^2 + c^2$ and $(p, c) = 1$, by (35) it follows that

$$p^2 \mid P_{s-1}^2 + 1. \tag{36}$$

But $P_{s-1}^2 + 1 = pQ_{s-1}^2$ and consequently from (36) we obtain $p \mid Q_{s-1}$, $Q_{s-1} = y_0$. The proof of the Theorem 2 is complete.

From Theorem 1 we obtain the following:

Corollary. *Let $\langle x_0, y_0\rangle$ be fundamtental solution of the equation $x^2 - py^2 = -1$, where $p \equiv 1 \pmod 4$ is a prime such that $p = b^2 + c^2$ and let $\sqrt{p} = [q_0; \overline{q_1, q_2, \dots, q_s}]$, $s = 2r + 1$ be the representation of $\sqrt{p}$ as a simple continued fraction. If $p \mid y_0$ then $\mathrm{ord}_p(cQ_r - bQ_{r-1}) = 1$ or $\mathrm{ord}_p(bQ_r - cQ_{r-1}) = 1$.*

Proof. If $p \mid y_0$ then by the Theorem 1 it follows that $\alpha = \mathrm{ord}_p(cQ_r + bQ_{r-1}) \geq 1$ and $\beta = \mathrm{ord}_p(bQ_r - cQ_{r-1}) \geq 1$. Suppose that $\alpha \geq 2$ and $\beta \geq 2$. Then we have

$$p^2 \mid cQ_r + bQ_{r-1}, \qquad p^2 \mid bQ_r - cQ_{r-1}. \tag{37}$$

From (37) we obtain $p^2 \mid c^2Q_r + bcQ_{r-1}$ and $p^2 \mid b^2Q_r - bcQ_{r-1}$. Hence

$$p^2 \mid (b^2 + c^2)Q_r. \tag{38}$$

Since $p = b^2 + c^2$ then by (38) it follows that $p \mid Q_r$. By $y_0 = Q_{s-1} = Q_r^2 + Q_{r-1}^2$ and virtue of $p \mid y_0$, $p \mid Q_r$ we get $p \mid Q_{r-1}$. On the other hand from Lemma 2 we have $P_r = bQ_r + cQ_{r-1}$ and therefore we obtain $p \mid P_r$. Hence we have $p \mid P_r$ and $p \mid Q_r$, which is impossible because $(P_r, Q_r) = 1$. The proof is complete.

Remark. If the representation of $\sqrt{d}$ as a simple continued fraction has the period $s = 3$ then $d \nmid y_0$, where $\langle x_0, y_0\rangle$ is the fundamental solution of the non-Pellian equation $x^2 - dy^2 = -1$. Really, putting $s = 3$ in Lemma 3 we obtain

$$y_0 = Q_0^2 + Q_1^2 = 1 + q_1^2. \tag{39}$$

On the other hand it is well-known (see, [8]; Thm. 4, p. 323) that all natural numbers d, for which the representation of $\sqrt{d}$ as a simple continued fraction has the period $s = 3$ are given by the formula:

$$d\left(\left(q_1^2+1\right)k+\frac{q_1}{2}\right)^2+2q_1k+1, \tag{40}$$

where q_1 is an even natural number and $k = 1, 2, 3, \ldots$ Suppose that $d \mid y_0$, then we have $d \leq y_0$. By (39) and (40) it follows that $d > y_0$ and we get a contradiction.

From this observation follows that A-A-C conjecture is true for all primes $p \equiv 1 \pmod 4$, having the representation in the form (40).

References

[1] T. Agoh, A note on unit and class number of real quadratic fields *Acta Math. Sinica* **5** (1989), 281–288.

[2] N. C. Ankeny, E. Artin and S. Chowla, The class number of real quadratic number fields *Annals of Math.* **51** (1952), 479–483.

[3] N. C. Ankeny and S. Chowla, A note on the class number of real quadratic fields, *Acta Arith.* VI. (1960), 145–147.

[4] B. D. Beach, H. C. Willims and C. R. Zarnke, Some computer results on units in quadratic and cubic fields, *Proc. 25 Summer Mitting Canad. Math. Congr.* (1971), 609–649.

[5] L. J. Mordell, On a Pellian equation conjecture, *Acta Arith.* VI. (1960), 137–144.

[6] M. Sheingorn, Hyperbolic reflections on Pell's equation, Theory 33. (1989), 267–285.

[7] M. Sheingorn, The $\sqrt{p}$ Riemann surface, *Acta. Arith.* LXIII. 3. (1993), 255–266.

[8] W. Sierpinski, Elementary Thory of Numbers, PWN-Warszawa, (1987)

Aleksander Grytczuk
Intitute of Mathematics
Department of Algebra and Number Theory
T. Kotarbiński Pedagogical University
Pl. Słowiański 9, 65-069 Zielona Góra
Poland

Some congruences concerning second order linear recurrences

JAMES P. JONES and PÉTER KISS*

Abstract. Let U_n and V_n (n=0,1,2,...) be sequences of integers satisfying a second order linear recurrence relation with initial terms U_0=0, U_1=1, V_0=2, V_1=A. In this paper we investigate the congruence properties of the terms U_{nk} and V_{nk}, where the moduli are powers of U_n and V_n.

Let U_n and V_n ($n = 0, 1, 2, \ldots$) be second order linear recursive sequences of integers defined by

$$U_n = AU_{n-1} - BU_{n-2} \quad (n > 1)$$

and

$$V_n = AV_{n-1} - BV_{n-2} \quad (n > 1),$$

where A and B are nonzero rational integers and the initial terms are $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, $V_1 = A$. Denote by α, β the roots of the characteristic equation $x^2 - Ax + B = 0$ and suppose $D = A^2 - 4B \neq 0$ and hence that $\alpha \neq \beta$. In this case, as it is well known, the terms of the sequences can be expressed as

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n \tag{1}$$

for any $n \geq 0$.

Many identities and congruence properties are known for the sequences U_n and V_n (see, e.g. [1], [4], [5] and [6]). Some congruence properties are also known when the modulus is a power of a term of the sequences (see [2], [3], [7] and [8]). In [3] we derived some congruences where the moduli was U_n^3, V_n^2 or V_n^3. Among other congruences we proved that

$$U_{nk} \equiv kB^{n\frac{k-1}{2}} U_n \pmod{U_n^3}$$

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16975 and 020295.

when k is odd and a similar congruence for even k. In this paper we extend the results of [3]. We derive congruences in which the moduli are product of higher powers of U_n and V_n.

Theorem. *Let U_n and V_n be second order linear recurrences defined above and let $D = A^2 - 4B$ be the discriminant of the characteristic equation. Then for positive integers n and k we have*

1. $U_{nk} \equiv kB^{\frac{k-1}{2}n}U_n + \frac{k(k^2-1)}{24}DB^{\frac{k-3}{2}n}U_n^3 \pmod{D^2U_n^5}$, k odd,
2. $U_{nk} \equiv \frac{k}{2}B^{\frac{k-2}{2}n}V_nU_n + \frac{k(k^2-4)}{48}DB^{\frac{k-4}{2}n}V_nU_n^3 \pmod{D^2V_nU_n^5}$, k even,
3. $V_{nk} \equiv k(-1)^{\frac{k-1}{2}}B^{\frac{k-1}{2}n}V_n + \frac{k(k^2-1)}{24}(-1)^{\frac{k-3}{2}}B^{\frac{k-3}{2}n}V_n^3 \pmod{V_n^5}$, k odd,
4. $V_{nk} \equiv 2(-1)^{\frac{k}{2}}B^{\frac{k}{2}n} + \frac{k^2}{4}(-1)^{\frac{k-2}{2}}B^{\frac{k-2}{2}n}V_n^2 \pmod{V_n^4}$, k even,
5. $U_{nk} \equiv U_n(-1)^{\frac{k-1}{2}}B^{\frac{k-1}{2}n} + \frac{k^2-1}{8}(-1)^{\frac{k-3}{2}}B^{\frac{k-3}{2}n}U_nV_n^2 \pmod{U_nV_n^4}$, k odd,
6. $U_{nk} \equiv \frac{k}{2}(-1)^{\frac{k-2}{2}}B^{\frac{k-2}{2}n}U_nV_n + \frac{k(k^2-4)}{48}(-1)^{\frac{k-4}{2}}B^{\frac{k-4}{2}n}U_nV_n^3 \pmod{U_nV_n^5}$, k even,
7. $V_{nk} \equiv B^{\frac{k-1}{2}n}V_n + \frac{k^2-1}{8}DB^{\frac{k-3}{2}n}V_nU_n^2 \pmod{D^2V_nU_n^4}$, k odd,
8. $V_{nk} \equiv 2B^{\frac{k}{2}n} + \frac{k^2}{4}B^{\frac{k-2}{2}n}DU_n^2 \pmod{D^2U_n^4}$, k even.

We note that the congruences of [3] follow as consequences of this theorem.

For the proof of the Theorem we need some auxiliary results which are known (see e.g. [6]) but we show short proofs for them. In the followings we suppose that $A > 0$ and hence that

$$\alpha = \frac{A+\sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{A-\sqrt{D}}{2},$$

so that $\alpha - \beta = \sqrt{D}$, $\alpha + \beta = A$, $\alpha\beta = B$ and hence by (1)

$$U_n = \frac{\alpha^n - \beta^n}{\sqrt{D}} \tag{2}$$

Lemma 1. *For any integer $n \geq 0$ we have*

$$U_{3n} = 3U_nB^n + DU_n^3.$$

Proof. By (2), using that $\alpha\beta = B$, we have to prove that

$$\frac{\alpha^{3n} - \beta^{3n}}{\sqrt{D}} = 3 \cdot \frac{\alpha^n - \beta^n}{\sqrt{D}}(\alpha\beta)^n + D\left(\frac{\alpha^n - \beta^n}{\sqrt{D}}\right)^3,$$

which follows from $\alpha^{3n} - \beta^{3n} = 3(\alpha^n - \beta^n)\alpha^n\beta^n + (\alpha^n - \beta^n)^3$.

Lemma 2. *For any non-negative integers m and n we have*

$$U_{m+2n} = V_n U_{m+n} - B^n U_m.$$

Proof. Similarly as in the proof of Lemma 1,

$$\frac{\alpha^{m+2n} - \beta^{m+2n}}{\sqrt{D}} = (\alpha^n + \beta^n)\frac{\alpha^{m+n} - \beta^{m+n}}{\sqrt{D}} - (\alpha\beta)^n \frac{\alpha^m - \beta^m}{\sqrt{D}}$$

is an identity which by (1) and (2), implies the lemma.

Lemma 3. *For any $n \geq 0$ we have*

$$V_{2n} = 2B^n + DU_n^2 = V_n^2 - 2B^n \quad \text{and} \quad U_{2n} = U_n V_n.$$

Proof. The identities

$$\alpha^{2n} + \beta^{2n} = 2(\alpha\beta)^n + D\left(\frac{\alpha^n - \beta^n}{\sqrt{D}}\right)^2 \quad \text{and} \quad \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{D}} = \frac{\alpha^n - \beta^n}{\sqrt{D}}(\alpha^n + \beta^n)$$

prove the lemma.

Proof of the Theorem. We prove the first congruence of the Theorem by double induction on k. For $k = 1$ and $k = 3$, by Lemma 1, the congruence is an identity. Suppose the congruence holds for k and $k + 2$, where $k \geq 1$ is odd. Then by Lemma 2 and 3 we have

$$\begin{aligned} U_{n(k+4)} &= U_{nk+4n} = V_{2n}U_{nk+2n} - B^{2n}U_{nk} \\ &= (2B^n + DU_n^2)U_{n(k+2)} - B^{2n}U_{nk} \\ &\equiv (2B^n + DU_n^2)Q - B^{2n}R \pmod{D^2U_n^5}, \end{aligned} \tag{3}$$

where

$$Q = (k+2)B^{\frac{k+1}{2}n}U_n + \frac{(k+2)((k+2)^2-1)}{24}DB^{\frac{k-1}{2}n}U_n^3 \tag{4}$$

and

$$R = kB^{\frac{k-1}{2}n}U_n + \frac{k(k^2-1)}{24}DB^{\frac{k-3}{2}n}U_n^3. \tag{5}$$

After some calculation (3), (4) and (5) imply

$$U_{n(k+4)} \equiv U_n T + U_n^3 S \pmod{D^2 U_n^5}, \tag{6}$$

where

$$T = (2(k+2)-k)\, B^{\frac{k+3}{2}n} = (k+4)B^{\frac{(k+4)-1}{2}}$$

and

$$S = (k+2)DB^{\frac{k+1}{2}n} + 2\frac{(k+2)\left((k+2)^2-1\right)}{24}DB^{\frac{k+1}{2}n} - \frac{k(k^2-1)}{24}DB^{\frac{k+1}{2}n} = \frac{(k+4)\left((k+4)^2-1\right)}{24}DB^{\frac{(k+4)-3}{2}n},$$

and so by (6),

$$U_{n(k+4)} \equiv (k+4)B^{\frac{(k+4)-1}{2}}U_n + \frac{(k+4)\left((k+4)^2-1\right)}{24}DB^{\frac{(k+4)-3}{2}n}U_n^3 \pmod{D^2U_n^5}.$$

Hence the congruence holds also for $k+4$ and for any odd positive integer k.

The other congruences in the Theorem can be proved similarly using Lemma 1, 2, 3 and the identities

$$\begin{aligned}
U_{2n} &= V_n U_n,\\
V_{2n} &= V_n^2 - 2B^n = 2B^n + DU_n^2,\\
U_{3n} &= U_n V_n^2 - B^n U_n,\\
V_{3n} &= V_n^3 - 3B^n V_n = B^n V_n + DV_n U_n^2,\\
U_{4n} &= U_n V_n^3 - 2B^n U_n V_n,\\
V_{4n} &= V_n^4 - 4B^n V_n^2 + 2B^{2n}.
\end{aligned}$$

References

[1] D. JARDEN, Recurring sequences, *Riveon Lematematika*, Jerusalem (Israel), 1973.

[2] J. P. JONES AND P. KISS, Some identities and congruences for a special family of second order recurrences, *Acta Acad. Paed. Agriensis, Sect. Math.* **23** (1995–96), 3–9.

[3] J. P. JONES AND P. KISS, Some new identities and congruences for Lucas sequences, *Discuss Math.*, to appear.

[4] D. H. LEHMER, On the multiple solutions of the Pell Equation, *Annals of Math.* **30** (1928), 66–72.

[5] D. H. LEHMER, An extended theory of Lucas' functions, *Annals of Math.* **31** (1930), 419–448.

[6] E. LUCAS, Theorie des functions numériques simplement périodiques, *American Journal of Mathematics*, vol. **1** (1878), 184–240, 289–321. English translation: Fibonacci Association, Santa Clara Univ., 1969.

[7] S. VAJDA, Fibonacci & Lucas numbers, and the golden section, *Ellis Horwood Limited Publ.*, New York-Toronto, 1989.

[8] C. R. WALL, Some congruences involving generalized Fibonacci numbers, *The Fibonacci Quarterly* **17.1** (1979), 29–33.

JAMES P. JONES
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF CALGARY
CALGARY, ALBERTA T2N 1N4
CANADA

PÉTER KISS
ESZTERHÁZY KÁROLY TEACHERS' TRAINING COLLEGE
DEPARTMENT OF MATHEMATICS
LEÁNYKA U. 4.
3301 EGER, PF. 43.
HUNGARY
E-mail: kissp@gemini.ektf.hu

Pure powers in recurrence sequences

KÁLMÁN LIPTAI* and TIBOR TÓMÁCS

Abstract. Let G be a linear recursive sequence of order k satisfying the recursion $G_n = A_1 G_{n-1} + \cdots + A_k G_{n-k}$. In the case $k=2$ it is known that there are only finitely many perfect powers in such a sequence.

Ribenboim and McDaniel proved for sequences with $k=2$, $G_0=0$ and $G_1=1$ that in general for a term G_n there are only finitely many terms G_m such that $G_n G_m$ is a perfect square. P. Kiss proved that for any n there exists a number q_0, depending on G and n, such that the equation $G_n G_x = w^q$ in positive integers x, w, q has no solution with $x > n$ and $q > q_0$. We show that for any n there are only finitely many $x_1, x_2, \ldots, x_k, x, w, q$ positive integers such that $G_n G_{x_1} \cdots G_{x_k} G_x = w^q$ and some conditions hold.

Let $R = R(A, B, R_0, R_1)$ be a second order linear recursive sequence defined by

$$R_n = AR_{n-1} + BR_{n-2} \qquad (n > 1),$$

where A, B, R_0 and R_1 are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. α/β is not a root of unity, where α and β denote the roots of the polynomial $x^2 - Ax - B$.

The special cases $R(1,1,0,1)$ and $R(2,1,0,1)$ of the sequence R is called Fibonacci and Pell sequence, respectively.

Many results are known about relationship of the sequences R and perfect powers. For the Fibonacci sequence Cohn [2] and Wylie [23] showed that a Fibonacci number F_n is a square only when $n = 0, 1, 2$ or 12. Pethő [12], furthermore London and Finkelstein [9,10] proved that F_n is full cube only if $n = 0, 1, 2$ or 6. From a result of Ljunggren [8] it follows that a Pell number is a square only if $n = 0, 1$ or 7 and Pethő [12] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results was showed by McDaniel and Ribenboim [11], Robbins [19,20] Cohn [3,4,5] and Pethő [15]. Shorey and Stewart [21] showed, that any non degenerate binary recurrence sequence contains only finitely many perfect powers which can be effictively determined. This results follows also from a result of Pethő [14].

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16975 and 020295.

Another type of problems was studied by Ribenboim and McDaniel. For a sequence R we say that the terms R_m, R_n are in the same square-class if there exist non zero integers x, y such that

$$R_m x^2 = R_n y^2,$$

or equivalently

$$R_m R_n = t^2,$$

where t is a positive rational integer.

A square-class is called trivial if it contains only one element. Ribenboim [16] proved that in the Fibonacci sequence the square-class of a Fibonacci number F_m is trivial, if $m \neq 1, 2, 3, 6$ or 12 and for the Lucas sequence $L(1,1,2,1)$ the square-class of a Lucas number L_m is trivial if $m \neq 0, 1, 3$ or 6. For more general sequences $R(A, B, 0, 1)$, with $(A, B) = 1$, Ribenboim and McDaniel [17] obtained that each square class is finite and its elements can be effectively computed (see also Ribenboim [18]).

Further on we shall study more general recursive sequences.

Let $G = G(A_1, \ldots, A_k, G_0, \ldots, G_{k-1})$ be a k^{th} order linear recursive sequence of rational integers defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \qquad (n > k-1),$$

where $A_1, \ldots, A_k$ and $G_0, \ldots, G_{k-1}$ are not all zero integers. Denote by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ the distinct zeros of the polynomial $x^k - A_1 x^{k-1} - A_2 x^{k-2} - \cdots - A_k$. Assume that $\alpha, \alpha_2, \ldots, \alpha_s$ has multiplicity $1, m_2, \ldots, m_s$ respectively and $|\alpha| > |\alpha_i|$ for $i = 2, \ldots, s$. In this case, as it is known, the terms of the sequence can be written in the form

$$G_n = a\alpha^n + r_2(n)\alpha_2^n + \cdots + r_s(n)\alpha_s^n \qquad (n \geq 0), \tag{1}$$

where $r_i (i = 2, \ldots, s)$ are polynomials of degree $m_i - 1$ and the coefficients of the polynomials and a are elements of the algebraic number field $\mathbf{Q}(\alpha, \alpha_2, \ldots, \alpha_s)$. Shorey and Stewart [21] prowed that the sequence G does not contain q^{th} powers if q is large enough. This result follows also from [7] and [22], where more general theorems where showed.

Kiss [6] generalized the square-class notion of Ribenboim and McDaniel. For a sequence G we say that the terms G_m and G_n are in the same q^{th}-power class if $G_m G_n = w^q$, where w, q rational integers and $q \geq 2$.

In the above mentioned paper Kiss proved that for any term G_n of the sequence G there is no terms G_m such that $m > n$ and G_n, G_m are elements of the same q^{th}-power class if q sufficiently large.

The purpose of this paper to generalize this result. We show that the under certain conditions the number of the solutions of equation

$$G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q$$

where n is fixed, are finite.

We use a well known result of Baker [1].

Lemma. *Let $\gamma_1, \dots, \gamma_v$ be non-zero algebraic numbers. Let $M_1, \dots, M_v$ be upper bounds for the heights of $\gamma_1, \dots, \gamma_v$, respectively. We assume that M_v is at least 4. Further let $b_1, \dots, b_{v-1}$ be rational integers with absolute values at most B and let b_v be a non-zero rational integer with absolute value at most B'. We assume that B' is at least three. Let L defined by*

$$L = b_1 \log \gamma_1 + \cdots + b_v \log \gamma_v,$$

where the logarithms are assumed to have their principal values. If $L \neq 0$, then

$$|L| > \exp(-C(\log B' \log M_v + B/B')),$$

where C is an effectively computable positive number depending on only the numbers $M_1, \dots, M_{v-1}$, $\gamma_1, \dots, \gamma_v$ and v (see Theorem 1 of [1] with $\delta = 1/B'$).

Theorem. *Let G be a k^{th} order linear recursive sequence satisfying the above conditions. Assume that $a \neq 0$ and $G_i \neq a\alpha^i$ for $i > n_0$. Then for any positive integer n, k and K there exists a number q_0, depending on n, G, K and k, such that the equation*

$$G_n G_{x_1} G_{x_2} \cdots G_{x_k} G_x = w^q \qquad (n \leq x_1 \leq \cdots \leq x_k < x) \tag{2}$$

in positive integer $x_1, x_2, \dots, x_k, x, w, q$ has no solution with $x_k < Kn$ and $q > q_0$.

Proof of the theorem. We can assume, without loss of generality, that the terms of the sequence G are positive. We can also suppose that $n > n_0$ and n sufficiently large since otherwise our result follows from [20] and [7].

Let $x_1, x_2, \dots, x_k, x, w, q$ positive integers satisfying (2) with the above conditions. Let ε_m be defined by

$$\varepsilon_m := \frac{1}{a} r_2(m) \Big(\frac{\alpha_2}{\alpha}\Big)^m + \frac{1}{a} r_3(m) \Big(\frac{\alpha_3}{\alpha}\Big)^m + \cdots + \frac{1}{a} r_s(m) \Big(\frac{\alpha_s}{\alpha}\Big)^m \qquad (m \geq 0).$$

By (1) we have

$$(1+\varepsilon_n)(1+\varepsilon_x)\prod_{i=1}^{k}(1+\varepsilon_{x_i})\,a^{k+2}\alpha^{n+x+x_1+\cdots+x_k}=w^q$$

from which

$$\begin{aligned} q\log w &= (k+2)\log a+\left(n+x+\sum_{i=1}^{k}x_i\right)\log\alpha+\log(1+\varepsilon_n)\\ &\quad+\log(1+\varepsilon_x)+\sum_{i=1}^{k}\log(1+\varepsilon_{x_i}) \end{aligned} \tag{3}$$

follows. It is obvious that $x < n+x+\sum_{i=1}^{k} x_i < (k+2)x$. Using that $\log|1+\varepsilon_m|$ is bounded and $\lim_{m\to\infty} \frac{1}{a}r_i(m)\left(\frac{\alpha_i}{\alpha}\right)^m = 0 \quad (i=2,\dots,s)$, we have

$$c_1\frac{x}{q} < \log w < c_2\frac{x}{q} \tag{4}$$

where c_1 and c_2 are constants.

Let L be defined by

$$L := \left|\log\frac{w^q}{G_nG_{x_1}G_{x_2}\cdots G_{x_k}a\alpha^x}\right| = |\log(1+\varepsilon_x)|\,.$$

By the definition of ε_x and the properties of logarithm function there exists a constant c_3 that

$$L < e^{-c_3x}. \tag{5}$$

On the other hand, by the Lemma with $v=k+4, M_{k+4}=w, B'=q$ and $B=x$ we obtain the estimation

$$L=\left|q\log w-\log G_n-\sum_{i=1}^{k}\log G_{x_i}-\log a-x\log\alpha\right|>e^{-C(\log q\log w+x/q)} \tag{6}$$

where C depends on heights. By $x_k < Kn$ heights depend on $G_n,\dots,G_{Kn}$, i.e. on n, K, k and on the parameters of the recurrence. By (4), (5) and (6) we have $c_3x < C(\log q\log w + x/q) < c_4\log q\log w$, i.e.

$$x < c_5\log q\log w \tag{7}$$

with some c_3, c_4, c_5. Using (4) and (7) we get $c_6 q \log w < x < c_5 \log q \log w$, i.e. $q < c_7 \log q$, where c_6 and c_7 are constants. But this inequality does not hold if $q > q_0 = q_0(G, n, K, k)$, which proves the theorem.

References

[1] A. BAKER, A sharpening of the bounds for linear forms in logarithms II, *Acta Arithm.* **24** (1973), 33–36.

[2] J. H. E. COHN, On square Fibonacci numbers, *J. London Math. Soc.* **39** (1964), 537–540.

[3] J. H. E. COHN, Squares in some recurrent sequences, *Pacific J. Math.* **41** (1972), 631–646.

[4] J. H. E. COHN, Eight Diophantine equations, *Proc. London Math. Soc.* **16** (1966), 153–166.

[5] J. H. E. COHN, Five Diophatine equations, *Math. Scand.* **21** (1967), 61–70.

[6] P. KISS, Pure powers and power classes in recurrence sequences, (to appear).

[7] P. KISS, Differences of the terms of linear recurrences, *Studia Sci. Math. Hungar.* **20** (1985), 285–293.

[8] W. LJUNGGREN, Zur Theorie der Gleichung $x^2 + 1 = Dy^4$, *Avh. Norske Vid Akad. Oslo* **5** (1942).

[9] J. LONDON and R. FINKELSTEIN, On Fibonacci and Lucas numbers which are perfect powers, *Fibonacci Quart.* **7** (1969) 476–481, 487, errata ibid **8** (1970) 248.

[10] J. LONDON and R. FINKELSTEIN, On Mordell's equation $y^2 - k = x^3$, *Bowling Green University Press* (1973).

[11] W. L. MCDANIEL and P. RIBENBOIM, Squares and double-squares in Lucas sequences, *C. R. Math. Acad. Sci. Soc. R. Canada* **14** (1992), 104–108.

[12] A. PETHŐ, Full cubes in the Fibonacci sequence, *Publ. Math. Debrecen* **30** (1983), 117–127.

[13] A. PETHŐ, The Pell sequence contains only trivial perfect powers, *Coll. Math. Soc. J. Bolyai, 60 sets, Graphs and Numbers,* Budapest, (1991), 561–568.

[14] A. PETHŐ, Perfect powers in second order linear recurrences, *J. Number Theory* **15** (1982), 5–13.

[15] A. PETHŐ, Perfect powers in second order recurrences, *Topics in Classical Number Theory,* Akadémiai Kiadó, Budapest, (1981), 1217–1227.

[16] P. RIBENBOIM, Square classes of Fibonacci and Lucas numbers, *Portugaliae Math.* **46** (1989), 159–175.

[17] P. RIBENBOIM and W. L. MCDANIEL, Square classes of Fibonacci and Lucas sequences, *Portugaliae Math.*, **48** (1991), 469–473.

[18] P. RIBENBOIM, Square classes of $(a^n-1)/(a-1)$ and a^n+1, *Sichuan Daxue Xunebar* **26** (1989), 196–199.

[19] N. ROBBINS, On Fibonacci numbers of the form px^2, where p is prime, *Fibonacci Quart.* **21** (1983), 266–271.

[20] N. ROBBINS, On Pell numbers of the form PX^2, where P is prime, *Fibonacci Quart.* **22** (1984), 340–348.

[21] T. N. SHOREY and C. L. STEWART, On the Diophantine equation $ax^{2t}+bx^ty+cy^2=d$ and pure powers in recurrence sequences, *Math. Scand.* **52** (1983), 24–36.

[22] T. N. SHOREY and C. L. STEWART, Pure powers in recurrence sequences and some related Diophatine equations, *J. Number Theory* **27** (1987), 324–352.

[23] O. WYLIE, In the Fibonacci series $F_1=1, F_2=1, F_{n+1}=F_n+F_{n-1}$ the first, second and twelvth terms are squares, *Amer. Math. Monthly* **71** (1964), 220–222.

KÁLMÁN LIPTAI and TIBOR TÓMÁCS
ESZTERHÁZY KÁROLY TEACHERS' TRAINING COLLEGE
DEPARTMENT OF MATHEMATICS
LEÁNYKA U. 4.
3301 EGER, PF. 43.
HUNGARY

E-mail: liptaik@gemini.ektf.hu
tomacs@gemini.ektf.hu

A generalization of an approximation problem concerning linear recurrences

BÉLA ZAY*

Abstract. Let $\{G_n\}$ be a linear recursive sequence of order $t(\geq 2)$ defined by $G_n = A_1 G_{n-1} + \cdots + A_t G_{n-t}$ for $n \geq t$, where $A_1, \ldots, A_t$ and $G_0, \ldots, G_{t-1}$ are given rational integers. Denote by $\alpha_1, \alpha_2, \ldots, \alpha_t$ the roots of the polynomial $x^t - A_1 x^{t-1} - \cdots - A_t$ and suppose that $|\alpha_1| > |\alpha_i|$ for $2 \leq i < t$. It is known that $\lim_{n\to\infty} \frac{G_{n+s}}{G_n} = \alpha_1^s$, where s is a positive integer. The quality of the approximation of α_1 by rational numbers $\frac{G_{n+s}}{G_n}$ in the case $s=1$ was investigated in several papers. Extending the earlier results we show that the inequality

$$\left|\alpha_1^s - \frac{G_{n+s}}{G_n}\right| < \frac{1}{cG_n^r}$$

holds for infinitely many positve integers n with some constant c if and only if

$$r \leq 1 - \frac{\log|\alpha_2|}{\log|\alpha_1|}.$$

Let $\{G_n\}_{n=1}^{\infty}$ be a k^{th} order $(k \geq 2)$ linear recursive sequence defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \cdots + A_k G_{n-k} \quad \text{for} \quad n \geq k,$$

where $A_1, \ldots, A_k$, and $G_1, \ldots, G_k$ are given rational integers with $A_k \neq 0$ and $G_0^2 + \cdots + G_{k-1}^2 \neq 0$. Denote by $\alpha_1, \ldots, \alpha_t$ the distinct roots of the characteristic polinomial

$$f(x) = x^k - A_1 x^{k-1} - \cdots - A_k = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_t)^{m_t}.$$

Using the well known explicite form of the terms of linear recursive sequences, G_n can be expressed by

$$(1) \qquad G_n = \sum_{i=1}^{t}\left(\sum_{j=1}^{m_i} a_{ij} n^{j-1}\right)\alpha_i^n = \sum_{i=1}^{t} P_i(n)\alpha_i^n \quad (n \geq 0)$$

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16975 and 020295.

where the coefficients a_{ij} of polinomials $P_i(n)$ are elements of the algebraic number field $Q(\alpha_1, \dots, \alpha_t)$. We assume that the sequence G is a non degenerate one, i.e. $a_{11}, a_{21}, \dots, a_{t_1}$ are non zero algebraic numbers and α_i/α_j is not a root of unity for any $1 \le i < j \le t$. We can also assume that $G_n \neq 0$ for $n > 0$ since the sequence have only finitely many zero terms and after a movement of indices this condition will be fulfilled. If $|\alpha_i| < \alpha_1$ for $i = 2, 3, \dots, t$ than from (1) it follows that $\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \alpha_1$. In the case $k = 2$ the quality of the approximation of α_1 by rational numbers G_{n+1}/G_n was investigated some earlier papers (e.g. set [2], [3], [4] and [5]). In the general case P. Kiss ([1]) proved the following result. Let G be a t^{th} order linear recurrence with conditions $|\alpha_1| > |\alpha_2| \ge |\alpha_3| > \dots > |\alpha_t|$, where $m_1 = \dots = m_t = 1$). Then

$$\left|\alpha_1 - \frac{G_{n+1}}{G_n}\right| < \frac{1}{cG_n^k}$$

holds for infinitely many positive integers n with some constant c if and only if $k \le k_0$, where

$$k_0 = 1 - \frac{\log|\alpha_2|}{\log|\alpha_1|} \le 1 + \frac{1}{t-1}$$

and the equation $k_0 = 1 + \frac{1}{t-1}$ can be held only if $|A_t| = 1$ and $|\alpha_1| > |\alpha_2| = \dots = |\alpha_t|$.

In [1] the following lemma was also proved.

Lemma. *Let β and γ be complex algebraic numbers for which $|\beta| = |\gamma| = 1$ and γ is not a root of unity. Then there are positive numbers δ and n_0 depending only on β and γ such that*

$$|1 + \beta\gamma^n| > e^{\delta \log n}$$

for any $n > n_0$.

In the case $|\alpha_1| > \alpha_i$ $(2 \le i \le t)$ it is clear that $\lim_{n\to\infty} \frac{G_{n+s}}{G_n} = \alpha_1^s$ for any fixed positive integer s.

The purpose of this paper is the investigation of the quality of the approximation of α_1^s by rational numbers $\frac{G_{n+s}}{G_n}$ and to prove an extension of P. Kiss's theorem.

Theorem. *Let G be a non degenerate k^{th} order linear recurrence sequence with conditions:*

$$|\alpha_1| > |\alpha_2| \ge |\alpha_3| > |\alpha_4| \ge \dots \ge |\alpha_t|, \quad m_1 = m_2 = 1, \quad \sum_{i=1}^{t} m_i = k$$

(where m_i is the multiplicity of α_i in the characteristic polinomial of G) and $G_n > 0$ for $n > 0$. Then

$$\left|\alpha_1^s - \frac{G_{n+s}}{G_n}\right| < \frac{1}{cG_n^r} \tag{2}$$

holds for infinitely many positive integers n with some positive constant c if and only if

$$r \le r_0 = 1 - \frac{\log|\alpha_2|}{\log|\alpha_1|}. \tag{3}$$

We remark that in the case of $s = 1$, $m_1 = \cdots = m_t = 1$ we get the result of P. Kiss ([1]). In the next proof we shall use similar arguments wich was used by P. Kiss.

Proof of the Theorem. Since $m_1 = m_2 = 1$ the polinomials $P_1(n)$ and $P_2(n)$ are non zero constants (denoted by a_{11} and a_{21} respectively) and so by (1) we have

$$\begin{aligned}\left|\alpha_1^s - \frac{G_{n+s}}{G_n}\right| &= \left|\alpha_1^s - \frac{P_1(n+s)\alpha_1^{n+s} + \cdots + P_t(n+s)\alpha_t^{n+s}}{P_1(n)\alpha_1^n + \cdots + P_t(n)\alpha_t^n}\right| \\ &= \left|G_n^{-1}\right| \left|a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n + \sum_{i=3}^{t}(\alpha_1^s p_i(n) - \alpha_i^s P_i(n+s))\alpha_i^n\right| \\ &= \left|G_n^{-1} a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n\right| H_3(n)\end{aligned}$$

where

$$H_3(n) = \left|1 + \sum_{i=3}^{t} \frac{(\alpha_1^s P_i(n) - \alpha_i^s P_i(n+s))\,\alpha_i^n}{a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n}\right|.$$

Since $G_n = a_{11}\alpha_1^n(1 + d_n)$, where $\lim_{n\to\infty} d_n = 0$, (2) holds if and only if

$$\begin{aligned}&c\left|a_{21}(\alpha_1^s - \alpha_2^s)\alpha_2^n G_n^{r-1}\right| H_3(n) \\ &= c\left|a_{11}^{r-1}a_{21}(\alpha_1^s - \alpha_2^s)(1 + d_n)^{r-1}\right| \left|\alpha_2\alpha_1^{r-1}\right|^n H_3(n) < 1.\end{aligned}$$

Denoting the second and the third factors of the last product by $H_1(n)$ and $H_2(n)$ respectively, (2) holds if and only if

$$cH_1(n)H_2(n)H_3(n) < 1. \tag{4}$$

It is easy to see that

$$e^{c_1} < H_1(n) < e^{c_2}$$

holds with suitable real numbers c_1, c_2.

From this it follows that

$$ce^{hn+c_1} < cH_1(n)H_2(n) < ce^{hn+c_2} \tag{5}$$

where $h = \log \alpha_2 + (r-1)\log \alpha_1$.

If we assume that $|\alpha_2| > |\alpha_3|$ then $\lim_{n\to\infty} cH_1(n)H_3(n) = cc_0$, where $c_0 = \left|a_{11}^{r-1}a_{21}(\alpha_1^s - \alpha_2^s)\right|$.

Using the well known fact

$$\lim_{n\to\infty} H_2(n) = \lim_{n\to\infty} \left|\alpha_1\alpha_2^{r-1}\right|^n = \begin{cases} 0, & \text{if } r < r_0 = 1 - \frac{\log|\alpha_2|}{\log|\alpha_1|} \\ 1, & \text{if } r = r_0 \\ \infty, & \text{if } r > r_0 \end{cases}$$

it is clear that (4) (and so (2), too) holds for infinitely many positive integers n with some positive constant c $(0 < c \le c_0^{-1})$ if and only if $r \le r_0$. Now we assume that

$$|\alpha_1| > |\alpha_2| = |\alpha_3| > |\alpha_4| \ge \cdots \ge |\alpha_t|.$$

Since α_1 is real and α_3/α_2 is not a root of unity α_3 and α_2 are (not real) conjugate complex numbers and $m_2 = m_3$ (i.e. $m_1 = m_2 = 1 = m_3$ and $P_3(n) = P_3(n+s) = a_{31}$). Furthermore a_{21} and a_{31} also are conjugate numbers since they are solutions of the system of linear equations

$$G_n = \sum_{i=1}^{t}\left(\sum_{j=1}^{m_i} a_{ij}n^{j-1}\right)\alpha_i^n, \quad 0 \le n \le k-1.$$

Hence $\frac{a_{3,1}(\alpha_1^s - \alpha_3^s)}{a_{2,1}(\alpha_1 - \alpha_2^s)}$ and $\frac{\alpha_3}{\alpha_2}$ are algebraic numbers with absolute value 1 and so using the Lemma (proved by P. Kiss in [1]), we obtain the estimation

$$\left|1 + \frac{a_{31}(\alpha_1^s - \alpha_3^s)}{a_{21}(\alpha_1^s - \alpha_2^s)}\left(\frac{\alpha_3}{\alpha_2}\right)^n\right| > e^{-\delta \log n}$$

with some positive real δ.

But $|\alpha_i| < |\alpha_2|$ for $i \geq 4$, so by the last inequality

$$e^{-c_3 \log n} < \left| 1 + \frac{a_3(\alpha_1^s - \alpha_3^s)}{(\alpha_1^s - \alpha_2^s)} \left(\frac{\alpha_3}{\alpha_2}\right)^n \right. \tag{6}$$
$$\left. + \sum_{i=4}^{t} \frac{\alpha_1^s P_i(n) - \alpha_i^s P_i(n+s)}{a_{21}(\alpha_1^s - \alpha_2^s)} \left(\frac{\alpha_i}{\alpha_2}\right)^n \right| = H_3(n) < 3$$

with some $c_3 > 0$ if n is large enough.

By (5) and (6) we have

$$ce^{hn - c_3 \log n + c_1} < cH_1(n)H_2(n)H_3(n) < ce^{hn + c_2 + \log 3}. \tag{7}$$

(7) holds for infinitely many positive integers if and only if $h \leq 0$, which is equivalent to $r \leq r_0$.

This completes the proof of the theorem.

References

[1] P. Kiss, An approximation problem concerning linear recurrences, to appear.

[2] P. Kiss, A Diophantine approximative property of the second order linear recurrences, *Period. Math. Hungar.* **11** (1980), 281–287.

[3] P. Kiss and Zs. Sinka, On the ratios of the terms of second order linear recurrences, *Period. Math. Hungar.* **23** (1991), 139–143.

[4] P. Kiss and R. F. Tichy, A discrepancy problem with applications to linear recurrences I., *Proc. Japan Acad.* **65** (ser A), No 5. (1989), 135–138.

[5] P. Kiss and R. F. Tichy, A discrepancy problem with applications to linear recurrences II., *Proc. Japan Acad.* **65** (ser A), No 5. (1989), 131–194.

Béla Zay
Eszterházy Károly Teachers' Training College
Department of Mathematics
Leányka u. 4.
3301 Eger, Pf. 43.
Hungary

A note on the products of the terms of linear recurrences

LÁSZLÓ SZALAY

Abstract. For an integer $\nu>1$ let $G^{(i)}$ $(i=1,\dots,\nu)$ be linear recurrences defined by

$$G_n^{(i)}=A_1^{(i)}G_{n-1}^{(i)}+\dots+A_{k_i}^{(i)}G_{n-k_i}\quad (n\geq k_i).$$

In the paper we show that the equation

$$dG_{x_1}^{(1)}\cdots G_{x_\nu}^{(\nu)}=sw^q,$$

where d,s,w,q,x_i are positive integers satisfying some conditions, implies the inequality $q<q_0$ with some effectively computable constant q_0. This result generalizes some earlier results of Kiss, Pethő, Shorey and Stewart.

1. Introduction

Let $G^{(i)}=\{G_n^{(i)}\}_{n=0}^{\infty}$ $(i=1,2,\dots,\nu)$ be linear recurrences of order k_i $(k_i\geq 2)$ defined by

$$(1)\qquad G_n^{(i)}=A_1^{(i)}G_{n-1}^{(i)}+\dots+A_{k_i}^{(i)}G_{n-k_i}^{(i)}\quad (n\geq k_i),$$

where the initial values $G_j^{(i)}$ $(j=0,1,\dots,k_i-1)$ and the coefficients $A_l^{(i)}$ $(l=1,2,\dots,k_i)$ of the sequences are rational integers. We suppose, that $A_{k_i}^{(i)}\neq 0$ and there is at least one non-zero initial value for any recurrences.

By $\alpha_1^{(i)}=\gamma_i,\alpha_2^{(i)},\dots,\alpha_{t_i}^{(i)}$ we denote the distinct roots of the characteristic polynomial

$$p_i(x)=x^{k_i}-A_1^{(i)}x^{k_i-1}-\dots-A_{k_i}^{(i)}$$

of the sequence $G^{(i)}$, and we assume that $t_i>1$ and $|\gamma_i|>|\alpha_j^{(i)}|$ for $j>1$. Consequently $|\gamma_i|>1$. Suppose that the multiplicity of the roots γ_i are 1. Then the terms of the sequences $G^{(i)}$ $(i=1,2,\dots,\nu)$ can be written in the form

$$(2)\qquad G_n^{(i)}=a_i\gamma_i^n+p_2^{(i)}(n)\left(\alpha_2^{(i)}\right)^n+\dots+p_{t_i}^{(i)}(n)\left(\alpha_{t_i}^{(i)}\right)^n\quad (n\geq 0),$$

where $a_i \neq 0$ are fixed numbers and $p_j^{(i)}$ $(j = 1, 2 \ldots, t_i)$ are polynomials of

$$\mathbf{Q}(\gamma_i, \alpha_2^{(i)}, \ldots, \alpha_{t_i}^{(i)})[x]$$

(see e.g. [8]).

A. Pethő [4,5,6], T. N. Shorey and C. L. Stewart [7] showed that a sequence $G(= G^{(i)})$ does not contain q-th powers if q is large enough. Similar result was obtained by P. Kiss in [2]. In [3] we investigated the equation

$$G_x H_y = w^q \tag{3}$$

where G and H are linear recurrences satisfying some condititons, and showed that if x and y are not too far from each other then q is (effectively computable) upper bounded: $q < q_0$.

2. Theorem

Now we shall investigate the generalization of equation (3). Let $d \in \mathbf{Z}$ be a fixed non-zero rational integer, and let $p_1, \ldots, p_t$ be given rational primes. Denote by S the set of all rational integers composed of $p_1, \ldots, p_t$:

$$S = \{s \in \mathbf{Z} \colon s = \pm p_1^{e_1} \cdots p_t^{e_t},\ e_i \in \mathbf{N}\}. \tag{4}$$

In particular $1 \in S$ $(e_1 = \cdots = e_t = 0)$. Let

$$\mathcal{G}(x_1, \ldots, x_\nu) = G_{x_1}^{(1)} \ldots G_{x_\nu}^{(\nu)} \tag{5}$$

be a function defined on the set $\mathbf{N}^\nu$. By the definitions of the sequences $G^{(i)}$'s $\mathcal{G}$ takes integer values. With a given d let us consider the equation

$$d\mathcal{G}(x_1, \ldots, x_\nu) = sw^q$$

in positive integers $w > 1$, q, x_i $(i = 1, 2, \ldots, \nu)$ and $s \in S$. We will show under some conditions for $\mathcal{G}$ that $q < q_0$ is also fulfilled if q satisfies the equation above. Exactly, using the Baker-method, we will prove the following

Theorem. *Let $\mathcal{G}(x_1, \ldots, x_\nu)$ be the function defined in (5). Futher let $0 \neq d \in \mathbf{Z}$ be a fixed integer, and let δ be a real number with $0 < \delta < 1$. Assume that $G(x_1, \ldots, x_\nu) \neq \prod_{i=1}^{\nu} a_i \gamma_i^{x_i}$ if $x_i > n_0$ $(i = 1, 2, \ldots, \nu)$. Then the equation*

$$d\mathcal{G}(x_1, \ldots, x_\nu) = sw^q \tag{6}$$

in positive integers $w > 1, q, x_1, \ldots, x_\nu$ and $s \in S$ for which $x_j > \delta \max_i\{x_i\}$ $(j = 1, 2, \ldots, \nu)$, implies that $q < q_0$, where q_0 is an effectively computable number depending on $n_0, \delta, G^{(1)}, \ldots, G^{(\nu)}$.

3. Lemmas

In the proof of our Theorem we need a result due to A. Baker [1].

Lemma 1. *Let $\pi_1, \pi_2, \ldots, \pi_r$ be non-zero algebraic numbers of heights not exceeding $M_1, M_2, \ldots, M_r$ respectively ($M_r \geq 4$). Further let $b_1, b_2, \ldots, b_{r-1}$ be rational integers with absolute values at most B and let b_r be a non-zero rational integer with absolute value at most B' ($B' \geq 3$). Suppose, that $\sum_{i=1}^r b_i \log \pi_i \neq 0$. Then there exists an effectively computable constant $C = C(r, M_1, \ldots, M_{r-1}, \pi_1, \ldots, \pi_r)$ such that*

$$\left|\sum_{i=1}^{r} b_i \log \pi_i\right| > e^{-C\left(\log M_r \log B' + \frac{B}{B'}\right)}, \tag{7}$$

where logarithms have their principal values.

We need the following auxiliary result.

Lemma 2. *Let $c_1, \ldots, c_k$ be positive real numbers and $0 < \delta < 1$ be an arbitrary real number. Further let $x_1, \ldots, x_k$ be natural numbers with maximum value $x_m = \max_i\{x_i\}$ ($m \in \{1, \ldots, k\}$). If $x_j > \delta x_m$ ($j = 1, \ldots, k$) and $x_m > x_0$ then there exists a real number $c > 0$, which depends on $k, \delta, \max_i\{c_i\}$ and x_0, for which*

$$\sum_{i=1}^{k} e^{-c_i x_i} < e^{-c(x_1 + \cdots + x_k)} = e^{-cx}, \tag{8}$$

where $x = x_1 + \cdots + x_k$.

Proof of Lemma 2. Using the conditions of the lemma we have

$$\sum_{i=1}^{k} e^{-c_i x_i} < \sum_{i=1}^{k} e^{-c_i \delta x_m} = \sum_{i=1}^{k} e^{-d_i x_m},$$

where $d_i = \delta c_i$. If $d_m = \min_i\{d_i\}$ then

$$\sum_{i=1}^{k} e^{-d_i x_m} \leq k e^{-d_m x_m} = e^{\log k - d_m x_m}.$$

Since $x_m \geq x_0$, it follows that

$$e^{\log k - d_m x_m} \leq e^{-d_m^\star x_m} = e^{-ckx_m} \leq e^{-cx}$$

with a suitable constant $d_m^\star$ and $c = \frac{d_m^\star}{k}$.

4. Proof of the Theorem

By $c_1, c_2, \ldots$ we denote positive real numbers which are effectively computable. We may assert, without loss of generality, that the terms of the recurrences $G^{(i)}$ are positive, $d > 0$, $s > 0$ and the inequality

$$|\gamma_1| \geq |\gamma_2| \geq \cdots \geq |\gamma_\nu| \tag{9}$$

also holds.

Let us observe that it is sufficient to consider the case $x_i > n_0$ ($i = 1, 2, \ldots, \nu$). Otherwise, if we suppose that some $x_j \leq n_0$ ($j \in \{1, 2, \ldots, \nu\}$) then $x_m = \max_i\{x_i\}$ cannot be arbitrary large because of the assertion $x_j > \delta x_m$. It means that we have finitely many possibilities to choose the ν-tuples $(x_1, \ldots, x_\nu)$, and the range of $\mathcal{G}(x_1, \ldots, x_\nu)$ is finite. So with a fixed d, if inequality (6) is satisfied then q must be bounded.

In the sequel we suppose that $x_i > n_0$ ($i = 1, 2, \ldots, \nu$). Let $x_1, \ldots, x_\nu$, w, q and $s \in S$ be integers satisfying (6). We may assume that if

$$s = p_1^{e_1} \cdots p_t^{e_t} \tag{10}$$

then $e_j < q$, else a part of s can be joined to w^q. Using (2), from (6) we have

$$sw^q = d \prod_{i=1}^{\nu} a_i \left(\gamma_i\right)^{x_i} \left(1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i}\right)^{x_i} + \cdots\right). \tag{11}$$

A consequence of the assumptions $|\gamma_i| > |\alpha_j^{(i)}|$ $(1 < j \leq t_i)$ is that

$$\left(1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i}\right)^{x_i} + \cdots\right) \longrightarrow 1 \quad \text{whenever} \quad x_i \longrightarrow \infty. \tag{12}$$

Hence there exist real constants $0 < \varepsilon_1, \ldots, \varepsilon_\nu < 1$ such that

$$d \prod_{i=1}^{\nu} |a_i||\gamma_i|^{x_i}(1 - \varepsilon_i) < sw^q < d \prod_{i=1}^{\nu} |a_i||\gamma_i|^{x_i}(1 + \varepsilon_i),$$

and

$$c_1 \prod_{i=1}^{\nu} |\gamma_i|^{x_i} < sw^q < c_2 \prod_{i=1}^{\nu} |\gamma_i|^{x_i}.$$

As before, let $x = x_1 + \cdots + x_\nu$ and applying (9) we may write

$$\log c_1 + x \log |\gamma_\nu| < \log s + q \log w < \log c_2 + x \log |\gamma_1|.$$

Since $\log s \geq 0$, we have

$$\log c_3 + x \log |\gamma_\nu| < q \log w < \log c_2 + x \log |\gamma_1| \tag{13}$$

with $c_3 = \frac{c_1}{s}$. From (13) it follows that

$$c_4 \frac{x}{q} < \log w < c_5 \frac{x}{q} \tag{14}$$

with some positive constants c_4, c_5. Ordering the equality (11) and taking logarithms, by the definition of ε_i we obtain

$$Q = \left| \log \frac{sw^q}{d \prod_{i=1}^{\nu} |a_i| |\gamma_i|^{x_i}} \right| = \left| \log \prod_{i=1}^{\nu} \left| 1 + \frac{p_2^{(i)}(x_i)}{a_i} \left(\frac{\alpha_2^{(i)}}{\gamma_i} \right)^{x_i} + \cdots \right| \right| <$$

$$< \sum_{i=1}^{\nu} \log |1 + \varepsilon_i| \leq \sum_{i=1}^{\nu} e^{-c_i^\star x_i},$$

where $Q \neq 0$ if we assume, that $x_i > n_0$ for every $i = 1, 2, \ldots, \nu$, and $c_i^\star$ is a suitable positive constant ($i = 1, 2, \ldots, \nu$). Applying Lemma 2 and using the notation $x = x_1 + \cdots + x_\nu$, it yields that

$$Q < e^{-c_6(x_1 + \cdots + x_\nu)} = e^{-c_6 x}. \tag{15}$$

On the other hand

$$Q = \left| \log s + q \log w - \log d - \log \prod_{i=1}^{\nu} |a_i| - x_1 \log |\gamma_1| - \cdots - x_\nu \log |\gamma_\nu| \right|, \tag{16}$$

where $\log s = e_1 \log p_1 + \cdots + e_t \log p_t$ (see (10)). Now we may use Lemma 1 with $\pi_r = w = M_r$, since the ordinary heights of p_j ($j = 1, 2, \ldots, t$), d, $\prod_{i=1}^{\nu} |a_i|$ and $|\gamma_i|$ ($i = 1, 2, \ldots, \nu$) are constants. So $B' = q$. In comparison the absolute values of the integer coefficients of the logarithms in (16), we can choose B as $B = x$. So by (16) and Lemma 1 it follows that

$$Q > e^{-c_7\left(\log w \log q + \frac{x}{q}\right)}. \tag{17}$$

Combining (15) and (17) it yields the following inequality:

$$c_6 x < c_7 \left(\log w \log q + \frac{x}{q}\right), \tag{18}$$

and by (14) it follows that

$$c_6 x < c_7 \left(\log w \log q + \frac{1}{c_4} \log w\right) < c_8 \log w \log q \tag{19}$$

with some $c_8 > 0$. Applying (14) again, we conclude that $\frac{1}{c_5} q \log w < x$ and so by (19)

$$c_9 q < \log q \tag{20}$$

follows. But (20) implies that $q < q_0$, which proves the theorem.

References

[1] A. Baker, A sharpening of the bounds for linear forms in logarithms II., *Acta Arith.* **24** (1973), 33–36.

[2] P. Kiss, Pure powers and power classes in the recurrence sequences, *Math. Slovaca* **44** (1994), No. 5, 525–529.

[3] K. Liptai, L. Szalay, On products of the terms of linear recurrences, to appear.

[4] A. Pethő, Perfect powers in second order linear recurrences, *J. Num. Theory* **15** (1982), 5–13.

[5] A. Pethő, Perfect powers in second order linear recurrences, Topics in Classical Number Theory, Proceedings of the Conference in Budapest 1981, *Colloq. Math. Soc. János Bolyai* **34**, North Holland, Amsterdam, 1217–1227.

[6] A. PETHŐ, On the solution of the diophantine equation $G_n = p^z$, Proceedings of EUROCAL '85, Linz, *Lecture Notes in Computer Science* **204**, Springer-Verlag, Berlin, 503–512.

[7] T. N. SHOREY, C. L. STEWART, On the Diophantine equation $ax^{2t} + bx^t y + cy^2 = d$ and pure powers in recurrence sequences, *Math. Scand.* **52** (1987), 324–352.

[8] T. N. SHOREY, R. TIJDEMAN, Exponential diophantine equations, Cambridge, 1986.

LÁSZLÓ SZALAY
UNIVERSITY OF SOPRON
INSTITUTE OF MATHEMATICS
SOPRON, ADY U. 5.
H-9400, HUNGARY
E-mail: laszalay@efe.hu

The asymptotic behavior of the real roots of Fibonacci-like polynomials

FERENC MÁTYÁS*

Abstract. The Fibonacci-like polynomials $G_n(x)$ are defined by the recursive formula $G_n(x)=xG_{n-1}(x)+G_{n-2}(x)$ for $n\geq 2$, where $G_0(x)$ and $G_1(x)$ are given seed-polynomials. In this paper the non-zero accumulation points of the set of the real roots of Fibonacci-like polynomials are determined if either both of the seed-polynomials are constants or $G_0(x)=-a$ and $G_1(x)=x\pm a$ ($a\in\mathbf{R}\setminus\{0\}$). The theorems generalize the results of G. A. Moore and H. Prodinger who investigated this problem if $G_0(x)=-1$ and $G_1(x)=x-1$, furthermore we extend a result of Hongquan Yu, Yi Wang and Mingfeng He.

Introduction

The Fibonacci-like polynomials $G_n(x)$ are defined by the following manner. For $n \geq 2$

$$G_n(x) = xG_{n-1}(x) + G_{n-2}(x), \tag{1}$$

where $G_0(x)$ and $G_1(x)$ are fixed polynomials (so-called seed-polynomials) with real coefficients. If it is necessary to denote the seed-polynomials, then we will use the notation $G_n(x) = G_n(G_0(x), G_1(x), x)$, too. The polynomials $G_n(0, 1, x)$ are the original Fibonacci polynomials and the numbers $G_n(0, 1, 1)$ are the well-known Fibonacci numbers.

Recently, G. A. Moore [5] investigated the maximal real roots g'_n of the polynomials $G_n(-1, x-1, x)$ and proved that g'_n exists for every $n \geq 1$ and $\lim_{n\to\infty} g'_n = 3/2$. (These numbers g'_n are called as "golden numbers".) H. Prodinger [6] gave the asymptotic formula $g'_n \sim \frac{3}{2} + (-1)^n \frac{25}{12} 4^{-n}$. Hongquan Yu, Yi Wang and Mingfeng He [3] investigated the limit of the maximal real roots g'_n of polynomials $G_n(-a, x-a, x)$ if $a \in \mathbf{R}^+$.

For brevity let us introduce the following notations. B denotes the set of the real roots of polynomials $G_n(x)$ ($n = 0, 1, 2, \ldots$) and A denotes the set of the the accumulation points of set B. In [4] we investigated these sets.

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 020295.

Although, the main result of [4] is formulated for seed-polynomials with integer coefficients but it is true for seed-polynomials with real coefficients, too. Since we are going to apply it, therefore we cite it as a lemma.

Lemma 1. *Let $G_0(x)$ and $G_1(x)$ be two fixed polynomials with real coefficients, $G_0(0)\cdot G_1(0)\neq 0$ and $x_0\in\mathbf{R}$. $x_0\in A$ if and only if one of the following conditions holds:*
(i) $-\frac{G_1(x_0)}{G_0(x_0)}=\frac{1}{\alpha(x_0)}$ and $x_0>0$;
(ii) $-\frac{G_1(x_0)}{G_0(x_0)}=\frac{1}{\beta(x_0)}$ and $x_0<0$;
(iii) $x_0=0$,
where

$$\alpha(x)=\frac{x+\sqrt{x^2+4}}{2}\quad\text{and}\quad\beta(x)=\frac{x-\sqrt{x^2+4}}{2}. \tag{2}$$

The purpose of this paper is to investigate the asymptotic behavior of the elements of the set B in the cases of simple seed-polynomials. In our discussion we are going to use the following explicit formulae for the polynomial $G_n(x)=G_n\left(G_0(x),G_1(x),x\right)$. It is known that

$$G_n(x)=p(x)\alpha^n(x)-q(x)\beta^n(x) \tag{3}$$

for $n\geq 0$, where $\alpha(x)$ and $\beta(x)$ are defined in (2), while

$$p(x)=\frac{G_1(x)-\beta(x)G_0(x)}{\alpha(x)-\beta(x)}\quad\text{and}\quad q(x)=\frac{G_1(x)-\alpha(x)G_0(x)}{\alpha(x)-\beta(x)}.$$

These formulae can be obtained by standard methods or see in [2].

Since we want to investigate the roots of the polynomials $G_n(x)$, therefore it is worth rephasing the expression $G_n(x)=0$ as

$$\frac{p(x)}{q(x)}=\left(\frac{\beta(x)}{\alpha(x)}\right)^n,$$

that is

$$\frac{G_1(x)-\beta(x)G_0(x)}{G_1(x)-\alpha(x)G_0(x)}=\left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right)^n. \tag{4}$$

Let us consider the polynomial $G_n\left(G_0(x),G_1(x),x\right)$. It is obvious that $G_n(0,0,x)$ is identical to the zero polynomial for every $n\geq 0$. Using (3) the identities $G_n(0, G_1(x), x) = G_1(x) \cdot G_n(0,1,x)$ and $G_n(G_0(x), 0, x) = G_0(x) \cdot G_n(1,0,x)$ yield. But it is known from [2] and can be obtained easily from (4) that neither the Fibonacci polynomials $G_n(0,1,x)$ nor the polynomials $G_n(1,0,x)$ have real root x' except $x' = 0$ if n is even or odd, respectively. Therefore investigating the asymptotic behavior of the roots of polynomials $G_n(G_0(x), G_1(x), x)$ we can assume that the seed-polynomials differ from the zero polynomial and at least one of them is a monic polynomial (since one can simplify the left-hand side of (4) with the leading coefficient of the polynomial $G_1(x)$ or $G_0(x)$).

Theorems and Proofs

First of all we need the following lemma, which deals with the properties of the functions $\alpha(x)$ and $\beta(x)$ defined in (2).

Lemma 2. (a) *On the interval* $[0, \infty)$ *the function* $\frac{1}{\alpha(x)}$ *is continuous and strictly monotonically decreasing, its graph is convex and* $1 \geq \frac{1}{\alpha(x)} > 0$.

(b) *On the interval* $(-\infty, 0]$ *the function* $\frac{1}{\beta(x)}$ *is continuous and strictly monotonically decreasing, its graph is concave and* $0 > \frac{1}{\beta(x)} \geq -1$.

Proof. By (2) it is obvious that the functions $\frac{1}{\alpha(x)}$ and $\frac{1}{\beta(x)}$ are continuous on the above mentioned intervals. The rest of the statement can be proved easily using the methods of differential calculus.

Further on we deal with the set A if $G_0(x) = 1$ and $G_1(x) = a$. In this case, using Lemma 1, the set A can be determined in a very simple manner.

Theorem 1. *Let* $a \in \mathbf{R} \setminus \{0\}$ *and* $G_n(1, a, x)$ *be Fibonacci-like polinomials. If* $0 < |a| < 1$ *then* $A \setminus \{0\} = \left\{\frac{a^2-1}{a}\right\}$, *while in the case* $|a| \geq 1$ $A \setminus \{0\} = \emptyset$.

Proof. According to Lemma 1 to get the elements of the set $A \setminus \{0\}$ we have to solve the equations

$$-a = \frac{2}{x + \sqrt{x^2+4}} \quad \text{for} \quad x > 0 \tag{5}$$

and

$$-a = \frac{2}{x - \sqrt{x^2+4}} \quad \text{for} \quad x < 0. \tag{6}$$

By Lemma 2 the functions $\frac{1}{\alpha(x)} = \frac{2}{x+\sqrt{x^2+4}}$ and $\frac{1}{\beta(x)} = \frac{2}{x-\sqrt{x^2+4}}$ are continuous, $1 > \frac{1}{\alpha(x)} > 0$ for any $x > 0$ and $0 > \frac{1}{\beta(x)} > -1$ for any $x < 0$, therefore $0 < |a| < 1$ is a necessary and sufficient condition for the solvability of (5) and (6). Solving (5) and (6) we get that the single real root x_0 is $x_0 = \frac{a^2-1}{a}$, where $x_0 > 0$ if $-1 < a < 0$ and $x_0 < 0$ if $0 < a < 1$. This completes the proof.

In the following theorems we prove asymptotic formulae for those real roots g_n of the polynomials $G_n(-a, x \pm a, x)$ which do not tend to 0 if n tends to infinity.

Theorem 2. *Let $G_0(x) = -a$ and $G_1(x) = x - a$, where $a \in \mathbf{R} \setminus \{0\}$. If either $a > 0$ or $a < -2$ then $A \setminus \{0\} = \left\{\frac{a(a+2)}{a+1}\right\}$, while in the case $-2 \le a < 0$ we have $A \setminus \{0\} = \emptyset$. Furthermore for large n*

$$g_n \sim \frac{a(a+2)}{a+1} + (-1)^n \frac{a(a^2+2a+2)^2}{(a+1)^2(a+2)}(a+1)^{-2n}.$$

Proof. According to Lemma 1, $x_0 \in A \setminus \{0\}$ if and only if

$$\frac{x_0 - a}{a} = \frac{2}{x_0 + \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 > 0 \tag{7}$$

or

$$\frac{x_0 - a}{a} = \frac{2}{x_0 - \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 < 0 \tag{8}$$

holds. Using the statements of Lemma 2 one can verify that (7) has a solution for x_0 if and only if $a > 0$, while (8) has a solution for x_0 if and only if $a < -2$. Solving (7) and (8) we get that

$$x_0 = \frac{a(a+2)}{a+1}.$$

To determine the asymptotic behavior of g_n we apply (4), which in our case has the following form

$$\frac{2(g_n - a) + a\left(g_n - \sqrt{g_n^2+4}\right)}{2(g_n - a) + a\left(g_n + \sqrt{g_n^2+4}\right)} = \left(\frac{g_n - \sqrt{g_n^2+4}}{g_n + \sqrt{g_n^2+4}}\right)^n.$$

This will be much nicer when we substitute

(9) $$g_n = u - \frac{1}{u}.$$

Without loss of generality we can assume that $u > 0$ and we get the equality

(10) $$\frac{(au+u+1)(u-1)}{(a+1-u)(u+1)} = -(-u^2)^n.$$

Since $x_0 = u - \frac{1}{u}$ holds for $u = a+1$ and $u = -\frac{1}{a+1}$ therefore it is plain to see that, for large n, (9) can only hold if u is either close to $a+1$ or $-\frac{1}{a+1}$. In both cases this would mean that g_n is close to x_0.

Let us assume that u is close to $a+1$ and so $a > 0$ because of $u > 0$. It is clear from (10) that the cases when n is even or odd have to be distinguished.

We start with $n = 2m$ and rewrite (10) as

(11) $$a+1-u = -\frac{(au+u+1)(u-1)}{u+1} \cdot u^{-4m}.$$

We get the asymptotic behavior by a process known as "bootstrapping" which is explained in [1]. First we insert $u = a+1+\delta_1$ into the left-hand side of (11) and $u = a+1$ into the right-hand side of (11). So we get an approximation for δ_1. Then we insert $u = a+1+\delta_1+\delta_2$ into the left-hand side of (11) and $u = a+1+\delta_1$ into the rihgt-hand side of (11) and get an approximation for δ_2. This procedure can be repeated to get better and better estiamations for u. Now we determine only the number δ_1. From (11) we have

$$\delta_1 \sim \frac{a(a^2+2a+2)}{a+2}(a+1)^{-4m}$$

and so

$$u = a+1+\delta_1 \sim a+1+\frac{a(a^2+2a+2)}{a+2}(a+1)^{-4m}.$$

Substituting u into (9) we get that

(12) $$g_{2m} = a+1+\delta_1 - \frac{1}{a+1+\delta_1} \sim \frac{a(a+2)}{a+1} + \frac{(a(a^2+2a+2)^2}{(a+1)^2(a+2)}(a+1)^{-4m}.$$

If $n = 2m+1$ then (10) can be rewrite as

$$a+1-u = \frac{(au+u+1)(u-1)}{u^2(u+1)}u^{-4m}.$$

Using the "bootstrapping" method for $u = a + 1 + \delta_1'$ we get the estimation

$$\delta_1' \sim \frac{a(a^2 + 2a + 2)}{(a+2)(a+1)^2}(a+1)^{-4m},$$

which implies the following form:

$$\begin{aligned} g_{2m+1} &= a + 1 + \delta_1' - \frac{1}{a + 1 + \delta_1'} \\ &\sim \frac{a(a+2)}{a+1} - \frac{a(a^2+2a+2)^2}{(a+2)(a+1)^4}(a+1)^{-4m}. \end{aligned} \tag{13}$$

Comparing (12) and (13) the desired approximation yields since $a > 0$.

One can verify in the same manner that the estimation for g_n also holds when $a < -2$. This completes the proof.

Remark. From our proof one can see that for large n $g_n = g_n'$ if $a > 0$ while g_n is the minimal real root if $a < -2$.

A similar result can be proved for the polynomials $G_n(-a, x + a, x)$.

Theorem 3. *Let $G_0(x) = -a$ and $G_1(x) = x + a$ where $a \in \mathbf{R} \setminus \{0\}$. If etiher $a > 0$ or $a < -2$ then $A \setminus \{0\} = \left\{-\frac{a(a+2)}{a+1}\right\}$, while $A \setminus \{0\} = \emptyset$ if $-2 \le a < 0$. Furthermore for large n*

$$g_n \sim -\frac{a(a+2)}{a+1} + (-1)^n \frac{a(a^2+2a+2)^2}{(a+1)^2(a+2)}(a+1)^{-2n},$$

where $G_n(g_n) = 0$ and $\lim_{n\to\infty} g_n \neq 0$.

Proof. For a real number x_0, by our Lemma 2, $x_0 \in A \setminus \{0\}$ if and only if

$$\frac{x_0 + a}{a} = \frac{2}{x_0 + \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 > 0 \tag{14}$$

or

$$\frac{x_0 + a}{a} = \frac{2}{x_0 - \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 < 0 \tag{15}$$

holds. Substituting $-x_0$ for x_0 into (14) and (15) we get that

$$\frac{x_0 - a}{a} = \frac{2}{x_0 - \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 < 0 \tag{16}$$

and

$$(17) \qquad \frac{x_0 - a}{a} = \frac{2}{x_0 + \sqrt{x_0^2 + 4}} \quad \text{and} \quad x_0 > 0$$

Since (16) and (17) are identical to (8) and (7), respectively, therefore all of the statements of our theorem follows from the Theorem 2. Thus the theorem is proved.

Concluding Remarks

Using our Theorem 2 for $a = 1$ we get that $g_n = g_n' \sim \frac{3}{2} + (-1)^n \frac{25}{12} 4^{-n}$, which matches perfectly with the result of H. Prodinger.

On the other hand it is quite likely that similar results can be obtained for seed-polynomials $G_0(x) = x \pm a$ and $G_0(x) = a$ or for other polynomials. This could be the subject of further research work.

References

[1] D. GREENE and D. KNUTH, Mathematics for the Analysis of Algorithms, Birkhäuser, 1981.

[2] V. E. HOGGAT, JR. and M. BICKNELL, Roots of Fibonacci Polynomials, *The Fibonacci Quarterly* **11.3** (1973), 271–274.

[3] HONGQUAN YU, YI WANG and MINGFENG HE, On the Limit of Generalized Golden Numbers, *The Fibonacci Quarterly* **34.4** (1996), 320–322.

[4] F. MÁTYÁS, Real Roots of Fibonacci-like Polynomials, *Proceedings of Number Theory Conference*, Eger (1996) (to appear)

[5] G. A. MOORE, The Limit of the Golden Numbers is 3/2, *The Fibonacci Quarterly* **32.3** (1994), 211–217.

[6] H. PRODINGER, The Asymptotic Behavior of the Golden Numbers, *The Fibonacci Quarterly* **35.3** (1996), 224–225.

FERENC MÁTYÁS
ESZTERHÁZY KÁROLY TEACHERS' TRAINING COLLEGE
DEPARTMENT OF MATHEMATICS
LEÁNYKA U. 4.
3301 EGER, PF. 43.
HUNGARY
E-mail: matyas@gemini.ektf.hu

Két kombinatorikai identitás általánosítása

FEHÉR ZOLTÁN

Abstract. (A generalization of two combinatorial identities) In this paper we formulate and prove two combinatorial identities for Gauss' binomial coefficients which are generalized forms of known combinatorial identities.

Legyen n természetes szám és $i = 0, 1, \ldots, n-1$, akkor érvényesek az

$$\sum_{k=i}^{n} \binom{n}{k} \binom{k}{i} = \binom{n}{i} 2^{n-i}, \tag{1}$$

$$\sum_{k=i}^{n} (-1)^{k-i} \binom{n}{k} \binom{k}{i} = 0 \tag{2}$$

egyenlőségek (lásd [2], 6. old., lásd [3]).

Ez a cikk az (1) és (2) identitások általánosítását tartalmazza a Gauss-féle binomiális együtthatók felhasználásával.

Gauss-féle binomiális együtthatónak (lásd. [1], 35. old.) nevezzük az

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q-1)(q^2-1) \cdots (q^k - 1)}$$

kifejezést, ahol n és k a $0 < k \leq n$ feltételt kielégítő egész számok. A q olyan valós szám, melyre $q^\alpha - 1 \neq 0$ ($\alpha = 1, 2, \ldots, k$). Továbbá $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ és $\begin{bmatrix} n \\ k \end{bmatrix} = 0$, ha nem teljesül a $0 \leq k \leq n$ egyenlőtlenség. A Gauss-féle binomiális együtthatókra teljesül, hogy ha $q \to 1$, akkor $\begin{bmatrix} n \\ k \end{bmatrix} \to \binom{n}{k}$.

A következő két tétel általánosítja az (1) és (2)-t, mert ha $q \to 1$, akkor a tételben szereplő állítások megegyeznek a Newton-féle binomiális együtthatóra fent megadott két kombinatorikai identitással.

1. Tétel. *Legyen n természetes szám. Akkor minden $1 \le i < n$ természetes számra érvényes*

$$(3)\qquad \sum_{k=i}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} q^{(k-i)(k-i-1)/2} = \begin{bmatrix} n \\ i \end{bmatrix} \prod_{j=1}^{n-i} (1 + q^{j-1}).$$

Bizonyítás. A definíció segítségével könnyen belátható, hogy

$$\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n-i \\ k-i \end{bmatrix}.$$

Ebből adódik, hogy

$$\sum_{k=i}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \cdot q^{(k-i)(k-i-1)/2} = \sum_{k=i}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n-i \\ k-i \end{bmatrix} q^{(k-i)(k-i-1)/2} =$$
$$= \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{n-i} \begin{bmatrix} n-i \\ j \end{bmatrix} q^{j(j-1)/2}.$$

Mivel

$$(1+x)(x+qx)\cdots(1+q^{n-1}x) = \begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} x + \cdots + \begin{bmatrix} n \\ n \end{bmatrix} q^{n(n-1)/2} x^n$$

(lásd [1], 36. old.), akkor $x = 1$ választással kapjuk a (3) egyenlőséget.

A 2. tétel bizonyításához felhasználjuk a következő segédtételt.

Lemma. *Legyen k nemnegatív egész szám és x valós szám. Legyen*

$$P_k(x) = \begin{cases} (x-1)(x-q)\cdots(x-q^{k-1}), & \text{ha } k \ge 1, \\ 1, & \text{ha } k = 0. \end{cases}$$

Akkor

$$(4)\qquad P_k(x) = \sum_{i=0}^{k} (-1)^{k-i} \begin{bmatrix} k \\ i \end{bmatrix} q^{(k-i)(k-i-1)/2} x^i.$$

Bizonyítás. Legyen $P_k(x) = p_{k,0} + p_{k,1}x + \cdots + p_{k,k}x^k$ és határozzuk meg a $p_{k,i}$ $(i = 0, 1, \ldots, k)$ együtthatókat.

Felhasználjuk az

$$(x - q^{k-1})P_k(qx) = (q^k x - q^{k-1})P_k(x)$$

egyenlőséget. Tehát

$$(x-q^{k-1})(p_{k,0}+p_{k,1}qx+\cdots+p_{k,k}q^kx^k)=$$
$$=(q^kx-q^{k-1})(p_{k,0}+p_{k,1}x+\cdots+p_{k,k}x^k).$$

Innen $i=1,2,\ldots,k$ esetben kapjuk

$$q^{i-1}p_{k,i-1}-q^{k+i-1}p_{k,i}=q^kp_{k,i-1}-q^{k-1}p_{k,i},$$

vagyis

$$p_{k,i}=\frac{q^{k-i+1}-1}{q^i-1}\left(-q^{i-k}\right)p_{k,i-1}.$$

Ezzel a rekurziós formulával valamennyi együttható meghatározható az abszolút tagból kiindulva. Az abszolút tagot a polinom felírásából kapjuk meg. Tehát

$$p_{k,0}=(-1)^kq^{k(k-1)/2}$$

és minden $i=1,2,\ldots,k-2$ számra

$$p_{k,i}=\frac{q^{k-i+1}-1}{q^i-1}(-q^{i-k})\begin{bmatrix}k\\i-1\end{bmatrix}(-1)^{k-i+1}q^{(k-i+1)(k-i)/2}=$$
$$=\begin{bmatrix}k\\i\end{bmatrix}(-1)^{k-i}q^{(k-i)(k-i-1)/2}.$$

Továbbá $p_{k,k}=1$ és $p_{k,k-1}=-\begin{bmatrix}k\\k-1\end{bmatrix}$. Ezzel a (4) egyenlőséget igazoltuk.

2. Tétel. *Legyen n tetszőleges természetes szám és i olyan egész szám, hogy $0\le i<n$. Akkor*

$$\sum_{k=i}^{n}(-1)^{k-i}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}q^{(k-i)(k-i-1)/2}=0. \tag{5}$$

Bizonyítás. Jelöljük (5) bal oldalán álló összeget $a_{n,i}$-vel, akkor

$$a_{n,i}=\sum_{k=i}^{n}(-1)^{k-i}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\i\end{bmatrix}q^{(k-i)(k-i-1)/2}=$$
$$=\sum_{k=i}^{n}(-1)^{k-i}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n-i\\k-i\end{bmatrix}q^{(k-i)(k-i-1)/2}=$$
$$=\begin{bmatrix}n\\i\end{bmatrix}\sum_{j=0}^{n-i}(-1)^{j}\begin{bmatrix}n-i\\j\end{bmatrix}q^{j(j-1)/2}.$$

Vegyük észre, hogy $i = 0$ esetben

$$a_{n,0} = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ 0 \end{bmatrix} q^{k(k-1)/2} = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2}. \tag{6}$$

Ezt felhasználva kapjuk, hogy

$$a_{n,i} = \begin{bmatrix} n \\ i \end{bmatrix} \sum_{j=0}^{n-i} \begin{bmatrix} n-i \\ j \end{bmatrix} (-1)^j q^{j(j-1)/2} = \begin{bmatrix} n \\ i \end{bmatrix} a_{n-i,0}, \quad \text{ahol} \quad 0 \le i < n.$$

Elegendő belátni, hogy $a_{k,0} = 0$, $k = 1, 2, \ldots, n$ esetben. Mivel

$$\begin{bmatrix} k \\ i \end{bmatrix} = \begin{bmatrix} k \\ k-i \end{bmatrix},$$

ezért a $P_k(x)$ polinom értéke $x = 1$ esetben

$$P_k(1) = \sum_{i=0}^{k} (-1)^{k-i} \begin{bmatrix} k \\ i \end{bmatrix} q^{(k-i)(k-i-1)/2} =$$

$$= \sum_{i=0}^{k} (-1)^{k-i} \begin{bmatrix} k \\ k-i \end{bmatrix} q^{(k-i)(k-i-1)/2} = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j-1)/2}.$$

A kapott kifejezést összehasonlítva (6)-tal, kapjuk $P_k(1) = a_{k,0}$. Viszont $x = 1$ értékre

$$P_k(1) = (1-1)(1-q)\cdots(1-q^{k-1}) = 0.$$

Tehát $a_{k,0} = P_k(1) = 0$, $(k = 1, 2, \ldots, n)$.

Irodalom

[1] PÓLYA GY.—SZEGŐ G.: Feladatok és tételek az analízis köréből I., Tankönyvkiadó, Budapest, 1980.

[2] M. KMET'OVÁ, T. ŠALÁT, Metóda mrežových bodov v kombinatorike, *Matematické Obzory* **40** (1993), 1–10.

[3] N. J. VILENKIN: Kombinatorika. (2. kiadás), Műszaki könyvkiadó, Budapest, 1987.

UNIVERSITY OF EDUCATION
DEPARTMENT OF MATHEMATICS
FARSKÁ 19
94974 NITRA, SLOVAKIA

Független metszőrendszerek II.

RÓKA SÁNDOR

Abstract. (Independent intersection systems) Definition 1 gives the meaning of independent intersection systems, which is similar of the meaning of separating systems (theorem 5), which is discussed by others (see for e.g. [2–8]).

If the number of the elements of the intersection systems on the n-element set is m, then $c_1 \log_2 n \leq m \leq c_2 n^2$. There bounds can not be sharpened. Theorems 3 and 4 are related to uniform systems.

1. Definíció. $A_1, A_2, \dots, A_m$ az n-elemű H halmaz részhalmazai *független metszőrendszert* alkotnak, ha

(1) $\forall x \in H$ előáll A_i metszeteként;

(2) Az $A_1, A_2, \dots, A_m$ rendszerből bármely A_i-t elhagyva (1) nem teljesül.

1. Tétel. *$A_1, A_2, \dots, A_m$ pontosan akkor független metszőrendszer, ha $\overline{A_1}, \overline{A_2}, \dots, \overline{A_m}$ is független metszőrendszer.*

Bizonyítás. Legyen $H = \{x_1, x_2, \dots, x_m\}$, $X_i = \{x_i\}$, $i = 1, 2, \dots, n$. Tegyük fel, hogy $A_1, A_2, \dots, A_m$ független metszőrendszer. Megmutatjuk, hogy például X_1 előállítható az $\overline{A_1}, \overline{A_2}, \dots, \overline{A_m}$ halmazokból néhánynak a metszeteként.

$$X_1 = H \setminus \{X_2 \cup X_3 \cdots \cup X_n\} = \overline{X_2 \cup X_3 \cup \cdots \cup X_n} =$$

$$= \overline{X_2} \cap \overline{X_3} \cap \cdots \cap \overline{X_n} = \overline{\left(\bigcap_{i \in I_2} A_i\right)} \cap \overline{\left(\bigcap_{i \in I_3} A_i\right)} \cap \cdots \cap \overline{\left(\bigcap_{i \in I_n} A_i\right)} =$$

$$= \left(\bigcup_{i \in I_2} \overline{A_i}\right) \cap \left(\bigcup_{i \in I_3} \overline{A_i}\right) \cap \cdots \cap \left(\bigcup_{i \in I_n} \overline{A_i}\right) = \cup(\overline{A_{i_2}} \cap \overline{A_{i_3}} \cap \cdots \cap \overline{A_{i_n}})$$

Mivel X_1 egyetlen elemből álló halmazt jelentett, így az unió tagjainak valamelyike egyenlő X_1-gyel.

Látható, hogy az $A_1, A_2, \dots, A_m$ és az $\overline{A_1}, \overline{A_2}, \dots \overline{A_m}$ rendszerek egyszerre rendelkeznek az (1) tulajdonsággal, és ebből adódóan a (2) tulajdonsággal is. ∎

Következmény. *Az $A_1, A_2, \dots, A_m$ független metszőrendszeren nem oldható meg az*

$$(3) \qquad A_l = \bigcap_{i\in I} A_i \quad I \subseteq \{1,2,\dots,m\} \quad |I| \neq 1$$

egyenlet.

Bizonyítás. A definícióból következik, hogy $A_l = \bigcup_{i\in I} A_i$, továbbá $I \subseteq \{1,2,\dots,m\}$ és az $|I| \neq 1$ egyenlet nem oldható meg, s így az előbbi tétel miatt nem oldható meg a (3) egyenlet sem. ∎

2. Tétel. *Az $A_1, A_2, \dots, A_m$ az n-elemű H halmazon független metszőrendszer, akkor $c_1 \log_2 n \leq m \leq c_2 n^2$, és ezek a korlátok nagyságrendjükben pontosak.*

Bizonyítás. Legyen $X_k = \bigcup_{i\in I_k} A_i$, $|X_k| = 1$, $k = 1,2,\dots,n$. Az $I_1, I_2, \dots, I_n \subseteq \{1,2,\dots,m\}$ rendszer *Sperner-rendszer*, így a *Sperner-lemma* [1] miatt $n \leq \binom{m}{[\frac{m}{2}]}$, azaz $m \geq c_1 \log_2 n$. Megmutatjuk, hogy az alsó korlát pontos, vagyis, ha $\binom{m}{[\frac{m}{2}]} \leq n$, akkor van egy legfeljebb m halmazból álló $A_1, A_2, \dots, A_m$ független metszőrendszer az n elemű H halmazon.

Tekintsük azt az A $m \times \binom{m}{[\frac{m}{2}]}$-es 0-1 mátrixot, amelynek oszlopai az $\{1,2,\dots,m\}$ halmaz egy-egy $[\frac{m}{2}]$-elemű részét reprezentálják. Az A mátrix soraival megadott m db halmazból néhányat választva az $\binom{m}{[\frac{m}{2}]}$ elemű halmaz bármely elemét kimetszhetjük.

A felső korlát igazolása. Legyen $x \in H$, s tekintsük azon A_i-ket, melyek szükségesek az (1) és (2) szerinti előállításhoz. A_i-kből hagyjuk el az x elemet. Az így kapott halmazokat jelölje A'_i. Ezen A'_i-k száma s. Most bármely $s-1$ db A'_i metszete $\neq \emptyset$, míg az összes A'_i metszete $= \emptyset$. Így minden A'_i-hez hozzárendelhetünk egy olyan elemet a $H \setminus \{x\}$ halmazból, mely a többi A'_j-nek nem eleme. Ezért $s \leq n-1$. Így leszámolva az $A_1, A_2, \dots, A_m$ halmazokat, azt kapjuk, hogy $m \leq n(n-1)$.

Megadunk olyan független metszőrendszert a $H = \{x_1, x_2, \dots, x_n\}$ halmazon, mely $c_2 n^2$ halmazból áll. Legyen $r = [\frac{n}{2}]$, $R = \{x_1, x_2, \dots, x_r\}$, $A_i = R \setminus \{x_i\}$, $i = 1,2,\dots,r$; $A_{i,j} = \{x_i\} \cup A_j$, $i = r+1, r+2, \dots, n$ $j = 1,2,\dots,r$.

Az $A_{i,j}$ halmazok független metszőrendszert alkotnak, s a halmazok száma több, mint $\frac{1}{4}n^2 - 1$. ∎

Vizsgáljuk most azon független metszőrendszereket, melyek uniform rendszerek.

Legyen $A_1, A_2, \dots, A_m$ az n-elemű H halmazon független metszőrendszer, és $|A_i| = k$, $i = 1, 2, \dots, m$. Jelölje m értékének legkisebb értékét $f_k(n)$, legnagyobb értékét $F_k(n)$.

Az 1. Tétel miatt $f_k(n) = f_{n-k}(n)$ és $F_k(n) = F_{n-k}(n)$. Ezért $f_k(n)$ értékét elegendő $1 \le k \le \frac{n}{2}$ esetekben meghatározni.

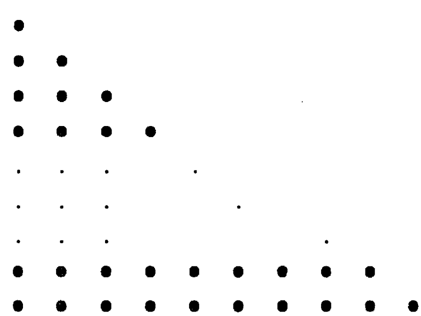
3. Tétel.
(a) $f_k(n) \le \frac{2n}{k}$, $1 \le k \le \frac{n}{2}$.
(b) *Ha* $k \le \sqrt{2n}$, $k \mid 2n$ *és* k *páros, akkor* $f_k(n) = \frac{2n}{k}$.
(c) $f_k\left(\frac{k(k+1)}{2}\right) = k + 1$, *azaz ha* $n = \frac{k(k+1)}{2}$, *akkor* $f_k(n) = \frac{2n}{k}$.

Bizonyítás. Legyen $A_1, A_2, \dots, A_m$ a $H = \{x_1, x_2, \dots, x_n\}$ halmazon független metszőrendszer, $|A_i| = k$, $i = 1, 2, \dots, m$, és $B_i = \{k : x_i \in A_k\}$, $i = 1, 2, \dots, n$. Ekkor $|B_i| \le 2$.

Mivel $mk = \sum_{i=1}^{m} |A_i| = \sum_{j=1}^{n} |B_j| \le 2n$, így $m \le \frac{2n}{k}$.

Belátjuk, ha $k \le \sqrt{2n}$, $k \mid 2n$ és k páros, akkor $f_k(n) \le \frac{2n}{k}$. Innen (a) felhasználásával adódik a (b) állítás.

Készítsünk egy gráfot. Legyen a G gráf csúcsainak száma $m = \frac{2n}{k}$, és számozzuk meg a gráf csúcsait az $1, 2, \dots, m$ számokkal. A gráf minden csúcsára k él illeszkedjen. Ekkor a gráfnak n éle van. Ilyen gráf az *Erdős—Gallai-tétel* [9] szerint készíthető. Álljon most az n elemű H halmaz a gráf éleiből, az $A_1, A_2, \dots, A_m$ halmazokat pedig a következő módon kapjuk: A_i a gráf azon éleiből áll, amelyek illeszkednek a gráf i-edik csúcsára.



Legyen most $n = \binom{k+1}{2}$. A H halmaz az ábrán látható pontokból áll, az $A_1, A_2, \dots, A_{k+1}$ halmazokat pedig a háromszög átlóján levő pontok határozzák meg, egy-egy ilyen ponttal közös halmazba tartoznak a vele egy sorban és a vele egy oszlopban levő pontok, valamint egy további halmaz van még, mely az átlón levő pontokból áll. Ezen konstrukció miatt $f_k(n) \le k+1$, ám (a) miatt $f_k(n) \ge k + 1$, ezzel igazoltuk (c)-t is. ∎

4. Tétel. $k(n-k) \leq F_k(n) \leq kn$, $1 \leq k \leq \frac{n}{2}$.

Bizonyítás. Legyen

$$H = \{x_1, x_2, \ldots, x_n\},\ R = \{x_1, x_2, \ldots, x_k\},\ A_i = R \setminus \{x_i\},\ i = 1, 2, \ldots, k;$$

$$A_{i,j} = \{x_i\} \cup A_j,\ i = k+1, k+2, \ldots, n,\ j = 1, 2, \ldots, k.$$

Az $A_{i,j}$ halmazok független metszőrendszert alkotnak, a halmazok száma $k(n-k)$, és $|A_{i,j}| = k$ így valóban $F_k(n) \geq k(n-k)$.

A felső korlát igazolása. Legyen $A_1, A_2, \ldots, A_m$ az n-elemű H halmazon független metszőrendszer, $|A_i| = k$, $i = 1, 2, \ldots, m$. Legyen $x \in H$, s tekintsük azon A_i-ket, melyek szükségesek az (1) és (2) szerinti előállításához. A_i-kből hagyjuk el az x elemet. Az így kapott halmazokat jelölje A_i'. Ezen A_i'-k száma s. Most bármely $s-1$ db A_i' metszete $\neq \emptyset$, míg az összes A_i' metszete $= \emptyset$. Így minden A_i'-hez megadható olyan elem, mely neki nem eleme, de a többi A_j'-nek igen. Rögzített A_j' mellet sorra véve a többi A_i'-t, A_j'-nek $s-1$ különböző eleme jelölhető így ki, azaz $s - 1 \leq |A_j'| = k - 1$. Tehát H egy tetszőlegesen kiválasztott x eleméhez legfeljebb k db A_i' rendelhető, így $m \leq nk$. ∎

Már eddig is több független metszőrendszerre láttunk példát, most még további két rendszert konstruálunk.

1. konstrukció. Tekintsük egy N pontú teljes gráfot, s minden élére helyezzünk újabb pontot. A H halmaz ezen „régi" és „új" pontokból áll, tehát $|H| = N + \binom{N}{2}$. A metszőrendszer a „fél-élekből" áll, tehát olyan kételemű halmazokból, melyek egyik eleme egy „régi" és egy „új" pont, és közös az az él a teljes gráfban, melyre ez a két pont illeszkedik. A metszőrendszer elemeinek m száma itt nagyságában $2|H|$-hoz közelít.

2. konstrukció. Vegyünk egy N elemű X halmazt, ennek elemeit nevezzük „régi" pontoknak, az „új" pontok az N elemű halmaz $\left[\frac{N}{2}\right]$-elemű részei. Így elkészítettük a H halmazt, mely ezen „régi" és „új" pontokból áll, tehát $|H| = N + \binom{m}{\left[\frac{m}{2}\right]}$. A metszőrendszer egy-egy halmaza egy „új" pontból és a hozzá tartozó $\left[\frac{N}{2}\right]$ „régi" pontból $\left[\frac{N}{2}\right] - 1$ választva — ezt az összes lehetséges módon megtesszük — áll. Itt $m = \left[\frac{N}{2}\right]\binom{N}{\left[\frac{N}{2}\right]}$, vagyis m nagyságában $|H|\log|H|$-hoz közelít.

A dolgozatban megadott független metszőrendszerek mindegyike *Sperner-rendszer* volt. Ez azonban nem szükségszerű, mint az alábbi példa mutatja.

A következő konstrukció mutatja, hogy egy független metszőrendszer nem feltétlenül *Sperner-rendszer*. (A táblázat sorai jelölik ki egy 6-elemű halmaz részhalmazait.)

•	•				
•	•	•			
	•	•	•		
		•	•	•	
•			•		
				•	•
•					•

Az pedig nyilvánvaló, hogy egy *Sperner-rendszer* többnyire nem lesz független metszőrendszer, még akkor sem, ha telített (egy *Sperner-rendszert* akkor nevezünk telítettnek, ha a rendszer nem bővíthető további halmazzal úgy, hogy a *Sperner*-tulajdonsága megmaradjon). Erre mutat példát a második ábra.

•	•	•	•	
•	•		•	•
		•		•

Várhatnánk, hogy ha S és $S^\star$ is *Sperner-rendszer*, akkor S független metszőrendszer. ($A = \{A_1, A_2, \dots, A_n\}$, $A_i \subseteq H$, ahol $H = \{x_1, x_2, \dots, x_n\}$. Ekkor $A^\star = \{B_1, B_2, \dots, B_n\}$, ahol $B_i = \{k : x_i \in A_k\}$. Látható, hogy $(A^\star)^\star = A$.)

•	•		
	•	•	
		•	•
•			•
•	•	•	

2. Definíció. $A_1, A_2, \dots, A_m$ az n-elemű H halmaz részhalmazai *független t-metszőrendszer* alkotnak, ha

(1') H bármely t-elemű része előáll néhány A_i metszeteként;

(2') Az $A_1, A_2, \dots, A_m$ rendszerből bármely A_i-t elhagyva (1') nem teljesül.

A független metszőrendszer fogalma kapcsolódik egy régen ismert fogalomhoz, a *kölcsönösen szeparáló rendszerekéhez.* Legyen H egy n elemű halmaz, $A_1, A_2, \ldots, A_m$ ennek részhalmazai. Ez utóbbi rendszer *szeparáló rendszer*, ha H bármely x, y elemeihez van olyan A_i, hogy vagy $x \in A_i$ és $y \notin A_i$, vagy $x \in A_i$ és $y \notin A_i$. Ezt a fogalmat Rényi [2] vezette be, és Katona [3] megmutatta, hogy $m \geq [\log_2 n]$. Dickson a szeparáló rendszer definícióját módosította. Az $A_1, A_2, \ldots, A_m$ rendszert *kölcsönösen szeparáló rendszernek* nevezi, ha H bármely x, y elemeihez van olyan A_i és A_j, hogy $x \in A_i$ és $y \notin A_i$, továbbá $x \notin A_j$ és $y \in A_j$. Dickson [4], majd Spencer [5] megmutatja, hogy erre a rendszerre is $m \geq \log_2 n$. Katona fogalmazta meg a *teljes szeparáló rendszer* fogalmát. Az $A_1, A_2, \ldots, A_m$ rendszert akkor tekintjük ilyennek, ha H bármely x, y elemeihez van olyan A_i és A_j, hogy $x \in A_i$ és $y \in A_j$, és $A_i \cap A_j = \emptyset$. Erre a rendszerre az előbbiekhez hasonló becslést ad Yao [7] és tőle függetlenül Cai Mao-cheng [8].

Látható, hogy egy teljesen független rendszer egyúttal kölcsönösen szeparáló, és egy kölcsönösen szeparáló pedig szeparáló.

5. Tétel. *A független metszőrendszer (1) feltétele azonos a kölcsönös szeparáló rendszer definíciójával.*

Bizonyítás. Ha egy halmazrendszerre teljesül az (1) feltétel, akkor az kölcsönösen szeparáló. Ugyanis vegyük H két x, y elemét. Ekkor x is y is előáll néhány A_i metszeteként. Ezért van olyan A_i melynek x eleme, de y nem, és van olyan A_j melynek y eleme, de x nem. Ha egy halmazrendszer kölcsönösen szeparáló, akkor kielégíti az (1) feltételt, hiszen már az a kikötés, hogy H bármely x, y eleméhez van olyan A_i, hogy $x \in A_i$ és $y \notin A_j$, garantálja azt, hogy x előállítható néhány A_i metszeteként. ∎

Ezt a tételt figyelembe véve nevezzük az n-elemű H halmaz $A_1, A_2, \ldots,$ A_m részhalmazait *t-szeparálónak*, ha kielégíti az (1') feltételt.

Látható, hogy egy t-szeparáló rendszer egyúttal t'-szeparáló is, ahol $t > t' \geq 1$.

Az is könnyen ellenőrizhető, ha a H halmaznak $A_1, A_2, \ldots, A_m$ 1-szeparáló rendszere, akkor az $\bigcup_{i \in I} A_i$, $i \subset \{1, 2, \ldots, m\}$, $|I| = t$ halmazokból álló rendszer t-szeparáló.

Ha az n elemű H halmazon $A_1, A_2, \ldots, A_m$ t-szeparáló rendszert alkot, akkor $\binom{m}{[\frac{m}{2}]} \geq \binom{n}{t}$. Ez ugyanúgy bizonyítható, mint ahogy a hasonló állítást igazoltuk a 2. Tétel esetén.

Hálával tartozom Katona Gyulának a tőle kapott segítségért és biztatásáért, melyet ezúton szeretnék megköszönni.

Irodalom

[1] E. SPERNER, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27** (1928), 544–548.

[2] A. RÉNYI, On Random Generating Elements of a Finite Boolean Algebra, *Acta Sci. Math.* (Szeged) **22** (1961), 75–81.

[3] G. KATONA, On Separating Systems of a Finite Set, *J. Combinatorial Theory* **1** (1966), 174–194.

[4] T. J. DICKSON, On a problem concerning Separating Systems of a Fintie Set, *J. Combinatorial Theory* **7** (1966), 191–196.

[5] J. SPENCER, Minimal completely Separating Systems, *J. Combiantorial Theory* **8** (1970), 446–447.

[6] G. O. H. KATONA, Combinatorial search problem, A Survey of Combiantorial Theory, North-Holland, Amsterdam, 1973, pp. 285–308.

[7] A. C.-C. YAO, On a Problem of Katona on Minimal Separating Systems, *Discrete Math.* **15** (1976), 193–199.

[8] CAI MAO-CHENG, Solutions to Edmonds' and Katona's problems on families of separating subsets, *Discrete Math.* **47** (1983), 13–21.

[9] ERDŐS—GALLAI, Gráfok előírt fokú pontokkal, *Matematikai Lapok* **11** (1960), 264—274.

[10] RÓKA SÁNDOR, Független metszőrendszerek, *Acta Academiae Paedagogicae Nyíregyháziensis* **12** (1990), 17—20.

Az eredmények egy része a [10] dolgozatban megtalálható, itt a teljesség kedvéért ismételtem meg az ott leírtakat. *Az új eredmények:* a 2. Tétel bizonyításában az utolsó konstrukció, a 3—4. Tétel, valamint a független metszőrendszer és a Sperner-rendszerek közti kapcsolat vizsgálata.

BESSENYEI COLLEGE
DEPT. OF MATH.
NYÍREGYHÁZA, P. O. BOX 166., H-4400,
HUNGARY

Residual Lie nilpotence of the augmentation ideal

BERTALAN KIRÁLY*

Abstract. In this paper we give necessary and sufficient conditions for the residual Lie nilpotence of the augmentation ideal for an arbitrary group ring RG except for the case when the derived group of G is with no generalized torsion elements with respect to the lower central series of G and the torsion subgroup of the additive group of R contains a non-trivial element of infinite height. From this results we get the residual Lie nilpotence of the augmentation ideal of the p-adic integer group rings.

1. Introduction

Let R be a commutative ring with identity, G a group and RG its group ring. The group ring RG may be considered as a Lie algebra, with the usual bracket operation. The study of this Lie algebra was initiated by I. B. S. Passi, D. S. Passman and S. K. Sehgal [5]. Additional results on the Lie structure of RG may be found in [4] and [6].

Let $A(RG)$ denote the *augmentation ideal* of RG, that is the kernel of the homomorphism RG onto R which sends each group element to 1. It is easy to see that as R-module $A(RG)$ is a free module with elements $g-1\,(g \in G)$ as a basis.

There are many problems and results relating to $A(RG)$ ([4], [6]). In particular, it is an interesting problem to characterize the group rings whose augmentation ideal satisfy some conditions. In this paper, we treat the Lie property.

The Lie powers $A^{[\lambda]}(RG)$ of $A(RG)$ are defined inductively: $A^{[1]}(RG) = A(RG)$, $A^{[\lambda+1]}(RG) = [A^{[\lambda]}(RG), A(RG)]{\cdot}RG$, if λ is not a limit ordinal, and for the limit ordinal λ, $A^{[\lambda]}(RG) = \cap_{\nu<\lambda} A^{[\nu]}(RG)$, where $[K, M]$ denotes the R-submodule of RG generated by $[k,m] = km - mk$ ($k \in K \subseteq RG$, $m \in M \subseteq RG$), and for $K \cdot RG$ denotes the right ideal generated by K in RG.

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16432 and 014279.

For the first limit ordinal ω we adopt the notation:

$$A^{[\omega]}(RG) = \bigcap_{i=1}^{\infty} A^{[i]}(RG).$$

The ideal $A(RG)$ of the group ring RG is said to be *residually Lie nilpotent* if $A^{[\omega]}(RG) = 0$.

In this paper we give necessary and sufficient conditions for the residual Lie nilpotence of the augmentation ideal for an arbitrary group ring RG except for the case when the derived group of G is with no generalized torsion elements with respect to the lower central series of G and the torsion subgroup of the additive group of R contains a non-trivial element of infinite height.

Our main results are given in section 3. These results (Theorem A, B and C) are rather technical so they are not stated in the introduction.

2. Notations and some known facts

If H is a normal subgroup of G, then $I(RH)$ (or $I(H)$ for short) denotes the ideal of RG generated by elements of the form $h-1, (h \in H)$. It is well known that $I(RH)$ is the kernel of the natural epimorphism $\overline{\phi}: RG \to RG/H$ induced by the group homomorphism ϕ of G onto G/H. It is clear that $I(RG) = A(RG)$.

Let F be a free group on the free generators x_i $(i \in I)$ and ZF be its integral group ring (Z denotes the ring of rational integers). Then every homomorphism $\phi: F \to G$ induces a ring homomorphism $\overline{\phi}: ZF \to RG$ by letting $\overline{\phi}(\sum n_y y) = \sum n_y \phi(y)$. If $f \in ZF$, we denote by $A_f(RG)$ the two-sided ideal of RG generated by the elements $\overline{\phi}(f)$, $\phi \in \mathrm{Hom}(F, G)$, the set of homomorphism from F to G. In other words $A_f(RG)$ is the ideal generated by the values of f in RG as the elements of G are substituted for the free generators x_i-s.

An ideal J of RG is called a *polynomial ideal* if $J = A_f(RG)$ for some $f \in ZF$. It is easy to see that the augmentation ideal $A(RG)$ is a polynomial ideal. Really, $A(RG)$ is generated as an R-module by elements $g-1$ $(g \in G)$, i.e. by the values of the polynomial $x-1$.

We also use the following

Lemma 2.1. ([4], Proposition 1.4., page 2.) *Let $f \in ZF$. Then f defines a polynomial ideal $A_f(RG)$ in every group ring RG. Further, if $\theta: RG \to KH$*

is a ring homomorphism induced by a group homomorphism $\phi: G \to H$ *and a ring homomorphism* $\psi: R \to K$*, then*

$$\theta(A_f(RG)) \subseteq A_f(KH).$$

(It is assumed here that $\psi(1_R) = 1_K$, where 1_R and 1_K are identities of rings R and K respectively.)

For every natural number n $A^{[n]}(RG)$ is a polynomial ideal (see in particular [4], Corollary 1.9., page 6.) and by Lemma 2.1.

$$\overline{\phi}(A^{[n]}(RG)) \subseteq A^{[n]}(RG/L)$$

for every n. From this inclusion it can be obtained easily that

$$\overline{\phi}(A^{[\omega]}(RG)) \subseteq A^{[\omega]}(RG/L). \tag{1}$$

If $\mathcal{K}$ denotes a class of groups we define the class $\mathbf{R}\mathcal{K}$ of residually-$\mathcal{K}$ groups by letting $G \in \mathbf{R}\mathcal{K}$ if and only if: whenever $1 \neq g \in G$, there exists a normal subgroup H_g of the group G such that $G/H_g \in \mathcal{K}$ and $g \notin H_g$. It is easy to see that $G \in \mathbf{R}\mathcal{K}$ if and only if there exists a family $\{H_i\}_{i \in I}$ of normal subgroups G such that $G/H_i \in \mathcal{K}$ for every $i \in I$ and $\cap_{i \in I} H_i = \langle 1 \rangle$.

A group G is said to be *discriminated* by $\mathcal{K}$ if for every finite set $g_1, g_2, \dots, g_n$ of distinct elements of G, there exists a group $H \in \mathcal{K}$ and a homomorphism $\phi: G \to H$ such that $\phi(g_i) \neq \phi(g_j)$ if $i \neq j, (1 \leq i, j \leq n)$.

Lemma 2.2. *Let a class of groups* $\mathcal{K}$ *be closed with respect to forming subgroups and finite direct products and let* G *be a residually-*$\mathcal{K}$ *group. Then* G *is discriminated by* $\mathcal{K}$*.*

The proof can be obtained easily.

It is easy to show that if G is discriminated by a class of groups $\mathcal{K}$ and if x is a non-zero element of RG, then there exists a group $H \in \mathcal{K}$ and a homomorphism ϕ of RG to RH such that $\phi(x) \neq 0$.

From this fact and from inclusion (1) we have

Lemma 2.3. *If* G *is discriminated by a class of groups* $\mathcal{K}$ *and for each* $H \in \mathcal{K}$ *the equation* $A^{[\omega]}(RH) = 0$ *holds, then* $A^{[\omega]}(RG) = 0$.

We use the following notations for standard group classes:

$\mathcal{D}_0$ — the class of those nilpotent groups whose derived groups are torsion-free.

$\mathcal{D}_p$ — the class of nilpotent groups whose derived groups are p-groups of bounded exponent.

$\mathcal{N}_0$ — the class of torsion-free nilpotent groups.
$\mathcal{N}_p$ — the class of nilpotent p-groups of bounded exponent.
$\mathcal{N}_\Omega = \cup_{p\in\Omega}\mathcal{N}_p$ and
$\mathcal{D}_\Omega = \cup_{p\in\Omega}\mathcal{D}_p$, where Ω is a subset of the set of primes.

The ideal $J_p(R)$ of a ring R is defined by $J_p(R) = \cap_{n=1}^{\infty} p^n R$.

Theorem 2.4. ([4], Theorem 2.13., page 85.) *Let G be a residually $\mathcal{D}_p$-group and $J_p(R) = 0$. Then $A^{[\omega]}(RG) = 0$.*

We shall use the following lemma, which gives some elementary properties of the Lie powers of $A(RG)$.

Lemma 2.5. ([4], Proposition 1.7., page 4.) *For arbitrary natural numbers n and m are true:*

(1) $I(\gamma_n(G)) \subseteq A^{[n]}(RG)$,
(2) $[A^{[n]}(RG), A^{[m]}(RG)] \subseteq A^{[n+m]}(RG)$,
(3) $A^{[n]}(RG)\cdot A^{[m]}(RG) \subseteq A^{[n+m-1]}(RG)$,
where $\gamma_n(G)$ is the nth term of the lower central series of G.

We write $D_{[n]}(RG)$ for the nth *Lie dimension subgroup* $D_{[n]}(RG)$ of G over R. That is

$$D_{[n]}(RG) = \{g \in G | g - 1 \in A^{[n]}(RG)\}.$$

By Lemma 2.5. it follows that for every natural number n the inclusion

$$\gamma_n(G) \subseteq D_{[n]}(RG)$$

holds.

We also use the following theorems

Theorem 2.6. ([1], Theorem 3.2.) *Let a group G contain a non-trivial generalized torsion element. Then $A(RG)$ is residually nilpotent if and only if there exists a non-empty subset Ω of the set of primes such that $\cap_{p\in\Omega} J_p(R) = 0$, G is discriminated by the class $\mathcal{N}_\Omega$ and for every proper subset Λ of the set Ω at least one of the conditions*

(1) $\cap_{p\in\Lambda} J_p(R) = 0$
(2) *G is discriminated by the class of groups $\mathcal{N}_{\Omega\setminus\Lambda}$*
holds.

Let $T(R^+)$ denote the torsion subgroup of the additive group R^+ of a ring R and let $A^\omega(RG) = \cap_{i=1}^{\infty} A^n(RG)$, where $A^n(RG)$ is the nth associative power of $A(RG)$.

Theorem 2.7. ([4], Theorem 2.7., page 87.) *If $G \in \mathbf{R}\mathcal{N}_0$ and R is a ring with identity such that its additive group R^+ is torsion-free, then $A^\omega(RG) = 0$.*

3. Residual Lie nilpotence

It is clear, that $A^{[2]}(RG) = 0$ if and only if G is an Abelian group. Therefore we may assume that the derived group $G' = \gamma_2(G)$ of G is non-trivial.

For a nilpotent group G the following inclusion is true

$$A^{[\omega]}(RG) \subseteq A^\omega(RG')RG \tag{2}$$

(see in particular [4]). For every natural number $i > 1$ we define the normal subgroup

$$L_i = \{g \in G' | g^k \in \gamma_i(G) \text{ for a suitable } k \geq 1\}$$

of G. It is easy to see that $\gamma_i(G) \subseteq L_i$ and also that $G/L_i \in \mathcal{D}_0$ for every $i > 1$.

An element g of a group G is called a *generalized torsion element* with respect to the lower central series of G if for every n the order of the elements $g\gamma_n(G)$ of the factor group $G/\gamma_n(G)$ is finite.

We recall that if the derived group G' of G contains no generalized torsion elements with respect to the lower central series of G, then G' has no generalized torsion elements with respect to the lower central series of G'.

Theorem A. *Let R be a commutative ring with identity, $T(R^+) = 0$ and let G' be with no generalized torsion elements with respect to the lower central series of G. Then $A^{[\omega]}(RG) = 0$ if and only if G is a residually-$\mathcal{D}_0$ group.*

Proof. Since G' is with no generalized torsion elements with respect to the lower central series of G, then $\cap_{i=2}^{\infty} L_i = \langle 1 \rangle$ and so, $G \in \mathbf{R}\mathcal{D}_0$.

Conversely. Let $G \in \mathbf{R}\mathcal{D}_0$ and $T(R^+) = 0$. Since class $\mathcal{D}_0$ is closed with respect to forming subgroups and finite direct products, by Lemmas 2.2. and 2.3. it is enough to show that $A^{[\omega]}(RG) = 0$ for all $G \in \mathcal{D}_0$. So let $G \in \mathcal{D}_0$. Then by (2)

$$A^{[\omega]}(RG) \subseteq A^\omega(RG')RG.$$

Because G' is a torsion-free nilpotent group, by Theorem 2.7. $A^\omega(RG') = 0$, and so, $A^{[\omega]}(RG) = 0$. The proof is completed.

Let p be a prime and n a natural number. Then G^{p^n} is the subgroup of G generated by all elements of the form g^{p^n}, $g \in G$.

For a prime p and a natural number k the normal subgroup $G_{[p,k]}$ of G is defined by

$$G_{[p,k]} = \bigcap_{n=1}^{\infty} (G')^{p^n} \gamma_k(G).$$

We have the following sequence

$$G = G_{[p,1]} \supseteq G_{[p,2]} \supseteq \dots \supseteq G_{[p]}$$

of normal subgroups $G_{[p,k]}$ of G, where

$$G_{[p]} = \bigcap_{k=1}^{\infty} G_{[p,k]}.$$

It is clear, that $G/(G')^{p^n}\gamma_k(G)$ are in $\mathcal{D}_p$, and $G/G_{[p,k]}$ and $G/G_{[p]}$ are residually-$\mathcal{D}_p$ groups for every k and n.

Lemma 3.1. *If $n \geq ks$ and $h \in (G')^{p^n}\gamma_k(G)$, then*

$$h - 1 \equiv p^s X(k,h) \pmod{A^{[k]}(RG)}$$

for a suitable $X(k,h) \in A^{[2]}(RG)$.

Proof. Let $h \in (G')^{p^n}\gamma_k(G)$. We can write element h as

$$h = h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} y_k$$

where $h_i \in G', y_k \in \gamma_k(G)$. Using the identity

$$ab - 1 = (a-1)(b-1) + (a-1) + (b-1) \tag{3}$$

to $h - 1$ we have that

$$h - 1 = (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1)(y_k - 1) + (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) + (y_k - 1).$$

By Lemma 2.5. $I(\gamma_k(G)) \subseteq A^{[k]}(RG)$ and hence $y_k - 1 \in A^{[k]}(RG)$. Therefore

$$h - 1 \equiv (h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1) \pmod{A^{[k]}(RG)}.$$

Applying identity (3) repeatedly to $(h_1^{p^n} h_2^{p^n} \cdots h_m^{p^n} - 1)$ from the previous congruence it follows that

$$h - 1 \equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv \sum_{i=1}^{m} \sum_{j=1}^{p^n} \binom{p^n}{j} (h_i - 1)^j b_i \pmod{A^{[k]}(RG)},$$

where $b_i \in RG$. Because $h_i \in G' = \gamma_2(G)$, from Lemma 2.5. (cases 1 and 3) we obtain that $(h_i - 1)^j \in A^{[j+1]}(RG)$ for every i and j. If $n \geq sk$, then p^s divides $\binom{p^n}{j}$ for every $j = 1, 2, \ldots, k-1$. Therefore

$$\begin{aligned} h - 1 &\equiv \sum_{i=1}^{m} (h_i^{p^n} - 1) b_i \equiv p^s \sum_{i=1}^{m} \sum_{j=1}^{k-1} d_j (h_i - 1)^j b_i \\ &\equiv p^s X(k, h) \pmod{A^{[k]}(RG)}, \end{aligned}$$

where $X(k,h) = \sum_{i=1}^{m} \sum_{j=k}^{p^n} d_j (h_i - 1)^j b_i$, $b_i \in RG$, $p^s d_j = \binom{p^n}{j}$. The Lemma is proved.

It is easy to show that if $g \in G'$ and $g^{p^n} \in D_{[k]}(RG)$ then

$$p^m (g - 1) \in A^{[k]}(RG) \tag{4}$$

for a large enough m.

Lemma 3.2. ([1], Lemma 3.6.) *Let $\mathcal{K}$ be a class of groups and $\{G_\alpha\}_{\alpha \in I}$ a family of normal subgroups of G such that for all α ($\alpha \in I$) the conditions*
(1) $G/G_\alpha \in \mathcal{K}$
(2) G_α *is torsion-free*
hold. If G is not discriminated by $\mathcal{K}$ then there exists a finite set of distinct elements $g_1, g_2, \ldots, g_s$ from G such that the non-zero element $y = (g_1 - 1)(g_2 - 1) \cdots (g_s - 1)$ lies in the ideal $\cap_{\alpha \in I} I(G_\alpha)$.

The torsion subgroup $T(R^+)$ of the additive group R^+ of a ring R is the direct sum of its p-primary components $S_p(R^+)$. Let Π be the set of those primes for which the p-primary components $S_p(R^+)$ of $T(R^+)$ are non-zero.

An element a of an additive Abelian group A is called an *element of infinite p-height* for a prime p, if the equation $p^n x = a$ has a solution in A for every natural number n.

Proposition 3.3. ([1], Theorem 3.3.) *Let $T(R^+) \neq 0$, and suppose that for some $p \in \Pi$ group $T(R^+)$ has no element of infinite p-height. Further* *let G be a group with no generalized torsion elements. Then $A^{\omega}(RG) = 0$ if and only if G is a residually-$\mathcal{N}_p$ group for all $p \in \Pi$.*

Theorem B. *Let $T(R^+) \neq 0$. If G' is with no generalized torsion elements with respect to the lower central series of G and $T(R^+)$ is with no non-trivial elements of infinite p-height then $A^{[\omega]}(RG) = 0$ if and only if G is a residually-$\mathcal{D}_p$ group for all $p \in \Pi$.*

Proof. Let p an arbitrary prime of Π, $A^{[\omega]}(RG) = 0$, and let p^s $(s \geq 1)$ be the order of element $a \in T(R^+)$. Since the equation

$$G_{[p]} = \bigcap_{k=1}^{\infty} G[p,k] = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} (G')^{p^n} \gamma_k(G)) = \langle 1 \rangle$$

implies that $G \in \mathbf{R}\mathcal{D}_p$, it is enough to show, that $G_{[p]} = \langle 1 \rangle$.

Suppose that $g \in G_{[p]}$. Then $g \in (G')^{p^n} \gamma_k(G)$ for every n and k and by Lemma 3.1. we have that

$$g - 1 \equiv p^s X(k,g) \pmod{A^{[k]}(RG)}$$

for every k. From $p^s a = 0$ it follows that $a(g-1) \in A^{[k]}(RG)$ for every k. Hence $a(g-1) \in A^{[\omega]}(RG)$ and $a(g-1) = 0$. This implies that $g = 1$. Consequently $G_{[p]} = \langle 1 \rangle$. This means that G is a residually-$\mathcal{D}_p$ group for all $p \in \Pi$.

Conversely. Let $G \in \mathbf{R}\mathcal{D}_p$ for $p \in \Pi$ and let $1 \neq g$ be an arbitrary element of G'. Then there exists a normal subgroup H of G such that $G/H \in \mathcal{D}_p$ and $g \notin H$. Since $G/H \in \mathcal{D}_p$ then $(G/H)' \in \mathcal{N}_p$. By the isomorphism $G'H/H \cong G'/H \cap G'$ we have that $\overline{g} = g(H \cap G') \neq \overline{1}$. This means that if $G \in \mathbf{R}\mathcal{D}_p$ then $G' \in \mathbf{R}\mathcal{N}_p$. Using Proposition 3.3. we have that $A^{\omega}(RG') = 0$ and from (2) it follows that $A^{[\omega]}(RG) = 0$.

Lemma 3.4. *Let*

$$y \in \bigcap_{p \in \Gamma} \bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} I((G')^{p^n} \gamma_j(G)).$$

Then for a prime $p \in \Gamma$ and arbitrary natural numbers k and s

$$y \equiv p^s Y(p,k,s,y) \pmod{A^{[k]}(RG)},$$

where $Y(p,k,s,y) \in RG$ and Γ is a subset of the set of prime numbers.

Proof. Let $p \in \Gamma$. For every natural n we can express y as

$$y = \sum_{i=1}^{l} \alpha_i z_i (h_i - 1),$$

where $h_i \in (G')^{p^n}\gamma_k(G)$, $\alpha_i \in R$ and every z_i is from a set of coset representatives of $(G')^{p^n}\gamma_k(G)$ in G. For a large enough n by Lemma 3.1.

$$h_i - 1 \equiv p^s X(k, h_i) \pmod{A^{[k]}(RG)}$$

for every i ($i = 1, 2, \ldots, l$) and the proof follow.

If $g \in G'$ is a generalized torsion element of a group G then Ω_g denotes the set of the prime divisors of the order of the elements $g\gamma_k(G) \in G/\gamma_k(G)$ for every $k = 2, 3, \ldots$.

Lemma 3.5. *Let $g \in G'$ be a generalized torsion element of a group G, Λ an arbitrary subset of Ω_g, $a \in \cap_{p\in\Lambda} J_p(R)$ and let*

$$x \in \bigcap_{p\in\Omega_g\setminus\Lambda} \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\infty} I((G')^{p^i}\gamma_k(G)).$$

Then one of the following statements

(1) *if Λ is a proper subset of Ω_g, then $a(g-1)x \in A^{[\omega]}(RG)$*
(2) *if $\Lambda = \Omega_g$, then $a(g-1) \in A^{[\omega]}(RG)$*
(3) *if $\Lambda = \emptyset$, then $(g-1)x \in A^{[\omega]}(RG)$*

holds.

Proof. It is enough to show that for an arbitrary natural number k the elements $a(g-1), (g-1)x, a(g-1)x$ are in the ideal $A^{[k]}(RG)$.

If $g \in \gamma_k(G)$ then by Lemma 2.5. $(g-1) \in A^{[k]}(RG)$, and the statements follow. Now let $g \notin \gamma_k(G)$ and let

$$n_k = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$$

be the prime factorization of the order of the elements $g\gamma_k(G)$ of the nilpotent group $G/\gamma_k(G)$. It is clear that $p_i \in \Omega_g$ for every $i = 1, 2, \ldots, s$. Let Λ a subset of Ω_g. With loss of generality we may assume that $p_1, p_2, \ldots, p_l \in \Lambda$ and $p_i \notin \Lambda$ for $i > l$.

Let $g = g_1 g_2 \cdots g_s \gamma_k(G)$ be the decomposition of the element $g\gamma_k(G)$ of the nilpotent group $G/\gamma_k(G)$ in the product of p_i-elements $g_i\gamma_k(G)$ ($i = 1, 2, \ldots, s$). Then

$$g = g_1 g_2 \cdots g_s y_k, \quad g_i \in G', i = 1, 2, \ldots, s$$

for a suitable $y_k \in \gamma_k(G)$. Then there exists m_i ($i = 1, 2, \ldots, s$) such that

$$g_i^{p_i^{m_i}} \in \gamma_k(G).$$

Using identity (3) repeatedly to $(g-1)$ we conclude that

$$g - 1 \equiv v + w + (y_k - 1) \equiv v + w \pmod{A^{[k]}(RG)},$$

where $v = \sum_{i=1}^{l} (g_i - 1)x_i$, $w = \sum_{i=l+1}^{s} (g_i - 1)x_i$ and $x_i \in RG$. In the case when $\Lambda \cap \{p_1, p_2, \ldots, p_s\} = \emptyset$ we assume that $v = 0$, and if $\Lambda \cap \{p_1, p_2, \ldots, p_s\} = \{p_1, p_2, \ldots, p_s\}$ we put $w = 0$. Because

$$g_i^{p_i^{m_i}} \in \gamma_k(G) \subseteq D_{[k]}(G)$$

and $g_i \in G'$ for every $i = 1, 2, \ldots, s$, we conclude from (4) that there exists a natural number r_i $(i = 1, 2, \ldots, s)$ such that

$$p_i^{r_i}(g_i - 1) \in A^{[k]}(RG). \tag{5}$$

Also, since

$$a \in \bigcap_{p \in \Lambda} J_p(R) \subseteq \bigcap_{i=1}^{l} J_p(R)$$

we can express a as $a = p_i^{r_i} a_i$ $(a_i \in R)$ for each $i \leq l$. Then by (5)

$$av \equiv \sum_{i=1}^{l} a_i p_i^{r_i}(g_i - 1)x_i \equiv 0 \pmod{A^{[k]}(RG)}.$$

Therefore

$$a(g-1) \equiv av + aw \equiv aw \pmod{A^{[k]}(RG)}. \tag{6}$$

If $\Lambda = \Omega_g$ then $w = 0$ and case 2) is proved.

By Lemma 3.4.

$$x \equiv p_i^{r_i} Y(p_i, k, r_i, x) \pmod{A^{[k]}(RG)},$$

and so,

$$wx \equiv \sum_{i=l+1}^{s} p_i^{r_i}(g_i - 1)x_i Y(p_i, k, r_i, x) \pmod{A^{[k]}(RG)}.$$

Hence by (5)

$$wx \equiv 0 \pmod{A^{[k]}(RG)}. \tag{7}$$

If $\Lambda = \emptyset$, then $v = 0$, and so,

$$(g-1)x \equiv vx + wx \equiv wx \equiv 0 \pmod{A^{[k]}(RG)}$$

and case 3) is proved.

Also, since

$$a(g-1)x \equiv avx + awx \pmod{A^{[k]}(RG)}$$

from congruences (6) and (7) the proof (of case 1)) follows.

We recall that for a prime p $\mathcal{N}_p$ denotes the class of nilpotent groups whose derived groups are p-groups of bounded exponent, and if Ω a subset of the set of primes, then $\mathcal{N}_\Omega = \cup_{p\in\Omega}\mathcal{N}_p$ and $\mathcal{D}_\Omega = \cup_{p\in\Omega}\mathcal{D}_p$.

Let a group G be discriminated by the class of groups $\mathcal{D}_\Gamma$ ($\Gamma \neq \emptyset$) and let $g_1, g_2, \ldots, g_n$ be a finite set of distinct elements of G'. Then there exists a normal subgroup H of G such that $g_iH \neq g_jH$ if $i \neq j$ and $G/H \in \mathcal{D}_\Gamma$. Therefore $(G/H)' \in \mathcal{N}_p$ for any prime $p \in \Gamma$. By the isomorphism $G'H/H \cong G'/H \cap G'$ we have $g_iH(\cap G') \neq g_j(H \cap G')$ if $i \neq j$ $(i,j = 1,2,\ldots,n)$. This means, that if G is discriminated by the class $\mathcal{D}_\Gamma$, then G' is discriminated by the class of groups $\mathcal{N}_\Gamma$.

Lemma 3.6. *Let Ω be a non-empty subset of the set of primes such that*
$\cap_{p\in\Omega} J_p(R) = 0$ and a group G is discriminated by the class of groups $\mathcal{D}_\Omega$. If for every proper subset Λ of the set Ω at least one of the conditions

(1) $\cap_{p\in\Lambda} J_p(R) = 0$

(2) *G is discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$*

holds, then $A^{[\omega]}(RG) = 0$.

Proof. Let

$$x = \sum_{i=1}^{n} \alpha_i g_i \in A^{[\omega]}(RG).$$

By Lemma 2.3. it is enough to show that $A^{[\omega]}(RG) = 0$ for all groups $G \in \mathcal{D}_\Omega$. So let $G \in \mathcal{D}_\Omega$. Then G is a nilpotent group and by (2)

$$A^{[\omega]}(RG) \subseteq A^{\omega}(RG')RG.$$

Clearly, $G' \in \mathcal{N}_\Omega$. If G is discriminated by the class of groups $\mathcal{D}_\Gamma$, where Γ is an arbitrary non-empty subset of Ω, then G' is discriminated by the clas $\mathcal{N}_\Gamma$, which was showed above. Then G' satisfies Theorem 2.6. and so, $A^{\omega}(RG') = 0$. Consequently $A^{[\omega]}(RG) = 0$.

Theorem C. *Let the derived group G' contain a generalized torsion element of G with respect to the lower central series of G. Then $A(RG)$ is residually Lie nilpotent if and only if there exists a non-empty subset Ω of the set of primes such that $\cap_{p\in\Omega} J_p(R) = 0$, G is discriminated by the class of groups $\mathcal{D}_\Omega$ and every proper subset Λ of the set Ω at least one of the conditions*

(1) $\cap_{p\in\Lambda} J_p(R) = 0$

(2) G is discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$

holds.

Proof. Let $A^{[\omega]}(RG) = 0$. Let us first consider the case when G' contains a non-trival torsion element. Then there exists a p-element g in G' with $p \in \Omega$. Then by (4) for every k there exists a natural number m such that

$$p^m(g-1) \in A^{[k]}(RG). \tag{8}$$

If $a \in J_p(R)$, then for each m we can write element a as $a = p^m a_m$ ($a_m \in R$). Therefore $a(g-1) \in A^{[k]}(RG)$ for every k, that is $a(g-1) \in A^{[\omega]}(RG)$. Hence $a(g-1) = 0$ and so, $a = 0$. Consequently $J_p(R) = 0$.

Now we show, that G is discriminated by $\mathcal{D}_{\{p\}}$. Let

$$h \in \bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty}(G')^{p^i}\gamma_k(G).$$

Then

$$h-1 \in \bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty} I((G')^{p^i}\gamma_k(G))$$

and by Lemma 3.4. for every k and m

$$h-1 \equiv p^m Y(p,k,m,h-1) \pmod{A^{[k]}(RG)}. \tag{9}$$

By (8) and (9) we have that

$$(g-1)(h-1) \equiv p^m(g-1)(h-1)Y(p,m,k,h-1) \pmod{A^{[k]}(RG)}$$

for every k. This implies that

$$(g-1)(h-1) \in A^{[\omega]}(RG) \text{ and so, } (g-1)(h-1) = 0.$$

From this equation we have that the characteristic of R is p $(=2)$ and from (9) it follows that $h-1 \in A^{[\omega]}(RG)$. Therefore $h = 1$ and so

$$\bigcap_{k=1}^{\infty}\bigcap_{i=1}^{\infty}(G')^{p^i}\gamma_k(G) = \langle 1\rangle.$$

For every k and i $G/(G')^{p^i}\gamma_k(G) \in \mathcal{D}_{\{p\}}$. The class $\mathcal{D}_{\{p\}}$ is closed with respect to forming subgroups and finite direct products, and by Lemma 2.2. G is discriminated by $\mathcal{D}_{\{p\}}$. Consequently we can choose the set $\Omega = \{p\}$.

Let us consider the case when G' is a torsion-free group and $1 \neq g \in G'$ is a generalized torsion element of G. We put $\Omega = \Omega_g$. From Lemma 3.5. (case 2) it follows that

$$\bigcap_{p\in\Omega} J_p(R) = 0.$$

From Lemma 3.2. (here we put $\{G_\alpha\}_{\alpha\in I} = \{(G')^{p^n}\gamma_k(G), k, n = 1, 2, \ldots\}_{p\in\Omega}$) and Lemma 3.5. (case 3) we have that G is discriminated by the class $\mathcal{D}_\Omega$.

Let Λ be an arbitrary subset of Ω and let $\cap_{p\in\Lambda} J_p(R) \neq 0$. If G is not discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$, then by Lemma 3.2. there exists a set of elements $g_1, g_2, \ldots, g_n$ ($g_i \in G$) of infinite orders such that

$$0 \neq (g_1 - 1)(g_2 - 1)\cdots(g_n - 1) \in \bigcap_{p\in\Omega\setminus\Lambda} \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\infty} I((G')^{p^i}\gamma_k(G)).$$

By Lemma 3.5. (case 1) for every element $a \in \cap_{p\in\Lambda} J_p(R)$

$$a(g - 1)(g_1 - 1)(g_2 - 1)\cdots(g_n - 1) \in A^{[\omega]}(RG).$$

Because $A^{[\omega]}(RG) = 0$ we have that

$$a(g - 1)(g_1 - 1)(g_2 - 1)\cdots(g_n - 1) = 0.$$

Since element g_i ($i = 1, 2, \ldots, n$) has infinite order and so has zero left (and right) annihilator in RG, then for g_n we have

$$a(g - 1)(g_1 - 1)(g_2 - 1)\cdots(g_{n-1} - 1) = 0.$$

Continuing this procedure for $i = n-1, n-2, \ldots, 1$ on the last step we get that

$$a(g - 1) = 0.$$

Since the element g has infinite order, its left annihilator is zero in RG, which implies $a = 0$. Consequently, if G is not discriminated by the class of groups $\mathcal{D}_{\Omega\setminus\Lambda}$, then $\cap_{p\in\Lambda} J_p(R) = 0$.

The sufficiency part is proved in Lemma 3.6.

Corollary. *Let $R = \widehat{Z}_p$, the ring of p-adic integers. Then $A^{[\omega]}(\widehat{Z}_p G) = 0$ if and only if either*

(1) *G is discriminated by the class* $\mathcal{D}_0$ *or*
(2) *G is discriminated by the class* $\mathcal{D}_p$.

Proof. If G' is with no generalized torsion elements (with respect to the lower central series of G), then by Theorem A $A^{[\omega]}(\widehat{Z}_pG) = 0$ if and only if G is discriminated by the class $\mathcal{D}_0$.

Let us consider the case when G' contains a generalized torsion element.

Let $A^{[\omega]}(\widehat{Z}_pG) = 0$. By Theorem C there exists a non-empty subset Ω of the set of primes, such that $\cap_{q\in\Omega}J_q(\widehat{Z}_p) = 0$. It is known that $J_p(\widehat{Z}_p) = 0$ and for a prime $q \neq p$, $J_q(\widehat{Z}_p) = \widehat{Z}_p$. Therefore $p \in \Omega$. If $\Omega = \{p\}$, then by the last theorem G is discriminated by $\mathcal{D}_p$. If Ω contains a prime $q \neq p$, then we choose $\Lambda \subseteq \Omega$ such that $\Omega \setminus \Lambda = \{p\}$. Then $\cap_{q\in\Lambda}J_q(\widehat{Z}_p) \neq 0$ and by Theorem C G is discriminated by the class $\mathcal{D}_p$.

Conversely. If G is discriminated by the class $\mathcal{D}_p$, we put $\Omega = \{p\}$, and the proof follows from Theorem C.

From Theorem A and C we also get the results of I. Musson and A. Weiss ([2], Theorem A).

References

[1] KIRÁLY B., The residual nilpotency of the augmentation ideal, *Publ. Math. Debrecen.*, **45** (1994), 133–144.

[2] MUSSON I., WEISS A., Integral group rings with residually nilpotent unit groups, *Arch. Math.*, **38** (1982), 514–530.

[3] PARMENTER, M. M., PASSI, I. B. S. and SEHGAL, S. K., Polynomial ideals in group rings, *Canad. J. Math.*, **25** (1973), 1174–1182.

[4] PASSI, I. B. S., Group ring and their augmentation ideals, *Lecture notes in Math.*, 715, Springer-Verlag, Berlin–Heidelberg–New York, 1979.

[5] PASSI, I. B. S., PASSMAN D. S. and *Sehgal S. K.*, Lie solvable group rings, *Canad. J. Math.* **25** (1973), 748–757.

[6] SEHGAL S. K., Topics in group rings, Marcel–Dekker Inc., New York–Basel, 1978.

BERTALAN KIRÁLY
ESZTERHÁZY KÁROLY TEACHERS' TRAINING COLLEGE
DEPARTMENT OF MATHEMATICS
LEÁNYKA U. 4.
3301 EGER, PF. 43.
HUNGARY
E-mail: kiraly@gemini.ektf.hu

Unitary subgroup of the Sylow 2-subgroup of the group of normalized units in an infinite commutatuve group ring

ATTILA SZAKÁCS*

Abstract. Let G be an abelian group, K a commutative ring with unity of prime characteristic p and let $V(KG)$ denote the group of normalized units of the group ring KG. An element $u=\sum_{g\in G}\alpha_g g\in V(KG)$ is called unitary if u^{-1} coincides with the element $u^*=\sum_{g\in G}\alpha_g g^{-1}$. The set of all unitary elements of the group $V(KG)$ forms a subgroup $V_*(KG)$.

S. P. Novikov had raised the problem of determining the invariants of the group $V_*(KG)$ when G has a p-power order and K is a finite field of characteristic p. This problem was solved by A. Bovdi and the author. We gave the Ulm–Kaplansky invariants of the unitary subgroup of the Sylow p-subgroup of $V(KG)$ whenever G is an arbitrary abelian group and K is a commutative ring with unity of odd prime characteristic p without nilpotent elements. Here we continue this works describing the unitary subgroup of the Sylow 2-subgroup of the group $V(KG)$ in case when G is an arbitrary abelian group and K is a commutative ring with unity of characteristic 2 without zero divisors.

Let G be an abelian group and K a commutative ring with unity of prime characteristic p. Let, further on, $V(KG)$ denote the group of normalized units (i.e. of augmentation 1) of the group ring KG and $V_p(KG)$ the Sylow p-subgroup of the group $V(KG)$. We say that for $x=\sum_{g\in G}\alpha_g g\in KG$ the element $x^*=\sum_{g\in G}\alpha_g g^{-1}$ is conjugate to x. Clearly, the map $x\to x^*$ is an anti-isomorphism (involution) of the ring KG. An element $u\in V(KG)$ is called unitary if $u^{-1}=u^*$. The set of all unitary elements of the group $V(KG)$ obviously forms a subgroup, which we therefore call the unitary subgroup of $V(KG)$, and we denote it by $V_*(KG)$.

Let G^p denote the subgroup $\{g^p : g\in G\}$ and λ an arbitrary ordinal. The subgroup G^{p^λ} of the group G is defined by transfinite induction in following way: $G^{p^0}=G$, for a non-limited ordinals $G^{p^{\lambda+1}}=\left(G^{p^\lambda}\right)^p$, and if λ is a limited ordinal, then $G^{p^\lambda}=\cap_{\nu<\lambda}G^{p^\nu}$.

---

* Research (partially) supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 16432 and 014279.

The subring K^{p^λ} of the ring K is defined similarly. The ring K is called p-divisible if $K^p = K$.

Let $G[p]$ denote the subgroup $\{g \in G : g^p = 1\}$ of G. Then the factorgroup $G^\lambda[p]/G^{\lambda+1}[p]$ can be considered as a vector space over $GF(p)$ the field of p elements and the cardinality of a basis of this vector space is called the λ-th Ulm–Kaplansky invariant $f_\lambda(G)$ of the group G concerning to p.

S. P. Novikov had raised the problem of determining the invariants of the group $V_\star(KG)$ when G has a p-power order and K is a finite field of characteristic p. This was solved by A. Bovdi and the author in [1]. In [2] we gave the Ulm–Kaplansky invariants of the unitary subgroup $W_p(KG)$ of the group $V_p(KG)$ whenever G is an arbitrary abelian group and K is a commutative ring of odd prime characteristic p without nilpotent elements. Here we continue this works describing the unitary subgroup $W_2(KG)$ of the Sylow 2-subgroup $V_2(KG)$ of the group $V(KG)$ in case when G is an arbitrary abelian group and K is a commutative ring with unity of characteristic 2 without zero divisors.

Note that for the odd primes p the problem of determining the Ulm–Kaplansky invariants of the group $W_p(KG)$ is based, in fact, in the following statement

$$W_p(KG) = \{x^{-1}x^\star : x \in V_p(KG)\}$$

(see [2]). But in case $p = 2$ this statement is not true and in the characterization of the group $W_2(KG)$ we must keep in mind the following lemma.

Lemma 1. *Let G be an abelian group of exponent 2^{n+1} $(n > 0)$ and K a commutative ring with unity of characteristic 2 without zero divisors. Then $(V_\star(KG))^{2^n} = G^{2^n}$.*

Proof. At first we shall prove the lemma for a finite group G. We shall use induction on the exponent of G.

Let $n = 1$, i.e. G is a group of exponent 4. We shall prove by induction on the order of G that $(V_\star(KG))^2 = G^2$.

Let $G = \langle a : a^4 = 1\rangle$. Then the element

$$x = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 \in V(KG)$$

is unitary if and only if

$$xx^\star = 1 + (\alpha_0 + \alpha_2)(\alpha_1 + \alpha_3)(a + a^3) = 1.$$

Hence $\alpha_0 = \alpha_2$ or $\alpha_1 = \alpha_3$. If $\alpha_1 = \alpha_3$ then, according to the condition $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$, the unitary element x has the form $x = 1 + \alpha_2(1 + a^2) + \alpha_1(a + a^3)$ and $x^2 = 1$. If $\alpha_0 = \alpha_2$ then $x = \alpha_0(1 + a^2) + \alpha_1 a + (1 + \alpha_1)a^3$.

Therefore $x^2 = a^2$ and the statement is proved for the cyclic group G of order 4.

Let G be a non-cyclic group of exponent 4 and order greater than 4. Then G can be presented as a direct product of a suitable group H and the cyclic group $\langle b \rangle$ which order divides 4.

Suppose that b is an element of second order. Then every $x \in V(KG)$ can be written in the form $x = x_0 + x_1 b$, where $x_0, x_1 \in KH$. If x is a unitary element then

$$xx^\star = x_0 x_0{}^\star + x_1 x_1{}^\star + (x_0{}^\star x_1 + x_0 x_1{}^\star)b = 1$$

and the equations $x_0 x_0{}^\star + x_1 x_1{}^\star = 1$, $x_0{}^\star x_1 + x_0 x_1{}^\star = 0$ hold. Hence $(x_0 + x_1)(x_0{}^\star + x_1{}^\star) = 1$ and $y = x_0 + x_1 \in V_\star(KH)$. By the induction hypothesis, $y^2 = h^2$ for some $h \in H$. Obviously $x^2 = h^2$.

Let b be an element of order 4. The element

$$x = x_0 + x_2 b^2 + (x_1 + x_3 b^2)b \quad (x_i \in KH, i = 0,1,2,3)$$

of the group $V(KG)$ is unitary if and only if

$$(1) \qquad \begin{cases} (x_0 + x_2 b^2)(x_0{}^\star + x_2{}^\star b^2) + (x_1 + x_3 b^2)(x_1{}^\star + x_3{}^\star b^2) = 1, \\ (x_0 + x_2 b^2)(x_1{}^\star + x_3{}^\star b^2) = 0. \end{cases}$$

Let $\chi(x_0 + x_2 b^2) = \gamma$ denote the sum of coefficients of the element $x_0 + x_2 b^2$. Then $\chi(x_1 + x_3 b^2) = 1 + \gamma$ and from the second equation of (1) we have that $\gamma(1 + \gamma) = 0$. Since K without zero divisors, it follows that $\gamma = 0$ or $\gamma = 1$ i.e. one of the elements $x_0 + x_2 b^2$ or $x_1 + x_3 b^2$ is invertible. Hence for the unitary element x either $x_0 = x_2 b^2$ or $x_1 = x_3 b^2$. If $x_0 = x_2 b^2$ then, by (1), the element $y = x_1 + x_3 b^2$ is unitary in the group ring of the group $\widetilde{H} = H \times \langle b^2 \rangle$. Then, by the induction hypothesis, $y^2 = h^2$ for some $h \in H$ and obviously $x^2 = y^2 b^2 = h^2 b^2 \in G^2$. If $x_1 = x_3 b^2$ then $y = x_0 + x_2 b^2 \in V_\star(K\widetilde{H})$ and $x^2 = y^2 \in G^2$. So $(V_\star(KG))^2 = G^2$ for a finite group G of exponent 4.

Suppose that G is a group of exponent 2^{n+1} $(n > 1)$ and the statement is proved for the groups of exponent less than 2^{n+1}. It is easy to see that $(V_\star(KG))^2 \subseteq V_\star(KG^2)$. From this, useing the induction hypothesis $(V_\star(KG^2))^{2^{n-1}} = (G^2)^{2^{n-1}}$, we have that $V_\star(KG)^{2^n} \subseteq G^{2^n}$. The reverse inclusion is obvious and the lemma is proved for a finite group G.

Let G be an infinite abelian group of exponent 2^{n+1} $(n > 0)$ and $x \in V_\star(KG)$. Then the subgroup $H = \langle \text{supp } x \rangle$ of the support of x is finite and, by the statement proved in above, $x^{2^n} \in H^{2^n}$. This completes the proof of the lemma.

Theorem. *Let λ be an arbitrary ordinal, K a commutative ring with unity of characteristic 2 without zero divisors, P the maximal divisible subgroup of the Sylow 2-subgroup S of an abelian group G, $G_\lambda = G^{2^\lambda}$, $S_\lambda = S^{2^\lambda}$, $K_\lambda = K^{2^\lambda}$. Let, further on, $V_2 = V_2(KG)$ denote the Sylow 2-subgroup of the group $V = V(KG)$ of normalized units in the group ring KG and $W = W(KG)$ the unitary subgroup of $V_2(KG)$. In case $P \neq 1$ we assume that the ring K is 2-divisible.*

If $G_\lambda \neq G_{\lambda+1}$, $S_\lambda \neq 1$ and at least one of the ordinals $|K_\lambda|$ and $|G_\lambda|$ is infinite, then the λ-th Ulm–Kaplansky invariant $f_\lambda(W)$ of the group W concerning to 2 is characterized in the following way:

$$f_\lambda(W) = \begin{cases} \max\{|G|, |K|\}, & \text{if } \lambda = 0, \\ f_\lambda(V_2) = \max\{|G_\lambda|, |K_\lambda|\}, & \text{if } \lambda > 0 \text{ and } G_{\lambda+1} \neq 1, \\ f_\lambda(G), & \text{if } \lambda > 0 \text{ and } G_{\lambda+1} = 1. \end{cases}$$

Proof. It is easy to prove the following statements (see [3]):

1) $|K^2| = |K|$;

2) if n a nonnegative integer and $J(G^{p^n}[p])$ the ideal of the ring $(KG)^{2^n}$ generated by the elements of the form $g-1$ $\left(g \in G^{2^n}[2]\right)$, then $V^{2^n}(KG)[2] = V(K_nG_n)[2] = 1 + J(G^{2^n}[2])$.

Note if $G_\lambda = G_{\lambda+1}$ or $S_\lambda = 1$ then, according to [3], $f_\lambda(V_2) = 0$ and hence $f_\lambda(W) = 0$.

At first we shall prove the theorem for a finite ordinal $\lambda = n$. Suppose that n is a nonnegative integer, the Sylow 2-subgroup S_n of the group G_n is not singular, $G_n \neq G_{n+1}$ and at least one of the ordinals $|K_n|$ and $|G_n|$ is infinite. Since

$$W^{2^n}[2] \subseteq V^{2^n} = V(K_nG_n),$$

it follows that

$$f_n(W) \leq |V^{2^n}| \leq \max\{|K_n|, |G_n|\} = \beta.$$

In the proof of the equation $f_n(W) = \beta$ we shall consider the following cases:

A) $|K_n| \geq |G_n|$,

B) $|G_n| > |K_n|$ and $S_n \neq S_{n+1}$,

C) $|G_n| > |K_n|$ and $S_n = S_{n+1}$,

and in each of this cases we shall construct a set $M \subseteq W^{2^n}(KG)[2]$ of cardinality $\beta = \max\{|K_n|, |G_n|\}$ (if, keeping in mind Lemma 1, it is possible) which elements belong to the different cosets of the group $V^{2^n}(KG)[2]$ by the subgroup $V^{2^{n+1}}(KG)[2]$. This will be sufficient for the proof of the lemma, because the elements of the such constructed set M can be considered as the representatives of the cosets of the group $W^{2^n}(KG)[2]$ by the subgroup $W^{2^{n+1}}(KG)[2]$. Note that the elements of the set M we shall choose in the form $yy^\star$ $\left(y \in V^{2^n}(KG)\right)$.

Let A) holds, i.e. $|K_n| \geq |G_n|$.

It is easy to prove that in this case the Sylow 2-subgroup S_n of the group G_n has such element g of order 2 and there exists an $a \in G_n$ that one of the following conditions holds:

A$_1$) $G_n \neq \langle g \rangle, a \notin \langle g \rangle$ and $a^2 \notin \langle g \rangle$,

A$_2$) $G_n \neq \langle g \rangle, a \notin \langle g \rangle$ and $a^2 \in \langle g \rangle$,

A$_3$) $G_n = \langle g \rangle$

and in cases A$_1$) and A$_2$) at least one of the elements a or g do not belong to the subgroup G_{n+1}. Indeed, if $g \in G_{n+1}$ then, by condition $G_n \neq G_{n+1}$, the set $G_n \setminus G_{n+1}$ has a proper element a.

Let A$_1$) holds. Let α be a nonzero element of the ring K_n and $y_\alpha = 1 + \alpha a(1+g)$. We shall prove that the set

$$M = \left\{x_\alpha = y_\alpha {y_\alpha}^\star = 1 + \alpha(a + a^{-1})(1+g) \colon 0 \neq \alpha \in K_n\right\}$$

has the above declared property. Really, since $a^2 \notin \langle g \rangle$, it follows that the elements a and a^{-1} belong to the different cosets of the group G_n by the subgroup $\langle g \rangle$. Hence $x_\alpha \neq 1$. It is easy to see that ${x_\alpha}^\star = x_\alpha = {x_\alpha}^{-1}$. Therefore x_α is a unitary element of second order of the group $V(K_nG_n)$. If $x_\alpha \in V^{2^{n+1}}$ then, from the condition $a^2 \notin \langle g \rangle$, it follows that the elements a and ag belong to the group G_{n+1}, but this contradicts to the choice of elements a and g. Therefore $x_\alpha \in W^{2^n}[2] \setminus W^{2^{n+1}}[2]$.

Suppose that the coset $x_\alpha V^{2^{n+1}}[2]$ coincides with $x_\nu V^{2^{n+1}}[2]$ for a different α and ν from K_n. Then $x_\alpha = x_\nu z$ for a suitable $z \in V^{2^{n+1}}$. Since ${x_\nu}^\star = {x_\nu}^{-1}$, it follows that

$$z = x_\alpha {x_\nu}^\star = 1 + (\alpha + \nu)(a + a^{-1})(1+g) = x_{\alpha+\nu}$$

and $x_{\alpha+\nu}$ belongs to the subgroup $V^{2^{n+1}}$ what contradicts it which was proved in above. Obviously $|M| = |K_n|$. Therefore the constructed set M has the above declared property.

Let A$_2$) holds.

It is easy to see that the elements of the set

$$M = \{x_\alpha = 1 + \alpha a(1+g) \colon 0 \neq \alpha \in K\}$$

belong to the different cosets of the group $V(KG)[2]$ by the subgroup $V^2(KG)[2]$. Indeed, if $x_\alpha \in V^2$ then $a \in G_1$ and $ag \in G_1$. But this contradicts to the choice of the elements a and g and hence $x_\alpha \in W[2] \setminus W^2[2]$.

The equation $x_\alpha = x_\nu z$ $\left(z \in V^2, \alpha \neq \nu\right)$ is impossible since from it follows that $z = x_\alpha x_\nu = 1 + (\alpha + \nu)a(1 + g) = x_{\alpha+\nu}$, and, by proved in above, $x_{\alpha+\nu} \notin V^2$. Obviously $|M| = |K|$ and therefore $f_0(W) = |K|$.

Let us contruct the set M in case $n > 0$.

Since, by Lemma 1, $f_n(W_2) = f_n(G)$ when $G_{n+1} = 1$, it follows that we can assume that $G_{n+1} \neq 1$. Let $|G_n| \neq 4$. Then the set $G_n \setminus G_{n+1}$ has neither element a, which order is not divisible by 2, or element b of order $2^r > 4$, or has a subgroup $\langle c : c^4 = 1\rangle \times \langle d : d^2 = 1\rangle$. Obviously in the first case $a^2 \notin \langle g\rangle$. If in the other cases we put $a = b, g = b^{2^{r-1}}$ or $a = c, g = d$ respectively then the condition $a^2 \notin \langle g\rangle$ holds and we have the above considered case A$_1$).

Let $G_n = \langle a : a^4 = 1\rangle$ and $y_\alpha = 1 + \alpha(a + 1)$. Obviously the element

$$x_\alpha = y_\alpha y_\alpha{}^\star = 1 + (\alpha + \alpha^2)(a + a^3)$$

is unitary. Let L denote a subset of K_n that has a unique representative in every subset of the form $\{\alpha, 1 + \alpha\} \subseteq K_n$. Then the elements of the set

$$M = \left\{x_\alpha = y_\alpha y_\alpha{}^\star = 1 + (\alpha + \alpha^2)(a + a^3) : 0 \neq \alpha \in L\right\}$$

belong to the different cosets of the group $W^{2^n}(KG)[2]$ by the subgroup $W^{2^{n+1}}(KG)[2]$. Really, if x_α coincides with x_ν $(\alpha, \nu \in L)$, then $\alpha + \alpha^2 = \nu + \nu^2$. Hence the equation $(\alpha + \nu)(1 + \alpha + \nu) = 0$ holds, but in the ring without zero divisors this is possible for the different α and ν only in the case $\nu = 1 + \alpha$, what contradicts to the choice of the elements of the set L. Obviously $|M| = |L| = |K_n|$. By Lemma 1, $W^{2^{n+1}} = \langle a^2\rangle$. If $x_\alpha W^{2^{n+1}} = x_\nu W^{2^{n+1}}$ $(x_\alpha \neq x_\nu)$ we get the contradictinally equation

$$1 + (\alpha + \alpha^2)(a + a^3) = a^2 + (\nu + \nu^2)(a + a^3).$$

Therefore $x_\alpha W^{2^{n+1}} \neq x_\nu W^{2^{n+1}}$ for $x_\alpha \neq x_\nu$ the case A$_2$) is considered.

Let A$_3$) holds, i.e. $G_n = \langle g\rangle$. Then $G_{n+1} = 1$. If $n = 0$ then $W(KG) = V_2(KG)$ and $f_0(W) = f_0(V_2) = |K|$. If $n > 0$ then, according to Lemma 1, $f_n(W) = f_n(G)$.

Therefore the case A) is fully considered.

Suppose now that B) holds, i.e. $|G_n| > |K_n|$ and the Sylow 2-subgroup S_n of the group G_n does not coincide with the Sylow 2-subgroup S_{n+1} of the group G_{n+1}. Then the set $S_n \setminus S_{n+1}$ has an element g of order $q = 2^r$. Let, further on, $\Pi = \Pi(G_n/\langle g\rangle)$ denote the full set of representatives of the cosets of the group G_n by the subgroup $\langle g\rangle$. Let us consider two disjunct subsets

$$\Pi_1 = \left\{a \in \Pi : a^2 \notin \langle g\rangle\right\} \quad \textit{and} \quad \Pi_2 = \left\{a \in \Pi : a^2 \in \langle g\rangle\right\}$$

of the set Π. Since G_n is infinite, it is easy to see that $|G_n| = |\Pi| = \max\{|\Pi_1|, |\Pi_2|\}$.

Let us suppose at first that $|G_n| = |\Pi_1|$. Without loss of generality we can assume that the representative of the coset $a^{-1}\langle g\rangle$ is the element a^{-1}. Let E denote the set which has a unique representative in every subset of the form $\{a, a^{-1}\} \subseteq \Pi_1$ and $y_a = 1 + a(1 + g + \cdots + g^{q-1})$. Then $|G_n| = |E|$ and the elements of the set

$$M = \{x_a = y_a y_a{}^{\star} = 1 + (a + a^{-1})(1 + g + \cdots + g^{q-1}) : a \in E\}$$

belong to the different cosets of the group $V^{2^n}[2]$ by the subgroup $V^{2^{n+1}}[2]$. Indeed, from the supposition $x_a \in V^{2^{n+1}}[2]$ it follows that $ag^i \in G_{n+1}$ for every $i = 0, 1, \ldots, q-1$, but this contradicts to the choice of the element $g \in G_n \setminus G_{n+1}$. It is easy to see that x_a is a unitary element and so $x_a \in W^{2^n}[2] \setminus W^{2^{n+1}}[2]$. Suppose that a and c are the distinct elements of the set E. If $x_a = x_c z$ for some $z \in V^{2^{n+1}}$ then

$$z = x_a x_c{}^{\star} = 1 + (a + a^{-1} + c + c^{-1})(1 + g + \cdots + g^{q-1}).$$

According to the choice of the elements of the set E we have that the elements a, a^{-1}, c, c^{-1} belong to the distinct cosets of the group G_n by the subgroup $\langle g\rangle$. Hence from the condition $z \in V^{2^{n+1}}$ it follows that $a \in G_{n+1}$, $ag \in G_{n+1}$, which contradicts to the choice of the element $g \in S_n \setminus S_{n+1}$.

Let be now $|G_n| = |\Pi_2|$. If $G^2 = 1$ then $W(KG) = V(KG)$ and $f_0(W) = f_0(V_2) = |G|$. If $n > 0$ and $G_{n+1} = 1$ then, by Lemma 1, $f_n(W) = f_n(G)$. Suppose that $G_{n+1} \neq 1$. Then the group G_n has such element v of order not equals to 2 that $\langle g\rangle \cap \langle v\rangle = 1$. If a such representative of the coset $a\langle g\rangle$ that $a^2 \in \langle g\rangle$ and $a^2 \neq 1$, then $a^2 = g^i \in G_{n+1}$ and, according to the choice of the element g, the integer i is divisible by 2. In this case in role of the representative of the coset $a\langle g\rangle$ in the set Π_2 we can choose the element $a_1 = ag^{-\frac{i}{2}}$. Therefore, we can assume that the set Π_2 consists of the elements of second order. Since $\langle g\rangle \cap \langle v\rangle = 1$, it follows that from the Π_2 we can choose a subset $\widetilde{\Pi}_2$ which elements belong to the distinct cosets of the group G_n by the subgroup $\langle g, v\rangle$ and $|G_n| = |\Pi_2|$. Let $y_a = 1 + av(1 + g + \cdots + g^{q-1})$. Then the set

$$M = \left\{x_a = y_a y_a{}^{\star} = 1 + a(v + v^{-1})(1 + g + \cdots + g^{q-1}) : a \in \widetilde{\Pi}_2\right\}$$

has the need property. Indeed, the cosets $x_a V^{2^{n+1}}$ and $x_c V^{2^{n+1}}$ coincide if and only if

$$x_a x_c = 1 + (a + c)(v + v^{-1})(1 + g + \cdots + g^{q-1}) \in V^{2^{n+1}}$$

Since the elements a and c belong to the distinct cosets of the group G_n by the subgroup $\langle g, v\rangle$, it follows that $av \in G_{n+1}$ and $avg \in G_{n+1}$, but this contradicts to the choice of the element $g \in G_n \setminus G_{n+1}$. So the case B) is fully considered.

Let C) holds, that is $|G_n| > |K_n|$ and the Sylow 2-subgroup S_n of the group G_n is 2-divisible.

Let us fix an element $g \in S_n[2]$ and choose such $v \in G_n \setminus G_{n+1}$ that 2 does not divide the order of element v. Since $|S_n| = [S_n : \langle g\rangle] \geq |\langle v\rangle|$ and $v \notin S_n$, it follows that the cardinality of the full set of representatives of the cosets $\Pi = \Pi(G_n/\langle g, v\rangle)$ of the group G_n by the subgroup $\langle g, v\rangle$ coincides with $|G_n|$. Obviously the set Π decomposes to the two disjunct subsets $\Pi_1 = \{a \in \Pi \colon a^2 \notin \langle v, g\rangle\}$ and $\Pi_2 = \{a \in \Pi \colon a^2 \in \langle v, g\rangle\}$.

Let $| G_n |=| \Pi_1 |$, E be the set which has a unique representative in every subset of the form $\{a, a^{-1}\} \subseteq \Pi_1$ and $y_a = 1 + a(1 + v + v^{-1}(1+g)$. Then the set M can be choosen in the following way:

$$M = \{x = y_a y_a{}^\star = 1 + (a + a^{-1})(1 + v + v^{-1})(1+g) \colon a \in E\}.$$

Indeed, from the equation $x_a = x_c z$ $\left(z \in V^{2^{n+1}},\ a \neq c\right)$ follows that

$$z = 1 + (a + a^{-1} + c + c^{-1})(1 + v + v^{-1})(1+g) \in V^{2^{n+1}}.$$

Hence, according to the construction of the set E, the elements a and av belong to the subgroup G_{n+1}, but this contradicts to the condition $v \notin G_{n+1}$.

Suppose now that $|G_n| = |\Pi_2|$. Then $v^2 \neq 1$. If $a^2 = v^2$ for some $a \in \Pi_2$, then from the condition $v \notin G_{n+1}$ it follows that i is an even number. Let us choose in the role of the representative of the coset $a\langle g, v\rangle$ the element $a_1 = av^{-\frac{i}{2}}$. Hence.we can assume that the set Π_2 of the representatives of the group G_n by the subgroup $\langle g, v\rangle$ consists of the elements of the group $S_n = S_{n+1}$. The set

$$M = \{x_a = 1 + a(v + v^{-1})(1+g) \colon \dot{a} \in \Pi_2\}$$

has the need property. Indeed, if $x_a = x_c z$ for the distinct $a, c \in \Pi_2$ and for some $z \in V^{p^{n+1}}$, then $z = x_a x_c = 1 + (a+c)(v+v^{-1})(1+g)$ and $av \in G_{n+1}$. Hence $v \in G_{n+1}$ because – by the choice – $\Pi_2 \subseteq S_{n+1}$, and so we get the contradiction.

Therefore the case C) is fully considered and the statement is proved for a finite ordinal $\lambda = n$.

Let us consider the case of infinite ordinal λ.

Let λ be an arbitrary infinite ordinal $R = K_\lambda, H = G_\lambda \neq G_{\lambda+1}$ and the Sylow 2-subgroup S_λ of the group G_λ is not singular. Then

$$W(KG)^{2^\lambda} \subseteq W(RH) \subseteq V_2(RH)$$

and by transfinite induction it is easy to prove the equation

$$V_2(KG)^{2^\lambda} = V_2(RH). \tag{2}$$

As compared to the group $V_2(RH)$ we can construct the set M as in the above shown cases $A), B)$ and $C)$. Since in every of this cases the set M consist of the elements of the form $x = y^{-1}y^\star$ and, by (2), y belongs to the group $V_2(RH) = V_2(KG)^{2^\lambda}$, it follows that the elements x are the representatives of the cosets of group $W^{2^\lambda}(KG)[2]$ by the subgroup $W^{2^{\lambda+1}}(KG)[2]$.

Therefore for an arbitrary infinite ordinal λ the Ulm–Kaplansky invariants of the group $W(KG)$ can be calculated in the above shown way for the case $\lambda = n$.

References

[1] A. A. Bovdi and A. A. Szakacs, The unitary subgroup of the group of units in a modular group algebra of a finite abelian p-group, *Math. Zametki* 6 **45** (1989), 23–29 (in Russian). (English translation *Math. Notes*, 5–6 **45** (1989), 445–450.)

[2] A. Szakács, Unitary subgroup of the Sylow p-subgroup of the group of normalized units in an infinite commutative group ring. *Acta Acad. Paed. Agriensis. Sec. Math.* **XXII** (1994) 85–93.

[3] A. A. Bovdi and Z. F. Patay, The structure of the centre of the multiplicative group of group ring of p-group over a ring of characteristic p. *Vesci Akad. Nauk. Bssr. Ser. Fiz. Math. Nauk.* (1978) No. 1, 5–11.

Attila Szakács
Kőrösi Csoma Sándor College
Institute of Business Finance
Department of Mathematics
5600 Békéscsaba, Bajza u. 33.
H-Hungary

Free-form curve design by neural networks

MIKLÓS HOFFMANN and LAJOS VÁRADY

Abstract. This paper gives a new approach of the two dimensional scattered data manipulation. The standard approximation and interpolation methods which can only be used for non-scattered data will also be applicable for scattered input with the help of the neural network. The Kohonen network produces an ordering of the scattered input points and here the B-spline curve is used for the approximation and interpolation.

Introduction

The interpolation and the approximation of two dimensional scattered data are interesting problems of computer graphics. By scattered data we mean a set of points without any predefined order. Unfortunately all the standard interpolation and approximation methods—like Hermite interpolation, Bezier curves or B-spline curves—need a sequence of points, hence if we want to apply these methods we have to order the data. A good survey of the scattered data interpolation can be found in [1]. In this paper a completely new approach is given where the self-organizing ability of the neural networks will be used to order the points. The Kohonen network [2,3] can be trained by scattered data, that is the points will form the input of the network, while the weights of the network and their connections give us a polygon, the vertices of which will be the input points. In this way the polygon we obtained can be used as the control polygon of a B-spline curve, so finally a standard approximation or interpolation method can be applied for the scattered data. We begin our discussion with the short definition of the B-spline curve and Kohonen's neural network.

The B-spline curve

The B-spline curve is the most common and widely used free-form representation method which can be used as an approximating and also as an interpolating curve [4]. If we have a sequence of points P_i $(i = 1, \ldots, n)$, then the curve approximating the plane polygon given by the points is defined as

Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA F 019395.

$$S_i(u) = \sum_{r=-1}^{2} P_{i+r} b_r(u) \quad u \in [0,1] \quad i = 2, \dots, n-2$$

where b_r are the well-known B-spline basis functions.

The Kohonen neural network

Neural networks can be divided into two classes, the supervised and the non-supervised learning or self organizing neural networks. Supervised learning neural nets have to be trained with training or test data sets, where the result of the task to be done has to be provided in advance. After training, the net is adapted to the problem by the test set and is able to generalize its behavior. Self organizing networks, however, organize the data during the learning phase where the result of the task is not required. Following the training rules, the network adapts its internal knowledge to the task. The Kohonen neural network is a two-layered non-supervised learning neural network.

Adaptation of the Kohonen net to the problem

Let a set of points P_i $(i = 1, \dots, n)$ (scattered data) be given on the plane. Our purpose is to fit (by interpolation or approximation) a B-spline curve to them. Thus our first task is to determine the order of the points for the interpolating or approximating methods.

The Kohonen net is used to order the points. The first layer of neurons is called input layer and contains the two input neurons which pick up the data. The input neurons are entirely interconnected to a second, competitive layer, which contains m neurons (where $m \geq n$). The weights associated with the connections are adjusted during training. Only one single neuron can be active at a time and this neuron represents the cluster which the input data set belongs to.

Let a set of two dimensional vectors $P_i(x_1, x_2)$ be given. These vectors are called input vectors. The coordinates of these vectors are submitted to the input layer which contains two neurons. When all the input vectors were presented to the input neurons, we restart at the first vector.

Let the output vectors $o_1, \dots, o_m$ be two dimensional vectors with the coordinates $(w_{1j}, w_{2j}), j = 1, \dots, m$, where w_{ij} denotes the weights between the input neuron i and the output neuron j. We use the terms "output vector" and "weights of the output neuron" interchangeably. Let the output map be one dimensional (see Figure 1).

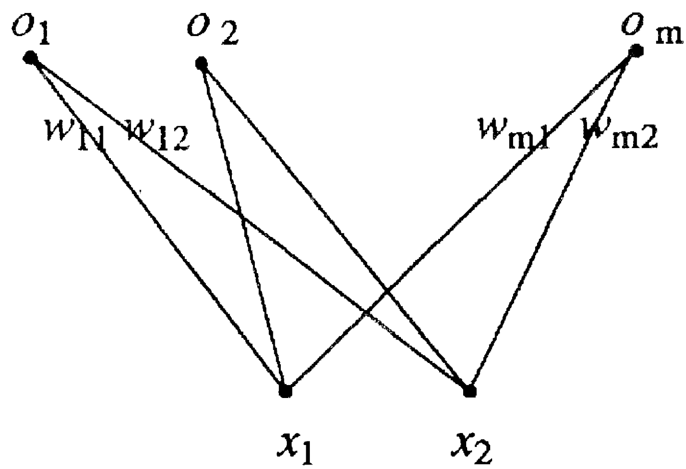


Figure 1.

The training of the network is figured out by presenting data vectors P_i to the input layer of the network whose connection weights w_{ij} are initially chosen as random values. Compute the Euclidean distance between the input point $P_1(x_1, x_2)$ and the output neurons $o_i(w_{i1}, w_{i2})$ with

$$d_i = \sqrt{\sum_{j=1}^{2}(x_j - w_{ij})^2}$$

The neuron c with the minimum distance will be activated, where $d_c = \min\{d_i\}$ $(i = 1, \ldots, m)$. The update of the weights w_{ij} associated to the neurons is only performed within a neighbourhood $N_c(t)$ of c. This neighbourhood is reduced with training time t. The update follows the equation

$$w_{ij}^{(t+1)} = w_{ij}^{(t)} + \Delta w_{ij}^{(t)}, \quad (i = 1, \ldots, m;\ j = 1, 2)$$

where

$$\Delta w_{ij}^{(t)} = \eta(t)(x_j - w_{ij}^{(t)})$$

and

$$\eta(t) = \eta_0 \left(1 - \frac{t}{T}\right), \quad \text{where} \quad t \in [0, T]$$

Here $\eta(t)$ represents a time-dependent learning rate which is decreasing in time. The term can be chosen as a Gaussian function.

After updating the weights w_{ij} a new input is presented and the next iteration starts. The algorithm determines (using the Euclidean distance) the closest output vector to the presented input vector. The coordinates i.e. the weights of this output vector and those of the vectors that are in a certain neighbourhood of the nearest output vector are updated so that these output vectors get closer to the presented input vector. The degree of the update depends on the gain term and the distance of the output vector and the presented input vector. When the radius (which specifies the neighborhood around an output vector) is large, many output vectors tend towards the presented input. For this reason, initially the output vectors move to places where the density of the input vectors is large, since more input vectors are presented from this areas. The radius (i.e. the size of the neighborhood) and the gain term is decreasing in time. The latter results in that after enough iterations the locations of the output vectors does not change significantly (if the gain term is almost zero then the change in the weights is negligible). The gain term should diminish only when the weights are already close to the input vectors.

A net is said to be convergent if for all the input vectors P_i $(i = 1, \ldots, n)$ there is an output vector o_j such that after a certain time t_0 the Euclidean distance of o_j and P_i is smaller than a predefined limit. A stronger convergence can be obtained if we require that the output vectors which do not converge to an input vector are on the line determined by its two neighbouring output vectors.

In the general case the convergence of the Kohonen net has not been proved yet. Kohonen proved the convergence only in a very simple case when the output is one dimensional and the inputs are the elements of an interval (see[2]).

The radius, the gain term and the number of the outputs can be adjusted so that the output vectors satisfy the stronger convergence mentioned above. This stronger convergence is important especially in term of the smoothness of the future curve.For the detailed description and evaluation of this problem see [5,6]. Let two converging outputs be o_i and o_{i+k} while the outputs which are between the converging outputs be $o_{i+1}, \ldots, o_{i+k-1}$. These outputs are in the neighborhoods of the outputs o_i and o_{i+k} (depending on the radius and k). Since these converged output vectors are close to some input vectors, the outputs $o_{i+1}, \ldots, o_{i+k-1}$ will move towards these outputs (and the input vectors). Since they will move to the common line of the ceonverged output vectors.

The Kohonen net retains the topological ordering of its output vectors. The weights of two output vectors will be close to each other if the vectors are close on the map. The same is true for the approximated input vectors.

Results and further possibilities

The following figures show the ordering of the input vectors and the approximating B-spline curve. There are 20 input vectors and 80 output nodes.

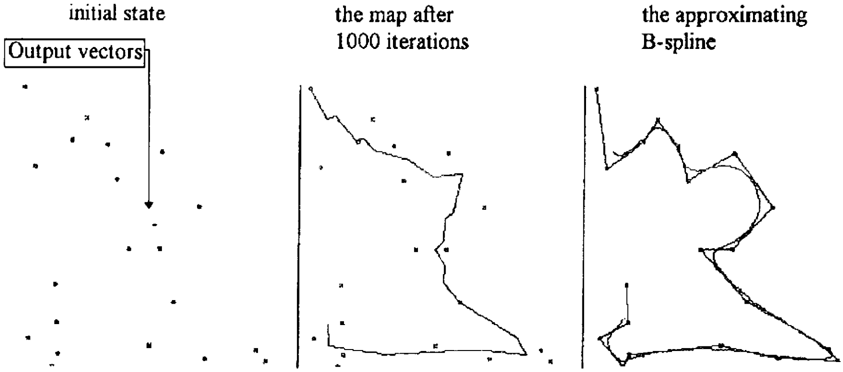


Figure 2.

We plan to generalize the method to three dimensional input points using the Kohonen net. In this case the output map is two dimensional and the input vectors and the weights are three dimensional. When the net converges, the grid approximates the input points and an interpolating or approximating surface can be fitted to the input points.

References

[1] W. Boehm, G. Farin and J. Kahmann, A survey of curve and surface methods in CAGD, *CAGD* **1** (1984), 1–60.

[2] T. Kohonen, Self-organization and associative memory, Springer Verlag, 1984.

[3] M. Alder, R. Togneri, E. Lai and Y. Attikiouzel, Kohonen's algorithm for the numerical parametrisation of manifolds, *Pattern Recognition Letters* **11** (1990), 313–319.

[4] L. D. Faux and M. J. Pratt, Computational Geometry for Design and Manufacture, Wiley & Sons, NY, 1979.

[5] L. VÁRADY, Analysis of the Dynamic Kohonen Network Used for Approximating Scattered Data, *Proceedings of the 7th ICECGDG*, Cracow, 1996, 433–436.

[6] M. HOFFMANN, Modified Kohonen Neural Network for Surface Reconstruction, *Publ. Math. Debrecen*, 1997 (to appear)

MIKLÓS HOFFMANN
ESZTERHÁZY KÁROLY TEACHERS' TRAINING COLLEGE
DEPARTMENT OF MATHEMATICS
LEÁNYKA U. 4.
3301 EGER, PF. 43.
HUNGARY
E-mail: hofi@gemini.ektf.hu

LAJOS VÁRADY
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN, P.O.BOX 12
HUNGARY
E-mail: lvarady@pc123.math.klte.hu

On the Fejér kernel functions with respect to the Walsh–Paley system

GYÖRGY GÁT*

Abstract. In this paper we prove some lemmas with respect to the Fejér kernels of the Walsh–Paley system. This lemmas give a new proof for the known a.e. convergence $\sigma_n f \to f$ $(n\to\infty,\ f\in L^1)$.

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$ and $I := [0,1)$ the unit interval. Denote the Lebesgue measure of any set $E \subset I$ by $|E|$. Denote the $L^p(I)$ norm of any function f by $\|f\|_p$ $(1 \le p \le \infty)$.

Denote the dyadic expansion of $n \in \mathbf{N}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbf{N}$ choose the expansion which terminates in zeros (these numbers are the dyadic rationals)). n_i, x_i are the i-th coordinates of n, x, respectively. Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} (x_j + y_j \ \mathrm{mod} 2) 2^{-j-1}.$$

The sets

$$I_n(x) := \{y \in I : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$$

for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbf{P}$ and $I_0(x) := I$ are the dyadic intervalls of I. Set $e_n := (0, \ldots, 0, 1, 0, \ldots)$ where the n-th coordinate of e_n is 1 the rest are zeros for all $n \in \mathbf{N}$. The dyadic rationals are the finite $0,1$ combinations of the elements of the set $\{e_n : n \in \mathbf{N}\}$ (which dense in I).

Let $(\omega_n, n \in \mathbf{N})$ represent the Walsh–Paley system ([2], [8]) that is,

$$\omega_n(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}, \quad n \in \mathbf{N},\ x \in I.$$

Denote by $D_n := \sum_{k=0}^{n-1} \omega_k$, the Walsh–Dirichlet kernels.

---

* Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA T 020334.

It is well-known that ([2], [8])

$$S_n f(y) = \int_I f(x) D_n(y+x)dx = f * D_n(y)$$

($y \in I,\ n \in \mathbf{P}$) the n-th partial sum of the Walsh–Fourier series. Moreover, ([8], p. 28.)

$$D_{2^n}(x) := \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

$$D_n(x) = \omega_n(x) \sum_{k=0}^{\infty} n_k (D_{2^{k+1}}(x) - D_{2^k}(x)) = \omega_n(x) \sum_{k=0}^{\infty} n_k (-1)^{x_k} D_{2^k}(x), \tag{2}$$

$n \in \mathbf{N}$, $x \in I$.

Define the n-th Fejér means [8] of function $f \in L^1(I)$ as

$$\sigma_n f(y) := \frac{1}{n} \sum_{k=0}^{n-1} S_k f(y)$$

for $y \in I$ and $n \in \mathbf{P}$ and define n-th Fejér kernel [8]

$$K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x)$$

for $x \in I$ and $n \in \mathbf{P}$. This gives

$$\sigma_n f(y) = \int_I f(x) K_n(x+y)dx = f * K_n(y) \quad (y \in I,\ n \in \mathbf{P}).$$

Set

$$K_{a,b} := \sum_{j=a}^{b-1} D_j \quad a,b \in \mathbf{N} \quad \text{and} \quad n^{(s)} := \sum_{i=s}^{\infty} n_i 2^i \quad (n, s \in \mathbf{N}).$$

Also set for $n \in \mathbf{N}$ $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. That is, $2^{|n|} \leq n < 2^{|n|+1}$. In this paper c denotes an absolute constant which may not be the same at different occurences. Then we have by an easy calculation that

Lemma 1. $nK_n = \sum_{s=0}^{|n|} n_s K_{n^{(s+1)},2^s}$ *for all* $n \in \mathbf{P}$. ∎

Lemma 2. *Suppose that $s,t,n \in \mathbf{N}$, $x \in I_t \setminus I_{t+1}$. If $s \le t \le |n|$, then* $\left|K_{n^{(s+1)},2^s}(x)\right| \le c2^{s+t}$. *On the other hand, if $t < s \le |n|$, we have*

$$K_{n^{(s+1)},2^s}(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x) 2^{s+t-1} & \text{if } x - x_t e_t \in I_s. \end{cases}$$

Proof. If $s \le t$, then for all $k \in \mathbf{N}$ by (1) and (2) we have $|D_k(x)| \le c\sum_{j=0}^{t} 2^j \le c2^t$, thus in this case $\left|K_{n^{(s+1)},2^s}(x)\right| \le c2^{s+t}$. On the other hand, let $|n| \ge s > t$. Then

$$\begin{aligned} D_{n^{(s+1)}+j}(x) &= \omega_{n^{(s+1)}+j}(x) \sum_{k=0}^{t} (n^{(s+1)} + j)_k r_k(x) \\ &= \omega_{n^{(s+1)}+j}(x) \left(\sum_{k=0}^{t-1} j_k 2^k - j_t 2^t \right). \end{aligned}$$

This implies that

$$\begin{aligned} K_{n^{(s+1)},2^s}(x) &= \sum_{j=0}^{2^s-1} D_{n^{(s+1)}+j}(x) \\ &= \omega_{n^{(s+1)}}(x) \sum_{j=0}^{2^s-1} \omega_j(x) \left(\sum_{k=0}^{t-1} j_k 2^k - j_t 2^t \right) =: \sum\nolimits_1 - \sum\nolimits_2. \end{aligned}$$

$$\begin{aligned} \sum\nolimits_1 &= \omega_{n^{(s+1)}}(x) \sum_{j_0,\ldots,j_{s-1}} \omega_j(x) \sum_{k=0}^{t-1} j_k 2^k \\ &= \sum_{j_i=0, i\ne t, i=0,\ldots,s-1}^{1} \sum_{k=0}^{t-1} j_k 2^k \sum_{j_t=0}^{1} \omega_j(x) = 0, \end{aligned}$$

since

$$\sum_{j_t=0}^{1} \omega_j(x) = \sum_{j_t=0}^{1} (-1)^{j_0 x_0 + \cdots + j_{t-1} x_{t-1} + j_{t+1} x_{t+1} + \cdots + j_{s-1} x_{s-1}} = 0.$$

That is,

$$\begin{aligned} K_{n^{(s+1)},2^s}(x) &= -\omega_{n^{(s+1)}}(x) \sum_{j=0}^{2^s-1} \omega_j(x) j_t 2^t \\ &= \begin{cases} 0 & \text{if } x - x_t e_t \notin I_s, \\ \omega_{n^{(s+1)}}(x) 2^{s+t-1} & \text{if } x - x_t e_t \in I_s. \ \blacksquare \end{cases} \end{aligned}$$

As a straightforward consequence of Lemma 2 we get

Lemma 3. $\int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \le c\sqrt{2^{s+t}}$, *where* $m \ge s,t \in \mathbf{N}$ *are fixed.*

Proof. If $s > t$, then by Lemma 2 it follows that

$$\int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx = \int_{I_s(e_t)} 2^{s+t-1} dx = 2^{t-1}.$$

On the other hand, if $s \le t$, then also by Lemma 2 we have

$$\int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \le c\int_{I_t\setminus I_{t+1}} c2^{s+t} \le c2^s. \ \blacksquare$$

Lemma 4. $\int_{I\setminus I_k} \sup_{|n|\ge A} |K_n(x)|\, dx \le c\sqrt{2^{k-A}}$, *for all* $A \ge k \in \mathbf{N}$.

Proof. By Lemma 1 we have

$$n\,|K_n| \le \sum_{s=0}^{|n|} \left|K_{n^{(s+1)},2^s}\right|,$$

consequently,

$$\begin{aligned}
\int_{I\setminus I_k} \sup_{|n|\ge A} |K_n(x)|\, dx &\le \sum_{t=0}^{k-1} \int_{I_t\setminus I_{t+1}} \sum_{m=A}^{\infty} \sup_{|n|=m} |K_n(x)|\, dx \\
&\le \sum_{t=0}^{k-1} \sum_{m=A}^{\infty} \frac{1}{2^m} \int_{I_t\setminus I_{t+1}} \sup_{|n|=m} n\,|K_n(x)|\, dx \\
&\le \sum_{t=0}^{k-1} \sum_{m=A}^{\infty} \frac{1}{2^m} \left(\sum_{s=0}^{t} \int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \right. \\
&\quad \left. + \sum_{s=t+1}^{m} \int_{I_t\setminus I_{t+1}} \sup_{|n|=m} \left|K_{n^{(s+1)},2^s}(x)\right| dx \right) \\
&\le c\sum_{t=0}^{k-1} \sum_{m=A}^{\infty} \frac{1}{2^m} \sum_{s=0}^{m} 2^{\frac{s+t}{2}} \le c\sum_{t=0}^{k-1} \sum_{m=A}^{\infty} 2^{\frac{t-m}{2}} \le c2^{\frac{k-A}{2}}. \ \blacksquare
\end{aligned}$$

The following Theorem shows that the maximal operator

$$Tf := \sup_{n\in\mathbf{P}} |\sigma_n f|$$

is quasi-local. The conception of quasi-locality is introduced by F. Schipp [8]. Let $f \in L^1(I)$, supp $f \subset I_k(x^0)$ for some $k \in \mathbf{N}, x^0 \in I$ and suppose that the integral of Tf on the set $I \setminus I_k(x^0)$ is bounded by $c\,|f|_1$. Then we call T quasi-local. That is, we prove

Theorem 5. $\int_{I\setminus I_k(x^0)} Tf \le c\,|f|_1$.

Proof. If $n < 2^k$, then $\hat{f}(n) = \int_I f\omega_n = \int_{I_k(x^0)} f\omega_n = \omega_n(x^0)\int_{I_k(x^0)} f = 0$, thus $S_n f = 0, \sigma_n f = 0$. That is, we have $Tf = \sup_{n\ge 2^k} |\sigma_n f|$. By Lemma 4 it follows

$$\int_{I\setminus I_k(x^0)} \sup_{n\ge 2^k} \left| \int_{I_k(x^0)} f(x)K_n(x+y)dx \right| dy$$
$$\le \int_{I_k(x^0)} |f(x)| \int_{I\setminus I_k(x^0)} \sup_{n\ge 2^k} |K_n(x+y)dy|\,dx$$
$$= \int_{I_k(x^0)} |f(x)| \int_{I\setminus I_k} \sup_{n\ge 2^k} |K_n(y)dy|\,dx \le c\,|f|_1 . \blacksquare$$

Define the Hardy space H as follows. Let $f^* := \sup_{n\in\mathbf{N}} |S_{2^n} f|$ be the maximal function of the integrable function $f \in L^1(I)$. Then,

$$H(I) := \{f \in L^1(I) : f^* \in L^1(I)\},$$

moreover H is a Banach space endowed with the norm $|f|_H := |f^*|_1$. By standard argument (see e.g. [8]) and by the help of Theorem 5 one can prove that the operator T is of type (H, L^1) which means that $|Tf|_1 \le c\,|f|_H$ for all $f \in H$. This result with respect to the Walsh system is due to Schipp [7] and Fujii [2]. With respect to bounded Vilenkin system it is proved by Simon [6]. The noncommutative case is discussed by the author ([4]).

Also by standard argument (see e.g. [8]) and by the help of Theorem 5 we have that for all $f \in L^1(I)$ the almost everywhere convergence $\sigma_n f \to f$ $(n \to \infty,\ f \in L^1(I))$ holds. This result with respect to the Walsh system is due to Fine [1]. With respect to bounded Vilenkin systems it is proved by Pál and Simon [5]. The so-called 2-adic integers and the noncommutative case are discussed by the author ([3], [4]).

References

[1] Fine, N. J., Cesàro summability of Walsh–Fourier series, *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 558–591.

[2] FUJII, N., A maximal inequality for H^1 functions on the generalized Walsh–Paley group, *Proc. Amer. Math. Soc.* **77** (1979), 111–116.

[3] GÁT, G., On the almost everywhere convergence of Fejér means of functions on the group of 2-adic integers, *Journal of Approx. Theory*, (1995) (to appear).

[4] GÁT, G., Pointwise convergence of Fejér means on compact totally disconnected groups, *Acta Sci. Math.* (Szeged) **60** (1995), 311–319.

[5] PÁL, J., SIMON, P., On a generalization of the concept of derivative, *Acta Math. Acad. Sci. Hungar.* **29** (1977), 155–164.

[6] SIMON, P., Investigations with respect to the Vilenkin system, *Annales Univ. Sci. Budapestiensis, Sectio Math.* **27** (1985), 87–101.

[7] SCHIPP, F., Über gewiessen Maximaloperatoren *Annales Univ. Sci. Budapestiensis, Sectio Math.* **18** (1975), 189–195.

[8] SCHIPP, F., WADE, W. R., SIMON, P., PÁL, J., *Walsh series: an introduction to dyadic harmonic analysis*, Adam Hilger, Bristol and New York, 1990.

[9] TAIBLESON, M. H., Fourier Series on the Ring of Integers in a p-series Field, *B.A.M.S* **73** (1967), 623–629.

BESSENYEI COLLEGE,
DEPT. OF MATH.,
NYÍREGYHÁZA, P.O.BOX 166.,
H–4400, HUNGARY
E-mail: gatgy@ny2.bgytf.hu

On the certain subsets of the space of metrics

S. ČERETKOVÁ, J. FULIER and J. T. TÓTH

Abstract. In this note we look at certain subsets of the metric space of metrics for an arbitrary given set X and show that in terms of cardinility these can be very large while being extremely small in the topological point of view.

Introduction

Let X be a given non-void set. Denote by $\mathcal{M}$ the set of all metrics on X endowed with the metric:

$$d^\star(d_1, d_2) = \min\{1, \sup_{x,y\in X} \{|d_1(x,y) - d_2(x,y)|\} \text{ for } d_1, d_2 \in \mathcal{M}\}.$$

First of all recall some basic definitions and notations.

Suppose $\alpha > 0$ and put

$$\mathcal{H}_\alpha = \{d \in \mathcal{M} : \forall_{\substack{x\neq y \\ x,y\in X}} d(x,y) \geq \alpha\} \text{ and } \mathcal{H} = \bigcup_{\alpha>0} \mathcal{H}_\alpha.$$

Results shown in [2] include $\mathcal{M}$ is a non-complete Baire space and $\mathcal{H}$ is an open and dense subset of $\mathcal{M}$, thus $\mathcal{M} \mid \mathcal{H}$ is nowhere dense in $\mathcal{M}$. Other results on the metric space of metrics may be found in [2], [3] and [4].

Let $\mathcal{A}$ and $\mathcal{B}$ denote the set of all metrics on X that are unbounded and bounded, respectively. It is proved in [2] (Theorem 5) that $\mathcal{A}, \mathcal{B}$ are non-empty, open subsets of the Baire space $(\mathcal{M}, d^\star)$ (of [2], Theorem 3) provided X is infinite. Thereby $\mathcal{A}, \mathcal{B}$ are sets of the 2-nd cathegory in $\mathcal{M}$, if is infinite. If X is finite, then $\mathcal{B} = \mathcal{M}$ and $\mathcal{A} = \emptyset$.

Now define the mapping

$$f\colon \mathcal{M} \to (0,1],\ g\colon \mathcal{M} \to [0,\infty) \text{ and } h\colon \mathcal{B} \to (0,+\infty)$$

as follows:

$$f(d) = \sup_{x,y\in X} \frac{d(x,y)}{1+d(x,y)} \quad \text{where} \quad d \in \mathcal{M},$$

$$g(d) = \inf_{\substack{x\neq y \\ x,y\in X}} d(x,y) \quad \text{where} \quad d \in \mathcal{M}, \quad \text{and}$$

$$h(d) = \sup_{x,y\in X} d(x,y) \quad \text{where} \quad d \in \mathcal{B}.$$

Obviously $f^{-1}(\{1\}) = \mathcal{A}$ and $g^{-1}(\{0\}) = \mathcal{M} \setminus \mathcal{H}$.

It is purpose of this paper to establish how large sets $f^{-1}(\{t\})$, $g^{-1}(\{t\})$, are given.

In what follows if $\mathcal{U} \subset \mathcal{M}$, then $\mathcal{U}$ is considered as a metrics space with the metric $d^\star |_{\mathcal{U}\times\mathcal{U}}$ (a metric subspace of $\mathcal{M}$).

Main results

Let $\varphi(t) = \frac{t}{1+t}$ for $t \in [0, +\infty)$. Then φ is increasing and continuous function on $[0, +\infty)$. Therefore $f(d) = \sup_{x,y\in X} \varphi(d(x,y))$ for $d \in \mathcal{M}$. The natural question arises wether f is continuous on $\mathcal{M}$, too. The answer of this question is positive. We have

Lemma. *The function f, g are uniformly continuous on $\mathcal{M}$ and the function h is uniformly continuous on $\mathcal{B}$.*

Proof. Let $0 < \varepsilon < 1$ and $d_1, d_2 \in \mathcal{M}$ such that $d^\star(d_1, d_2) < \varepsilon$. We show

$$|f(d_1) - f(d_2)| \le d^\star(d_1, d_2), \quad |g(d_1) - g(d_2)| \le d^\star(d_1, d_2).$$

We can simply count

$$\begin{aligned}\varphi(d_1(x,y)) &\le \varphi(d_2(x,y)) + |\varphi(d_1(x,y)) - \varphi(d_2(x,y))| \\ &\le \varphi(d_2(x,y)) + d^\star(d_1, d_2)\end{aligned}$$

because

$$\frac{|d_1(x,y) - d_2(x,y)|}{(1 + d_1(x,y))(1 + d_2(x,y))} \le d^\star(d_1, d_2).$$

Taking supremum in the previous inequality we obtain $f(d_1) \le f(d_2) + d^\star(d_1, d_2)$, therefore $f(d_1) - f(d_2) \le d^\star(d_1, d_2)$. From symetrics we have $f(d_2) - f(d_1) \le d^\star(d_1, d_2)$ and $|f(d_1) - f(d_2)| \le d^\star(d_1, d_2)$. From this we see that the function f is uniformly continuous on $\mathcal{M}$. Obviously for $x, y \in X$

$$|d_1(x,y) - d_2(x,y)| \ge d_1(x,y) - d_2(x,y) \ge g(d_1) - d_2(x,y).$$

Then

$$g(d_1) - g(d_2) \le \inf_{\substack{x,y\in X\\ x\ne y}} |d_1(x,y) - d_2(x,y)| \le d^\star(d_1, d_2). \tag{1}$$

According the inequality $|d_1(x,y) - d_2(x,y)| \ge d_2(x,y) - d_1(x,y)$, similarly to the previous we get

$$g(d_2) - g(d_1) \le d^\star(d_1, d_2). \tag{2}$$

Then we required inequality follows from (1) and (2).

Analogously $|h(d_1) - h(d_2)| \le d^\star(d_1, d_2)$. ∎

Remark 1. The function h can be continuosly continued on $\mathcal{M}$. Because the set $\mathcal{B}$ is closed in $\mathcal{M}$, the Hausdorff's function (see [1], p. 382) is continuous continuation of the function h on $\mathcal{M}$.

Remark 2. Because $\mathcal{A} \cup \mathcal{B} = \mathcal{M}$, $\mathcal{A} \cap \mathcal{B} = \emptyset$ and the set $\mathcal{B}$ is closed in $\mathcal{M}$, according the lemma of Uryshon there exists a function $G\colon \mathcal{M} \to [0,1]$ such that G is continuous on $\mathcal{M}$ and $G(\mathcal{A}) = \{0\}$, $G(\mathcal{B}) = \{1\}$. For this reason $G(\mathcal{M}) = \{0,1\}$.

Space $(\mathcal{M}, d^\star)$ is Bair's space, e.g. every non-empty open subset of the set $\mathcal{M}$ is of the 2-nd cathegory in $\mathcal{M}$. The set $\mathcal{A}$ is non-void and open subset in $\mathcal{M}$, then the set $f^{-1}(\{1\}) = \mathcal{A}$ is of the 2-nd cathegory in $\mathcal{M}$. One may ask: Is there any $t \in (0,1)$ such that the set $f^{-1}(\{t\})$ is of the 2-nd cathegory in $\mathcal{M}$? Similarly for $g^{-1}(\{t\})$ and $h^{-1}(\{t\})$. This question is answered in the next theorem.

Theorem 1. *We have*

(i) *For arbitratry $t \in (0,1)$ the set $f^{-1}(\{t\})$ is nowhere dense in $\mathcal{M}$.*

(ii) *For arbitrary $t \in [0, +\infty)$ the set $g^{-1}(\{t\})$ is nowhere dense in $\mathcal{M}$.*

(iii) *For arbitrary $t \in [0, +\infty)$ the set $h^{-1}(\{t\})$ is nowhere dense in $\mathcal{M}$.*

Proof. (i) Let $0 < t < 1$. According to lemma the set $f^{-1}(\{t\})$ is closed in $\mathcal{M}$. Therefore it is sufficient to prove that the set $\mathcal{M} \setminus f^{-1}(\{t\})$ is dense in $\mathcal{M}$. We will use inequality

$$\frac{t_2}{1+t_2} \ge \frac{t_1}{1+t_1} + \frac{t_2 - t_1}{(1+t_2)^2} \quad \text{for } \ 0 \le t_1 \le t_2 \tag{3}$$

(it is equivalent to $(t_2 - t_1)^2 \ge 0$).

Let $d \in f^{-1}(\{t\})$ and $0 < \varepsilon < 1$. Clearly $d \in \mathcal{B}$ and there exists a $K \in R^+$ such that

$$d(x,y) \le K \ \text{ for every } \ x, y \in X. \tag{4}$$

Choose $d' \in \mathcal{M}$ as follows

$$d'(x,y) = \begin{cases} d(x,y) + \frac{\varepsilon}{2}, & \text{if } x, y \in x, x \ne y \\ 0, & \text{if } x = y. \end{cases}$$

Then $d^\star(d, d') < \varepsilon$. We show that $d' \in \mathcal{M} \setminus f^{-1}(\{t\})$. From (3) and (4) for $x, y \in X (x \ne y)$ and $t_1 = d(x,y)$, $t_2 = d'(x,y)$ we have

$$\varphi(d'(x,y)) \ge \varphi(d(x,y)) + \frac{\frac{\varepsilon}{2}}{(1 + d'(x,y))^2} \ge \varphi(d(x,y)) + \frac{\frac{\varepsilon}{2}}{(1+K)^2}.$$

Then $f(d') > f(d)$, so $d' \notin f^{-1}(\{t\})$.

(ii) According to lemma the set $g^{-1}(\{t\})$ is closed in $\mathcal{B}$. It is enough to show that the set $\mathcal{B}\setminus g^{-1}(\{t\})$ is dense in $\mathcal{B}$. Let $d \in g^{-1}(\{t\})$ and $0 < \varepsilon < 1$. Define d' on X as follows:

$$d'(x,y) = \begin{cases} d(x,y) + \frac{\varepsilon}{2}, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Evidently $g(d') = t + \frac{\varepsilon}{2}$, therefore $g' \in \mathcal{B} \setminus g^{-1}(\{t\})$ and $d^*(d,d') < \varepsilon$.

(iii) We can prove similarly like (ii). ∎

From the above Theorem 1 we can see that the sets $f^{-1}(\{t\})$, $g^{-1}(\{t\})$, $h^{-1}(\{t\})$ are small from the topological point of view but on the other hand we show, that the cardinality of them is equal to the cardinality of the set $\mathcal{M}$.

In [2] is proved: $\text{card}(\mathcal{M}) = c$ if X is a finite set having at least two elements and card $(\mathcal{M}) = 2^{\text{card}(X)}$ if X is infinite set (c denotes the cardinality of the set of all real numbers).

Theorem 2. *Let X be an infinite set. Then we have:*

1. $\text{card}(f^{-1}(\{t\})) = 2^{\text{card}(X)}$ *for* $t \in (0,1]$
2. $\text{card}(g^{-1}(\{t\})) = 2^{\text{card}(X)}$ *for* $t \in [0,+\infty)$
3. $\text{card}(h^{-1}(\{t\})) = 2^{\text{card}(X)}$ *for* $t \in (0,+\infty)$.

Proof. 1. Let $0 < t < 1$ and $0 < \varepsilon < \frac{1}{2} \cdot \frac{t}{1-t}$. Let $B \subseteq X$ for which $\text{card}(B) \geq 2$. We define the metric on X as follows:

$$\sigma_B(x,y) = \begin{cases} 0, & \text{if } x = y \\ \frac{t}{1-t}, & \text{if } x,y \in B;\ x \neq y \\ \frac{t}{1-t} - \varepsilon, & \text{if } x \notin B \text{ or } y \notin B,\ x \neq y \end{cases}$$

It is to easy to verify that σ_B is a metric and that $\sigma_B \neq \sigma'_B$, if $B \neq B'$. Evidently $f(\sigma_B) = t$. There are $2^{\text{card}(X)}$ many choices for B so we can see

$$2^{\text{card}(X)} \leq \text{card}(f^{-1}(\{t\})) \leq \text{card}(\mathcal{M}) \leq 2^{\text{card}(X)}.$$

We get by the Cantor–Bernstein theorem that $\text{card}(f^{-1}(\{t\})) = 2^{\text{card}(X)}$.

Let now $t = 1$ and $X_0 = \{x_1 < x_2 < \cdots < x_n < \cdots\} \subset X$. Define the function $d_B \colon X \times X \to R$:

$$\begin{aligned} d_B(x_n, x_m) &= |n - m| \quad \text{for} \quad n,m = 1,2,\ldots \\ d_B(x, x_n) &= d_B(x_n, x) = n \quad \text{for} \quad x \notin X_0 \\ d_B(x,y) &= d_B(y,x) = 1 \quad \text{for} \quad x,y \notin X_0,\ x \neq y \\ d_B(x,x) &= 0 \quad \text{for} \quad x \in X. \end{aligned}$$

(The same function was used in [2], Theorem 5.) It can be easily verified that $d_B(x_n, x_1) \to \infty (n \to \infty)$, hence $f(d_B) = 1$. Thereby we have $2^{\mathrm{card}(X)}$ possibilities for choosing of B, we get that $\mathrm{card}(f^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$.

2. For $t = 0$ it has been proved in [4] (Theorem 1), that $\mathrm{card}(g^{-1}(\{t\})) = 2^{\mathrm{card}(X)}$. Let $t > 0$. Let $B \subset X$ is so, that $\mathrm{card}(B) \geq 2$. Define ρ_B on X as follows:

$$\rho_B(x,y) = \begin{cases} 0, & \text{for } x = y \\ t & \text{for } x, y \in B, x \neq y \\ t+1 & \text{otherwise.} \end{cases}$$

Then $\rho_B \in \mathcal{M}$ and $g(\rho_B) = t$.

3. Let $t > 0$ and $0 < \zeta < \frac{t}{2}$. Then the function τ_B defined on X by this way

$$\tau_B(x,y) = \begin{cases} 0, & \text{for } x = y \\ t, & \text{for } x, y \in B,\ x \neq y \\ t - \zeta & \text{otherwise,} \end{cases}$$

is a metric on X and $h(\tau_B) = t$. ∎

References

[1] R. Engelking, General Topology, PWN, Warszawa, 1977 (in russian).

[2] T. Šalat, J. Toth, L. Zsilinszky, Metric space of metrics defined on a given set, *Real Anal. Exch.*, **18** No. 1 (1992/93), 225–231.

[3] T. Šalat, J. Toth, L. Zsilinszky, On structure of the space of metrics defined on a given set, *Real Anal. Exch.*, **19** No. 1 (1993/94), 321–327.

[4] R. W. Vallin, More on the metric space of metrics, *Real. Anal. Exch.*, **21** No. 2. (1995/96), 739–742.

University of Education,
Department of Mathematics,
Farská 19, 949 74 Nitra, Slovakia
E-mail: toth@unitra.sk

General solution of the differential equation $y''(x) - \left(y'(x)\right)^2 + x^2 e^{y(x)} = 0$

KRYSTYNA GRYTCZUK

Abstract. In this note we prove that the general solution of the differential equation $y''(x) - \left(y'(x)\right)^2 + x^2 e^{y(x)} = 0$, $x > 0$ is the function $y(x) = -\ln W(x)$, where $W(x) = \frac{1}{12}x^4 + Ax + B$ and A, B are arbitrary constants.

1. Introduction

In this note we prove that the general solution of the differential equation

$$(1) \qquad y''(x) - \left(y'(x)\right)^2 + x^2 e^{y(x)} = 0, \quad x > 0$$

is the function

$$(2) \qquad y(x) = -\ln W(x), \quad \text{where} \quad W(x) = \frac{1}{12}x^4 + Ax + B$$

and A, B are arbitrary constants. First, we note that such type of differential equations as (1) are difficult to solve. For example, E. Y. RODIN (see [1], p. 474, Unsolved problems, SIAM 81-17) posed the following problem. Find the general solution of the differential equation:

$$(3) \qquad y''(x) + x^2 e^{y(x)} = 0, \quad x > 0.$$

We prove that (1) has general solution given by (2), however we can't find the general solution of (3).

2. The Result

We prove the following theorem:

Theorem. *The general solution of the differential equation (1) is the function*

$$y(x) = \ln\left(\frac{1}{12}x^2 + Ax + B\right)$$

where A, B are arbitrary constants.

Proof. Putting $y(x) = \ln z(x)$ we obtain

$$y'(x) = \frac{z'(x)}{z(x)} \tag{4}$$

and consequently we have

$$y''(x) = \frac{z''(x)}{z(x)} - \left(\frac{z'(x)}{z(x)}\right)^2. \tag{5}$$

Since $y(x) = \ln z(x)$, then $e^{y(x)} = z(x)$ and by (4) and (5) it follows that (1) can be reduced to the following form:

$$\frac{z''(x)}{z^2(x)} - 2\frac{\left(z'(x)\right)^2}{z^3(x)} = -x^2. \tag{6}$$

Integrating (6) with respect to x we obtain:

$$\int \left(\frac{z''(x)}{z^2(x)} - 2\frac{\left(z'(x)\right)^2}{z^3(x)}\right) dx = -\frac{1}{3}x^3 + C_1.$$

Denote by

$$f\left(z(x)\right) = \frac{z''(x)}{z^2(x)} - 2\frac{\left(z'(x)\right)^2}{z^3(x)}. \tag{7}$$

Then we see that the function

$$\frac{z'(x)}{z^2(x)} = F\left(z(x)\right) \tag{8}$$

satisfies the following condition

$$F'\left(z(x)\right) = \left(\frac{z'(x)}{z^2(x)}\right)' = \frac{z''(x)}{z^2(x)} - 2\frac{\left(z'(x)\right)^2}{z^3(x)} = f\left(\left(z(x)\right)\right)$$

and therefore by (7) and (8) it follows that

$$\frac{z'(x)}{z^2(x)} = -\frac{1}{3}x^3 + C_1. \tag{9}$$

Integrating the last equality with respect to x we obtain

$$\int \frac{z'(x)}{z^2(x)}dx = -\frac{1}{12}x^4 + C_1x + C_2. \tag{10}$$

On the other hand it easy to see that

$$\left(-\frac{1}{z(x)}\right)' = \frac{z'(x)}{z^2(x)} \tag{11}$$

and consequently by (10) and (11) it follows that

$$-\frac{1}{z(x)} = -\frac{1}{12}x^4 + C_1x + C_2$$

and we have

$$y(x) = \ln z(x) = -\ln\left(\frac{1}{12}x^4 + Ax + B\right) = -\ln W(x)$$

where $A = -C_1$, $B = -C_2$. The proof of the theorem is complete.

References

[1] S. RABINOWITZ, Index to Mathematical Problems, Westford, Massachusetts, USA, 1992.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
ZIELONA GÓRA, UL. PODGÓRNA 50
POLAND

Módszertani cikkek
Methodological papers

A tanulók viszonya a matematika tantárgy tanulásához

DR. OROSZ GYULÁNÉ

Abstract. (The motivation system of the mathematics learning) The students of the Department of Mathematics study methodology. Our main purpose is to make our students teach mathematics the help of the given methods and make their lessons more interesting. For realising they have to know the motivation system of the mathematics learning. This paper is about our experiences. The structure of this paper is as follows: General thought about motivation, mathematics learning in practice conslusions about our experiment.

A motivációkutatások szakirodalma napjainkban szinte könyvtárnyi anyagot tesz ki. Dolgozatunkban nem vállalkozhatunk ezek ismertetésére.

A következőkben néhány fontos, munkánkhoz kapcsolódó elvi-elméleti jelentőségü hazai eredményt ismertetünk, a [15] alapján.

Grastyán Endre és munkatársai azokat az organikus alapokat tárták fel, amelyekre az ember specifikus jellegét, társadalmi mivoltát meghatározó funckciók épülnek. Feltevésükből a motivációs folyamat két ellentétes előjelű, egymást feltételező rendszer eredőjeként értelmezhető, mely ugyanazon funkció gátolt és gátolatlan változata.

Barkóczi és Putnoky a motívumot, illetve a motivációt gyűjtőfogalomként értelmezi, mely minden belső cselekvésre, viselkedésre késztető tényezőt magába foglal.

Kozéki Béla foglalkozott a motiváció pedagógiai-pszichológiai elméletének átfogó kimunkálásával. A motiváció pedagógiai-pszichológiai vizsgálatához sajátos modellt alkotott. Nézete szerint a motiválás területei: érzelmi kapcsolatok (affektív), értelmi ösztönzés (kognitív), morális (effektív) jellemzők.

Rókusfalvi Pál a teljesítmény motivációs összetevőinek elemzésével a rubinsteini felfogást fejleszti tovább.

Juhász Ferenc a motívumok fejlesztésének nevelési vonatkozásaival foglalkozik.

Egyes részterületek vizsgálatával foglalkoztak többek között: Surányi Gábor (a tanulás motivációs hátterét tárta fel), Békési Imre és Zsolnai József az anyanyelvi oktatás hatékonyságnövelő pedagógiai eljárássorozatot állított össze: Molnár Dezsőné (tantárgyi attitűdvizsgálatokat végzett).

Igen gazdag az oktató-nevelő munka hatékonyságának növelésére irányuló nevelési és oktatási kísérletek száma.

Réthy Endréné foglalkozik a tanulási motiváció kérdésének oktatáselméleti hátterű elemzésével. A tanulás és a motiváció kapcsolatának áttekintése alapján megállapítja a következőket: a tanulási motiváció a tanulási tevékenységre serkentő belső feszültség, melyet a környezet motiváló hatásának minősége és a tanulási tapasztalatok határoznak meg.

Tág és nem könnyű a motivációval, a matematika tanulásával kapcsolatos kutatások kérdésköre. Pszichológiai vizsgálatok igazolják, hogy jelenleg sok az elvi-elméleti tisztázatlanság a tanulási motiváció terén. Eddig még kevés a matematikatanítási-tanulási folyamat motivációs lehetőségeinek feltárásával foglalkozó kutatások száma. Ezért is fontos ezek hátterének elemzése, fejlesztő, motiváló eljárások kidolgozása.

Kérdőíves módszerrel vizsgáltuk a matematika tanulásának motivációs rendszerét, választ keresve arra, hogy a jelen gyakorlatban milyen tényezők befolyásolják legerősebben ennek hatékonyságát. Az egyén-környezet-interakció hatásrendszerében folyamatosan és fokozatosan alakul ki a tanulási motiváció rendszere. Vizsgálataink e rendszer hatékonyságának feltárására irányultak.

A vizsgálat módszere

Eger város általános iskoláiban 350 tanulót vizsgáltunk, akik 13-14 évesek. A kérdőívet a University of Lancester, Department of Educational Resarch (1975) kérdéseit alapul véve állítottuk össze, adaptálva a matematika tantárgyra. A kérdőív 24 kérdését a tanulóknak egy ötfokozatú skálán jelölve kellett megválaszolniuk. A kérdőívet az 1. mellékletben ismertetjük. A kérdések három fontos területre irányultak.

1. A tanulás érzelmi-szociális dimenziója területén intenzív befolyásoló tényező az iskola empátiás, identifikációs és affiliatív késztetésrendszere.

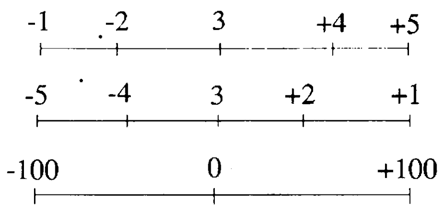
E területen az alábbi tényezőket vizsgáltuk: érzelmi viszony a matematika tantárgyhoz, a matematika tanulásához, a tanárokhoz, a tanulók teljesítménye.

2. A megismerési (kognitív) dimenziót tekintve a tanulók megismerési érdeklődését, aktivitását, kitartását, önállóságát elemeztük.

3. Az önintegrációs morális (effektív) dimenzióban a kötelességtudatot és az önértékelést vizsgáltuk.

Kérdéseinkkel arra kerestünk választ, hogy a matematikatanulási motivációra mely dimenzió hat a legerőteljesebben, s egy-egy dimenzión belül milyen e hatások rangsora. A feldolgozást az összehasonlító rangsorolás módszerével végeztük.

Minden kérdésnél, kérdéscsoportnál kiszámítottuk a transzformált átlagot, és ezt a számot tekintettük a rangsorolás mérőszámának csökkenő sorrendben. (A matematikai feldolgozást az EKTF számítástechnika szakos hallgatói végezték.) A kérdőíveken ötfokozatú diszkrét skála szerepelt, kérdésenként meg volt határozva az irányítottság, s minden skála alapja (semleges jellegű válasz) a középpont volt: Valamennyi skálát egy nulla bázisú, a $[-100, +100]$ intervallumra kiterjedő közös skálára transzformáltunk az alábbi leképezési séma szerint.



A transzformált átlag a transzformált alapadatok átlaga. A táblázatokban a csoportokat és kérdéseket a transzformált átlag csökkenő sorrendjében rendeztük.

A kérdésekhez négy adatot számoltunk ki:

transzformált átlag: a skálára vetített válaszok átlaga;

variancia: alapadatok varianciája (szórásnégyzete) a transzformált skála alapján;

szórás: a variancia négyzetgyöke;

átlag: alapadatok átlaga az eredeti skála szerint. (Az eltérő irányítottság miatt nem szerepel a táblázatokban).

A matematikai feldolgozás eredményének egy részét a következő oldalon lévő táblázatokban mutatjuk be.

Eredményrészlet és rövid elemzése

Az elkészített táblázatokat összehasonlítva megállapíthatjuk, hogy a matematika tanulásának motiváltságát legerősebben az érzelmi hatás befolyásolja, ezt követi az értelmi, végül az erkölcsi.

A táblázatok finomabb elemzésével egy-egy dimenzión belül feltárhatjuk a különböző hatások intenzitását is.

Az érzelmi dimenzión belül a legerősebb hatást a matematikaórák érdekessége jelenti. Ezt követi a matematikatanárhoz való viszony, majd a matematika tantárgyhoz való kötődés, s végül a teljesítmény és a szorongás.

Az empirikus vizsgálatok eredményei igen hasznosak lehetnek a matematikatanítás fejlesztő, motiváló modelljeinek kidolgozásához, amellyel a következő munkánkban szeretnénk foglalkozni.

1. A matematikaórák érdekessége Transzformált átlag: 38,75

Kérdés száma	Transzformált átlag	Variancia	Szórás
19.	53,50	25,60	5,06
13.	42,00	30,47	5,52
23.	38,50	19,80	4,45
12.	21,00	24,50	4,95

2. Viszony a matematikatanárhoz Transzformált átlag: 30,80

Kérdés száma	Transzformált átlag	Variancia	Szórás
9.	58,00	31,58	5,62
6.	50,50	24,60	4,96
5.	23,00	36,24	6,02
16.	16,00	26,42	5,14
20.	6,50	34,45	5,87

3. Viszony a matematika tantárgyhoz Transzformált átlag: 29,63

Kérdés száma	Transzformált átlag	Variancia	Szórás
3.	47,00	25,70	5,07
7.	31,00	46,92	6,85
4.	27,00	29,38	5,42
1.	13,50	12,46	3,53

4. Teljesítmény, szorongás Transzformált átlag: 27,00

Kérdés száma	Transzformált átlag	Variancia	Szórás
2.	42,50	38,94	6,24
10.	23,00	17,39	4,17
11.	22,00	41,99	6,48
24.	20,50	17,72	4,21

Melléklet

Név:

Iskola: Osztály:

Félévi osztályzatom matematikából:

Múlt év végi osztályzatom matematikából:

Húzd alá! Fiú Lány

A matematika tantárgyhoz kapcsolódó vizsgálathoz szeretnénk megtudni véleményeteket. Nincs jó vagy rossz válasz. A válaszadás úgy történik, hogy beírod azt a számot, amelyik legközelebb áll ahhoz, amit csinálsz vagy érzel.

1. egyáltalán nem
2. ritkán — alig
3. nem elég gyakran
4. gyakran
5. nagyon sokszor

1. Szívesen foglalkozol-e matematikával?
2. Fontos-e a matematikát tanulni?
3. Nehéz tantárgy-e a matematika?
4. Örülsz-e, ha megoldasz egy feladatot?
5. Kapsz-e a matematikatanártól külön feladatokat?
6. Örülsz-e, ha segít a tanár, ha nem bírkózol meg egy feladattal?
7. Szereted-e a matematikát?
8. Önállóan oldod-e meg a házi feladatot?
9. Segíted a matematikatanár munkáját az órán?
10. Elégedett vagy-e a matematikában elért eredményeddel?
11. Megteszel-e mindent, hogy jobb eredményt érj el?
12. Szívesen veszed-e, ha verseny van az órán?
13. Tetszenek-e a tréfás matematika feladatok?
14. Jársz-e matematika szakköre?
15. Voltál-e már matematikaversenyen?
16. Megkérdezed-e a tanártól, ha nem értesz valamit az órán?
17. Gyakorolsz-e, ha bizonytalan vagy valamiben?
18. Készülsz-e a matematika dolgozatokra?
19. Szereted-e a matematikai játékokat?
20. Kapsz-e dicséretet a matematikatanártól, ha jól dolgozol az órán?
21. Megoldod-e a szorgalmi feladatokat?
22. Bekapcsolódsz-e a matematika házi versenybe?
23. Kedveled-e az újszerű, szokatlan feladatokat?
24. Izgulsz-e a matematikaórákon?

Irodalom

[1] ATHINSON, I. W.—RAYNER, I. O.: Motivation and Achivement. Winston Sous, Washington, New York

[2] BÁBOSIK ISTVÁN: Személyiségformálás közvetett hatásokkal. Tankönyvkiadó, Budapest, 1982.

[3] DANYILOV, M. A.—BOLDIREV, N. I.: Pedagógiai metodológia és kutatásmódszertan. Tankönyvkiadó, Budapest, 1978.

[4] FALUS IVÁN: A tanári hatékonyságról és a tanárképzésről. *Pedagógiai Szemle*, 1972. 12. sz.

[5] FORRAI TIBORNÉ: Iskolai teljesítmény és szorongás. Akadémiai Kiadó, Budapest, 1968.

[6] HAJTMAN BÉLA: Bevezetés a matematikai statisztikába pszichológusok számára. Akadémiai Kiadó, Budapest, 1971.

[7] JUHÁSZ FERENC: A motiváció szerepe a nevelésben. Tankönyvkiadó, Budapest, 1969.

[8] KELEMEN LÁSZLÓ: Pedagógiai pszichológia. Tankönyvkiadó, Budapest, 1981.

[9] KISS ÁRPÁD: Mérés, értékelés, osztályozás. Korszerű nevelés. Tankönyvkiadó, Budapest, 1978.

[10] KOZÉKI BÉLA: A motiváció pedagógiai, pszichológiai fogalmáról. *Magyar Pszichológiai Szemle*, 1972. 3—4. sz.

[11] KOZÉKI BÉLA: Motiválás és motiváció. Tankönyvkiadó, Budapest, 1975.

[12] KOZÉKI BÉLA: A motiválás és a motiváció összefüggéseinek pedagógiai-pszichológiai vizsgálata. Akadémiai Kiadó, Budapest, 1980.

[13] NAGY SÁNDOR: Az oktatáselmélet alapkérdései. Tankönyvkiadó, Budapest, 1981.

[14] NAGY SÁNDOR, A tanulás pedagógiai kérdései. *OOK*, Veszprém, 1983.

[15] PIAGET, J.—FRAISSE, P.—REUCHLIN, M: A kísérleti pszichológia módszerei. Akadémiai Kiadó, Budapest, 1967.

[16] RÉTHY ENDRÉNÉ: A tanítás-tanulási folyamat motivációs lehetőségeinek elemzése. Akadémiai Kiadó, Budapest, 1988.

[17] RÉTHY ENDRÉNÉ: A tanítás-tanulás folyamat motivációs lehetőségeinek vizsgálata egy tantárgyi téma feldolgozása során. *Magyar Pedagógia*, 1974. 1. sz.

[18] RÉTHY ENDRÉNÉ: Az oktatási folyamat faktoranalízise. *Magyar Pedagógia*, 1978/a 3/4. sz.

[19] RÉTHY ENDRÉNÉ: Motiváció a tanítási órán. *Pedagógiai Közlemények* **19** Tankönyvkiadó, Budapest, 1978/b II.

[20] RÓKUSFALVY P.—STULLER GY.—KELEMENNÉ TÓTH É.: A pedagógusszemélyiség és tanárképzés. Tankönyvkiadó, Budapest, 1981.

[21] RUBINSTEIN, SZ. L.: Az általános pszichológia alapjai II. Akadémiai Kiadó, Budapest, 1964.

[22] SALAMON JENŐ: A Gelperin-féle „értelmi cselekvés" elmélete. *Pedagógiai Szemle*, 1966. 6. sz.

[23] SKEMP, P. R: A matematikatanulás pszichológiája. Gondolat, Budapest, 1975.

[24] SKINNER B. F.: A tanítás technológiája. Gondolat, Budapest, 1973.

DR. OROSZ GYULÁNÉ
ESZTERHÁZY KÁROLY TEACHERS' TRAINING COLLEGE
DEPARTMENT OF MATHEMATICS
LEÁNYKA U. 4.
3301 EGER, PF. 43.
HUNGARY

Vírusok a tanulók matematikai gondolkodásában

SZILÁK ALADÁRNÉ

Abstract. La skr bleciono demonstracias uni típan eraron en la matematika pensado de la lernantoj. Ofta eraro en la lecionosolvo, ke la lernantoj ne indikas ĉiû solvojn de la „duondiverĝaj" (diverĝaj) lecionoj. Per helpo de modelleciono sur ĝin demando serĉas respondon, kun kiaj metodoj eblas antaŭforigi, elimini la erarojn.

A matematika tantárgypedagógia a logikus (matematikai) gondolkodás fogalmát és nevelését igen összetetten fogalmazza meg. A korszerű matematikatanításban ezen gondolkodás elemei egyre hangsúlyozottabban jelennek meg. Erről tanúskodnak az utóbbi évek matematika tantervei és a NAT is. A Nemzeti alaptanterv külön tömbben (Gondolkodási módszerek) írja elő a matematikai gondolkodáshoz kapcsolódó tananyagot, az általános és speciális fejlesztési követelményeket. Több matematikus, tantárgypedagógus, didaktikus (Pólya György, Rubinstein, Nagy Sándor, Kelemen László, Mosonyi Kálmán, Czeglédy István) foglalkozott és foglalkozik ma is e fontos területen előforduló gondolkodási hibákkal (vírusokkal). Igen sok odafigyelést feltételez a „megelőzés" és a „gyógyítás" is: Egyrészt meg kellene találnunk az „okokat", másrészt olyan módszereket kellene kidolgoznunk, amelyek gátolnák a hibák létrejöttét, kialakulását. Nincs könnyű dolgunk, hiszen a gondolkodási hibák „tárháza" szinte kimeríthetetlen. E cikkben csupán egyetlen gondolkodási hibával szeretnénk foglalkozni részletesebben.

Többször tapasztaljuk a tanulók feladatmegoldásában azt az alapvető hiányosságot, hogy nem adják meg a feladat teljes (minden) megoldását. Megtalálnak egyet a lehetséges „eredmények" közül, és ezzel megelégedve befejezik a feladatot. Még a tehetséges tanulóknál is előfordul, hogy például matematikaversenyen azért veszítenek pontokat, mert nem hozzák a feladat minden eredményét. Az ilyen típusú hiba alapvető oka lehet a „féldivergens" (divergens) gondolkodás hiánya.

Az olyan feladatokat, amelyeknek egynél több, de véges számú megoldása van „féldivergens" feladatoknak nevezzük. (Az olyan feladatok, amelyeknek végtelen sok megoldása van divergensek.) Az ilyen típusú feladatokhoz kapcsolódó sajátos gondolkodás a „féldivergens" (divergens) gondolkodás, amely szoros összefüggésben van a tanulók kreativitásával is. E gondolkodás lényege az, hogy a feladat megoldása során minden esetet, minden lehetőséget meg kell vizsgálni. Hogyan lehet ezt elérni, azaz hogyan lehet valamennyi megoldást megtalálni? Erre egységes szabály nincs: Például valamilyen rendezőelv, egy algoritmus, diszkusszió, analógia, általánosítás, másféle megoldási módszer stb. is segíthet. Arra sem találunk általában utalást a feladatok szövegében, hogy több megoldás is van. (Természetesen ez nem hiba!)

A következő feladatsor mindegyik feladata „féldivergens", melyek megoldása során érdemes odafigyelni a címbeli felszólításra.

Keress meg minden megoldást!

1. Van-e olyan tízes számrendszerbeli háromjegyű szám, amelynek középső jegyét törölve, a belőle így nyert kétjegyű szám az eredetinek kilencede? (6. o.)

2. Egy háromszögről a következőket tudjuk:

— Oldalai hosszúságának mérőszámai egymást követő prímszámok.
— Kerületének mérőszáma 50-nél kisebb prímszám.

Mekkorák a háromszög oldalai? (7. o.)

3. Egy társaságban az angol, német, orosz nyelvek közül mindenki beszél legalább kettőt. Németül 24-en, angolul 26-an, oroszul 22-en tudnak:

a) Hány tagja van a társaságnak?
b) Hányan beszélnek németül és angolul; németül és oroszul; angolul és oroszul? (7—8. o.)

4. Szerkessz deltoidot, ha adott átlóinak és egyik oldalának hossza! (7. o.)

5. Összeadtunk néhány egymás után következő természetes számot, és eredményül 3000-et kaptunk. Mely számokat adtuk össze? (8. o.)

6. Szerkessz egyenlő szárú háromszöget, amelynek szárai 5 cm-esek és a szárakhoz tartozó magasság 2,5 cm! Mekkorák a háromszög szögei? (8. o.)

7. Egy négyzet alapú egyenes hasáb térfogatának és felszínének mérőszáma egyenlő. Minden él hossza egész szám. Add meg a hasáb adatait! (7. o.)

8. Szerkessz derékszögű háromszöget, ha adott az átfogójának hossza, továbbá tudjuk, hogy egyik szögének felezője úgy vágja ketté a háromszöget, hogy az egyik rész egyenlő szárú háromszög. (7—8. o.)

9. Egy térképvázlaton négy fa helyét jelölő pontok olyan rombuszt határoznak meg, amelynek egyik szöge $37,5°$, oldala pedig 6 cm. Szerkessz kör alakú utat, amely mindegyik fától egyenlő távolságra halad! (8. o.)

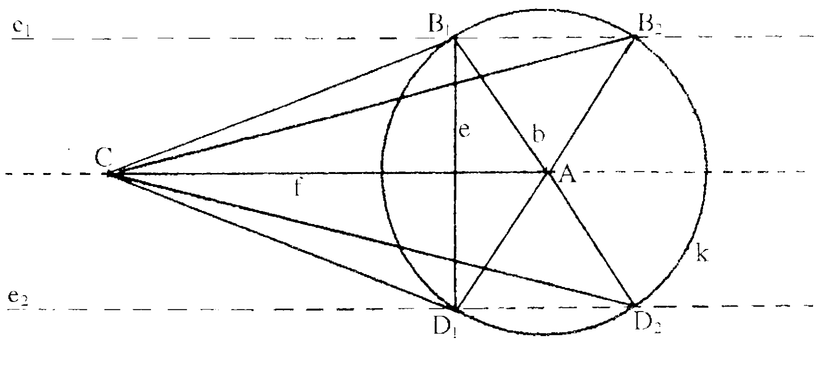
10. Adott egy egyenes és tőle 2 cm-re egy pont. Hol vannak azok a pontok, amelyek az egyenestől 4 cm-nél távolabb vannak, de a ponttól nincsenek messzebb 10 cm-nél? (6—7. o.)

A teljesség igénye nélkül nézzünk meg néhányat a fenti feladatok közül olyan szempontból, hogy a „féldivergens" gondolkodáshoz kapcsolódó hiányosságokat (gondolkodási hibákat) hogyan lehetne a tanulókkal kiküszöböltetni.

(a) Egy geometriai szerkesztési feladatnál a **diszkusszióban** szoktunk a megoldások számával (hány nem egybevágó, az adatoknak és a feltételeknek eleget tevő geometriai alakzat szerkeszthető) foglalkozni, mely általában az utolsó lépés. Ha már az összefüggések keresése és az elemzés közben is a szerkesztési alapelemeket (pl. pont, egyenes, kör) szintézisében (egészében) láttatjuk, és azok kölcsönös helyzetét is vizsgáljuk, akkor nagy valószínűséggel megtalálunk minden megoldást.

A 4. feladatban deltoidot kell szerkeszteni, ha adott az átlóinak (e, f) és egyik oldalának (b) hossza.

Vázlat:



Elemzés, összefüggések keresése:

— A deltoid $B_1(B_2, D_1, D_2)$ csúcsa az A-tól b távolságra van. (Azon tulajdonságú pontok halmaza, amelyek A-tól b távolságra vannak egy A középpontú b sugarú kör.)

— Az f átló egyenese a deltoid B_1 és D_1 (B_2 és D_2) csúcsaira illeszkedő párhuzamos egyenesek (e_1, e_2) középpárhuzamosa.

A fentiek alapján a szerkesztés lépései:

(1) Az $f(AC)$ szimmetriaátló felvétele.

(2) A középpontú b sugarú kör (k) szerkesztése.

(3) e_1 és e_2 szerkesztése.

(4) $k \cap e_1 = B_1$; $k \cap e_2 = D_1$; $k \cap e_1 = B_2$; $k \cap e_2 = D_2$.

(5) B_1 és D_1 összekötése A-val és C-vel (AB_1CD_1 konvex deltoid).

(6) B_2 és D_2 összekötése A-val és C-vel (AB_2CD_2 nem konvex deltoid)

Tipikus hibaként fordul elő a tanulók részéről, hogy a (2) és (3) lépést felcserélve (melyet természetesen meg lehet tenni) a körvonalnak csak egy részét (ívét) rajzolják meg, így a körnek az egyenesekkel egy-egy közös pontja lesz. A tanulók figyelme általában a konvex alakzatokra irányul, és így a nem konvex deltoid hiányozni fog a megoldásból.

Arra sem mindig gondolnak, hogy az e is lehet szimmetriaátló, és az előbbi szerkesztési lépéseket követve másik két deltoid is szerkeszthető. (Megjegyezzük, hogy itt most nem térünk ki az egybevágó megoldásokra és a speciális deltoidokra sem.)

Összegezve: a feladatnak 4 megoldása van (4 nem egybevágó deltoid szerkeszthető), és a szerkeszthetőség feltételei: ha f a szimmetriaátló, akkor $\frac{e}{2} < b$-nek kell teljesülni, ha pedig e a szimmetriaátló, akkor $\frac{f}{2} < b$ kell, hogy igaz legyen.

Természetesen a szerkesztés elvégezhető más összefüggések alkalmazásával is.

Hasonló gondolatmenet követhető a 6., 8., 9. feladatok megoldásakor.

(b) Több olyan feladat van, amelyet „ránézésből” is meg lehet oldani.

Ilyen például az 5. feladat. A 999, 1000, 1001 számokat adtuk össze (lehet a tanulók válasza). Ez viszont nem elég! Ahhoz, hogy a feladat minden megoldását megtaláljuk **általánosítanunk** kell a problémát:

$$m + (m+1) + (m+2) + \cdots + (m+k) = 3000.$$

A számtani sorozat összegének kiszámítására vonatkozó képletet alkalmazva a fenti összefüggést így írhatjuk:

$$\frac{m + (m+k)}{2}(k+1) = 3000.$$

Átalakítással az alábbi egyenlőséget kapjuk:

$$(2m + k)(k+1) = 6000.$$

A kéttényezős szorzat egyik tényezője páros, a másik páratlan, és $2m+k > k+1$.

A 6000 prímtényezős felbontásából ($6000 = 2^4 \cdot 3 \cdot 5^3$) a kapott feltételeket figyelembe véve a megoldások egy táblázatba felírhatók.

	$2m + k$	$k + 1$	m	k	Az összeadott számok
1.	2000	3	999	2	99, 1000, 1001
2.	1200	5	598	4	598, 599, ..., 602
3.	400	15	193	14	193, 194, ..., 207
4.	375	16	180	15	180, 181, ..., 195
5.	240	25	108	24	108, 109, ..., 132
6.	125	48	39	47	39, 40, ..., 86
7.	80	75	3	74	3, 4, ..., 77

Ha az 1., 3. és 7. feladatok megoldása során a fentihez hasonlóan általánosítunk, akkor biztosan megtalálunk minden megoldást.

(c) Az **algoritmizálás** is segíthet az összes megoldás megtalálásában, és a „féldivergens" gondolkodásmód kialakításában.

A 2. feladatot néhány évvel ezelőtt egy televíziós vetélkedőn részt vevő három tanuló közül egyik sem oldotta jól meg. Megtalálták ugyan a feladat feltételeinek megfelelő három számhármast, de külön-külön mindegyikük egyet-egyet. Célszerű lett volna a következő algoritmus lépései szerint eljárniuk:

(1) Írjuk fel az 50-nél kisebb prímszámokat!
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.
(2) $K = 2 + 3 + 5 = 10$ nem prímszám.
(3) $K = 3 + 5 + 7 = 15$ nem prímszám.
(4) $K = 5 + 7 + 11 = 23$ prímszám;
$5 + 7 > 11; 7 + 11 > 5; 5 + 11 > 7;$
az 5, 7, 11 számok lehetnek háromszög oldalainak mérőszámai.
(5) $K = 7 + 11 + 13 = 31$ prímszám;
$7 + 11 > 13$; $11 + 13 > 7$; $7 + 13 > 11$;
7, 11, 13 lehetnek a háromszög oldalainak mérőszámai.
(6) $K = 11 + 13 + 17 = 41$ prímszám;
$11 + 13 > 17$; $11 + 17 > 13$; $13 + 17 > 11$;
11, 13, 17 lehetnek a háromszög oldalainak mérőszámai.
(7) $K = 13 + 17 + 19 = 49$ nem prímszám.
(8) $K = 17 + 19 + 23 = 59$;
$59 > 50$, minden megoldást megtaláltunk.

A 3. és a 7. feladat egy-egy része megoldható algoritmus alkalmazásával is, mely az általánosításhoz képest más megoldási módszer.

(d) Korábban már szóltunk arról, hogy a több eredményre történő utalást általában a feladatok szövege nem tartalmazza.

Az 1. feladat „van-e olyan" kérdése például mintha egyetlen eredmény felé irányítaná a gondolkodásunkat. Hívjuk fel a tanulók figyelmét az így megfogalmazott feladatoknál arra, hogy ha találnak egy megoldást, akkor is keressenek további — a feltételeknek megfelelő — számokat, mert az összes szám jelenti a feladat teljes megoldását.

A 10. feladathoz hasonló példáknál gyakori hiba, hogy a megoldást csak a sík pontjaira adják meg, és nem gondolnak a térbeli megoldásokra. Ilyen feladatok esetében az **analógia** (sík-tér) segíti a „féldivergens" gondolkodást.

Összefoglalva: A tanulók gondolkodásának hibáit szinte lehetetlen differenciálni, mert amilyen összetett a logikus, matematikai gondolkodás, olyan összetettek a hibák is. A „féldivergens" (divergens) gondolkodást és más gondolkodási módszereket, műveleteket — mint láttuk — nem lehet egymástól elválasztani. Hangsúlyozottabban odafigyelhetünk bizonyos „vírusokra", és ha sikerül némi eredményeket elérni (például a fenti feladatok minden megoldását megtalálják a tehetségesebb tanulóink), akkor elégedettek lehetünk.

Irodalom

[1] Dr. Czeglédy István—Dr. Orosz Gyuláné—Dr. Szalontai Tibor—Sziläk Aladárné: Matematika tantárgypedagógia I., Calibra Kiadó, Budapest, 1994.

[2] Újvári I.: Matematikai gondolkodást fejlesztő feladatsorok, Pest Megyei Pedagógiai Intézet, Budapest, 1990.

[3] Matematika 7—8. (Feladatgyűjtemény), Szerkesztette: Hajdu Sándor, Tankönyvkiadó, Budapest, 1990.

Sziläk Aladárné
Eszterházy Károly Teachers' Training College
Department of Mathematics
Leányka u. 4.
3301 Eger, Pf. 43.
Hungary

Tartalom (Contents)

Molnár és Társa '2001' Kft. Nyomda Eger Tel.: 36/422-946