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FUNDAMENTAL SOLITONS FOR THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL

M.Sc. THESIS

Burcu Ece ALP

Mathematical Engineering Department

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DECEMBER 2016

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

$\mathscr{PT}$-SİMETRİK BİR POTANSİYEL İÇEREN DOĞRUSAL OLMAYAN YÜKSEK MERTEBE SCHRÖDINGER DENKLEMİNİN TEMEL SOLITONLARI

YÜKSEK LİSANS TEZİ

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Matematik Mühendisliği Anabilim Dalı

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ARALIK 2016

Burcu Ece ALP, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509131054 successfully defended the thesis entitled "FUNDAMENTAL SOLITONS FOR THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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	Assist. Prof. Dr. Güler KAYGUSUZOĞLU Namık Kemal University	....

Date of Submission : **25 November 2016**
Date of Defense : **22 December 2016**

To my country and mother,

FOREWORD

I would like to express my deep appreciation to my advisor, Assoc. Prof. Dr. Ilkay Bakırtaş for standing behind me throughout this project. She provided me the opportunity to work on one of the most modern and excellent subjects of applied mathematics. I would also like to thank my colleague, İzzet Göksel, for his assistance and fruitful discussions about numerical methods and solitons. Additionaly, I would like to thank my colleague, Hüseyin Tunç, for his contributions and advices on the Matlab programming. Last but not least, I would like to thank my family, especially my mother for their unconditional support.

December 2016 Burcu Ece ALP

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ABBREVIATIONS

NLS	**:** Nonlinear Schrödinger
CNLS	**:** Cubic Nonlinear Schrödinger
3OD	**:** Third Order Dispersion
4OD	**:** Fourth Order Dispersion
$\mathscr{PT}$	**:** Parity - Time
SR	**:** Spectral Renormalization
KdV	**:** Korteweg-de Vries

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FUNDAMENTAL SOLITONS FOR THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL

SUMMARY

In nature, a soliton is described as a type of nonlinear wave structure; it exhibits mathematical model for various field in science, such as fluid dynamics, biological systems, nonlinear optics. The form of the nonlinear waves are maintained while they spread at fixed velocity. The solutions of nonlinear wave-type partial differential equations, including KDV, sine-Gordon and NLS are represented by solitons.

In this thesis, the theoretical and numerical analysis of optical solitons of nonlinear Schrödinger equation with a fourth-order dispersion term and $\mathscr{PT}$-symmetric potential is explored.

$$iu_z + u_{xx} + \alpha|u|^2u + \gamma u_{xxxx} + V_{PT}u = 0 \tag{1}$$

In Section 1, the historical background of studies on the optical solitons are investigated. The application areas and the structure of the NLS 4OD equation are expressed. General information about $\mathscr{PT}$-symmetric potentials are given. The purpose of the thesis, required literature review and hypothesis of the thesis are given, respectively.

In Section 2, the numerical method which is used to find a localized soliton solution is explained and then modified in order to be applied to the NLS equation with a fourth order dispersion term and an external potential. In order to compute localized solutions, using the spectral renormalization method. This method used the ansatz $u(x,z) = f(x)e^{i\mu z}$ where $f(x)$ is a complex-valued function and μ is the propagation constant. For stability analysis Split-step Fourier method is introduced.

In Section 3, (1+1)D 4OD cubic NLS equation without an external potential is considered. Exact solution of the equation is analysed and numerically produced results are depicted. The cubic nonlinear Schrödinger equation with a fourth order dispersion term (4OD) is given as follows:

$$iu_z + \beta u_{xx} + \gamma u_{xxxx} + |u|^2u = 0 \tag{2}$$

The analytical and the numerical solutions are shown be consistent with each other by graphs. Finally, the nonlinear stability of the soliton solutions are investigated and the produced results are discussed for various parameters of the considered equation.

In Section 4, the exact soliton solution of the (1+1)D 4OD cubic NLS equation with a $\mathscr{PT}$-symmetric potential is studied. This $\mathscr{PT}$-symmetric potential is introduced and by the use of this potential, soliton solutions are obtained for various values of parameters. To obtain non-zero stationary solitons, the following ansatz is used:

$$u(x,z) = f(x)e^{i(\mu z + g(x))} \tag{3}$$

The $\mathscr{P}\mathscr{T}$-symmetric potential is obtained :

$$V_{PT} = [V_1 \operatorname{sech}^2(x) + V_2 \operatorname{sech}^4(x)] + i[W_\star \operatorname{sech}^3(x) \tanh(x)]. \tag{4}$$

Exact and numerical results are compared, the maximum amplitude of the solitons are compared in terms of the parameters of the equation and the effect of the eigenvalue of the numerical solutions are figured out. The nonlinear stability of the produced solitons are demonstrated and compared in terms of various parameters.

In Section 5, the results of the study are discussed. In this thesis all of the results produced by MATLAB2016a computer programme.

$\mathscr{PT}$-SİMETRİK BİR POTANSİYEL İÇEREN DOĞRUSAL OLMAYAN YÜKSEK MERTEBE SCHRÖDINGER DENKLEMİNİN TEMEL SOLITONLARI

ÖZET

Soliton, akışkanlar dinamiği, biyolojik sistemler ve doğrusal olmayan optik gibi çeşitli alanlarda fiziksel problemlerin çözümünde ortaya konan matematiksel modellerin çözümlerinde elde edilen kararlı bir yapı olarak tanımlanır. Ortaya çıktıkları sistemdeki nonlineerite ve dispersiyonun dengelenmesiyle oluşan bu dalgalar (solitonlar) ilerlerken kendi şekillerini korurlar. KDV, sine-Gordon ve NLS gibi doğrusal olmayan dalga tipindeki kısmi diferansiyel denklemlerin çözümleri solitonlar tarafından temsil edilir. Evrendeki bütün fiziksel gözlemler reel büyüklükler ile ifade edilmelidir. Kuantum mekaniğinde gözlemler matematiksel olarak operatörlerin özdeğerlerine karşılık gelir. Böylelikle operatörlerin özdeğerlerinin reel olmaları gerekir. Bunu garanti edebilmek için bütün gözlemlerin Hermityan operatörlerin özdeğerlerine karşılık gelmesi gerekir. Fakat son yıllarda yapılan bazı çalışmalarda bu gerekliliğin zayıflatılabileceği gözlenmiş ve operatörlerin uzay-zaman simetrisini ($\mathscr{PT}$-simetri) sağlamasının fiziksel sonuçları tartışılmıştır.

Burada potansiyel $\mathscr{PT}$-simetrik Hamiltoniyen özelliği taşır, yani $V(x) = V^*(-x)$ ilişkisini sağlayan karmaşık bir potansiyele sahip tek boyutlu bir Schrödinger operatörü içerir. Bu tür problemlerin bir kısmı sayısal ve analitik tekniklerle tanımlanmıştır. $\mathscr{PT}$-simetrik potansiyeli aşağıdaki gibi tanımlanmıştır:

$$V_{PT} = V(x) + iW(x) \tag{5}$$

Burada $V(x)$ ve $W(x)$, sırasıyla, $\mathscr{PT}$-simetrik kompleks potansiyelin reel ve imajiner bileşenleridir. Potansiyelin reel bileşeni çift fonksiyon özelliğine sahipken, imajiner bileşeni tek bir fonksiyondur.

Bu çalışmada, aşağıda ifade edilen, dördüncü mertebeden ve bir dış potansiyel içeren doğrusal olmayan Schrödinger denklemlerinin optik soliton çözümlerinin sayısal olarak varlığı ve kararlılık (stabilite) analizleri bu çalışmada incelenmiştir.

$$iu_z + u_{xx} + \alpha|u|^2u + \gamma u_{xxxx} + V_{PT}u = 0 \tag{6}$$

Verilen denklemde u kompleks değerli türevlenebilir fonksiyonu, u_{xx} kırılımı modelleyen terimi, α üçüncü mertebeden doğrusal olmayan terimin katsayısını, γ terimi dördüncü mertebeden dispersiyon teriminin katsayısını ve V_{PT} $\mathscr{PT}$-simetrisi özelliği sağlayan potansiyeli temsil eder. Bu tezin amacı, dördüncü mertebe dispersiyon teriminin (u_{xxxx}) ve V_{PT} $\mathscr{PT}$-simetrisi özelliği sağlayan potansiyelin, soliton çözümünde ve bu çözümlerin kararlılığında yarattığı etkiyi gözlemlemektir.

Bölüm 1'de, optik solitonlarla ilgili çalışmaların tarihsel gelişimleri anlatılmıştır. Dördüncü mertebeden dispersiyon terimi ve $\mathscr{PT}$-simetrik potansiyel içeren,

doğrusal olmayan Schrödinger denkleminin yapısı ve uygulama alanları anlatılmıştır. Denklemin çözümünde kullanılmış olan sayısal analiz metotları ve bunların tarihsel gelişimleri irdelenmiştir. Ayrıca, dördüncü mertebe dispersiyon terimi içeren NLS denkleminin soliton çözümlerini analitik ve sayısal olarak inceleyen çalışmalar analiz edilmiştir. $\mathscr{PT}$-simetrik potansiyelin fiziksel anlamı ve sağlaması gereken özellikler belirtilmiştir. Bu bölümde, spektral renormalizasyon (SR) metodunun temel yaklaşımı ve gelişimi açıklanıp, literatürde bu metodun kullanıldığı diğer problemlerden bahsedilmiştir. Tezin amacı, gerekli literatür taraması ve tezin hipotezi sırasıyla verilmiştir.

Bölüm 2'de dış potansiyel içeren ve dördüncü mertebeden dispersiyon terimi bulunan NLS denkleminin soliton çözümü elde etmek için Ablowitz ve Musslimani'nin ortaya koyduğu Spektral Renormalizasyon (SR) yöntemi uygulanmış ve denklemin sayısal çözümleri bu yöntemin bir modifikasyonu ile elde edilmiştir. Bu yöntem, $u(x,z) = f(x)e^{i\mu z}$ yaklaşımını kullanır ve $f(x)$ kompleks değerli fonksiyonunu Fourier uzayında iteratif olarak çözer. Daha sonra, Ayrık adımlı Fourier metodu (Split-step Fourier Method) kullanılarak, elde edilen solitonların kararlılık analizi araştırması yapılmıştır.

Bölüm 3'de potansiyelsiz halde, (1+1) boyutlu dördüncü mertebeden bir dispersiyon terimi içeren, kübik NLS denklemi ele alınmıştır. Bu denklem aşağıdaki gibi verilir:

$$iu_z + \beta u_{xx} + \gamma u_{xxxx} + |u|^2 u = 0 \tag{7}$$

Literatürde, bu denklemin analitik çözümleri

$$u(x,z) = \sqrt{\frac{3\beta^2}{10\gamma}} \sec h^2 \left(\frac{x}{\sqrt{20\gamma/\beta}} \right) \exp \left(i \frac{4\beta^2}{25\gamma} z \right) \tag{8}$$

formunda elde edilmiş ve belli parametreler için bu soliton tipi çözümler incelenmiştir. Uyguladığımız sayısal algoritmanın doğruluğunu test etmek ve bu çözümleri daha derinlemesine inceleyebilmek için, Spektral renormalizasyon (SR) metodu ile bu denklem çözülmüş ve literatürde var olan analitik çözümler ile, SR algoritmasından elde edilen çözümler karşılaştırılarak, bu iki çözümünün üst üste düştüğü gösterilmiştir. Bu çözümlerin soliton yapılarının dördüncü mertebeden dispersiyon teriminin büyüklüğü ile olan değişimi analiz edilmiş ve daha sonra bu solitonların kararlılıkları incelenmiştir.

Bölüm 4'de (1+1) boyutlu dördüncü mertebeden dispersiyon terimi ve $\mathscr{PT}$-simetrik potansiyel içeren kübik NLS denklemi ele alınmıştır. Analitik çözümleri üretebilmek için $u(x,z) = f(x)e^{i(\mu z + g(x))}$ çözüm önerisi yapılmıştır. Burada $f(x)$ ve $g(x)$ henüz yapısı belli olmayan reel değerli fonksiyonlar olarak kabul edilmiştir. Bu çözüm önerisi denklemde yerine konarak, $\mathscr{PT}$-simetrik potansiyelin yapısı aşağıdaki şekilde elde edilmiştir

$$V_{PT} = [V_1 \mathrm{sech}^2(x) + V_2 \mathrm{sech}^4(x)] + i[W_3 \mathrm{sech}^3(x) \tanh(x)]. \tag{9}$$

Potansiyelin yapısındaki katsayıların soliton çözümüne olan etkisi sayısal olarak incelenmiş ve etkileri tartışılmıştır. Maksimum genlik ile dispersiyon teriminin katsayısı olan (γ) ve özdeğer (μ) arasındaki değişimler yine sayısal olarak incelenmiş ve çeşitli grafiklerle elde edilen sonuçlar gösterilmiştir. Spektral renormalizasyon metodu ile üretilen soliton çözümleri, üretilen analitik çözüm ile farklı parametreler için karşılaştırılmış ve bu çözümlerin üstüste düştüğü gösterilmiştir. Daha sonra, elde edilen bu soliton çözümlerinin kararlılık analizleri yapılmış ve elde edilen sonuçlar grafikler üzerinde gösterilmiş ve tartışılmıştır. Dispersiyon teriminin katsayısı (γ) ve özdeğerinin(μ) solitonun kararlılığı üzerindeki etkisi incelenip, kararlı olan ve olmayan solitonlar gösterilip nedenleri tartışılmıştır.

Bölüm 5'de tezde elde edilen tüm sonuçlar ayrıntılı olarak açıklanmıştır. Potansiyelsiz denklemde ve bir dış potansiyel içeren denklemdeki sonuçlar özetlenip, sisteme eklenen dış potansiyelin etkisi tartışılmıştır.

Bu tezde sunulan bütün çözümler MATLAB2016a bilgisayar programı kullanılarak üretilmiştir.

1. INTRODUCTION

In the field of optics, a soliton denote to any optical field that does not change its shape during propagation because of the sensitive balance between linear and nonlinear effects in the medium [1]. Optical solitons have become the main area for studying solitons' interactions in the last few years.

Study of the solitary wave solution of the nonlinear Schrödinger (NLS) equation is one of the important works in nonlinear science [2]. There have been many studies attempting to find out the analytical solution of the NLS equation. These results have important scientific values and application prospects. The nonlinear evolution of short pulses in an optical fiber is usually described by the NLS equation. It is well known that the NLS equation properly describes the nonlinear dynamics of pulses on a picosecond time-scale Mathematical and numerical analysis of the considered equation with application areas can be found in the reference [3].

The propagation of an optical pulse in optical materials is described by the nonlinear Schrödinger equation:

$$iu_z + u_{xx} + \alpha|u|^2 u = 0. \tag{1.1}$$

In optics, u corresponds to the differentiable complex valued, slowly varying amplitude of the electric field; z is a scaled propagation distance; u_{xx} corresponds to diffraction; the coefficients α represents the cubic nonlinearities of the medium.

It is known that, NLS equation Eq. (1.1), does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses are short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasimonochromaticity is no longer valid and consequently higher order dispersion terms are needed. If the group velocity dispersion is close to zero, one needs to consider the third order dispersion for performance enhancement along trans-oceanic and trans-continental distances. Also, for short pulse widths where the group velocity dispersion changes, within the spectral bandwidth of the signal, can no longer be neglected. This reasoning leads to the inclusion of the fourth dispersion terms to the model equation.

In this thesis, we consider the model equation, which includes the higher-order dispersion terms, is called fourth-order dispersion (4OD) nonlinear Schrödinger equation with a $\mathscr{PT}$-symmetric optical potential:

$$iu_z + u_{xx} + \alpha|u|^2 u + \gamma u_{xxxx} + V_{PT} u = 0. \tag{1.2}$$

Here γ is a fourth-order diffraction coupling constant taken to be negative and V_{PT} is a $\mathscr{PT}$-symmetric external potential.

Aim of this thesis is to explore the effect of fourth order dispersion term γu_{xxxx} on the soliton properties and their stabilities. Assuming the case of 4OD cubic nonlinear Schrödinger equation without a $\mathscr{PT}$-symmetric potential, as it is stated [4], this equation represent the mathematical model of the evolution of the ultrashort optical pulses in fibers. In [4], detailed explanations of the effects of the both second and negative fourth order dispersion terms are given. Additionally, the properties of the soliton solutions, Hamiltonian forms and stability analysis of considered case are studied in [5] and [6]. Note that the sign of the parameter γ has importance in existence and stability sense and it will be discussed in Chapter 4.

Any measurement of a physical observable in our universe obviously yields a real quantity. In quantum mechanics, observables correspond to eigenvalues of operators. Hence, the reality requires all the eigenvalues of operators to be real. To guarantee a real spectrum, it was postulated that all observables corresponded to eigenvalues of Hermitian (i.e. self adjoint) operators. In fact, a Hermitian Hamiltonian ensures a real energy spectrum. However, in the past decade there has seen considerable attention [7–10] in a weaker version of Hermicity axiom which requires that the Hamiltonian instead only space time reflection symmetry or $\mathscr{PT}$-symmetry. Furthermore, they showed in many cases a threshold value above which the spectrum becomes complex. This threshold is the boundary between the $\mathscr{PT}$-symmetric and broken symmetry phases and in literature, the transition is referred to as spontaneous $\mathscr{PT}$-symmetry breaking. $\mathscr{PT}$-symmetric is defined by means parity operator $\hat{P}$ and the time operator $\hat{T}$ whose actions are given by $\hat{P}: \hat{p} \rightarrow -\hat{p}, \hat{x} \rightarrow \hat{x}, i \rightarrow -i$, where $\hat{p}$ is the momentum operator, $\hat{x}$ is the position operator and i is the imaginary unit [11]. A Hamiltonian

$\hat{H} = \hat{p}^2 + V(x)$ is said to be $\mathscr{P}\mathscr{T}$-symmetric if it has the same eigenfunctions, as the $\hat{P}\hat{T}$ operator and satisfies the commutativity $\hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T}$, namely $V(x) = V^*(-x)$ [12]. One speaks of broken $\mathscr{P}\mathscr{T}$-symmetry if the latter is satisfied but the same eigenfunctions are not shared. $\mathscr{P}\mathscr{T}$ symmetric structures have been realized in optical models governed by NLS type equations in which the propagation distance z replaces time in quantum mechanics [8].

Assuming the case of 4OD cubic nonlinear Schrödinger equation with a $\mathscr{P}\mathscr{T}$ -symmetric potential, we will consider the following $\mathscr{P}\mathscr{T}$-symmetric potential

$$V_{PT} = V(x) + iW(x) \tag{1.3}$$

where $V(x)$ and $W(x)$ are the real and imaginary components of the complex $\mathscr{P}\mathscr{T}$-symmetric potential, respectively. Evidently, the real part of a $\mathscr{P}\mathscr{T}$ potential must be a symmetric function of a position whereas the imaginary component should be anti-symmetric. In Figure 1.1, we plotted the real and the imaginary parts of the $\mathscr{P}\mathscr{T}$-symmetric potential that is derived in Chapter 4.

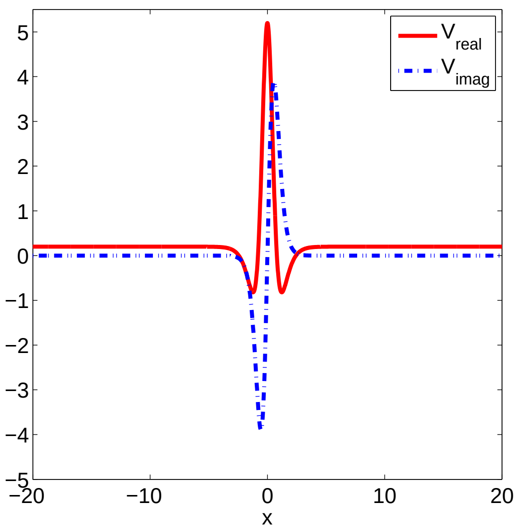

Figure 1.1 : Real and imaginary parts of $\mathscr{P}\mathscr{T}$-symmetric potential plotted on top of each other.

In this thesis, we will use spectral renormalization scheme with which we can compute localized solutions in nonlinear waveguides [13]. The main idea of this method is to transform the considered equation into Fourier space and find out a nonlinear integral equation coupled to an algebraic equation. The main advantages and detailed explanation of the spectral renormalization method can be found in

[13]. Implementation procedure of this method to the nonlinear Schrödinger equation explained in Chapter 2.

1.1 Purpose of Thesis

In this thesis, we aim to investigate the effect of the external potential on the soliton solutions of the 4OD cubic nonlinear Schrödinger equation. The cases of 4OD cubic nonlinear Schrödinger equation without a potential and with a special type of $\mathscr{PT}$-symmetric potential are compared to understand this effect.

1.2 Literature Review

Optical solitons have been the objects of extensive theoretical and experimental studies in recent years because of their applications to telecommunication and ultrafast signal routing systems [14]. They evolve from a nonlinear change in the refracive index of a material induced by the light intensity distribution [15]. In the picosecond regime, the main nonlinear equation governing the pulse evolution is the nonlinear Schrödinger equation (NLS) [16].

The NLS equation represents the mathematical models of various physical problems. The nonlinear evolution of short pulses in an optical fiber is usually denoted by the nonlinear Schrödinger (NLS) equation [17]. The equation appears in the studies of the propagation of plane-diffracted wave beams in the focusing regions of the ionosphere [18], decay problem of the ps degenerate soliton and the effect of the frequency downshift [19,20]. Additionally, wave propagation in nonlinear media [21], surface waves on deep waters [22] and signal propagation in optical fibers [14,23–26] are denoted by NLS equation [1].

Mainly, cubic nonlinear Schrödinger equation with fourth order dispersion term considered in this study. NLS 4OD equation represent the mathematical model of the optical pulses in fibers [4]. There have been many studies to find out analytical and numerical solutions of the considered equation. In the study of the [27], the method of solitary wave ansatze is used to produce exact solution of the NLS 4OD equation. The method of $sine-cosine$ and the method of $tanh$ are analyzed mathematically and applied to the considered 4OD NLS equation to obtain exact solutions [28]. Also in [29], the NLS equation including both third and fourth order dispersion terms investigated and analytical results are figured out.

Spectral renormalization method is a numerical approach for constructing localized solutions of a nonlinear system based on transforming to Fourier space, first introduced by [30]. The method is used to produce numerical results of the various nonlinear systems and partial differential equations. In order to find out localized solution of KDV equation [31], dispersion-managed systems [32], discrete diffraction-managed systems [33, 34] and NLS equation [35] the spectral renormalization method is used. (2+1)D and (1+1)D NLS equation with an external potential was solved using spectral renormalization method and the produced results are shown in [36, 37] and [38], respectively.

1.3 Hypothesis

The effect of the external potential and its type on the existence and stability of fundamental solitons is crucial. The maximum amplitude and the change in solitons are affected by the dispersion coefficient γ. The selection of this parameter has great importance in terms of existence and stability of the soliton solutions.

2. NUMERICAL METHODS

2.1 Spectral Renormalization Method

In order to compute localized solutions (i.e., soliton solutions) to nonlinear evolution equations, various techniques have been used. Numerical solutions to Eq. (1.1) are the sought by means of the Spectral renormalization method which is essentially a Fourier iteration method. The idea of this method was proposed by Petviashvili in [30].

Later, this method was improved by Ablowitz and Musslimani [13] a generalized numerical scheme for computing solitons in nonlinear wave guides (SR). The essence of the method is to transform the governing equation Fourier space and find a nonlinear nonlocal integral equation coupled to an algebraic equation. The coupling prevents the numerical scheme from diverging.

The optical mode is then obtained from an iteration scheme, which converges rapidly. This method can efficiently be applied to a large class of problems including higher order nonlinear terms with different homogenetic.

In this section, numerical solution to the NLS 4OD cubic equation with an external potential given in Eq. (1.2) will be obtained by the spectral renormalization method.

The method is configured so that it can be applied mainly to the (1+1)D NLS 4OD cubic equation with $\mathscr{P}\mathscr{T}$-symmetric potential as follows:

$$iu_z + u_{xx} + \alpha|u|^2u + \gamma u_{xxxx} + V_{PT}u = 0. \tag{2.1}$$

Using the ansatz $u(x,z) = f(x)e^{i\mu z}$ where $f(x)$ is a complex-valued function and μ is the propagation constant (or eigenvalue) leads one to the following expressions:

$$\begin{aligned} u_z &= i\mu f e^{i\mu z} \\ u_{xx} &= f_{xx}e^{i\mu z} \\ u_{xxxx} &= f_{xxxx}e^{i\mu z} \\ u^* &= f e^{-i\mu z} \\ |u|^2 &= |f|^2. \end{aligned} \tag{2.2}$$

Substituting the set of the terms in Eq. (2.2) into Eq. (2.1), the following nonlinear equation for f is obtained

$$-\mu f e^{i\mu z} + f_{xx}e^{i\mu z} + \alpha|f|^2 f e^{i\mu z} + \gamma f_{xxxx}e^{i\mu z} + V_{PT} f e^{i\mu z} = 0. \tag{2.3}$$

simplifying these equations we obtain

$$-\mu f + f_{xx} + \alpha|f|^2 f + \gamma f_{xxxx} + V_{PT} f = 0. \tag{2.4}$$

After applying Fourier transformation to Eq. (2.4)

$$\mathscr{F}\{-\mu f\} + \mathscr{F}\{f_{xx}\} + \mathscr{F}\{\alpha|f|^2 f\} + \mathscr{F}\{\gamma f_{xxxx}\} + \mathscr{F}\{V_{PT} f\} = \mathscr{F}\{0\}. \tag{2.5}$$

where $\mathscr{F}$ denotes Fourier transformation and considering the properties of this transformation, following equation is obtained

$$-\mu \hat{f} + (-ik_x)^2 \hat{f} + \alpha \mathscr{F}\{|f|^2 f\} + \gamma(ik_x)^4 \hat{f} + \mathscr{F}\{(V+iW)f\} = 0 \tag{2.6}$$

where $\mathscr{F}(f) = \hat{f}$ and k_x are Fourier variables. Solving Eq. (2.6) for the $\hat{f}$ yields

$$\hat{f} = \frac{\alpha \mathscr{F}\{|f|^2 f\} + \mathscr{F}\{(V+iW)f\}}{[\mu + {k_x}^2 - \gamma {k_x}^4]} \tag{2.7}$$

This equation could be indexed and utilized as an iteration to find $f(x)$, but the scheme does not converge. However, introducing a new field variable $f(x) = \lambda w(x)$ with $\lambda \in R^+$ where λ is a parameter to be determined. The system with the new variable can be written as

$$\lambda \hat{w} = \frac{\alpha \mathscr{F}\{|w|^2|\lambda|^2 w\lambda + (V+iW)\lambda w\}}{\mu + {k_x}^2 - \gamma {k_x}^4} \tag{2.8}$$

simplifying this equation, we get

$$\hat{w} = \frac{\alpha \mathscr{F}\{|w|^2|\lambda|^2 w\} + \mathscr{F}\{(V+iW)w\}}{\mu + {k_x}^2 - \gamma {k_x}^4} \tag{2.9}$$

Eq. (2.9) can be utilized in an iterative method in order to find out w. For this purpose, $\hat{w}$ can be calculated using the following iteration approach:

$$\hat{w}_{n+1} = \frac{\alpha|\lambda|^2 \mathscr{F}\{|w_n|^2 w_n\} + \mathscr{F}\{(V+iW)w_n\}}{\mu + {k_x}^2 - \gamma {k_x}^4}, \quad n \in N \tag{2.10}$$

with the initial condition taken as a Gaussian type function

$$w_0 = e^{-x^2} \tag{2.11}$$

where our convergence criterions are $|w_{n+1} - w_n| < 10^{-12}$. Multiplying both sides of Eq. (2.9) by $(\mu + {k_x}^2 - \gamma {k_x}^4)$ yields to

$$(\mu + {k_x}^2 - \gamma {k_x}^4)\hat{w} = |\lambda|^2 \alpha \mathscr{F}\{|w|^2 w\} + \mathscr{F}\{(V+iW)w\}. \tag{2.12}$$

Taking all terms of Eq. (2.12) to the left side lead to following equation

$$(\mu + {k_x}^2 - \gamma {k_x}^4)\hat{w} - |\lambda|^2 \alpha \mathscr{F}\{|w|^2 w\} + \mathscr{F}\{(V+iW)w\} = 0. \tag{2.13}$$

Multiplying Eq. (2.13) by the conjugate of $\hat{w}$, i.e. by $\hat{w}^*$ yields

$$(\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2 - |\lambda|^2 \alpha \mathscr{F}\{|w|^2 w\}\hat{w}^* + \mathscr{F}\{(V+iW)w\}\hat{w}^* = 0. \tag{2.14}$$

Hence, integrating Eq. (2.14) leads to

$$\begin{aligned} &\int_{-\infty}^{\infty} (\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2 dk - |\lambda|^2 \int_{-\infty}^{\infty} \alpha \mathscr{F}\{|w|^2 w\}\hat{w}^* dk \\ &+ \int_{-\infty}^{\infty} \mathscr{F}\{(V+iW)w\}\hat{w}^* dk = 0 \end{aligned} \tag{2.15}$$

or in a more compact form

$$\begin{aligned} &-\int_{-\infty}^{\infty} \left[\mathscr{F}\{(V+iW)w\}\hat{w}^* + (\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2\right] dk \\ &+ |\lambda|^2 \int_{-\infty}^{\infty} \alpha \mathscr{F}\{|w|^2 w\}\hat{w}^* dk = 0. \end{aligned} \tag{2.16}$$

Eq. (2.16) is a second order polynomial of λ in the form $P(\lambda) = a\lambda^2 + b$ then λ can be calculated exactly by the imposing following formula:

$$\lambda_{1;2} = \pm\sqrt{\frac{b}{a}} \tag{2.17}$$

where

$$a = \alpha \int_{-\infty}^{\infty} \mathscr{F}\{|w|^2 w\}\hat{w}^* dk \tag{2.18}$$

$$b = -\int_{-\infty}^{\infty} [\mathscr{F}\{(V+iW)w\}\hat{w}^* + (\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2] dk. \tag{2.19}$$

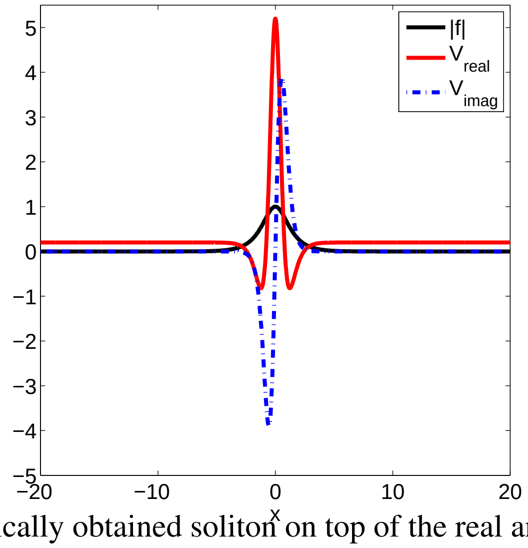

Figure 2.1 : Numerically obtained soliton on top of the real and imaginary parts of $\mathscr{PT}$-symmetric potential.

When the iteration convergence, the required soliton is $f(x) = \lambda(wx) = \lambda \mathscr{F}^{-1}(\hat{w})$.

In Fig. 2.1, the soliton obtained by the method described above is plotted on top of the real and the imaginary parts of the specific $\mathscr{PT}$-symmetric potential which is derived in Chapter 4.

2.2 Nonlinear Stability Analysis

A soliton should preserve its shape, location and maximum amplitude during direct simulations, in order to be considered as nonlinearity stable. In order to investigate the nonlinear stability of solitons, we directly compute Eq. (1.2) over a long distance. For this purpose, split-step Fourier method is employed to advance in z [39].

3. NLS 4OD EQUATION WITHOUT AN EXTERNAL POTENTIAL

3.1 Exact and Numerical Solutions

3.1.1 Exact solution

Modern fiber manufacturing techniques provide experimentalists with fibers having an extensive range of dispersive behaviour. When studying ultrashort pulses in such fibers we realize that not only the second order dispersion is important, but also its slope (third order dispersion, 3OD) and curvature (fourth order dispersion, 4OD) become significant. Specifically, at the frequency of ω_0 of minimum/maximum group velocity dispersion, the third order dispersion (3OD) vanishes and 4OD becomes the next higher-order dispersion. This situation was studied by Karlsson and Höök [4] for positive fourth order dispersion and it was found that pulses in such media will always loose power by radiation. However, the case of negative 4OD dispersion leads one to new solitary wave structures. In above mentioned work, the exact stationary solution of 4OD NLS with negative fourth order dispersion is also given.

In this section, we will use solve 4OD NLS equation without an external potential by the use of spectral renormalization method and investigate the soliton properties in order to compare the analytical solution given in [4] with the numerical solution. We also investigate the effect of the fourth order dispersion term by the use of the numerical method (SR).

The cubic nonlinear Schrödinger equation with a fourth order dispersion term (4OD) is given as follows:

$$iu_z + \beta u_{xx} + \gamma u_{xxxx} + |u|^2 u = 0 \tag{3.1}$$

where $x, z \in R$ and $u = u(x,z)$ is a complex-valued function. Eq. (3.1) was introduced in [4] and stationary solution of this equation is given as

$$u(x,z) = \sqrt{\frac{3\beta^2}{10\gamma}} \sec h^2 \left(\frac{x}{\sqrt{20\gamma/\beta}} \right) \exp\left(i \frac{4\beta^2}{25\gamma} z \right) \tag{3.2}$$

When we compared to the general NLS-soliton, this solution has no free parameter, and it cannot be given a relative velocity. It should be noted that, this type of 'fixed parameter' solutions have been found earlier [40, 41]. Additionally, the particular solution Eq. (3.2) could very well belong to a class of solutions with an amplitude-width relation similar to that of the NLS soliton [4]. These solutions should have the same $\sec h^2$-shape as the pulse in Eq. (3.2).

Eq. (3.1) is also investigated in [42] and [43] in connection with the nonlinear fiber optics and the theory of optical solitons in gyrotropic media.

3.2 Numerical Illustrations

In this subsection we will numerically demonstrate the soliton solution of Eq. (3.1) for various values of γ (40D term's coefficient) and the propagation constant μ.

First we show the exact soliton solution and the numerically obtained soliton solution of Eq. (3.1) in Fig. 3.1. As can be seen from this figure, our numerical algorithm converges to the exact solution for the parameters $\gamma = -1$, $\mu = 0.16$.

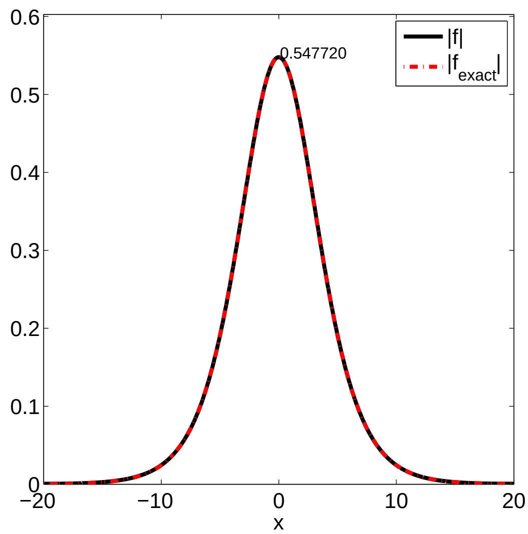

Figure 3.1 : Exact and numerical solutions of 4OD NLS equation on top of each other for $\gamma = -1, \mu = 0.16$

In order to investigate the effect of the fourth order dispersion and the eigenvalue on the soliton properties, we plotted solitons for various values of γ and μ.

In Fig. 3.2, now we fixed the coefficient of the 4OD dispersion as $\gamma = -1$ and then increased the eigenvalue to $\mu = 1$. In this case, we observe a larger increment in the maximum amplitude of the soliton comparing to $\gamma = -1, \mu = 0.16$ case, namely

$max|f| = 1.33$ and the soliton becomes more steep. One interesting observation about this soliton is the difference in its tails. This phenomenon was also observed by Karpman in [42] and he called these higher order solitons as "solitons with oscillating tails".

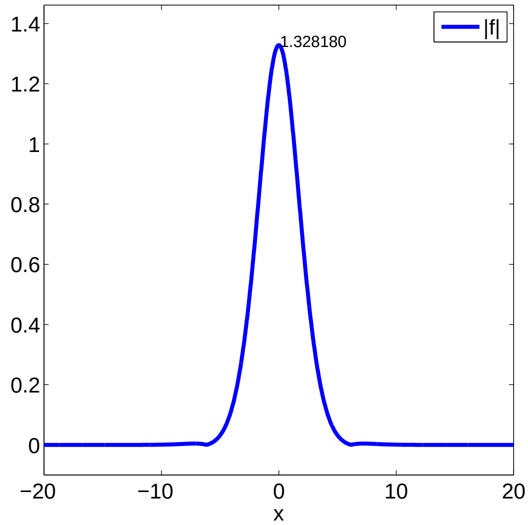

Figure 3.2 : Numerically obtained higher order mode for $\gamma = -1$, $\mu = 1$.

In Fig. 3.3, we increased the eigenvalue to $\mu = 2$ and this figure reveals that, the tails become more pronounced and the maximum amplitude increases to $max|f| = 1.86$.

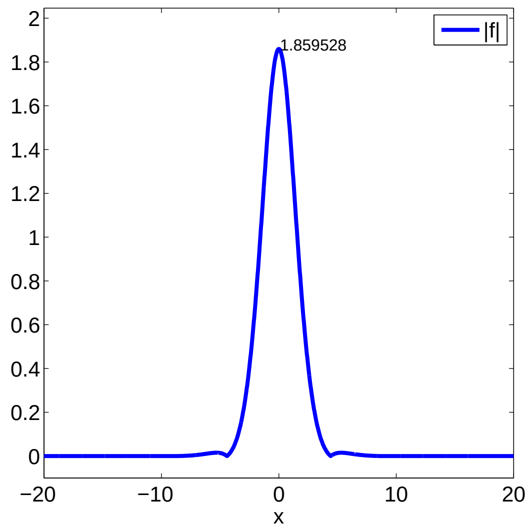

Figure 3.3 : Numerically obtained higher order mode for $\gamma = -1$, $\mu = 2$.

In order to observe the effect of the increasing eigenvalue μ, in the next figures, we decreased the effect of the 4OD dispersion by setting $\gamma = -0.1$ and gradually increased the eigenvalue μ to $\mu = 0.16$ and $\mu = 2$ respectively.

In Fig. 3.4, we observe a small increment in the maximum amplitude of the soliton comparing to larger 4OD effect case shown in In Fig. 3.1 but the shape of the soliton is more or less the same with that figure.

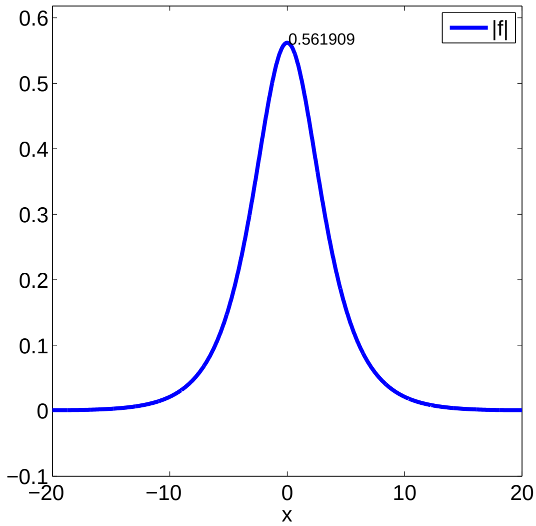

Figure 3.4 : Numerically obtained higher order mode for $\gamma = -0.1$, $\mu = 0.16$.

In Fig. 3.5, we observe an increment in the maximum amplitude of the soliton comparing to larger 4OD effect case shown in Fig. 3.3 and oscillating tails seem to disappear.

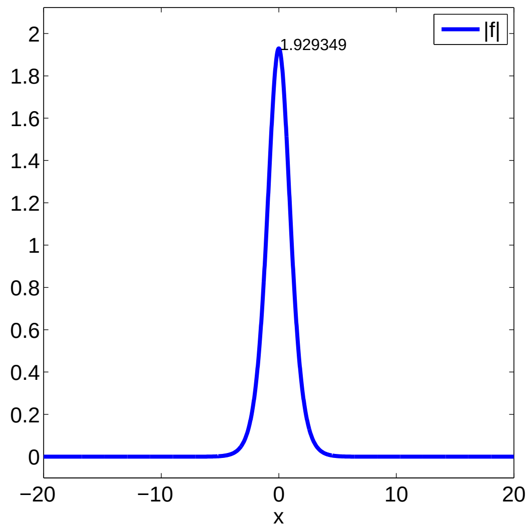

Figure 3.5 : Numerically obtained higher order mode for $\gamma = -0.1$, $\mu = 2$.

3.3 Nonlinear Stability

In this section, we will numerically demonstrate how the shape and the maximum amplitude of a higher order soliton effect its nonlinear stability properties. In order to investigate this, obtained solitons are computed over a long distance and changes in the shape, their maximum amplitudes and locations during the evolution are monitored. We investigated the previously obtained solitons in the same order.

First we took the soliton solution shown in Fig. 3.1 and evolved it for $z = 50$. The results are shown in Fig. 3.6. This figure reveals that, this soliton is nonlinearly stable as it preserves its shape and maximum amplitude during the evolution.

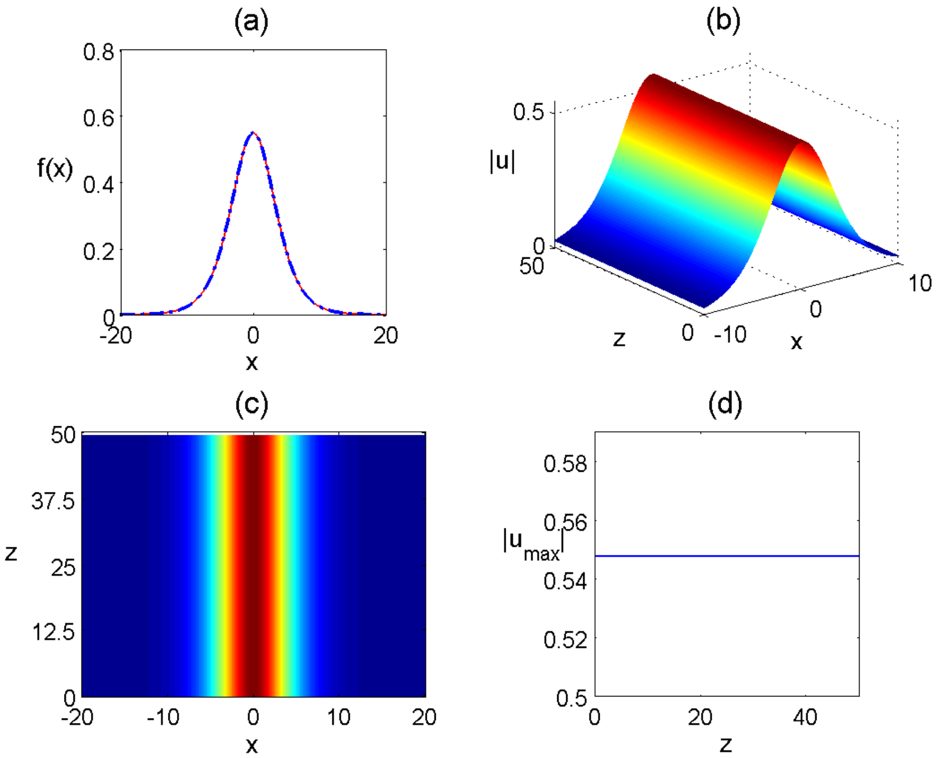

Figure 3.6 : Nonlinear stability of a higher order soliton for $\gamma = -1, \mu = 0.16$;(a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In Fig. 3.7, we show a nonlinear unstable higher order soliton obtained for $\gamma = -1, \mu = 1$. It is seen that, around $z = 36$, the maximum amplitude starts to decrease dramatically as a result of the deterioration in the shape of the soliton.

In Fig. 3.8, we show a nonlinear unstable higher order soliton obtained for $\gamma = -1, \mu = 2$. It is seen from the figure that, around $z = 40$, the maximum amplitude starts to decrease dramatically as a result of the deterioration in the shape of the soliton.

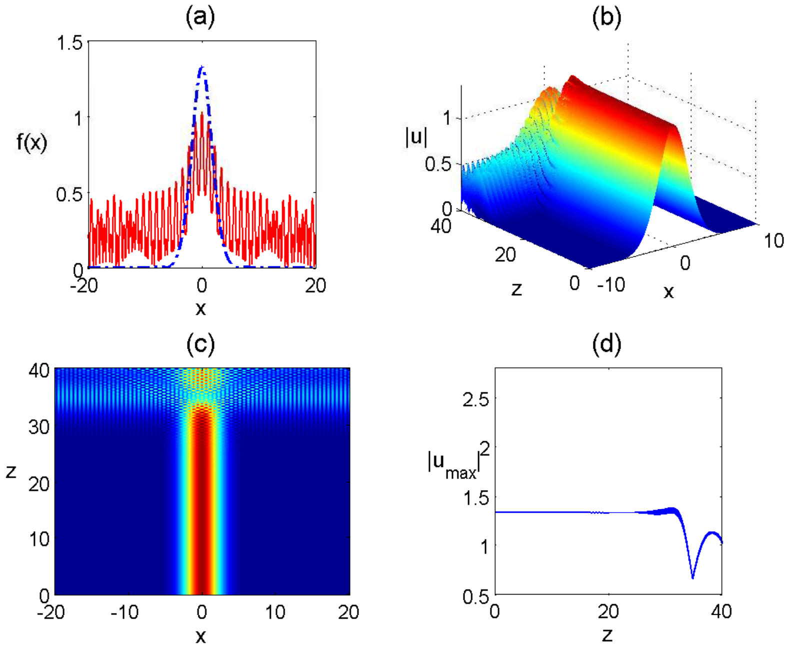

Figure 3.7 : Nonlinear instability of a higher order soliton for $\gamma = -1, \mu = 1$;(a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

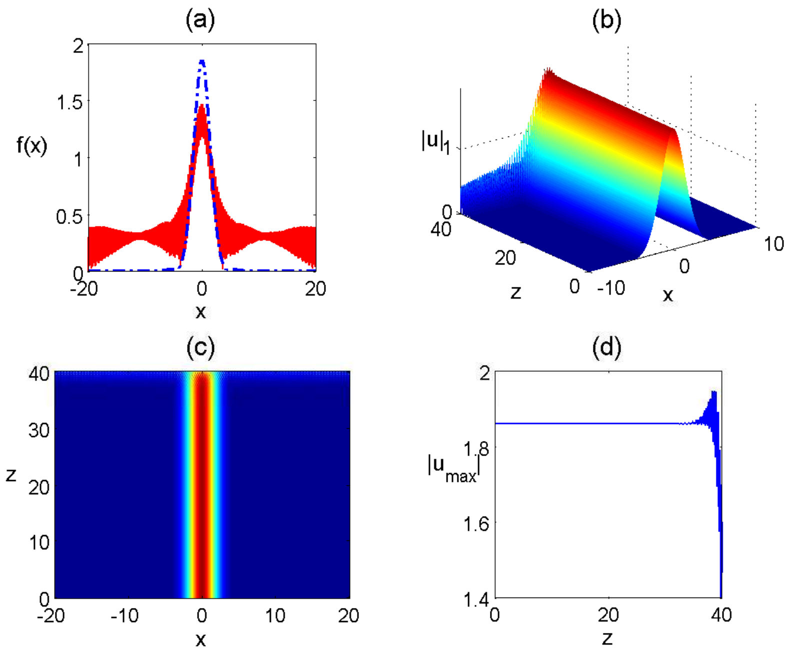

Figure 3.8 : Nonlinear instability of a higher order soliton $\gamma = -1, \mu = 2$;(a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

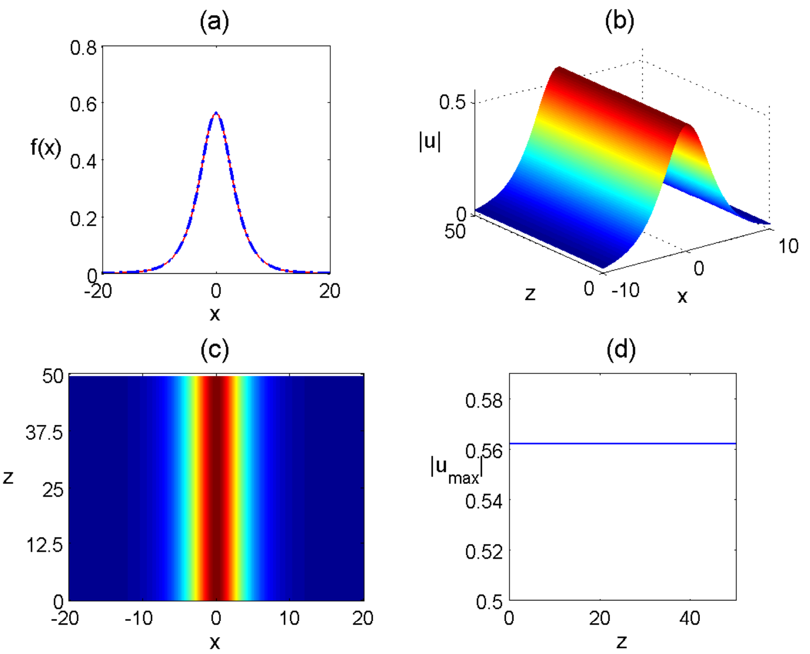

Figure 3.9 : Nonlinear stability of a higher order soliton for $\gamma = -0.1, \mu = 0.16$;(a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

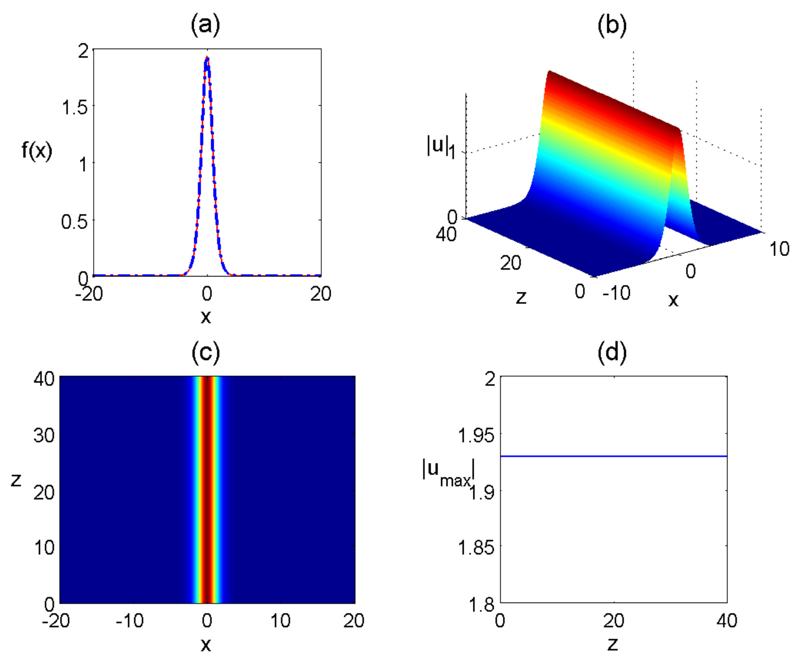

Figure 3.10 : Nonlinear stability of a higher order soliton for $\gamma = -0.1, \mu = 2$; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In Fig. 3.9, we investigated the case given in Fig. 3.4. It can be seen from the figure that, this soliton is nonlinearly stable as well since it preserves its shape and maximum amplitude during the evolution.

To observe the effect of the higher eigenvalue in this law-effect 4OD case, now we plotted the evolution of the soliton given in Fig. 3.5 and in Fig. 3.10 it is shown that, even for a higher eigenvalue of $\mu = 2$, the stability properties of the soliton is conserved.

As a result of the numerical observations we made in this section, we conclude that: (*i*) The higher order solitons obtained for 4OD NLS equation may have oscillating tails and these solitons are found to be nonlinearly *unstable* (cases obtained for both large 4OD effect and large eigenvalue μ); (*ii*) The higher order solitons obtained for 4OD cubic NLS equation are found to be nonlinearly *stable* for the cases when the effect of the 4OD effect is either small or the eigenvalue is μ is small (even if the 4OD effect is large).

4. NLS 4OD EQUATION WITH AN EXTERNAL POTENTIAL

4.1 Exact and Numerical Solutions

4.1.1 Exact solution

Exact solutions are useful to understand the mechanism of the complicated nonlinear physical phenomena which are related to wave propagation in a higher-order CNLS equation with $\mathscr{PT}$-symmetric potential.

Consider the following (1+1)D 4OD cubic NLS equation with a $\mathscr{PT}$-symmetric potential:

$$iu_z + u_{xx} + \alpha|u|^2 u + \gamma u_{xxxx} + V_{PT} u = 0. \tag{4.1}$$

Here $u = 0$ is an trivial solution of Eq. (4.1). To get non-zero solutions, set $u \neq 0$. Dividing Eq. (4.1) by u and by use of Eq. (1.3) yields

$$i\frac{u_z}{u} + \frac{u_{xx}}{u} + \alpha|u|^2 + \gamma\frac{u_{xxxx}}{u} + V + iW = 0. \tag{4.2}$$

To obtain non-zero stationary solitons, the following ansatz is used:

$$u(x,z) = f(x)e^{i(\mu z + g(x))} \tag{4.3}$$

where u is a function of x and z to be determined, $f(x)$ and $g(x)$ are real-valued functions different than zero and μ is the propagation constant. Taking derivatives of Eq. (4.3) with respect to z and x, leads to following equations,

$$u_z = f(x)i\mu e^{i(\mu z + g(x))} \tag{4.4}$$

$$u_{xx} = [f''(x) + 2if'(x)g'(x) + if(x)g''(x) - f(x)(g'(x))^2]e^{i(\mu z + g(x))} \tag{4.5}$$

$$|u|^2 = f(x)e^{i(\mu z + g(x))} f(x)e^{-i(\mu z + g(x))} = (f(x))^2 \tag{4.6}$$

$$\begin{aligned} u_{xxxx} = [&f''''(x) + 4if'''(x)g'(x) + 6if''(x)g''(x) + 4if'(x)g'''(x) + if(x)g''''(x) \\ &-6f''(x)(g'(x))^2 - 12f'(x)g'(x)g''(x) - 3f(x)(g''(x))^2 - 4f(x)g'(x)g'''(x) \\ &-4if'(x)(g'(x))^3 - 6if(x)(g'(x))^2 g''(x) + f(x)(g'(x))^4]. \end{aligned} \tag{4.7}$$

Substituting Eq. (4.4)-Eq. (4.7) into Eq. (4.2) yields

$$\begin{aligned}
&[-\mu+\alpha f^2(x)\frac{f''(x)}{f(x)}+\gamma\frac{f''''(x)}{f}-(g'(x))^2-6\gamma\frac{f''(x)}{f(x)}(g'(x))^2+\gamma(g'(x))^4\\
&-3\gamma(g''(x))^2-12\gamma\frac{f'(x)}{f(x)}g'(x)g''(x)-4\gamma g'(x)g'''(x)+V(x)]\\
&+i[-4\gamma\frac{f'(x)}{f(x)}(g'(x))^3+4\gamma\frac{f'(x)}{f(x)}g'''(x)+2\frac{f''(x)}{f(x)}g''(x)+6\gamma\frac{f''(x)}{f(x)}g''(x)\\
&+4\gamma\frac{f'''(x)}{f(x)}g'(x)+g''(x)-6\gamma(g'(x))^2g''(x)+\gamma\frac{g''''(x)}{f}+W(x)]=0.
\end{aligned} \tag{4.8}$$

To obtain soliton solutions, the following ansatz is used

$$f(x)=f_0\sec h^p(x),\quad g'(x)=g_0\sec h^q(x) \tag{4.9}$$

where f_0 and g_0 are non-zero real constants and $p\in N$. We need to evaluate the derivatives of the functions f and g to construct simple form of Eq. (4.8). By using Eq. (4.9) we obtain

$$f'(x)=-fp\tanh(x) \tag{4.10}$$

$$f''(x)=f[p^2-(p^2+p)\sec h^2(x)] \tag{4.11}$$

$$f'''(x)=f[-p^3+(p^3+3p^2+2p)\sec h^2(x)]\tanh(x) \tag{4.12}$$

$$\begin{aligned}
f''''(x)=f[&p^4-(2p^4+6p^3+8p^2+4p)\sec h^2(x)\\
&+(p^4+6p^3+11p^2+6p)\sec h^4(x)]
\end{aligned} \tag{4.13}$$

$$g'(x)=g_0\mathrm{sech}^q(x) \tag{4.14}$$

$$g''(x)=g_0q\sec h^q(x)\tanh(x) \tag{4.15}$$

$$g'''(x)=g_0q^2\sec h^q(x)-g_0(q^2+q)\sec h^{q+2}(x) \tag{4.16}$$

$$g''''(x)=-g_0q^3\sec h^q(x)\tanh(x)+g_0(q^3+3q^2+2p)\sec h^{q+2}(x)\tanh(x). \tag{4.17}$$

Substituting Eq. (4.10)-Eq. (4.17) into Eq. (4.8) we obtain

$$\begin{aligned}
&-\mu+p^2+\gamma p^4+\sec h^2(x)[-p^2-p-2\gamma p^4+6p^3+8p^2+4p]\\
&+\sec h^4(x)[\gamma(p^4+6p^3)]+\sec h^{2p}(x)(\gamma\alpha f_0{}^2)+\sec h^{4p}(x)[\gamma f_0{}^4]\\
&+\sec h^{2q}(x)[-g_0{}^2-6\gamma g_0{}^2p^2-7\gamma g_0{}^2q^2-12\gamma g_0{}^2pq]+\sec h^{4q}(x)[\gamma g_0{}^4]\\
&+\sec h^{2q+2}(x)[6\gamma g_0{}^2p^2+6\gamma g_0{}^2p+7\gamma g_0{}^2q^2+4\gamma g_0{}^2q+12\gamma g_0{}^2pq]+V\\
&+i[\sec h^q(x)\tanh(x)[-2pg_0-qg_0-4\gamma p^3g_0-6\gamma p^2qg_0-\gamma p^3g_0]\\
&+\sec h^{q+2}(x)\tanh(x)[4\gamma(p^3+3p^2+2p)g_0+6\gamma(p^2+p)qg_0+4\gamma p(q^2+q)g_0\\
&+\gamma(p^3+3p^2+2p)g_0]+\sec h^{3q}(x)\tanh(x)[4\gamma pg_0{}^3+6\gamma qg_0{}^3]+W]=0.
\end{aligned} \tag{4.18}$$

If we partition Eq. (4.18) into real and imaginary parts, we obtain the following expressions for the real and imaginary parts of the $\mathscr{PT}$-symmetric potential as:

Real Part

The real part of the Eq. (4.18) can be written as,

$$\begin{aligned}
&(-\mu+p^2+\gamma p^4)+\operatorname{sech}^2(x)[-p^2-p-\gamma(2p^4+6p^3+8p^2+4p)] \\
&+\operatorname{sech}^4(x)[\gamma(p^4+6p^3)]+\operatorname{sech}^{2p}(x)(\alpha {f_0}^2) \\
&+\operatorname{sech}^{2q}(x)[-{g_0}^2-6\gamma {g_0}^2p^2-7\gamma {g_0}^2q^2-12\gamma {g_0}^2pq] \\
&+\operatorname{sech}^{4q}(x)[\gamma {g_0}^4]+\operatorname{sech}^{2q+2}(x)[6\gamma {g_0}^2p^2+6\gamma {g_0}^2p \\
&+7\gamma {g_0}^2q^2+4\gamma {g_0}^2q+12\gamma {g_0}^2pq]+V=0.
\end{aligned} \tag{4.19}$$

The real part of the $\mathscr{PT}$-symmetric potential is obtained as

$$\begin{aligned}
V(x)=V_0&+V_1\operatorname{sech}^2(x)+V_2\operatorname{sech}^4(x)+V_3\operatorname{sech}^{2p}(x)+V_4\operatorname{sech}^{2q}(x) \\
&+V_5\operatorname{sech}^{4q}(x)+V_6\operatorname{sech}^{2q+2}(x)
\end{aligned} \tag{4.20}$$

where

$$V_0=-\mu+p^2+\gamma p^4 \tag{4.21}$$

$$V_1=-p^2-p-\gamma(2p^4+6p^3+8p^2+4p) \tag{4.22}$$

$$V_2=\gamma(p^4+6p^3+11p^2+6p) \tag{4.23}$$

$$V_3=\alpha {f_0}^2 \tag{4.24}$$

$$V_4=-{g_0}^2-6\gamma {g_0}^2p^2-7\gamma {g_0}^2q^2-12\gamma {g_0}^2pq \tag{4.25}$$

$$V_5=\gamma {g_0}^4 \tag{4.26}$$

$$V_6=6\gamma {g_0}^2p^2+6\gamma {g_0}^2p+7\gamma {g_0}^2q^2+4\gamma {g_0}^2q+12\gamma {g_0}^2pq \tag{4.27}$$

For the sake of simplicity, set $\mu=p^2+\gamma p^4$ to get rid of coefficient V_0. $V(x)$ is indeed an even function given in the following form,

$$\begin{aligned}
V(-x)&=V_1\operatorname{sech}^2(-x)+V_2\operatorname{sech}^4(-x)+V_3\operatorname{sech}^{2p}(-x) \\
&\quad+V_4\operatorname{sech}^{2q}(-x)+V_5\operatorname{sech}^{4q}(-x)+V_6\operatorname{sech}^{2q+2}(-x) \\
&=V_1\operatorname{sech}^2(x)+V_2\operatorname{sech}^4(x)+V_3\operatorname{sech}^{2p}(x) \\
&\quad+V_4\operatorname{sech}^{2q}(x)+V_5\operatorname{sech}^{4q}(x)+V_6\operatorname{sech}^{2q+2}(x) \\
&=V(x).
\end{aligned} \tag{4.28}$$

Now, $V(x)$ can be simplified by equating the powers of $sech(x)$. By assuming the case of $p = q = 1$, then Eq. (4.20) can be rewritten as,

$$\begin{aligned} V(x) = {} & [2+20\gamma-\alpha f_0{}^2+25\gamma g_0{}^2+g_0{}^2]\text{sech}^2(x) \\ & -(24\gamma+35\gamma g_0{}^2+\gamma g_0{}^4)\text{sech}^4(x) \end{aligned} \quad (4.29)$$

In order to find out even function $V(x)$, assuming $\gamma = -0.2$ leads one to

$$V(x) = -(2+\alpha f_0{}^2+4g_0{}^2)\text{sech}^2(x)+(0.2g_0{}^4+7g_0{}^2+4.8)\text{sech}^4(x). \quad (4.30)$$

where

$$V_1 = -2-\alpha f_0{}^2-4g_0{}^2 \quad (4.31)$$

$$V_2 = 0.2g_0{}^4+7g_0{}^2+4.8. \quad (4.32)$$

Considering the case of $f_0 = 1, \quad g_0 = 1$ we get

$$V(x) = -(\alpha+6)\text{sech}^2(x)+12\text{sech}^4(x). \quad (4.33)$$

Imaginary Part

The complex part of the Eq. (4.18) can be written as

$$\begin{aligned} & \text{sech}^q(x)\tanh(x)[-2pg_0-qg_0-4\gamma p^3g_0-6\gamma p^2qg_0-4\gamma pq^2g_0-\gamma q^3g_0] \\ & +\text{sech}^{q+2}(x)\tanh(x)[4\gamma(p^3+3p^2+2p)g_0+6\gamma(p^2+p)qg_0+4\gamma p(q^2+q)g_0 \\ & +\gamma(p^3+3q^2+2q)g_0]+\text{sech}^{3q}(x)\tanh(x)[4\gamma pg_0{}^3+6\gamma qg_0{}^3]+W(x) = 0 \end{aligned} \quad (4.34)$$

Then the imaginary part of the $\mathscr{PT}$-symmetric potential is obtained as

$$W(x) = W_0\text{sech}^q(x)\tanh(x)+W_1\text{sech}^{q+2}(x)\tanh(x)+W_2\sec h^{3q}(x)\tanh(x) \quad (4.35)$$

where

$$W_0 = g_0[2p+q+\gamma(4p^3+6p^2q+4pq^2+q^3)] \quad (4.36)$$

$$W_1 = -\gamma g_0[4p^3+12p^2+8p+6p^2q+10pq+4pq^2+q^3+3q^2+2q] \quad (4.37)$$

$$W_2 = -\gamma g_0{}^3[4p+6q]. \quad (4.38)$$

W(x) is indeed an odd function which can be expressed as follows

$$\begin{aligned} W(-x) = {} & W_0\text{sech}^q(-x)\tanh(-x)+W_1\text{sech}^{q+2}(-x)\tanh(-x) \\ & +W_2\text{sech}^{3q}(-x)\tanh(-x) \\ = {} & W_0\text{sech}^q(x)(-\tanh(x))+W_1\text{sech}^{q+2}(x)(-\tanh(x)) \\ & +W_2\text{sech}^{3q}(x)(-\tanh(x)) \\ = {} & -W(x). \end{aligned} \quad (4.39)$$

By assuming the case of $p=q=1$, Eq. (4.34) can be rewritten as,

$$W(x)=3g_0(5\gamma+1)\sec h(x)\tanh(x)-\gamma g_0(10{g_0}^2+50)\sec h^3(x)\tanh(x) \tag{4.40}$$

by taking $\gamma=-0.2$ we get

$$W(x)=2g_0({g_0}^2+5)\text{sech}^3(x)\tanh(x). \tag{4.41}$$

where

$$W_\star=2g_0({g_0}^2+5). \tag{4.42}$$

Taking $g_0=1$ into Eq. (4.41) leads one to

$$W(x)=12\text{sech}^3(x)\tanh(x). \tag{4.43}$$

Note that in case of $p=q=1$, by considering Eq. (4.29) and Eq. (4.40) the analytical solution of the problem can be stated as

$$u(x,z)=f_0\sec h(x)e^{i[\mu z+g_0\arctan h(x)\sinh(x)]}. \tag{4.44}$$

In conclusion, the $\mathscr{PT}$-symmetric potential in Eq. (1.2) with the real and the imaginary parts in Eq. (4.30) and Eq. (4.41) can be given as

$$V_{PT}=[V_1\text{sech}^2(x)+V_2\text{sech}^4(x)]+i[W_\star\text{sech}^3(x)\tanh(x)]. \tag{4.45}$$

Eq. (4.45) can be seen as an extension of the so-called complexified Scarf II potential $[V_0\text{sech}^2(x)+iW_0\text{sech}(x)\tanh(x)]$ for Kerr media with cubic nonlinearity.

4.1.2 Numerical illustrations

In this section, we demonstrate the produced results under the dependence of μ,γ,f_0 and g_0 respectively. As it will be seen Fig. 4.1 and Fig. 4.2, the selection of f_0 affects the maximum amplitude of the soliton. We plot the real and imaginary parts of the functions f and V in Fig. 4.1 with the parameters $f_0=1.5$, $g_0=1$, $\gamma=-0.2$ and $\mu=1$. In Fig. 4.2 and Fig. 4.3, we demonstrate the effect of the f_0 on the maximum amplitude of the soliton with the same parameters as in the Fig. 4.1, for $0\le f_0\le 1.55$. As it is seen in Fig. 4.2 the relation between the considered quantities is almost linear.

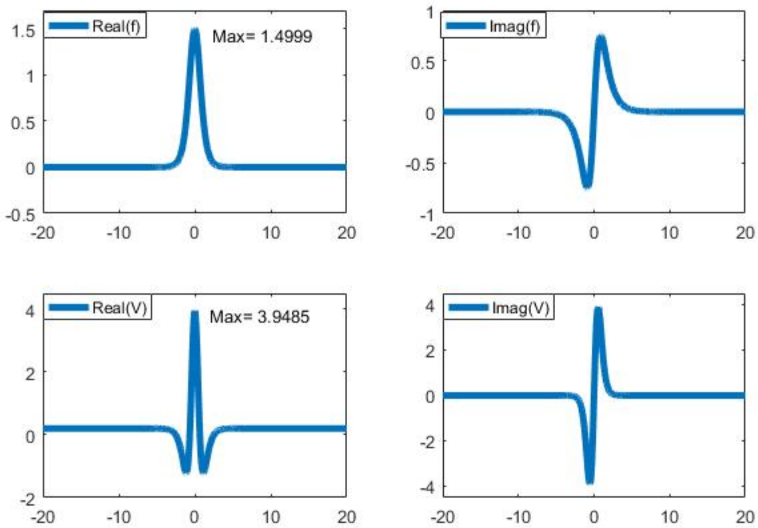

Figure 4.1 : Real and imaginary part of the soliton and potential for $f_0 = 1.5$, $g_0 = 1$, $\mu = 1$ and $\gamma = -0.2$.

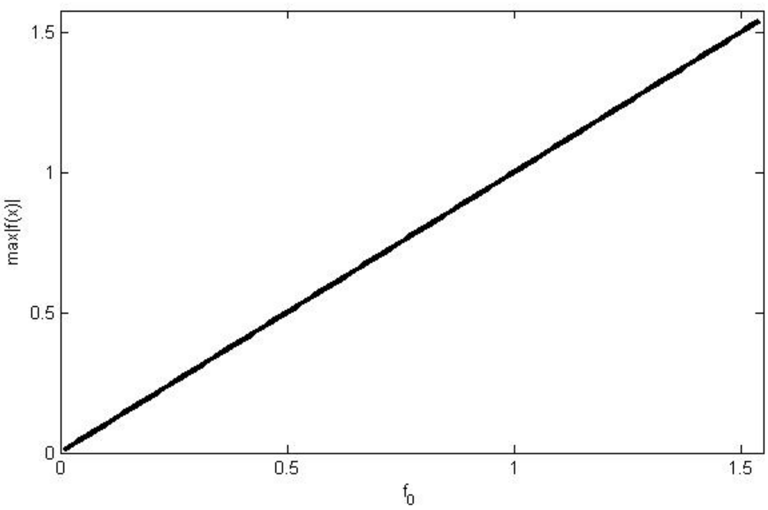

Figure 4.2 : Numerically obtained solitons for various values of f_0 for $\mu = 1$, $\gamma = -0.2$ and $g_0 = 1$.

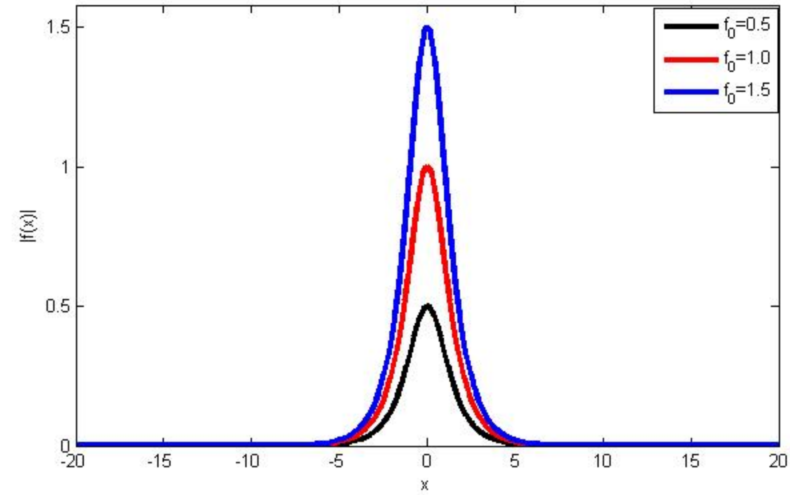

Figure 4.3 : Numerically obtained solitons for various values of f_0 for $\mu = 1$, $\gamma = -0.2$ and $g_0 = 1$.

In Fig. 4.4 and Fig. 4.5, we demonstrate the effect of the 4OD constant γ to the shape and to the maximum amplitude of the $\mathscr{PT}$-symmetric potential.

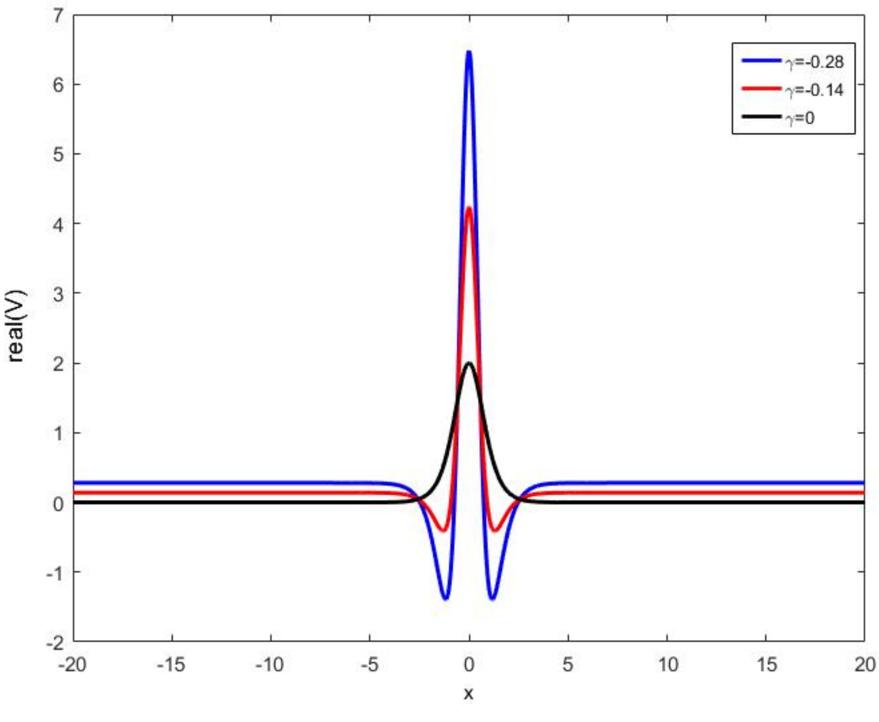

Figure 4.4 : Numerically obtained solitons for various values of γ for $\mu = 1$, $f_0 = 1$ and $g_0 = 1$.

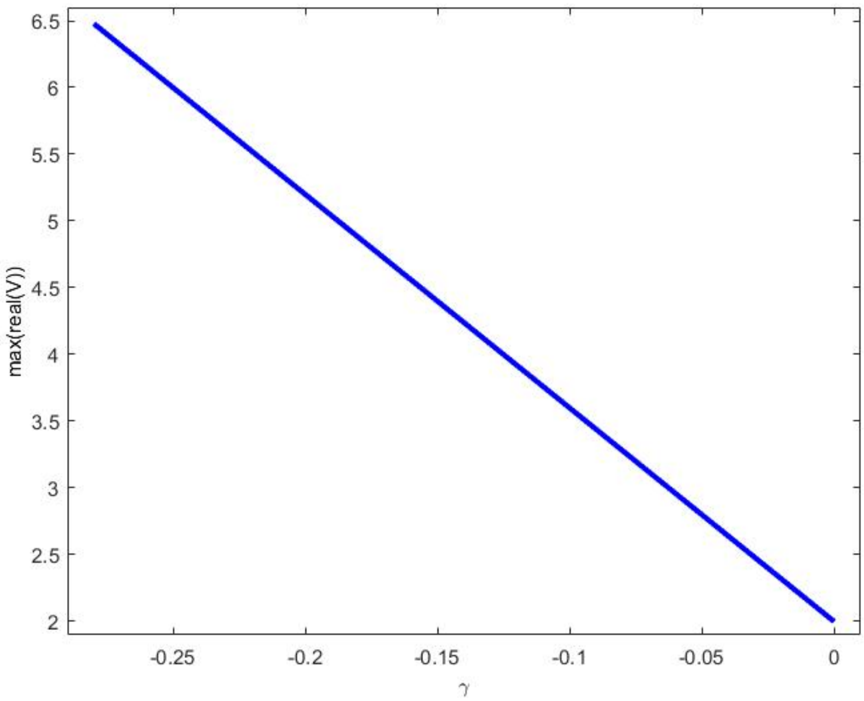

Figure 4.5 : Numerically obtained solitons for various values of γ for $\mu = 1$, $f_0 = 1$ and $g_0 = 1$.

In Fig. 4.6, the effect of the eigenvalue μ on the numerical solution f and $\mathscr{PT}$-symmetric potential are figured out.

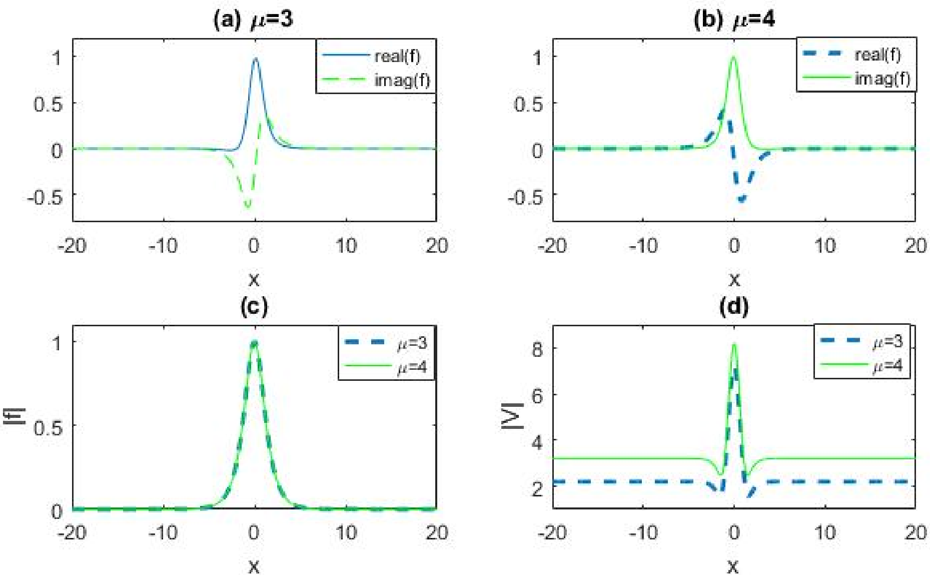

Figure 4.6 : (a) Real and imaginary parts of numerical solution f with $\mu = 3$, (b) real and imaginary parts of numerical solution f with $\mu = 4$, (c) $|f|$ for $\mu = 3$ and $\mu = 4$, (d) absolute values of $\mathscr{PT}$-symmetric potential for $\mu = 3$ and $\mu = 4$, by considering $\gamma = -0.2$, $f_0 = 1$ and $g_0 = 1$.

4.2 Nonlinear Stability

In order to be nonlinearly stable, solitons must maintain their shape, position and maximum amplitude during direct simulations. To understand their nonlinear stability, solitons are evolved over a long distance. To do this, split-step Fourier method is implemented to advance in z. In Fig. 4.7 we plotted that numerically produced soliton of the (1+1)D 4OD NLS equation with a $\mathscr{PT}$-symmetric potential, nonlinear evolution of the soliton, the view from top to $|u(x,z)|$ and maximum values of $|u|$ along with the z, respectively. It can be easily seen that the soliton maintain its shape but the maximum amplitude decays with variable z. We used the parameters $\mu = 1$, $\gamma = -0.2$, $f_0 = 1$ and $g_0 = 1$ to plot Fig. 4.7. In Fig. 4.7 the maximum amplitude of soliton decays up to $|u_{max}| = 0.75$. The dispersion coefficient γ is an another important parameter for the stability of the obtained results. It is obvious that the decaying constant and amplitude of the $|u_{max}|$ depends strongly on dispersion coefficient γ. In Fig. 4.8 we demonstrate the case of $\gamma = 0$ and we assumed that the other parameters are as in the Fig. 4.7. We also deal with the effect of potential depth on the maximum amplitude. As it is seen in Fig. 4.9 the maximum amplitude of the soliton less than the case of Fig. 4.7. The results in Fig. 4.9 obtained by considering $\mu = 1$, $\gamma = -0.2$, $f_0 = 0.5$ and $g_0 = 1$. In Fig. 4.10, we demonstrate the effects of positive dispersion coefficient $\gamma = 0.15$. As it is stated in Fig. 4.10-d the change in the maximum amplitude is less than or equal to 10^{-2}, then one can conclude that the soliton more conservative than the case of Fig. 4.9 in maximum amplitude sense.

As a result of the numerical observations we made in this section, we conclude that:
(i) increasing dispersion coefficient term γ decreases the maximum amplitude of the $\mathscr{PT}$-symmetric potential
(ii) If eigenvalue μ decreases or increases, then the maximum amplitude of the soliton is not affected but the shape of the real and imaginary parts of the solitons are changed;
(iii) The maximum amplitude of the solitons are direct proportional with the parameter f_0.

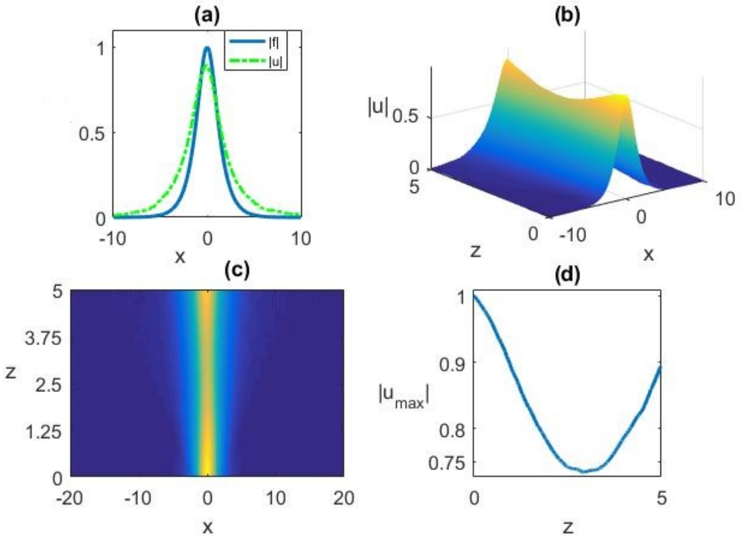

Figure 4.7 : Nonlinear instability of a higher order soliton for $\gamma = -0.2, \mu = 1$ $f_0 = 1$ and $g_0 = 1$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced higher order nonlinear soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z.

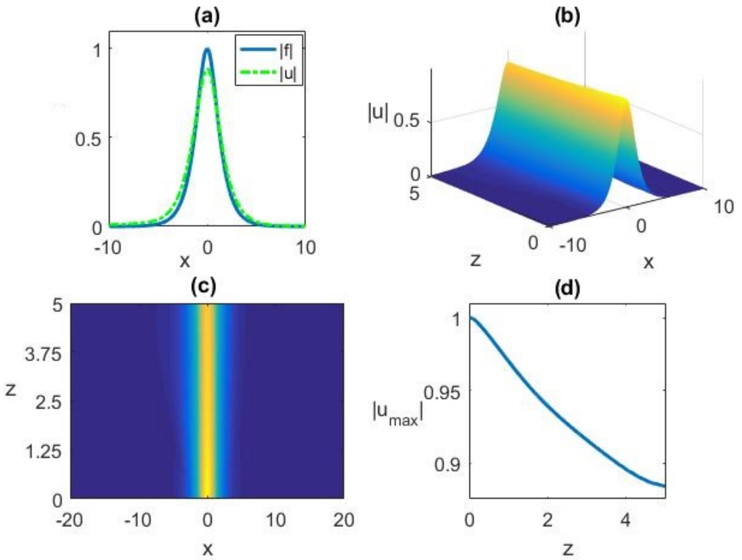

Figure 4.8 : Nonlinear instability of a higher order soliton for $\gamma = 0, \mu = 1$, $f_0 = 1$ and $g_0 = 1$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c) The view from top and
(d) Maximum amplitude as a function of the propagation distance z.

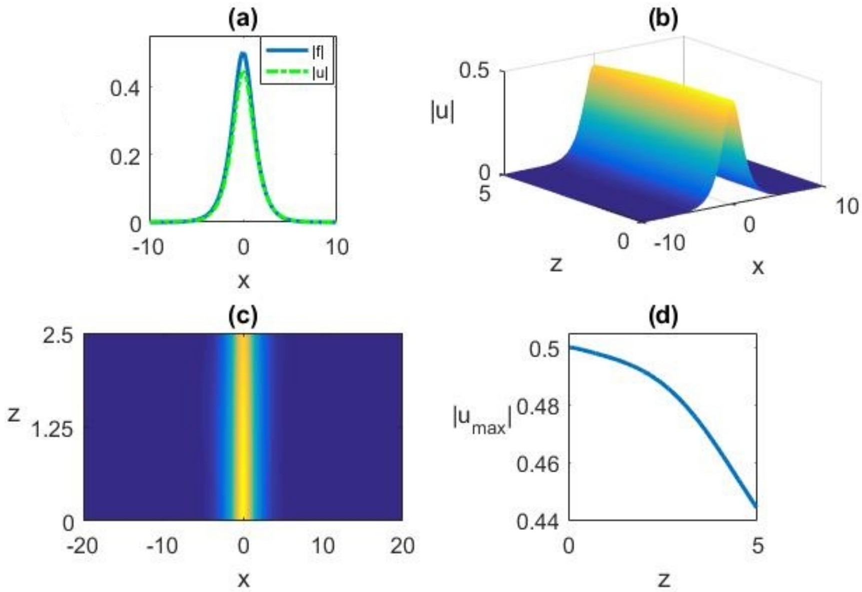

Figure 4.9 : Nonlinear instability of a higher order soliton for $\gamma = -0.2, \mu = 1$, $f_0 = 0.5$ and $g_0 = 1$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

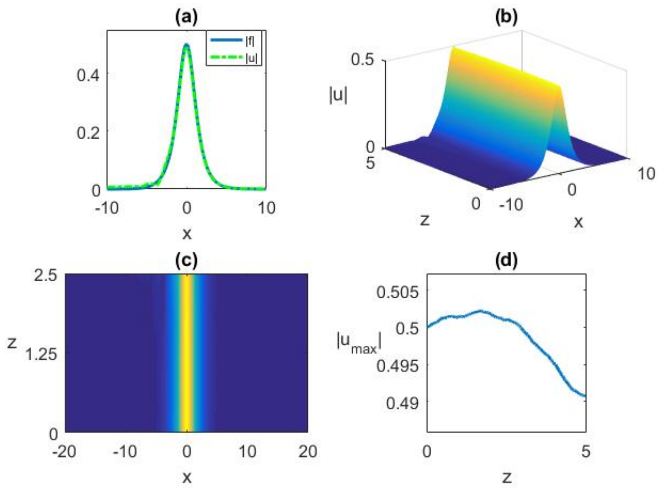

Figure 4.10 : Nonlinear instability of a higher order soliton for $\gamma = 0.15, \mu = 1$ $f_0 = 0.5$ and $g_0 = 1$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced higher order nonlinear soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

5. CONCLUSION

The purpose of this study is to investigate the existence and nonlinear stability of the (1+1)D 4OD cubic NLS equation with and without an external potential. Firstly, the numerical methods, which are spectral renormalization method and split-step Fourier method, are introduced to find out numerical solutions of the considered equations and to analyze stability of the fundamental solitons, respectively. The considered numerical approach implemented to the (1+1)D 4OD cubic NLS equations with and without $\mathscr{PT}$-symmetric potential. After that, we numerically illustrate the soliton solution of (1+1)D 4OD cubic NLS equation without an external potential for various values of γ (40D term's coefficient) and the propagation constant μ. The stability of the soliton solutions is analyzed by considering the effects of the parameters on the stability. Produced stable and unstable cases are depicted for various values of the dispersion coefficient γ and the eigenvalue μ. As a result of the numerical observations, we conclude that:

(*i*) The higher order solitons obtained for 4OD NLS equation may have oscillating tails and these solitons are found to be nonlinearly *unstable* (cases obtained for both large 4OD effect and large eigenvalue μ);

(*ii*) The higher order solitons obtained for 4OD cubic NLS equation are found to be nonlinearly *stable* for the cases when the effect of the 4OD effect is either small or the eigenvalue is μ is small (even if the 4OD effect is large).

Finally, the obtained numerical results are compared with exact solutions for the case with a $\mathscr{PT}$-symmetric potential. Numerical illustrations of the obtained solitons by SR method and stability of the produced results are shown for various values of the problem parameters. By considering various values solitons parameters the maximum amplitudes of the obtained solitons are explored and illustrated. In each case, the shape of the produced solitons, the change in maximum amplitude of the solitons are investigated and the results depicted. The effect of f_0, the eigenvalue μ and the dispersion coefficient term γ on the maximum amplitude of the solitons are depicted and following concluding remarks can be expressed:

(i) increasing dispersion coefficient term γ decreases the maximum amplitude of the $\mathscr{P}\mathscr{T}$-symmetric potential

(ii) increasing f_0 increases the maximum amplitude of the soliton

(iii) the change in the eigenvalue μ does not effect the maximum amplitude but effects the shape of the real and imaginary parts of the solitons.

As it is seen in illustrations, numerical results are satisfactory in terms of accuracy and stability. That means considered numerical method is suitable for the solution of (1+1)D 4OD cubic NLS equation to obtain physically acceptable solutions. In future, by considering different potential parameters V_1, V_2 and W_0, one may try to find stable soliton solutions and inclusion of third order dispersion to the problem can be discussed.

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APPENDICES

APPENDIX A.1 : Fourier Transform

APPENDIX A.1

Fourier Transform

For a continuous, smooth and absolutely integrable function $f(x)$, the integral transform

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(k_x)x} dx \qquad \textbf{(A.1)}$$

is called *the Fourier transform of f (x)* and conversely, the transform

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k_x)x} \mathrm{d}x \qquad \textbf{(A.2)}$$

is called *the inverse Fourier transform of* $F(k_x)$.

The Fourier transform of f is denoted by $\mathscr{F}(f) = \hat{f}$, the inverse Fourier transform of $\hat{f}$ is denoted by $\mathscr{F}^{-1}(\hat{f})$ and clearly $\mathscr{F}^{-1}(\hat{f}) = \mathscr{F}^{-1}(\mathscr{F}(\hat{f}))$.
Integral transform methods are very useful for solving partial differential equations because of their properties such as linearity, shifting, scaling, etc.
Suppose that $f(x)$ tends to zero as x tends to infinity. Then,

$$\begin{aligned}\mathscr{F}\left(f'(x)\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\prime(x) e^{i(k_x)x} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} f(x) e^{i(k_x)} \Big|_{-\infty}^{\infty} - ik_x \int_{-\infty}^{\infty} f(x) e^{i} k_x x \mathrm{d}x \\ &= -ik_x \mathscr{F}(f(x)) \qquad \textbf{(A.3)}\end{aligned}$$

This result can be extended to obtain the differentiation property of the Fourier transform:

$$\mathscr{F}(f'(x)) = (-ik_x)^n (f(x)) = (-ik_x)^n \hat{f}, \qquad n \in N \qquad \textbf{(A.4)}$$

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