



## Arithmetic properties of $\ell$ -regular overpartition pairs

Megadahalli Siddanaika Mahadeva NAIKA\*, Chandrappa SHIVASHANKAR

Department of Mathematics, Bangalore University, Central College Campus, Bangalore, Karnataka, India

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**Abstract:** In this paper, we investigate the arithmetic properties of  $\ell$ -regular overpartition pairs. Let  $\overline{B}_\ell(n)$  denote the number of  $\ell$ -regular overpartition pairs of  $n$ . We will prove the number of Ramanujan-like congruences and infinite families of congruences modulo 3, 8, 16, 36, 48, 96 for  $\overline{B}_3(n)$  and modulo 3, 16, 64, 96 for  $\overline{B}_4(n)$ . For example, we find that for all nonnegative integers  $\alpha$  and  $n$ ,  $\overline{B}_3(3^\alpha(3n+2)) \equiv 0 \pmod{3}$ ,  $\overline{B}_3(3^\alpha(6n+4)) \equiv 0 \pmod{3}$ , and  $\overline{B}_4(8n+7) \equiv 0 \pmod{64}$ .

**Key words:** Congruences, theta function, overpartition pair, regular partition

### 1. Introduction

An overpartition of a nonnegative integer  $n$  is a nonincreasing sequence of natural numbers whose sum is  $n$ , and where the first occurrence of a number may be overlined. For example, eight overpartitions of the integer 3 are  $3$ ,  $\overline{3}$ ,  $2+1$ ,  $\overline{2}+1$ ,  $2+\overline{1}$ ,  $\overline{2}+\overline{1}$ ,  $1+1+1$ , and  $\overline{1}+1+1$ . Due to Corteel and Lovejoy [9], the generating function of overpartition is:

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^n)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + 40q^6 + \dots \quad (1.1)$$

where  $\overline{p}(n)$  denotes the number of overpartitions of  $n$ . Many mathematicians extensively studied the arithmetic properties of  $\overline{p}(n)$ . For more information, see [12, 13, 17, 18, 23, 31].

An overpartition pair of a positive integer  $n$  was defined by Lovejoy [21] as a pair of overpartitions  $(\mu, \lambda)$  where the sum of all the parts is  $n$ . Lovejoy denoted the number of overpartition pairs of a positive integer  $n$  by  $\overline{pp}(n)$ ; for convenience, define  $\overline{pp}(0) = 1$ . The generating function for overpartition pairs is

$$\sum_{n=0}^{\infty} \overline{pp}(n)q^n = \prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2} = 1 + 4q + 12q^2 + 32q^3 + 76q^4 + 168q^5 + 352q^6 + \dots \quad (1.2)$$

Bringmann and Lovejoy [6] defined a rank for overpartition pairs to study the congruence properties of  $\overline{pp}(n)$  and they found many Ramanujan-type congruences for  $\overline{pp}(n)$ . For example,  $\overline{pp}(3n+2) \equiv 0 \pmod{3}$ . For more details on arithmetic properties of overpartition pairs one can see [8, 15, 19].

\*Correspondence: [mmsnaika@rediffmail.com](mailto:mmsnaika@rediffmail.com)

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For a positive integer  $\ell > 1$ , a partition is called  $\ell$ -regular if none of the parts are divisible by  $\ell$ . Let  $b_\ell(n)$  denote the number of  $\ell$ -regular partitions of  $n$ . The generating function for  $b_\ell(n)$  satisfies

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}, \tag{1.3}$$

where

$$f_k := (q^k; q^k)_\infty \quad \text{and} \quad (a; q)_\infty := \prod_{n=1}^{\infty} (1 - aq^{n-1}), \quad |q| < 1.$$

The regular partitions have been studied by a number of mathematicians; see [10, 14, 16, 24, 27, 30].

Recently, Lovejoy [22] studied the function  $\overline{A}_\ell(n)$ , which counts the number of overpartitions of  $n$  with no parts divisible by  $\ell$ , and Andrews [2] defined a new function  $\overline{C}_{k,i}(n)$  as the number of overpartitions of  $n$  in which no part is divisible by  $k$  and only parts  $\equiv \pm i \pmod{k}$  may be overlined. The generating function of  $\overline{C}_{k,i}(n)$  is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k; q^k)_\infty (-q^i; q^k)_\infty (-q^{k-i}; q^k)_\infty}{(q; q)_\infty}, \tag{1.4}$$

for  $k \geq 3$  and  $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ . With a note that  $\overline{A}_3(n) = \overline{C}_{3,1}(n)$ . Andrews proved the following two congruences:

$$\overline{C}_{3,1}(9n + 3) \equiv \overline{C}_{3,1}(9n + 6) \equiv 0 \pmod{3}. \tag{1.5}$$

Chen et al. [7] and Ahmed and Baruah [1] proved many infinite families of congruences of  $\overline{C}_{k,i}(n)$  for different values of  $k$  and  $i$ .

Following these works, more recently, Shen [25] proved a number of arithmetic properties of  $\overline{A}_\ell(n)$  and gave a combinatorial interpretation for  $\overline{A}_3(9n + 3)$  and  $\overline{A}_3(9n + 6)$  being divisible by 3. Shen called  $\overline{A}_\ell(n)$  the number of  $\ell$ -regular overpartitions of  $n$ . The generating function of  $\overline{A}_\ell(n)$  is

$$\sum_{n=0}^{\infty} \overline{A}_\ell(n)q^n = \frac{f_2}{f_1^2} \frac{f_\ell^2}{f_{2\ell}} = \frac{\varphi(-q^\ell)}{\varphi(-q)}. \tag{1.6}$$

In this paper, we shall study the arithmetic properties of  $\ell$ -regular overpartition pairs of  $n$ . An  $\ell$ -regular overpartition pair of  $n$  is a pair of  $\ell$ -regular overpartitions  $(\mu, \lambda)$  where the sum of all the parts is  $n$ . We have the generating function

$$\sum_{n=0}^{\infty} \overline{B}_\ell(n)q^n = \frac{f_2^2}{f_1^4} \frac{f_\ell^4}{f_{2\ell}^2} = \frac{\varphi(-q^\ell)^2}{\varphi(-q)^2}, \tag{1.7}$$

where  $\overline{B}_\ell(n)$  denotes the number of  $\ell$ -regular overpartitions of  $n$ .

We will prove the number of arithmetic properties of the partition function  $\overline{B}_\ell(n)$  for  $\ell = 3, 4$ .

**2. Preliminary results**

We define Ramanujan’s general theta function  $f(a, b)$  for  $|ab| < 1$  as

$$\begin{aligned}
 f(a, b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\
 &= (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}.
 \end{aligned}
 \tag{2.1}$$

The important cases of  $f(a, b)$  are

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{f_2^5}{f_1^2 f_4^2},
 \tag{2.2}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{f_2^2}{f_1},
 \tag{2.3}$$

$$f(-q) := f(-q^2, -q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_{\infty},
 \tag{2.4}$$

and

$$\chi(-q) := (q; q^2)_{\infty} = \frac{f_1}{f_2}.
 \tag{2.5}$$

The following dissection formula of Hirschhorn and Sellers [13] plays an important role in our work.

**Lemma 2.1** *We have*

$$\frac{1}{\varphi(-q)} = \frac{\varphi(-q^9)}{\varphi(-q^3)^4} (\varphi(-q^9)^2 + 2q\varphi(-q^9)A(q^3) + 4q^2A(q^3)^2)
 \tag{2.6}$$

$$= \frac{1}{\varphi(-q^4)^4} (\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3),
 \tag{2.7}$$

where

$$A(q) = \frac{f_1 f_6^2}{f_2 f_3}.$$

**Lemma 2.2** *We have*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9),
 \tag{2.8}$$

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}),
 \tag{2.9}$$

$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2.
 \tag{2.10}$$

**Proof** For the proofs of (2.8) and (2.9), see [5, p. 49, Corollary]. Adding identities (v) and (vi) of [5, p. 40, Entry 25.], we can easily get (2.10). □

**Lemma 2.3** [3] For  $1 \leq k \leq 7$ , we have

$$r_k(8n + k) = 2^{k-1} \left( 2 + \binom{k}{4} \right) t_k(n), \tag{2.11}$$

where  $r_k(n)$  denotes the number of representations of  $n$  as sum of  $k$  squares, and  $t_k(n)$  denotes the number of representations of  $n$  as the sum of  $k$  triangular numbers for nonnegative integers  $n$  and  $k$ .

**Lemma 2.4** The following 2-dissection holds:

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.12}$$

Xia and Yao [29] proved (2.12) by employing an addition formula for theta functions.

**Lemma 2.5** The following 2-dissections hold:

$$\frac{f_3^3}{f_1^3} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}, \tag{2.13}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}. \tag{2.14}$$

**Proof** Hirschhorn et al. [11] proved equation (2.13). In the same paper the authors also proved

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q \frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \tag{2.15}$$

Replacing  $q$  by  $-q$  in (2.15) and using  $(-q; -q)_\infty = \frac{f_2^3}{f_1 f_4}$ , we obtain (2.14). □

### 3. Congruence results for $\overline{B}_3(n)$

In this section, we establish several congruences for  $\overline{B}_3(n)$ .

**Lemma 3.1** We have

$$\sum_{n=0}^{\infty} \overline{B}_3(3n)q^n = \frac{\varphi(-q^3)^6}{\varphi(-q)^6} + 16q \frac{\varphi(-q^3)^3}{\varphi(-q)^6} A(q)^3, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{B}_3(3n + 1)q^n = 4 \frac{\varphi(-q^3)^5}{\varphi(-q)^6} A(q) + 16q \frac{\varphi(-q^3)^2}{\varphi(-q)^6} A(q)^4, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \overline{B}_3(3n + 2)q^n = 12 \frac{\varphi(-q^3)^4}{\varphi(-q)^6} A(q)^2. \tag{3.3}$$

**Proof** Setting  $\ell = 3$  in (1.7),

$$\sum_{n=0}^{\infty} \overline{B}_3(n)q^n = \frac{f_2^2 f_3^4}{f_1^4 f_6^2} = \frac{\varphi(-q^3)^2}{\varphi(-q)^2}. \tag{3.4}$$

Substituting (2.6) into (3.4), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{B}_3(n)q^n &= \frac{\varphi(-q^9)^2}{\varphi(-q^3)^6} (\varphi(-q^9)^2 + 2q\varphi(-q^9)A(q^3) + 4q^2A(q^3)^2)^2 \\ &= \frac{\varphi(-q^9)^2}{\varphi(-q^3)^6} (\varphi(-q^9)^4 + 4q^2\varphi(-q^9)^2A(q^3)^2 + 16q^4A(q^3)^4 \\ &\quad + 4q\varphi(-q^9)^3A(q^3) + 8q^2\varphi(-q^9)^2A(q^3)^2 + 16q^3\varphi(-q^9)A(q^3)^3). \end{aligned}$$

The lemma follows by extracting the terms involving  $q^{3n+j}$  for  $j = 0, 1, 2$ . □

**Theorem 3.1** For all integers  $n \geq 0$  and  $\alpha \geq 0$ ,

$$\overline{B}_3(3^{\alpha+1}n) \equiv \overline{B}_3(n) \pmod{3}, \tag{3.5}$$

$$\overline{B}_3(3^\alpha(3n + 2)) \equiv 0 \pmod{3}, \tag{3.6}$$

$$\overline{B}_3(3^\alpha(6n + 4)) \equiv 0 \pmod{3}, \tag{3.7}$$

$$\overline{B}_3(3^\alpha(8n + 4)) \equiv 0 \pmod{3}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} \overline{B}_3(6n + 1)q^n \equiv f_1f_3 \pmod{3}. \tag{3.9}$$

**Proof** For any prime  $p$ , we have

$$(q; q)^p \equiv (q^p; q^p) \pmod{p}. \tag{3.10}$$

It is easy to see that

$$\varphi(-q)^3 \equiv \varphi(-q^3) \pmod{3}. \tag{3.11}$$

In view of (3.4) and (3.11), we see that

$$\sum_{n=0}^{\infty} \overline{B}_3(n)q^n \equiv \frac{\varphi(-q)^6}{\varphi(-q)^2} \equiv \varphi(-q)\varphi(-q^3) \pmod{3}. \tag{3.12}$$

Employing (2.9) in (3.12), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(n)q^n \equiv \varphi(-q^3)\varphi(-q^9) - 2q\varphi(-q^3)f(-q^3, -q^{15}) \pmod{3}, \tag{3.13}$$

which implies that

$$\overline{B}_3(3n + 2) \equiv 0 \pmod{3}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} \overline{B}_3(3n)q^n \equiv \varphi(-q)\varphi(-q^3) \pmod{3}, \tag{3.15}$$

and

$$\sum_{n=0}^{\infty} \overline{B}_3(3n + 1)q^n \equiv \varphi(-q)f(-q, -q^5) \pmod{3}. \tag{3.16}$$

Congruences (3.5) and (3.6) follow from (3.12), (3.14), and (3.15) and by induction on  $\alpha$ .

From [4, Eq. (3.17)], we have

$$\varphi(-q)f(-q, -q^5) = \psi(q^2)f(q^2, q^4) - 3q \frac{f_{12}^3}{f_4}. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\sum_{n=0}^{\infty} \bar{B}_3(3n+1)q^n \equiv \psi(q^2)f(q^2, q^4) \pmod{3},$$

which yields

$$\bar{B}_3(6n+4) \equiv 0 \pmod{3}$$

and

$$\sum_{n=0}^{\infty} \bar{B}_3(6n+1)q^n \equiv \psi(q)f(q, q^2) \pmod{3}. \tag{3.18}$$

However,

$$f(q, q^2) = \frac{f_2 f_3^2}{f_1 f_6} \tag{3.19}$$

and for any integer  $m \geq 1$ , we have

$$f_{3m} \equiv f_m^3 \pmod{3}. \tag{3.20}$$

Combining (2.3), (3.18), and (3.19), we see that

$$\sum_{n=0}^{\infty} \bar{B}_3(6n+1)q^n \equiv \frac{f_2^3 f_3^2}{f_1^2 f_6} \pmod{3}. \tag{3.21}$$

Congruence (3.9) follows from (3.20) and (3.21).

Again, from equation (3.12), we have

$$\sum_{n=0}^{\infty} \bar{B}_3(n)q^n \equiv \varphi(-q)^4 \equiv \sum_{n=0}^{\infty} (-1)^n r_4(n)q^n \pmod{3}, \tag{3.22}$$

which implies that

$$\bar{B}_3(n) \equiv (-1)^n r_4(n) \pmod{3}.$$

Utilizing (2.11) in the above equation with  $k = 4$ , we obtain

$$\bar{B}_3(8n+4) \equiv 0 \pmod{3}. \tag{3.23}$$

Congruence (3.8) follows from (3.23) and (3.5). □

**Theorem 3.2** For any prime  $p \equiv 5 \pmod{6}$ ,  $\alpha \geq 1$ , and  $n \geq 0$ , we have

$$\overline{B}_3(6p^{2\alpha}n + (6i + p)p^{2\alpha-1}) \equiv 0 \pmod{3}, \tag{3.24}$$

where  $i = 1, 2, \dots, p - 1$ .

**Proof** Utilizing (2.4), we can rewrite (3.9) as

$$\sum_{n=0}^{\infty} \overline{B}_3(6n + 1)q^n \equiv \sum_{k,m=-\infty}^{\infty} (-1)^{k+m} q^{k(3k+1)/2+3m(3m+1)/2} \pmod{3}. \tag{3.25}$$

Consider the congruence equation

$$\frac{3k^2 + k}{2} + 3 \cdot \frac{3m^2 + m}{2} \equiv \frac{4p^2 - 4}{24} \pmod{p}, \tag{3.26}$$

which is equivalent to

$$(6k + 1)^2 + 3 \cdot (6m + 1)^2 \equiv 0 \pmod{p}.$$

Since  $\left(\frac{-3}{p}\right) = -1$  for  $p \equiv 5 \pmod{6}$ , we deduce that

$$6k + 1 \equiv 6m + 1 \equiv 0 \pmod{p}.$$

If  $p \equiv 1 \pmod{6}$ , then  $k \equiv m \equiv \frac{p-1}{6} \pmod{p}$ . Letting

$$k = rp + \frac{p-1}{6} \quad \text{and} \quad m = sp + \frac{p-1}{6},$$

we have

$$\frac{3k^2 + k}{2} + 3 \cdot \frac{3m^2 + m}{2} = \frac{p^2 - 1}{6} + p^2 \left(\frac{3r^2 + r}{2}\right) + 3p^2 \left(\frac{3s^2 + s}{2}\right).$$

If  $p \equiv -1 \pmod{6}$ , then  $k \equiv m \equiv \frac{-p-1}{6} \pmod{p}$ . Letting

$$k = -rp - \frac{p+1}{6} \quad \text{and} \quad m = -sp - \frac{p+1}{6},$$

we have

$$\frac{3k^2 + k}{2} + 3 \cdot \frac{3m^2 + m}{2} = \frac{p^2 - 1}{6} + p^2 \left(\frac{3r^2 + r}{2}\right) + 3p^2 \left(\frac{3s^2 + s}{2}\right).$$

By the above analysis, extracting the terms in which the powers of  $q$  are  $pn + \frac{p^2-1}{6}$  from (3.25) and dividing the resulting equation by  $q^{\frac{p^2-1}{6}}$ , we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(6pn + p^2)q^{pn} \equiv \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{p^2(3r^2+r)/2+3p^2(3s^2+s)/2} \pmod{3},$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(6p^2n + p^2)q^n \equiv f_1 f_3 \pmod{3}, \tag{3.27}$$

and for  $n \geq 0$ ,

$$\overline{B}_3(6p^2n + 6pi + p^2) \equiv 0 \pmod{3}, \tag{3.28}$$

where  $i$  is an integer and  $1 \leq i \leq p - 1$ . Combining (3.27) and (3.9), we see that for  $n \geq 0$ ,

$$\overline{B}_3(6p^2n + p^2) \equiv \overline{B}_3(6n + 1) \pmod{3}. \tag{3.29}$$

By (3.29) and mathematical induction, we deduce that for  $n \geq 0$  and  $\alpha \geq 0$ ,

$$\overline{B}_3(6p^{2\alpha}n + p^{2\alpha}) \equiv \overline{B}_3(6n + 1) \pmod{3}. \tag{3.30}$$

Replacing  $n$  by  $p^2n + pi + \frac{p^2-1}{6}$  in (3.30) and using (3.28), we deduce that for  $n \geq 0$  and  $\alpha \geq 0$ ,

$$\overline{B}_3(6p^{2\alpha+2}n + 6p^{2\alpha+1}i + p^{2\alpha+2}) \equiv 0 \pmod{3}. \tag{3.31}$$

Congruence (3.24) follows from (3.31). □

**Theorem 3.3** For all nonnegative integers  $\alpha$  and  $n$ ,

$$\overline{B}_3(6 \cdot 4^{\alpha+1}n + 2 \cdot 4^{\alpha+1}) \equiv \overline{B}_3(6n + 2) \pmod{36}, \tag{3.32}$$

$$\overline{B}_3(6 \cdot 4^{\alpha+1}n + 5 \cdot 4^{\alpha+1}) \equiv 0 \pmod{36}. \tag{3.33}$$

**Proof** Using definitions of  $\varphi(-q)$  and  $A(q)$ , we can rewrite (3.3) as

$$\sum_{n=0}^{\infty} \overline{B}_3(3n + 2)q^n = 12 \frac{f_2^4 f_3^6}{f_1^{10}}. \tag{3.34}$$

Applying (3.20) in (3.34), we see that

$$\sum_{n=0}^{\infty} \overline{B}_3(3n + 2)q^n \equiv 12 \frac{f_2^4 f_3^3}{f_1} \pmod{36}.$$

Substituting (2.13) into the above equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(3n + 2)q^n \equiv 12 \frac{f_2^2 f_4^3 f_6^2}{f_{12}} + 12q \frac{f_2^4 f_{12}^3}{f_4} \pmod{36},$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(6n + 2)q^n \equiv 12 \frac{f_1^2 f_2^3 f_3^2}{f_6} \equiv 12 \frac{f_3^3}{f_1} \pmod{36}. \tag{3.35}$$

Substituting (2.13) into (3.35), we find that

$$\sum_{n=0}^{\infty} \overline{B}_3(6n + 2)q^n \equiv 12 \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + 12q \frac{f_{12}^3}{f_4} \pmod{36}, \tag{3.36}$$



which yields

$$\sum_{n=0}^{\infty} \overline{B}_3(12n + 8)q^n \equiv 12 \frac{f_6^3}{f_2} \pmod{36},$$

which implies that

$$\overline{B}_3(24n + 8) \equiv \overline{B}_3(6n + 2) \pmod{36} \tag{3.37}$$

and

$$\overline{B}_3(24n + 20) \equiv 0 \pmod{36}. \tag{3.38}$$

Congruences (3.32) and (3.33) follow from (3.37), (3.38), and induction on  $\alpha$ . □

**Theorem 3.4** For any prime  $p \equiv 5 \pmod{6}$ ,  $\alpha \geq 1$ , and  $n \geq 0$ , we have

$$\overline{B}_3(12p^{2\alpha}n + (12i + 2p)p^{2\alpha-1}) \equiv 0 \pmod{36}, \tag{3.39}$$

where  $i = 1, 2, \dots, p - 1$ .

**Proof** Utilizing (3.20) in (3.36), we find that

$$\sum_{n=0}^{\infty} \overline{B}_3(12n + 2)q^n \equiv 12f_1f_3 \pmod{36}. \tag{3.40}$$

We omit the remaining part of the proof because it is exactly similar to the proof of (3.24). □

**Theorem 3.5** For all integers  $\alpha \geq 0$  and  $n \geq 0$ ,

$$\overline{B}_3(3 \cdot 4^{\alpha+1}n + 11 \cdot 4^\alpha) \equiv 0 \pmod{48}, \tag{3.41}$$

$$\overline{B}_3(24n + 14) \equiv 0 \pmod{48}, \tag{3.42}$$

$$\overline{B}_3(24n + 23) \equiv 0 \pmod{96}, \tag{3.43}$$

$$\overline{B}_3(48n + 14) \equiv 0 \pmod{96}. \tag{3.44}$$

**Proof** By mathematical induction it is easy to see that

$$f_{2m} \equiv f_m^2 \pmod{2}, \tag{3.45}$$

$$f_{2m}^2 \equiv f_m^4 \pmod{2^2}, \tag{3.46}$$

$$f_{2m}^4 \equiv f_m^8 \pmod{2^3}. \tag{3.47}$$

Employing (3.47) in (3.34), we see that

$$\sum_{n=0}^{\infty} \overline{B}_3(3n + 2)q^n \equiv 12 \frac{f_3^6}{f_1^2} \equiv 12 \sum_{n=0}^{\infty} A_3(n)q^n \pmod{96}, \tag{3.48}$$

which implies that

$$\overline{B}_3(3n + 2) \equiv 12A_3(n) \pmod{96}, \tag{3.49}$$

where  $A_3(n)$  denotes the number of bipartitions with 3-cores of  $n$ . Congruence (3.41) follows from Theorem 2.3 of [20]. Congruence (3.43) follows from Theorem 2.6 (Eq. 2.16) of [20]. Finally, congruences (3.42) and (3.44) follow from Theorem 1.5 of [28].  $\square$

**Theorem 3.6** For all nonnegative integers  $\alpha$  and  $n$ ,

$$\overline{B}_3(4 \cdot 3^{2\alpha+2}n + 3^{2\alpha+2}) \equiv \overline{B}_3(4n + 1) \pmod{2^4}, \tag{3.50}$$

$$\overline{B}_3(4 \cdot 3^{2\alpha+2}n + 7 \cdot 3^{2\alpha+1}) \equiv 0 \pmod{2^4}, \tag{3.51}$$

$$\overline{B}_3(4 \cdot 3^{2\alpha+2}n + 11 \cdot 3^{2\alpha+1}) \equiv 0 \pmod{2^4}. \tag{3.52}$$

**Proof** Setting  $\ell = 3$  in (1.7), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(n)q^n = \frac{f_2^2 f_3^4}{f_1^4 f_6^2}. \tag{3.53}$$

Substituting (2.12) into (3.53), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{B}_3(n)q^n &= \frac{f_2^2}{f_6^2} \left( \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \right)^2 \\ &= \frac{f_4^8 f_{12}^4}{f_2^8 f_8^2 f_{24}^2} + 4q^2 \frac{f_4^2 f_6^2 f_8^2 f_{24}^2}{f_2^6 f_{12}^2} + 4q \frac{f_4^5 f_6 f_{12}}{f_2^7}. \end{aligned} \tag{3.54}$$

Extracting the terms involving  $q^{2n+1}$  from both sides of the above equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(2n + 1)q^n = 4 \frac{f_2^5 f_3 f_6}{f_1^7}, \tag{3.55}$$

but

$$\frac{f_2^5 f_3 f_6}{f_1^7} \equiv \frac{f_2^3 f_3 f_6}{f_1^3} \pmod{2^2}. \tag{3.56}$$

Combining (3.55) and (3.56), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(2n + 1)q^n \equiv 4 \frac{f_2^3 f_3 f_6}{f_1^3} \pmod{2^4}. \tag{3.57}$$

Substituting (2.14) into (3.57), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(2n + 1)q^n \equiv 4 \frac{f_4^6 f_6^4}{f_2^6 f_{12}^2} + 12q \frac{f_4^2 f_6^2 f_{12}^2}{f_2^4} \pmod{2^4}, \tag{3.58}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(4n + 1)q^n \equiv 4 \frac{f_2^6 f_3^4}{f_1^6 f_6^2} \pmod{2^4}, \tag{3.59}$$

but

$$\frac{f_2^6 f_3^4}{f_1^6 f_6^2} \equiv \frac{f_2^4}{f_1^2} \equiv \psi(q)^2 \pmod{2^2}. \tag{3.60}$$

In view of (3.59) and (3.60), we have

$$\sum_{n=0}^{\infty} \bar{B}_3(4n+1)q^n \equiv 4\psi(q)^2 \pmod{2^4}. \tag{3.61}$$

Substituting (2.8) into (3.61), we obtain

$$\sum_{n=0}^{\infty} \bar{B}_3(4n+1)q^n \equiv 4f(q^3, q^6)^2 + 4q^2\psi(q^9)^2 + 8qf(q^3, q^6)\psi(q^9) \pmod{2^4}, \tag{3.62}$$

which implies that

$$\sum_{n=0}^{\infty} \bar{B}_3(12n+9)q^n \equiv 4\psi(q^3)^2 \pmod{2^4}, \tag{3.63}$$

which yields

$$\bar{B}_3(36n+9) \equiv \bar{B}_3(4n+1) \pmod{2^4}, \tag{3.64}$$

$$\bar{B}_3(36n+21) \equiv 0 \pmod{2^4}, \tag{3.65}$$

and

$$\bar{B}_3(36n+33) \equiv 0 \pmod{2^4}. \tag{3.66}$$

Congruences (3.50), (3.51), and (3.52) follow from (3.64), (3.65), (3.66), and induction on  $\alpha$ . □

**Theorem 3.7** *For any prime  $p \equiv 3 \pmod{4}$ ,  $\alpha \geq 1$ , and  $n \geq 0$ , we have*

$$\bar{B}_3(12p^{2\alpha}n + (12i+p)p^{2\alpha-1}) \equiv 0 \pmod{2^4}, \tag{3.67}$$

where  $i = 1, 2, \dots, p-1$ .

**Proof** Extracting the terms involving  $q^{3n}$  from both sides of (3.62), we have

$$\sum_{n=0}^{\infty} \bar{B}_3(12n+1)q^n \equiv 4f(q, q^2)^2 \pmod{2^4}, \tag{3.68}$$

but

$$f(q, q^2)^2 = \frac{f_2^2 f_3^4}{f_1^2 f_6^2} \equiv f_1^2 \pmod{4}. \tag{3.69}$$

Combining (3.68) and (3.69), we see that

$$\sum_{n=0}^{\infty} \bar{B}_3(12n+1)q^n \equiv 4f_1^2 \pmod{2^4}. \tag{3.70}$$

Utilizing (2.4), we can rewrite (3.70) as

$$\sum_{n=0}^{\infty} \bar{B}_3(12n+1)q^n \equiv 4 \sum_{k,m=-\infty}^{\infty} (-1)^{k+m} q^{k(3k+1)/2+m(3m+1)/2} \pmod{2^4}. \tag{3.71}$$

Consider the congruence equation

$$\frac{3k^2+k}{2} + \frac{3m^2+m}{2} \equiv \frac{p^2-1}{12} \pmod{p}, \tag{3.72}$$

which is equivalent to

$$(6k+1)^2 + (6m+1)^2 \equiv 0 \pmod{p}.$$

Since  $(\frac{-1}{p}) = -1$  for  $p \equiv 3 \pmod{4}$ , we deduce that

$$6k+1 \equiv 6m+1 \equiv 0 \pmod{p}.$$

If  $p \equiv 1 \pmod{6}$ , then  $k \equiv m \equiv \frac{p-1}{6} \pmod{p}$ . Letting

$$k = rp + \frac{p-1}{6} \quad \text{and} \quad m = sp + \frac{p-1}{6},$$

we have

$$\frac{3k^2+k}{2} + \frac{3m^2+m}{2} = \frac{p^2-1}{12} + p^2 \left( \frac{3r^2+r}{2} \right) + p^2 \left( \frac{3s^2+s}{2} \right).$$

If  $p \equiv -1 \pmod{6}$ , then  $k \equiv m \equiv \frac{-p-1}{6} \pmod{p}$ . Letting

$$k = -rp - \frac{p+1}{6} \quad \text{and} \quad m = -sp - \frac{p+1}{6},$$

we have

$$\frac{3k^2+k}{2} + \frac{3m^2+m}{2} = \frac{p^2-1}{12} + p^2 \left( \frac{3r^2+r}{2} \right) + p^2 \left( \frac{3s^2+s}{2} \right).$$

By the above analysis, extracting the terms in which the powers of  $q$  are  $pn + \frac{p^2-1}{12}$  from (3.71) and dividing the resulting equation by  $q^{\frac{p^2-1}{12}}$ , we obtain

$$\sum_{n=0}^{\infty} \bar{B}_3(12pn+p^2)q^{pn} \equiv 4 \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{p^2(3r^2+r)/2+p^2(3s^2+s)/2} \pmod{2^4},$$

which implies that

$$\sum_{n=0}^{\infty} \bar{B}_3(12p^2n+p^2)q^n \equiv 4f_1^2 \pmod{2^4}, \tag{3.73}$$

and for  $n \geq 0$ ,

$$\bar{B}_3(12p^2n+12pi+p^2) \equiv 0 \pmod{2^4}, \tag{3.74}$$

where  $i$  is an integer and  $1 \leq i \leq p - 1$ . Combining (3.73) and (3.70), we see that for  $n \geq 0$ ,

$$\overline{B}_3(12p^2n + p^2) \equiv \overline{B}_3(12n + 1) \pmod{2^4}. \tag{3.75}$$

By (3.75) and mathematical induction, we deduce that for  $n \geq 0$  and  $\alpha \geq 0$ ,

$$\overline{B}_3(12p^{2\alpha}n + p^{2\alpha}) \equiv \overline{B}_3(12n + 1) \pmod{2^4}. \tag{3.76}$$

Replacing  $n$  by  $p^2n + pi + \frac{p^2-1}{6}$  in (3.76) and using (3.74), we deduce that for  $n \geq 0$  and  $\alpha \geq 0$ ,

$$\overline{B}_3(12p^{2\alpha+2}n + 12p^{2\alpha+1}i + p^{2\alpha+2}) \equiv 0 \pmod{2^4}. \tag{3.77}$$

Congruence (3.67) follows from (3.77). □

**Theorem 3.8** For all nonnegative integers  $\alpha$  and  $n$ :

$$\overline{B}_3(4 \cdot 3^{\alpha+1}n + 2 \cdot 3^{\alpha+1}) \equiv \overline{B}_3(4n + 2) \pmod{2^3}, \tag{3.78}$$

$$\overline{B}_3(4 \cdot 3^{\alpha+1}n + 10 \cdot 3^\alpha) \equiv 0 \pmod{2^3}, \tag{3.79}$$

$$\overline{B}_3(48n + 2) \equiv \begin{cases} 4 \pmod{2^3}, & \text{if } n = P_k, \\ 0 \pmod{2^3}, & \text{otherwise.} \end{cases} \tag{3.80}$$

$$\overline{B}_3(12(4n + i) + 2) \equiv 0 \pmod{2^3}, \tag{3.81}$$

where  $i = 1, 2, 3$  and  $P_k$  is either of the generalized pentagonal numbers  $k(3k \pm 1)/2$ .

**Proof** From equation (3.54), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(2n)q^n = \frac{f_2^8 f_6^4}{f_1^8 f_4^2 f_{12}^2} + 4q \frac{f_2^2 f_3^2 f_4^2 f_{12}^2}{f_1^6 f_6^2}.$$

Applying (3.45) and (3.47) in the above equation, we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(2n)q^n \equiv \frac{f_2^4 f_6^4}{f_4^2 f_{12}^2} + 4q \frac{f_4^2 f_{12}^2}{f_2 f_6} \pmod{2^3},$$

which implies that

$$\sum_{n=0}^{\infty} \overline{B}_3(4n + 2)q^n \equiv 4 \frac{f_2^2 f_6^2}{f_1 f_3} \equiv 4\psi(q)\psi(q^3) \pmod{2^3}. \tag{3.82}$$

By substituting (2.8) into (3.82), we find that

$$\sum_{n=0}^{\infty} \overline{B}_3(4n + 2)q^n \equiv 4(f(q^3, q^6)\psi(q^3) + q\psi(q^3)\psi(q^9)) \pmod{2^3}, \tag{3.83}$$

which yields

$$\overline{B}_3(12n + 6) \equiv \overline{B}_3(4n + 2) \pmod{2^3}, \tag{3.84}$$

$$\overline{B}_3(12n + 10) \equiv 0 \pmod{2^3}. \tag{3.85}$$

Congruences (3.78) and (3.79) follow from (3.84), (3.85), and induction on  $\alpha$ .

Again, from (3.83), we have

$$\sum_{n=0}^{\infty} \overline{B}_3(12n + 2)q^n \equiv 4\psi(q)f(q, q^2) \pmod{2^3}. \tag{3.86}$$

Using the definitions of  $\psi(q)$  and  $f(q, q^2)$ , we see that

$$\psi(q)f(q, q^2) = \frac{f_2^3 f_3^2}{f_1^2 f_6} \equiv f_4 \pmod{2}. \tag{3.87}$$

By combining (3.86) and (3.87), we obtain

$$\sum_{n=0}^{\infty} \overline{B}_3(12n + 2)q^n \equiv 4f_4 \pmod{2^3}. \tag{3.88}$$

Congruences (3.80) and (3.81) follow from (3.88). □

#### 4. Congruence results for $\overline{B}_4(n)$

In this section, we prove several congruences for  $\overline{B}_4(n)$ .

**Theorem 4.1** *For any nonnegative integer  $n$ :*

$$\overline{B}_4(12n + 2) \equiv 0 \pmod{3}, \tag{4.1}$$

$$\overline{B}_4(12n + 4) \equiv 0 \pmod{3}, \tag{4.2}$$

$$\overline{B}_4(12n + 8) \equiv 0 \pmod{3}, \tag{4.3}$$

$$\overline{B}_4(12n + 10) \equiv 0 \pmod{3}, \tag{4.4}$$

$$\overline{B}_4(12n + 7) \equiv 0 \pmod{96}, \tag{4.5}$$

$$\overline{B}_4(12n + 11) \equiv 0 \pmod{96}, \tag{4.6}$$

$$\overline{B}_4(36n + 15) \equiv 0 \pmod{96}, \tag{4.7}$$

$$\overline{B}_4(36n + 27) \equiv 0 \pmod{96}. \tag{4.8}$$

**Proof** Setting  $\ell = 4$  in (1.7), we have

$$\sum_{n=0}^{\infty} \overline{B}_4(n)q^n = \frac{f_2^2 f_4^4}{f_1^4 f_8^2} = \frac{\varphi(-q^4)^2}{\varphi(-q)^2}. \tag{4.9}$$

Substituting (2.7) into (4.9), we see that

$$\sum_{n=0}^{\infty} \overline{B}_4(n)q^n = \frac{1}{\varphi(-q^4)^6} (\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3)^2.$$

After simplification, choosing the terms for which the powers of  $q$  are the form  $q^{4n+i}$  for  $i = 0, 1, 2, 3$ , we find that

$$\sum_{n=0}^{\infty} \overline{B}_4(4n)q^n = \frac{\varphi(q)^6}{\varphi(-q)^6} + 48q \frac{\varphi(q)^2\psi(q^2)^4}{\varphi(-q)^6}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+1)q^n = 4 \frac{\varphi(q)^5\psi(q^2)}{\varphi(-q)^6} + 64q \frac{\varphi(q)\psi(q^2)^5}{\varphi(-q)^6}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+2)q^n = 12 \frac{\varphi(q)^4\psi(q^2)^2}{\varphi(-q)^6} + 64q \frac{\psi(q^2)^6}{\varphi(-q)^6}, \tag{4.12}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+3)q^n = 32 \frac{\varphi(q)^3\psi(q^2)^3}{\varphi(-q)^6}. \tag{4.13}$$

With aid of (3.10), we have

$$\sum_{n=0}^{\infty} \overline{B}_4(4n)q^n \equiv \frac{\varphi(q^3)^2}{\varphi(-q^3)^2} \pmod{3}, \tag{4.14}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+2)q^n \equiv q \frac{\psi(q^6)^2}{\varphi(-q^3)^2} \pmod{3}, \tag{4.15}$$

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+3)q^n \equiv 32 \frac{\varphi(q^3)\psi(q^6)}{\varphi(-q^3)^2} \pmod{96}. \tag{4.16}$$

Equating the coefficients of  $q^{3n+1}$  and  $q^{3n+2}$  of (4.14), we obtain the congruences (4.2) and (4.3). Extracting the terms containing  $q^{3n}$  and  $q^{3n+2}$  from (4.15), we arrive at (4.1) and (4.4). Similarly, equating the coefficients of  $q^{3n+1}$  and  $q^{3n+2}$  of (4.16), we obtain the congruences (4.5) and (4.6).

Again from (4.16), we have

$$\sum_{n=0}^{\infty} \overline{B}_4(12n+3)q^n \equiv 32 \frac{\varphi(q)\psi(q^2)}{\varphi(-q)^2} \equiv 32 \frac{f_6^2}{f_3^2} \pmod{96}.$$

Congruences (4.7) and (4.8) follow from the above equation. □

**Theorem 4.2** For all integers  $\alpha \geq 0$  and  $n \geq 0$ ,

$$\overline{B}_4(8n+7) \equiv 0 \pmod{64}, \tag{4.17}$$

$$\overline{B}_4(8 \cdot 7^{2\alpha}n + 3 \cdot 7^{2\alpha}) \equiv \overline{B}_4(8n+3) \pmod{64}, \tag{4.18}$$

$$\overline{B}_4(8 \cdot 7^{2\alpha+2}n + j \cdot 7^{2\alpha+1}) \equiv 0 \pmod{64}, \tag{4.19}$$

where  $j \in \{5, 13, 29, 37, 45, 53\}$ .

**Proof** By definitions of  $\varphi(q)$  and  $\psi(q)$ , we can rewrite (4.13) as

$$\sum_{n=0}^{\infty} \overline{B}_4(4n+3)q^n = 32 \frac{f_2^{18}}{f_1^{18}}. \tag{4.20}$$

From (3.10) with  $p = 2$ , we have

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(4n+3)}{32} q^n \equiv f_2^9 \pmod{2}. \tag{4.21}$$

This yields congruence (4.17) by equating  $q^{2n+1}$ . Again, by equating  $q^{2n}$  from both sides of (4.21), we see that

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8n+3)}{32} q^n \equiv f_1^9 \equiv \left(\frac{f_2^2}{f_1}\right)^3 \pmod{2}. \tag{4.22}$$

Combining (2.3) and (4.22), we have

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8n+3)}{32} q^n \equiv \psi(q)^3 \pmod{2}. \tag{4.23}$$

From Entry 17(iv) on page 303 in Berndt’s book [5], we have the 7-dissection

$$\psi(q) = f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3 f(q^7, q^{42}) + q^6 \psi(q^{49}). \tag{4.24}$$

Employing (4.24) in (4.23), we obtain

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8n+3)}{32} q^n \equiv (f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3 f(q^7, q^{42}) + q^6 \psi(q^{49}))^3 \pmod{2}.$$

After simplification, extracting the terms containing  $q^{7n+4}$  from both sides of the resulting equation, we get

$$\sum_{n=0}^{\infty} \frac{\overline{B}_4(8 \cdot 7n + 8 \cdot 4 + 3)}{32} q^n \equiv q^2 \psi(q^7)^3 \pmod{2},$$

which implies that

$$\overline{B}_4(8 \cdot 7^2 n + 8 \cdot 18 + 3) \equiv \overline{B}_4(8n + 3) \pmod{64}, \tag{4.25}$$

and

$$\overline{B}_4(8 \cdot 7(7n + i) + 8 \cdot 4 + 3) \equiv 0 \pmod{64} \tag{4.26}$$

for  $i = 0, 1, 3, 4, 5, 6$ .

Congruences (4.18) and (4.19) follow from (4.25), (4.26), and induction on  $\alpha$ . □

**Theorem 4.3** For all nonnegative integers  $\alpha$  and  $n$ :

$$\overline{B}_4(4 \cdot 3^{2\alpha+2} n + 3^{2\alpha+2}) \equiv \overline{B}_4(4n + 1) \pmod{2^4}, \tag{4.27}$$

$$\overline{B}_4(4 \cdot 3^{2\alpha+2} n + 7 \cdot 3^{2\alpha+1}) \equiv 0 \pmod{2^4}, \tag{4.28}$$

$$\overline{B}_4(4 \cdot 3^{2\alpha+2} n + 11 \cdot 3^{2\alpha+1}) \equiv 0 \pmod{2^4}. \tag{4.29}$$



**Proof** Following (4.11), we have

$$\sum_{n=0}^{\infty} \bar{B}_4(4n+1)q^n \equiv 4 \frac{f_2^{30}}{f_1^{22} f_4^8} \pmod{2^4}, \tag{4.30}$$

but

$$\frac{f_2^{30}}{f_1^{22} f_4^8} \equiv \frac{f_2^4}{f_1^2} \pmod{4}. \tag{4.31}$$

Combining (4.30) and (4.31), we see that

$$\sum_{n=0}^{\infty} \bar{B}_4(4n+1)q^n \equiv 4\psi(q)^2 \pmod{2^4}.$$

Substituting (2.8) in the above equation, simplifying, and extracting the terms of the form  $q^{3n+2}$  from both sides of the resulting equation, we get

$$\sum_{n=0}^{\infty} \bar{B}_4(12n+9)q^n \equiv 4\psi(q^3)^2 \pmod{2^4},$$

which implies that

$$\bar{B}_4(36n+9) \equiv 4\bar{B}_4(4n+1) \pmod{2^4}, \tag{4.32}$$

$$\bar{B}_4(36n+21) \equiv 0 \pmod{2^4}, \tag{4.33}$$

and

$$\bar{B}_4(36n+33) \equiv 0 \pmod{2^4}. \tag{4.34}$$

The theorem follows from congruences (4.32), (4.33), and (4.34) and by induction on  $\alpha$ . □

### 5. Congruence result for $\bar{B}_5(n)$

In this section, we establish one congruence relation for  $\bar{B}_5(n)$ .

**Theorem 5.1** *For each nonnegative integer  $\alpha$  and  $n$ , we have*

$$\bar{B}_5(5^\alpha n) \equiv \bar{B}_5(n) \pmod{5}. \tag{5.1}$$

**Proof** Setting  $\ell = 5$  in (1.7), we have

$$\sum_{n=0}^{\infty} \bar{B}_5(n)q^n = \frac{\varphi(-q^5)^2}{\varphi(-q)^2}. \tag{5.2}$$

Following (3.10), it is easy to see that

$$\varphi(-q)^5 \equiv \varphi(-q^5) \pmod{5}. \tag{5.3}$$

From (5.3), (5.2) becomes

$$\sum_{n=0}^{\infty} \overline{B}_5(n)q^n \equiv \varphi(-q)^8 \equiv \sum_{n=0}^{\infty} (-1)^n r_8(n)q^n \pmod{5}. \tag{5.4}$$

From [26, Lemma 3], for any  $p \geq 3$ , we have

$$r_8(pn) \equiv r_8(n) \pmod{p^3}. \tag{5.5}$$

The theorem follows from (5.4) and (5.5) with  $p = 5$ . □

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### References

- [1] Ahmed Z, Baruah ND. New congruences for Andrews' singular overpartitions. *Int J Number Theory* 2015; 11: 2247-2264.
- [2] Andrews GE. Singular overpartitions. *Int J Number Theory* 2015; 11: 1523-1533.
- [3] Barrucand P, Cooper S, Hirschhorn MD. Relations between squares and triangles. *Discrete Math* 2002; 242: 245-247.
- [4] Baruah ND, Nath K. Some results on 3-cores. *P Am Math Soc* 2014; 142: 441-448.
- [5] Berndt BC. Ramanujan's Notebooks. Part III. New York, NY, USA: Springer-Verlag, 1991.
- [6] Bringmann K, Lovejoy J. Rank and congruences for overpartition pairs. *Int J Number Theory* 2008; 4: 303-322.
- [7] Chen SC, Hirschhorn MD, Sellers JA. Arithmetic properties of Andrews' singular overpartitions. *Int J Number Theory* 2015; 11: 1463-1476.
- [8] Chen WYC, Lin BLS. Arithmetic properties of overpartition pairs. *Acta Arith* 2012; 151: 263-277.
- [9] Corteel S, Lovejoy J. Overpartitions. *T Am Math Soc* 2004; 356: 1623-1635.
- [10] Cui SP, Gu NSS. Arithmetic properties of  $\ell$ -regular partitions. *Adv Appl Math* 2013; 51: 507-523.
- [11] Hirschhorn MD, Garvan F, Borwein J. Cubic analogs of the Jacobian cubic theta function  $\theta(z, q)$ . *Canad J Math* 1993; 45: 673-694.
- [12] Hirschhorn MD, Sellers JA. An infinite family of overpartition congruences modulo 12. *Integers* 2005; 5: A20.
- [13] Hirschhorn MD, Sellers JA. Arithmetic relations for overpartitions. *J Combin Math Combin Comput* 2005; 53: 65-73.
- [14] Hirschhorn MD, Sellers JA. Elementary proofs of parity results for 5-regular partitions. *Bull Aust Math Soc* 2010; 81: 58-63.
- [15] Keister D, Sellers JA, Vary R. Some arithmetic properties of overpartition k-tuples. *Integers* 2009; 9: A17.
- [16] Keith WJ. Congruences for 9-regular partitions modulo 3. *Ramanujan J* 2014; 35: 157-164.
- [17] Kim B. The overpartition function modulo 128. *Integers* 2008; 8: A38.
- [18] Kim B. A short note on the overpartition function. *Discrete Math* 2009; 309: 2528-2532.
- [19] Kim B. Overpartition pairs modulo powers of 2. *Discrete Math* 2011; 311: 835-840.
- [20] Lin BLS. Some results on bipartitions with 3-core. *J Number Theory* 2014; 139: 44-52.

- [21] Lovejoy J. Overpartition pairs. *Ann Inst Fourier* 2006; 56: 781-794.
- [22] Lovejoy J. Gordon's theorem for overpartitions. *J Comb Theory A* 2003; 103: 393-401.
- [23] Mahlburg K. The overpartition function modulo small powers of 2. *Discrete Math* 2004; 286: 263-267.
- [24] Penniston D. Arithmetic of  $\ell$ -regular partition functions. *Int J Number Theory* 2008; 4: 295-302.
- [25] Shen EYY. Arithmetic properties of  $\ell$ -regular overpartition. *Int J Number Theory* 2016; 12: 841-852.
- [26] Wang L. A short proof of a conjecture by Hirschhorn and Sellers on overpartitions. arXiv:1407.5430 [math.NT].
- [27] Webb JJ. Arithmetic of the 13-regular partition function modulo 3. *Ramanujan J* 2011; 25: 49-56.
- [28] Xia EXW. Arithmetic properties of bipartitions with 3-cores. *Ramanujan J* 2015; 38: 529-548.
- [29] Xia EXW, Yao OXM. Analogues of Ramanujan's partition identities. *Ramanujan J* 2013; 31: 373-396.
- [30] Xia EXW, Yao OXM. Parity results for 9-regular partitions. *Ramanujan J* 2014; 34: 109-117.
- [31] Yao OXM, Xia EXW. New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions. *J Number Theory* 2013; 133: 1932-1949.