DOI: 10.1142/S179304211750049X

Arithmetic properties of 5-regular bipartitions

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Received 15 June 2015
Accepted 21 April 2016
Published 26 October 2016

Let $B_t(n)$ denote the number of t-regular bipartitions of n. In this work, we establish several infinite families of congruences modulo powers of 2 and 5 for $B_5(n)$. For example, we find that for all nonnegative integers n, i and j and $r \in \{23, 47\}$,

$$B_5\left(2^{2i+4} \cdot 5^{2j+1} n + \frac{r \cdot 2^{2i+1} \cdot 5^{2j} - 1}{3}\right) \equiv 0 \pmod{2^4}.$$

Keywords: Partition; regular bipartition; congruence.

Mathematics Subject Classification 2010: 11P83, 05A17

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. We denote the number of partitions of n by $p(n)$.

Ramanujan proved that for every nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7 \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11}.$$

Throughout this paper, we will use the notation

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad f_k := (q^k;q^k)_\infty.$$

Recall that for an integer $t > 1$, a t-regular partition is a partition none of whose parts is divisible by t. We denote the number of t-regular partitions of n by $b_t(n)$ and assume $b_t(0) = 1$ by convention. The generating function for $b_t(n)$ satisfies

$$\sum_{n=0}^{\infty} b_t(n) q^n = \frac{f_t}{f_1}.$$

Let $f(a,b)$ be Ramanujan's general theta function given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}.$$

In this notation, Jacobi's triple product identity takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Thus,

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.1}$$

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{1.2}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty = f_1. \tag{1.3}$$

Equality (1.3) is a statement of Euler's famous pentagonal number theorem [1, pp. 9–12]. After Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty.$$

Recently, the arithmetic of t-regular partition functions has been studied by a number of authors. Calkin *et al.* [5] examined the 5-regular partition function modulo 2 and the 13-regular partition function modulo 2 and 3 using the theory of modular forms. Hirschhorn and Sellers [7] obtained stronger results for the 5-regular partition function using only Jacobi's triple product identity. For example, for all $m \geq 0$

$$b_5(4p^2m + 4u(pr-7)+1) \equiv 0 \pmod 2,$$

where p is any prime greater than 3 such that -10 is a quadratic nonresidue modulo p, u is the reciprocal of 24 modulo p^2, and $r \not\equiv 0 \pmod p$. By studying p-dissection of $f(-q)$ and $\psi(q)$, Cui and Gu [6] have derived several infinite families of congruences modulo 2 for $b_t(n)$, where $t \in \{2,4,5,8,13,16\}$. For example, for any prime $p \geq 5$ such that -10 is a quadratic nonresidue modulo p,

$$b_5\left(4p^{2\alpha+2}n + \frac{(24i+7p)p^{2\alpha+1}-1}{6}\right) \equiv 0 \pmod 2$$

and

$$b_5\left(4\cdot 5^{2\alpha+1}n + \frac{r\cdot 5^{2\alpha}-1}{6}\right) \equiv 0 \pmod 2$$

for all $\alpha, n \geq 0, 1 \leq i \leq p-1$ and $r \in \{31,79\}$.

A bipartition of n is an ordered pair of partitions (π_1, π_2) such that the sum of all of the parts equals n, where π_1 and π_2 are allowed to be the empty partition. Let $p_{-2}(n)$ denote the number of bipartitions of n. The generating function for $p_{-2}(n)$ satisfies

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{f_1^2}.$$

Several mathematicians have studied the function $p_{-2}(n)$. For example, Ramanathan [12] established the following analogs of Ramanujan's classical congruences for $p(n)$ (see also [2]):

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod 5. \tag{1.4}$$

For $t > 1$, a bipartition is said to be t-regular if none of its parts is divisible by t. Let $B_t(n)$ denote the number of t-regular bipartitions of n. Then the generating function of $B_t(n)$ satisfies

$$\sum_{n=0}^{\infty} B_t(n)q^n = \frac{f_t^2}{f_1^2}. \tag{1.5}$$

Lin [8] proved that

$$B_4\left(3^{2\alpha+2}n + \frac{11 \cdot 3^{2\alpha+1} - 1}{4}\right) \equiv 0 \pmod 3,$$

for all $\alpha, n \geq 0$, and using Ramanujan's modular equations of degree seven [9] showed that

$$B_7\left(3^{\alpha+2}n + \frac{5 \cdot 3^{\alpha+1} - 1}{2}\right) \equiv 0 \pmod 3.$$

In [10] the same author proved that

$$B_{13}(9n+5) \equiv 0 \pmod 3$$

and

$$B_{13}(3n+2) \equiv B_{13}(9n+8) \pmod 3.$$

The aim of this paper is to study the arithmetic properties of 5-regular bipartitions in the spirit of Ramanujan's congruences for the partition function $p(n)$.

The main results of this paper can be stated as follows.

Theorem 1.1. *Let $r \in \{29, 53, 77, 101\}$. Then, for all nonnegative integers α and n, we have*

$$B_5\left(2^{2\alpha+1}n + \frac{2^{2\alpha+2} - 1}{3}\right) \equiv B_5(2n+1) \pmod{2^2}, \tag{1.6}$$

$$B_5\left(2^{2\alpha+3}n+\frac{11\cdot 2^{2\alpha+1}-1}{3}\right)\equiv 0 \pmod{2^2}, \tag{1.7}$$

$$B_5\left(2^{2\alpha+4}n+\frac{17\cdot 2^{2\alpha+1}-1}{3}\right)\equiv 0 \pmod{2^2}, \tag{1.8}$$

$$B_5\left(5\cdot 2^{2\alpha+4}n+\frac{r\cdot 2^{2\alpha+1}-1}{3}\right)\equiv 0 \pmod{2^2}, \tag{1.9}$$

$$B_5\left(2^{2\alpha+4}n+\frac{7\cdot 2^{2\alpha+2}-1}{3}\right)\equiv 0 \pmod{2^2} \tag{1.10}$$

and

$$B_5\left(2^{2\alpha+5}n+\frac{13\cdot 2^{2\alpha+2}-1}{3}\right)\equiv 0 \pmod{2^2}. \tag{1.11}$$

Moreover,

$$\begin{aligned} &B_5\left(5\cdot 2^{2\alpha+4}n+\frac{5\cdot 2^{2\alpha+1}-1}{3}\right)\\ &\quad\equiv B_5\left(2^{2\alpha+5}n+\frac{2^{2\alpha+2}-1}{3}\right)\\ &\quad\equiv \begin{cases} 2 \pmod{2^2} & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 \pmod{2^2} & \text{otherwise.}\end{cases} \end{aligned} \tag{1.12}$$

Theorem 1.2. *For all nonnegative integers α and n, we have*

$$B_5\left(2^{2\alpha+2}n+\frac{2^{2\alpha+2}-1}{3}\right)\equiv B_5(4n+1) \pmod{2^3}, \tag{1.13}$$

$$B_5(16n+15)\equiv 0 \pmod{2^3}, \tag{1.14}$$

$$B_5\left(2^{2\alpha+5}n+\frac{19\cdot 2^{2\alpha+2}-1}{3}\right)\equiv 0 \pmod{2^3} \tag{1.15}$$

and

$$B_5\left(2^{2\alpha+6}n+\frac{23\cdot 2^{2\alpha+3}-1}{3}\right)\equiv 0 \pmod{2^3}. \tag{1.16}$$

Theorem 1.3. *Let $r\in\{83,107\}$ and $s\in\{31,79\}$. Then, for all nonnegative integers α, j and n, we have*

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+4}\cdot 5^{2j}n+\frac{11\cdot 2^{2\alpha+1}\cdot 5^{2j}-1}{3}\right)q^n\equiv 4f_1f_5^2 \pmod{2^3}, \tag{1.17}$$

$$B_5\left(2^{2\alpha+4}\cdot 5^{2j+1}n+\frac{r\cdot 2^{2\alpha+1}\cdot 5^{2j}-1}{3}\right)\equiv 0 \pmod{2^3}, \tag{1.18}$$

$$B_5\left(2^{2\alpha+4}\cdot 5^{2j+2}n+\frac{s\cdot 2^{2\alpha+1}\cdot 5^{2j+1}-1}{3}\right)\equiv 0 \pmod{2^3}, \tag{1.19}$$

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}\cdot 5^{2j}n+\frac{7\cdot 2^{2\alpha+2}\cdot 5^{2j}-1}{3}\right)q^n\equiv 4f_1^2f_5 \pmod{2^3}, \tag{1.20}$$

$$B_5\left(2^{2\alpha+5}\cdot 5^{2j+1}n+\frac{s\cdot 2^{2\alpha+2}\cdot 5^{2j}-1}{3}\right)\equiv 0 \pmod{2^3} \tag{1.21}$$

and

$$B_5\left(2^{2\alpha+5}\cdot 5^{2j+2}n+\frac{r\cdot 2^{2\alpha+2}\cdot 5^{2j+1}-1}{3}\right)\equiv 0 \pmod{2^3}. \tag{1.22}$$

Theorem 1.4. *For all nonnegative integers α and n, we have*

$$B_5\left(2^{2\alpha+4}n+\frac{2^{2\alpha+4}-1}{3}\right)\equiv B_5(16n+5) \pmod{2^4}. \tag{1.23}$$

Theorem 1.5. *Let $r\in\{23,47\}$. Then, for all nonnegative integers α, j and n, we have*

$$\sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2j+k}n+\frac{23\cdot 2^{\alpha+1}\cdot 5^{2j+k}-1}{3}\right)q^n\equiv 8q^2f_1f_5^{14} \pmod{2^4} \tag{1.24}$$

and

$$B_5\left(2^{\alpha+4}\cdot 5^{2j+1+k}n+\frac{r\cdot 2^{\alpha+1}\cdot 5^{2j+k}-1}{3}\right)\equiv 0 \pmod{2^4}, \tag{1.25}$$

where

$$k=\begin{cases}0 & \text{if } \alpha \text{ is even},\\ 1 & \text{if } \alpha \text{ is odd}.\end{cases}$$

Theorem 1.6. *For all nonnegative integers α and n, we have*

$$B_5(5n+2)\equiv B_5(5n+3)\equiv B_5(5n+4)\equiv 0 \pmod{5}, \tag{1.26}$$

$$B_5\left(2^{2\alpha}n+\frac{2^{2\alpha}-1}{3}\right)\equiv 2^{\alpha}\ B_5(n) \pmod{5} \tag{1.27}$$

and

$$B_5\left(2^{2\alpha+2}n+\frac{2^{2\alpha+1}\cdot 5-1}{3}\right)\equiv 0 \pmod{5}. \tag{1.28}$$

Theorem 1.7. *Let $r\in\{11,29\}$ and $s\in\{7,13,25\}$. Then, for all nonnegative integers α and n, we have*

$$B_5\left(5\cdot 2^{2\alpha+2}n+\frac{r\cdot 2^{2\alpha+1}-1}{3}\right)\equiv 0 \pmod{5^2} \tag{1.29}$$

and

$$B_5\left(5^2\cdot 2^{2\alpha+2}n+\frac{5s\cdot 2^{2\alpha+1}-1}{3}\right)\equiv 0 \pmod{5^2}. \tag{1.30}$$

The rest of the paper is organized as follows. In Sec. 2 we establish some preliminary results, and prove our main results in Secs. 3–5.

2. Preliminaries

In this section, we give three lemmas which are helpful in proving our main results.

Lemma 2.1. *The following* 2*-dissections hold*:

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{2.1}$$

and

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{2.2}$$

Proof. Equation (2.1) was proved by Hirschhorn and Sellers in [7] (see also [3]). Replacing q by $-q$ in (2.1) and using the relation

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain (2.2). □

Lemma 2.2. *The following* 2*-dissections hold*:

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.3}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.4}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2} \tag{2.5}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.6}$$

Proof. Lemma 2.2 is an immediate consequence of dissection formulas of Ramanujan, collected in Berndt's book [4, Entry 25, p. 40]. □

Lemma 2.3. *We have*

$$f_1 f_5^3 = 2q^2 f_4 f_{20}^3 + f_2^3 f_{10} - 2q^3\frac{f_4^4 f_{40}^2 f_{10}}{f_2 f_8^2} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} \tag{2.7}$$

and

$$f_1^3 f_5 = 2q^2\frac{f_4^6 f_{40}^2 f_{10}}{f_2 f_8^2 f_{20}^2} + \frac{f_4 f_{10}^2 f_2^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3. \tag{2.8}$$

Proof. In [11] the authors and Sumanth Bharadwaj employ theta function identities of Ramanujan to show that

$$(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 - (q;q^2)_\infty(q^5;q^{10})_\infty^3 = 4q^3\frac{f_4^4f_{40}^2}{f_2^2f_8^2f_{10}^2} + 2q\frac{f_2f_{20}}{f_4f_{10}}, \quad (2.9)$$

$$(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 + (q;q^2)_\infty(q^5;q^{10})_\infty^3 = 4q^2\frac{f_4f_{20}^3}{f_2f_{10}^3} + 2\frac{f_2^2}{f_{10}^2}, \quad (2.10)$$

$$(-q;q^2)_\infty^3(-q^5;q^{10})_\infty - (q;q^2)_\infty^3(q^5;q^{10})_\infty = 10q\frac{f_{10}^2}{f_2^2} - 4q\frac{f_4^3f_{20}}{f_2^3f_{10}} \quad (2.11)$$

and

$$(-q;q^2)_\infty^3(-q^5;q^{10})_\infty + (q;q^2)_\infty^3(q^5;q^{10})_\infty = 4q^2\frac{f_4^6f_{40}^2}{f_2^4f_8^2f_{20}^2} + 2\frac{f_4f_{10}}{f_2f_{20}}. \quad (2.12)$$

Subtracting (2.9) from (2.10), and then using the relation

$$(q;q^2)_\infty = \frac{f_1}{f_2},$$

we obtain (2.7). Similarly, (2.8) follows from (2.11) and (2.12). □

3. Proofs of Theorems 1.1–1.5

Setting $t = 5$ in (1.5) yields

$$\sum_{n=0}^{\infty} B_5(n)q^n = \frac{f_5^2}{f_1^2}. \quad (3.1)$$

Combining (2.1) and (3.1), we see that

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n = 2\frac{f_2^3f_{10}f_5}{f_1^5}. \quad (3.2)$$

Substituting (2.1) and (2.6) into (3.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_5(2n+1)q^n &= 2f_2^3f_{10}\frac{f_5}{f_1}\frac{1}{f_1^4}\\ &= 2f_2^3f_{10}\left(\frac{f_8f_{20}^2}{f_2^2f_{40}} + q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right),\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n = 2\frac{f_5f_2^{14}f_{10}^2}{f_1^{13}f_{20}f_4^3} + 8q\frac{f_2^5f_4^3f_{20}f_5^2}{f_1^{10}f_{10}}. \quad (3.3)$$

By the binomial theorem, it is easy to see that for any positive integer k,

$$f_k^2 \equiv f_{2k} \pmod 2, \quad (3.4)$$

$$f_k^4 \equiv f_{2k}^2 \pmod{2^2} \quad (3.5)$$

and

$$f_k^8 \equiv f_{2k}^4 \pmod{2^3}. \tag{3.6}$$

From (3.6) and (3.4), it follows that

$$\frac{f_5 f_2^{14} f_{10}^2}{f_1^{13} f_{20} f_4^3} \equiv \frac{f_2^2 f_{10}^2 f_4 f_5}{f_{20} f_1^5} \pmod{2^3} \tag{3.7}$$

and

$$\frac{f_2^5 f_4^3 f_{20} f_5^2}{f_1^{10} f_{10}} \equiv f_4^3 f_{20} \pmod{2}. \tag{3.8}$$

In view of (3.7) and (3.8), we can rewrite (3.3) as

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 2\frac{f_2^2 f_{10}^2 f_4}{f_{20}} \frac{f_5}{f_1} \frac{1}{f_1^4} + 8q f_4^3 f_{20} \pmod{2^4}. \tag{3.9}$$

By (3.5) and (3.9), we see that

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 2\frac{f_{10}^2 f_4 f_5}{f_{20} f_1} \pmod{2^3}. \tag{3.10}$$

Substituting (2.1) and (2.6) into (3.9), we obtain

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 2\frac{f_2^2 f_{10}^2 f_4}{f_{20}} \left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \right) \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}} \right) + 8q f_4^3 f_{20} \pmod{2^4},$$

which yields

$$\sum_{n=0}^{\infty} B_5(8n+5)q^n \equiv 2\frac{f_2^{18} f_5^3 f_{20}}{f_1^{15} f_4^5 f_{10}^2} + 8\frac{f_4^5 f_2^3 f_{10} f_5^2}{f_1^{10} f_{20}} + 8 f_2^3 f_{10} \pmod{2^4}. \tag{3.11}$$

It follows from (3.6) and (3.4) that

$$\frac{f_2^{18} f_5^3 f_{20}}{f_1^{15} f_4^5 f_{10}^2} \equiv \frac{f_2^2 f_{20} f_{10}^2 f_1}{f_4 f_5^5} \pmod{2^3} \tag{3.12}$$

and

$$\frac{f_4^5 f_2^3 f_{10} f_5^2}{f_1^{10} f_{20}} \equiv f_4^4 \pmod{2}. \tag{3.13}$$

In view of (3.12) and (3.13), we can rewrite (3.11) as

$$\sum_{n=0}^{\infty} B_5(8n+5)q^n \equiv 2\frac{f_2^2 f_{20} f_{10}^2}{f_4} \frac{f_1}{f_5} \frac{1}{f_5^4} + 8 f_4^4 + 8 f_2^3 f_{10} \pmod{2^4}, \tag{3.14}$$

and by (3.4) we also have

$$\frac{f_2^2 f_{20} f_{10}^2 f_1}{f_4 f_5^5} \equiv \frac{f_2^3 f_{10} f_5}{f_1^5} \pmod{2}.$$

From the above two identities, we arrive at

$$\sum_{n=0}^{\infty} B_5(8n+5)q^n \equiv 2\frac{f_2^3 f_{10} f_5}{f_1^5} \pmod{2^2}. \tag{3.15}$$

Using (3.2) and (3.15), we find that for $n \geq 0$,

$$B_5(8n+5) \equiv B_5(2n+1) \pmod{2^2}. \tag{3.16}$$

By (3.16) and mathematical induction, we find that (1.6) is true.

Using (3.6), we can rewrite (3.2) as

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n \equiv 2\frac{f_{10}}{f_2} f_1^3 f_5 \pmod{2^4}. \tag{3.17}$$

Employing (2.8) in (3.17) and then extracting the terms involving q^{2n+1} from both sides of the resulting identity, we arrive at

$$\sum_{n=0}^{\infty} B_5(4n+3)q^n \equiv 4f_2^3 f_{10}\frac{f_5}{f_1} - 10 f_5^4 \pmod{2^4}, \tag{3.18}$$

which implies that

$$\sum_{n=0}^{\infty} B_5(4n+3)q^n \equiv 2f_{20} \pmod{2^2}. \tag{3.19}$$

From (3.19), it is easy to see that for all $n \geq 0$ and $1 \leq i \leq 4$,

$$B_5(8n+7) \equiv 0 \pmod{2^2}, \tag{3.20}$$

$$B_5(16n+11) \equiv 0 \pmod{2^2}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} B_5(16n+3)q^n \equiv 2f_5 \pmod{2^2} \tag{3.22}$$

and

$$B_5(80n+16i+3) \equiv 0 \pmod{2^2}. \tag{3.23}$$

Congruences (1.7)–(1.9) follow from (3.20), (3.21), (3.23) and (1.6).

We can also rewrite (3.9) as

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 2\frac{f_{10}^2 f_4}{f_{20} f_2^2} f_1^3 f_5 + 8q f_4^3 f_{20} \pmod{2^4}. \tag{3.24}$$

Applying (2.8) in (3.24), then extracting the terms involving q^{2n} from both sides, we find that

$$\sum_{n=0}^{\infty} B_5(8n+1)q^n \equiv 4q\frac{f_{20}^2 f_2^7 f_5^3}{f_{10}^3 f_4^2 f_1^3} + 2\frac{f_2^2 f_5^4}{f_{10}^2} \pmod{2^4}, \tag{3.25}$$

and by (3.4) we also have

$$\frac{f_2^2 f_5^4}{f_{10}^2} \equiv f_4 \pmod{2}. \tag{3.26}$$

In view of (3.25) and (3.26), we have

$$\sum_{n=0}^{\infty} B_5(8n+1)q^n \equiv 2f_4 \pmod{2^2}. \tag{3.27}$$

From (3.27) we deduce that

$$B_5(16n+9) \equiv 0 \pmod{2^2} \tag{3.28}$$

and

$$B_5(32n+17) \equiv 0 \pmod{2^2} \tag{3.29}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} B_5(32n+1)q^n \equiv 2f_1 \pmod{2^2}. \tag{3.30}$$

Congruences (1.10) and (1.11) follow from (3.28), (3.29) and (1.6).

From (3.22) and (3.30), we see that

$$\sum_{n=0}^{\infty} B_5(80n+3)q^n \equiv \sum_{n=0}^{\infty} B_5(32n+1)q^n \equiv 2f_1 \pmod{2^2}, \tag{3.31}$$

and by (1.3) we have

$$f_1 = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}. \tag{3.32}$$

Combining (3.31) and (3.32), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} B_5(80n+3)q^n &\equiv \sum_{n=0}^{\infty} B_5(32n+1)q^n \\ &\equiv 2\sum_{k=-\infty}^{\infty} q^{k(3k-1)/2} \pmod{2^2}. \end{aligned} \tag{3.33}$$

By (3.33) and (1.6), we find that (1.12) is true.

Invoking (2.2) and (2.6) in (3.14), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} B_5(8n+5)q^n \equiv{}& 2\frac{f_2^2 f_{20} f_{10}^2}{f_4}\left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\right)\left(\frac{f_{20}^{14}}{f_{10}^{14} f_{40}^4} + 4q^5\frac{f_{20}^2 f_{40}^4}{f_{10}^{10}}\right) \\ &+ 8f_4^4 + 8f_2^3 f_{10} \pmod{2^4}, \end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} B_5(16n+5)q^n \equiv 2\frac{f_1^3 f_{10}^{18} f_4}{f_5^{15} f_2^2 f_{20}^5} - 8q^3\frac{f_1^2 f_2 f_{10}^3 f_{20}^5}{f_4 f_5^{10}} + 8f_2^4 + 8f_1^3 f_5 \pmod{2^4}. \tag{3.34}$$

From (3.6) and (3.4), it follows that

$$\frac{f_1^3 f_{10}^{18} f_4}{f_5^{15} f_2^2 f_{20}^5} \equiv \frac{f_2^2 f_4 f_{10}^2 f_5}{f_{20} f_1^5} \pmod{2^3} \tag{3.35}$$

and

$$\frac{f_1^2 f_2 f_{10}^3 f_{20}^5}{f_4 f_5^{10}} \equiv f_{20}^4 \pmod{2}. \tag{3.36}$$

From (3.35) and (3.36), (3.34) can be rewritten as

$$\sum_{n=0}^{\infty} B_5(16n+5)q^n \equiv 2\frac{f_2^2 f_4 f_{10}^2}{f_{20}}\frac{f_5}{f_1}\frac{1}{f_1^4} - 8q^3 f_{20}^4 + 8f_2^4 + 8f_1^3 f_5 \pmod{2^4}. \tag{3.37}$$

It follows that

$$\sum_{n=0}^{\infty} B_5(16n+5)q^n \equiv 2\frac{f_4 f_{10}^2 f_5}{f_{20} f_1} \pmod{2^3}. \tag{3.38}$$

In view of (3.10) and (3.38), we deduce that for all $n \geq 0$,

$$B_5(16n+5) \equiv B_5(4n+1) \pmod{2^3}. \tag{3.39}$$

By (3.39) and mathematical induction, we can deduce (1.13).

Substituting (2.1) and (2.5) in (3.18), we find that

$$\sum_{n=0}^{\infty} B_5(4n+3)q^n \equiv 4f_2^3 f_{10}\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right) - 10\left(\frac{f_{20}^{10}}{f_{10}^2 f_{40}^4} - 4q^5\frac{f_{10}^2 f_{40}^4}{f_{20}^2}\right) \pmod{2^4},$$

which yields

$$\sum_{n=0}^{\infty} B_5(8n+7)q^n \equiv 4\frac{f_2^3 f_{20} f_5^2}{f_4 f_{10}} + 8q^2\frac{f_{20}^4 f_5^2}{f_{10}^2} \pmod{2^4}. \tag{3.40}$$

Since

$$\frac{f_2^3 f_{20} f_5^2}{f_4 f_{10}} \equiv f_2 f_{10}^2 \pmod{2}, \tag{3.41}$$

it follows that

$$\sum_{n=0}^{\infty} B_5(8n+7)q^n \equiv 4f_2 f_{10}^2 \pmod{2^3}. \tag{3.42}$$

Congruence (1.14) follows from (3.42).

Using (3.5), we can rewrite (3.25) as

$$\sum_{n=0}^{\infty} B_5(8n+1)q^n \equiv 4q\frac{f_{10}^3 f_4^2}{f_2^3}\frac{f_1}{f_5} + 2\frac{f_2^2}{f_{10}^2} f_5^4 \pmod{2^4}. \tag{3.43}$$

Now, employing (2.2) and (2.5) in (3.43), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} B_5(8n+1)q^n &\equiv 4q\frac{f_{10}^3 f_4^2}{f_2^3}\left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\right) \\ &\quad + 2\frac{f_2^2}{f_{10}^2}\left(\frac{f_{20}^{10}}{f_{10}^2 f_{40}^4} - 4q^5\frac{f_{10}^2 f_{40}^4}{f_{20}^2}\right) \pmod{2^4},\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} B_5(16n+9)q^n \equiv 4\frac{f_2 f_4 f_{10}^3}{f_{20}}\frac{1}{f_1^2} + 8q^2 f_2 f_{20}^3 \pmod{2^4}. \tag{3.44}$$

By (3.44) and (3.4), we find that

$$\sum_{n=0}^{\infty} B_5(16n+9)q^n \equiv 4f_2^2 f_{10} \pmod{2^3}. \tag{3.45}$$

From (3.45) we deduce that

$$B_5(32n+25) \equiv 0 \pmod{2^3} \tag{3.46}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} B_5(32n+9)q^n \equiv 4f_1^2 f_5 \pmod{2^3}. \tag{3.47}$$

Congruence (1.15) follows from (3.46) and (1.13).

Using (3.6), we can rewrite (3.14) as

$$\sum_{n=0}^{\infty} B_5(8n+5)q^n \equiv 2\frac{f_2^2 f_{20}}{f_{10}^2 f_4} f_1 f_5^3 + 8f_4^4 + 8f_2^3 f_{10} \pmod{2^4}. \tag{3.48}$$

Invoking (2.7) in (3.48) and then extracting the terms involving q^{2n+1} from both sides of the resulting identity, we deduce that

$$\sum_{n=0}^{\infty} B_5(16n+13)q^n \equiv -4q\frac{f_2^3 f_{10} f_{20}^2}{f_4^2}\frac{f_1}{f_5} - 2\frac{f_{10}^2}{f_2^2} f_1^4 \pmod{2^4}. \tag{3.49}$$

Substituting (2.2) and (2.5) in (3.49), then extracting the terms involving q^{2n+1} from both sides, we obtain

$$\sum_{n=0}^{\infty} B_5(32n+29)q^n \equiv -4\frac{f_{10}^5 f_4}{f_2 f_{20}}\frac{1}{f_5^2} + 8\frac{f_4^4}{f_2^2} f_5^2 \pmod{2^4}. \tag{3.50}$$

It follows from (3.50) and (3.4) that

$$\sum_{n=0}^{\infty} B_5(32n+29)q^n \equiv 4f_2f_{10}^2 \pmod{2^3}. \tag{3.51}$$

From (3.51) we see that

$$B_5(64n+61) \equiv 0 \pmod{2^3} \tag{3.52}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} B_5(64n+29)q^n \equiv 4f_1f_5^2 \pmod{2^3}. \tag{3.53}$$

Congruence (1.16) follows from (3.52) and (1.13).

Identity (3.42) implies that

$$\sum_{n=0}^{\infty} B_5(16n+7)q^n \equiv 4f_1f_5^2 \pmod{2^3}. \tag{3.54}$$

From (1.13) and (3.53), we can easily see that for all integers $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+6}n + \frac{11\cdot 2^{2\alpha+3}-1}{3}\right)q^n \equiv 4f_1f_5^2 \pmod{2^3}. \tag{3.55}$$

In view of (3.54) and (3.55), we see that congruence (1.17) is true for $j=0$.

Now suppose that (1.17) holds for some $j \geq 0$, and recall Ramanujan's 5-dissection [13, p. 212]

$$f_1 = f_{25}(a(q) - q - q^2a^{-1}(q)), \tag{3.56}$$

where $a(q) := \frac{f(-q^{15},-q^{10})}{f(-q^{20},-q^5)}$.

Utilizing (3.56) in (1.17) and then extracting the terms involving q^{5n+1} from both sides of the resulting congruence, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} & B_5\left(2^{2\alpha+4}\cdot 5^{2j}(5n+1) + \frac{11\cdot 2^{2\alpha+1}\cdot 5^{2j}-1}{3}\right)q^n \\ &= \sum_{n=0}^{\infty} B_5\left(2^{2\alpha+4}\cdot 5^{2j+1}n + \frac{7\cdot 2^{2\alpha+1}\cdot 5^{2j+1}-1}{3}\right)q^n \\ &\equiv 4f_5f_1^2 \\ &\equiv 4f_5f_{25}^2(a^2(q)+q^2+q^4a^{-2}(q)) \pmod{2^3}.\end{aligned} \tag{3.57}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} & B_5\left(2^{2\alpha+4}\cdot 5^{2j+1}(5n+2) + \frac{7\cdot 2^{2\alpha+1}\cdot 5^{2j+1}-1}{3}\right)q^n \\ &= \sum_{n=0}^{\infty} B_5\left(2^{2\alpha+4}\cdot 5^{2(j+1)}n + \frac{11\cdot 2^{2\alpha+1}\cdot 5^{2(j+1)}-1}{3}\right)q^n \\ &\equiv 4f_1f_5^2 \pmod{2^3}.\end{aligned}$$

Thus, (1.17) is true for $j+1$. Hence, by mathematical induction congruence (1.17) holds for all $j \geq 0$, and thanks to (3.56), congruences (1.18) and (1.19) follow easily from (1.17) and (3.57), respectively.

It follows from (3.47) and (1.13) that for all integers $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+5}n + \frac{7\cdot 2^{2\alpha+2}-1}{3}\right)q^n \equiv 4f_1^2 f_5 \pmod{2^3},$$

which is the $j=0$ case of (1.20). The rest of the proof by mathematical induction is similar to that of (1.17), so we omit the details. Congruences (1.21) and (1.22) follow immediately from the proof of (1.20).

Substituting (2.1), (2.6) and (2.8) in (3.37) and then extracting the terms involving q^{2n+1} from both sides of the resulting identity, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} B_5(32n+21)q^n &\equiv 2\frac{f_2^{18}f_5^3f_{20}}{f_1^{15}f_4^5f_{10}^2} + 8\frac{f_4^5f_2^3f_{10}f_5^2}{f_1^{10}f_{20}} - 8qf_{10}^4 + 8f_1f_5^3 \\ &\equiv 2\frac{f_2^2f_{20}f_{10}^2}{f_4}\frac{f_1}{f_5}\frac{1}{f_5^4} + 8f_4^4 - 8qf_{10}^4 + 8f_1f_5^3 \pmod{2^4}.\end{aligned} \tag{3.58}$$

Now, employing (2.2), (2.6) and (2.7) in (3.58) and then extracting the terms which involve q^{2n}, we arrive at

$$\sum_{n=0}^{\infty} B_5(64n+21)q^n \equiv 2\frac{f_2^2f_4f_{10}^2f_5}{f_{20}f_1^5} - 8q^3f_{20}^4 + 8f_2^4 + 8f_1^3f_5 \pmod{2^4}. \tag{3.59}$$

By (3.37) and (3.59), we deduce that for all $n \geq 0$,

$$B_5(64n+21) \equiv B_5(16n+5) \pmod{2^4}. \tag{3.60}$$

Congruence (1.23) follows from (3.60) and mathematical induction.

Invoking (2.3) in (3.40), we see that

$$\sum_{n=0}^{\infty} B_5(8n+7)q^n \equiv 4\frac{f_{20}f_2^3}{f_{10}f_4}\left(\frac{f_{10}f_{40}^5}{f_{20}^2f_{80}^2} - 2q^5\frac{f_{10}f_{80}^2}{f_{40}}\right) + 8q^2\frac{f_{20}^4}{f_{10}} \pmod{2^4},$$

which implies that

$$\sum_{n=0}^{\infty} B_5(16n+15)q^n \equiv -8q^2\frac{f_{10}f_1^3f_{40}^2}{f_2f_{20}} \equiv 8q^2f_1f_5^{14} \pmod{2^4}. \tag{3.61}$$

Employing (2.4) in (3.44), then extracting terms of the form q^{2n+1} and using (3.56), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} B_5(32n+25)q^n &\equiv 8\frac{f_2^3f_8^2f_5^3}{f_{10}f_4f_1^4} \\ &\equiv 8f_1^{14}f_5 \\ &\equiv 8f_5f_{25}^{14}\left(a(q) - q - q^2a^{-1}(q)\right)^{14} \pmod{2^4}.\end{aligned} \tag{3.62}$$

Expanding the right-hand side of (3.62) and then extracting the terms involving q^{5n+4}, we see that

$$\sum_{n=0}^{\infty} B_5(160n+153)q^n = \sum_{n=0}^{\infty} B_5\left(2^5\cdot 5n + \frac{23\cdot 2^2\cdot 5-1}{3}\right)q^n$$
$$\equiv 8q^2 f_1 f_5^{14} \pmod{2^4}. \tag{3.63}$$

Employing (2.4) in (3.50), we deduce that

$$\sum_{n=0}^{\infty} B_5(64n+61)q^n = \sum_{n=0}^{\infty} B_5\left(2^6 n + \frac{23\cdot 2^3-1}{3}\right)q^n$$
$$\equiv 8q^2\frac{f_{10}^3 f_{40}^2 f_1}{f_5^4 f_{20}} \equiv 8q^2 f_1 f_5^{14} \pmod{2^4}. \tag{3.64}$$

Utilizing (3.6), we can rewrite (3.37) as

$$\sum_{n=0}^{\infty} B_5(16n+5)q^n \equiv 2\frac{f_4 f_{10}^2}{f_2^2 f_{20}} f_1^3 f_5 - 8q^3 f_{20}^4 + 8f_2^4 + 8f_1^3 f_5 \pmod{2^4}. \tag{3.65}$$

Substituting (2.8) in (3.65), we obtain

$$\sum_{n=0}^{\infty} B_5(32n+5)q^n \equiv 4q\frac{f_2^7 f_{20}^2 f_5^3}{f_4^2 f_{10}^3 f_1^3} + 2\frac{f_2^2 f_5^4}{f_{10}^2} \pmod{2^4}. \tag{3.66}$$

By (3.25) and (3.66), we see that for $n\ge 0$,

$$B_5(32n+5) \equiv B_5(8n+1) \pmod{2^4}. \tag{3.67}$$

In view of (1.23), congruence (3.67) implies that for all $\alpha\ge 0$,

$$B_5\left(2^{2\alpha+5}(20n+19) + \frac{2^{2\alpha+4}-1}{3}\right) \equiv B_5(8(20n+19)+1) \pmod{2^4}. \tag{3.68}$$

By (3.63), we rewrite (3.68) as

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+7}\cdot 5n + \frac{23\cdot 2^{2\alpha+4}\cdot 5-1}{3}\right)q^n \equiv 8q^2 f_1 f_5^{14} \pmod{2^4}. \tag{3.69}$$

Using (3.6), we can rewrite (3.58) as

$$\sum_{n=0}^{\infty} B_5(32n+21)q^n \equiv 2\frac{f_2^2 f_{20}}{f_4 f_{10}^2} f_1 f_5^3 + 8f_4^4 - 8qf_{10}^4 + 8f_1 f_5^3 \pmod{2^4}. \tag{3.70}$$

Now, employing (2.7) in (3.70), we deduce that

$$\sum_{n=0}^{\infty} B_5(64n+53)q^n \equiv -4q\frac{f_1 f_2^3 f_{10} f_{20}^2}{f_4^2 f_5} - 2\frac{f_1^4 f_{10}^2}{f_2^2} \pmod{2^4}. \tag{3.71}$$

It follows from (3.49) and (3.71) that for $n\ge 0$,

$$B_5(64n+53) \equiv B_5(16n+13) \pmod{2^4}.$$

Therefore,

$$B_5(256n+245) \equiv B_5(64n+61) \pmod{2^4}. \tag{3.72}$$

In view of (1.23), (3.64) and (3.72), we find that

$$\sum_{n=0}^{\infty} B_5\left(2^{2\alpha+8}n + \frac{23\cdot 2^{2\alpha+5}-1}{3}\right) q^n \equiv 8q^2 f_1 f_5^{14} \pmod{2^4}. \tag{3.73}$$

Combining (3.61), (3.63), (3.64), (3.69) and (3.73), we obtain

$$\sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^k n + \frac{23\cdot 2^{\alpha+1}\cdot 5^k-1}{3}\right) q^n \equiv 8q^2 f_1 f_5^{14} \pmod{2^4}, \tag{3.74}$$

which is the $j=0$ case of (1.24).

Now suppose that (1.24) holds for some $j \geq 0$. Using (3.56), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2j+k} n + \frac{23\cdot 2^{\alpha+1}\cdot 5^{2j+k}-1}{3}\right) q^n \\
&\quad \equiv 8q^2 f_1 f_5^{14} \\
&\quad \equiv 8q^2 f_5^{14} f_{25}\left(a(q) - q - q^2 a^{-1}(q)\right) \pmod{2^4}.
\end{aligned} \tag{3.75}$$

Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2j+k}(5n+3) + \frac{23\cdot 2^{\alpha+1}\cdot 5^{2j+k}-1}{3}\right) q^n \\
&\quad = \sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2j+k+1} n + \frac{19\cdot 2^{\alpha+1}\cdot 5^{2j+k+1}-1}{3}\right) q^n \\
&\quad \equiv 8 f_5 f_1^{14} \\
&\quad \equiv 8 f_5 f_{25}^{14}\left(a(q) - q - q^2 a^{-1}(q)\right)^{14} \pmod{2^4},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2j+k+1}(5n+4) + \frac{19\cdot 2^{\alpha+1}\cdot 5^{2j+k+1}-1}{3}\right) q^n \\
&\quad = \sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2(j+1)+k} n + \frac{23\cdot 2^{\alpha+1}\cdot 5^{2(j+1)+k}-1}{3}\right) q^n \\
&\quad \equiv 8q^2 f_1 f_5^{14} \pmod{2^4}.
\end{aligned}$$

So (1.24) holds. Also, from (3.75) we see that

$$\sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2j+k}(5n) + \frac{23\cdot 2^{\alpha+1}\cdot 5^{2j+k}-1}{3}\right) q^n \equiv 0 \pmod{2^4} \tag{3.76}$$

and

$$\sum_{n=0}^{\infty} B_5\left(2^{\alpha+4}\cdot 5^{2j+k}(5n+1)+\frac{23\cdot 2^{\alpha+1}\cdot 5^{2j+k}-1}{3}\right)q^n \equiv 0 \pmod{2^4}. \quad (3.77)$$

Congruence (1.25) follows from (3.76) and (3.77). This completes the proof.

4. Proof of Theorem 1.6

Congruence (1.26) follows readily from (1.4) and (3.1).

For any positive integer k, it is easy to see that

$$f_k^5 \equiv f_{5k} \pmod{5}. \quad (4.1)$$

Utilizing (4.1), one can rewrite (3.2) as

$$\sum_{n=0}^{\infty} B_5(2n+1)q^n \equiv 2\frac{f_{10}^2}{f_2^2} \pmod{5},$$

which yields

$$B_5(4n+3) \equiv 0 \pmod{5} \quad (4.2)$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 2\frac{f_5^2}{f_1^2} \pmod{5}. \quad (4.3)$$

It follows from (3.1) and (4.3) that for $n \geq 0$,

$$B_5(4n+1) \equiv 2\ B_5(n) \pmod{5}. \quad (4.4)$$

Congruence (1.27) follows from (4.4) and mathematical induction. Replacing n by $4n+3$ in (1.27) and employing (4.2), we deduce (1.28).

5. Proof of Theorem 1.7

One of the Ramanujan's modular equations of degree five [4, p. 259] can be written in the equivalent form

$$5\frac{\phi^2(-q^5)}{\phi^2(-q)} - 1 = 4\frac{\chi(-q^5)}{\chi^5(-q)}. \quad (5.1)$$

By manipulating the q-products, one can easily arrive at

$$\phi(-q) = \frac{f_1^2}{f_2}, \quad \chi(-q) = \frac{f_1}{f_2}. \quad (5.2)$$

In the view of (5.1) and (5.2), we see that

$$4\frac{f_5}{f_1^5} = 5\frac{f_5^4}{f_1^4 f_{10} f_2^3} - \frac{f_{10}}{f_2^5}. \quad (5.3)$$

Thanks to (5.3) and (4.1), we can rewrite (3.2) as

$$\sum_{n=0}^{\infty} 2B_5(2n+1)q^n = 5\frac{f_5^4}{f_1^4} - \frac{f_{10}^2}{f_2^2}$$
$$\equiv 5f_1f_5^3 - \frac{f_{10}^2}{f_2^2} \pmod{5^2}. \tag{5.4}$$

From (3.56), (3.1) and (5.4),

$$\sum_{n=0}^{\infty} 2B_5(2n+1)q^n \equiv 5f_5^3f_{25}(a(q) - q - q^2a^{-1}(q))$$
$$-\sum_{n=0}^{\infty} B_5(n)q^{2n} \pmod{5^2}. \tag{5.5}$$

Equating the terms involving q^{10n+3}, q^{10n+9} and q^{5n+1} on both sides of (5.5), we find that

$$B_5(20n+7) \equiv B_5(20n+19) \equiv 0 \pmod{5^2} \tag{5.6}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} 2B_5(10n+3)q^n \equiv -5f_5f_1^3 - \sum_{n=0}^{\infty} B_5(5n+3)q^{2n+1}$$
$$\equiv -5f_5f_{25}^3(a(q) - q - q^2a^{-1}(q))^3$$
$$-\sum_{n=0}^{\infty} B_5(5n+3)q^{2n+1} \pmod{5^2}. \tag{5.7}$$

Now, equating coefficients of q^{10n+2}, q^{10n+4} and q^{10n+8} in (5.7), we obtain

$$B_5(100n+23) \equiv B_5(100n+43) \equiv B_5(100n+83) \equiv 0 \pmod{5^2}. \tag{5.8}$$

Employing (2.7) in (5.4) and then extracting the terms involving even powers of q, we arrive at

$$\sum_{n=0}^{\infty} 2B_5(4n+1)q^n \equiv 5(2qf_2f_{10}^3 + f_1^3f_5) - \frac{f_5^2}{f_1^2} \pmod{5^2}.$$

Using (3.1), (3.56) and (4.1) in the above congruence, we obtain

$$\sum_{n=0}^{\infty} B_5(4n+1)q^n \equiv 5qf_2f_{10}^3 + 2\sum_{n=0}^{\infty} B_5(n)q^n$$
$$\equiv 5qf_{10}^3f_{50}(a(q^2) - q^2 - q^4a^{-1}(q^2))$$
$$+2\sum_{n=0}^{\infty} B_5(n)q^n \pmod{5^2}. \tag{5.9}$$

Thus,

$$B_5(20n+9) \equiv 2B_5(5n+2) \pmod{5^2} \tag{5.10}$$

and

$$B_5(20n+17) \equiv 2B_5(5n+4) \pmod{5^2} \tag{5.11}$$

for all $n \geq 0$ and

$$\begin{aligned}\sum_{n=0}^{\infty} B_5(20n+13)q^n &\equiv -5f_2^3 f_{10} + 2\sum_{n=0}^{\infty} B_5(5n+3)q^n \\ &\equiv -5f_{10}f_{50}^3(a(q^2) - q^2 - q^4 a^{-1}(q^2))^3 \\ &\quad + 2\sum_{n=0}^{\infty} B_5(5n+3)q^n \pmod{5^2}.\end{aligned} \tag{5.12}$$

From (5.12) it follows that

$$B_5(100n+33) \equiv 2B_5(25n+8) \pmod{5^2}, \tag{5.13}$$

$$B_5(100n+73) \equiv 2B_5(25n+18) \pmod{5^2} \tag{5.14}$$

and

$$B_5(100n+93) \equiv 2B_5(25n+23) \pmod{5^2}. \tag{5.15}$$

By (5.10), (5.11) and mathematical induction, we see that for all $n \geq 0$ and $\alpha \geq 0$,

$$B_5\left(5 \cdot 2^{2\alpha}n + \frac{13 \cdot 2^{2\alpha} - 1}{3}\right) \equiv 2^{\alpha} B_5(5n+4) \pmod{5^2} \tag{5.16}$$

and

$$B_5\left(5 \cdot 2^{2\alpha}n + \frac{7 \cdot 2^{2\alpha} - 1}{3}\right) \equiv 2^{\alpha} B_5(5n+2) \pmod{5^2}. \tag{5.17}$$

Congruence (1.29) follows from (5.16), (5.17) and (5.6).

Now, by (5.13) and mathematical induction, we deduce that

$$B_5\left(5^2 \cdot 2^{2\alpha}n + \frac{25 \cdot 2^{2\alpha} - 1}{3}\right) \equiv 2^{\alpha} B_5(25n+8) \pmod{5^2}. \tag{5.18}$$

Also, by (5.14), (5.15) and mathematical induction, we have

$$B_5\left(5^2 \cdot 2^{2\alpha}n + \frac{55 \cdot 2^{2\alpha} - 1}{3}\right) \equiv 2^{\alpha} B_5(25n+18) \pmod{5^2} \tag{5.19}$$

and

$$B_5\left(5^2 \cdot 2^{2\alpha}n + \frac{35 \cdot 2^{2\alpha+1} - 1}{3}\right) \equiv 2^{\alpha} B_5(25n+23) \pmod{5^2}. \tag{5.20}$$

Congruence (1.30) readily follows from (5.18), (5.19), (5.20) and (5.8). This completes the proof.

Acknowledgments

We would like to thank the anonymous referee for his/her careful reading of our manuscript and many helpful comments and suggestions. First author would like to thank DST for financial support through project no. SR/S4/MS:739/11 and second author would like to thank UGC for financial support through JRF, ref. no F.17-58/2008(SA-I).

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