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GENERALIZATION OF NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING 1-POINTS

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Abstract

In this paper, we generalize two theorems on the uniqueness of nonlinear differential polynomials sharing 1-points, which improves a result of Lahiri and Pal [7].

1. Introduction, Definitions and Main Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let k be a positive integer or infinity and $a \in \{\infty\} \cup \mathbb{C}$. We denote by $E_{k)}(a; f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. If for some $a \in \{\infty\} \cup \mathbb{C}$, $\mathbb{E}_{\infty)}(a, f) = E_{\infty)}(a; g)$ we say that f, g share the value a CM (counting multiplicities).

In [4], the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points was studied.

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Regarding the nonlinear differential polynomials the following question was asked in [4]: What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM? Some works have already been done in this direction [1, 2, 8, 9]. Recently Fang and Fang [2] and Lin and Yi [9] proved the following result.

Theorem A. *Let f and g be two nonconstant meromorphic functions and* $n\,(\geq 13)$ *be an integer. If* $f^n(f-1)^2 f'$ *and* $g^n(g-1)^2 g'$ *share the value* 1 *CM, then* $f \equiv g$.

In 2006, Lahiri and Pal [7] investigated the uniqueness problem of meromorphic functions when two nonlinear differential polynomials share the value 1 and proved the following two theorems, the first of which improves Theorem A.

Theorem B. *Let f and g be two nonconstant meromorphic functions and* $n\,(\geq 13)$ *be an integer. If* $E_{3)}(1;\, f^n(f-1)^2 f') = E_{3)}(1;\, g^n(g-1)^2 g')$, *then* $f \equiv g$.

Theorem C. *Let f and g be two nonconstant meromorphic functions and* $n\,(\geq 14)$ *be an integer. If* $E_{3)}(1;\, f^n(f^3-1) f') = E_{3)}(1;\, g^n(g^3-1) g')$, *then* $f \equiv g$.

In this paper, we generalize and improve Theorems A, B and C and obtain the following results.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions and* $n\,(\geq m+11)$ *be an integer. If*

$$E_{3)}(1;\, f^n(f-1)^m f') = E_{3)}(1;\, g^n(g-1)^m g'),$$

then $f \equiv g$.

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions and* $n\,(\geq m+11)$ *be an integer. If*

$$E_{3)}(1;\ f^{n}(f^{m}-1)f') = E_{3)}(1;\ g^{n}(g^{m}-1)g'),$$

then $f \equiv g$.

Remark. (1) If $m = 2$ in Theorem 1.1, then Theorem 1.1 reduces to Theorems A and B.

(2) If $m = 3$ in Theorem 1.2, then Theorem 1.2 reduces to Theorem C.

Though for the standard notations and definition of value distribution theory we refer [3], in the following definition we explain a notation used in the paper.

Definition 1.1. Let *f* be a meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer *p* we denote by $N_p(r, a;\ f)$ the counting function of *a*-points of *f*, where an *a*-point of multiplicity *m* is counted *m* times if $m \leq p$ and is counted *p* times if $m > p$.

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. *Let f and g be two nonconstant meromorphic functions. Then* $f^{n}(f-1)^{m} f' g^{n}(g-1)^{m} g' \not\equiv 1$, *where n is an integer.*

Proof. If possible let $f^{n}(f-1)^{m} f' g^{n}(g-1)^{m} g' \equiv 1$. Let z_0 be an 1-point of *f* with multiplicity $p\,(\geq 1)$. Then z_0 is a pole of *g* with multiplicity $q\,(\geq 1)$ such that

$$mp + p - 1 = (n + m + 1)q + 1 \geq n + m + 2$$

and so $p \geq \dfrac{n + m + 3}{m + 1}$.

Let z_1 be a zero of *f* with multiplicity $p\,(\geq 1)$ and it be a pole of *g* with multiplicity $q\,(\geq 1)$. Then

$$np + p - 1 = nq + mq + q + 1,$$

i.e.,

$$(n+1)(p-q) = mq + 2.$$

Hence $p \geq \frac{n+m-1}{m}$. Since a pole of f is either a zero of $g(g-1)$ or a zero of g', we get

$$\begin{aligned}\overline{N}(r, \infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, 1; g) + \overline{N}_0(r, 0; g') \\ &\leq \frac{m}{n+m-1} N(r, 0; g) + \frac{m+1}{n+m+3} N(r, 1; g) + \overline{N}(r, 0; g') \\ &\leq \left(\frac{m}{n+m-1} + \frac{m+1}{n+m+3}\right) T(r, g) + \overline{N}_0(r, 0; g'),\end{aligned}$$

where $\overline{N}_0(r, 0; g')$ is the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)$.

By the second fundamental theorem, we obtain

$$\begin{aligned}T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 1; f) - \overline{N}_0(r, 0; f') + S(r, f) \\ &\leq \frac{m}{n+m-1} N(r, 0; f) + \frac{m+1}{n+m+3} N(r, 1; f) \\ &\quad + \left(\frac{m}{n+m-1} + \frac{m+1}{n+m+3}\right) T(r, g) \\ &\quad + \overline{N}_0(r, 0; g') - \overline{N}_0(r, 0; f') + S(r, f),\end{aligned}$$

i.e.,

$$\begin{aligned}\left(1 - \frac{m}{n+m-1} - \frac{m+1}{n+m+3}\right) T(r, f) &\leq \left(\frac{m}{n+m-1} + \frac{m+1}{n+m+3}\right) T(r, g) \\ &\quad + \overline{N}_0(r, 0; g') - \overline{N}_0(r, 0; f') \\ &\quad + S(r, f). \qquad (2.1)\end{aligned}$$

Similarly, we get

$$\left(1-\frac{m}{n+m-1}-\frac{m+1}{n+m+3}\right)T(r,\ g) \le \left(\frac{m}{n+m-1}+\frac{m+1}{n+m+3}\right)T(r,\ f)$$
$$+\ \overline{N}_0(r,\ 0;\ f') - \overline{N}_0(r,\ 0;\ g')$$
$$+\ S(r,\ g). \tag{2.2}$$

Adding (2.1) and (2.2), we get

$$\left(1-\frac{2m}{n+m-1}-\frac{2(m+1)}{n+m+3}\right)\{T(r,\ f)+T(r,\ g)\} \le S(r,\ f)+S(r,\ g),$$

which is a contradiction. This proves the lemma.

Lemma 2.2 [10]. *Let f be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, *where* $a_0, a_1, a_2, \ldots, a_n$ *are constants and* $a_m \neq 0$. *Then*

$$T(r,\ P(f)) = nT(r,\ f) + S(r,\ f).$$

Lemma 2.3. *Let*

$$F = f^{n+1}\left[\frac{m_{c_0}}{n+m+1} f^m - \frac{m_{c_1}}{n+m} f^{n+m-1} + \cdots + (-1)^m \frac{1}{n+1}\right],$$

$$G = g^{n+1}\left[\frac{m_{c_0}}{n+m+1} g^m - \frac{m_{c_1}}{n+m} g^{n+m-1} + \cdots + (-1)^m \frac{1}{n+1}\right],$$

where $n\,(> m+3)$ *is an integer. Then* $F' \equiv G'$ *implies* $F \equiv G$.

Proof. Let $F' \equiv G'$. Then $F \equiv G + c$, where c is a constant. If possible, let $c \neq 0$. Then by the second fundamental theorem, we get

$$T(r,\ F) \le \overline{N}(r,\ 0;\ F) + \overline{N}(r,\ c;\ F) + \overline{N}(r,\ \infty;\ F) + S(r,\ F)$$
$$\le \overline{N}(r,\ 0;\ f) + \overline{N}(r,\ \infty;\ f) + \overline{N}\left(r,\ \frac{m_{c_0}}{n+m+1};\ f^m\right)$$

$$+\ \overline{N}(r,\ 0;\ g) + \overline{N}\left(r,\ \frac{m_{c_0}}{n+m+1};\ g^m\right) + S(r,\ f)$$

$$\leq 2T(r,\ f) + mT(r,\ f) + T(r,\ g) + mT(r,\ g) + S(r,\ f).$$

Since by Lemma 2.2,

$$T(r,\ F) = (n+m+1)T(r,\ f) + S(r,\ f),$$

it follows that

$$(n+m+1)T(r,\ f) \leq (2+m)T(r,\ f) + (m+1)T(r,\ g) + S(r,\ g). \quad (2.3)$$

Similarly, we get

$$(n+m+1)T(r,\ g) \leq (2+m)T(r,\ g) + (m+1)T(r,\ f) + S(r,\ f). \quad (2.4)$$

Adding (2.3) and (2.4), we obtain

$$(n-m-2)\{T(r,\ f) + T(r,\ g)\} \leq S(r,\ f) + S(r,\ g)$$

which is a contradiction. So $c = 0$ and the lemma is proved.

Lemma 2.4 [9]**.** *Let F and G be given as in Lemma* 2.3. *Then* $F \equiv G$ *implies* $f \equiv g$.

Lemma 2.5 [6]**.** *Let f, g are nonconstant meromorphic functions and* $E_{3)}(1;\ f) = E_{3)}(1;\ g)$ *then one of the following cases holds*:

(i) $T(r,\ f) + T(r,\ g) \leq 2\{N_2(r,\ 0;\ f) + N_2(r,\ 0;\ g) + N_2(r,\ \infty;\ f)$

$$+\ N_2(r,\ \infty;\ g)\} + S(r,\ f) + S(r,\ g);$$

(ii) $f \equiv g$;

(iii) $fg \equiv 1$.

Lemma 2.6 [5]**.** *Let f be a nonconstant meromorphic function and k be a positive integer*. *Then*

$$N_2(r,\ 0;\ f^{(k)}) \leq k\overline{N}(r,\ \infty;\ f) + N_{2+k}(r,\ 0;\ f) + S(r,\ f).$$

Lemma 2.7. *Let F and G be given as in Lemma* 2.3. *Then*

(i) $T(r, F) \leq T(r, F') + N(r, 0; f) + N(r, b_1; f) + N(r, b_2; f)$

$$+ \cdots + N(r, b_m; f) - N(r, c_1; f) - N(r, c_2; f)$$

$$- \cdots - N(r, c_m; f) - N(r, 0; f') + S(r, f);$$

(ii) $T(r, G) \leq T(r, G') + N(r, 0; g) + N(r, b_1; g) + N(r, b_2; g)$

$$+ \cdots + N(r, b_m; g) - N(r, c_1; g) - N(r, c_2; g)$$

$$- \cdots - N(r, c_m; g) - N(r, 0; g') + S(r, g).$$

Proof. By the Nevanlinna's first fundamental theorem and Lemma 2.2, we get

$$T(r, F) = T\left(r, \frac{1}{F}\right) + O(1)$$

$$= N(r, 0; F) + m\left(r, \frac{1}{F}\right) + O(1)$$

$$\leq N(r, 0; F) + m\left(r, \frac{F'}{F}\right) + m(r, 0; F') + O(1)$$

$$= T(r, F') + N(r, 0; F) - N(r, 0; F') + S(r, F)$$

$$\leq T(r, F') + N(r, 0; f) + N(r, b_1; f) + N(r, b_2; f)$$

$$+ \cdots + N(r, b_m; f) - N(r, c_1; f) - N(r, c_2; f)$$

$$- \cdots - N(r, c_m; f) - N(r, 0; f') + S(r, f).$$

Similarly, we get $T(r, G)$.

This proves the lemma.

Lemma 2.8. *Let f and g be two nonconstant meromorphic functions. Then* $f^n(f^m - 1) f' g^n (g^m - 1) g' \not\equiv 1$, *where n is a positive integer*.

Proof. If possible let $f^n(f^m - 1) f' g^n (g^m - 1) g' \equiv 1$. Let z_0 be a 1-point of f with multiplicity p. Then z_0 is a pole of g with multiplicity q, say, such that $(m-1)p-1=(n+m+1)q+1 \geq n+m+2$, i.e., $p \geq \frac{n+m+3}{m-1}$.

Hence $\Theta(1; f) > 1-\frac{m-1}{n+m+3}$.

Similarly, we can now show that

$$\Theta(\omega; f) \geq 1-\frac{m-1}{n+m+3}$$

and

$$\Theta(\omega^2; f) \geq 1-\frac{m-1}{n+m+3},$$

where ω is the imaginary cube root of unity.

Therefore

$$\Theta(1; f)+\Theta(\omega; f)+\Theta(\omega^2; f) \geq 3-\frac{3(m-1)}{n+m+3} > 2,$$

a contradiction. This proves the lemma.

Lemma 2.9. *Let*

$$F_1 = f^{n+1}\left[\frac{f^m}{n+m+1}-\frac{1}{n+1}\right],$$

$$G_1 = g^{n+1}\left[\frac{g^m}{n+m+1}-\frac{1}{n+1}\right],$$

where $n \geq 2$ *is an integer. If* $F_1 \equiv G_1$, *then* $f \equiv g$.

Proof. Let $h=\frac{g}{f}$. If possible, suppose that h is nonconstant. Since $F_1 \equiv G_1$, it follows that

$$f^m = \frac{n+m+1}{n+1} \cdot \frac{h^{n+1}-1}{h^{n+m+1}-1}.$$

Since f^m has no simple pole, it follows that $h - u_k = 0$ has no simple root for $k = 1, 2, ..., n+3,$ where $u_k = e^{\left(\frac{2\pi ik}{n+m+1}\right)}.$

Hence $\Theta(u_k; h) > \frac{1}{2}$ for $k = 1, 2, ..., n+3,$ which is impossible. Therefore h is a constant. If $h \neq 1,$ it follows that f is a constant, which is not the case. So $h = 1$ and hence $f \equiv g.$ This proves the lemma.

Lemma 2.10. *If* F_1 *and* G_1 *be defined as in Lemma* 2.9. *Then*

$$\text{(i)}\ T(r, F_1) \leq T(r, F_1') + N(r, 0; f) + N\left(r, \frac{n+m+1}{n+1}; f^m\right)$$

$$- N(r, 1; f^m) - N(r, 0; f') + S(r, f),$$

$$\text{(ii)}\ T(r, G_1) \leq T(r, G_1') + N(r, 0; g) + N\left(r, \frac{n+m+1}{n+1}; g^m\right)$$

$$- N(r, 1; g^m) - N(r, 0; g') + S(r, g).$$

The lemma can be proved in the line of the proof of Lemma 2.7.

Lemma 2.11. *Let* F_1 *and* G_1 *be defined as in Lemma* 2.9, *where* $n (\geq m+2)$ *is an integer. Then* $F_1' \equiv G_1'$ *implies* $F_1 \equiv G_1.$ *The proof is similar to that of Lemma* 2.3.

3. Proof of the Theorems

In this section, we present the proofs of the main results.

Proof of Theorem 1.1. Let F and G be defined as in Lemma 2.3. If possible, suppose that

$$T(r, F') + T(r, G') \leq 2\{N_2(r, 0; F') + N_2(r, 0; G') + N_2(r, \infty; F')$$

$$+ N_2(r, \infty; G')\} + S(r, F') + S(r, G').$$

Then by Lemmas 2.2, 2.6 and 2.7, we get

$$
\begin{aligned}
& T(r, F) + T(r, G) \\
\le\; & T(r, F') + N(r, 0; f) + N(r, b_1; f) + N(r, b_2; f) \\
& + \cdots + N(r, b_m; f) - N(r, c_1; f) - N(r, c_2; f) \\
& - \cdots - N(r, c_m; f) - N(r, 0; f') + T(r, G') \\
& + N(r, 0; g) + N(r, b_1; g) + N(r, b_2; g) + \cdots + N(r, b_m; g) \\
& - N(r, c_1; g) - N(r, c_1; g) - \cdots - N(r, c_m; g) - N(r, 0; g') \\
\le\; & 2\{N_2(r, 0; F') + N_2(r, 0; G') + N_2(r, \infty; F') + N_2(r, \infty; G')\} \\
& + N(r, 0; f) + N(r, b_1; f) + N(r, b_2; f) + \cdots + N(r, b_m; f) \\
& - N(r, c_1; f) - N(r, c_2; f) - \cdots - N(r, c_m; f) - N(r, 0; f') \\
& + N(r, 0; g) + N(r, b_1; g) + N(r, b_2; g) + \cdots + N(r, b_m; g) \\
& - N(r, c_1; g) - N(r, c_2; g) - \cdots - N(r, c_m; g) - N(r, 0; g') \\
& + S(r, f) + S(r, g) \\
\le\; & 4\overline{N}(r, 0; f) + 2N(r, 0; (f-1)^m) + 2N_2(r, 0; f') + 4\overline{N}(r, 0; g) \\
& + 2N(r, 0; (g-1)^m) + 2N_2(r, 0; g') + 4\overline{N}(r, \infty; f) \\
& + 4\overline{N}(r, \infty; g) + N(r, 0; f) + N(r, b_1; f) \\
& + N(r, b_2; f) + \cdots + N(r, b_m; f) - N(r, c_1; f) - N(r, c_2; f) \\
& - \cdots - N(r, c_m; f) - N(r, 0; f') + N(r, 0; g) + N(r, b_1; g) \\
& + N(r, b_2; g) + \cdots + N(r, b_m; g) - N(r, c_1; g) - N(r, c_2; g) \\
& - \cdots - N(r, c_m; g) - N(r, 0; g') + S(r, f) + S(r, g),
\end{aligned}
$$

$$(n+m+1)T(r,\ f)$$

$$\le 11T(r,\ f)+2mT(r,\ f)+11T(r,\ g)+2mT(r,\ g)+S(r,\ f)+S(r,\ g)$$

$$\le (11+2m)T(r,\ f)+(11+2m)T(r,\ g)+S(r,\ f)+S(r,\ g).$$

So by Lemma 2.2, we get

$$(n-m-10)\{T(r,\ f)+T(r,\ g)\}\le S(r,\ f)+S(r,\ g),$$

which is a contradiction.

Hence by Lemma 2.5 either $F'\equiv G'$ or $F'G'\equiv 1.$ Since by Lemma 2.1 $F'G'\not\equiv 1,$ it follows by Lemma 2.3 and Lemma 2.4 $f\equiv g.$ This proves the theorem.

Proof of Theorem 1.2. Let F_1 and G_1 be defined as in Lemma 2.9. If possible suppose that

$$T(r,\ F_1')+T(r,\ G_1')\le 2\{N_2(r,\ 0;\ F_1')+N_2(r,\ 0;\ G_1')+N_2(r,\ \infty;\ F_1')$$

$$+N_2(r,\ \infty;\ G_1')\}+S(r,\ F_1')+S(r,\ G_1').$$

Then by Lemmas 2.2, 2.6 and 2.10, we get

$$T(r,\ F_1)+T(r,\ G_1)$$

$$\le 2\{N_2(r,\ 0;\ F_1')+N_2(r,\ 0;\ G_1')+N_2(r,\ \infty;\ F_1')+N_2(r,\ \infty;\ G_1')\}$$

$$+N(r,\ 0;\ f)+N\left(r,\ \frac{n+m+1}{n+1};\ f^m\right)-N(r,\ 0;\ f')$$

$$-N(r,1;\ f^m)+N(r,\ 0;\ g)+N\left(r,\ \frac{n+m+1}{n+1};\ g^m\right)$$

$$-N(r,\ 0;\ g')-N(r,1;\ g^m)+S(r,\ f)+S(r,\ g)$$

$$\le 4N(r,\ 0;\ f)+2N_2(r,1;\ f^m)+2N_2(r,\ 0;\ f')$$

$$+4N(r,\ 0;\ g)+2N_2(r,1;\ g^m)+2N_2(r,\ 0;\ g')$$

$$+ 4\overline{N}(r, \infty; f) + 4\overline{N}(r, \infty; g) + N(r, 0; f)$$

$$+ N\left(r, \frac{n+m+1}{n+1}; f^m\right) - N(r, 1; f^m) - N(r, 0; f')$$

$$+ N(r, 0; g) + N\left(r, \frac{n+m+1}{n+1}; g^m\right) - N(r, 1; g^m)$$

$$- N(r, 0; g') + S(r, f) + S(r, g)$$

$$\leq (11 + 2m)T(r, f) + (11 + 2m)T(r, g) + S(r, f) + S(r, g)$$

and so by Lemma 2.2, we get

$$(n - m - 10)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which is a contradiction.

Hence by Lemma 2.5 either $F_1' \equiv G_1'$ or $F_1'G_1' \equiv 1$. Since by Lemma 2.8 $F_1'G_1' \not\equiv 1$, it follows by Lemmas 2.9 and 2.11 that $f \equiv g$. This proves the theorem.

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