SFU DEPARTMENT OF MATHEMATICS

CALCULUS
Early Transcendentals

Differential & Multi-Variable Calculus for Social Sciences

SFU DEPARTMENT OF MATHEMATICS

 Examples, in particular applications, have been added throughout the textbook to further illustrate concepts and to reflect social science calculus content; exercises have been added throughout the textbook to provide more extensive practice, and information throughout the book, as applicable, has been revised to reflect social science calculus content and modified to visually align definitions, theorems and guidelines established in the textbook.
The following additions have been made to these chapters:

Chapter 2:

- Transformation of Trigonometric Functions
- Symmetry
- Economic Models (Demand and Supply Functions, Cost, Revenue and Profit functions)

Chapter 3:

- Distinction between Limit at Infinity and Infinite Limits

Chapter 4:

- Secant & Tangent Lines, Tangent Line Equation
- Differentiating x and y as Functions of t
- Logarithmic Differentiation

Chapter 5:

- Elasticity of Demand
- Error Approximations
- Indeterminate Forms
- Reorganization of Newton's Method and Curve Sketching

Chapter 7 formerly Chapter 13:

- Second-Order Partial Derivatives

The following deletions have been made: Hyperbolic Functions, Integration Calculus, Sequences and Series, Differential Equations, Polar Coordinates, Parametric Equations, Vector Calculus.

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Open Educational Resources (OER) Support: Corrections and Suggestions

Please support the OER initiative. In an effort to improve the content of this textbook, contact Petra Menz at pmenz@sfu.ca with your suggestions for improvements, new content, or errata.

Dedication

To my son Eli, so that his access to learning remains open as he unfolds his wings to explore life. May his roots be strong and anchor him in all of his pursuits.

Petra Menz

Contents

Introduction

This course is designed for students specializing in business or the social sciences. Topics include a refresher in basic functions, their transformation and algebra; introduction to logarithmic exponential and trigonometric functions; the differential calculus topics limit, growth rate, derivative and differentiation applications such as related, rates, curve sketching and optimization as well as their application to business and economics; approximation methods such as linearization, differentials and Newton's Method; and functions of several variables including partial derivatives and extrema.

The following *Recommendations for Success in Mathematics* are excerpts taken from the same named document published by Petra Menz in order to provide strategies grouped into categories to all students who are thinking about their well-being, learning, and goals, and who want to be successful academically.

How to Take Lecture Notes:

Listen to the Instructor, who

- explains the concepts;
- draws connections;
- demonstrates examples;
- emphasizes material.

Copy the presented lecture material

- by arriving to the lecture prepared;
- using telegraphic writing, i.e. packing as much information into the smallest possible number of words/ symbols (do you really need to copy all the algebraic/manipulative steps?).

Mark up your notes immediately while listening and copying using a system such as offered here:

! pay attention (possible exam material)

? confusing (read course notes or visit ACW)

-> practice (using course notes and online assignments)

__ underline/highlight key concepts

Habits of a Successful Student: for detailed description of each item see the document *Recommendations for Success in Mathematics*

- Acts responsibly
- Sets goals
- Is reflective
- Is inquisitive
- Can communicate
- Enjoys learning
- Is resourceful
- Is organized
- Manages time effectively
- Is involved

Problem Solving Strategies

The emphasis in this course is on problems–doing calculations and applications. To master problem solving one needs a tremendous amount of practice doing problems. The more problems you do the better you will be at doing them, as patterns will start to emerge in both the problems and in successful approaches to them. You will learn quickly and effectively if you devote some time to doing problems every day.

Typically the most difficult problems are applications, since they require some effort before you can begin calculating. Here are some pointers for doing applications:

1. Carefully read each problem twice before writing anything.
2. Assign letters to quantities that are described only in words; draw a diagram if appropriate.
3. Decide which letters are constants (invariants) and which are variables (variants). A letter stands for a constant if its value remains the same throughout the problem. A letter stands for a variable if its value varies throughout the problem.
4. Using mathematical notation, write down what you know and then write down what you want to find.
5. Decide what category of problem it is (this might be obvious if the problem comes at the end of a particular chapter, but will not necessarily be so obvious if it comes on an exam covering several chapters).
6. Double check each step as you go along; don't wait until the end to check your work.
7. Use common sense; if an answer is out of the range of practical possibilities, then check your work to see where you went wrong

1. Review

Success in calculus depends on your background in algebra, trigonometry, analytic geometry and functions. In this chapter, we review many of the concepts you will need to know to succeed in this course.

1.1 Algebra

1.1.1. Sets and Number Systems

A **set** can be thought of as any collection of *distinct* objects considered as a whole. Typically, sets are represented using **set-builder notation** and are surrounded by braces. Recall that $(,)$ are called **parentheses** or **round brackets**; $[,]$ are called **square brackets**; and $\{,\}$ are called **braces** or **curly brackets**.

Example 1.1: Sets

The collection $\{a,b,1,2\}$ *is a set. It consists of the collection of four distinct objects, namely,* a, b, 1 *and* 2.

Let S be any set. We use the notation $\mathbf{x} \in \mathbf{S}$ to mean that x is an element *inside* of the set S, and the notation $\mathbf{x} \notin \mathbf{S}$ to mean that x is *not* an element of the set S.

Example 1.2: Set Membership

If $S = \{a,b,c\}$, *then* $a \in S$ *but* $d \notin S$.

The **intersection** between two sets S and T is denoted by $S \cap T$ and is the collection of all elements that belong to *both* S and T. The **union** between two sets S and T is denoted by $S \cup T$ and is the collection of all elements that belong to *either* S or T (or both).

Example 1.3: Union and Intersection

Let $S = \{a,b,c\}$ *and* $T = \{b,d\}$. *Then* $S \cap T = \{b\}$ *and* $S \cup T = \{a,b,c,d\}$. *Note that we do* not *write the element* b *twice in* $S \cup T$ *even though* b *is in both* S *and* T.

Numbers can be classified into sets called **number systems**.

Symbol	*Description*	*Set notation*
$\mathbb{N}$	*the* **natural** *numbers*	$\{1,2,3,\ldots\}$
$\mathbb{Z}$	*the* **integers**	$\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$
$\mathbb{Q}$	*the* **rational** *numbers*	*Ratios of integers:* $\left\{\frac{p}{q} : p,q \in \mathbb{Z}, q \neq 0\right\}$
$\mathbb{R}$	*the* **real** *numbers*	*Can be written using a finite or infinite decimal expansion*
$\mathbb{C}$	*the* **complex** *numbers*	*These allow us to solve equations such as* $x^2+1=0$

In the table, the set of rational numbers is written using set-builder notation. The colon, :, used in this manner means *such that*. Often times, a vertical bar | may also be used to mean *such that*. The expression $\left\{\frac{p}{q} : p,q \in \mathbb{Z}, q \neq 0\right\}$ can be read out loud as "the set of all fractions p over q such that p and q are both integers and q is not equal to zero".

Example 1.4: Rational Numbers

The numbers $-\frac{3}{4}$, 2.647, 17, $0.\bar{7}$ *are all rational numbers. You can think of rational numbers as fractions of one integer over another. Note that 2.647 can be written as a fraction:*

$$2.647 = 2.647 \times \frac{1000}{1000} = \frac{2647}{1000}.$$

Also note that in the expression $0.\bar{7}$*, the bar over the* 7 *indicates that the* 7 *is repeated forever:*

$$0.77777777\ldots = \frac{7}{9}.$$

All rational numbers are real numbers with the property that their decimal expansion either *terminates* after a finite number of digits or begins to *repeat* the same finite sequence of digits over and over. Real numbers that are not rational are called **irrational**.

Example 1.5: Irrational Numbers

Some of the most common irrational numbers include:

- $\sqrt{2}$. *Can you prove this is irrational? (The proof uses a technique called contradiction.)*
- π. *Recall that* π *(***pi***) is defined as the ratio of the circumference of a circle to its diametre and can be approximated by* 3.14159265.
- e. *Sometimes called Euler's number,* e *can be approximated by* 2.718281828459. *We will review the definition of* e *in a later chapter.*

Let S and T be two sets. If every element of S is also an element of T, then we say S is a **subset** of T and write $S \subseteq T$. Furthermore, if S is a subset of T but not equal to T, we often write $S \subset T$. The five sets of numbers in the table give an increasing sequence of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

That is, all natural numbers are also integers, all integers are also rational numbers, all rational numbers are also real numbers, and all real numbers are also complex numbers.

1.1.2. Law of Exponents

The Law of Exponents is a set of rules for simplifying expressions that governs the combination of exponents (powers). Recall that $\sqrt[n]{\ }$ denotes the n-th root. For example $\sqrt[3]{8} = 2$ represents that the cube root of 8 is equal to 2.

Definition 1.6: Law of Exponents

Definitions

If m, n are positive integers, then:

1. $x^n = x \cdot x \cdot \ldots \cdot x$ (n times).
2. $x^0 = 1$, for $x \neq 0$.
3. $x^{-n} = \dfrac{1}{x^n}$, for $x \neq 0$.
4. $x^{m/n} = \sqrt[n]{x^m}$ or $\left(\sqrt[n]{x}\right)^m$, for $x \geq 0$.

Combining

1. $x^a x^b = x^{a+b}$.
2. $\dfrac{x^a}{x^b} = x^{a-b}$, for $x \neq 0$.
3. $(x^a)^b = x^{ab} = x^{ba} = \left(x^b\right)^a$.

Distributing

1. $(xy)^a = x^a y^a$, for $x \geq 0$, $y \geq 0$.
2. $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$, for $x \geq 0$, $y > 0$.

In the next example, the word *simplify* means *to make simpler* or to write the expression more compactly.

Example 1.7: Laws of Exponents

Simplify the following expression as much as possible assuming $x, y > 0$:

$$\frac{3x^{-2}y^3x}{y^2\sqrt{x}}.$$

Solution. Using the Law of Exponents, we have:

$$\begin{aligned}
\frac{3x^{-2}y^3x}{y^2\sqrt{x}} &= \frac{3x^{-2}y^3x}{y^2x^{\frac{1}{2}}}, && \text{since } \sqrt{x}=x^{\frac{1}{2}}, \\
&= \frac{3x^{-2}yx}{x^{\frac{1}{2}}}, && \text{since } \frac{y^3}{y^2}=y, \\
&= \frac{3y}{x^{\frac{3}{2}}}, && \text{since } \frac{x^{-2}x}{x^{\frac{1}{2}}}=\frac{x^{-1}}{x^{\frac{1}{2}}}=x^{-\frac{3}{2}}=\frac{1}{x^{\frac{3}{2}}}, \\
&= \frac{3y}{\sqrt{x^3}}, && \text{since } x^{\frac{3}{2}}=\sqrt{x^3}.
\end{aligned}$$

An answer of $3yx^{-3/2}$ is equally acceptable, and such an expression may prove to be computationally simpler, although a positive exponent may be preferred. ♣

1.1.3. The Quadratic Formula and Completing the Square

The technique of **completing the square** allows us to solve quadratic equations and also to determine the center of a circle/ellipse or the vertex of a parabola.

The main idea behind completing the square is to turn:

$$ax^2+bx+c$$

into

$$a(x-h)^2+k.$$

One way to complete the square is to use the following formula:

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+c.$$

But this formula is a bit complicated, so some students prefer following the steps outlined in the next example.

Example 1.8: Completing the Square

Solve $2x^2+12x-32=0$ *by completing the square.*

Solution. In this instance, we will *not* divide by 2 first (usually you would) in order to demonstrate what you should do when the 'a' value is not 1.

$2x^2 + 12x - 32 = 0$	Start with original equation.
$2x^2 + 12x = 32$	Move the number over to the other side.
$2(x^2 + 6x) = 32$	Factor out the a from the $ax^2 + bx$ expression.
$6 \;\rightarrow\; \frac{6}{2} = 3 \;\rightarrow\; 3^2 = 9$	Take the number in front of x, **divide by** 2, then **square** it.
$2(x^2 + 6x + 9) = 32 + 2 \cdot 9$	Add the result to both sides, taking $a = 2$ into account.
$2(x+3)^2 = 50$	Factor the resulting perfect square trinomial.
	You have now completed the square!
$(x+3)^2 = 25 \;\rightarrow\; x = 2 \text{ or } x = -8$	To solve for x, simply divide by $a = 2$ and take square roots.

♣

Suppose we want to solve for x in the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$. The solution(s) to this equation are given by the **quadratic formula**.

The Quadratic Formula

The solutions to $ax^2 + bx + c = 0$ (with $a \neq 0$) are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof. To prove the quadratic formula we use the technique of *completing the square*. The general technique involves taking an expression of the form $x^2 + rx$ and trying to find a number we can add so that we end up with a perfect square (that is, $(x+n)^2$). It turns out if you add $(r/2)^2$ then you can factor it as a perfect square.

For example, suppose we want to solve for x in the equation $ax^2 + bx + c = 0$, where $a \neq 0$. Then we can move c to the other side and divide by a (remember, $a \neq 0$ so we can divide by it) to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

To write the left side as a perfect square we use what was mentioned previously. We have $r = (b/a)$ in this case, so we must add $(r/2)^2 = (b/2a)^2$ to both sides

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

We know that the left side can be factored as a perfect square

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

The right side simplifies by using the exponent rules and finding a common denominator

$$\left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}.$$

Taking the square root we get

$$x+\frac{b}{2a}=\pm\sqrt{\frac{-4ac+b^2}{4a^2}},$$

which can be rearranged as

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

In essence, the quadratic formula is just completing the square. ♣

1.1.4. Inequalities, Intervals and Solving Basic Inequalities

Inequality Notation

Recall that we use the symbols $<, >, \leq, \geq$ when writing an inequality. In particular,

- $a < b$ means a is to the *left* of b (that is, a is *strictly less* than b),
- $a \leq b$ means a is to the *left of or the same* as b (that is, a is *less than or equal* to b),
- $a > b$ means a is to the *right* of b (that is, a is *strictly greater* than b),
- $a \geq b$ means a is to the *right of or the same* as b (that is, a is *greater than or equal* to b).

To keep track of the difference between the symbols, some students use the following mnemonic.

Mnemonic

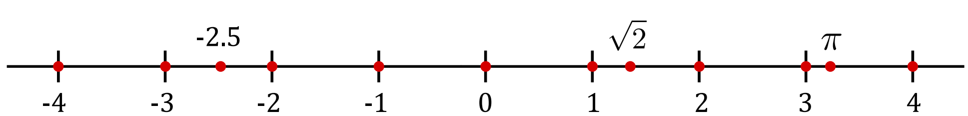
The $<$ symbol looks like a slanted L which stands for "*L*ess than".

Example 1.9: Inequalities

The following expressions are true:

$$1<2, \quad -5<-2, \quad 1\leq 2, \quad 1\leq 1, \quad 4\geq \pi>3, \quad 7.23\geq -7.23.$$

The real numbers are ordered and are often illustrated using the **real number line**:



Intervals

Assume a, b are real numbers with $a < b$ (i.e., a is strictly less than b). An **interval** is a set of every real number between two indicated numbers and may or may not contain the two numbers themselves. When describing intervals we use both round brackets and square brackets.

(1) Use of round brackets in intervals: (,). The notation **(a,b)** is what we call the **open interval from a to b** and consists of all the numbers between a and b, but does *not* include a or b. Using set-builder notation we write this as:

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}.$$

We read $\{x \in \mathbb{R} : a < x < b\}$ as "the set of real numbers x such that x is greater than a and less than b" On the real number line we represent this with the following diagram:

Note that the circles on a and b are not shaded in, we call these **open circles** and use them to denote that a, b are *omitted* from the set.

(2) Use of square brackets in intervals: [,]. The notation **[a,b]** is what we call the **closed interval from a to b** and consists of all the numbers between a and b and *including* a and b. Using set-builder notation we write this as

$$[a,b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}.$$

On the real number line we represent this with the following diagram:

Note that the circles on a and b are shaded in, we call these **closed circles** and use them to denote that a and b are *included* in the set.

To keep track of when to shade a circle in, you may find the following mnemonic useful:

Mnemonic

The round brackets (,) and non-shaded circle both form an "O" shape which stands for "Open and Omit".

Taking combinations of round and square brackets, we can write different possible types of intervals (we assume $a < b$):

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$ 	$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ 	$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$ 
$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ 	$(a,\infty) = \{x \in \mathbb{R} : x > a\}$ 	$[a,\infty) = \{x \in \mathbb{R} : x \geq a\}$ 
$(-\infty,b) = \{x \in \mathbb{R} : x < b\}$ 	$(-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$ 	$(-\infty,\infty) = \mathbb{R} =$ all real numbers

Note: Any set which is bound at positive and/or negative infinity is an open interval.

Inequality Rules

Before solving inequalities, we start with the properties and rules of inequalities.

Inequality Rules

Add/subtract a number to both sides:

- If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

Adding two inequalities of the same type:

- If $a < b$ and $c < d$, then $a + c < b + d$.
  Add the left sides together, add the right sides together.

Multiplying by a positive number:

- Let $c > 0$. If $a < b$, then $c \cdot a < c \cdot b$.

Multiplying by a negative number:

- Let $c < 0$. If $a < b$, then $c \cdot a > c \cdot b$.
  Note that we reversed the inequality symbol!

Similar rules hold for each of $\leq$, $>$ and $\geq$.

Solving Basic Inequalities

We can use the inequality rules to solve some simple inequalities.

Example 1.10: Basic Inequality

Find all values of x satisfying

$$3x + 1 > 2x - 3.$$

Write your answer in both interval and set-builder notation. Finally, draw a number line indicating your solution set.

Solution. Subtracting $2x$ from both sides gives $x + 1 > -3$. Subtracting 1 from both sides gives $x > -4$. Therefore, the solution is the interval $(-4, \infty)$. In set-builder notation the solution may be written as $\{x \in \mathbb{R} : x > -4\}$. We illustrate the solution on the number line as follows:

♣

Sometimes we need to split our inequality into two cases as the next example demonstrates.

Example 1.11: Double Inequalities

Solve the inequality

$$4 > 3x - 2 \geq 2x - 1.$$

Solution. We need both $4 > 3x - 2$ and $3x - 2 \geq 2x - 1$ to be true:

$$\begin{array}{rcl} 4 > 3x - 2 & \textit{and} & 3x - 2 \geq 2x - 1, \\ 6 > 3x & \textit{and} & x - 2 \geq -1, \\ 2 > x & \textit{and} & x \geq 1, \\ x < 2 & \textit{and} & x \geq 1. \end{array}$$

Thus, we require $x \geq 1$ but also $x < 2$ to be true. This gives all the numbers between 1 and 2, including 1 but not including 2. That is, the solution to the inequality $4 > 3x - 2 \geq 2x - 1$ is the interval $[1,2)$. In set-builder notation this is the set $\{x \in \mathbb{R} : 1 \leq x < 2\}$. ♣

Example 1.12: Positive Inequality

Solve $4x - x^2 > 0$.

Solution. We provide two methods to solve this inequality.

Method 1: Factor $4x - x^2$ as $x(4-x)$. The product of two numbers is positive when either both are positive or both are negative, i.e., if either $x > 0$ and $4 - x > 0$, or else $x < 0$ and $4 - x < 0$. The latter alternative is impossible, since if x is negative, then $4 - x$ is greater than 4, and so cannot be negative. As for the first alternative, the condition $4 - x > 0$ can be rewritten (adding x to both sides) as $4 > x$, so we need: $x > 0$ and $4 > x$ (this is sometimes combined in the form $4 > x > 0$, or, equivalently, $0 < x < 4$). In interval notation, this says that the solution is the interval $(0,4)$.

Method 2: Write $4x - x^2$ as $-(x^2 - 4x)$, and then complete the square, obtaining

$$-\left((x-2)^2 - 4\right) = 4 - (x-2)^2.$$

For this to be positive we need $(x-2)^2 < 4$, which means that $x - 2$ must be less than 2 and greater than -2: $-2 < x - 2 < 2$. Adding 2 to everything gives $0 < x < 4$.

Both of these methods are equally correct; you may use either in a problem of this type. ♣

We next present another method to solve more complicated looking inequalities. In the next example we will solve a rational inequality by using a number line and test points. We follow the guideline below.

Guideline for Solving Rational Inequalities

1. Move everything to *one side* to get a 0 on the other side.
2. If needed, combine terms using a *common denominator*.
3. *Factor* the numerator and denominator.
4. Identify points where either the numerator or denominator is 0. Such points are called **split points**.
5. Draw a *number line* and indicate your split points on the number line. Draw *closed/open circles* for each split point depending on if that split point satisfies the inequality (division by zero is not allowed).
6. The split points will split the number line into subintervals. For each subinterval pick a *test point* and see if the expression in Step 3 is positive or negative. Indicate this with a $+$ or $-$ symbol on the number line for that subinterval.
7. Now write your answer in set-builder notation. Use the union symbol $\cup$ if you have multiple intervals in your solution.

Example 1.13: Rational Inequality

Write the solution to the following inequality using interval notation:

$$\frac{2-x}{2+x} \geq 1.$$

Solution. One method to solve this inequality is to multiply both sides by $2+x$, but because we do not know if $2+x$ is positive or negative we must split it into two cases (*Case 1:* $2+x>0$ and *Case 2:* $2+x<0$).

Instead we follow the guideline for solving rational inequalities:

Start with original problem: $$\frac{2-x}{2+x} \geq 1$$

Move everything to one side: $$\frac{2-x}{2+x} - 1 \geq 0$$

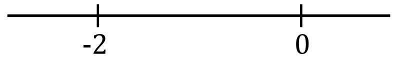
Find a common denominator: $$\frac{2-x}{2+x} - \frac{2+x}{2+x} \geq 0$$

Combine fractions: $$\frac{(2-x)-(2+x)}{2+x} \geq 0$$

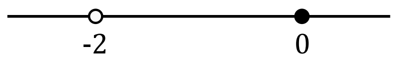
Expand numerator: $$\frac{2-x-2-x}{2+x} \geq 0$$

Simplify numerator: $$\frac{-2x}{2+x} \geq 0 \quad (*)$$

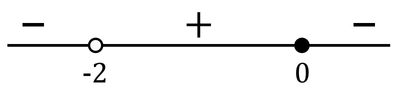
Now we have the numerator and denominator in fully factored form. The split points are $x = 0$ (makes the numerator 0) and $x = -2$ (makes the denominator 0). Let us draw a number line with the split points indicated on it:



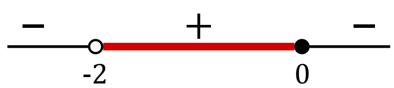
The point $x = 0$ is included since if we sub $x = 0$ into (*) we get $0 \geq 0$ which is true. The point $x = -2$ is not included since we cannot divide by zero. We indicate this with open/closed circles on the number line (remember that open means omit):



Now choosing a test point from each of the three subintervals we can determine if the expression $\frac{-2x}{2+x}$ is positive or negative. When $x = -3$, it is negative. When $x = -1$, it is positive. When $x = 1$, it is negative. Indicating this on the number line gives:



Since we wish to solve $\frac{-2x}{2+x} \geq 0$, we look at where the $+$ signs are and shade that area on the number line:



Since there is a closed circle at 0, we include it. Therefore, the solution is $(-2, 0]$. ♣

Example 1.14: Rational Inequality

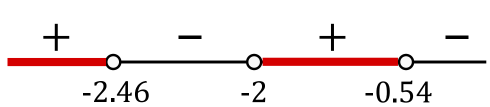
Write the solution to the following inequality using interval notation:

$$\frac{2}{x+2} > 3x+3.$$

Solution. We provide a brief outline of the solution. By subtracting $(3x+3)$ from both sides and using a common denominator of $x+2$, we can collect like terms and simplify to get:

$$\frac{-(3x^2+9x+4)}{x+2} > 0.$$

The denominator is zero when $x = -2$. Using the quadratic formula, the numerator is zero when $x = \frac{-9\pm\sqrt{33}}{6}$ (these two numbers are approximately -2.46 and -0.54). Since the inequality uses "$>$" and $0 > 0$ is false, we do not include any of the split points in our solution. After choosing suitable test points and determining the sign of $\frac{-(3x^2+9x+4)}{x+2}$ we have



Looking where the $+$ symbols are located gives the solution:

$$\left(-\infty, \frac{-9-\sqrt{33}}{6}\right) \bigcup \left(-2, \frac{-9+\sqrt{33}}{6}\right).$$

When writing the final answer we use *exact* expressions for numbers in mathematics, not approximations (unless stated otherwise). ♣

1.1.5. The Absolute Value

The **absolute value** of a number x is written as $|x|$ and represents the *distance* x is from zero. Mathematically, we define it as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Thus, if x is a negative real number, then $-x$ is a positive real number. The absolute value does *not* just turn minuses into pluses. That is, $|2x-1| \neq 2x+1$. You should be familiar with the following properties.

Absolute Value Properties

1. $|x| \geq 0$.
2. $|xy| = |x||y|$.
3. $|1/x| = 1/|x|$ when $x \neq 0$.
4. $|-x| = |x|$.
5. $|x+y| \leq |x| + |y|$. This is called the **triangle inequality**.
6. $\sqrt{x^2} = |x|$.

Example 1.15: $\sqrt{x^2} = |x|$

Observe that $\sqrt{(-3)^2}$ *gives an answer of* 3, *not* -3.

When solving inequalities with absolute values, the following are helpful.

Case 1: $a > 0$**.**

- $|x| = a$ has solutions $x = \pm a$.
- $|x| \leq a$ means $x \geq -a$ **and** $x \leq a$ (that is, $-a \leq x \leq a$).
- $|x| < a$ means $x < -a$ **and** $x < a$ (that is, $-a < x < a$).
- $|x| \geq a$ means $x \leq -a$ **or** $x \geq a$.

- $|x| > a$ means $x < -a$ **or** $x > a$.

Case 2: $a < 0$.

- $|x| = a$ has no solutions.
- Both $|x| \leq a$ and $|x| < a$ have no solutions.
- Both $|x| \geq a$ and $|x| > a$ have solution set $\{x | x \in \mathbb{R}\}$.

Case 3: $a = 0$.

- $|x| = 0$ has solution $x = 0$.
- $|x| < 0$ has no solutions.
- $|x| \leq 0$ has solution $x = 0$.
- $|x| > 0$ has solution set $\{x \in \mathbb{R} | x \neq 0\}$.
- $|x| \geq 0$ has solution set $\{x | x \in \mathbb{R}\}$.

1.1.6. Solving Inequalities that Contain Absolute Values

We start by solving an equality that contains an absolute value. To do so, we recall that if $a \geq 0$ then the solution to $|x| = a$ is $x = \pm a$. In cases where we are not sure if the right side is positive or negative, we must perform a check at the end.

Example 1.16: Absolute Value Equality

Solve for x in $|2x+3| = 2-x$.

Solution. This means that either:

$$\begin{array}{ccc} 2x+3 = +(2-x) & \textit{or} & 2x+3 = -(2-x) \\ 2x+3 = 2-x & \textit{or} & 2x+3 = -2+x \\ 3x = -1 & \textit{or} & x = -5 \\ x = -1/3 & \textit{or} & x = -5 \end{array}$$

Since we do not know if the right side "$a = 2-x$" is positive or negative, we must perform a check of our answers and omit any that are incorrect.

If $x = -1/3$, then we have $LS = |2(-1/3)+3| = |-2/3+3| = |7/3| = 7/3$ and $RS = 2-(-1/3) = 7/3$. In this case $LS = RS$, so $x = -1/3$ is a solution.

If $x = -5$, then we have $LS = |2(-5)+3| = |-10+3| = |-7| = 7$ and $RS = 2-(-5) = 2+5 = 7$. In this case $LS = RS$, so $x = -5$ is a solution. ♣

We next look at absolute values and inequalities.

Example 1.17: Absolute Value Inequality

Solve $|x-5| < 7$.

Solution. This simply means $-7 < x-5 < 7$. Adding 5 to each gives $-2 < x < 12$. Therefore the solution is the interval $(-2,12)$. ♣

In some questions you must be careful when multiplying by a negative number as in the next problem.

Example 1.18: Absolute Value Inequality

Solve $|2-z| < 7$.

Solution. This simply means $-7 < 2-z < 7$. Subtracting 2 gives: $-9 < -z < 5$. Now multiplying by -1 gives: $9 > z > -5$. *Remember to reverse the inequality signs!* We can rearrange this as $-5 < z < 9$. Therefore the solution is the interval $(-5,9)$. ♣

Example 1.19: Absolute Value Inequality

Solve $|2-z| \geq 7$.

Solution. Recall that for $a > 0$, $|x| \geq a$ means $x \leq -a$ or $x \geq a$. Thus, either $2-z \leq -7$ *or* $2-z \geq 7$. Either $9 \leq z$ *or* $-5 \geq z$. Either $z \geq 9$ *or* $z \leq -5$. In interval notation, either z is in $[9,\infty)$ *or* z is in $(-\infty,-5]$. All together, we get our solution to be: $(-\infty,-5] \cup [9,\infty)$. ♣

In the previous two examples the *only* difference is that one had $<$ in the question and the other had $\geq$. Combining the two solutions gives the *entire* real number line!

Example 1.20: Absolute Value Inequality

Solve $0 < |x-5| \leq 7$.

Solution. We split this into two cases.

(1) For $0 < |x-5|$ note that we always have that an absolute value is positive or zero (i.e., $0 \leq |x-5|$ is always true). So, for this part, we need to avoid $0 = |x-5|$ from occurring. Thus, x *cannot* be 5, that is, $x \neq 5$.

(2) For $|x-5| \leq 7$, we have $-7 \leq x-5 \leq 7$. Adding 5 to each gives $-2 \leq x \leq 12$. Therefore the solution to $|x-5| \leq 7$ is the interval $[-2,12]$.

To combine (1) and (2) we need combine $x \neq 5$ with $x \in [-2,12]$. Omitting 5 from the interval $[-2,12]$ gives our solution to be: $[-2,5) \cup (5,12]$. ♣

Exercises for Section 1.1

Exercise 1.1.1 *Simplify the following expressions as much as possible assuming $x, y > 0$:*

(a) $\dfrac{x^3 y^{-1/3}}{\sqrt[3]{y^2}x^2}$

(b) $\dfrac{3x^{-1/3}y^{-2}\sqrt[3]{x^4}}{\sqrt{9xy^{-3}}}$

(c) $\left(\dfrac{16x^2y}{x^4}\right)^{1/2}\dfrac{\sqrt[3]{x^2}}{2\sqrt{y}}$

Exercise 1.1.2 *Find the constants a, b, c if the expression*

$$\frac{4x^{-1}y^2\sqrt[3]{x}}{2x\sqrt{y}}$$

is written in the form ax^by^c.

Exercise 1.1.3 *Find the roots of the quadratic equation*

$$x^2 - 2x - 24 = 0.$$

Exercise 1.1.4 *Solve the equation*

$$\frac{x}{4x-16} - 2 = \frac{1}{x-3}.$$

Exercise 1.1.5 *Solve the following inequalities. Write your answer as a union of intervals.*

(a) $3x + 1 > 6$

(b) $0 \leq 7x - 1 < 1$

(c) $\dfrac{x^2(x-1)}{(x+2)(x+3)^3} \leq 0$

(d) $x^2 + 1 > 0$

(e) $x^2 + 1 < 0$

(f) $x^2 + 1 > 2x$

(g) $x^3 > 4x$

(h) $x^3 \geq 4x^2$

(i) $\dfrac{1}{x} > 2$

(j) $\dfrac{x}{x+2} \leq \dfrac{2}{x-1}$

Exercise 1.1.6 *Solve the equation $|6x + 2| = 1$.*

Exercise 1.1.7 *Find solutions to the following absolute value inequalities. Write your answer as a union of intervals.*

(a) $|x| \geq 2$

(b) $|x-3| \leq 1$

(c) $|2x+5| \geq 4$

(d) $|x+2| < 3x-6$

(e) $|2x+5|+4 \geq 1$

(f) $5 < |x+1| < 8$

Exercise 1.1.8 *Solve the equation* $\sqrt{1-x}+x=1$.

1.2 Analytic Geometry

In what follows, we use the notation (x_1, y_1) to represent a point in the (x, y) coordinate system, also called the x-y-plane. Previously, we used (a, b) to represent an open interval. Notation often gets reused and abused in mathematics, but thankfully, it is usually clear from the context what we mean.

In the (x, y) coordinate system we normally write the x-axis horizontally, with positive numbers to the right of the origin, and the y-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive x-direction and "upward" to be the positive y-direction. In a purely mathematical situation, we normally choose the same scale for the x- and y-axes. For example, the line joining the origin to the point (a, a) makes an angle of $45°$ with the x-axis (and also with the y-axis).

In applications, often letters other than x and y are used, and often different scales are chosen in the horizontal and vertical directions.

Example 1.21: Data Plot

Suppose you drop a coin from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter t denote the time (the number of seconds since the object was released) and to let the letter h denote the height. For each t (say, at one-second intervals) you have a corresponding height h. This information can be tabulated, and then plotted on the (t, h) coordinate plane, as shown in Figure 1.1.

We use the word "quadrant" for each of the four regions into which the plane is divided by the axes: the first quadrant is where points have both coordinates positive, or the "northeast" portion of the plot, and the second, third, and fourth quadrants are counted off counterclockwise, so the second quadrant is the northwest, the third is the southwest, and the fourth is the southeast.

Suppose we have two points A and B in the x-y-plane. We often want to know the change in x-coordinate (also called the "horizontal distance") in going from A to B. This is often written Δx, where the meaning of Δ (a capital delta in the Greek alphabet) is "change in". Thus, Δx can be read as "change in x" although it usually is read as "delta x". The point is that Δx denotes a single number, and should not be interpreted as "delta times x". Similarly, the "change in y" is written Δy and represents the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B. Using the symbol Δ in mathematical expression is referred to as **delta notion**.

seconds	0	1	2	3	4
metres	80	75.1	60.4	35.9	1.6

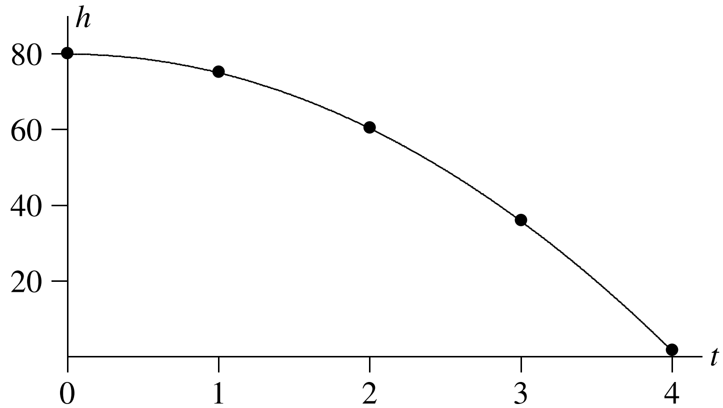


Figure 1.1: A data plot, height versus time.

Example 1.22: Change in x and y

If $A = (2,1)$ *and* $B = (3,3)$ *the change in* x *is*

$$\Delta x = 3 - 2 = 1$$

while the change in y *is*

$$\Delta y = 3 - 1 = 2.$$

The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) are:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that either or both of these might be negative.

1.2.1. Lines

If we have two *distinct* points $A(x_1, y_1)$ and $B(x_2, y_2)$, then we can draw one and only one straight line through both points. By the **slope** of this line we mean the ratio of Δy to Δx. The slope is often denoted by the letter m.

Slope Formula

The slope of the line joining the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{\Delta y}{\Delta x} = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{\text{rise}}{\text{run}}.$$

Example 1.23: Slope of a Line Joining Two Points

The line joining the two points $(1,-2)$ *and* $(3,5)$ *has slope* $m = \dfrac{5-(-2)}{3-1} = \dfrac{7}{2}$.

The most familiar form of the equation of a straight line is:

$$y = mx + b.$$

Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m; and if you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the **y-intercept**, because it is where the line crosses the y-axis (when $x = 0$). If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of either point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the **"point-slope" form** of the equation of a straight line: start with $(y - y_1)/(x - x_1) = m$ and then multiply to get

$$(y - y_1) = m(x - x_1),$$

the point-slope form. Of course, this may be further manipulated to get $y = mx - mx_1 + y_1$, which is essentially the **"**$y = mx + b$**" form**.

It is possible to find the equation of a line between two points directly from the relation $m = (y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$, which says "the slope measured between the point (x_1, y_1) and the point (x_2, y_2) is the same as the slope measured between the point (x_1, y_1) and any other point (x, y) on the line." For example, if we want to find the equation of the line joining our earlier points $A(2,1)$ and $B(3,3)$, we can use this formula:

$$m = \frac{y-1}{x-2} = \frac{3-1}{3-2} = 2, \qquad \text{so that} \qquad y - 1 = 2(x-2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

Of course, this is really just the point-slope formula, except that we are not computing m in a separate step. We summarize the three common forms of writing a straight line below:

Slope-Intercept Form of a Straight Line

An equation of a line with slope m and y-intercept b is:

$$y = mx + b.$$

Point-Slope Form of a Straight Line

An equation of a line passing through (x_1, y_1) and having slope m is:

$$y - y_1 = m(x - x_1).$$

General Form of a Straight Line

Any line can be written in the form

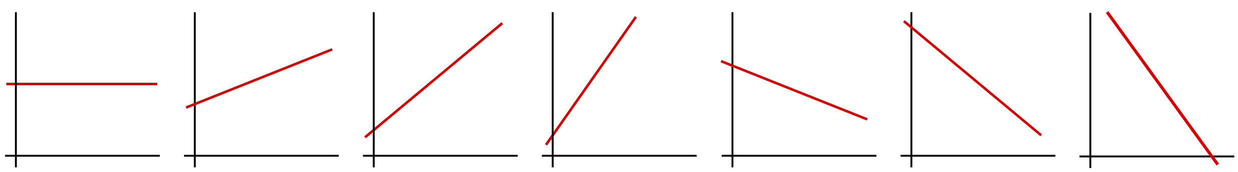
$$Ax+By+C=0,$$

where A,B,C are constants and A,B are not both 0.

The slope m of a line in the form $y=mx+b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline, while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward. If m is negative but small in absolute value, then it points only a little downward.

If $m=0$, then the line is horizontal and its equation is simply $y=b$.

All of these possibilities are illustrated below.



There is one type of line that cannot be written in the form $y=mx+b$, namely, vertical lines. A vertical line has an equation of the form $x=a$. Sometimes one says that a vertical line has an "infinite" slope.

It is often useful to find the x-intercept of a line $y=mx+b$. This is the x-value when $y=0$. Setting $mx+b$ equal to 0 and solving for x gives: $x=-b/m$.

Example 1.24: Finding x-intercepts

To find x-intercept(s) of the line $y=2x-3$ we set $y=0$ and solve for x:

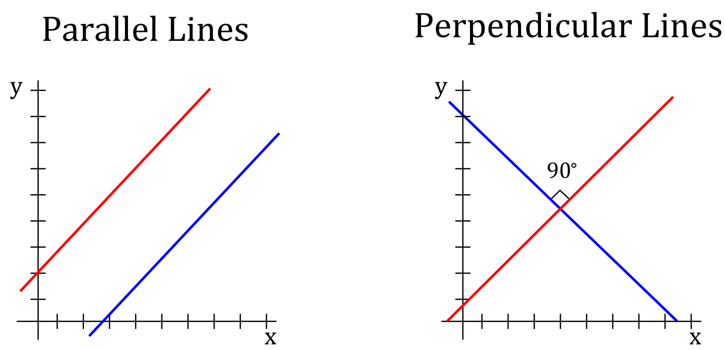
$$0=2x-3 \qquad \rightarrow \qquad x=\frac{3}{2}.$$

Thus, the line has an x-intercept of $3/2$.

It is often necessary to know if two lines are parallel or perpendicular. Let m_1 and m_2 be the slopes of the nonvertical lines L_1 and L_2. Then:

- L_1 and L_2 are **parallel** if and only if $m_1=m_2$.
- L_1 and L_2 are **perpendicular** if and only if $m_2=\dfrac{-1}{m_1}$. (Equivalently, $m_1=\frac{-1}{m_2}$).

In the case of perpendicular lines, we say their slopes are *negative reciprocals*. Below is a visual representation of a pair of parallel lines and a pair of perpendicular lines.



Example 1.25: Equation of a Line

For each part below, find an equation of a line satisfying the requirements:

(a) *Through the two points* $(0,3)$ *and* $(-2,4)$.

(b) *With slope* 7 *and through point* $(1,-2)$.

(c) *With slope* 2 *and* y*-intercept* 4.

(d) *With* x*-intercept* 8 *and* y*-intercept* -3.

(e) *Through point* $(5,3)$ *and parallel to the line* $2x+4y+2=0$.

(f) *With* y*-intercept* 4 *and perpendicular to the line* $y=-\frac{2}{3}x+3$.

Solution. (a) We use the *slope formula* on $(x_1,y_1)=(0,3)$ and $(x_2,y_2)=(-2,4)$ to find m:

$$m=\frac{(4)-(3)}{(-2)-(0)}=\frac{1}{-2}=-\frac{1}{2}.$$

Now using the *point-slope formula* we get an equation to be:

$$y-3=-\frac{1}{2}(x-0) \quad \rightarrow \quad y=-\frac{1}{2}x+3.$$

(b) Using the *point-slope formula* with $m=7$ and $(x_1,y_1)=(1,-2)$ gives:

$$y-(-2)=7(x-1) \quad \rightarrow \quad y=7x-9.$$

(c) Using the *slope-intercept formula* with $m=2$ and $b=4$ we get $y=2x+4$.

(d) Note that the intercepts give us two points: $(x_1,y_1)=(8,0)$ and $(x_2,y_2)=(0,-3)$. Now follow the steps in part (a):

$$m=\frac{-3-0}{0-8}=\frac{3}{8}.$$

Using the *point-slope formula* we get an equation to be:

$$y-(-3)=\frac{3}{8}(x-0) \quad \rightarrow \quad y=\frac{3}{8}x-3.$$

(e) The line $2x+4y+2=0$ can be written as:

$$4y=-2x-2 \quad \rightarrow \quad y=-\frac{1}{2}x-\frac{1}{2}.$$

This line has slope $-1/2$. Since our line is *parallel* to it, we have $m=-1/2$. Now we have a point $(x_1,y_1)=(5,3)$ and slope $m=-1/2$, thus, the *point-slope formula* gives:

$$y-3=-\frac{1}{2}(x-5).$$

(f) The line $y=-\frac{2}{3}x+3$ has slope $m=-2/3$. Since our line is perpendicular to it, the slope of our line is the *negative reciprocal*, hence, $m=3/2$. Now we have $b=4$ and $m=3/2$, thus by the *slope-intercept formula*, an equation of the line is

$$y=\frac{3}{2}x+4.$$

♣

Example 1.26: Parallel and Perpendicular Lines

Are the two lines $7x+2y+3=0$ and $6x-4y+2=0$ perpendicular? Are they parallel? If they are not parallel, what is their point of intersection?

Solution. The first line is:

$$7x+2y+3=0 \quad \rightarrow \quad 2y=-7x-3 \quad \rightarrow \quad y=-\frac{7}{2}x-\frac{3}{2}.$$

It has slope $m_1=-7/2$. The second line is:

$$6x-4y+2=0 \quad \rightarrow \quad -4y=-6x-2 \quad \rightarrow \quad y=\frac{3}{2}x+\frac{1}{2}.$$

It has slope $m_2=3/2$. Since $m_1 \cdot m_2 \neq -1$ (they are not negative reciprocals), the lines are not perpendicular. Since $m_1 \neq m_2$ the lines are not parallel.

We find points of intersection by setting y-values to be the same and solving. In particular, we have

$$-\frac{7}{2}x-\frac{3}{2}=\frac{3}{2}x+\frac{1}{2}.$$

Solving for x gives $x=-2/5$. Then substituting this into either equation gives $y=-1/10$. Therefore, the lines intersect at the point $(-2/5,-1/10)$. ♣

1.2.2. Distance between Two Points and Midpoints

Given two points (x_1, y_1) and (x_2, y_2), recall that their horizontal distance from one another is $\Delta x = x_2 - x_1$ and their vertical distance from one another is $\Delta y = y_2 - y_1$. Actually, the word "distance" normally denotes "positive distance". Δx and Δy are *signed* distances, but this is clear from context. The (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs $|\Delta x|$ and $|\Delta y|$, as shown in Figure 1.2. The Pythagorean Theorem states that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

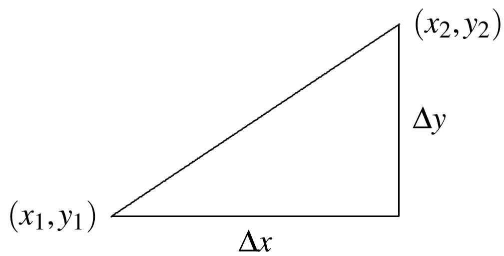


Figure 1.2: Distance between two points (here, Δx and Δy are positive).

Distance Formula

The distance between points (x_1, y_1) and (x_2, y_2) is

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1.27: Distance Between Two Points

The distance, d, between points $A(2,1)$ and $B(3,3)$ is

$$d = \sqrt{(3-2)^2 + (3-1)^2} = \sqrt{5}.$$

As a special case of the distance formula, suppose we want to know the distance of a point (x, y) to the origin. According to the distance formula, this is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

A point (x, y) is at a distance r from the origin if and only if $\sqrt{x^2 + y^2} = r$, or, if we square both sides: $x^2 + y^2 = r^2$. As we will see, this is the equation of the circle of radius, r, centered at the origin.

Furthermore, given two points we can determine the **midpoint** of the line segment joining the two points.

Midpoint Formula

The midpoint of the line segment joining two points (x_1,y_1) and (x_2,y_2) is the point with coordinates:

$$\text{midpoint} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Example 1.28: Midpoint of a Line Segment

Find the midpoint of the line segment joining the given points: $(1,0)$ *and* $(5,-2)$.

Solution. Using the *midpoint formula* on $(x_1,y_1) = (1,0)$ and $(x_2,y_2) = (5,-2)$ we get:

$$\left(\frac{(1)+(5)}{2}, \frac{(0)+(-2)}{2}\right) = (3,-1).$$

Thus, the midpoint of the line segment occurs at $(3,-1)$. ♣

1.2.3. Conics

In this section we review equations of parabolas, circles, ellipses and hyperbolas. We will give the equations of various conics in **standard form** along with a sketch. A useful mnemonic is the following.

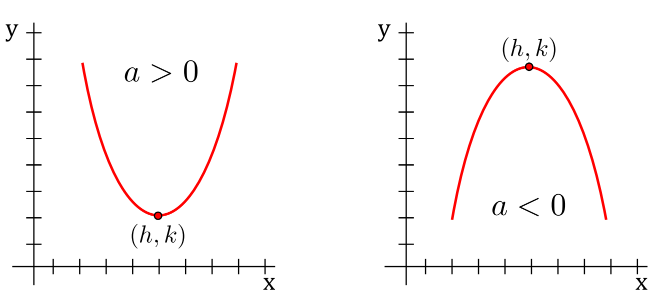
Mnemonic

In each conic formula presented, the terms '$x-h$' and '$y-k$' will always appear. The point (h,k) will alway represent either the centre or vertex of the particular conic.

Note that h or k (or both) may equal 0.

Vertical Parabola: The equation of a vertical parabola is:

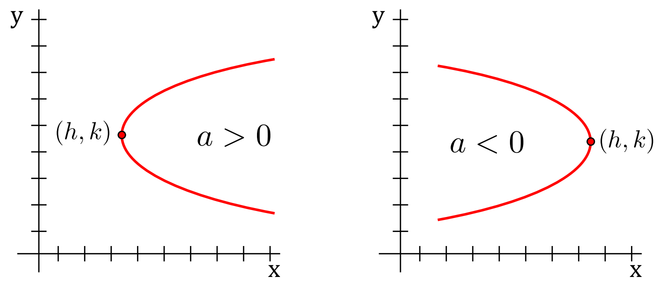
$$y-k = a(x-h)^2$$



- (h,k) is the *vertex* of the parabola.
- a is the vertical *stretch factor*.
- If $a > 0$, the parabola opens *upward*.
- If $a < 0$, the parabola opens *downward*.

Horizontal Parabola: The equation of a horizontal parabola is:

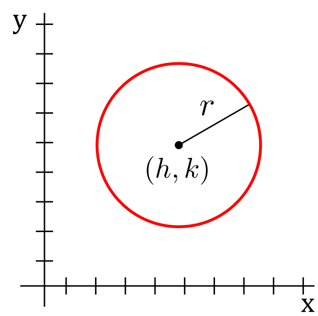
$$x - h = a(y - k)^2$$



- (h,k) is the *vertex* of the parabola.
- a is the horizontal *stretch factor*.
- If $a > 0$, the parabola opens *right*.
- If $a < 0$, the parabola opens *left*.

Circle: The equation of a circle is:

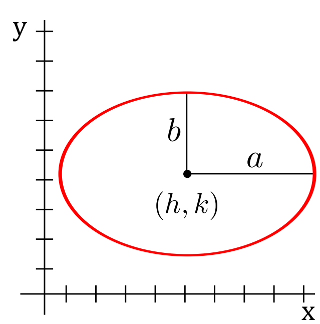
$$(x - h)^2 + (y - k)^2 = r^2$$



- (h,k) is the *centre* of the circle.
- r is the *radius* of the circle.

Ellipse: The equation of an ellipse is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



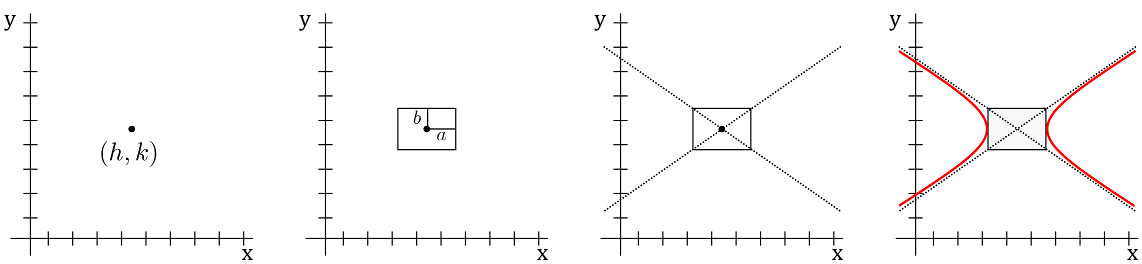
- (h,k) is the *centre* of the ellipse.
- a is the *horizontal distance* from the centre to the edge of the ellipse.
- b is the *vertical distance* from the centre to the edge of the ellipse.

Horizontal Hyperbola: The equation of a horizontal hyperbola is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- (h,k) is the *centre* of the hyperbola.
- a is the *horizontal distance* from the centre to the edge of the box.
- a,b are the *reference box* values. The box has a centre of (h,k).
- b is the *vertical distance* from the centre to the edge of the box.

Given the equation of a horizontal hyperbola, one may sketch it by first placing a dot at the point (h,k). Then draw a box around (h,k) with horizontal distance a and vertical distance b to the edge of the box. Then draw dotted lines (called the **asymptotes** of the hyperbola) through the corners of the box. Finally, sketch the hyperbola in a horizontal direction as illustrated below.

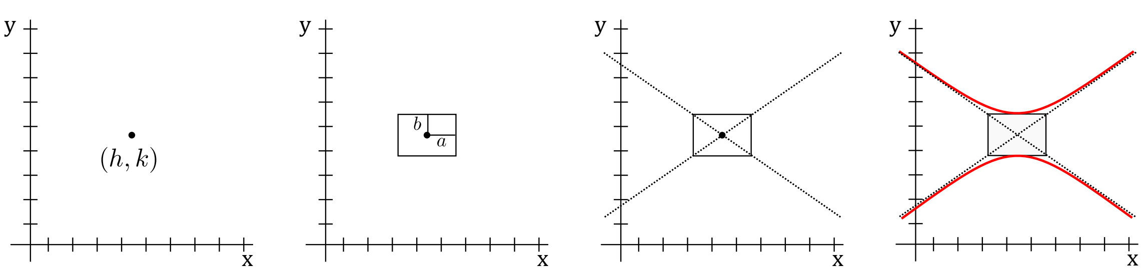


Vertical Hyperbola: The equation of a vertical hyperbola is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$$

- (h,k) is the *centre* of the hyperbola.
- a is the *horizontal distance* from the centre to the edge of the box.
- a,b are the *reference box* values. The box has a centre of (h,k).
- b is the *vertical distance* from the centre to the edge of the box.

Given the equation of a vertical hyperbola, one may sketch it by following the same steps as with a horizontal hyperbola, but sketching the hyperbola going in a vertical direction.



Determining the Type of Conic

An equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

gives rise to a graph that can be generated by performing a conic section (parabolas, circles, ellipses, hyperbolas). Note that the Bxy term involves conic rotation. The Dx, Ex, and F terms affect the vertex and centre. For simplicity, we omit the Bxy term. To determine the type of graph we focus our analysis on the values of A and C.

- If $A = C$, the graph is a *circle*.
- If $AC > 0$ (and $A \neq C$), the graph is an *ellipse*.
- If $AC = 0$, the graph is a *parabola*.
- If $AC < 0$, the graph is a *hyperbola*.

Example 1.29: Center and Radius of a Circle

Find the centre and radius of the circle $y^2 + x^2 - 12x + 8y + 43 = 0$.

Solution. We need to complete the square twice, once for the x terms and once for the y terms. We'll do both at the same time. First let's collect the terms with x together, the terms with y together, and move the number to the other side.

$$(x^2 - 12x) + (y^2 + 8y) = -43$$

We add 36 to both sides for the x term ($-12 \to \frac{-12}{2} = -6 \to (-6)^2 = 36$), and 16 to both sides for the y term ($8 \to \frac{8}{2} = 4 \to (4)^2 = 16$):

$$(x^2 - 12x + 36) + (y^2 + 8y + 16) = -43 + 36 + 16$$

Factoring gives:

$$(x-6)^2 + (y+4)^2 = 3^2.$$

Therefore, the centre of the circle is $(6, -4)$ and the radius is 3. ♣

Example 1.30: Type of Conic

What type of conic is $4x^2 - y^2 - 8x + 8 = 0$*? Put it in standard form.*

Solution. Here we have $A = 4$ and $C = -1$. Since $AC < 0$, the conic is a hyperbola. Let us complete the square for the x and y terms. First let's collect the terms with x together, the terms with y together, and move the number to the other side.

$$(4x^2 - 8x) - y^2 = -8$$

Now we factor out 4 from the x terms.

$$4(x^2 - 2x) - y^2 = -8$$

Notice that we don't need to complete the square for the y terms (it is already completed!). To complete the square for the x terms we add 1 since $-2 \to \frac{-2}{2} = -1 \to (-1)^2 = 1$, taking into consideration that the a value is 4:

$$4(x^2 - 2x + 1) - y^2 = -8 + 4 \cdot 1$$

Factoring gives:

$$4(x-1)^2 - y^2 = -4$$

A hyperbola in standard form has ± 1 on the right side and a positive x^2 on the left side, thus, we must divide by 4:

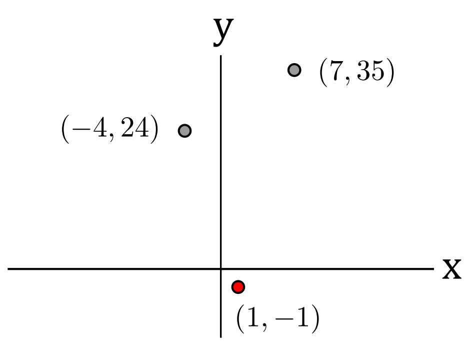
$$(x-1)^2 - \frac{y^2}{4} = -1$$

Now we can see that the equation represents a vertical hyperbola with centre $(1,0)$ (and with a value $\sqrt{1} = 1$, and b value $\sqrt{4} = 2$). ♣

Example 1.31: Equation of Parabola

Find an equation of the parabola with vertex $(1,-1)$ that passes through the points $(-4,24)$ and $(7,35)$.

Solution. We first need to determine if it is a vertical parabola or horizontal parabola. See below for a sketch of the three points $(1,-1)$, $(-4,24)$ and $(7,35)$ in the x-y-plane.



Note that the vertex is $(1,-1)$. Given the location of the vertex, the parabola cannot open downwards. It also cannot open left or right (because the vertex is between the other two points - if it were to open to the right, every other point would need to be to the right of the vertex; if it were to open to the left, every other point would need to be to the left of the vertex). Therefore, the parabola must open upwards and it is a vertical parabola. It has an equation of

$$y - k = a(x-h)^2.$$

As the vertex is $(h,k) = (1,-1)$ we have:

$$y - (-1) = a(x-1)^2$$

To determine a, we substitute one of the points into the equation and solve. Let us substitute the point $(x,y) = (-4,24)$ into the equation:

$$24 - (-1) = a(-4-1)^2 \quad \to \quad 25 = 25a \quad \to \quad a = 1.$$

Therefore, the equation of the parabola is:

$$y+1=(x-1)^2.$$

Note that if we substituted $(7,35)$ into the equation instead, we would also get $a=1$. ♣

Exercises for Section 1.2

Exercise 1.2.1 *Find the equation of the line in the form $y=mx+b$:*

(a) *through* $(1,1)$ *and* $(-5,-3)$

(b) *through* $(-1,2)$ *with slope* -2

(c) *through* $(-1,1)$ *and* $(5,-3)$

(d) *through* $(2,5)$ *and parallel to the line* $3x+9y+6=0$

(e) *with x-intercept 5 and perpendicular to the line* $y=2x+4$

Exercise 1.2.2 *Change the following equations to the form $y=mx+b$, graph the line, and find the y-intercept and x-intercept.*

(a) $y-2x=2$

(b) $x+y=6$

(c) $x=2y-1$

(d) $3=2y$

(e) $2x+3y+6=0$

Exercise 1.2.3 *Determine whether the lines $3x+6y=7$ and $2x+4y=5$ are parallel.*

Exercise 1.2.4 *Suppose a triangle in the x-y-plane has vertices $(-1,0)$, $(1,0)$ and $(0,2)$. Find the equations of the three lines that lie along the sides of the triangle in $y=mx+b$ form.*

Exercise 1.2.5 *Let x stand for temperature in degrees Celsius (centigrade), and let y stand for temperature in degrees Fahrenheit. A temperature of $0°C$ corresponds to $32°F$, and a temperature of $100°C$ corresponds to $212°F$. Find the equation of the line that relates temperature Fahrenheit y to temperature Celsius x in the form $y=mx+b$. Graph the line, and find the point at which this line intersects $y=x$. What is the practical meaning of this point?*

Exercise 1.2.6 *A car rental firm has the following charges for a certain type of car: \$25 per day with 100 free miles included, \$0.15 per mile for more than 100 miles. Suppose you want to rent a car for one day, and you know you'll use it for more than 100 miles. What is the equation relating the cost y to the number of miles x that you drive the car?*

Exercise 1.2.7 *A photocopy store advertises the following prices: 5c per copy for the first 20 copies, 4c per copy for the 21st through 100th copy, and 3c per copy after the 100th copy. Let x be the number of copies, and let y be the total cost of photocopying. (a) Graph the cost as x goes from 0 to 200 copies. (b) Find the equation in the form* $y = mx + b$ *that tells you the cost of making x copies when x is more than 100.*

Exercise 1.2.8 *Market research tells you that if you set the price of an item at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Let x be the number of items you can sell, and let P be the price of an item. (a) Express P linearly in terms of x, in other words, express P in the form* $P = mx + b$*. (b) Express x linearly in terms of P.*

Exercise 1.2.9 *An instructor gives a 100-point final exam, and decides that a score 90 or above will be a grade of 4.0, a score of 40 or below will be a grade of 0.0, and between 40 and 90 the grading will be linear. Let x be the exam score, and let y be the corresponding grade. Find a formula of the form* $y = mx + b$ *which applies to scores x between 40 and 90.*

Exercise 1.2.10 *Find the distance between the pairs of points:*

(a) $(-1,1)$ *and* $(1,1)$.

(b) $(5,3)$ *and* $(-7,-2)$.

(c) $(1,1)$ *and the origin.*

Exercise 1.2.11 *Find the midpoint of the line segment joining the point* $(20,-10)$ *to the origin.*

Exercise 1.2.12 *Find the equation of the circle of radius 3 centered at:*

(a) $(0,0)$

(b) $(5,6)$

(c) $(-5,-6)$

(d) $(0,3)$

(e) $(0,-3)$

(f) $(3,0)$

Exercise 1.2.13 *For each pair of points* $A(x_1,y_1)$ *and* $B(x_2,y_2)$ *find an equation of the circle with center at A that goes through B.*

(a) $A(2,0)$, $B(4,3)$

(b) $A(-2,3)$, $B(4,3)$

Exercise 1.2.14 *Determine the type of conic and sketch it.*

(a) $x^2 + y^2 + 10y = 0$

(b) $9x^2 - 90x + y^2 + 81 = 0$

(c) $6x + y^2 - 8y = 0$

Exercise 1.2.15 *Find the standard equation of the circle passing through* $(-2,1)$ *and tangent to the line* $3x - 2y = 6$ *at the point* $(4,3)$*. Sketch. (Hint: The line through the center of the circle and the point of tangency is perpendicular to the tangent line.)*

1.3 Trigonometry

In this section we review the definitions of trigonometric functions.

1.3.1. Angles and Sectors of Circles

Mathematicians tend to deal mostly with **radians** and we will see later that some formulas are more elegant when using radians (rather than degrees). The relationship between degrees and radians is:

$$\pi \text{ rad} = 180^\circ.$$

Using this formula, some common angles can be derived:

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Example 1.32: Degrees to Radians

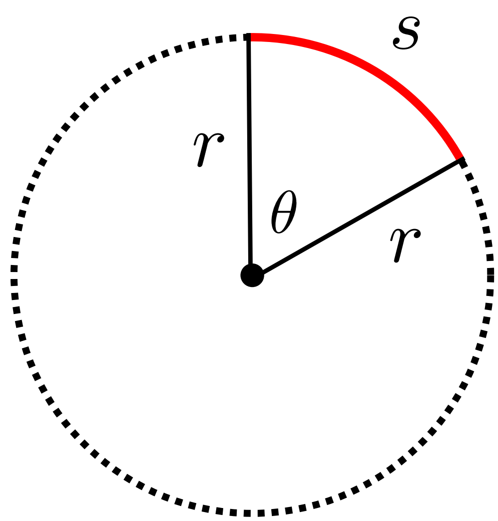
To convert 45° *to radians, multiply by* $\frac{\pi}{180^\circ}$ *to get* $\frac{\pi}{4}$.

Example 1.33: Radians to Degrees

To convert $\frac{5\pi}{6}$ *radians to degrees, multiply by* $\frac{180^\circ}{\pi}$ *to get* 150°.

From now on, unless otherwise indicated, we will *always* use radian measure.

In the diagram below is a sector of a circle with **central angle** θ and radius r **subtending** an arc with length s.



When θ is measure in radians, we have the following formula relating θ, s and r:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta.$$

Sector Area

The area of the sector is equal to:

$$\text{Sector Area} = \frac{1}{2}r^2\theta.$$

Example 1.34: Angle Subtended by Arc

If a circle has radius 3 cm, then an angle of 2 rad is subtended by an arc of 6 cm ($s = r\theta = 3 \cdot 2 = 6$).

Example 1.35: Area of Circle

If we substitute $\theta = 2\pi$ (a complete revolution) into the sector area formula we get the area of a circle:

$$A = \frac{1}{2}r^2(2\pi) = \pi r^2$$

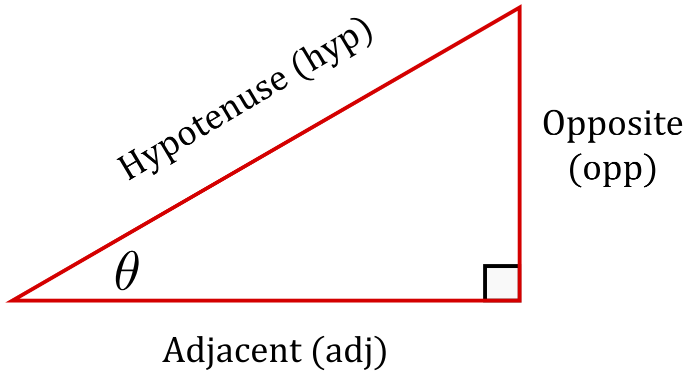
1.3.2. Trigonometric Functions

There are six basic trigonometric functions:

- Sine (abbreviated by sin)
- Cosine (abbreviated by cos)
- Tangent (abbreviated by tan)
- Cosecant (abbreviated by csc)

- Secant (abbreviated by sec)
- Cotangent (abbreviated by cot)

We first describe trigonometric functions in terms of ratios of two sides of a *right angle triangle* containing the angle θ.



With reference to the above triangle, for an acute angle θ (that is, $0 \leq \theta < \pi/2$), the six trigonometric functions can be described as follows:

Definition 1.36: Basic Six Trigonometric Functions

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

where opp stands for opposite, adj for adjacent, and hyp for hypotenuse.

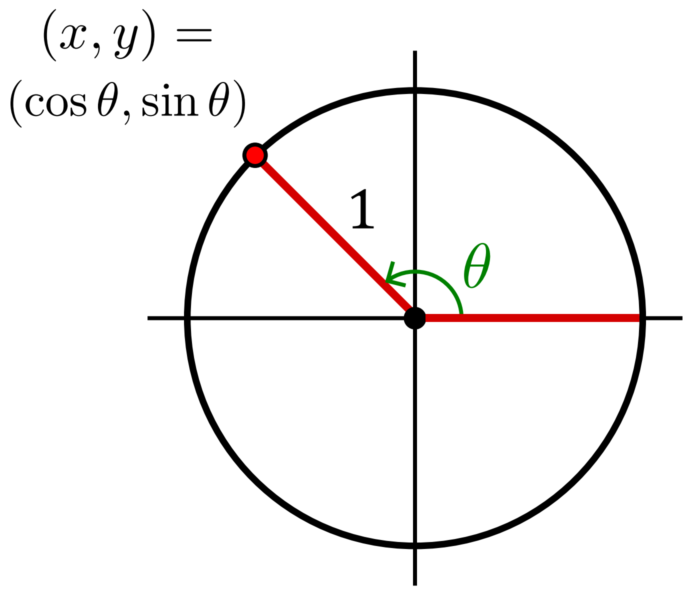
Notice that sine is the ratio of the **o**pposite and **h**ypotenuse. We use the mnemonic SOH to remember this ratio. Similarly, CAH and TOA remind us of the cos and tan ratios.

Mnemonic

The mnemonic *SOH CAH TOA* is useful in remembering how trigonometric functions of acute angles relate to the sides of a right triangle.

This description does not apply to *obtuse* or *negative angles*. To define the six basic trigonometric functions we first define sine and cosine as the lengths of various line segments from a unit circle, and then we define the remaining four basic trigonometric functions in terms of sine and cosine.

Take a line originating at the origin (making an angle of θ with the positive half of the x-axis) and suppose this line intersects the unit circle at the point (x,y). The x- and y-coordinates of this point of intersection are equal to $\cos\theta$ and $\sin\theta$, respectively.



For angles greater than 2π or less than -2π, simply continue to rotate around the circle. In this way, sine and cosine become periodic functions with period 2π:

$$\sin\theta = \sin(\theta + 2\pi k) \quad \cos\theta = \cos(\theta + 2\pi k)$$

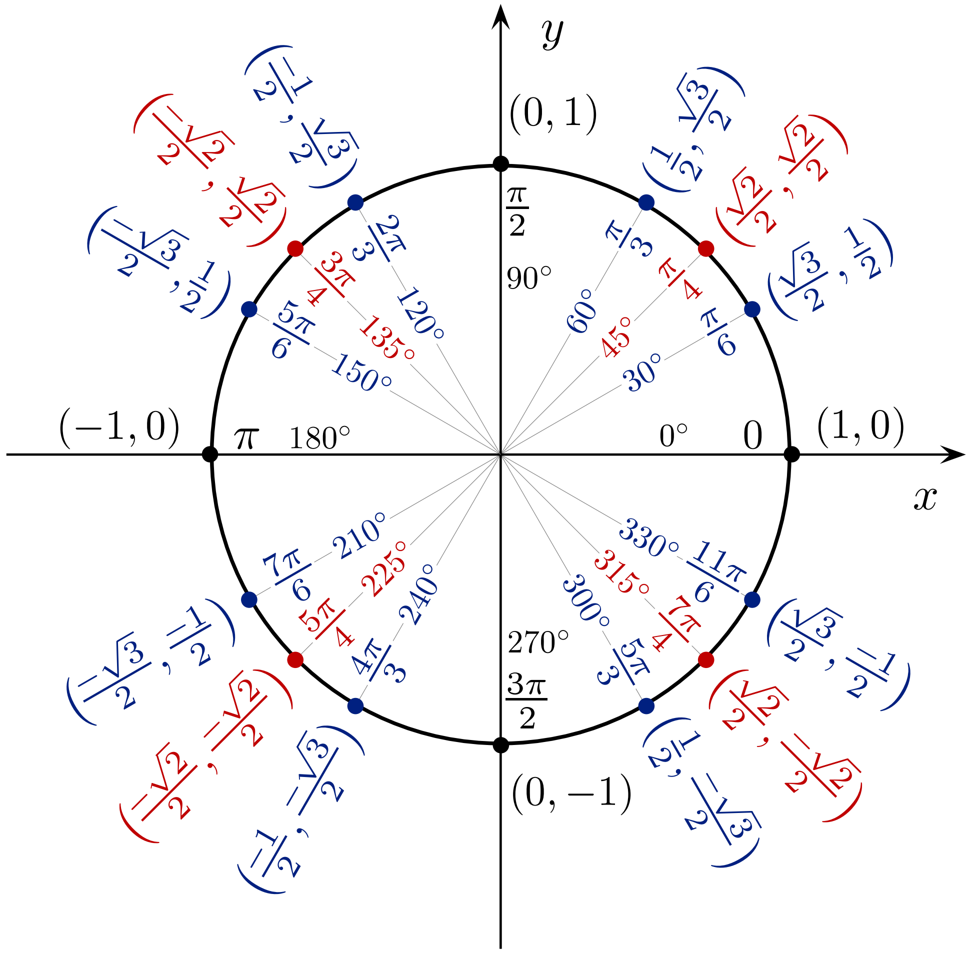
for any angle θ and any integer k.

Above, only sine and cosine were defined directly by the circle. We now define the remaining four basic trigonometric functions in terms of the functions $\sin\theta$ and $\cos\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

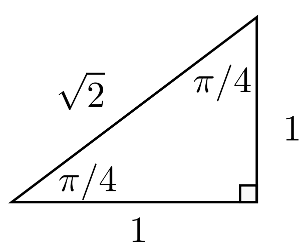
1.3.3. Computing Exact Trigonometric Ratios

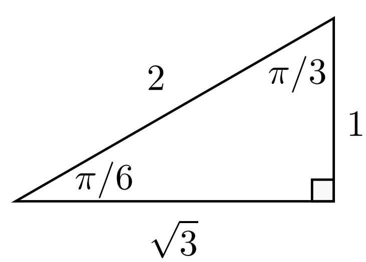
The **unit circle** is often used to determine the *exact* value of a particular trigonometric function.



Reading from the unit circle one can see that $\cos 5\pi/6 = -\sqrt{3}/2$ and $\sin 5\pi/6 = 1/2$ (remember the that the x-coordinate is $\cos\theta$ and the y-coordinate is $\sin\theta$). However, we don't always have access to the unit circle. In this case, we can compute the exact trigonometric ratios for $\theta = 5\pi/6$ by using **special triangles** and the **CAST Rule** described below.

The first special triangle has angles of $45°, 45°, 90°$ (i.e., $\pi/4, \pi/4, \pi/2$) with side lengths $1, 1, \sqrt{2}$, while the second special triangle has angles of $30°, 60°, 90°$ (i.e., $\pi/6, \pi/3, \pi/2$) with side lengths $1, 2, \sqrt{3}$. They are classically referred to as the 1-1-$\sqrt{2}$ triangle, and the 1-2-$\sqrt{3}$ triangle, respectively, shown below.





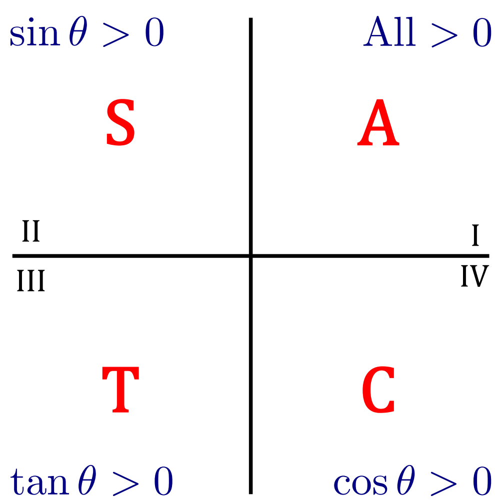
Mnemonic

The first triangle should be easy to remember. To remember the second triangle, place the largest number (2) across from the largest angle ($90^\circ = \pi/2$). Place the smallest number (1) across from the smallest angle ($30^\circ = \pi/6$). Place the middle number ($\sqrt{3} \approx 1.73$) across from the middle angle ($60^\circ = \pi/3$). Double check using the Pythagorean Theorem that the sides satisfy $a^2 + b^2 = c^2$.

The special triangles allow us to compute the exact value (excluding the sign) of trigonometric ratios, but to determine the sign, we can use the CAST Rule.

The CAST Rule

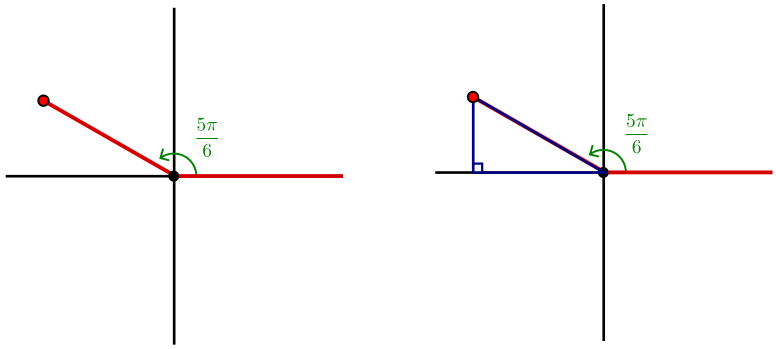
The CAST Rule says that in quadrant I all three of $\sin\theta$, $\cos\theta$, $\tan\theta$ are positive. In quadrant II, only $\sin\theta$ is positive, while $\cos\theta$, $\tan\theta$ are negative. In quadrant III, only $\tan\theta$ is positive, while $\sin\theta$, $\cos\theta$ are negative. In quadrant IV, only $\cos\theta$ is positive, while $\sin\theta$, $\tan\theta$ are negative. To remember this, simply label the quadrants by the letters C-A-S-T starting in the bottom right and labelling counter-clockwise.



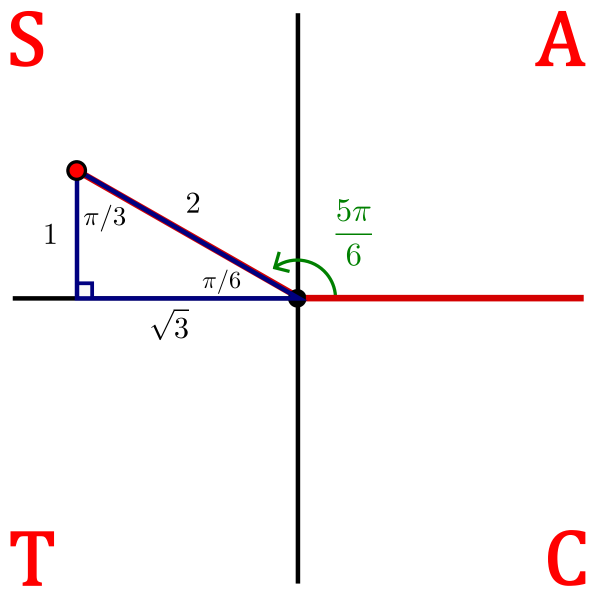
Example 1.37: Determining Trigonometric Ratios Without Unit Circle

Determine $\sin 5\pi/6$, $\cos 5\pi/6$, $\tan 5\pi/6$, $\sec 5\pi/6$, $\csc 5\pi/6$ *and* $\cot 5\pi/6$ *exactly by using the special triangles and CAST Rule.*

Solution. We start by drawing the x-y-plane and indicating our angle of $5\pi/6$ in standard position (positive angles rotate *counterclockwise* while negative angles rotate *clockwise*). Next, we drop a perpendicular to the x-axis (never drop it to the y-axis!).



Notice that we can now figure out the angles in the triangle. Since $180^\circ = \pi$, we have an interior angle of $\pi - 5\pi/6 = \pi/6$ inside the triangle. As the *angles of a triangle add up to* $180^\circ = \pi$, the other angle must be $\pi/3$. This gives one of our special triangles. We label it accordingly and add the CAST Rule to our diagram.



From the above figure we see that $5\pi/6$ lies in quadrant II where $\sin\theta$ is positive and $\cos\theta$ and $\tan\theta$ are negative. This gives us the *sign* of $\sin\theta$, $\cos\theta$ and $\tan\theta$. To determine the *value* we use the special triangle and SOH CAH TOA.

Using $\sin\theta = opp/hyp$ we find a value of $1/2$. Since $\sin\theta$ is positive in quadrant II, we have

$$\sin\frac{5\pi}{6} = +\frac{1}{2}.$$

Using $\cos\theta = adj/hyp$ we find a value of $\sqrt{3}/2$. But $\cos\theta$ is negative in quadrant II, therefore,

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}.$$

Using $\tan\theta = opp/adj$ we find a value of $1/\sqrt{3}$. But $\tan\theta$ is negative in quadrant II, therefore,

$$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}.$$

To determine $\sec\theta$, $\csc\theta$ and $\cot\theta$ we use the definitions:

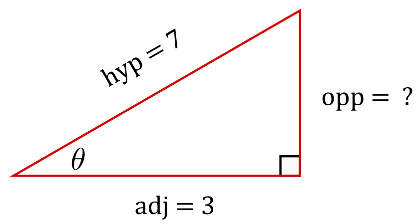
$$\csc\frac{5\pi}{6} = \frac{1}{\sin\frac{5\pi}{6}} = +2, \qquad \sec\frac{5\pi}{6} = \frac{1}{\cos\frac{5\pi}{6}} = -\frac{2}{\sqrt{3}}, \qquad \cot\frac{5\pi}{6} = \frac{1}{\tan\frac{5\pi}{6}} = -\sqrt{3}.$$

♣

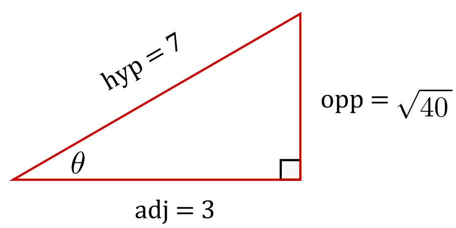
Example 1.38: CAST Rule

If $\cos\theta = 3/7$ and $3\pi/2 < \theta < 2\pi$, then find $\cot\theta$.

Solution. We first draw a right angle triangle. Since $\cos\theta = adj/hyp = 3/7$, we let the adjacent side have length 3 and the hypotenuse have length 7.



Using the Pythagorean Theorem, we have $3^2 + (\text{opp})^2 = 7^2$. Thus, the opposite side has length $\sqrt{40}$.



To find $\cot\theta$ we use the definition:

$$\cot\theta = \frac{1}{\tan\theta}.$$

Since we are given $3\pi/2 < \theta < 2\pi$, we are in the fourth quadrant. By the CAST Rule, $\tan\theta$ is negative in this quadrant. As $\tan\theta = opp/adj$, it has a value of $\sqrt{40}/3$, but by the CAST Rule it is negative, that is,

$$\tan\theta = -\frac{\sqrt{40}}{3}.$$

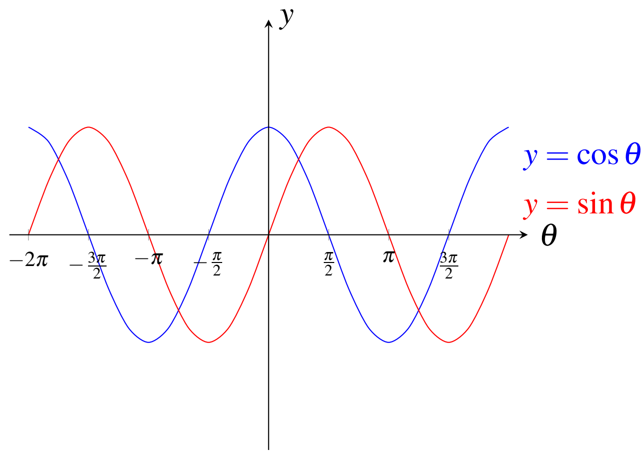
Therefore,

$$\cot\theta = -\frac{3}{\sqrt{40}}.$$

♣

1.3.4. Graphs of Trigonometric Functions

The graph of the functions $\sin x$ and $\cos x$ can be visually represented as:

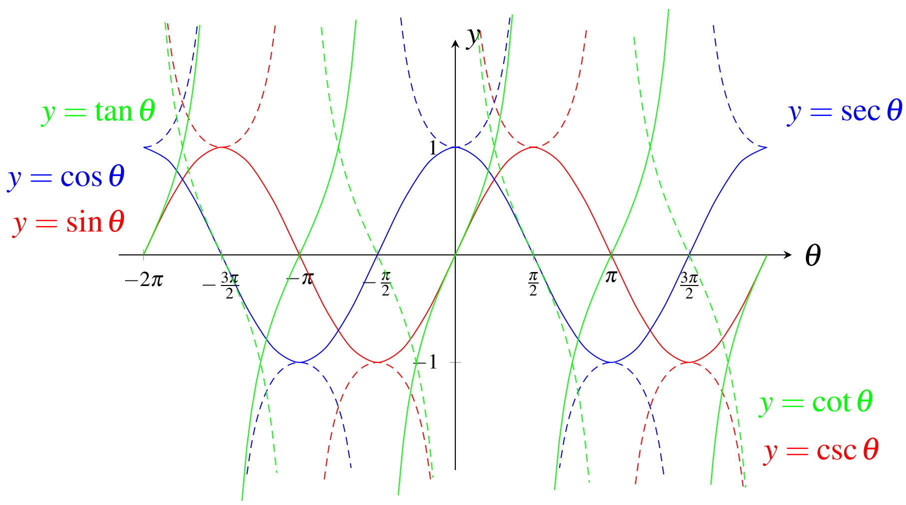


Both $\sin x$ and $\cos x$ have domain $(-\infty,\infty)$ and range $[-1,1]$. That is,

$$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1.$$

The zeros of $\sin x$ occur at the integer multiples of π, that is, $\sin x = 0$ whenever $x = n\pi$, where n is an integer. Similarly, $\cos x = 0$ whenever $x = \pi/2 + n\pi$, where n is an integer.

The six basic trigonometric functions can be visually represented as:



Both tangent and cotangent have range $(-\infty,\infty)$, whereas cosecant and secant have range $(-\infty,-1]\cup[1,\infty)$. Each of these functions is periodic. Tangent and cotangent have period π, whereas sine, cosine, cosecant and secant have period 2π.

1.3.5. Trigonometric Identities

There are numerous trigonometric identities, including those relating to shift/periodicity, Pythagoras type identities, double-angle formulas, half-angle formulas and addition formulas. We list these below.

Shifts and Periodicity

$\sin(\theta+2\pi)=\sin\theta$	$\cos(\theta+2\pi)=\cos\theta$	$\tan(\theta+2\pi)=\tan\theta$
$\sin(\theta+\pi)=-\sin\theta$	$\cos(\theta+\pi)=-\cos\theta$	$\tan(\theta+\pi)=\tan\theta$
$\sin(-\theta)=-\sin\theta$	$\cos(-\theta)=\cos\theta$	$\tan(-\theta)=-\tan\theta$
$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$	$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$	$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

Pythagoras Type Formulas

$$\sin^2\theta+\cos^2\theta=1$$

$$\tan^2\theta+1=\sec^2\theta$$

$$1+\cot^2\theta=\csc^2\theta$$

Double-angle Formulas

$$\sin(2\theta)=2\sin\theta\cos\theta$$

$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$$

Half-angle Formulas

$$\cos^2\theta=\frac{1+\cos(2\theta)}{2} \qquad \sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

Addition Formulas

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$$

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

$$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$$

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$$

Example 1.39: Double Angle

Find all values of x with $0\le x\le\pi$ such that $\sin 2x=\sin x$.

Solution. Using the double-angle formula $\sin 2x = 2\sin x \cos x$ we have:

$$2\sin x\cos x = \sin x$$

$$2\sin x\cos x - \sin x = 0$$

$$\sin x(2\cos x - 1) = 0$$

Thus, either $\sin x = 0$ or $\cos x = 1/2$. For the first case when $\sin x = 0$, we get $x = 0$ or $x = \pi$. For the second case when $\cos x = 1/2$, we get $x = \pi/3$ (use the special triangles and CAST rule to get this). Thus, we have three solutions: $x = 0$, $x = \pi/3$, $x = \pi$. ♣

Exercises for Section 1.3

Exercise 1.3.1 *Find all values of* θ *such that* $\sin(\theta) = -1$*; give your answer in radians.*

Exercise 1.3.2 *Find all values of* θ *such that* $\cos(2\theta) = 1/2$*; give your answer in radians.*

Exercise 1.3.3 *Compute the following:*

(a) $\sin(3\pi)$

(b) $\sec(5\pi/6)$

(c) $\cos(-\pi/3)$

(d) $\csc(4\pi/3)$

(e) $\tan(7\pi/4)$

(f) $\cot(13\pi/4)$

Exercise 1.3.4 *If* $\sin\theta = \frac{3}{5}$ *and* $\frac{\pi}{2} < \theta < \pi$*, then find* $\sec\theta$.

Exercise 1.3.5 *Suppose that* $\tan\theta = x$ *and* $\pi < \theta < \frac{3\pi}{2}$*, find* $\sin\theta$ *and* $\cos\theta$ *in terms of* x.

Exercise 1.3.6 *Find an angle* θ *such that* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ *and* $\sin\theta = \sin\frac{23\pi}{7}$.

Exercise 1.3.7 *Use an angle sum identity to compute* $\cos(\pi/12)$.

Exercise 1.3.8 *Use an angle sum identity to compute* $\tan(5\pi/12)$.

Exercise 1.3.9 *Verify the following identities*

(a) $\cos^2(t)/(1-\sin(t)) = 1+\sin(t)$

(b) $2\csc(2\theta) = \sec(\theta)\csc(\theta)$

(c) $\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)$

Exercise 1.3.10 *Sketch the following functions:*

(a) $y = 2\sin(x)$

(b) $y = \sin(3x)$

(c) $y = \sin(-x)$

Exercise 1.3.11 *Find all of the solutions of* $2\sin(t) - 1 - \sin^2(t) = 0$ *in the interval* $[0, 2\pi]$.

1.4 Additional Exercises

These problems require a comprehensive knowledge of the skills reviewed in this chapter. They are not in any particular order. A proficiency in these skills will help you a long way as your learn the calculus material in the following chapters.

Exercise 1.4.1 *Rationalize the denominator for each of the following expressions. That is, re-write the expression in such a way that no square roots appear in the denominator. Also, simplify your answers if possible.*

(a) $\dfrac{1}{\sqrt{2}}$

(b) $\dfrac{3h}{\sqrt{x+h+1} - \sqrt{x+1}}$

Exercise 1.4.2 *Solve the following equations.*

(a) $2 - 5(x-3) = 4 - 10x$

(b) $2x^2 - 5x = 3$

(c) $x^2 - x - 3 = 0$

(d) $x^2 + x + 3 = 0$

(e) $\sqrt{x^2+9} = 2x$

Exercise 1.4.3 *By means of counter-examples, show why it is wrong to say that the following equations hold for all real numbers for which the expressions are defined.*

(a) $(x-2)^2 = x^2 - 2^2$

(b) $\dfrac{1}{x+h} = \dfrac{1}{x} + \dfrac{1}{h}$

(c) $\sqrt{x^2+y^2} = x + y$

Exercise 1.4.4 *Find an equation of the line passing through the point* $(-2,5)$ *and parallel to the line* $x+3y-2=0$.

Exercise 1.4.5 *Solve* $\dfrac{x^2-1}{3x-1} \leq 1$.

Exercise 1.4.6 *Explain why the following expression never represents a real number (for any real number x):* $\sqrt{x-2}+\sqrt{1-x}$.

Exercise 1.4.7 *Simplify the expression* $\dfrac{\left[3(x+h)^2+4\right]-\left[3x^2+4\right]}{h}$ *as much as possible.*

Exercise 1.4.8 *Simplify the expression* $\dfrac{\frac{x+h}{2(x+h)-1}-\frac{x}{2x-1}}{h}$ *as much as possible.*

Exercise 1.4.9 *Simplify the expression* $-\sin x(\cos x+3\sin x)-\cos x(-\sin x+3\cos x)$.

Exercise 1.4.10 *Solve the equation* $\cos x = \frac{\sqrt{3}}{2}$ *on the interval* $0 \leq x \leq 2\pi$.

Exercise 1.4.11 *Find an angle* θ *such that* $0 \leq \theta \leq \pi$ *and* $\cos\theta = \cos\frac{38\pi}{5}$.

Exercise 1.4.12 *What can you say about* $\dfrac{|x|+|4-x|}{x-2}$ *when x is a large (positive) number?*

Exercise 1.4.13 *Find an equation of the circle with centre in* $(-2,3)$ *and passing through the point* $(1,-1)$.

Exercise 1.4.14 *Find the centre and radius of the circle described by* $x^2+y^2+6x-4y+12=3$.

Exercise 1.4.15 *If* $y=9x^2+6x+7$, *find all possible values of y.*

Exercise 1.4.16 *Simplify* $\left(\dfrac{3x^2y^3z^{-1}}{18x^{-1}yz^3}\right)^2$.

Exercise 1.4.17 *If* $y=\dfrac{3x+2}{1-4x}$, *then what is x in terms of y?*

Exercise 1.4.18 *Divide* x^2+3x-5 *by* $x+2$ *to obtain the quotient and the remainder. Equivalently, find polynomial* $Q(x)$ *and constant R such that*

$$\frac{x^2+3x-5}{x+2} = Q(x) + \frac{R}{x+2}.$$

2. Functions

2.1 All About Functions

A **function** $y = f(x)$ is a rule for determining y when we're given a value of x. For example, the rule $y = f(x) = 2x + 1$ is a function. Any line $y = mx + b$ is called a **linear** function. The graph of a function looks like a curve above (or below) the x-axis, where for any value of x the rule $y = f(x)$ tells us how far to go above (or below) the x-axis to reach the curve.

Functions can be defined in various ways: by an algebraic formula or several algebraic formulas, by a graph, or by an experimentally determined table of values. In the latter case, the table gives a bunch of points in the plane, which we might then interpolate with a smooth curve, if that makes sense.

Given a value of x, a function must give at most one value of y. Thus, vertical lines are not functions. For example, the line $x = 1$ has infinitely many values of y if $x = 1$. It is also true that if x is any number (not 1) there is no y which corresponds to x, but that is not a problem—only multiple y values is a problem.

One test to identify whether or not a curve in the (x, y) coordinate system is a function is the following.

Theorem 2.1: The Vertical Line Test

A curve in the (x, y) coordinate system represents a function if and only if no vertical line intersects the curve more than once.

In addition to lines, another familiar example of a function is the parabola $y = f(x) = x^2$. We can draw the graph of this function by taking various values of x (say, at regular intervals) and plotting the points $(x, f(x)) = (x, x^2)$. Then connect the points with a smooth curve. (See Figure 2.1.)

The two examples $y = f(x) = 2x + 1$ and $y = f(x) = x^2$ are both functions which can be evaluated at *any* value of x from negative infinity to positive infinity. For many functions, however, it only makes sense to take x in some interval or outside of some "forbidden" region. The interval of x-values at which we're allowed to evaluate the function is called the **domain** of the function.

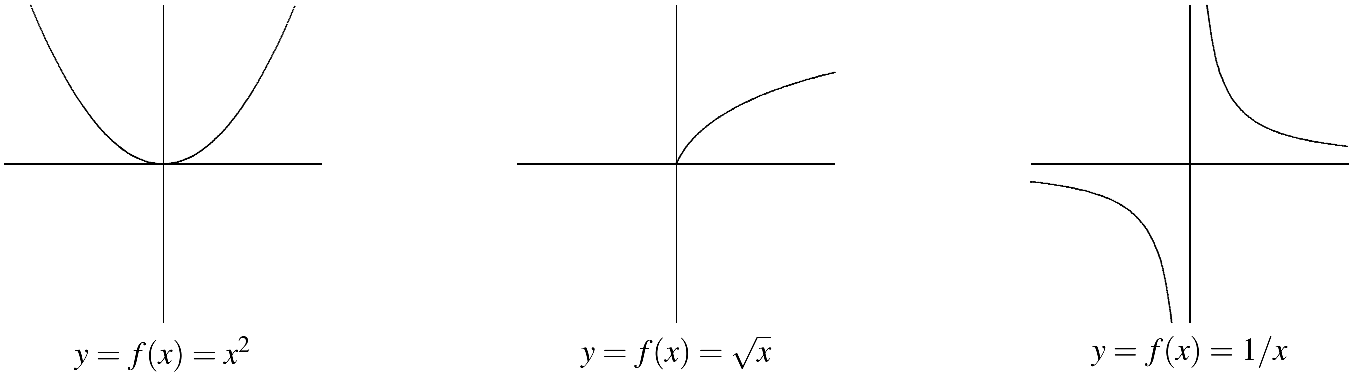


Figure 2.1: Graphs of Functions

Example 2.2: Domain of the Square-Root Function

The square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an x-value, take the nonnegative number whose square is x. This rule only makes sense if $x \geq 0$. We say that the domain of this function is $x \geq 0$, or more formally $\{x \in \mathbb{R} : x \geq 0\}$. Alternately, we can use interval notation, and write that the domain is $[0, \infty)$. The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function (see Figure 2.1) we have points (x, y) only above x-values on the right side of the x-axis.

Another example of a function whose domain is not the entire x-axis is: $y = f(x) = 1/x$, the reciprocal function. We cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero x, so we take the domain to be: $\{x \in \mathbb{R} : x \neq 0\}$. The graph of this function does not have any point (x, y) with $x = 0$. As x gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an **asymptote**.

To summarize, two reasons why certain x-values are excluded from the domain of a function are the following.

Restrictions for the Domain of a Function

1. We cannot divide by zero, and
2. We cannot take the square root of a negative number.

When the domain of a function is restricted, we say that the function is undefined at that point. We will encounter some other ways in which functions might be undefined later.

Another reason why the domain of a function might be restricted is that in a given situation the x-values outside of some range might have no practical meaning. For example, if y is the area of a square of side x, then we can write $y = f(x) = x^2$. In a purely mathematical context the domain of the function $y = x^2$ is all of $\mathbb{R}$. However, in the story-problem context of finding areas of squares, we restrict the domain to positive values of x, because a square with negative or zero side makes no sense.

In a problem in pure mathematics, we usually take the domain to be all values of x at which the formulas can be evaluated. However, in a story problem there might be further restrictions on the domain because only certain values of x are of interest or make practical sense.

In a story problem, we often use letters other than x and y. For example, the volume V of a sphere is a function of the radius r, given by the formula $V = f(r) = \frac{4}{3}\pi r^3$. Also, letters different from f may be used. For example, if y is the velocity of something at time t, we may write $y = v(t)$ with the letter v (instead of f) standing for the velocity function (and t playing the role of x).

The letter playing the role of x is called the **independent variable**, and the letter playing the role of y is called the **dependent variable** (because its value "depends on" the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables. If only words and no letters are given, then we have to decide which letters to use. Some letters are traditional. For example, almost always, t stands for time.

Example 2.3: Open Box

An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side x from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume V of the box as a function of x, and find the domain of this function.

Solution. The box we get will have height x and rectangular base of dimensions $a - 2x$ by $b - 2x$. Thus,

$$V = f(x) = x(a-2x)(b-2x).$$

Here a and b are constants, and V is the variable that depends on x, i.e., V is playing the role of y.

This formula makes mathematical sense for any x, but in the story problem the domain is much less. In the first place, x must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

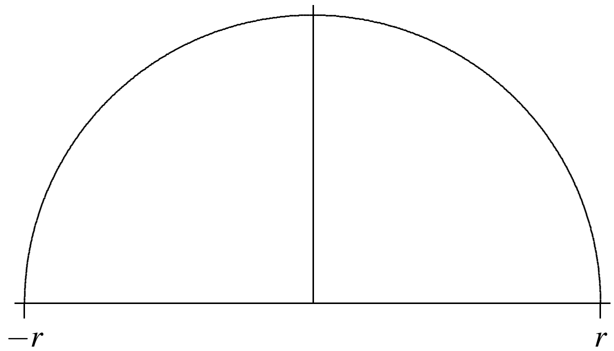
$$\left\{ x \in \mathbb{R} : 0 < x < \frac{1}{2}(\text{minimum of } a \text{ and } b) \right\}.$$

In interval notation we write: the domain is the interval $(0, \min(a,b)/2)$. You might think about whether we could allow 0 or (the minimum of a and b) to be in the domain. They make a certain physical sense, though we normally would not call the result a box. If we were to allow these values, what would the corresponding volumes be? Does that volume make sense? ♣

Example 2.4: Circle of Radius r Centered at the Origin

Is the circle of radius r centered at the origin the graph of a function?

Solution. The equation for this circle is usually given in the form $x^2 + y^2 = r^2$. To write the equation in the form $y = f(x)$ we solve for y, obtaining $y = \pm\sqrt{r^2 - x^2}$. But *this is not a function*, because when we substitute a value in the interval $(-r, r)$ for x there are two corresponding values of y. To get a function, we must choose one of the two signs in front of the square root. If we choose the positive sign, for example, we get the upper semicircle $y = f(x) = \sqrt{r^2 - x^2}$ (see graph below). The domain of this function is the interval $[-r, r]$, i.e., x must be between $-r$ and r (including the endpoints). If x is outside of that interval, then $r^2 - x^2$ is negative, and we cannot take the square root. In terms of the graph, this just means that there are no points on the curve whose x-coordinate is greater than r or less than $-r$.



♣

Example 2.5: Domain

Find the domain of

$$y = f(x) = \frac{1}{\sqrt{4x - x^2}}.$$

Solution. To answer this question, we must rule out the x-values that make $4x - x^2$ negative (because we cannot take the square root of a negative number) and also the x-values that make $4x - x^2$ zero (because if $4x - x^2 = 0$, then when we take the square root we get 0, and we cannot divide by 0). In other words, the domain consists of all x for which $4x - x^2$ is strictly positive. The inequality $4x - x^2 > 0$ was solved in Example 1.12. In interval notation, the domain is the interval $(0, 4)$. ♣

A function does not always have to be given by a single formula as the next example demonstrates.

Example 2.6: Piecewise Velocity

Suppose that $y = v(t)$ is the velocity function for a car which starts out from rest (zero velocity) at time $t = 0$; then increases its speed steadily to 20 m/sec, taking 10 seconds to do this; then travels at constant speed 20 m/sec for 15 seconds; and finally applies the brakes to decrease speed steadily to 0, taking 5 seconds to do this. The formula for $y = v(t)$ is different in each of the three time intervals: first $y = 2x$, then $y = 20$, then $y = -4x + 120$. The graph of this function is shown in Figure 2.2.

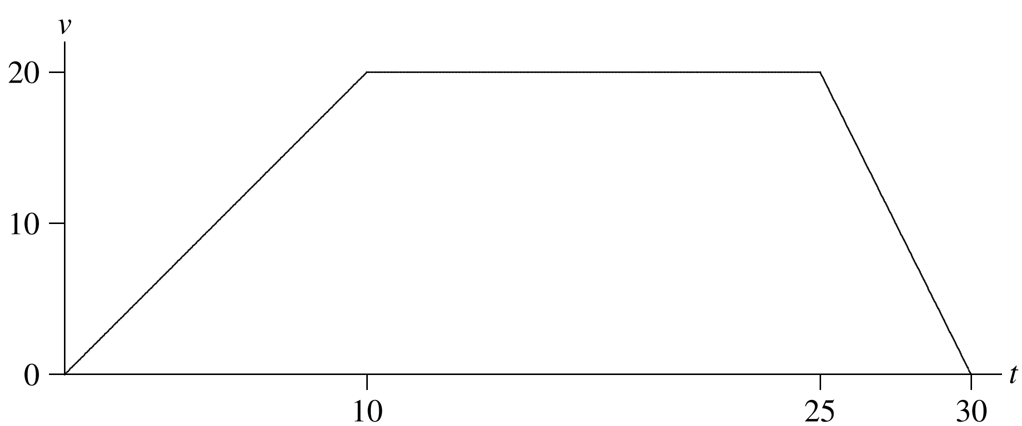


Figure 2.2: A velocity function.

This example leads to the following definition of a piecewise defined function.

Definition 2.7: Piecewise Defined Function

A **piecewise defined function** f *is defined by more than one rule:*

$$f(x) = \begin{cases} f_1(x) & x \in \mathscr{D}_{f_1} \\ f_2(x) & x \in \mathscr{D}_{f_2} \\ \vdots \\ f_n(x) & x \in \mathscr{D}_{f_n} \end{cases}$$

where $\mathscr{D}_{f_i}$, $1 \leq i \leq n$, *are mutually exclusive domains.*

Example 2.8: Piecewise Defined Function

Sketch the graph of the function f *defined by*

$$f(x) = \begin{cases} -x & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$$

Solution. The function f is defind in a piecewise fashion on the set of all real numbers. In the subdomain $(-\infty, 0)$, the rule for f is given by $f(x) = -x$. The equation $y = -x$ is a linear equation in the slope-intercept form (with slope -1 and intercept 0). Therefore, the graph of f corresponding to the subdomain $(-\infty, 0)$ is the half-line shown in Figure 2.3. Next, in the subdomain $[0, \infty)$, the rule for f is given by $f(x) = \sqrt{x}$. The values of $f(x)$ corresponding to $x = 0, 1, 2, 3, 4, 9, 16$ are shown in the following table:

x	0	1	2	3	4	9	16
$f(x)$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	3	4

Using these values, we sketch the graph of the function f as shown in Figure 2.3. ♣

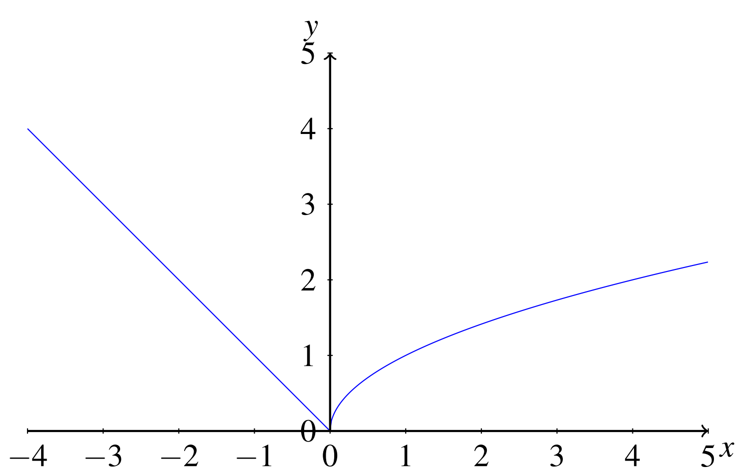


Figure 2.3: The graph of $y = f(x)$ is obtained by graphing $y = -x$ over $(-\infty, 0)$ and $y = \sqrt{x}$ over $[0, \infty)$.

Our focus in this textbook will be on **elementary functions**. Examples of elementary functions include both *algebraic* functions, such as polynomials, rational functions, and powers, and *transcendental* functions, such as exponential, logarithmic, and trigonometric functions.

Exercises for Section 2.1

Exercise 2.1.1 *Find the domain of each of the following functions:*

(a) $y = x^2 + 1$

(b) $y = f(x) = \sqrt{2x-3}$

(c) $y = f(x) = 1/(x+1)$

(d) $y = f(x) = 1/(x^2-1)$

(e) $y = f(x) = \sqrt{-1/x}$

(f) $y = f(x) = \sqrt[3]{x}$

(g) $y = f(x) = \sqrt{r^2-(x-h)^2}$, *where* r *and* h *are positive constants.*

(h) $y = f(x) = \sqrt[4]{x}$

(i) $y = \sqrt{1-x^2}$

(j) $y = f(x) = \sqrt{1-(1/x)}$

(k) $y = f(x) = 1/\sqrt{1-(3x)^2}$

(l) $y = f(x) = \sqrt{x} + 1/(x-1)$

(m) $y = f(x) = 1/(\sqrt{x}-1)$

Exercise 2.1.2 *A farmer wants to build a fence along a river. He has 500 feet of fencing and wants to enclose a rectangular pen on three sides (with the river providing the fourth side). If* x *is the length of the side perpendicular to the river, determine the area of the pen as a function of* x*. What is the domain of this function?*

Exercise 2.1.3 *A can in the shape of a cylinder is to be made with a total of 100 square centimetres of material in the side, top, and bottom; the manufacturer wants the can to hold the maximum possible volume. Write the volume as a function of the radius* r *of the can; find the domain of the function.*

Exercise 2.1.4 *A can in the shape of a cylinder is to be made to hold a volume of one liter (1000 cubic centimetres). The manufacturer wants to use the least possible material for the can. Write the surface area of the can (total of the top, bottom, and side) as a function of the radius* r *of the can; find the domain of the function.*

Exercise 2.1.5 *Let* f *be the function defined by*

$$f(x) = \begin{cases} -\frac{1}{2}x^2 + 3 & x < 1 \\ 2x^2 + 1 & x \geq 1 \end{cases}$$

Find $f(-1)$, $f(0)$, $f(1)$, *and* $f(2)$.

Exercise 2.1.6 *For each of the following functions, sketch the graph and find the domain and range.*

(a)

$$f(x) = \begin{cases} x & x < 0 \\ 2x+1 & x \geq 0 \end{cases}$$

(b)

$$f(x) = \begin{cases} -x+1 & x \leq 1 \\ x^2-1 & x > 1 \end{cases}$$

Exercise 2.1.7 *In* 2006, *the Canadian postage for domestic lettermail was* \$0.51 *for the first* 30 *grams or a fraction thereof,* \$0.89 *over* 30g *and up to* 50g, \$1.05 *over* 50g *and up to* 100g, \$1.78 *over* 100g *and up to* 200g, *and* \$2.49 *over* 200g *and up to* 500g. *Any lettermail not exceeding* \$500g *may be sent by domestic mail. Letting* x *denote the weight of a parcel in grams and* $f(x)$ *the postage in dollars, complete the following description of the "lettermail function"* f*:*

$$f(x) = \begin{cases} 0.51 & 0 < x \leq 30 \\ \vdots & \\ 2.49 & 200 < x \leq 500 \end{cases}$$

Sketch the graph of f *and state the domain.*

2.2 Symmetry, Transformations and Compositions

2.2.1. Symmetry

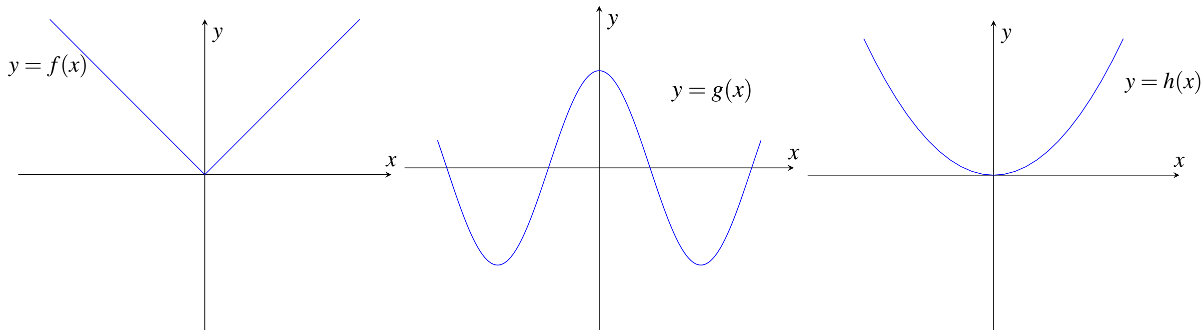
When graphing functions, we can sometimes make use of their inherit **symmetry** with respect to the coordinate axes to ease geometric interpretation. Not all functions exhibit symmetry, but for those that do, we differentiate between **even** and **odd** symmetry as defined below.

Definition 2.9: Function Symmetry

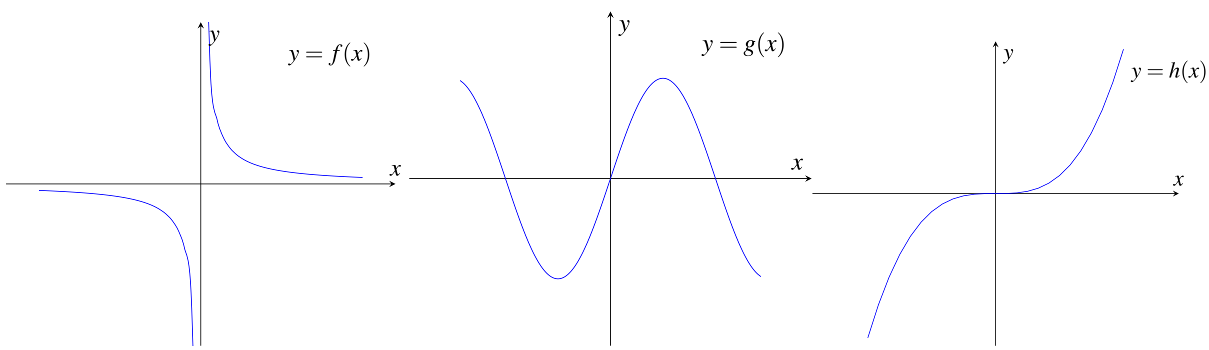
A function $y = f(x)$ *is called:* **even** *if* $f(x) = f(-x)$, *and*
odd *if* $f(x) = -f(-x)$, *and*
neither *otherwise.*

Note:

1. The graph of an even function stays the same when reflected in the y-axis. Examples of such functions are $f(x) = |x|$, $g(x) = \cos(x)$ and $h(x) = x^2$ as shown below.



2. The graph of an odd function stays the same when reflected in the x-axis and the y-axis, or alternatively, rotated by 180° around the origin. Examples of such functions are $f(x) = 1/x$, $g(x) = \sin(x)$, and $h(x) = x^3$ as shown below.



2.2.2. Transformations

Transformations are operations we can apply to a function in order to obtain a *new* function. The most common transformations include translations (shifts), stretches and reflections. We summarize these below.

Function	*Conditions*	*How to graph $F(x)$ given the graph of $f(x)$*
$F(x) = f(x) + c$	$c > 0$	*Shift $f(x)$ upwards by c units*
$F(x) = f(x) - c$	$c > 0$	*Shift $f(x)$ downwards by c units*
$F(x) = f(x + c)$	$c > 0$	*Shift $f(x)$ to the left by c units*
$F(x) = f(x - c)$	$c > 0$	*Shift $f(x)$ to the right by c units*
$F(x) = -f(x)$		*Reflect $f(x)$ about the x-axis*
$F(x) = f(-x)$		*Reflect $f(x)$ about the y-axis*
$F(x) = \|f(x)\|$		*Take the part of the graph of $f(x)$ that lies below the x-axis and reflect it about the x-axis*

For horizontal and vertical stretches, different resources use different terminology and notation. Use the one you are most comfortable with! Below, both a, b are positive numbers. Note that we only use the term *stretch* in this case:

Function	*Conditions*	*How to graph $F(x)$ given the graph of $f(x)$*
$F(x) = af(x)$	$a > 0$	*Stretch $f(x)$ vertically by a factor of a*
$F(x) = f(bx)$	$b > 0$	*Stretch $f(x)$ horizontally by a factor of $1/b$*

In the next case, we use both the terms *stretch* and *shrink*. We also split up vertical stretches into two cases ($0 < a < 1$ and $a > 1$), and split up horizontal stretches into two cases ($0 < b < 1$ and $b > 1$). Note that having $0 < a < 1$ is the same as having $1/c$ with $c > 1$. Also note that *stretching by a factor of* $1/c$ is the same as *shrinking by a factor* c.

Function	*Conditions*	*How to graph $F(x)$ given the graph of $f(x)$*
$F(x) = cf(x)$	$c > 1$	*Stretch $f(x)$ vertically by a factor of c*
$F(x) = (1/c)f(x)$	$c > 1$	*Shrink $f(x)$ vertically by a factor of c*
$F(x) = f(cx)$	$c > 1$	*Shrink $f(x)$ horizontally by a factor of c*
$F(x) = f(x/c)$	$c > 1$	*Stretch $f(x)$ horizontally by a factor of c*

Some resources keep the condition $0 < c < 1$ rather than using $1/c$. This is illustrated in the next table.

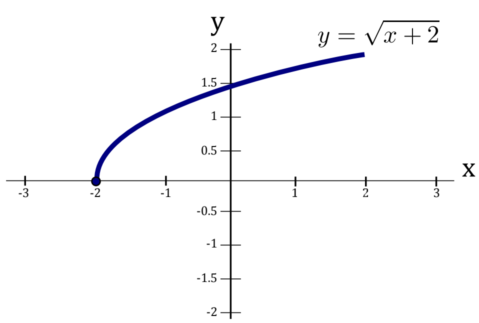
Function	*Conditions*	*How to graph $F(x)$ given the graph of $f(x)$*
$F(x) = df(x)$	$d > 1$	*Stretch $f(x)$ vertically by a factor of d*
$F(x) = df(x)$	$0 < d < 1$	*Shrink $f(x)$ vertically by a factor of $1/d$*
$F(x) = f(dx)$	$d > 1$	*Shrink $f(x)$ horizontally by a factor of d*
$F(x) = f(dx)$	$0 < d < 1$	*Stretch $f(x)$ horizontally by a factor of $1/d$*

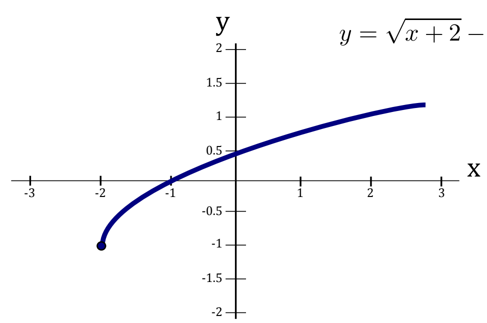
Example 2.10: Transformations and Graph Sketching

In this example we will use appropriate transformations to sketch the graph of the function

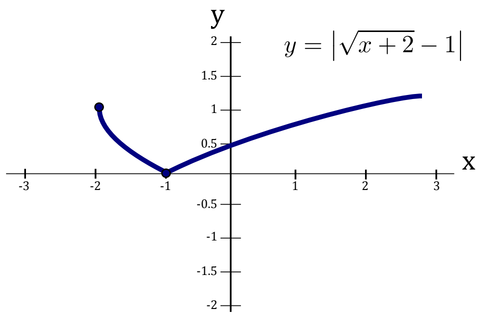
$$y = |\sqrt{x+2} - 1| - 1$$

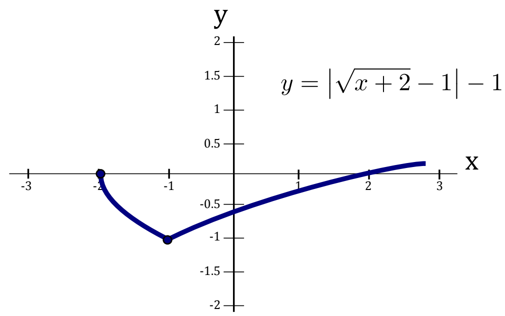
Solution. We start with the graph of a function we know how to sketch, in particular, $y = \sqrt{x}$: To obtain the graph of the function $y = \sqrt{x+2}$ from the graph $y = \sqrt{x}$, we must shift $y = \sqrt{x}$ to the left by 2 units. To obtain the graph of the function $y = \sqrt{x+2} - 1$ from the graph $y = \sqrt{x+2}$, we must shift $y = \sqrt{x+2}$ downwards by 1 unit.





To obtain the graph of the function $y = |\sqrt{x+2} - 1|$ from the graph $y = \sqrt{x+2} - 1$, we must take the part of the graph of $y = \sqrt{x+2} - 1$ that lies below the x-axis and reflect it (upwards) about the x-axis. Finally, to obtain the graph of the function $y = |\sqrt{x+2} - 1| - 1$ from the graph $y = |\sqrt{x+2} - 1|$, we must shift $y = |\sqrt{x+2} - 1|$ downwards by 1 unit:





♣

2.2.3. Combining Two Functions

Let f and g be two functions. Then we can form new functions by adding, subtracting, multiplying, or dividing. These new functions, $f+g$, $f-g$, fg and f/g, are defined in the usual way.

Operations on Functions

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Suppose D_f is the domain of f and D_g is the domain of g. Then the domains of $f+g$, $f-g$ and fg are the same and are equal to the intersection $D_f \cap D_g$ (that is, everything that is in *common* to both the domain of f and the domain of g). Since division by zero is *not allowed*, the domain of f/g is $\{x \in D_f \cap D_g : g(x) \neq 0\}$.

Another way to combine two functions f and g together is a procedure called composition.

Function Composition

Given two functions f and g, the **composition** of f and g, denoted by $f \circ g$, is defined as:

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is $\{x \in D_g : g(x) \in D_f\}$, that is, it contains all values x in the domain of g such that $g(x)$ is in the domain of f.

Example 2.11: Domain of a Composition

Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find the domain of $f \circ g$.

Solution. The domain of f is $D_f = \{x \in \mathbb{R}\}$. The domain of g is $D_g = \{x \in \mathbb{R} : x \geq 0\}$. The function $(f \circ g)(x) = f(g(x))$ is:

$$f(g(x)) = \left(\sqrt{x}\right)^2 = x.$$

Typically, $h(x) = x$ would have a domain of $\{x \in \mathbb{R}\}$, but since it came from a **composed function**, we must consider $g(x)$ when looking at the domain of $f(g(x))$. Thus, the domain of $f \circ g$ is $\{x \in \mathbb{R} : x \geq 0\}$.

♣

Example 2.12: Combining Two Functions

Let $f(x) = x^2 + 3$ and $g(x) = x - 2$. Find $f + g$, $f - g$, fg, f/g, $f \circ g$ and $g \circ f$. Also, determine the domains of these new functions.

Solution. For $f + g$ we have:

$$(f+g)(x) = f(x) + g(x) = (x^2+3) + (x-2) = x^2 + x + 1.$$

For $f - g$ we have:

$$(f-g)(x) = f(x) - g(x) = (x^2+3) - (x-2) = x^2 + 3 - x + 2 = x^2 - x + 5.$$

For fg we have:

$$(fg)(x) = f(x) \cdot g(x) = (x^2+3)(x-2) = x^3 - 2x^2 + 3x - 6.$$

For f/g we have:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2+3}{x-2}.$$

For $f \circ g$ we have:

$$(f \circ g)(x) = f(g(x)) = f(x-2) = (x-2)^2 + 3 = x^2 - 4x + 7.$$

For $g \circ f$ we have:

$$(g \circ f)(x) = g(f(x)) = g(x^2+3) = (x^2+3) - 2 = x^2 + 1.$$

The domains of $f+g$, $f-g$, fg, $f \circ g$ and $g \circ f$ is $\{x \in \mathbb{R}\}$, while the domain of f/g is $\{x \in \mathbb{R} : x \neq 2\}$.

♣

As in the above problem, $f \circ g$ and $g \circ f$ are generally different functions.

Exercises for Section 2.2

Exercise 2.2.1 *Starting with the graph of $y = \sqrt{x}$, the graph of $y = 1/x$, and the graph of $y = \sqrt{1-x^2}$ (the upper unit semicircle), sketch the graph of each of the following functions:*

(a) $f(x) = \sqrt{x-2}$

(b) $f(x) = -1 - 1/(x+2)$

(c) $f(x) = 4 + \sqrt{x+2}$

(d) $y = f(x) = x/(1-x)$

(e) $y = f(x) = -\sqrt{-x}$

(f) $f(x) = 2 + \sqrt{1-(x-1)^2}$

(g) $f(x) = -4 + \sqrt{-(x-2)}$

(h) $f(x) = 2\sqrt{1-(x/3)^2}$

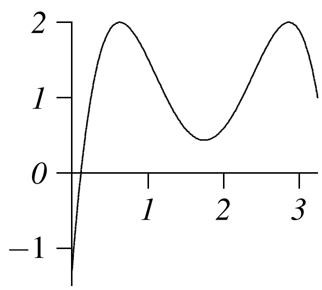
(i) $f(x) = 1/(x+1)$

(j) $f(x) = 4 + 2\sqrt{1-(x-5)^2/9}$

(k) $f(x) = 1 + 1/(x-1)$

(l) $f(x) = \sqrt{100-25(x-1)^2} + 2$

Exercise 2.2.2 *The graph of $f(x)$ is shown below. Sketch the graphs of the following functions.*



(a) $y = f(x-1)$

(b) $y = 1 + f(x+2)$

(c) $y = 1 + 2f(x)$

(d) $y = 2f(3x)$

(e) $y = 2f(3(x-2)) + 1$

(f) $y = (1/2)f(3x-3)$

(g) $y = f(1 + x/3) + 2$

(h) $y = |f(x) - 2|$

Exercise 2.2.3 *Suppose $f(x) = 3x - 9$ and $g(x) = \sqrt{x}$. What is the domain of the composition $(g \circ f)(x)$?*

Exercise 2.2.4 *Let $f(x) = 2x^2 - x + 4$. Find and simplify*

$$\frac{f(a+h) - f(a)}{h} \quad (h \neq 0)$$

.

Exercise 2.2.5 *Let $h(x) = \dfrac{2+x}{\sqrt{x^2+4x}}$. Find two functions f and g (not necessarily unique) such that,*

(a) $h(x) = (f+g)(x)$

(b) $h(x) = (fg)(x)$

(c) $h(x) = (f \circ g)(x)$

2.3 Exponential Functions

An **exponential function** is a function of the form $f(x) = a^x$, where a is a constant. Examples are 2^x, 10^x and $(1/2)^x$. To more formally define the exponential function we look at various kinds of input values.

It is obvious that $a^5 = a \cdot a \cdot a \cdot a \cdot a$ and $a^3 = a \cdot a \cdot a$, but when we consider an exponential function a^x we can't be limited to substituting integers for x. What does $a^{2.5}$ or $a^{-1.3}$ or a^{π} mean? And is it really true that $a^{2.5}a^{-1.3} = a^{2.5-1.3}$? The answer to the first question is actually quite difficult, so we will evade it; the answer to the second question is "yes."

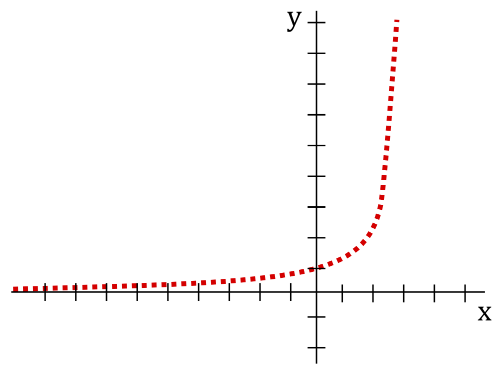
We'll evade the full answer to the hard question, but we have to know something about exponential functions. You need first to understand that since it's not "obvious" what 2^x should mean, we are really free to make it mean whatever we want, so long as we keep the behavior that *is* obvious, namely, when x is a positive integer. What else do we want to be true about 2^x? We want the properties of the previous two paragraphs to be true for all exponents: $2^x 2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$.

After the positive integers, the next easiest number to understand is 0: $2^0 = 1$. You have presumably learned this fact in the past; why is it true? It is true precisely because we want $2^a 2^b = 2^{a+b}$ to be true about the function 2^x. We need it to be true that $2^0 2^x = 2^{0+x} = 2^x$, and this only works if $2^0 = 1$. The same argument implies that $a^0 = 1$ for any a.

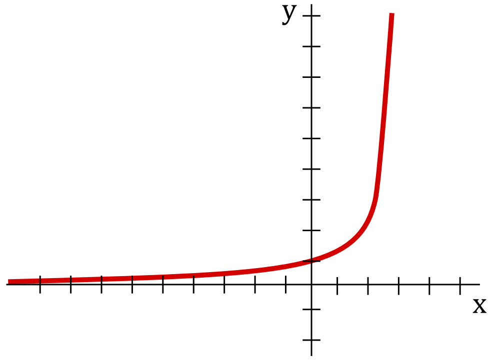
The next easiest set of numbers to understand is the negative integers: for example, $2^{-3} = 1/2^3$. We know that whatever 2^{-3} means it must be that $2^{-3}2^3 = 2^{-3+3} = 2^0 = 1$, which means that 2^{-3} must be $1/2^3$. In fact, by the same argument, once we know what 2^x means for some value of x, 2^{-x} must be $1/2^x$ and more generally $a^{-x} = 1/a^x$.

Next, consider an exponent $1/q$, where q is a positive integer. We want it to be true that $(2^x)^y = 2^{xy}$, so $(2^{1/q})^q = 2$. This means that $2^{1/q}$ is a q-th root of 2, $2^{1/q} = \sqrt[q]{2}$. This is all we need to understand that $2^{p/q} = (2^{1/q})^p = (\sqrt[q]{2})^p$ and $a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p$.

What's left is the hard part: what does 2^x mean when x cannot be written as a fraction, like $x = \sqrt{2}$ or $x = \pi$? What we know so far is how to assign meaning to 2^x whenever $x = p/q$. If we were to graph a^x (for some $a > 1$) at points $x = p/q$ then we'd see something like this:

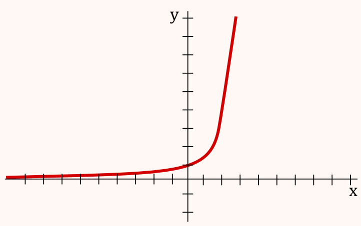


This is a poor picture, but it illustrates a series of individual points above the rational numbers on the x-axis. There are really a lot of "holes" in the curve, above $x = \pi$, for example. But (this is the hard part) it is possible to prove that the holes can be "filled in", and that the resulting function, called a^x, really does have the properties we want, namely that $a^x a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$. Such a graph would then look like this:

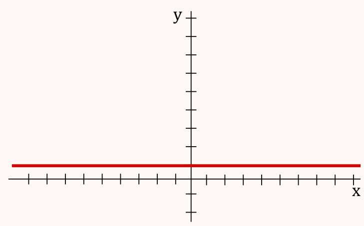


Three Types of Exponential Functions

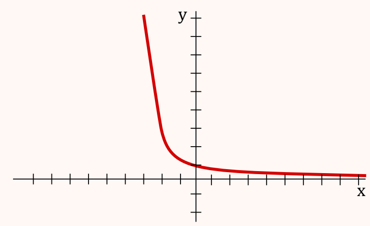
There are *three kinds* of exponential functions $f(x) = a^x$ with x real depending on whether $a > 1$, $a = 1$ or $0 < a < 1$:



$f(x) = a^x$
$a > 1$



$f(x) = 1^x$



$f(x) = a^x$
$0 < a < 1$

Properties of Exponential Functions

The first thing to note is that if $a < 0$ then problems can occur. Observe that if $a = -1$ then $(-1)^x$ is not defined for every x. For example, $x = 1/2$ is a square root and gives $(-1)^{1/2} = \sqrt{-1}$ which is not a real number.

Exponential Function Properties

- *Only defined for positive a:* a^x is only defined for all real x if $a > 0$
- *Always positive:* $a^x > 0$, for all real x
- *Exponent rules:* If $a, b > 0$ and x, y real numbers, then

  1. $a^x a^y = a^{x+y}$
  2. $\dfrac{a^x}{a^y} = a^{x-y}$
  3. $(a^x)^y = a^{xy} = a^{yx} = (a^y)^x$
  4. $a^x b^x = (ab)^x$

- *Long-term behaviour:* If $a > 1$, then $a^x \to \infty$ as $x \to \infty$ and $a^x \to 0$ as $x \to -\infty$.

The last property can be observed from the graph. If $a > 1$, then as x gets larger and larger, so does a^x. On the other hand, as x gets large and negative, the function approaches the x-axis, that is, a^x approaches 0.

Example 2.13: Reflection of Exponential

Determine an equation of the function after reflecting $y = 2^x$ about the line $x = -2$.

Solution. First reflect about the y-axis to get $y = 2^{-x}$. Now shift by $2 \times 2 = 4$ units to the *left* to get $y = 2^{-(x+4)}$. Side note: Can you see why this sequence of transformations is the same as reflection in the line $x = -2$? Can you come up with a general rule for these types of reflections? ♣

Example 2.14: Determine the Exponential Function

Determine the exponential function $f(x) = ka^x$ that passes through the points $(1,6)$ and $(2,18)$.

Solution. We substitute our two points into the equation to get:

$$x = 1, y = 6 \rightarrow 6 = ka^1$$

$$x = 2, y = 18 \rightarrow 18 = ka^2$$

This gives us $6 = ka$ and $18 = ka^2$. The first equation is $k = 6/a$ and subbing this into the second gives: $18 = (6/a)a^2$. Thus, $18 = 6a$ and $a = 3$. Now we can see from $6 = ka$ that $k = 2$. Therefore, the exponential function is

$$f(x) = 2 \cdot 3^x.$$

♣

There is one base that is so important and convenient that we give it a special symbol. This number is denoted by $e = 2.71828\ldots$ (and is an irrational number). Its *importance* stems from the fact that it simplifies many formulas of Calculus and also shows up in other fields of mathematics.

Example 2.15: Domain of Function with Exponential

Find the domain of $f(x) = \dfrac{1}{\sqrt{e^x + 1}}$.

Solution. For domain, we cannot divide by zero or take the square root of negative numbers. Note that one of the properties of exponentials is that they are always positive! Thus, $e^x + 1 > 0$ (in fact, as $e^x > 0$ we actually have that $e^x + 1$ is at least one). Therefore, $e^x + 1$ is never zero nor negative, and gives no restrictions on x. Thus, the domain is $\mathbb{R}$. ♣

Special Base e

A question of interest in calculus is the following: What base of an exponential function has the property that at the point $(0,1)$ the slope of the tangent line is one? We will answer this informally here. Consider the function $f(x) = a^x$ and a tangent line at the point $(0,1)$. If $a = 2$, the slope of the tangent line is approximately 0.7, see the left graph in Figure 2.4. If $a = 3$ the slope of the tangent line is approximately 1.1, see the right graph in Figure 2.4. It turns out that when the base is

$$2.71828182845904523536028747135266249775724709369995...$$

the slope of the tangent line is exactly equal to one! This number is denoted by e**=2.71828...** and is an irrational number. It is sometimes called **Euler's constant** named after the mathematician Leonhard Euler. Its importance stems from the fact that it simplifies many formulas of Calculus and also shows up in other fields of mathematics.

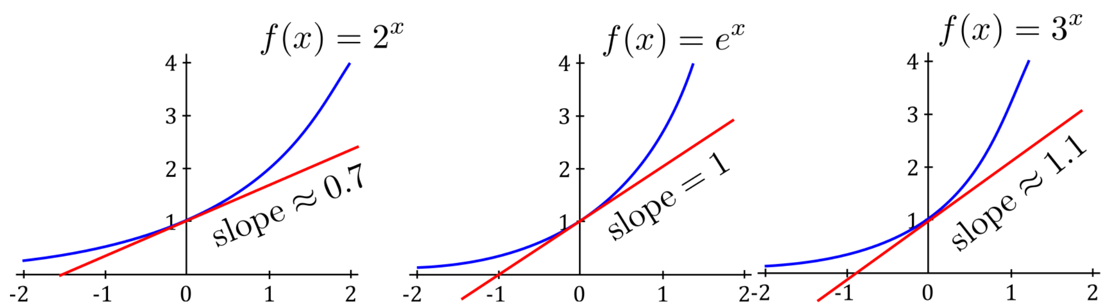


Figure 2.4

Exercises for Section 2.3

Exercise 2.3.1 *Determine an equation of the function $y = a^x$ passing through the point* $(3,8)$.

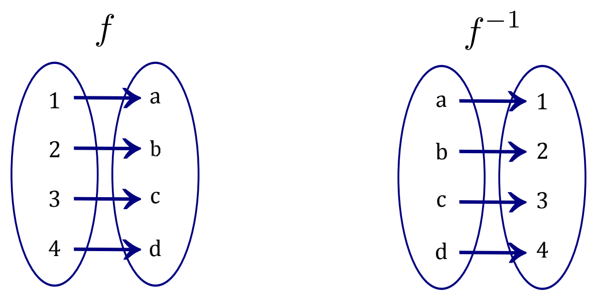
Exercise 2.3.2 *Find the y-intercept of $f(x) = 4^x + 6$.*

Exercise 2.3.3 *Find the y-intercept of $f(x) = 2\left(\frac{1}{2}\right)^x$.*

Exercise 2.3.4 *Find the domain of $y = e^{-x} + e^{\frac{1}{x}}$.*

2.4 Inverse Functions

In mathematics, an *inverse* is a function that serves to "undo" another function. That is, if $f(x)$ produces y, then putting y into the inverse of f produces the output x. A function f that has an inverse is called invertible and the inverse is denoted by f^{-1}. It is best to illustrate inverses using an arrow diagram:



Notice how f maps 1 to a, and f^{-1} undoes this, that is, f^{-1} maps a back to 1. Don't confuse $f^{-1}(x)$ with exponentiation: the inverse f^{-1} is *different* from $\frac{1}{f(x)}$.

Not every function has an inverse. It is easy to see that if a function $f(x)$ is going to have an inverse, then $f(x)$ *never* takes on the same value twice. We give this property a special name.

A function $f(x)$ is called **one-to-one** if every element of the range corresponds to *exactly* one element of the domain. Similar to the Vertical Line Test (VLT) for functions, we have the Horizontal Line Test (HLT) for the one-to-one property.

Theorem 2.16: The Horizontal Line Test

A function is one-to-one if and only if there is no horizontal line that intersects its graph more than once.

Example 2.17: Parabola is Not One-to-one

The parabola $f(x) = x^2$ it not one-to-one because it does not satisfy the Horizontal Line Test. For example, the horizontal line $y = 1$ intersects the parabola at two points, when $x = -1$ and $x = 1$.

We now formally define the inverse of a function.

Definition 2.18: Inverse of a Function

Let $f(x)$ and $g(x)$ be two one-to-one functions. If $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ then we say that $f(x)$ and $g(x)$ are **inverses** *of each other. We denote $g(x)$ (the inverse of $f(x)$) by $g(x) = f^{-1}(x)$.*

Thus, if f maps x to y, then f^{-1} maps y back to x. This gives rise to the *cancellation formulas*:

$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in the domain of } f(x),$$

$$f(f^{-1}(x)) = x, \quad \text{for every } x \text{ in the domain of } f^{-1}(x).$$

Example 2.19: Finding the Inverse at Specific Values

If $f(x) = x^9 + 2x^7 + x + 1$, find $f^{-1}(5)$ and $f^{-1}(1)$.

Solution. Rather than trying to compute a formula for f^{-1} and then computing $f^{-1}(5)$, we can simply find a number c such that f evaluated at c gives 5. Note that subbing in some simple values ($x = -3,-2,1,0,1,2,3$) and evaluating $f(x)$ we eventually find that $f(1) = 1^9 + 2(1^7) + 1 + 1 = 5$ and $f(0) = 1$. Therefore, $f^{-1}(5) = 1$ and $f^{-1}(1) = 0$. ♣

To compute the equation of the inverse of a function we use the following *guideline*.

Guideline for Computing Inverses

1. Write down $y = f(x)$.
2. Solve for x in terms of y.
3. Switch the x's and y's.
4. The result is $y = f^{-1}(x)$.

Example 2.20: Finding the Inverse Function

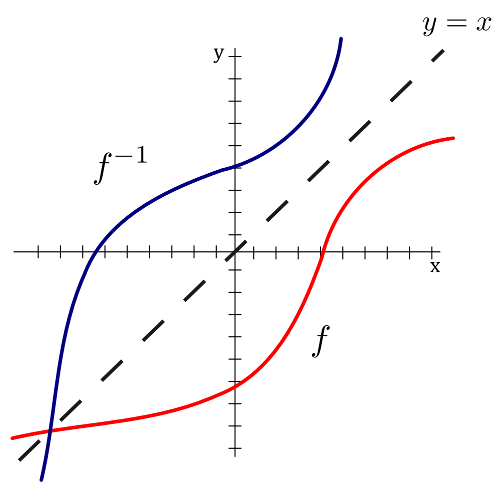
We find the inverse of the function $f(x) = 2x^3 + 1$.

Solution. Starting with $y = 2x^3 + 1$ we solve for x as follows:

$$y - 1 = 2x^3 \quad \rightarrow \quad \frac{y-1}{2} = x^3 \quad \rightarrow \quad x = \sqrt[3]{\frac{y-1}{2}}.$$

Therefore, $f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$. ♣

This example shows how to find the inverse of a function *algebraically*. But what about finding the inverse of a function *graphically*? Step 3 (switching x and y) gives us a good graphical technique to find the inverse, namely, for each point (a,b) where $f(a) = b$, sketch the point (b,a) for the inverse. More formally, to obtain the graph of $f^{-1}(x)$ *reflect* the graph of $f(x)$ about the line $y = x$.



Exercises for Section 2.4

Exercise 2.4.1 *Is the function* $f(x) = |x|$ *one-to-one?*

Exercise 2.4.2 *If* $h(x) = e^x + x + 1$*, find* $h^{-1}(2)$.

Exercise 2.4.3 *Find a formula for the inverse of the function* $f(x) = \dfrac{x+2}{x-2}$.

Exercise 2.4.4 *Determine whether or not the following pairs are inverse functions.*

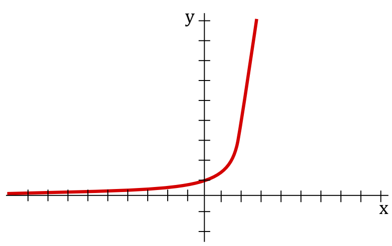
(a) $g(x) = \dfrac{ax+b}{cx-d}$ *and* $f(x) = \dfrac{b+dx}{cx-a}$

(b) $g(x) = x^2$ *and* $f(x) = \sqrt{x}$ *for* $x \geq 0$

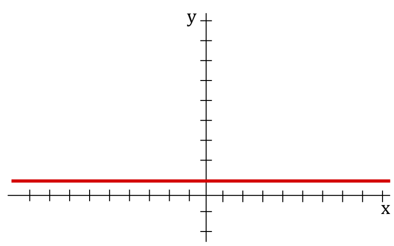
(c) $g(x) = \frac{1}{2}x^5$ *and* $f(x) = \sqrt[5]{\frac{1}{2}x}$

2.5 Logarithmic Functions

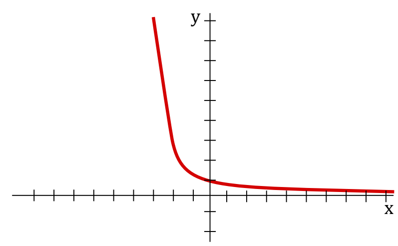
Recall the *three kinds* of exponential functions $f(x) = a^x$ depending on whether $0 < a < 1$, $a = 1$ or $a > 1$:



$f(x) = a^x$
$a > 1$



$f(x) = 1^x$



$f(x) = a^x$
$0 < a < 1$

So long as $a \neq 1$, the function $f(x) = a^x$ satisfies the Horizontal Line Test and therefore has an inverse. We call the *inverse of* a^x the **logarithmic function with base a** and denote it by $\log_a$. In particular,

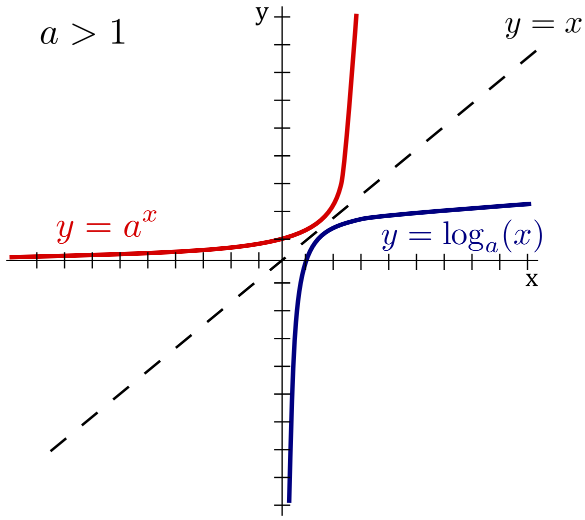
$$\log_a x = y \iff a^y = x.$$

The *cancellation formulas* for logs are:

$$\log_a(a^x) = x, \quad \text{for every } x \in \mathbb{R},$$

$$a^{\log_a(x)} = x, \quad \text{for every } x > 0.$$

Since the function $f(x) = a^x$ for $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, the logarithmic function has domain $(0, \infty)$ and range $\mathbb{R}$. For the most part, we only focus on logarithms with a base larger than 1 (i.e., $a > 1$) as these are the most important.



Notice that every logarithm passes through the point $(1, 0)$ in the same way that every exponential function passes through the point $(0, 1)$.

Some properties of logarithms are as follows.

Logarithm Properties

Let A, B be positive numbers and $b > 0$ ($b \neq 1$) be a base.

- $\log_b(AB) = \log_b A + \log_b B$,
- $\log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$,
- $\log_b(A^n) = n \log_b A$, where n is any real number.

Example 2.21: Compute Lorarithms

To compute $\log_2(24) - \log_2(3)$ *we can do the following:*

$$\log_2(24) - \log_2(3) = \log_2\left(\frac{24}{3}\right) = \log_2(8) = 3,$$

since $2^3 = 8$.

The Natural Logarithm

As mentioned earlier for exponential functions, the number $e \approx 2.71828\ldots$ is the most convenient base to use in Calculus. For this reason we give the logarithm with base e a special name: **the natural logarithm**.

We also give it special notation:

$$\log_e x = \ln x.$$

You may pronounce ln as either: "el - en", "lawn", or refer to it as "natural log". The above properties of logarithms also apply to the natural logarithm.

Often we need to turn a logarithm (in a different base) into a natural logarithm. This gives rise to the *change of base formula*.

Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}.$$

Example 2.22: Combine Logarithms

Write $\ln A + 2\ln B - \ln C$ *as a single logarithm.*

Solution. Using properties of logarithms, we have,

$$\begin{aligned} \ln A + 2\ln B - \ln C &= \ln A + \ln B^2 - \ln C \\ &= \ln(AB^2) - \ln C \\ &= \ln \frac{AB^2}{C} \end{aligned}$$

♣

Example 2.23: Solve Exponential Equations using Logarithms

If $e^{x+2} = 6e^{2x}$*, then solve for* x*.*

Solution. Taking the natural logarithm of both sides and noting the cancellation formulas (along with $\ln e = 1$), we have:

$$\begin{aligned} e^{x+2} &= 6e^{2x} \\ \ln e^{x+2} &= \ln(6e^{2x}) \\ x+2 &= \ln 6 + \ln e^{2x} \\ x+2 &= \ln 6 + 2x \\ x &= 2 - \ln 6 \end{aligned}$$

♣

Example 2.24: Solve Logarithm Equations using Exponentials

If $\ln(2x-1) = 2\ln(x)$, *then solve for* x.

Solution. "Taking e" of both sides and noting the cancellation formulas, we have:

$$\begin{aligned} e^{\ln(2x-1)} &= e^{2\ln(x)} \\ (2x-1) &= e^{\ln(x^2)} \\ 2x-1 &= x^2 \\ x^2-2x+1 &= 0 \\ (x-1)^2 &= 0 \end{aligned}$$

Therefore, the solution is $x = 1$. ♣

Exercises for Section 2.5

Exercise 2.5.1 *Expand* $\log_{10}((x+45)^7(x-2))$.

Exercise 2.5.2 *Expand* $\log_2 \dfrac{x^3}{3x-5+(7/x)}$.

Exercise 2.5.3 *Write* $\log_2 3x + 17\log_2(x-2) - 2\log_2(x^2+4x+1)$ *as a single logarithm.*

Exercise 2.5.4 *Solve* $\log_2(1+\sqrt{x}) = 6$ *for* x.

Exercise 2.5.5 *Solve* $2^{x^2} = 8$ *for* x.

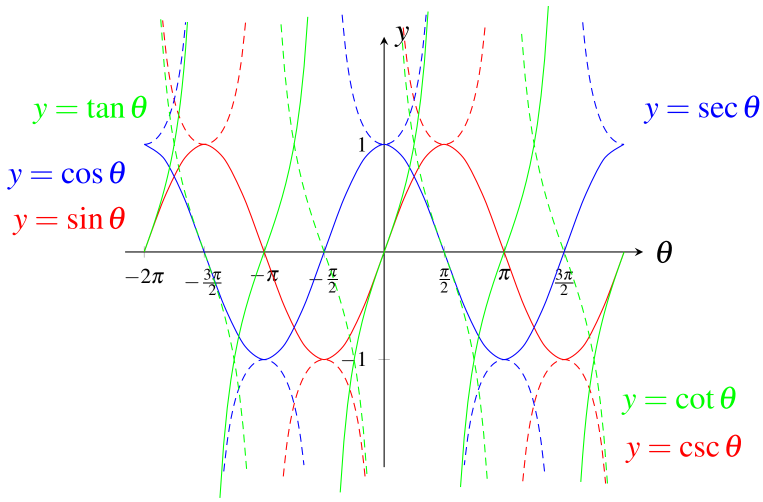
Exercise 2.5.6 *Solve* $\log_2(\log_3(x)) = 1$ *for* x.

Exercise 2.5.7 *Solve* $a^{2x} - 4a^x + 4 = 0$ *for* x *(a constant).*

2.6 Trigonometric Functions

2.6.1. Transformation of Trigonometric Functions

Recall the graphs of the six basic trigonometric functions reviewed in Section 1.3.4:



We will now look at a specific mathematical model of a phenomenon exhibiting cyclical behaviour – the so-called **predator-prey population model**.

Example 2.25: The predator-prey population model

The population of owls (predators) in a certain region over a 2-year period is estimated to be

$$P(t) = 1000 + 100\sin\left(\frac{\pi t}{12}\right)$$

in month t, and the population of mice (prey) in the same area at time t is given by

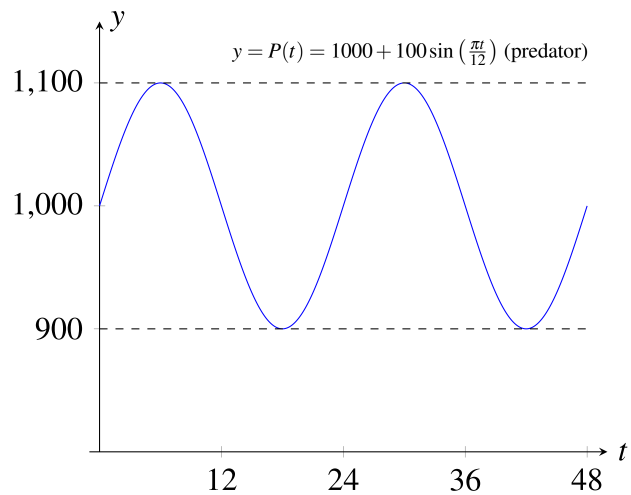
$$p(t) = 20,000 + 4000\cos\left(\frac{\pi t}{12}\right).$$

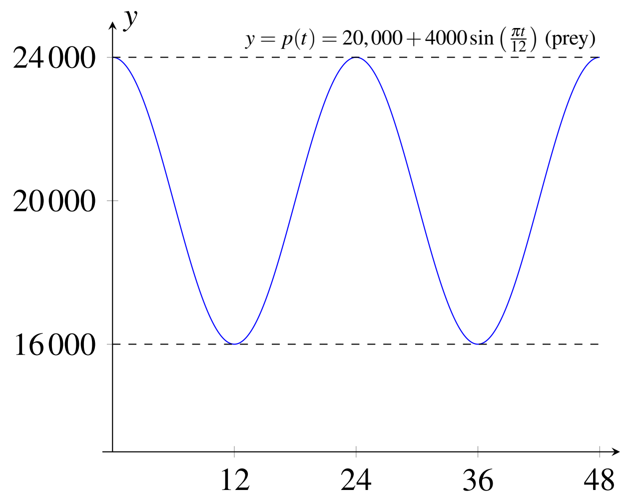
Sketch the graphs of these two functions and explain the relationship between the sizes of the two populations.

Solution. We first observe that both of the given functions are periodic with period 24 months. To see this, recall that both the sine and cosine functions are periodic with period 2π. The smallest value of $t > 0$ such that $\sin(\pi t/12) = 0$ is then obtained by solving the equation

$$t = \frac{2\pi}{\pi/12},$$

giving $t = 24$ as the period of $\sin(\pi t/12)$. Since $P(t+24) = P(t)$, we see that the function P is periodic with period 24. Similarly, one verifies that the function p is also periodic with period 24. Next, recall that both the sine and cosine functions oscillate between -1 and $+1$ so that $P(t)$ is seen to oscillate between $[1000+100(-1)]$, or 900, and $[1000+100(1)]$, or 1100, while $p(t)$ oscillates between $[20,000+4000(-1)]$, or 16,000, and $[20,000+4000(1)]$, or 24,000. Finally, plotting a few points on each graph for –say, $t = 0, 2, 3$, and so on – we obtain the graphs of the functions P and p as shown below.





From the graphs, we see that at time $t = 0$, the predator population stands at 1000 owls. As the predator population increases, the prey population decreases from $24,000$ at that instant. Eventually, this decrease in the food supply causes the predator population to decrease, which in turn allows for an increase in the prey population. But as the prey population increases, resulting in an increase in food supply, the predator population once again increases. The cycle is complete and starts all over again. ♣

For the solution of Example 2.25, we described how to plot sine and cosine functions when they have been transformed by some translations,reflections, and stretchings in the horizontal and vertical directions. Recall from Section 2.2 that this has already been described for functions in general. However, since trigonometric functions are special in that they have periodic behaviour, we will summarize transformations of sine and cosine functions. This information will help when graphing these functions in general, if we know the graphs of the basic functions $y = \sin x$ and $y = \cos x$. The tangent function can be dealt with in a similar manner; just remember that it has period π.

Transformation of the Sine and Cosine Functions

Given the function

$$F(x) = Af(B(x - C)) + D$$

with real constants $A \neq 0, B \neq 0, C$ and D, and $f(x) = \sin x$ or $f(x) = \cos x$, then the graph of F is transformed as follows from f:

The constant $|A|$ is called the **amplitude** and stretches f vertically. Furthermore, if $A < 0$ then f reflects in the x-axis.

The constant $\frac{2\pi}{|B|}$ is called the **period** and stretches f horizontally. Furthermore, if $B < 0$ then f reflects in the y-axis.

If $C > 0$ then f shifts to the right, otherwise to the left.

If $D > 0$ then f shifts up, otherwise down.

Transformation of the Tangent Function

Given the function

$$F(x) = Af(B(x-C)) + D$$

with real constants $A \neq 0, B \neq 0, C$ and D, and $f(x) = \tan x$, then the graph of F is transformed as follows from f:

The constant $|A|$ stretched f vertically and in this case is *not* an amplitude. Furthermore, if $A < 0$ then f reflects in the x-axis.

The constant $\frac{\pi}{|B|}$ is called the **period** and stretches f horizontally. Furthermore, if $B < 0$ then f reflects in the y-axis.

If $C > 0$ then f shifts to the right, otherwise to the left.

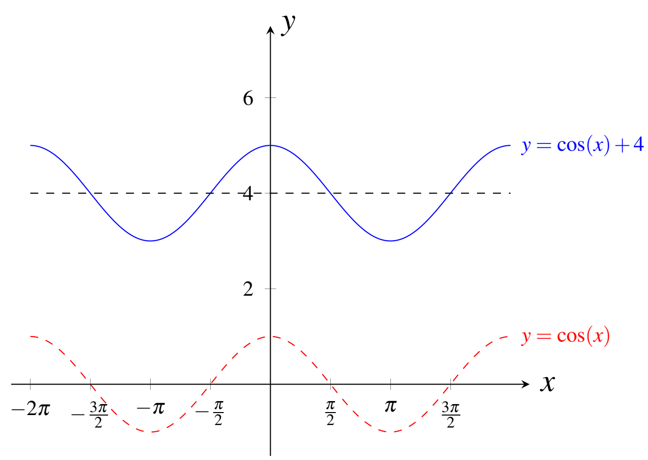
If $D > 0$ then f shifts up, otherwise down.

Note: Remember the order in which to perform the translations to produce the correct graph: first reflections and stretchings, then translations.

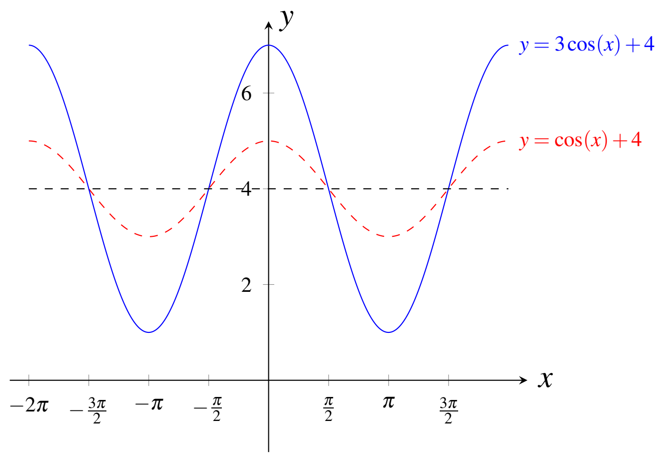
Example 2.26:

Graph the function defined by $f(x) = 3\cos(2x) + 4$.

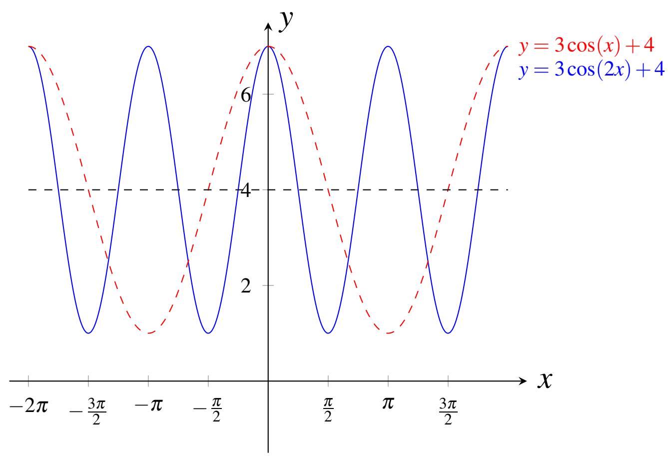
Solution. The graph of the basic function $y = \cos(x)$ is translated vertically up by 4 units:



The amplitude of this cosine function is 3, which means we now stretch the graph of the cosine function vertically by 3 units:



Lastly, the period is $\frac{2\pi}{2} = \pi$, which means the graph of the cosine function is compressed by a factor of $\frac{1}{2}$:

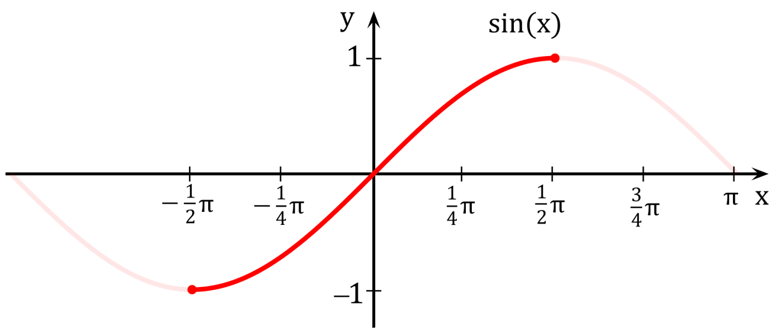


This results in the graph of f as desired (in blue). There are no horizontal or vertical reflections, since both $3 > 0$ and $2 > 0$. ♣

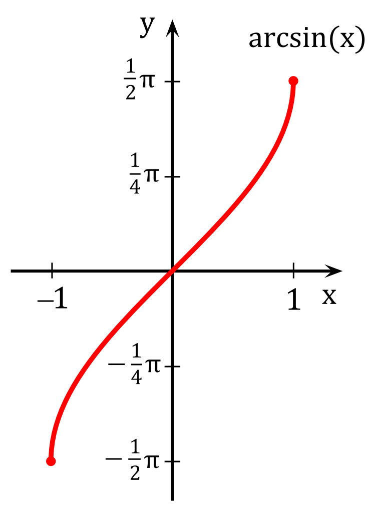
2.6.2. Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn't actually have an inverse that reliably "undoes" the sine function. If you know that $\sin x = 0.5$, you can't reverse this to discover x, that is, you can't solve for x, as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function does have a proper inverse.

The sine takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$.



If we truncate the sine, keeping only the interval $[-\pi/2, \pi/2]$, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write it in one of two common notation: $y = \arcsin(x)$, or $y = \sin^{-1}(x)$.



Definition 2.27: Inverse Sine Function

Given the restricted sine funtion $y = \sin(x)$ *with domain* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *the* **inverse sine function** *(or* **arcsine function***) is denoted by*

$$y = \sin^{-1}(x) \qquad (\text{or, } y = \arcsin(x)),$$

with domain $[-1, 1]$ *and range* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Recall that a function and its inverse undo each other in either order, for example, $(\sqrt[3]{x})^3 = x$ and $\sqrt[3]{x^3} = x$. This does not work with the sine and the "inverse sine" because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that $\sin(\arcsin(x)) = x$, that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, $\sin(5\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$, so doing first the sine then the arcsine does not get us back where we started. This is because $5\pi/6$ is not in the domain of the truncated sine. If we start with an angle between $-\pi/2$ and $\pi/2$ then the arcsine does reverse the sine: $\sin(\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$.

Example 2.28: Arcsine of Common Values

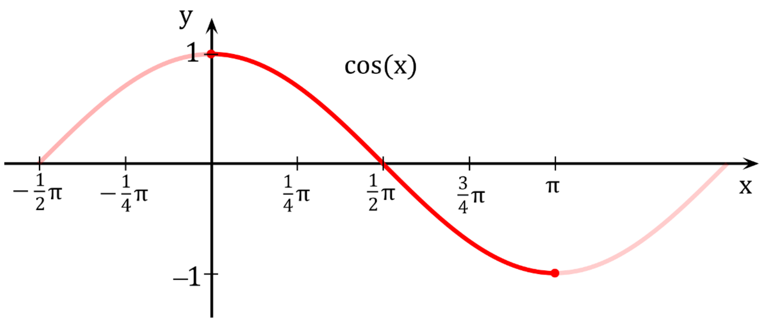
Compute $\sin^{-1}(0)$, $\sin^{-1}(1)$ *and* $\sin^{-1}(-1)$.

Solution. These come directly from the graph of $y = \arcsin x$:

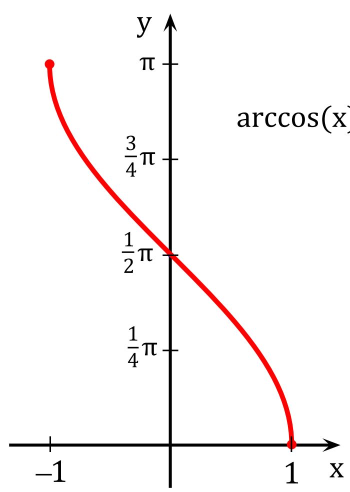
$$\sin^{-1}(0) = 0 \qquad \sin^{-1}(1) = \frac{\pi}{2} \qquad \sin^{-1}(-1) = -\frac{\pi}{2}$$

♣

We can do something similar for the cosine function. As with the sine, we must first truncate the cosine so that it can be inverted, in particular, we use the interval $[0, \pi]$.



Note that the truncated cosine uses a different interval than the truncated sine, so that if $y = \arccos(x)$ we know that $0 \leq y \leq \pi$.



Example 2.29: Arccosine of Common Values

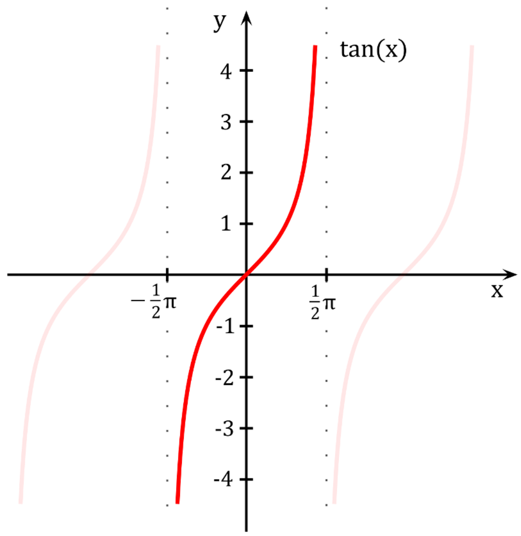
Compute $\cos^{-1}(0)$, $\cos^{-1}(1)$ *and* $\cos^{-1}(-1)$.

Solution. These come directly from the graph of $y = \arccos x$:

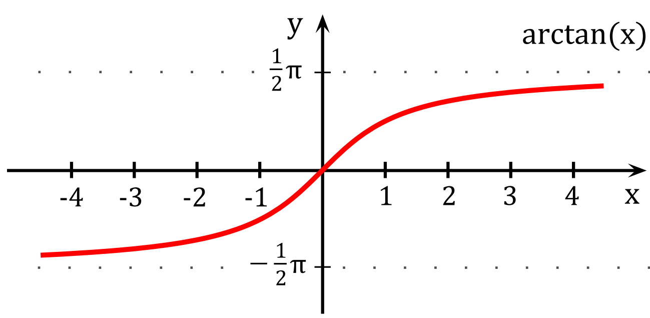
$$\cos^{-1}(0) = \frac{\pi}{2} \qquad \cos^{-1}(1) = 0 \qquad \cos^{-1}(-1) = \pi$$

♣

The truncated tangent uses an interval of $(-\pi/2, \pi/2)$.



Reflecting the truncated tangent in the line $y = x$ gives the arctangent function.



Example 2.30: Arctangent of Common Values

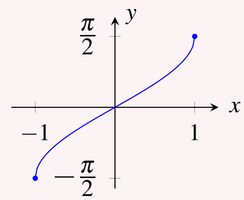
Compute $\tan^{-1}(0)$. *What value does* $\tan^{-1} x$ *approach as* x *gets larger and larger? What value does* $\tan^{-1} x$ *approach as* x *gets large (and negative)?*

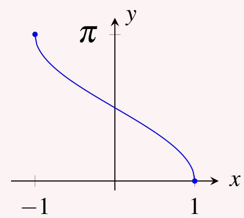
Solution. These come directly from the graph of $y = \arctan x$. In particular, $\tan^{-1}(0) = 0$. As x gets larger and larger, $\tan^{-1} x$ approaches a value of $\frac{\pi}{2}$, whereas, as x gets large but negative, $\tan^{-1} x$ approaches a value of $-\frac{\pi}{2}$. ♣

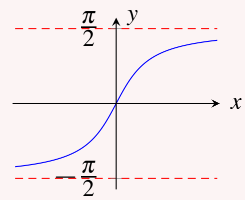
The following definition summarizes the inverse trigonometric functions and their respective domains and restricted ranges.

Definition 2.31: Inverses of the Primary Trigonometric Functions

notation:	$y = \sin^{-1}(x)$	$y = \cos^{-1}(x)$	$y = \tan^{-1}(x)$
domain:	$[-1,1]$	$[-1,1]$	$(-\infty,\infty)$
range:	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$[0,\pi]$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



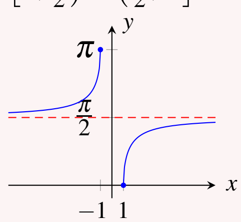


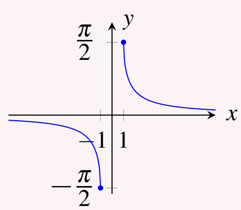


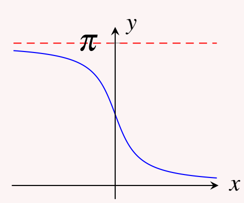
While the inverses for the sine, cosine and tangent are enough for most purposes, we state the remaining trigonometric inverses for completeness.

Definition 2.32: Inverses of the Secondary Trigonometric Functions

notation:	$y = \csc^{-1}(x)$	$y = \sec^{-1}(x)$	$y = \cot^{-1}(x)$
domain:	$(-\infty,-1]\cup[-1,\infty)$	$(-\infty,-1]\cup[1,\infty)$	$(-\infty,\infty)$
range:	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$	$(0,\pi)$







Note:

1. We use both $\sin^{-1}(x)$ and $\arcsin(x)$ to represent inverse sine.
2. $\sin^{-1}(x) \neq (\sin x)^{-1} \neq \sin(x^{-1})$ since $\sin^{-1}(x)$ denotes the inverse sine function, $(\sin x)^{-1} = \frac{1}{\sin x}$ denotes the reciprocal of sine, and $\sin x^{-1} = \sin\frac{1}{x}$ denotes the sine of the reciprocal of x.

The Cancellation Rules are tricky since we restricted the domains of the trigonometric functions in order to obtain inverse trig functions:

Cancellation Rules

$$\sin(\sin^{-1} x) = x, \quad x \in [-1,1] \qquad \sin^{-1}(\sin x) = x, \quad x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\cos(\cos^{-1} x) = x, \quad x \in [-1,1] \qquad \cos^{-1}(\cos x) = x, \quad x \in [0,\pi]$$

$$\tan(\tan^{-1} x) = x, \quad x \in (-\infty,\infty) \qquad \tan^{-1}(\tan x) = x, \quad x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Example 2.33: Arcsine

Find $\sin^{-1}(1/2)$.

Solution. Since $\sin^{-1}(x)$ outputs values in $[-\pi/2,\ \pi/2]$, the answer must be in this interval. Let $\theta = \sin^{-1}(1/2)$. We need to compute θ. Take the sine of both sides to get $\sin\theta = \sin(\sin^{-1}(1/2)) = 1/2$ by the Cancellation Rule. There are many angles θ that work, but we want the one in the interval $[-\pi/2,\ \pi/2]$. Thus, $\theta = \pi/6$ and hence, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. ♣

Example 2.34: Arccosine

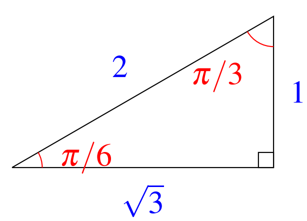
Find $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution. **Trick:** Let $\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. Need to compute θ.
Take cos of both sides:

$$\cos\theta = \cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \frac{\sqrt{3}}{2} \quad \text{by the Cancellation Rule.}$$

There are many angles θ that work, but we need the one in the interval $[0,\ \pi]$.
By the special triangle shown and the definition of cosine:

$$\theta = \frac{\pi}{6} \quad \text{and so} \quad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$



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Example 2.35: Arctangent

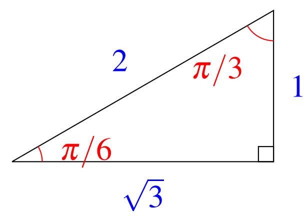
Find $\tan^{-1}(\sqrt{3})$.

Solution. **Trick:** Let $\theta = \tan^{-1}(\sqrt{3})$. Need to compute θ.
Take tan of both sides:

$$\tan\theta = \tan(\tan^{-1}(\sqrt{3}) = \sqrt{3} \quad \text{by the Cancellation Rule.}$$

There are many angles θ that work, but we need the one in the interval $(-\pi/2,\ \pi/2)$.
Thus, $\theta = \pi/3$ by the special triangle shown and the definition of tangent:

$$\theta = \frac{\pi}{3} \quad \text{and so} \quad \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$



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Example 2.36: Cancellation Rule

Find $\cos^{-1}(\cos(-5\pi/3))$.

Solution. Range: $\cos^{-1}(x)$ outputs values in $[0, \pi]$, thus the answers must be in this interval.
We can **not** cancel yet. Instead, we add/subtract multiples of 2π until we get a number in this range.
By the periodicity of $\cos x$, we have

$$\cos\left(-\frac{5\pi}{3}\right) = \cos\left(2\pi - \frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right).$$

By the Cancellation Rule:

$$\cos^{-1}\left(\cos\left(-\frac{5\pi}{3}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}.$$

♣

Example 2.37: Arccosine and the Cancellation Rule

Compute $\cos^{-1}(\cos(0)), \quad \cos^{-1}(\cos(\pi)), \quad \cos^{-1}(\cos(2\pi)), \quad \cos^{-1}(\cos(3\pi))$.

Solution. Since $\cos^{-1}(x)$ outputs values in $[0, \pi]$, the answers must be in this interval. The first two we can cancel using the Cancellation Rules:

$$\cos^{-1}(\cos(0)) = 0 \qquad \text{and} \qquad \cos^{-1}(\cos(\pi)) = \pi.$$

The third one we cannot cancel since $2\pi \notin [0, \pi]$:

$$\cos^{-1}(\cos(2\pi)) \quad \text{is NOT equal to} \quad 2\pi.$$

But we know that cosine is a 2π-periodic function, so $\cos(2\pi) = \cos(0)$:

$$\cos^{-1}(\cos(2\pi)) = \cos^{-1}(\cos(0)) = 0$$

Similarly with the fourth one, we can NOT cancel yet since $3\pi \notin [0, \pi]$. Using $\cos(3\pi) = \cos(3\pi - 2\pi) = \cos(\pi)$:

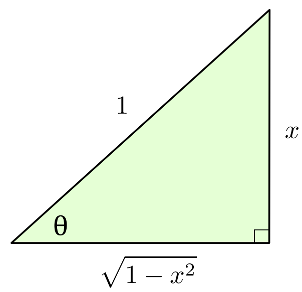
$$\cos^{-1}(\cos(3\pi)) = \cos^{-1}(\cos(\pi)) = \pi.$$

♣

Example 2.38: The Triangle Technique

Rewrite the expression $\cos(\sin^{-1} x)$ *without trig functions. Note that the domain of this function is all* $x \in [-1,1]$.

Solution. Let $\theta = \sin^{-1} x$. We need to compute $\cos\theta$. Taking the sine of both sides gives $\sin\theta = \sin(\sin^{-1}(x)) = x$ by the Cancellation Rule. We then draw a right triangle using $\sin\theta = x/1$:



If z is the remaining side, then by the Pythagorean Theorem:

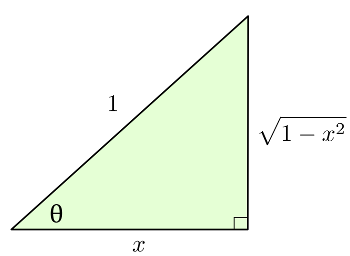
$$z^2 + x^2 = 1 \qquad \to \qquad z^2 = 1 - x^2 \qquad \to \qquad z = \pm\sqrt{1-x^2}$$

and hence $z = +\sqrt{1-x^2}$ since $\theta \in [-\pi/2,\ \pi/2]$. Thus, $\cos\theta = \sqrt{1-x^2}$ by SOH CAH TOA, so, $\cos(\sin^{-1} x) = \sqrt{1-x^2}$. ♣

Example 2.39: The Triangle Technique 2

For $x \in (0,1)$, *rewrite the expression* $\sin(2\cos^{-1} x)$. *Compute* $\sin(2\cos^{-1}(1/2))$.

Solution. Let $\theta = \cos^{-1} x$ so that $\cos\theta = x$. The question now asks for us to compute $\sin(2\theta)$. We then draw a right triangle using $\cos\theta = x/1$:



To find $\sin(2\theta)$ we use the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$. But $\sin\theta = \sqrt{1-x^2}$, for $\theta \in [0,\pi]$, and $\cos\theta = x$. Therefore, $\sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$. When $x = 1/2$ we have $\sin(2\cos^{-1}(1/2)) = \dfrac{\sqrt{3}}{2}$. ♣

Exercises for Section 2.6

Exercise 2.6.1 *Use transformations to sketch the graphs of the following functions.*

(a) $y = \sin(2x)$ *over the interval* $[0, 2\pi]$

(b) $y = -\sin(x)$ *over the interval* $[0, 2\pi]$

(c) $y = 3\sin(4x) - 5$ *over the interval* $[0, 2\pi]$

(d) $y = 2\sin\left(x - \frac{\pi}{2}\right) + 1$ *over the interval* $[0, 2\pi]$

(e) $y = \tan\left(\frac{x}{3}\right)$ *over the interval* $[-2\pi, 2\pi]$

(f) $y = -\tan(x) + 2$ *over the interval* $[-2\pi, 2\pi]$

(g) $y = \tan\left(x - \frac{\pi}{2}\right)$ *over the interval* $[-2\pi, 2\pi]$

Exercise 2.6.2 *Sales of large kitchen appliances such as ovens and fridges are usually subject to seasonal fluctuations. Everything Kitchen's sales of fridge models from the beginning of* 2001 *to the end of* 2002 *can be approximated by*

$$S(x) = \frac{1}{10}\sin\left(\frac{\pi}{2}(x+1)\right) + \frac{1}{2}$$

where x *is time in quarters,* $x = 1$ *represents the end of the first quarter of* 2001*, and* S *is measured in millions of dollars.*

(a) What are the maximum and minimum quarterly revenues for fridges?

(b) Find the values of x *where the quarterly revenues are highest and lowest.*

Exercise 2.6.3 *Compute the following:*

(a) $\sin^{-1}(\sqrt{3}/2)$

(b) $\cos^{-1}(-\sqrt{2}/2)$

Exercise 2.6.4 *Compute the following:*

(a) $\sin^{-1}(\sin(\pi/4))$

(b) $\sin^{-1}(\sin(17\pi/3))$

(c) $\cos\left(\cos^{-1}(1/3)\right)$

(d) $\tan\left(\cos^{-1}(-4/5)\right)$

Exercise 2.6.5 *Rewrite the expression* $\tan\left(\cos^{-1} x\right)$ *without trigonometric functions. What is the domain of this function?*

Exercise 2.6.6 *Let* $\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$*. Evaluate:*

(a) $\sin\theta$

(b) $\cos\theta$

(c) $\sec\theta$

(d) $\tan\theta$

(e) $\cot\theta$

2.7 Economic Models

2.7.1. Demand and Supply Functions

In a market economy that has few or no restrictions and regulations on buyers and sellers, the consumer demand for a particular commodity is dependent on the commodity's unit price. The relationship between a unit price and the quantity demanded is articulated by a so-called **demand equation** and its graph is referred to as a **demand curve**. In general, the quantity demanded of a commodity increases as the commodity's unit price decreases, and vice versa.

Definition 2.40: Demand Function

A **demand function** *is defined by $p = f(x)$, where p measures the unit price and x measures the number of units of the commodity in question, and is generally characterized as a decreasing function of x; that is, $p = f(x)$ decreases as x increases. Since both x and p assume only nonnegative values, the demand curve is that part of the graph of $f(x)$ that lies in the first quadrant (Figure 2.5).*

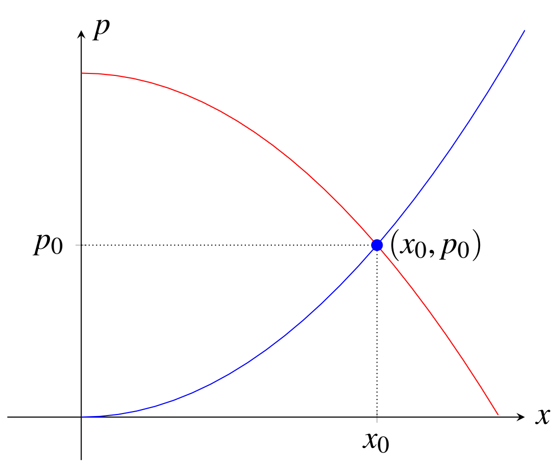


Figure 2.5: Example of a supply curve (in blue) and a demand curve (in red). The point of intersection (x_0, p_0) corresponds to market equilibrium.

In a market in which a large number of producers compete with each other to satisfy the wants and needs of a large number of consumers, the unit price of a commodity is dependent on the commodity's availability in the market. The relationship between a unit price and the quantity supplied is articulated by a so-called **supply equation** and its graph is referred to as a **supply curve**. In general, an increase or decrease in the commodity's unit price induces the producer to respectively increase or decrease the supply of the commodity.

Definition 2.41: Supply Function

A **supply function** *defined by $p = f(x)$ with p and x as before is generally characterized as an increasing function of x; that is, $p = f(x)$ increases as x increases. Since both x and p assume only nonnegative values, the supply curve is that part of the graph of $f(x)$ that lies in the first quadrant (Figure 2.5)*

In a competitive market, the price of a commodity will eventually settle at a certain level, namely when the supply of the commodity will be equal to the demand for it. Simply put, if the price is too high, then the consumer will not buy as much of the product as is being supplied; conversely, if the price is too low, then the supplier will not produce as much of the product as is being demanded. When the quantity produced is equal to the quantity demanded, then a so-called **market equilibrium** is achieved. The number of units at which demand equals supply is the **equilibrium quantity**, the corresponding price per unit is the **equilibrium price**, and the corresponding ordered pair gives the equilibrium point. f

Definition 2.42: Equilibrium Point

The **equilibrium point** (x, p) *is defined by*

$$(x, p) = (\textbf{equilibrium quantity}, \textbf{equilibrium price})$$

and provides the quantity x and price p at which the demand equals the supply.

Since the supply of a product is equal to the demand for it under market equilibrium, this market equilibrium must be the point at which the demand curve and the supply curve intersect. In Figure 5.2, the point (x_0, p_0) represents (equilibrium quantity, equilibrium price). Since the point (x_0, p_0) lies on the supply curve as well as the demand curve, it satisfies both the supply equation and the demand equation. Thus, to find the market equilibrium, we solve the demand and supply equations simultaneously for x and p, which will yield the point (x_0, p_0). Mathematically speaking one or both of the values may be negative when solving the system of two equations; however, for the solutions to be meaningful, both x_0 and p_0 must be positive.

Example 2.43: Supply-Demand

The demand function for a certain commercial product is given by

$$p = d(x) = -0.01x^2 - 0.2x + 8$$

and the corresponding supply function is given by

$$p = s(x) = 0.01x^2 + 0.1x + 3$$

where p is expressed in dollars and x is measured in units of a hundred. Find the market equilibrium.

Solution. We solve the following system of equations:

$$p = -0.01x^2 - 0.2x + 8$$
$$p = 0.01x^2 + 0.1x + 3$$

Subtracting the first equation from the second equation, we obtain

$$-0.01x^2 - 0.2x + 8 = 0.01x^2 + 0.1x + 3$$

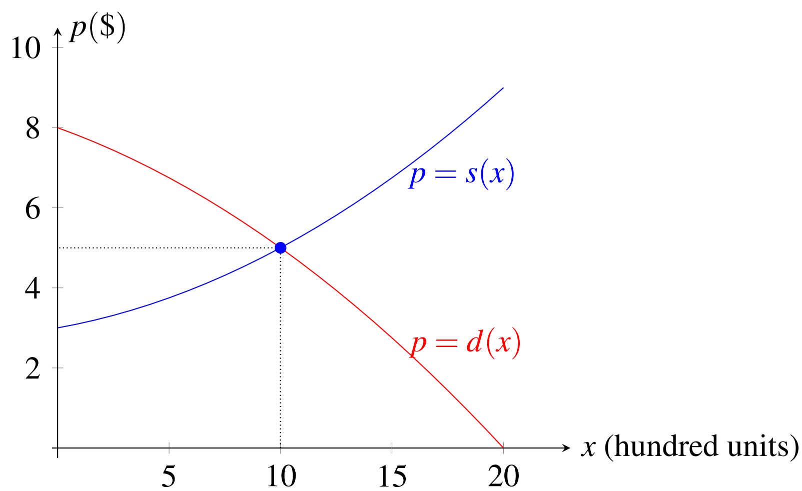
which is equivalent to

$$\begin{aligned} 0.02x^2 + 0.3x - 5 &= 0 \\ 2x^2 + 30x - 500 &= 0 \\ x^2 + 15x - 250 &= 0 \\ (x+25)(x-10) &= 0 \end{aligned}$$

Thus, $x = -25$ or $x = 10$. Since the number of units x must be nonnegative, the root $x = -25$ is rejected. Therefore, the equilibrium quantity is 1000 units. The equilibrium price is given by

$$p = 0.01(10^2) + 0.1(10) + 3 = 5$$

or \$5 per unit. The graphs of the demand and supply functions and their intersection point are shown below.



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2.7.2. Cost, Revenue and Profit Functions

When a problem arising from a practical situation is being modelled mathematically, then this often leads to an expression that involves the combination of functions. For example, modelling the costs incurred in running a business. Here, we differentiate between costs that remain more or less constant regardless of the firm's level of activity, namely the **fixed costs** or **constant costs**, and costs that vary with production or sales, namely the **variable costs** or **marginal costs**. Rental fees and executive salaries are examples of fixed costs, while wages and purchases of raw materials are examples of variable costs. Therefore, in order to consider the **total cost** of operating a business, typically denoted by C, we must sum the variable costs and the fixed costs.

Definition 2.44: Linear Cost Function

The **linear cost function** *C is given by*

$$C(x) = mx + b$$

where m is the **variable cost** *per unit, b is the overall* **fixed cost***, and x is the number of units produced.*

The **average cost** of manufacturing x units of a certain product is obtained by dividing the total production cost by the number of units manufactured.

Definition 2.45: Average Cost Function

Suppose $C(x)$ is the total cost function. Then the **average cost function***, denoted by $\overline{C}(x)$ – read "C bar of x" – is given by*

$$\overline{C}(x) = \frac{C(x)}{x}.$$

The revenue realized by a company from the sale of x units of a certain commodity is given by the so-called **revenue function**, typically denoted by R. Then a simple model is *revenue = (price per unit)·(number of units produced or sold)*. However, the market in which the company operates dictates the price that the company can demand for the product. Recall that in a competitive market environment the price is determined by market equilibrium. In other words, if the company is one of many, then none of them is able to dictate the price of the commodity. But in a monopolistic market, the company is the sole supplier of the product and can therefore manipulate the price of the commodity by controlling the supply. The unit selling price p of the commodity and the quantity x of the commodity demanded are related to each other by the demand function. This leads to the following definition of the revenue function.

Definition 2.46: Revenue Function

The **Revenue function** *R is given by*

$$R(x) = px = xf(x)$$

where p is the unit selling price of the commodity, x is the quantity of the commodity demanded, and f is the demand function.

The **profit** realized by a company in operating a business is the difference between the total revenue realized and the total cost incurred.

Definition 2.47: Profit Function

The **Profit function** *P is given by*

$$P(x) = R(x) - C(x)$$

where R is the revenue function, C is the cost function, and x is the quantity of the commodity sold.

Example 2.48: Cost Functions

Suppose a certain company has a monthly fixed cost of \$10,000 and a variable cost of

$$-0.0001x^2 + 12x \quad 0 \leq x \leq 40,000$$

dollars, where x denotes the number of units manufactured per month. Find a function C that gives the total cost incurred by the company in the manufacture of x units.

Solution. The company's monthly fixed cost is \$10,000, regardless of the level of production. The total cost will be the sum of the variable and fixed costs,

$$C(x) = -0.0001x^2 + 12x + 10000 \quad 0 \leq x \leq 40,000$$

♣

Example 2.49: Profit Functions

Suppose the total revenue realized by the same company from Example 2.48 from the sale of x units is given by the total revenue function

$$R(x) = -0.0005x^2 + 22x \quad 0 \leq x \leq 40,000$$

(a) Find the total profit function – that is, the function that descrbies the total profit the company realizes in manufacturing and selling x units per month.

(b) Find the profit when the level of production is 10,000 units.

Solution.

(a) We obtain the total profit function by taking the difference between the total revenue realized and the total cost incurred:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-0.0005x^2 + 22x) - (-0.0001x^2 + 12x + 10000) \\ &= -0.0004x^2 + 10x - 10000 \end{aligned}$$

(b) When the production level is 10000 units per month, the company's profit is given by

$$P(10000) = -0.0004(10000)^2 + 10(10000) - 10000 = 50000$$

or \$50,000 per month.

♣

Example 2.50: Revenue

If exactly 1000 people sign up for a charter flight, a certain travel agency charges 600 per person. However, if more than 1000 people sign up for the flight (assume this is the case), then each fare is reduced by \$5 for each additional person. Letting x denote the number of passengers above 1000, find a function giving the revenue realized by the company.

Solution. If there are x passengers above 1000, then the number of passengers signing up for the flight in total is $1000 + x$. The fare will be $\$(600 - 5x)$ per passenger. Therefore, the revenue will be

$$\begin{aligned} R(x) &= (\text{number of passengers}) \times (\text{fare per passenger}) \\ &= (1000 + x)(600 - 5x) \\ &= -5x^2 - 4400x + 600{,}000 \end{aligned}$$

Clearly, x must be nonnegative, and $600 - 5x \geq 0$. So the final revenue function is $R(x) = -5x^2 - 4400x + 600{,}000$ for x in the interval $[0, 120]$. ♣

A company can realize a profit only if the revenue received from its customers exceeds the cost of producing and selling its products. It is therefore of interest to find out when revenue equals cost, which is referred to as the **break-even point**. The number of units at which revenue equals cost is the **break-even quantity**, the corresponding price is the **break-even price**, and the corresponding ordered pair gives the break-even point.

Definition 2.51: Break-Even Point

The **break-even point** (x, p) *is defined by*

$$(x, p) = (\textbf{break-even quantity}, \textbf{break-even price})$$

and provides the quantity x and price p at which revenue equals cost.

Example 2.52: Break-Even Point

A certain firm determines that the total cost $C(x)$ in dollars of producing and selling x units is given by

$$C(x) = 20x + 100.$$

Management plans to charge \$24 per unit.

(a) How many units must be sold for the firm to break even?

(b) What is the profit if 100 units of feed are sold?

(c) How many units must be sold to produce a profit of \$900?

Solution.

(a) The firm will break even as long as revenue just equals cost, or $R(x) = C(x)$. From the given information,

$$R(x) = 24x.$$

Substituting for $R(x)$ and $C(x)$ gives

$$24x = 20x + 100,$$

from which we get that $x = 25$. The firm therefore breaks even by selling 25 units (this is the break-even quantity). If the company sells more than 25 units, it makes a profit. If it sells fewer than 25 units, it loses money, as shown in the figure below.

(b) First, we find the formula for the profit, $P(x)$.

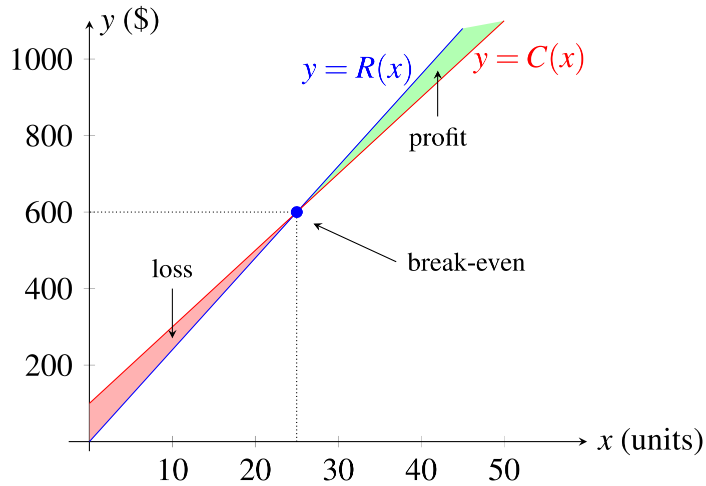
$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 24x - (20x + 100) \\ &= 4x - 100 \end{aligned}$$

Thus, $P(100) = 4(100) - 100 = 300$. The firm will make a profit of \$300 from the sale of 100 units.

(c) Let $P(x) = 900$ in the equation $P(x) = 4x - 100$.

$$\begin{aligned} 900 &= 4x - 100 \\ 1000 &= 4x \\ x &= 250 \end{aligned}$$

Sales of 250 units will produce a profit of \$900.



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Exercises for Section 2.7

Exercise 2.7.1 *For the following demand equations, where x is the quantity demanded in units of a thousand and p is the unit price in dollars, sketch the demand curve and determine the quantity demanded at the given unit price.*

(a) $p = -x^2 + 36$; $p = 11$

(b) $p = \sqrt{9 - x^2}$; $p = 2$

Exercise 2.7.2 *For the following supply equations, where x is the quantity supplied in units of a thousand and p is the unit price in dollars, sketch the supply curve and determine the price at the given number of units.*

(a) $p = 2x^2 + 18$; $x = 2000$

(b) $p = x^3 + x + 10$; $x = 2000$

Exercise 2.7.3 *For the following pairs of supply and demand equations, where x represents the quantity demanded in units of a thousand and p is the unit price in dollars, find the equilbrium point.*

(a) $0 = -2x^2 - p + 80$ *and* $0 = 15x - p + 30$

(b) $11p + 3x = 66$ *and* $2p^2 + p - x = 10$

Exercise 2.7.4 *A production company has a monthly fixed cost of* $\$100,000$ *and a variable cost of* $\$14$ *for each unit produced. The commodity sells for* $\$20$ *per unit.*

(a) *What is the cost function?*

(b) *What is the revenue function?*

(c) *What is the profit function?*

(d) *Compute the profit (or loss) corresponding to manufacturing levels of* 12,000 *and* 20,000 *units.*

Exercise 2.7.5 *The price of ivory is determined by a variety of legal market sources and illegal ivory trades. The World Wildlife Fund, an international non-governmental organization trying to preserve wilderness, determined that the ivory price can be approximated by the piecewise-defined function*

$$p(t) = \begin{cases} 8.37t + 7.44 & \text{if } 0 \leq t \leq 8 \\ 2.84t + 51.68 & \text{if } 8 < t \leq 30 \end{cases}$$

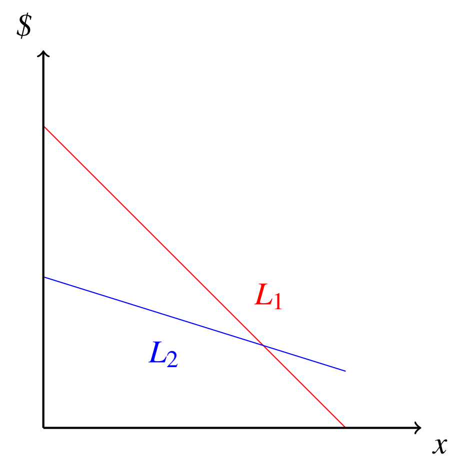
where t *is measured in years, with* $t = 0$ *corresponding to the beginning of the year* 1970*, and* $p(t)$ *is measured in dollars per kilogram.*

(a) *Sketch the graph of* $p(t)$.

(b) *What was the price of ivory at the beginning of* 1970*?*

(c) *What was the price of ivory at the beginning of* 1990*?*

Exercise 2.7.6 *In the accompanying figure,* L_1 *is the demand curve for the version X of cell phones, and* L_2 *is the demand curve for the version Y of cell phones. Which line has greater slope? Interpret your results.*



Exercise 2.7.7 *The demand function for a certain product is given by*

$$p = \frac{3000}{2x^2 + 100} \quad (0 \leq x \leq 10)$$

where x (measured in units of a thousand) is the quantity demanded per week and p is the unit price in dollars. Sketch the graph of the demand function. What is the price that corresponds to a quantity demanded of $10,000$ *units?*

Exercise 2.7.8 *A certain manufacturer has determined that the weekly demand and supply functions for their product is given by*

$$p(q) = 144 - q^2$$
$$p(q) = 48 + \frac{1}{2}q^2$$

respectively, where p is measured in dollars and q is measured in units of a thousand. Find the equilibrium point.

Exercise 2.7.9 *Suppose that the demand and price for a certain model of wristwatch are related by*

$$p = D(q) = 16 - 1.25q$$

where p is the price in dollars and q is the demand in units of hundreds. The price and supply of the watch are related by

$$p = S(q) = 0.75q$$

where p is the price in dollars and q is the supply of watches in units of hundreds.

(a) *Find the price when the demand is* 0 *watches*

(b) 400 *watches*

(c) 800 *watches*

(d) *Find the demand when the price is* \$8

(e) \$10

(f) \$12

(g) *Sketch $D(q)$*

(h) *Find the supply when the price is* \$0

(i) \$10

(j) \$20

(k) *Sketch $S(q)$ on the same plot as for part (g)*

(l) *Find the equilibrium quantity and the equilibrium price*

Exercise 2.7.10 *Joanne sells T-shirts at community festivals and creaft fairs. Her marginal cost to produce one T-shirt is \$3.50. Her total cost to produce* 60 *T-shirts is \$300, and she sells them for \$9 each.*

(a) *Find the linear cost function for Joanne's T-shirt production.*

(b) How many T-shirts must she produce and sell in order to break even?

(c) How many T-shirts must she produce and sell to make a profit of \$500?

Exercise 2.7.11 *You are the manager of a firm. You are considering the manufacture of a new product, so you ask the accounting department for cost estimates and the sales deparment for sales estimates. After you recieve the data, you must decide whether to go ahead with production of the new product. Analyze the given data (find a break-even quantity) and then decide what you would do in each case. Include the profit function.*

(a) $C(x) = 105x + 6000$; $R(x) = 250x$; *no more than* 400 *units can be sold.*

(b) $C(x) = 1000x + 5000$; $R(x) = 900x$.

2.8 Additional Exercises

Exercise 2.8.1 *If* $f(x) = \dfrac{1}{x-1}$, *then which of the following is equal to* $f\left(\dfrac{1}{x}\right)$?

(a) $f(x)$

(b) $-f(x)$

(c) $xf(x)$

(d) $-xf(x)$

(e) $\dfrac{f(x)}{x}$

(f) $-\dfrac{f(x)}{x}$

Exercise 2.8.2 *If* $f(x) = \dfrac{x}{x+3}$, *then find and simplify* $\dfrac{f(x)-f(2)}{x-2}$.

Exercise 2.8.3 *If* $f(x) = x^2$, *then find and simplify* $\dfrac{f(3+h)-f(3)}{h}$.

Exercise 2.8.4 *What is the domain of*

(a) $f(x) = \dfrac{\sqrt{x-2}}{x^2-9}$?

(b) $g(x) = \dfrac{\sqrt[3]{x-2}}{x^2-9}$?

Exercise 2.8.5 *Suppose that $f(x) = x^3$ and $g(x) = x$. What is the domain of $\dfrac{f}{g}$?*

Exercise 2.8.6 *Suppose that $f(x) = 3x-4$. Find a function g such that $(g \circ f)(x) = 5x+2$.*

Exercise 2.8.7 *Which of the following functions is one-to-one?*

(a) $f(x) = x^2 + 4x + 3$

(b) $g(x) = |x| + 2$

(c) $h(x) = \sqrt[3]{x+1}$

(d) $F(x) = \cos x,\ -\pi \leq x \leq \pi$

(e) $G(x) = e^x + e^{-x}$

Exercise 2.8.8 *What is the inverse of $f(x) = \ln\left(\dfrac{e^x}{e^x-1}\right)$? What is the domain of f^{-1}?*

Exercise 2.8.9 *Solve the following equations.*

(a) $e^{2-x} = 3$

(b) $e^{x^2} = e^{4x-3}$

(c) $\ln(1+\sqrt{x}) = 2$

(d) $\ln(x^2-3) = \ln 2 + \ln x$

Exercise 2.8.10 *Find the exact value of* $\sin^{-1}\left(-\sqrt{2}/2\right) - \cos^{-1}\left(-\sqrt{2}/2\right)$.

Exercise 2.8.11 *Find* $\sin^{-1}(\sin(23\pi/5))$.

Exercise 2.8.12 *It can be proved that $f(x) = x^3 + x + e^{x-1}$ is one-to-one. What is the value of $f^{-1}(3)$?*

Exercise 2.8.13 *Sketch the graph of* $f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ \tan^{-1} x & \text{if } x > 0 \end{cases}$

3. Limits and Continuity

3.1 The Limit

The value a function f approaches as its input x approaches some value is said to be the limit of f. Limits are essential to the study of calculus and, as we will see, are used in defining continuity, derivatives, and integrals.

Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Notice that $x = 1$ does not belong to the domain of $f(x)$. Regardless, we would like to know how $f(x)$ behaves close to the point $x = 1$. We start with a table of values:

x	0.5	0.9	0.99	1.01	1.1	1.5
$f(x)$	1.5	1.9	1.99	2.01	2.1	2.5

It appears that for values of x close to 1 we have that $f(x)$ is close to 2. In fact, we can make the values of $f(x)$ as close to 2 as we like by taking x sufficiently close to 1. We express this by saying *the limit of the function $f(x)$ as x approaches* 1 *is equal to* 2 and use the notation:

$$\lim_{x \to 1} f(x) = 2.$$

Definition 3.1: Limit (Useable Definition)

In general, we write

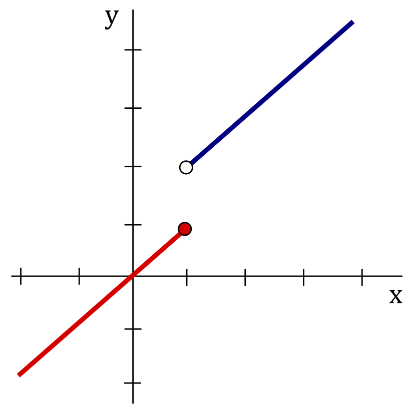
$$\lim_{x \to a} f(x) = L,$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

We read the expression $\lim_{x \to a} f(x) = L$ as "the limit of $f(x)$ as x approaches a is equal to L". When evaluating a limit, you are essentially answering the following question: What number does the function *approach* while x gets closer and closer to a (but *not equal* to a)? The phrase *but not equal to a* in the definition of a limit means that when finding the limit of $f(x)$ as x approaches a we never actually consider $x = a$. In fact, as we just saw in the example above, a may not even belong to the domain of f. All that matters for limits is what happens to f close to a, not necessarily what happens to f at a.

One-sided limits

Consider the following piecewise defined function:



$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$$

Observe from the graph that as x gets closer and closer to 1 from the *left*, then $f(x)$ approaches $+1$. Similarly, as x gets closer and closer 1 from the *right*, then $f(x)$ approaches $+2$. We use the following notation to indicate this:

$$\lim_{x \to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 2.$$

The symbol $x \to 1^-$ means that we only consider values of x sufficiently close to 1 which are less than 1. Similarly, the symbol $x \to 1^+$ means that we only consider values of x sufficiently close to 1 which are greater than 1.

Definition 3.2: Left and Right-Hand Limit (Useable Definition)

In general, we write

$$\lim_{x \to a^-} f(x) = L,$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a. This is called the **left-hand limit** *of $f(x)$ as x approaches a. Similarly, we write*

$$\lim_{x \to a^+} f(x) = L,$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a. This is called the **right-hand limit** *of $f(x)$ as x approaches a.*

We note the following fact:

Theorem 3.3: Connection between Limit and Left & Right Limits

$$\lim_{x \to a} f(x) = L \qquad \textit{if and only if} \qquad \lim_{x \to a^-} f(x) = L \qquad \textit{and} \qquad \lim_{x \to a^+} f(x) = L.$$

Or more concisely:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

A consequence of this fact is that if the one-sided limits are *different*, then the two-sided limit $\lim_{x \to a} f(x)$ does not exist, often denoted as DNE.

Exercises for Section 3.1

Exercise 3.1.1 *Use a calculator to estimate* $\lim_{x \to 0} \frac{\sin x}{x}$*, where x is in radians.*

Exercise 3.1.2 *Use a calculator to estimate* $\lim_{x \to 0} \frac{\tan(3x)}{\tan(5x)}$*, where x is in radians.*

Exercise 3.1.3 *Use a calculator to estimate* $\lim_{x \to 1^+} \frac{|x-1|}{1-x^2}$ *and* $\lim_{x \to 1^-} \frac{|x-1|}{1-x^2}$.

3.2 Precise Definition of a Limit

The definition given for a limit previously is more of a working definition. In this section we pursue the actual, official definition of a limit.

Definition 3.4: Precise Definition of Limit

Suppose f is a function. We say that $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| < \delta$, $|f(x) - L| < \varepsilon$.

The ε and δ here play exactly the role they did in the preceding discussion. The definition says, in a very precise way, that $f(x)$ can be made as close as desired to L (that's the $|f(x) - L| < \varepsilon$ part) by making x close enough to a (the $0 < |x-a| < \delta$ part). Note that we specifically make no mention of what must happen if $x = a$, that is, if $|x-a| = 0$. This is because in the cases we are most interested in, substituting a for x doesn't make sense.

Make sure you are not confused by the names of important quantities. The generic definition talks about $f(x)$, but the function and the variable might have other names. The x was the variable of the original function; when we were trying to compute a slope or a velocity, x was essentially a fixed quantity, telling us at what point we wanted the slope. In the velocity problem, it was literally a fixed quantity, as we focused on the time $t = 2$. The quantity a of the definition in all the examples was zero: we were always interested in what happened as Δx became very close to zero.

Armed with a precise definition, we can now prove that certain quantities behave in a particular way. The bad news is that even proofs for simple quantities can be quite tedious and complicated. The good news is that we rarely need to do such proofs, because most expressions act the way you would expect, and this can be proved once and for all.

Example 3.5: Epsilon Delta

Let's show carefully that $\lim_{x\to 2} x+4=6$.

Solution. This is not something we "need" to prove, since it is "obviously" true. But if we couldn't prove it using our official definition there would be something very wrong with the definition.

As is often the case in mathematical proofs, it helps to work backwards. We want to end up showing that under certain circumstances $x+4$ is close to 6; precisely, we want to show that $|x+4-6|<\varepsilon$, or $|x-2|<\varepsilon$. Under what circumstances? We want this to be true whenever $0<|x-2|<\delta$. So the question becomes: can we choose a value for δ that guarantees that $0<|x-2|<\delta$ implies $|x-2|<\varepsilon$? Of course: no matter what ε is, $\delta=\varepsilon$ works. ♣

So it turns out to be very easy to prove something "obvious," which is nice. It doesn't take long before things get trickier, however.

Example 3.6: Epsilon Delta

It seems clear that $\lim_{x\to 2} x^2=4$. *Let's try to prove it.*

Solution. We will want to be able to show that $|x^2-4|<\varepsilon$ whenever $0<|x-2|<\delta$, by choosing δ carefully. Is there any connection between $|x-2|$ and $|x^2-4|$? Yes, and it's not hard to spot, but it is not so simple as the previous example. We can write $|x^2-4|=|(x+2)(x-2)|$. Now when $|x-2|$ is small, part of $|(x+2)(x-2)|$ is small, namely $(x-2)$. What about $(x+2)$? If x is close to 2, $(x+2)$ certainly can't be too big, but we need to somehow be precise about it. Let's recall the "game" version of what is going on here. You get to pick an ε and I have to pick a δ that makes things work out. Presumably it is the really tiny values of ε I need to worry about, but I have to be prepared for anything, even an apparently "bad" move like $\varepsilon=1000$. I expect that ε is going to be small, and that the corresponding δ will be small, certainly less than 1. If $\delta\le 1$ then $|x+2|<5$ when $|x-2|<\delta$ (because if x is within 1 or 2, then x is between 1 and 3 and $x+2$ is between 3 and 5). So then I'd be trying to show that $|(x+2)(x-2)|<5|x-2|<\varepsilon$. So now how can I pick δ so that $|x-2|<\delta$ implies $5|x-2|<\varepsilon$? This is easy: use $\delta=\varepsilon/5$, so $5|x-2|<5(\varepsilon/5)=\varepsilon$. But what if the ε you choose is not small? If you choose $\varepsilon=1000$, should I pick $\delta=200$? No, to keep things "sane" I will never pick a δ bigger than 1. Here's the final "game strategy": when you pick a value for ε, I will pick $\delta=\varepsilon/5$ or $\delta=1$, whichever is smaller. Now when $|x-2|<\delta$, I know both that $|x+2|<5$ and that $|x-2|<\varepsilon/5$. Thus $|(x+2)(x-2)|<5(\varepsilon/5)=\varepsilon$.

This has been a long discussion, but most of it was explanation and scratch work. If this were written down as a proof, it would be quite short, like this:

Proof that $\lim_{x\to 2} x^2=4$. Given any ε, pick $\delta=\varepsilon/5$ or $\delta=1$, whichever is smaller. Then when $|x-2|<\delta$, $|x+2|<5$ and $|x-2|<\varepsilon/5$. Hence $|x^2-4|=|(x+2)(x-2)|<5(\varepsilon/5)=\varepsilon$. ♣

It probably seems obvious that $\lim_{x\to 2} x^2=4$, and it is worth examining more closely why it seems obvious. If we write $x^2=x\cdot x$, and ask what happens when x approaches 2, we might say something like, "Well, the first x approaches 2, and the second x approaches 2, so the product must approach $2\cdot 2$." In fact this is pretty much right on the money, except for that word "must." Is it really true that if x approaches a and y approaches b then xy approaches ab? It is, but it is not really obvious, since x and y might be quite complicated. The good news is that we can see that this is true once and for all, and then we don't have to worry about it ever again. When we say that x might be "complicated" we really mean that in practice it might be a function. Here is then what we want to know:

Theorem 3.7: Limit Product

Suppose $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then* $\lim_{x\to a} f(x)g(x) = LM$.

Proof. We must use the Precise Definition of a Limit to prove the Produce Law for Limits. So given any ε we need to find a δ so that $0 < |x-a| < \delta$ implies $|f(x)g(x) - LM| < \varepsilon$. What do we have to work with? We know that we can make $f(x)$ close to L and $g(x)$ close to M, and we have to somehow connect these facts to make $f(x)g(x)$ close to LM.

We use, as is often the case, a little algebraic trick:

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x)-M) + (f(x)-L)M| \\ &\leq |f(x)(g(x)-M)| + |(f(x)-L)M| \\ &= |f(x)||g(x)-M| + |f(x)-L||M|.\end{aligned}$$

This is all straightforward except perhaps for the "$\leq$". That is an example of the *triangle inequality*, which says that if a and b are any real numbers then $|a+b| \leq |a| + |b|$. If you look at a few examples, using positive and negative numbers in various combinations for a and b, you should quickly understand why this is true. We will not prove it formally.

Suppose $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there is a value δ_1 such that $0 < |x-a| < \delta_1$ implies $|f(x) - L| < \dfrac{\varepsilon}{2(1+|M|)}$. This means that $0 < |x-a| < \delta_1$ implies $|f(x)-L||M| < |f(x)-L|(1+|M|) < \varepsilon/2$.

Now we focus our attention on the other term in the inequality, $|f(x)||g(x)-M|$. We can make $|g(x) - M|$ smaller than any fixed number by making x close enough to a; unfortunately, $\varepsilon/(2f(x))$ is not a fixed number, since x is a variable. Here we need another little trick, just like the one we used in analyzing x^2. We can find a δ_2 so that $|x-a| < \delta_2$ implies that $|f(x)-L| < 1$, meaning that $L-1 < f(x) < L+1$. This means that $|f(x)| < N$, where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x)-M| < \varepsilon/(2N)$. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that $|f(x)-L| < \varepsilon/(2(1+|M|))$, $|f(x)| < N$, and $|g(x)-M| < \varepsilon/(2N)$. Then

$$\begin{aligned}|f(x)g(x) - LM| &\leq |f(x)||g(x)-M| + |f(x)-L||M| \\ &< N\frac{\varepsilon}{2N} + \frac{\varepsilon}{2(1+|M|)}(1+|M|) \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This is just what we needed, so by the official definition, $\lim_{x\to a} f(x)g(x) = LM$. ♣

The concept of a **one-sided limit** can also be made precise.

Definition 3.8: One-sided Limit

Suppose that $f(x)$ is a function. We say that $\lim_{x\to a^-} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < a - x < \delta$, $|f(x) - L| < \varepsilon$. We say that $\lim_{x\to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < x - a < \delta$, $|f(x) - L| < \varepsilon$.

Exercises for Section 3.2

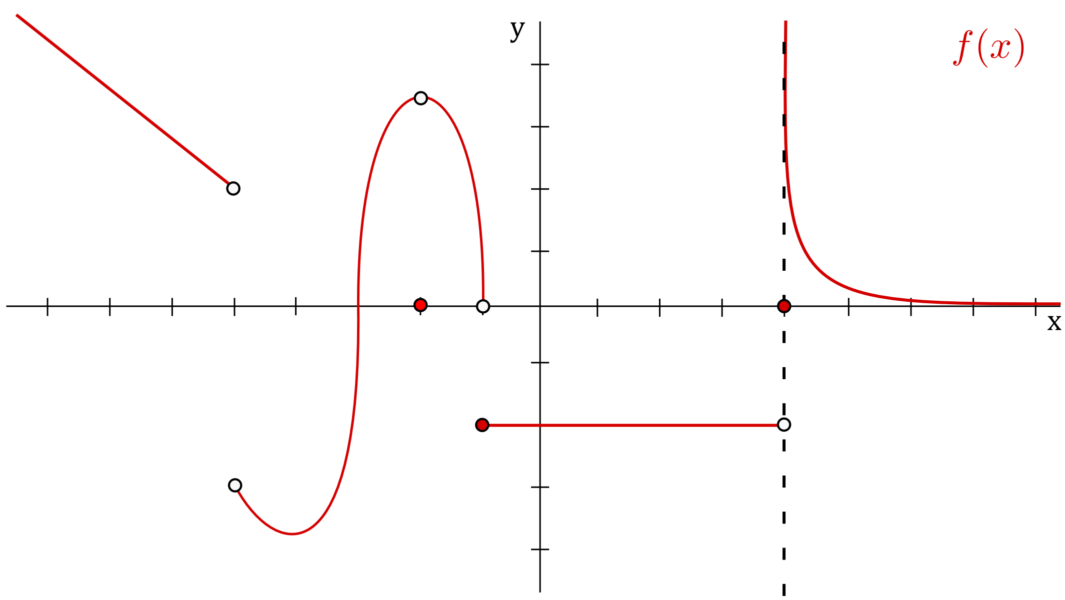
Exercise 3.2.1 *Give an ε–δ proof of the fact that $\lim_{x\to 4}(2x - 5) = 3$.*

Exercise 3.2.2 *Let ε be a small positive real number. How close to 2 must we hold x in order to be sure that $3x + 1$ lies within ε units of 7?*

3.3 Computing Limits: Graphically

In this section we look at an example to illustrate the concept of a limit *graphically*.

The graph of a function $f(x)$ is shown below. We analyze the behaviour of $f(x)$ around $x = -5$, $x = -2$, $x = -1$ and $x = 0$, and $x = 4$.



Observe that $f(x)$ is indeed a function (it passes the Vertical Line Test). We now analyze the function at each point separately.

$x=-5$: Observe that at $x=-5$ there is no closed circle, thus $f(-5)$ is undefined. From the graph we see that as x gets closer and closer to -5 from the left, then $f(x)$ approaches 2, so

$$\lim_{x\to-5^-} f(x)=2.$$

Similarly, as x gets closer and closer -5 from the right, then $f(x)$ approaches -3, so

$$\lim_{x\to-5^+} f(x)=-3.$$

As the right-hand limit and left-hand limit are not equal at -5, we know that

$$\lim_{x\to-5} f(x) \quad \text{does not exist.}$$

$x=-2$: Observe that at $x=-2$ there is a closed circle at 0, thus $f(-2)=0$. From the graph we see that as x gets closer and closer to -2 from the left, then $f(x)$ approaches 3.5, so

$$\lim_{x\to-2^-} f(x)=3.5.$$

Similarly, as x gets closer and closer -2 from the right, then $f(x)$ again approaches 3.5, so

$$\lim_{x\to-2^+} f(x)=3.5.$$

As the right-hand limit and left-hand limit are both equal to 3.5, we know that

$$\lim_{x\to-2} f(x)=3.5.$$

Do not be concerned that the limit does not equal 0. This is a discontinuity, which is completely valid, and will be discussed in a later section.

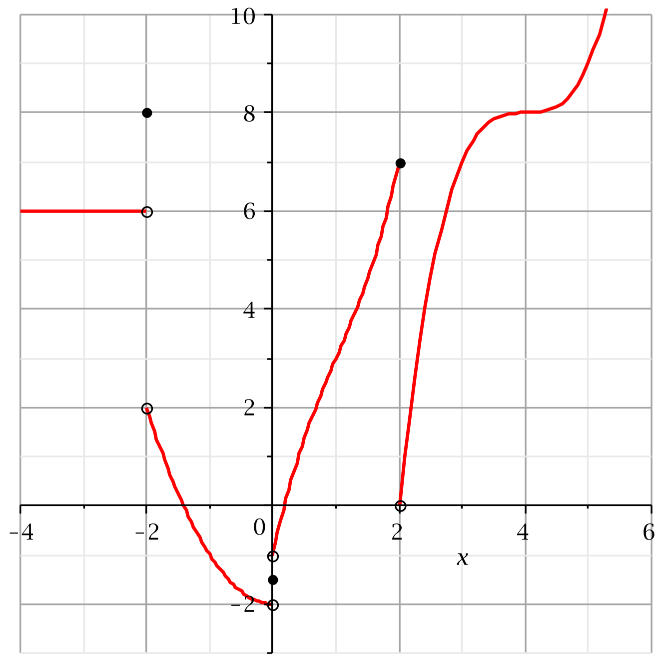
We leave it to the reader to analyze the behaviour of $f(x)$ for x close to -1 and 0.

Summarizing, we have:

$f(-5)$ *is undefined*	$f(-2)=0$	$f(-1)=-2$	$f(0)=-2$
$\lim_{x\to-5^-} f(x)=2$	$\lim_{x\to-2^-} f(x)=3.5$	$\lim_{x\to-1^-} f(x)=0$	$\lim_{x\to0^-} f(x)=-2$
$\lim_{x\to-5^+} f(x)=-3$	$\lim_{x\to-2^+} f(x)=3.5$	$\lim_{x\to-1^+} f(x)=-2$	$\lim_{x\to0^+} f(x)=-2$
$\lim_{x\to-5} f(x)=DNE$	$\lim_{x\to-2} f(x)=3.5$	$\lim_{x\to-1} f(x)=DNE$	$\lim_{x\to0} f(x)=-2$

Exercises for Section 3.3

Exercise 3.3.1 *Evaluate the expressions by reference to this graph:*



(a) $\lim_{x\to 4} f(x)$

(b) $\lim_{x\to -3} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to 0^-} f(x)$

(e) $\lim_{x\to 0^+} f(x)$

(f) $f(-2)$

(g) $\lim_{x\to 2^-} f(x)$

(h) $\lim_{x\to -2^-} f(x)$

(i) $\lim_{x\to 0} f(x+1)$

(j) $f(0)$

(k) $\lim_{x\to 1^-} f(x-4)$

(l) $\lim_{x\to 0^+} f(x-2)$

3.4 Computing Limits: Algebraically

Properties of limits

We begin by deriving a handful of theorems to give us the tools to compute many limits without explicitly working with the precise definition of a limit.

Theorem 3.9: Limit Properties

Suppose that $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *and* k *is some constant. Then*

- $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL$
- $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L+M$
- $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L-M$
- $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = LM$
- $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$, *if* M *is not 0*

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 3.10: Limit Properties

Compute $\lim_{x\to 1}\frac{x^2-3x+5}{x-2}$.

Solution. If we apply the theorem in all its gory detail, we get

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1}(x^2-3x+5)}{\lim_{x\to 1}(x-2)} \\
&= \frac{(\lim_{x\to 1} x^2)-(\lim_{x\to 1} 3x)+(\lim_{x\to 1} 5)}{(\lim_{x\to 1} x)-(\lim_{x\to 1} 2)} \\
&= \frac{(\lim_{x\to 1} x)^2-3(\lim_{x\to 1} x)+5}{(\lim_{x\to 1} x)-2} \\
&= \frac{1^2-3\cdot 1+5}{1-2} \\
&= \frac{1-3+5}{-1} = -3
\end{aligned}$$

♣

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. However, 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the values of the function (height of the graph) as x approaches 1.

We're primarily interested in limits that aren't so easy, namely, limits in which a denominator approaches zero. There are a handful of algebraic tricks that work on many of these limits.

We first look at two cases when the limit of the quotient $\dfrac{f(x)}{g(x)}$ as x approaches a takes on the **indeterminate form** $\dfrac{0}{0}$.

Example 3.11: Zero Denominator

Compute $\lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x-1}$.

Solution. We can't simply plug in $x = 1$ because that makes the denominator zero. However:

$$\begin{aligned}\lim_{x\to 1} \frac{x^2+2x-3}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} \\ &= \lim_{x\to 1}(x+3) = 4\end{aligned}$$

♣

The technique used to solve the previous example can be referred to as *factor and cancel*. Its validity comes from the fact that we are allowed to cancel $x-1$ from the numerator and denominator. Remember in Calculus that we have to make sure we don't cancel zeros, so we require $x-1 \neq 0$ in order to cancel it. But looking back at the definition of a limit using $x \to 1$, the key point for this example is that we are taking values of x close to 1 but *not* equal to 1. This is exactly what we wanted ($x \neq 1$) in order to cancel this common factor.

Example 3.12: Another zero denominator

Compute

$$\lim_{x\to 0} \frac{x}{x^3}$$

Solution. We can't simply plug in $x = 0$ because that makes the denominator zero. Let's simplify algebraically first to get

$$\lim_{x\to 0} \frac{x}{x^3} = \lim_{x\to 0} \frac{1}{x^2}.$$

As x approaches zero from either the left or right side, the value of x^2 approaches zero and so $\dfrac{1}{x^2}$ approaches infinity. In other words, $\lim_{x\to 0} \dfrac{x}{x^3}$ does not exist since it does not approach a real number. ♣

The last two examples show that limits of **indeterminate form** $\dfrac{0}{0}$ either exist or do not, which is why this type of limit is called indeterminate.

While Theorem 3.9 is very helpful, we need a bit more to work easily with limits. Since the theorem applies when some limits are already known, we need to know the behavior of some functions that cannot themselves be constructed from the simple arithmetic operations of the theorem, such as $\sqrt{x}$. Also, there is one other extraordinarily useful way to put functions together: composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x)=\sqrt{x}$ and $g(x)=x^2+5$, then $f(g(x))=\sqrt{x^2+5}$ and $g(f(x))=(\sqrt{x})^2+5=x+5$. Here is a companion to Theorem 3.9 for composition:

Theorem 3.13: Limit of Composition

Suppose that $\lim\limits_{x\to a} g(x)=L$ *and* $\lim\limits_{x\to L} f(x)=f(L)$. *Then*

$$\lim_{x\to a} f(g(x))=f(L).$$

Note the special form of the condition on f: it is not enough to know that $\lim\limits_{x\to L} f(x)=M$, though it is a bit tricky to see why. We have included an example in the exercise section to illustrate this tricky point for those who are interested. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

Theorem 3.14: Continuity of Roots

Suppose that n *is a positive integer. Then*

$$\lim_{x\to a} \sqrt[n]{x}=\sqrt[n]{a},$$

provided that a *is positive if* n *is even.*

This theorem is not too difficult to prove from the definition of limit.

Another of the most common algebraic tricks is called *rationalization*. Rationalizing makes use of the difference of squares formula $(a-b)(a+b)=a^2-b^2$. Here is an example.

Example 3.15: Rationalizing

Compute $\lim\limits_{x\to -1} \dfrac{\sqrt{x+5}-2}{x+1}$.

Solution.

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}\cdot\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\
&= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)}
\end{aligned}$$

$$= \lim_{x \to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}$$

At the very last step we have used Theorems 3.13 and 3.14. ♣

Example 3.16: Left and Right Limit

Evaluate $\lim_{x \to 0^+} \frac{x}{|x|}$.

Solution. The function $f(x) = x/|x|$ is undefined at 0; when $x > 0$, $|x| = x$ and so $f(x) = 1$; when $x < 0$, $|x| = -x$ and $f(x) = -1$. Thus

$$\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} -1 = -1$$

while

$$\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} 1 = 1.$$

The limit of $f(x)$ must be equal to both the left and right limits; since they are different, the limit $\lim_{x \to 0} \frac{x}{|x|}$ does not exist. ♣

Example 3.17: Limit of Piecewise-defined Function

Let

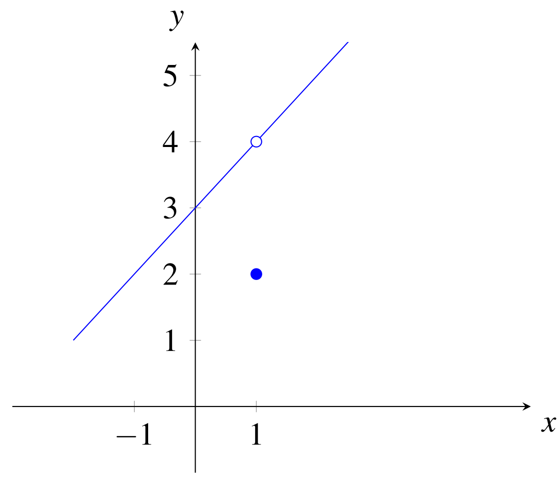
$$g(x) = \begin{cases} x+3 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

Evaluate $\lim_{x \to 1} g(x)$.

Solution. The domain of g is the set of all real numbers. From the graph shown below, we see that $g(x)$ can be made as close to 4 as we please by taking x sufficiently close to 1. Therefore,

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x+3) = 4$$

Observe that $g(1) = 2$, which does not equal the value of the limit of the function g as x approaches 1. The value of $g(x)$ at $x = 1$ has no bearing on the existence or value of the limit as g approaches 1.



♣

Exercises for Section 3.4

Exercise 3.4.1 *Compute the limits. If a limit does not exist, explain why.*

(a) $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

(b) $\lim_{x\to 1} \frac{x^2+x-12}{x-3}$

(c) $\lim_{x\to -4} \frac{x^2+x-12}{x-3}$

(d) $\lim_{x\to 2} \frac{x^2+x-12}{x-2}$

(e) $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$

(f) $\lim_{x\to 0^+} \sqrt{\frac{1}{x}+2}-\sqrt{\frac{1}{x}}$

(g) $\lim_{x\to 2} 3$

(h) $\lim_{x\to 4} 3x^3-5x$

(i) $\lim_{x\to 0} \frac{4x-5x^2}{x-1}$

(j) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(k) $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x}$

(l) $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x+1}$

(m) $\lim_{x\to a} \frac{x^3-a^3}{x-a}$

(n) $\lim_{x\to 2} (x^2+4)^3$

Exercise 3.4.2 *Let* $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ *and* $g(x) = 0$*. What are the values of* $L = \lim_{x\to 0} g(x)$ *and* $M = \lim_{x\to L} f(x)$*? Is it true that* $\lim_{x\to 0} f(g(x)) = M$*? What are some noteworthy differences between this example and Theorem 3.13?*

Exercise 3.4.3 *Sketch the graph of the given function f and evaluate $\lim_{x\to a} f(x)$, if it exists, for the given values of a.*

(a)

$$f(x) = \begin{cases} x+1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}, \ a = 0$$

(b)

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ -1 & \text{if } x = 1 \\ -x+2 & \text{if } x > 1 \end{cases}, \ a = 1$$

(c)

$$f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}, \ a = 0$$

Exercise 3.4.4 *Find the indicated limit given that*

$$\lim_{x\to a} p(x) = 3 \ \text{ and } \ \lim_{x\to a} q(x) = 4.$$

(a) $\lim_{x\to a} [p(x) - q(x)]$

(b) $\lim_{x\to a} \sqrt{q(x)}$

(c) $\lim_{x\to a} \left[\frac{2p(x) - q(x)}{p(x)q(x)}\right]$

Exercise 3.4.5 *Find the indicated limit, if it exists.*

(a) $\lim_{x\to 0} \frac{x^3 - x^2}{x^2}$

(b) $\lim_{x\to 1} \frac{x^2}{x(x-1)}$

(c) $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$

(d) $\lim_{x\to -1} \frac{x+1}{x^3+1}$

(e) $\lim_{x\to 1} \frac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}$

(f) $\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$

Exercise 3.4.6 *Find the indicated one-sided limit, if it exists.*

(a) $\lim_{x\to 1^-}(2x+4)$

(b) $\lim_{x\to 2^+}\frac{x-3}{x+2}$

(c) $\lim_{x\to 0^-}\frac{1}{x}$

(d) $\lim_{x\to 0^-}\frac{x-1}{x^2+1}$

(e) $\lim_{x\to 1^+}\frac{1+x}{1-x}$

(f) $\lim_{x\to 0^+}f(x)$ *and* $\lim_{x\to 0^-}f(x)$ *where*

$$f(x)=\begin{cases}2x & \text{if } x\geq 0\\ x^2 & \text{if } x<0\end{cases}$$

(g) $\lim_{x\to 0^+}f(x)$ *and* $\lim_{x\to 0^-}f(x)$ *where*

$$f(x)=\begin{cases}-x+1 & \text{if } x>0\\ 2x+3 & \text{if } x\leq 0\end{cases}$$

3.5 Limits at Infinity, Infinite Limits and Asymptotes

3.5.1. Limits at Infinity

We occasionally want to know what happens to some quantity when a variable gets very large or "goes to infinity".

Example 3.18: Limit at Infinity

What happens to the function $\cos(1/x)$ *as* x *goes to infinity? It seems clear that as* x *gets larger and larger,* $1/x$ *gets closer and closer to zero, so* $\cos(1/x)$ *should be getting closer and closer to* $\cos(0)=1$.

As with ordinary limits, this concept of "limit at infinity" can be made precise. Roughly, we want $\lim_{x\to\infty}f(x)=L$ to mean that we can make $f(x)$ as close as we want to L by making x large enough.

Definition 3.19: Limit at Infinity

In general, we write

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough. If this limits exists, we say that the function f has the limit L as x increases without bound.
Similarly, we write

$$\lim_{x\to-\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x to be negative and sufficiently large in absolute value. If this limit exists, we say that the function f has the limit L as x decreases without bound.

Example 3.20: Limit at Infinity

Let f and g be the functions

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases} \text{ and } g(x) = \frac{1}{x^2}$$

Evaluate:

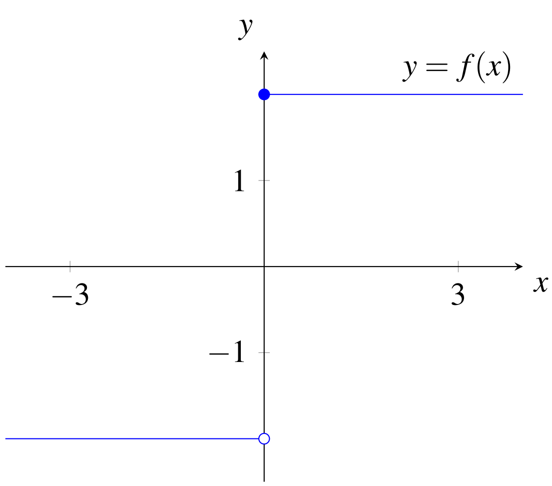
(a) $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$.

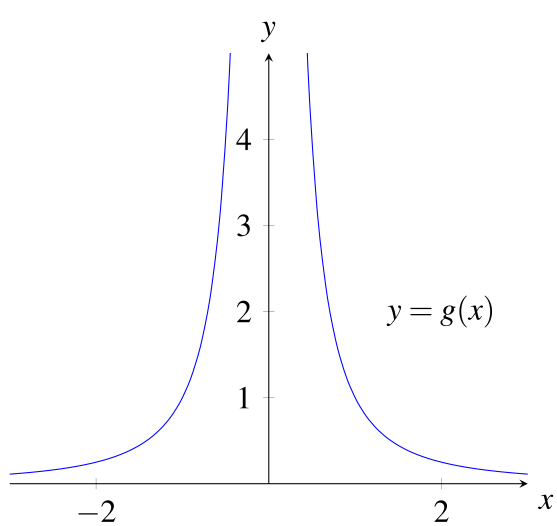
(b) $\lim_{x\to\infty} g(x)$ *and* $\lim_{x\to-\infty} g(x)$.

Solution. Referring to the graphs of $f(x)$ and $g(x)$ shown below, we see that

(a) $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = -2$.

(b) $\lim_{x\to\infty} g(x) = 0$ and $\lim_{x\to-\infty} g(x) = 0$.





♣

Definition 3.21: Limit at Infinity (Formal Definition)

If f is a function, we say that $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is an $N > 0$ so that whenever $x > N$, $|f(x) - L| < \varepsilon$. We may similarly define $\lim_{x\to-\infty} f(x) = L$.

We include this definition for completeness, but we will not explore it in detail. Suffice it to say that such limits behave in much the same way that ordinary limits do; in particular there is a direct analog of Theorem 3.9.

Example 3.22: Limit at Infinity

Compute $\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}$.

Solution. As x goes to infinity both the numerator and denominator go to infinity. We divide the numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{7}{x^2}}{1 + \frac{47}{x} + \frac{1}{x^2}}.$$

Now as x approaches infinity, all the quotients with some power of x in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2,

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$$

♣

In the previous example, we *divided by the highest power of x that occurs in the denominator* in order to evaluate the limit. We illustrate another technique similar to this.

Example 3.23: More Limits at Infinity

Compute the following limit:

$$\lim_{x\to\infty} \frac{2x^2 + 3}{5x^2 + x}.$$

Solution. As x becomes large, both the numerator and denominator become large, so it isn't clear what happens to their ratio. The highest power of x in the denominator is x^2, therefore we will divide every term in both the numerator and denominator by x^2 as follows:

$$\lim_{x\to\infty} \frac{2x^2 + 3}{5x^2 + x} = \lim_{x\to\infty} \frac{2 + 3/x^2}{5 + 1/x}.$$

We can also apply limit laws to infinite limits instead of arguing as we did in Example 3.23.

$$= \frac{\lim\limits_{x\to\infty} 2 + 3\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty}\frac{1}{x}} = \frac{2+3(0)}{5+0} = \frac{2}{5}.$$

Note that we used the theorem above to get that $\lim\limits_{x\to\infty}\frac{1}{x} = 0$ and $\lim\limits_{x\to\infty}\frac{1}{x^2} = 0$.

A shortcut technique is to analyze only the *leading terms* of the numerator and denominator. A leading term is a term that has the highest power of x. If there are multiple terms with the same exponent, you must include all of them.
Top: The leading term is $2x^2$.
Bottom: The leading term is $5x^2$.
Now only looking at leading terms and ignoring the other terms we get:

$$\lim_{x\to\infty}\frac{2x^2+3}{5x^2+x} = \lim_{x\to\infty}\frac{2x^2}{5x^2} = \frac{2}{5}.$$

♣

Example 3.24: Application of Limits at Infinity

A certain manufacturer makes a line of luxurious chairs. It is estimated that the total cost of making x luxurious chairs is

$$C(x) = 350x + 200{,}000$$

dollars per year. Thus, the average cost of making x chairs is given by

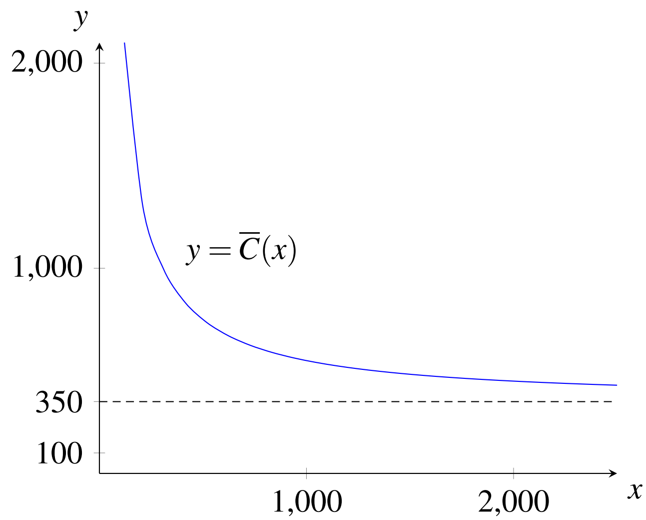
$$\overline{C(x)} = \frac{C(x)}{x} = \frac{350x + 200{,}000}{x} = 350 + \frac{200{,}000}{x}$$

dollars per chair. Evaluate $\lim\limits_{x\to\infty}\overline{C(x)}$ and interpret your results.

Solution.

$$\begin{aligned}\lim_{x\to\infty}\overline{C(x)} &= \lim_{x\to\infty}\left(350 + \frac{200{,}000}{x}\right)\\ &= \lim_{x\to\infty} 350 + \lim_{x\to\infty}\frac{200{,}000}{x}\\ &= 350\end{aligned}$$

A sketch of the graph of the function $\overline{C(x)}$ is shown below. The result we obtained is fully expected if we consider its economic implications. Note that as the level of production increases, the fixed cost per chair produced, represented by the term $\frac{200{,}000}{x}$, drops steadily. The average cost should approach a constant unit cost of production – \$350 in this case.



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3.5.2. Infinite Limits

We next look at functions whose limit at $x = a$ does not exist, but whose values increase or decrease without bound as x approaches a from the left or right.

Definition 3.25: Infinite Limit (Useable Definition)

In general, we write

$$\lim_{x \to a} f(x) = \infty$$

if we can make the value of $f(x)$ arbitrarily large by taking x to be sufficiently close to a (on either side of a) but not equal to a. Similarly, we write

$$\lim_{x \to a} f(x) = -\infty$$

if we can make the value of $f(x)$ arbitrarily large and negative by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Note:

1. We want to emphasize that by the proper definition of limits, the above limits do not exist, since they are not real numbers. However, writing $\pm\infty$ provides us with more information than simply writing DNE.

2. This definition can be modified for one-sided limits as well as limits with $x \to a$ replaced by $x \to \infty$ or $x \to -\infty$.

Example 3.26: Simple Infinite Limit

Compute the following limit: $\lim_{x\to 0} \frac{1}{x^2}$.

Solution. We refer to the graph below. Let's first look at the limit as $x \to 0^+$, and notice that $\frac{1}{x^2}$ increases without bound. Therefore,

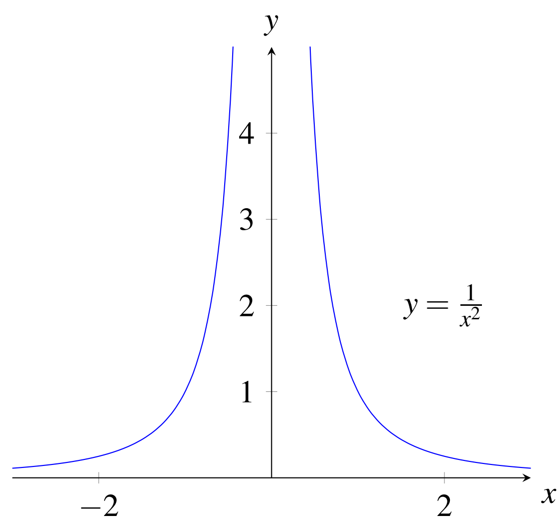
$$\lim_{x\to 0^+} \frac{1}{x^2} = +\infty.$$

As $x \to 0^-$, we again see that $\frac{1}{x^2}$ increases without bound:

$$\lim_{x\to 0^-} \frac{1}{x^2} = +\infty.$$

We conclude that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$



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Example 3.27: Infinite Limit

Compute the following limit: $\lim_{x\to\infty} (x^3 - x)$.

Solution. One might be tempted to write:

$$\lim_{x\to\infty} x^3 - \lim_{x\to\infty} x = \infty - \infty,$$

however, we do not know what $\infty - \infty$ is, as ∞ is not a real number and so cannot be treated like one. Incidentally, the expression $\infty - \infty$ is another indeterminate form.

We instead write:

$$\lim_{x\to\infty}(x^3-x)=\lim_{x\to\infty}x(x^2-1).$$

As x becomes arbitrarily large, then both x and x^2-1 become arbitrarily large, and hence their product $x(x^2-1)$ will also become arbitrarily large. Thus we see that

$$\lim_{x\to\infty}(x^3-x)=\infty.$$

♣

Example 3.28: More Infinite Limit

Let

$$f(x)=\frac{5x^3-3x^2+1}{x^2+2x+4}.$$

Evaluate

(a) $\lim_{x\to\infty} f(x)$

(b) $\lim_{x\to-\infty} f(x)$

Solution.

(a) Dividing the numerator and the denominator of the rational expression by x^2, we obtain

$$\lim_{x\to\infty}\frac{5x^3-3x^2+1}{x^2+2x+4}=\lim_{x\to\infty}\frac{5x-3+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{4}{x^2}}$$

Since the numerator becomes arbitrarily large whereas the denominator approaches 1 as x tends to infinity, we see that the quotient $f(x)$ gets larger and larger as x approaches infinity. In other words, the limit does not exist. We indicate this by writing

$$\lim_{x\to\infty}\frac{5x^3-3x^2+1}{x^2+2x+4}=\infty.$$

(b) Once again, dividing both the numerator and the denominator by x^2, we obtain

$$\lim_{x\to-\infty}\frac{5x^3-3x^2+1}{x^2+2x+4}=\lim_{x\to-\infty}\frac{5x-3+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{4}{x^2}}$$

In this case, the numerator becomes arbitrarily large in magnitude, bu negative in sign, whereas the denominator approaches 1 as x appraoches negative infinity. Therefore, the quotient $f(x)$ decreases without bound, and the limit does not exists. We indicate this by writing

$$\lim_{x\to-\infty}\frac{5x^3-3x^2+1}{x^2+2x+4}=-\infty.$$

♣

Example 3.29: Limit at Infinity, Infinite Limit and Basic Functions

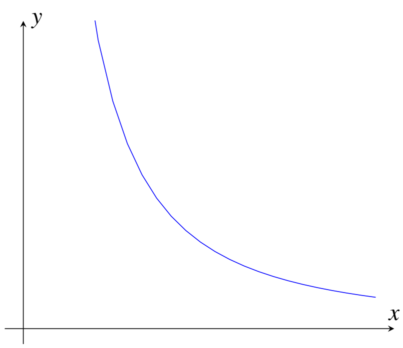
Find the following limits by observing the behaviour of the graph of each function.

(a) $\lim_{x\to\infty} \frac{6}{\sqrt{x^3}}$ *(b)* $\lim_{x\to-\infty} \left(x-x^2\right)$ *(c)* $\lim_{x\to\infty} \left(x^3+x\right)$

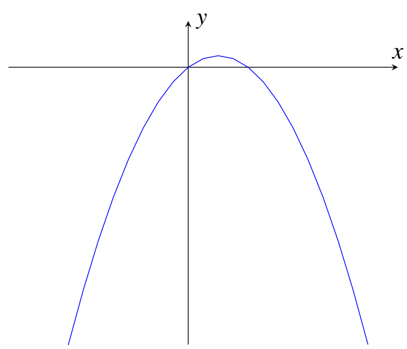
(d) $\lim_{x\to\infty} \cos(x)$ *(e)* $\lim_{x\to\infty} e^x$ *(f)* $\lim_{x\to-\infty} e^x$

(g) $\lim_{x\to 0^+} \ln x$ *(h)* $\lim_{x\to 0} \cos(1/x)$

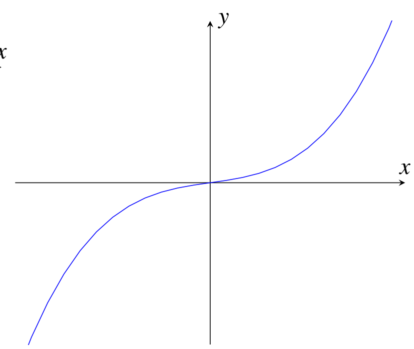
Solution. We can easily evaluate the following limits by observation:



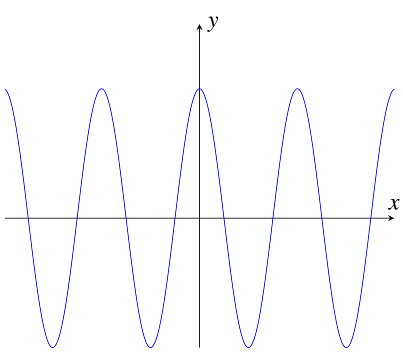
(a) $\lim_{x\to\infty} \frac{6}{\sqrt{x^3}} = 0$



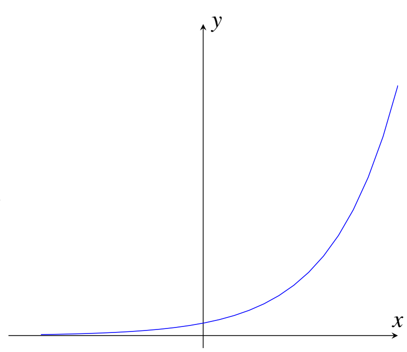
(b) $\lim_{x\to\infty} \left(x-x^2\right) = -\infty$



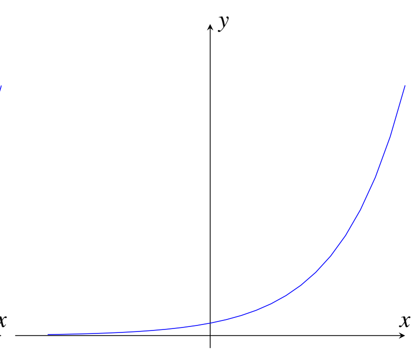
(c) $\lim_{x\to\infty} \left(x^3+x\right) = \infty$



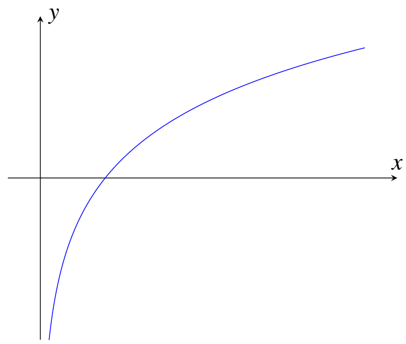
(d) $\lim_{x\to\infty} \cos x$ *DNE*



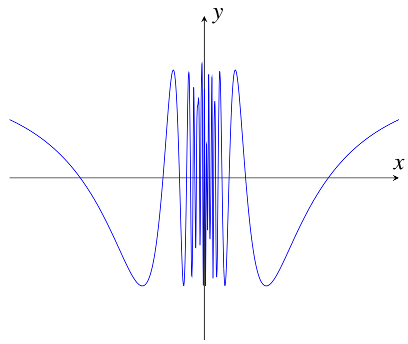
(e) $\lim_{x\to\infty} e^x = \infty$



(f) $\lim_{x\to-\infty} e^x = 0$



(g) $\lim_{x \to 0^+} \ln x = -\infty$



(h) $\lim_{x \to 0} \cos(1/x) = DNE$

♣

Note: Often, the shorthand notation $\frac{1}{0^+} = +\infty$ and $\frac{1}{0^-} = -\infty$ is used to represent the following two limits respectively:

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Making use of the results from Example 3.29 we can compute the following limits.

Example 3.30: More Limit at Infinity, Infinite Limit and Basic Functions

Compute $\lim_{x \to 0^+} e^{1/x}$, $\lim_{x \to 0^-} e^{1/x}$ *and* $\lim_{x \to 0} e^{1/x}$.

Solution. We have:

$$\lim_{x \to 0^+} e^{\frac{1}{x}} = e^{\frac{1}{0^+}} = e^{+\infty} = \infty.$$

$$\lim_{x \to 0^-} e^{\frac{1}{x}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0.$$

Thus, as left-hand limit $\neq$ right-hand limit,

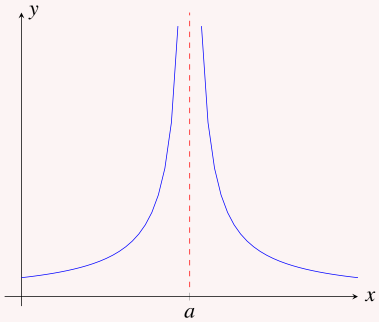
$$\lim_{x \to 0} e^{\frac{1}{x}} = \text{DNE}.$$

♣

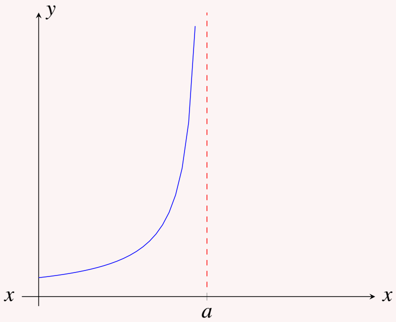
3.5.3. Vertical Asymptotes

Definition 3.31: Vertical Asymptote

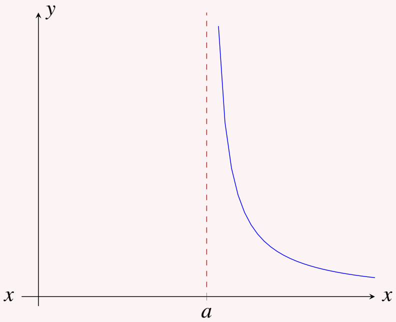
The line $x = a$ is called a **vertical asymptote** *of $f(x)$ if at least one of the following is true:*



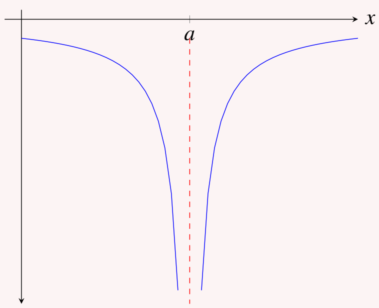
(a) $\lim_{x\to a} f(x) = \infty$



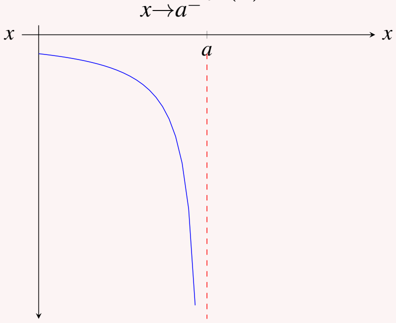
(b) $\lim_{x\to a^-} f(x) = \infty$



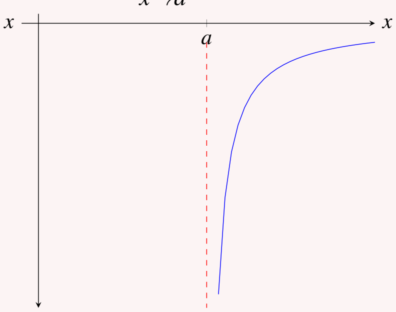
(c) $\lim_{x\to a^+} f(x) = \infty$



(d) $\lim_{x\to a} f(x) = -\infty$



(e) $\lim_{x\to a^-} f(x) = -\infty$



$\lim_{x\to a^+} f(x) = -\infty$

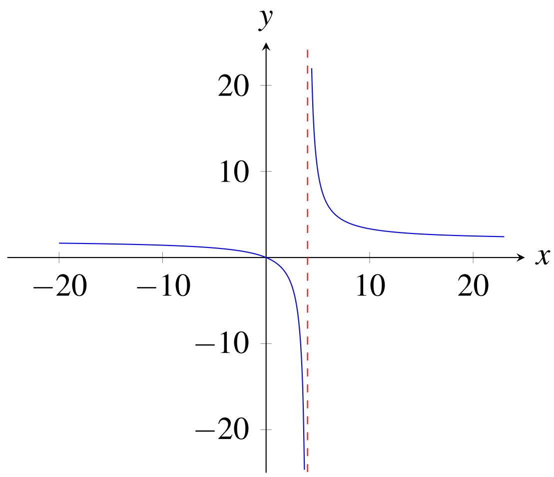
Example 3.32: Vertical Asymptotes

Find the vertical asymptotes of $f(x) = \dfrac{2x}{x-4}$.

Solution. In the definition of vertical asymptotes we need a certain limit to be $\pm\infty$. Candidates would be to consider values not in the domain of $f(x)$, such as $a = 4$. As x approaches 4 but is larger than 4 then $x-4$ is a small positive number and $2x$ is close to 8, so the quotient $2x/(x-4)$ is a large positive number. Thus we see that

$$\lim_{x\to 4^+} \frac{2x}{x-4} = \infty.$$

Thus, at least one of the conditions in the definition above is satisfied. Therefore $x = 4$ is a vertical asymptote, as shown below.



♣

3.5.4. Horizontal Asymptotes

Definition 3.33: Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** *of $f(x)$ if either*

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Example 3.34: Horizontal Asymptotes

Find the horizontal asymptotes of $f(x) = \dfrac{|x|}{x}$.

Solution. We must compute two infinite limits. First,

$$\lim_{x\to\infty} \frac{|x|}{x}.$$

Notice that for x arbitrarily large that $x > 0$, so that $|x| = x$. In particular, for x in the interval $(0,\infty)$ we have

$$\lim_{x\to\infty} \frac{|x|}{x} = \lim_{x\to\infty} \frac{x}{x} = 1.$$

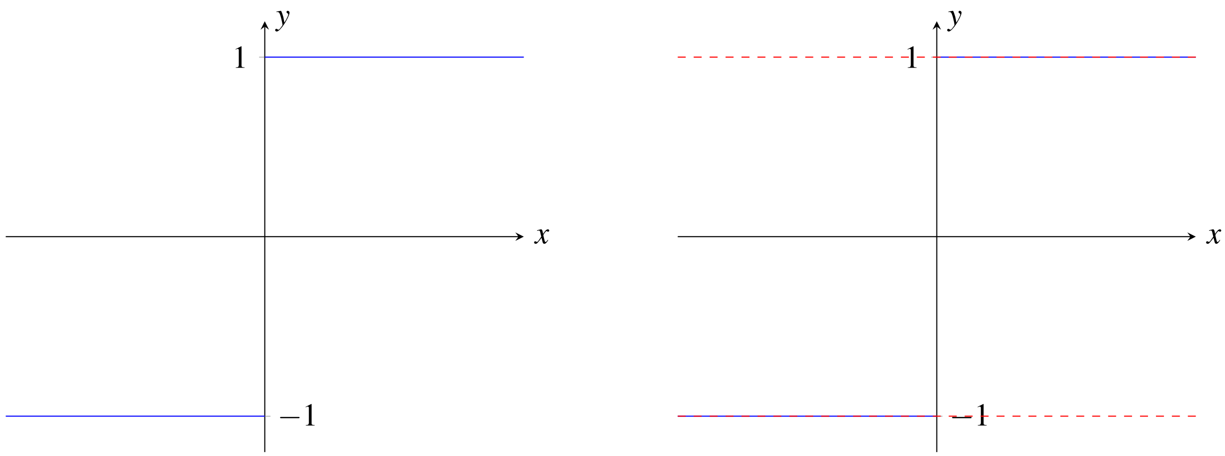
Second, we must compute

$$\lim_{x\to-\infty} \frac{|x|}{x}.$$

Notice that for x arbitrarily large negative that $x < 0$, so that $|x| = -x$. In particular, for x in the interval $(-\infty,0)$ we have

$$\lim_{x\to-\infty} \frac{|x|}{x} = \lim_{x\to-\infty} \frac{-x}{x} = -1.$$

Therefore there are two horizontal asymptotes, namely, $y = 1$ and $y = -1$, as shown to the right.



The graph of $y = f(x) = \frac{|x|}{x}$. *The horizontal asymptotes of* f *at* $y = 1$ *and* $y = -1$.

♣

3.5.5. Slant Asymptotes

Some functions may have slant (or *oblique*) asymptotes, which are neither vertical nor horizontal.

Definition 3.35: Slant Asymptote

The line $y = mx + b$ *is a* **slant asymptote** *of* $f(x)$ *if either*

$$\lim_{x\to\infty}[f(x) - (mx+b)] = 0 \quad \text{or}$$

$$\lim_{x\to-\infty}[f(x) - (mx+b)] = 0,$$

as shown in Figure 3.1.

Visually, the vertical distance between $f(x)$ and $y = mx + b$ is decreasing towards 0 and the curves do not intersect or cross at any point as x approaches infinity.

Example 3.36: Slant Asymptote in a Rational Function

Find the slant asymptotes of $f(x) = \dfrac{-3x^2+4}{x-1}$.

Solution. Note that this function has no horizontal asymptotes since $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.

In rational functions, slant asymptotes occur when the degree in the numerator is one greater than in the denominator. We use long division to rearrange the function:

$$\frac{-3x^2+4}{x-1} = -3x - 3 + \frac{1}{x-1}.$$

The part we're interested in is the resulting polynomial $-3x-3$. This is the line $y = mx+b$ we were seeking, where $m = -3$ and $b = -3$. Notice that

$$\lim_{x\to\infty} \frac{-3x^2+4}{x-1} - (-3x-3) = \lim_{x\to\infty} \frac{1}{x-1} = 0$$

and

$$\lim_{x\to-\infty} \frac{-3x^2+4}{x-1} - (-3x-3) = \lim_{x\to-\infty} \frac{1}{x-1} = 0.$$

Thus, $y = -3x-3$ is a slant asymptote of $f(x)$, as shown in Figure 3.1 below. ♣

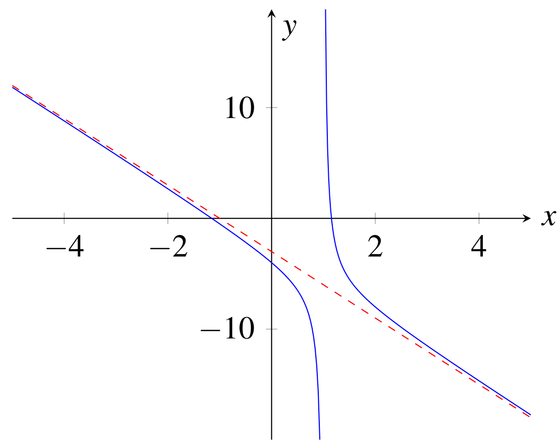


Figure 3.1: $y = f(x) = \frac{-3x^2+4}{x-1}$ **has slant asymptote** $y = -3x-3$**.**

Although rational functions are the most common type of function we encounter with slant asymptotes, there are other types of functions we can consider that present an interesting challenge.

Example 3.37: Slant Asymptote

Show that $y = 2x+4$ is a slant asymptote of $f(x) = 2x-3^x+4$.

Solution. This is because

$$\lim_{x\to-\infty} [f(x)-(2x+4)] = \lim_{x\to-\infty} (-3^x) = 0.$$

♣

We note that $\lim_{x\to\infty}[f(x)-(2x+4)] = \lim_{x\to\infty}(-3^x) = -\infty$. So, the vertical distance between $y = f(x)$ and the line $y = 2x+4$ decreases toward 0 only when $x \to -\infty$ and not when $x \to \infty$. The graph of f approaches the slant asymptote $y = 2x+4$ only at the far left and not at the far right. One might ask if $y = f(x)$ approaches a slant asymptote when $x \to \infty$. The answer turns out to be no, but we will need to know something about the relative growth rates of the exponential functions and linear functions in order to prove this. Specifically, one can prove that when the base is greater than 1 the exponential functions grows faster than any power function as $x \to \infty$. This can be phrased like this: For any $a > 1$ and any $n > 0$,

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \infty \text{ and } \lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

These facts are most easily proved with the aim of something called the L'Hôpital's Rule.

3.5.6. End Behaviour and Comparative Growth Rates

Let us now look at the last two subsections and go deeper. In the last two subsections we looked at horizontal and slant asymptotes. Both are special cases of the end behaviour of functions, and both concern situations where the graph of a function approaches a straight line as $x \to \infty$ or $-\infty$. But not all functions have this kind of end behaviour. For example, $f(x) = x^2$ and $f(x) = x^3$ do not approach a straight line as $x \to \infty$ or $-\infty$. The best we can say with the notion of limit developed at this stage are that

$$\begin{aligned}\lim_{x\to\infty} x^2 &= \infty, \ \lim_{x\to-\infty} x^2 = \infty, \\ \lim_{x\to\infty} x^3 &= \infty, \ \lim_{x\to-\infty} x^3 = -\infty.\end{aligned}$$

Similarly, we can describe the end behaviour of transcendental functions such as $f(x) = e^x$ using limits, and in this case, the graph approaches a line as $x \to -\infty$ but not as $x \to \infty$.

$$\lim_{x\to-\infty} e^x = 0, \ \lim_{x\to\infty} e^x = \infty.$$

People have found it useful to make a finer distinction between these end behaviours all thus far captured by the symbols ∞ and $-\infty$. Specifically, we will see that the above functions have different growth rates at infinity. Some increases to infinity faster than others. Specifically,

Definition 3.38: Comparative Growth Rates

Suppose that f and g are two functions such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. We say that $f(x)$ grows faster than $g(x)$ as $x \to \infty$ if the following holds:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

or equivalently,

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

Here are a few obvious examples:

Example 3.39: Growth Rate in Polynomial Functions

Show that if $m > n$ are two positive integers, then $f(x) = x^m$ grows faster than $g(x) = x^n$ as $x \to \infty$.

Solution. Since $m > n$, $m - n$ is a positive integer. Therefore,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty.$$

♣

Example 3.40: Growth Rate in Polynomial Functions

Show that if $m > n$ are two positive integers, then any monic polynomial $P_m(x)$ of degree m grows faster than any monic polynomial $P_n(x)$ of degree n as $x \to \infty$. [Recall that a polynomial is monic if its leading coefficient is 1.]

Solution. By assumption, $P_m(x) = x^m +$ terms of degrees less than $m = x^m + a_{m-1}x^{m-1} + \ldots$, and $P_n(x) = x^n +$ terms of degrees less than $n = x^n + b_{n-1}x^{n-1} + \ldots$. Dividing the numerator and denominator by x^n, we get

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{m-n} + a_{m-1}x^{m-n-1} + \ldots}{1 + \frac{b_{n-1}}{x} + \ldots} = \lim_{x\to\infty} x^{m-n}\left(\frac{1 + \frac{a_{m-1}}{x} + \ldots}{1 + \frac{b_{n-1}}{x} + \ldots}\right) = \infty,$$

since the limit of the bracketed fraction is 1 and the limit of x^{m-n} is ∞, as we showed in Example 3.39.

♣

Example 3.41: Growth Rate in Polynomial Functions

Show that a polynomial grows exactly as fast as its highest degree term as $x \to \infty$ or $-\infty$. That is, if $P(x)$ is any polynomial and $Q(x)$ is its highest degree term, then both limits

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} \text{ and } \lim_{x\to-\infty} \frac{P(x)}{Q(x)}$$

are finite and nonzero.

Solution. Suppose that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$. Then the highest degree term is $Q(x) = a_n x^n$. So,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) = a_n \neq 0.$$

♣

Let's state a theorem we mentioned when we discussed the last example in the last subsection:

Theorem 3.42: Growth Comparison between Exponential and Power Functions

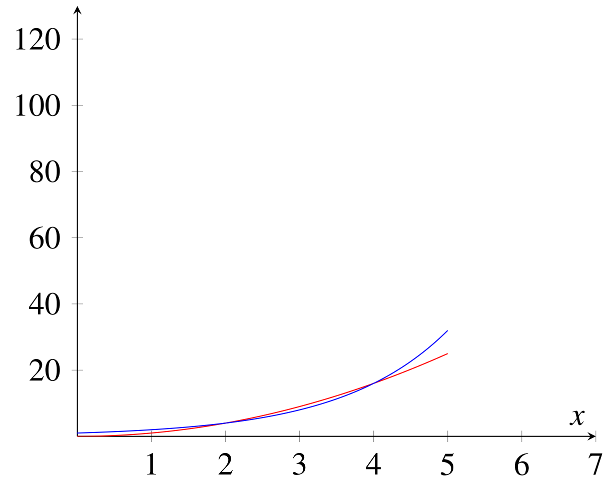
Let a and n be positive real numbers with $a > 1$. Then $f(x) = a^x$ grows faster than $g(x) = x^n$ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

In particular,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

The easiest way to prove this is to use the L'Hôpital's Rule, which we will introduce in a later chapter. For now, one can plot and compare the graphs of an exponential function and a power function. Here is a comparison between $f(x) = x^2$ and $g(x) = 2^x$:



Notice also that as $x \to -\infty$, x^n grows in size but e^x does not. More specifically, $x^n \to \infty$ or $-\infty$ according as n is even or odd, while $e^x \to 0$. So, it is meaningless to compare their "growth" rates, although we can still calculate the limit

$$\lim_{x \to -\infty} \frac{e^x}{x^n} = 0.$$

Let's see an application of our theorem.

Example 3.43: Horizontal Asymptotes

Find the horizontal asymptote(s) of $f(x) = \dfrac{x^3 + 2e^x}{e^x - 4x^2}$.

Solution. To find horizontal asymptotes, we calculate the limits of $f(x)$ as $x \to \infty$ and $x \to -\infty$. For $x \to \infty$, we divide the numerator and the denominator by e^x, and then we take limit to get

$$\lim_{x \to \infty} \frac{x^3 + 2e^x}{e^x - 4x^2} = \lim_{x \to \infty} \frac{\frac{x^3}{e^x} + 2}{1 - 4\frac{x^2}{e^x}} = \frac{0 + 2}{1 - 4(0)} = 2.$$

For $x \to -\infty$, we divide the numerator and the denominator by x^2 to get

$$\lim_{x \to -\infty} \frac{x^3 + 2e^x}{e^x - 4x^2} = \lim_{x \to -\infty} \frac{x + 2\frac{e^x}{x^2}}{\frac{e^x}{x^2} - 4}.$$

The denominator now approaches $0 - 4 = -4$. The numerator has limit $-\infty$. So, the quotient has limit ∞:

$$\lim_{x \to -\infty} \frac{x + 2\frac{e^x}{x^2}}{\frac{e^x}{x^2} - 4} = \infty.$$

So, $y = 2$ is a horizontal asymptote. The function $y = f(x)$ approaches the line $y = 2$ as $x \to \infty$. And this is the only horizontal asymptote, since the function $y = f(x)$ does not approach any horizontal line as $x \to -\infty$. ♣

Since the growth rate of a polynomial is the same as that of its leading term, the following is obvious:

Example 3.44: Growth Rate in Polynomial and Exponential Functions

If $P(x)$ is any polynomial, then

$$\lim_{x\to\infty}\frac{P(x)}{e^x}=0.$$

Also, if r is any real number, then we can place it between two consecutive integers n and $n+1$. For example, $\sqrt{3}$ is between 1 and 2, e is between 2 and 3, and π is between 3 and 4. Then the following is totally within our expectation:

Example 3.45: Growth Rate in Exponential Functions

Prove that if $a>1$ is any basis and $r>0$ is any exponent, then $f(x)=a^x$ grows faster than $g(x)=x^r$ as $x\to\infty$.

Solution. Let r be between consecutive integers n and $n+1$. Then for all $x>1$, $x^n\le x^r\le x^{n+1}$. Dividing by a^x, we get

$$\frac{x^n}{a^x}\le\frac{x^r}{a^x}\le\frac{x^{n+1}}{a^x}.$$

Since

$$\lim_{x\to\infty}\frac{x^n}{a^x}=0.$$

♣

What about exponential functions with different bases? We recall from the graphs of the exponential functions that for any base $a>1$,

$$\lim_{x\to\infty}a^x=\infty.$$

So, the exponential functions with bases greater than 1 all grow to infinity as $x\to\infty$. How do their growth rates compare?

Theorem 3.46: More Growth Comparison between Exponential Functions

If $1<a<b$, then $f(x)=b^x$ grows faster than $g(x)=a^x$ as $x\to\infty$.

Proof. Proof. Since $a<b$, we have $\frac{b}{a}>1$. So,

$$\lim_{x\to\infty}\frac{b^x}{a^x}=\lim_{x\to\infty}\left(\frac{b}{a}\right)^x=\infty.$$

♣

Another function that grows to infinity as $x\to\infty$ is $g(x)=\ln x$. Recall that the natural logarithmic function is the inverse of the exponential function $y=e^x$. Since e^x grows very fast as x increases, we should expect $\ln x$ to grow very slowly as x increases. The same applies to logarithmic functions with any basis $a>1$. This is the content of the next theorem.

Theorem 3.47: Growth Comparison between Logarithmic and Power Functions

Let a and n be any positive real numbers with $a > 1$. Then $f(x) = x^n$ grows faster than $g(x) = \log_a x$ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{x^n}{\log_a x} = \infty \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\log_a x}{x^n} = 0.$$

In particular,

$$\lim_{x\to\infty} \frac{x^n}{\ln x} = \infty \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\ln x}{x^n} = 0.$$

Proof.

1. We use a change of variable. Letting $t = \ln x$, then $x = e^t$. So, $x \to \infty$ if and only if $t \to \infty$, and

$$\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{t\to\infty} \frac{t}{(e^t)^r} = \lim_{t\to\infty} \frac{t}{(e^r)^t}.$$

   Now, since $r > 0$, $a = e^r > 1$. So, a^t grows as t increases, and it grows faster than t as $t \to \infty$. Therefore,

$$\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{t\to\infty} \frac{t}{(e^r)^t} = \lim_{t\to\infty} \frac{t}{a^t} = 0.$$

2. The change of base identity $\log_a x = \dfrac{\ln x}{\ln a}$ implies that $\log_a x$ is simply a constant multple of $\ln x$. The result now follows from (a).

♣

Exercises for Section 3.5

Exercise 3.5.1 *Compute the following limits.*

(a) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$

(b) $\displaystyle\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}$

(c) $\displaystyle\lim_{t\to 1^+} \frac{(1/t)-1}{t^2-2t+1}$

(d) $\displaystyle\lim_{t\to\infty} \frac{t+5-2/t-1/t^3}{3t+12-1/t^2}$

(e) $\displaystyle\lim_{y\to\infty} \frac{\sqrt{y+1}+\sqrt{y-1}}{y}$

(f) $\displaystyle\lim_{x\to 0^+} \frac{3+x^{-1/2}+x^{-1}}{2+4x^{-1/2}}$

(g) $\displaystyle\lim_{x\to\infty} \frac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$

(h) $\displaystyle\lim_{t\to\infty} \frac{1-\sqrt{\frac{t}{t+1}}}{2-\sqrt{\frac{4t+1}{t+2}}}$

(i) $\displaystyle\lim_{t\to\infty} \frac{1-\frac{t}{t-1}}{1-\sqrt{\frac{t}{t-1}}}$

(j) $\displaystyle\lim_{x\to -\infty} \frac{x+x^{-1}}{1+\sqrt{1-x}}$

(k) $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{x}}{x-1}$

(l) $\displaystyle\lim_{x\to\infty} \frac{x^{-1}+x^{-1/2}}{x+x^{-1/2}}$

(m) $\displaystyle\lim_{x\to\infty} \frac{x+x^{-2}}{2x+x^{-2}}$

(n) $\displaystyle\lim_{x\to\infty} \frac{5+x^{-1}}{1+2x^{-1}}$

(o) $\lim_{x\to\infty} \frac{4x}{\sqrt{2x^2+1}}$

(p) $\lim_{x\to\infty} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(q) $\lim_{x\to 0^+} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(r) $\lim_{x\to 2} \frac{x^3-6x-2}{x^3-4x}$

(s) $\lim_{x\to -\infty} \frac{3x^3+x^2+1}{x^3+1}$

(t) $\lim_{x\to -\infty} \frac{x^4+1}{x^3-1}$

(u) $\lim_{x\to\infty} \frac{x^5-x^3+x-1}{x^6+2x^2+1}$

Exercise 3.5.2 *The function* $f(x)=\frac{x}{\sqrt{x^2+1}}$ *has two horizontal asymptotes. Find them and give a rough sketch of* f *with its horizontal asymptotes.*

Exercise 3.5.3 *Find the vertical asymptotes of* $f(x)=\frac{\ln x}{x-2}$.

Exercise 3.5.4 *Suppose that a falling object reaches velocity* $v(t)=50(1-e^{-t/5})$ *at time* t*, where distance is measured in m and time s. What is the object's terminal velocity, i.e. the value of* $v(t)$ *as* t *goes to infinity?*

Exercise 3.5.5 *Find the slant asymptote of* $f(x)=\frac{x^2+x+6}{x-3}$.

Exercise 3.5.6 *Compute the following limits.*

(a) $\lim_{x\to -\infty} (2x^3-x)$

(b) $\lim_{x\to\infty} \tan^{-1}(e^x)$

(c) $\lim_{x\to -\infty} \tan^{-1}(e^x)$

(d) $\lim_{x\to\infty} \frac{e^x+x^4}{x^3+5\ln x}$

(e) $\lim_{x\to\infty} \frac{2^x+5(3^x)}{3(2^x)-3^x}$

(f) $\lim_{x\to -\infty} \frac{2^x+5(3^x)}{3(2^x)-3^x}$

Exercise 3.5.7 *Due to the mining of some oil sands, a river flowing nearby is found to be contaminated with bitumen, a dense and extremely viscous form of petroleum. Since this river is part of the water supply to a large farm region downstream from it, a proposal submitted to the farm council indicates that the cost, measured in millions of dollars, of removing x% of the toxic pollutant is given by*

$$C(x)=\frac{0.5x}{100-x} \quad (0<x<100)$$

(a) *Find the cost of removing* 50%, 60%, 70%, 80%, 90% *and* 95% *of the pollutant.*

(b) *Evaluate*

$$\lim_{x\to 100}\frac{0.5x}{100-x}$$

and interpret your results.

Exercise 3.5.8 *The total sales from a Pulitzer-prized book is approximated by the function*

$$T(x)=\frac{120x^2}{x^2+4}$$

where $T(x)$ *is measured in millions of dollars and* x *is the number of months since the book's release.*

(a) *What are the sales after the first month? The second month? the third month?*

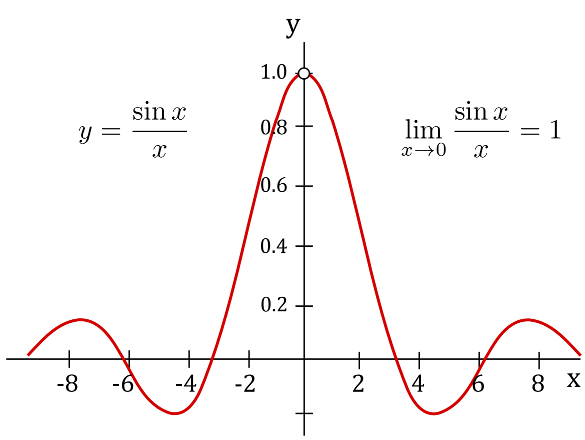
(b) *What will the book gross in the long run?*

3.6 The Squeeze Theorem

In this section we aim to compute the limit:

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

We start by analyzing the graph of $y=\frac{\sin x}{x}$:



Notice that $x=0$ is not in the domain of this function. Nevertheless, we can look at the limit as x approaches 0. From the graph we find that the limit is 1 (there is an open circle at $x=0$ indicating 0 is not in the domain). We just convinced you this limit formula holds true based on the graph, but how does one attempt to prove this limit more formally? To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the **Squeeze Theorem**.

Theorem 3.48: Squeeze Theorem

Suppose that $g(x) \le f(x) \le h(x)$ *for all* x *close to* a *but not equal to* a*. If* $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$*, then* $\lim_{x\to a} f(x) = L$*.*

This theorem can be proved using the official definition of limit. We won't prove it here, but point out that it is easy to understand and believe graphically. The condition says that $f(x)$ is trapped between $g(x)$ below and $h(x)$ above, and that at $x = a$, both g and h approach the same value. This means the situation looks something like Figure 3.2.

For example, imagine the blue curve is $f(x) = x^2 \sin(\pi/x)$, the upper (red) and lower (green) curves are $h(x) = x^2$ and $g(x) = -x^2$. Since the sine function is always between -1 and 1, $-x^2 \le x^2 \sin(\pi/x) \le x^2$, and it is easy to see that $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$. It is not so easy to see directly (i.e. algebraically) that $\lim_{x\to 0} x^2 \sin(\pi/x) = 0$, because the π/x prevents us from simply plugging in $x = 0$. The Squeeze Theorem makes this "hard limit" as easy as the trivial limits involving x^2.

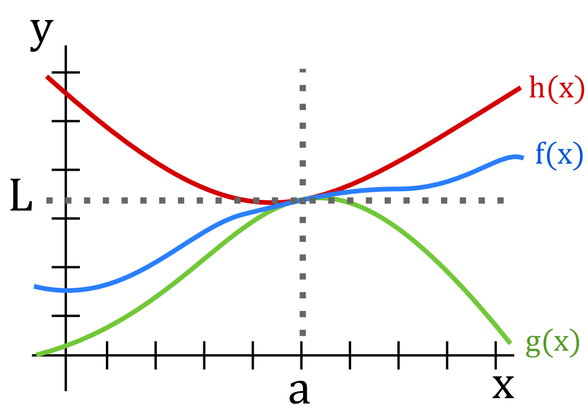


Figure 3.2: The Squeeze Theorem.

To compute $\lim_{x\to 0}(\sin x)/x$, we will find two simpler functions g and h so that $g(x) \le (\sin x)/x \le h(x)$, and so that $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x)$. Not too surprisingly, this will require some trigonometry and geometry. Referring to Figure 3.3, x is the measure of the angle in radians. Since the circle has radius 1, the coordinates of point A are $(\cos x, \sin x)$, and the area of the small triangle is $(\cos x \sin x)/2$. This triangle is completely contained within the circular wedge-shaped region bordered by two lines and the circle from $(1,0)$ to point A. Comparing the areas of the triangle and the wedge we see $(\cos x \sin x)/2 \le x/2$, since the area of a circular region with angle θ and radius r is $\theta r^2/2$. With a little algebra this turns into $(\sin x)/x \le 1/\cos x$, giving us the h we seek.

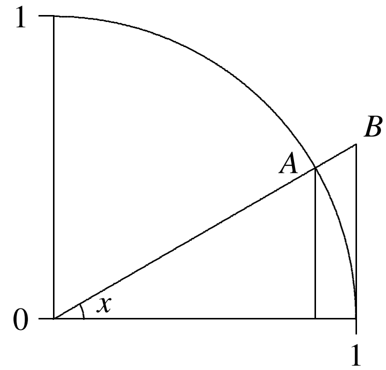


Figure 3.3: Visualizing $\sin x/x$**.**

To find g, we note that the circular wedge is completely contained inside the larger triangle. The height of the triangle, from $(1,0)$ to point B, is $\tan x$, so comparing areas we get $x/2 \le (\tan x)/2 = \sin x/(2\cos x)$. With a little algebra this becomes $\cos x \le (\sin x)/x$. So now we have

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$

Finally, the two limits $\lim_{x\to 0}\cos x$ and $\lim_{x\to 0} 1/\cos x$ are easy, because $\cos(0) = 1$. By the Squeeze Theorem, $\lim_{x\to 0}(\sin x)/x = 1$ as well.

Using the above, we can compute a similar limit:

$$\lim_{x\to 0}\frac{\cos x - 1}{x}.$$

This limit is just as hard as $\sin x/x$, but closely related to it, so that we don't have to do a similar calculation; instead we can do a bit of tricky algebra.

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{x}\frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = -\frac{\sin x}{x}\frac{\sin x}{\cos x + 1}.$$

To compute the desired limit it is sufficient to compute the limits of the two final fractions, as x goes to 0. The first of these is the hard limit we've just done, namely 1. The second turns out to be simple, because the denominator presents no problem:

$$\lim_{x\to 0}\frac{\sin x}{\cos x + 1} = \frac{\sin 0}{\cos 0 + 1} = \frac{0}{2} = 0.$$

Thus,

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Theorem 3.49: Two Special Trig Limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \textit{and} \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Example 3.50: Limit of Other Trig Functions

Compute the following limit $\lim_{x\to 0} \frac{\sin 5x \cos x}{x}$.

Solution. We have

$$\begin{aligned} \lim_{x\to 0} \frac{\sin 5x \cos x}{x} &= \lim_{x\to 0} \frac{5 \sin 5x \cos x}{5x} \\ &= \lim_{x\to 0} 5 \cos x \left(\frac{\sin 5x}{5x} \right) \\ &= 5 \cdot (1) \cdot (1) = 5 \end{aligned}$$

since $\cos(0) = 1$ and $\lim_{x\to 0} \frac{\sin 5x}{5x} = 1$. ♣

Let's do a harder one now.

Example 3.51: Limit of Other Trig Functions

Compute the following limit: $\lim_{x\to 0} \frac{\tan^3 2x}{x^2 \sin 7x}$.

Solution. Recall that the $\tan^3(2x)$ means that $\tan(2x)$ is being raised to the third power.

$$\begin{aligned} \lim_{x\to 0} \frac{\tan^3(2x)}{x^2 \sin(7x)} &= \lim_{x\to 0} \frac{(\sin(2x))^3}{x^2 \sin(7x) \cos^3(2x)} && \text{Rewrite in terms of sin and cos} \\ &= \lim_{x\to 0} \frac{(2x)^3 \left(\frac{\sin(2x)}{2x}\right)^3}{x^2 (7x) \left(\frac{\sin(7x)}{7x}\right) \cos^3(2x)} && \text{Make sine terms look like: } \frac{\sin\theta}{\theta} \\ &= \lim_{x\to 0} \frac{8x^3 (1)^3}{7x^3 (1)(1^3)} && \text{Replace } \lim_{x\to 0} \frac{\sin nx}{nx} \text{ with 1. Also, } \cos(0) = 1. \\ &= \lim_{x\to 0} \frac{8}{7} && \text{Cancel } x^3\text{'s.} \\ &= \frac{8}{7}. \end{aligned}$$

♣

Example 3.52: Applying the Squeeze Theorem

Compute the following limit: $\lim_{x\to 0^+} x^3 \cos\left(\frac{1}{\sqrt{x}}\right)$.

Solution. We use the Squeeze Theorem to evaluate this limit. We know that $\cos\alpha$ satisfies $-1 \le \cos\alpha \le 1$ for any choice of α. Therefore we can write:

$$-1 \le \cos\left(\frac{1}{\sqrt{x}}\right) \le 1$$

Since $x \to 0^+$ implies $x > 0$, multiplying by x^3 gives:

$$-x^3 \le x^3 \cos\left(\frac{1}{\sqrt{x}}\right) \le x^3.$$

$$\lim_{x\to 0^+}(-x^3) \le \lim_{x\to 0^+}\left(x^3\cos\left(\frac{1}{\sqrt{x}}\right)\right) \le \lim_{x\to 0^+} x^3.$$

But using our rules we know that

$$\lim_{x\to 0^+}(-x^3) = 0, \qquad \lim_{x\to 0^+} x^3 = 0$$

and the Squeeze Theorem says that the only way this can happen is if

$$\lim_{x\to 0^+} x^3\cos\left(\frac{1}{\sqrt{x}}\right) = 0.$$

♣

When solving problems using the Squeeze Theorem it is also helpful to have the following theorem.

Theorem 3.53: Monotone Limits

If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

Exercises for Section 3.6

Exercise 3.6.1 *Compute the following limits.*

(a) $\lim_{x\to 0} \frac{\sin(5x)}{x}$

(b) $\lim_{x\to 0} \frac{\sin(7x)}{\sin(2x)}$

(c) $\lim_{x\to 0} \frac{\cot(4x)}{\csc(3x)}$

(d) $\lim_{x\to 0} \frac{\tan x}{x}$

(e) $\lim_{x\to\pi/4} \dfrac{\sin x - \cos x}{\cos(2x)}$

Exercise 3.6.2 *For all* $x \geq 0$, $4x-9 \leq f(x) \leq x^2 - 4x + 7$. *Find* $\lim_{x\to 4} f(x)$.

Exercise 3.6.3 *For all* x, $2x \leq g(x) \leq x^4 - x^2 + 2$. *Find* $\lim_{x\to 1} g(x)$.

Exercise 3.6.4 *Use the Squeeze Theorem to show that* $\lim_{x\to 0} x^4 \cos(2/x) = 0$.

Exercise 3.6.5 *Find the value of* $\lim_{x\to\infty} \dfrac{3x+\sin x}{x+\cos x}$. *Justify your steps carefully.*

3.7 Continuity and IVT

3.7.1. Continuity

The graph shown in Figure 3.4(a) represents a **continuous** function. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near $x = 1$ on the graph in Figure 3.4(b) which is not continuous at that location.

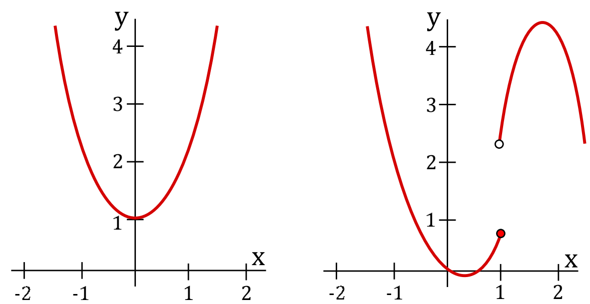


Figure 3.4: (a) A continuous function. (b) A function with a discontinuity at $x = 1$.

Definition 3.54: Continuous at a Point

A function f *is* **continuous at a point** a *if*

$$\lim_{x\to a} f(x) = f(a).$$

Some readers may prefer to think of continuity at a point as a three part definition. We provide the following guideline for determining the continuity of a function.

Guideline for Checking Continuity at a Point

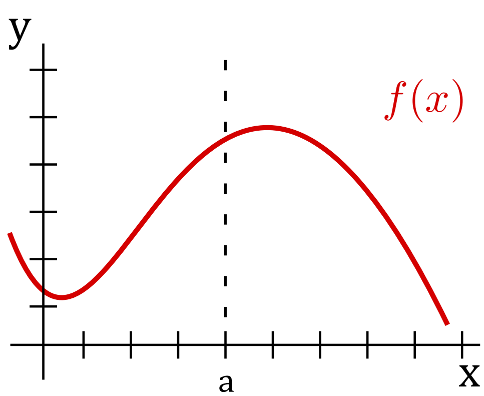
A function $f(x)$ is continuous at $x = a$ if the following three conditions hold:

(i) $f(a)$ is defined (that is, a belongs to the domain of f),

(ii) $\lim_{x\to a} f(x)$ exists (that is, left-hand limit = right-hand limit),

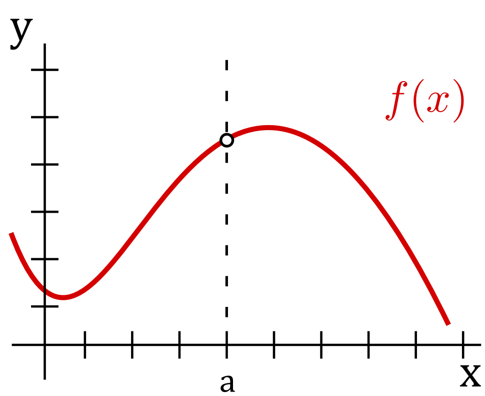
(iii) $\lim_{x\to a} f(x) = f(a)$ (that is, the numbers from (i) and (ii) are equal).

The figures below show graphical examples of functions where either (i), (ii) or (iii) can fail to hold.



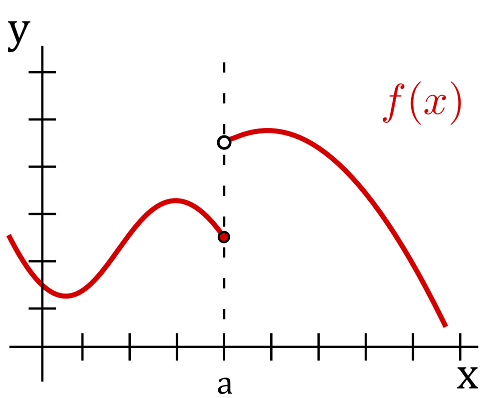
continuous at x = a

$\left(\lim_{x\to a} f(x) = f(a)\right)$



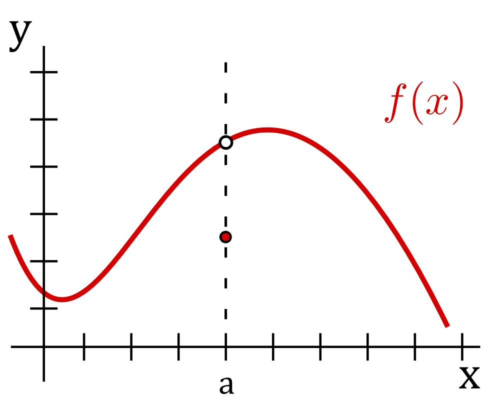
f(a) not defined

(i) fails to hold



$\lim_{x\to a} f(x)$ does not exist

(ii) fails to hold



$\lim_{x\to a} f(x) \neq f(a)$

(iii) fails to hold

On the other hand, if f is defined on an open interval containing a, except perhaps at a, we can say that f is **discontinuous** at a if f is not continuous at a.

Graphically, you can think of continuity as being able to draw your function without having to lift your pencil off the paper. If your pencil has to jump off the page to continue drawing the function, then the function is not continuous at that point. This is illustrated in Figure 3.4(b) where if we tried to draw the function (from left to right) we need to lift our pencil off the page once we reach the point $x = 1$ in order to be able to continue drawing the function.

Definition 3.55: Continuity on an Open Interval

A function f is **continuous on an open interval** (a,b) *if it is continuous at every point in the interval.*

Furthermore, a function is **everywhere continuous** *if it is continuous on the entire real number line* $(-\infty,\infty)$..

Recall the function graphed in a previous section as shown in Figure 3.5.

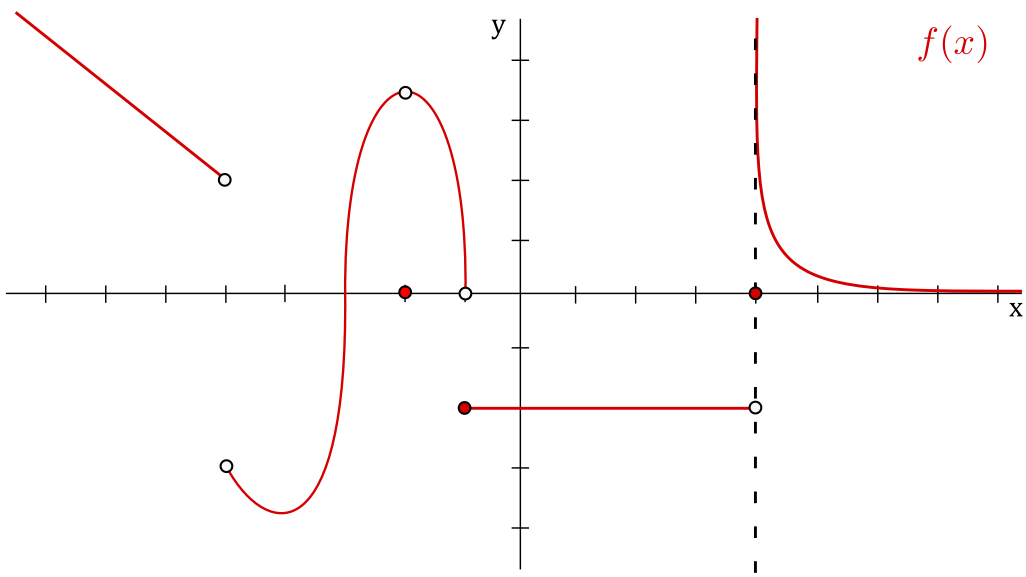
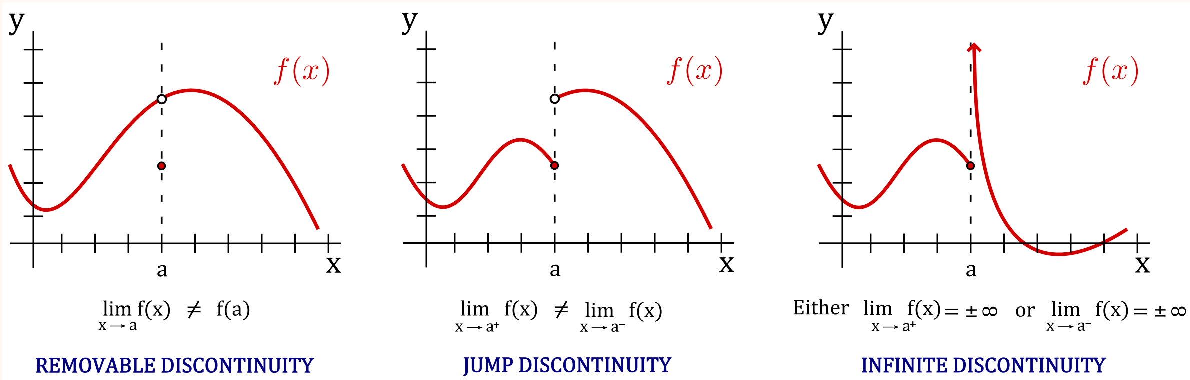


Figure 3.5: A function with discontinuities at $x = -5$, $x = -2$, $x = -1$ and $x = 4$.

We can draw this function without lifting our pencil *except* at the points $x = -5$, $x = -2$, $x = -1$, and $x = 4$. Thus, $f(x)$ is *continuous* at every real number *except* at these four numbers. At $x = -5$, $x = -2$, $x = -1$, and $x = 4$, the function $f(x)$ is *discontinuous*.

At $x = -2$ we have a **removable discontinuity** because we could remove this discontinuity simply by redefining $f(-2)$ to be 3.5. Formally, we say $f(x)$ has a **removable discontinuity** at $x = a$ if $\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$. Note that we do not require $f(a)$ to be defined in this case, that is, a need not belong to the domain of $f(x)$. At $x = -5$ and $x = -1$ we have **jump discontinuities** because the function jumps from one value to another. From the right of $x = 4$, we have an **infinite discontinuity** because the function goes off to infinity. These types of discontinuities are summarized below.

Summary of Discontinuities



Example 3.56: Continuous at a Point

What value of c will make the following function $f(x)$ continuous at 2?

$$f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ c & \text{if } x = 2 \end{cases}$$

Solution. In order to be continuous at 2 we require

$$\lim_{x\to 2} f(x) = f(2)$$

to hold. We use the three part definition listed previously to check this.

1. First, $f(2) = c$, and c is some real number. Thus, $f(2)$ is defined.

2. Now, we must evaluate the limit. Rather than computing both one-sided limits, we just compute the limit directly. For x close to 2 (but not equal to 2) we can replace $f(x)$ with $\frac{x^2-x-2}{x-2}$ to get:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3.$$

Therefore the limit exists and equals 3.

3. Finally, for f to be continuous at 2, we need that the numbers in the first two items to be equal. Therefore, we require $c = 3$. Thus, when $c = 3$, $f(x)$ is continuous at 2, for any other value of c, $f(x)$ is discontinuous at 2. ♣

For continuity on a closed interval, we consider the **one-sided limits** of a function. Recall that $x \to a^-$ means x approaches a from values less than a.

Definition 3.57: Continuous from the Right and from the Left

A function f is **left continuous at a point a** *if*

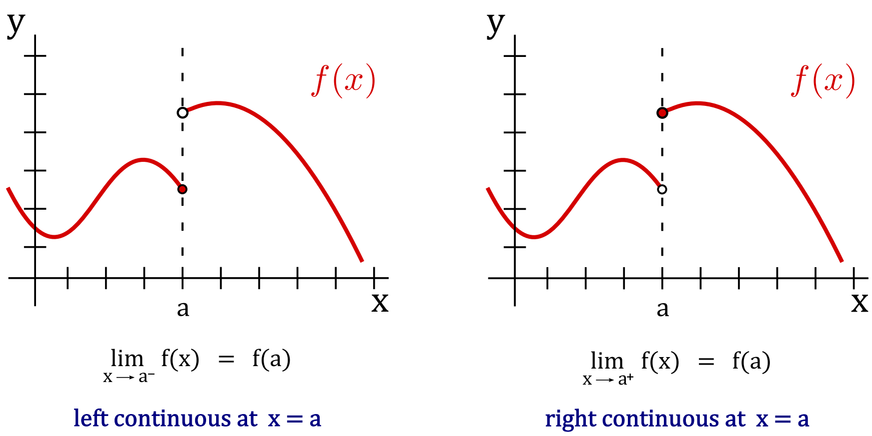
$$\lim_{x\to a^-} f(x) = f(a)$$

and **right continuous at a point a** *if*

$$\lim_{x\to a^+} f(x) = f(a).$$

If a function f is continuous at a, then it is both left and right continuous at a.

The above definition regarding left (or right) continuous functions is illustrated with the following figure:



One-sided limits allows us to extend the definition of continuity to closed intervals. The following definition means a function is continuous on a closed interval if it is continuous in the interior of the interval and possesses the appropriate one-sided continuity at the endpoints of the interval.

Definition 3.58: Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a,b]$ *if:*

(i) it is continuous on the open interval (a,b);

(ii) it is left continuous at point a:

$$\lim_{x\to a^-} f(x) = f(a);$$

and

(iii) it is right continuous at point b:

$$\lim_{x\to b^+} f(x) = f(b).$$

This definition can be extended to continuity on half-open intervals such as $(a,b]$ and $[a,b)$, and unbounded intervals.

Example 3.59: Continuity on Other Intervals

The function $f(x) = \sqrt{x}$ is continuous on the (closed) interval $[0, \infty)$.

The function $f(x) = \sqrt{4-x}$ is continuous on the (closed) interval $(-\infty, 4]$.

The continuity of functions is preserved under the operations of addition, subtraction, multiplication and division (in the case that the function in the denominator is nonzero).

Theorem 3.60: Operations of Continuous Functions

If f and g are continous at a, and c is a constant, then the following functions are also continuous at a:

(i) $f \pm g$;

(ii) cf;

(iii) fg;

(iv) f/g *(provided* $g(a) \neq 0$*).*

Below we list some common functions that are known to be continuous on every interval inside their domains.

Example 3.61: Common Types of Continuous Functions

- *Polynomials (for all x), e.g., $y = mx + b$, $y = ax^2 + bx + c$.*
- *Rational functions (except at points x which gives division by zero).*
- *Root functions $\sqrt[n]{x}$ (for all x if n is odd, and for $x \geq 0$ if n is even).*
- *Trigonometric functions*
- *Inverse trigonometric functions*
- *Exponential funtions*
- *Logarithmic functions*

For rational functions with removable discontinuities as a result of a zero, we can define a new function filling in these gaps to create a piecewise function that is continuous everywhere.

Continuous functions are where the *direct substitution property* hold. This fact can often be used to compute the limit of a continuous function.

Example 3.62: Evaluate a Limit

Evaluate the following limit: $\displaystyle\lim_{x \to \pi} \frac{\sqrt{x} + \sin x}{1 + x + \cos x}$.

Solution. We will use a continuity argument to justify that direct substitution can be applied. By the list above, $\sqrt{x}$, $\sin x$, 1, x and $\cos x$ are all continuous functions at π. Then $\sqrt{x} + \sin x$ and $1 + x + \cos x$ are both continuous at π. Finally,

$$\frac{\sqrt{x}+\sin x}{1+x+\cos x}$$

is a continuous function at π since $1+\pi+\cos\pi \neq 0$. Hence, we can directly substitute to get the limit:

$$\lim_{x\to\pi}\frac{\sqrt{x}+\sin x}{1+x+\cos x}=\frac{\sqrt{\pi}+\sin\pi}{1+\pi+\cos\pi}=\frac{\sqrt{\pi}}{\pi}=\frac{1}{\sqrt{\pi}}.$$

♣

Continuity is also preserved under the composition of functions.

Theorem 3.63: Continuity of Function Composition

If g is continuous at a and f is continuous at $g(a)$, then the composition function $f\circ g$ is continuous at a.

Example 3.64: Continuity with Composition of Functions

Determine where the following functions is continuous:

(a) $h(x)=\cos(x^2)$

(b) $H(x)=\ln(1+\sin(x))$

Solution.

(a) The functions that make up the composition $h(x)=f(g(x))$ are $g(x)=x^2$ and $f(x)=\cos(x)$. The function g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Therefore, $h(x)=(f\circ g)(x)$ is continuous on $\mathbb{R}$ by Theorem 3.63.

(b) We know from Example 3.61 that $f(x)=\ln x$ is continuous and $g(x)=1+\sin x$ are continuous. Thus by Theorem 3.63, $H(x)=f(g(x))$ is continuous wherever it is defined. Now $\ln(1+\sin x)$ is defined when $1+\sin x>0$. Recall that $-1\leq\sin x\leq 1$, so $1+\sin x>0$ except when $\sin x=-1$, which happens when $x=\pm 3\pi/2,\pm 7\pi/2,\ldots$. Therefore, H has discontinuities when $x=3\pi n/2$, $n=1,2,3,\ldots$ and is continuous on the intervals between these values.

♣

3.7.2. The Intermediate Value Theorem

Whether or not an equation *has* a solution is an important question in mathematics. Consider the following two questions:

Example 3.65: Motivation for the Intermediate Value Theorem

(a) Does $e^x + x^2 = 0$ have a solution?

(b) Does $e^x + x = 0$ have a solution?

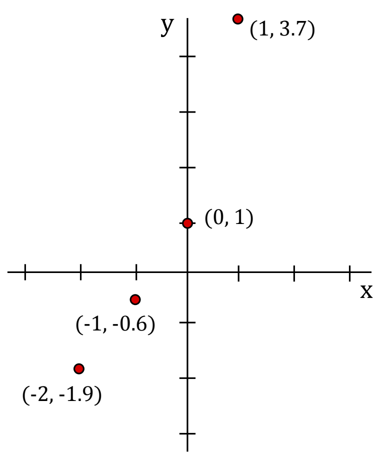
Solution.

(a) The first question is easy to answer since for any exponential function we know that $a^x > 0$, and we also know that whenever you square a number you get a nonnegative answer: $x^2 \geq 0$. Hence, $e^x + x^2 > 0$, and thus, is never equal to zero. Therefore, the first equation has no solution.

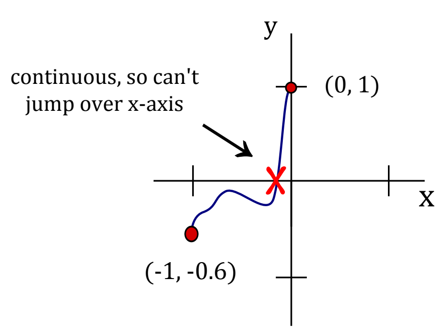
(b) For the second question, it is difficult to see if $e^x + x = 0$ has a solution. If we tried to solve for x, we would run into problems. Let's make a table of values to see what kind of values we get (recall that $e \approx 2.7183$):

x	-2	-1	0	1
$e^x + x$	$e^{-2} - 2 \approx -1.9$	$e^{-1} - 1 \approx 1$	$e^0 + 0 = 1$	$e + 1 \approx 3.7$

Sketching this gives:



Let $f(x) = e^x + x$. Notice that if we choose $a = -1$ and $b = 0$ then we have $f(a) < 0$ and $f(b) > 0$. A point where the function $f(x)$ crosses the x-axis gives a solution to $e^x + x = 0$. Since $f(x) = e^x + x$ is continuous (both e^x and x are continuous), then the function *must* cross the x-axis somewhere between -1 and 0:



Therefore, our equation has a solution.

Note that by looking at smaller and smaller intervals (a, b) with $f(a) < 0$ and $f(b) > 0$, we can get a better and better approximation for a solution to $e^x + x = 0$. For example, taking the interval

$(-0.4, -0.6)$ gives $f(-0.4) < 0$ and $f(-0.6) > 0$, thus, there is a solution to $f(x) = 0$ between -0.4 and -0.6. It turns out that the solution to $e^x + x = 0$ is $x \approx -0.56714$.

♣

We now generalize the argument used in the previous example. In that example we had a continuous function that went from negative to positive and hence, had to cross the x-axis at some point. In fact, we don't need to use the x-axis, any line $y = N$ will work so long as the function is continuous and below the line $y = N$ at some point and above the line $y = N$ at another point. This is known as the Intermediate Values Theorem and it is formally stated as follows:

Theorem 3.66: Intermediate Value Theorem

If f is continuous on the interval $[a,b]$ and N is between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there is a number c in (a,b) such that $f(c) = N$.

The Intermediate Value Theorem guarantees that if $f(x)$ is continuous and $f(a) < N < f(b)$, the line $y = N$ intersects the function at some point $x = c$. Such a number c is between a and b and has the property that $f(c) = N$ (see Figure 3.6(a)). We can also think of the theorem as saying if we draw the line $y = N$ between the lines $y = f(a)$ and $y = f(b)$, then the function cannot jump over the line $y = N$. On the other hand, if $f(x)$ is *not* continuous, then the theorem may *not* hold. See Figure 3.6(b) where there is no number c in (a,b) such that $f(c) = N$. Finally, we remark that there may be multiple choices for c (i.e., lots of numbers between a and b with y-coordinate N). See Figure 3.6(c) for such an example.

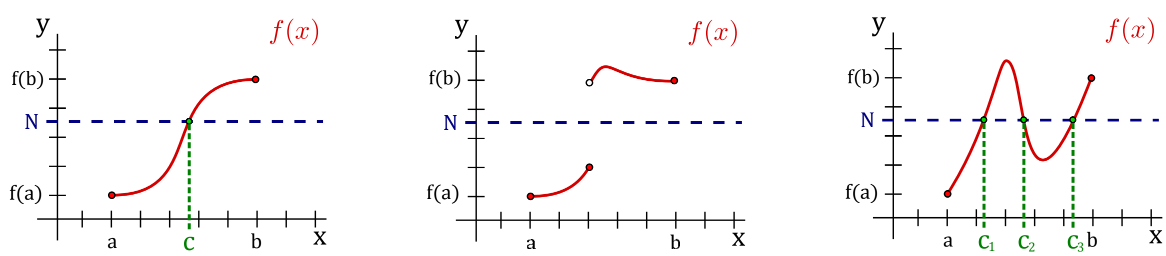


Figure 3.6: (a) A continuous function where IVT holds for a single value c. (b) A discontinuous function where IVT fails to hold. (c) A continuous function where IVT holds for multiple values in (a,b).

The Intermediate Value Theorem is most frequently used for $N = 0$.

Example 3.67: Intermediate Value Theorem

Show that there is a solution of $\sqrt[3]{x} + x = 1$ in the interval $(0,8)$.

Solution. Let $f(x) = \sqrt[3]{x} + x - 1$, $N = 0$, $a = 0$, and $b = 8$. Since $\sqrt[3]{x}$, x and -1 are continuous on $\mathbb{R}$, and the sum of continuous functions is again continuous, we have that $f(x)$ is continuous on $\mathbb{R}$, thus in particular, $f(x)$ is continuous on $[0,8]$. We have $f(a) = f(0) = \sqrt[3]{0} + 0 - 1 = -1$ and $f(b) = f(8) = \sqrt[3]{8} + 8 - 1 = 9$. Thus $N = 0$ lies between $f(a) = -1$ and $f(b) = 9$, so the conditions of the Intermediate Value Theorem are satisfied. So, there exists a number c in $(0,8)$ such that $f(c)=0$. This means that c satisfies $\sqrt[3]{c}+c-1=0$, in otherwords, is a solution for the equation given.

Alternatively we can let $f(x)=\sqrt[3]{x}+x$, $N=1$, $a=0$ and $b=8$. Then as before $f(x)$ is the sum of two continuous functions, so is also continuous everywhere, in particular, continuous on the interval $[0,8]$. We have $f(a)=f(0)=\sqrt[3]{0}+0=0$ and $f(b)=f(8)=\sqrt[3]{8}+8=10$. Thus $N=1$ lies between $f(a)=0$ and $f(b)=10$, so the conditions of the Intermediate Value Theorem are satisfied. So, there exists a number c in $(0,8)$ such that $f(c)=1$. This means that c satisfies $\sqrt[3]{c}+c=1$, in otherwords, is a solution for the equation given. ♣

Example 3.68: Roots of Function

Explain why the function $f=x^3+3x^2+x-2$ has a root between 0 and 1.

Solution. By theorem 3.9, f is continuous. Since $f(0)=-2$ and $f(1)=3$, and 0 is between -2 and 3, there is a $c\in(0,1)$ such that $f(c)=0$. ♣

This example also points the way to a simple method for approximating roots.

Example 3.69: Approximating Roots

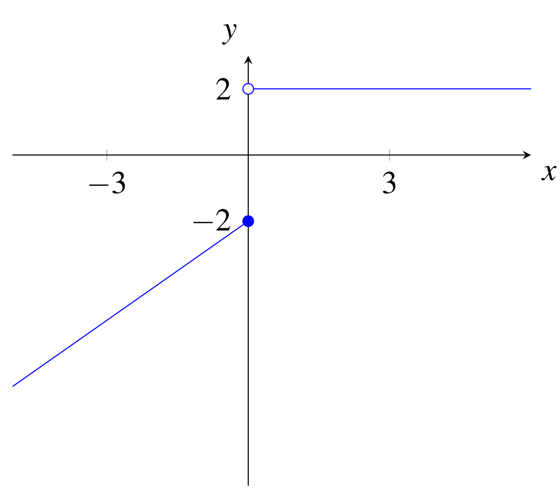
Approximate the root of the previous example to one decimal place.

Solution. If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6)<0$ and $f(0.7)>0$, so by the Intermediate Value Theorem, f has a root between 0.6 and 0.7. Repeating the process with $f(0.61)$, $f(0.62)$, and so on, we find that $f(0.61)<0$ and $f(0.62)>0$, so f has a root between 0.61 and 0.62, and the root is 0.6 rounded to one decimal place. ♣

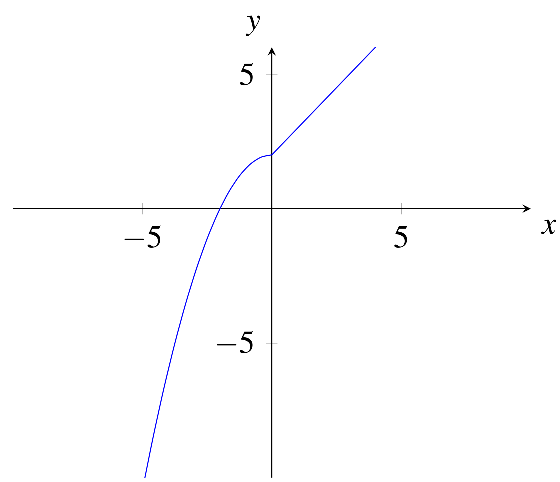
Exercises for Section 3.7

Exercise 3.7.1 *Determine the values of x, if any, at which each function is discontinuous. At each point of discontinuity, state the condition(s) for continuity which are violated.*

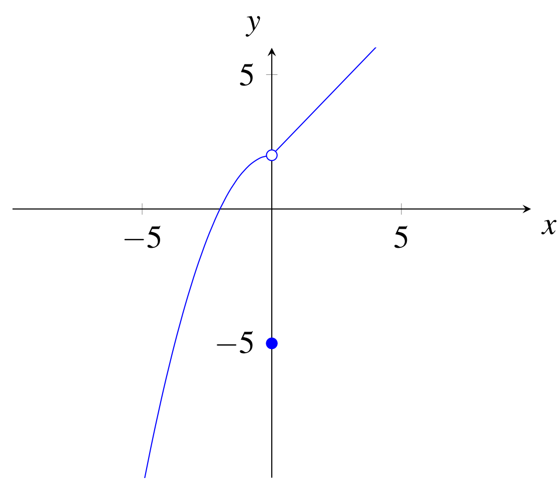
(a) $$f(x)=\begin{cases} x-2 & x\le 0 \\ 2 & x>0 \end{cases}$$



(b) $f(x) = \begin{cases} x+2 & x>0 \\ -\frac{1}{2}x^2+2 & x \leq 0 \end{cases}$



(c) $f(x) = \begin{cases} x+2 & x>0 \\ -5 & x=0 \\ -\frac{1}{2}x^2+2 & x>0 \end{cases}$



Exercise 3.7.2 *Determine the values of x for which each function is continuous.*

(a) $f(s) = \dfrac{2}{s^2+1}$

(b) $g(t) = \dfrac{2t+1}{t^2+t-2}$

(c)

$$h(u) = \begin{cases} \dfrac{u^2-1}{u-1} & u \neq 1 \\ 2 & u=1 \end{cases}$$

Exercise 3.7.3 *Consider the function*

$$h(x) = \begin{cases} 2x-3, & \text{if } x<1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x=0$. Is h a continuous function?

Exercise 3.7.4 *Find the values of a that make the function $f(x)$ continuous for all real numbers.*

$$f(x)=\begin{cases} 4x+5, & \text{if } x\geq -2, \\ x^2+a, & \text{if } x<-2. \end{cases}$$

Exercise 3.7.5 *Find the values of the constant c so that the function $g(x)$ is continuous on $(-\infty,\infty)$, where*

$$g(x)=\begin{cases} 2-2c^2x, & \text{if } x<-1, \\ 6-7cx^2, & \text{if } x\geq -1. \end{cases}$$

Exercise 3.7.6 *A data plan at an internet café charges \$1.00 for the first minute and \$0.50 for each additional minute or part thereof, subject to a maximum of \$3.00. Derive a function f relating the data charges to the length of time x spent at the café. Sketch the graph of f and determine the values of x for which the function is discontinuous.*

Exercise 3.7.7 *Approximate a root of $f=x^3-4x^2+2x+2$ to one decimal place.*

Exercise 3.7.8 *Approximate a root of $f=x^4+x^3-5x+1$ to one decimal place.*

Exercise 3.7.9 *Show that the equation $\sqrt[3]{x}+x=1$ has a solution in the interval $(0,8)$.*

4. Derivatives

4.1 The Rate of Change of a Function

Before we embark on setting the groundwork for the derivative of a function, let's review some terminology and concepts. Remember that the slope of a line is defined as the quotient of the difference in y-values and the difference in x-values. Recall from Section 1.2 that a difference between two quantities is often denoted by the Greek symbol Δ - read "delta" as shown next, where **delta notation** is being used when calculating and interpreting the slope of a line.

Calculating and Interpreting the Slope of a Line

Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line of a linear function $y = f(x)$. Then the slope of the line is calculated by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

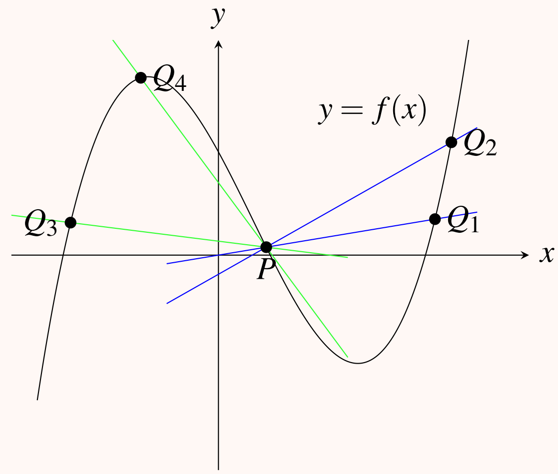
We can interpret this equation by saying that the slope m measures the change in y per unit change in x. In other words, the slope m provides a **measure of sensitivity** .

For example, if $y = 100x + 5$, a small change in x corresponds to a change one hundred times as large in y, so y is quite sensitive to changes in x.

Next, we introduce the properties of two special lines, the **tangent line** and the **secant line**, which are pertinent for the understanding of a derivative.

Secant Line

Secant is a Latin word meaning *to cut*, and in mathematics a **secant line** cuts an arbitrary curve described by $y = f(x)$ through two points P and Q. The figure shows two such secant lines of the curve f to the right and to the left of the point P, respectively.



Since by necessity the secant line goes through two points on the curve of $y = f(x)$, we can readily calculate the slope of this secant line.

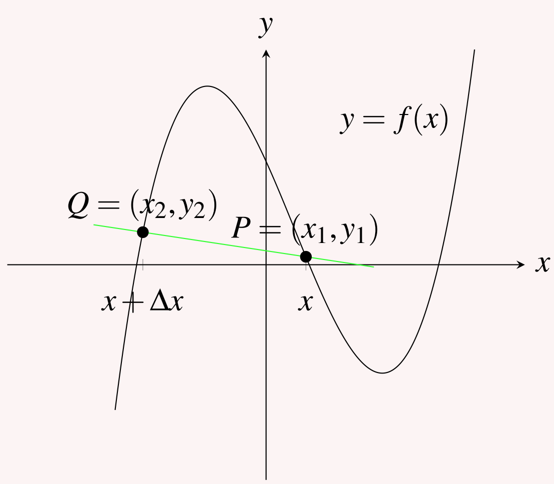
Definition 4.1: Slope of Secant Line - Average Rate of Change

Suppose we are given two points (x_1, y_1) *and* (x_2, y_2) *on the secant line of the curve described by the function* $y = f(x)$ *as shown. Then the slope of the secant line is calculated by*

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Note that we may also be given the change in x *directly as* Δx*, i.e the two points are given as* $(x, f(x))$ *and* $(x + \Delta x, f(x + \Delta x))$*, and so*

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

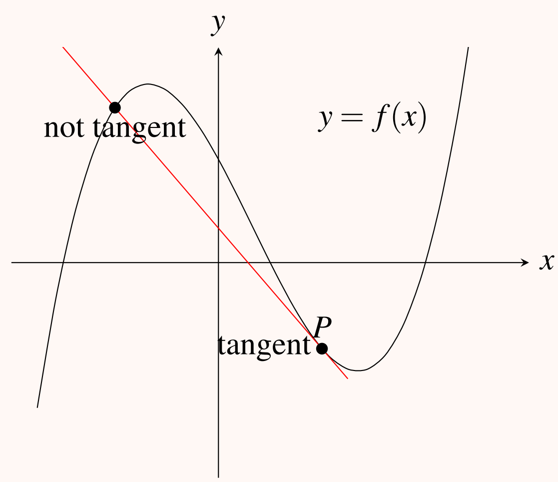


Note:

1. In the above figure, the value of Δx must be negative since it is on the left side of x.
2. The slope of the secant line is also referred to as the **average rate of change** of f over the interval $[x, x+\Delta x]$.
3. The expression $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ is referred to as the **difference quotient**.

Tangent Line

Tangent is a Latin word meaning *to touch*, and in mathematics a **tangent line** touches an arbitrary curve described by $y = f(x)$ at a point P but not any other points nearby as shown.



Since by definition the tangent line only touches one point on the curve of $y = f(x)$, we cannot calculate the slope of this tangent line with our slope formula for a line. In fact, up to now, we do not have any way of calculating this slope unless we are able to use some geometry.

Suppose that y is a function of x, say $y = f(x)$. Since it is often useful to know how sensitive the value of y is to small changes in x, let's explore this concept via an example, and see how this will inform us about the calculation of the slope of the tangent line.

Example 4.2: Small Changes in x

Consider $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin), and let's compute the changes of y resulting from small changes of x around $x = 7$.

Solution. When $x = 7$, we find that $y = \sqrt{625-49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

Let us look at the ratio $\Delta y/\Delta x$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\Delta y = f(x+\Delta x) - f(x) = f(7.1) - f(7)$$

$$= \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2}$$
$$\approx 23.9706 - 24 = -0.0294.$$

Thus, $\Delta y / \Delta x \approx -0.0294/0.1 = -0.294$. This means that y changes by less than one third the change in x, so apparently y is not very sensitive to changes in x at $x = 7$. We say "apparently" here because we don't really know what happens between 7 and value at 7. This is not in fact the case for this particular function, but we don't yet know why. ♣

The quantity $\Delta y / \Delta x \approx -0.294$ may be interpreted as the slope of the secant line through $(7, 24)$ and $(7.1, 23.9706)$. In general, if we draw the secant line from the point $(7, 24)$ to a nearby point on the semicircle $(7 + \Delta x, f(7 + \Delta x))$, the slope of this secant line is the so-called **difference quotient**

$$\frac{f(7+\Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625 - (7+\Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the secant line) is approximately equal to $(23.997081 - 24)/0.01 = -0.2919$. This is slightly different than for the secant line from $(7, 24)$ to $(7.1, 23.9706)$.

As Δx is made smaller (closer to 0), $7 + \Delta x$ gets closer to 7 and the secant line joining $(7, f(7))$ to $(7 + \Delta x, f(7 + \Delta x))$ shifts slightly, as shown in Figure 4.1. The secant line gets closer and closer to the **tangent line** to the circle at the point $(7, 24)$. (The tangent line is the line that just grazes the circle at that point, i.e., it doesn't meet the circle at any second point.) Thus, as Δx gets smaller and smaller, the slope $\Delta y / \Delta x$ of the secan line gets closer and closer to the slope of the tangent line. This is actually quite difficult to see when Δx is small, because of the scale of the graph. The values of Δx used for the figure are 1, 5, 10 and 15, not really very small values. The tangent line is the one that is uppermost at the right hand endpoint.

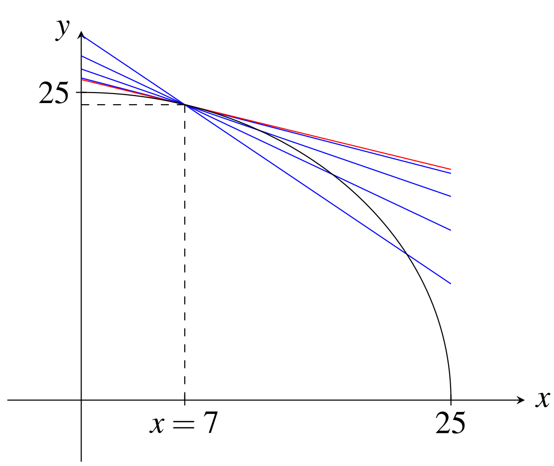


Figure 4.1: Secant lines approximating the tangent line.

So far we have found the slopes of two secant lines that should be close to the slope of the tangent line, but what is the slope of the tangent line exactly? Since the tangent line touches the circle at just one point, we will never be able to calculate its slope directly, using two "known" points on the line. What we need is a way to capture what happens to the slopes of the secant lines as they get "closer and closer" to the tangent line.

Instead of looking at more particular values of Δx, let's see what happens if we do some algebra with the difference quotient using just Δx. The slope of a secant line from $(7,24)$ to a nearby point $(7+\Delta x, f(7+\Delta x))$ is given by

$$\begin{aligned}
\frac{f(7+\Delta x)-f(7)}{\Delta x} &= \frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x} \\
&= \left(\frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x}\right)\left(\frac{\sqrt{625-(7+\Delta x)^2}+24}{\sqrt{625-(7+\Delta x)^2}+24}\right) \\
&= \frac{625-(7+\Delta x)^2-24^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{49-49-14\Delta x-\Delta x^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{625-(7+\Delta x)^2-24^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{49-49-14\Delta x-\Delta x^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{\Delta x(-14-\Delta x)}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{-14-\Delta x}{\sqrt{625-(7+\Delta x)^2}+24}
\end{aligned}$$

Now, can we tell by looking at this last formula what happens when Δx gets very close to zero? The numerator clearly gets very close to -14 while the denominator gets very close to $\sqrt{625-7^2}+24=48$. The fraction is therefore very close to $-14/48=-7/24 \cong -0.29167$. In fact, the slope of the tangent line is exactly $-7/24$.

What about the slope of the tangent line at $x=12$? Well, 12 can't be all that different from 7; we just have to redo the calculation with 12 instead of 7. This won't be hard, but it will be a bit tedious. What if we try to do all the algebra without using a specific value for x? Let's copy from above, replacing 7 by x.

$$\begin{aligned}
\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x} \\
&= \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x}\frac{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}} \\
&= \frac{625-(x+\Delta x)^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{625-x^2-2x\Delta x-\Delta x^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{\Delta x(-2x-\Delta x)}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})}
\end{aligned}$$

$$= \frac{-2x - \Delta x}{\sqrt{625 - (x+\Delta x)^2} + \sqrt{625 - x^2}}$$

Now what happens when Δx is very close to zero? Again it seems apparent that the quotient will be very close to

$$\frac{-2x}{\sqrt{625 - x^2} + \sqrt{625 - x^2}} = \frac{-2x}{2\sqrt{625 - x^2}} = \frac{-x}{\sqrt{625 - x^2}}.$$

Replacing x by 7 gives $-7/24$, as before, and now we can easily do the computation for 12 or any other value of x between -25 and 25.

So now we have a single expression, $\frac{-x}{\sqrt{625 - x^2}}$, that tells us the slope of the tangent line for any value of x. This slope, in turn, tells us how sensitive the value of y is to small changes in the value of x.

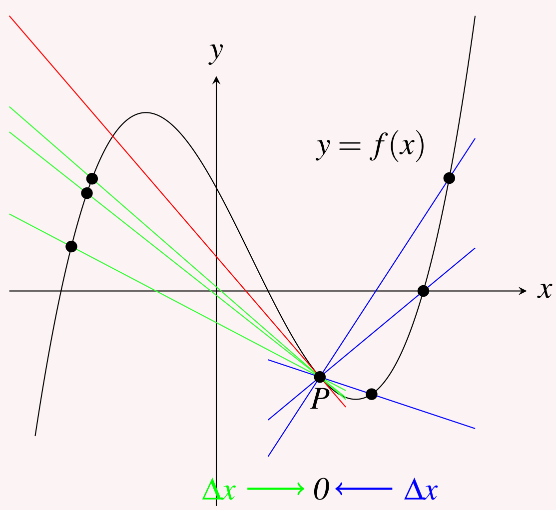
To summarize, we computed the **slope of the tangent line** at a point $P = (x, f(x))$ on the curve of a function $y = f(x)$ by forming the difference quotient and figuring out what happens when Δx gets very close to 0. At this point, we should note that the idea of letting get closer and closer to 0 is precisely the idea of a limit that we discussed in the last chapter. This leads us to the following definition.

Definition 4.3: Slope of Tangent Line - Instantaneous Rate of Change

The slope of the tangent line to the graph of a function $y = f(x)$ at the point $P = (x, f(x))$ is given by

$$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

provided this limit exists.



Note: The slope of the tangent line is also referred to as the **insantaneous rate of change** of f at x.

The expression $\frac{-x}{\sqrt{625 - x^2}}$ defines a new function called the **derivative** (see Section 4.2) of the original function (since it is derived from the original function). If the original is referred to as f or y then the derivative is often written as f' or y', read "f prime" or "y prime". We also write

$$f'(x) = \frac{-x}{\sqrt{625-x^2}} \text{ or } y' = \frac{-x}{\sqrt{625-x^2}}.$$

At a particular point, say $x = 7$, we write $f'(7) = -7/24$ and we say that "f prime of 7 is $-7/24$" or "the derivative of f at 7 is $-7/24$."

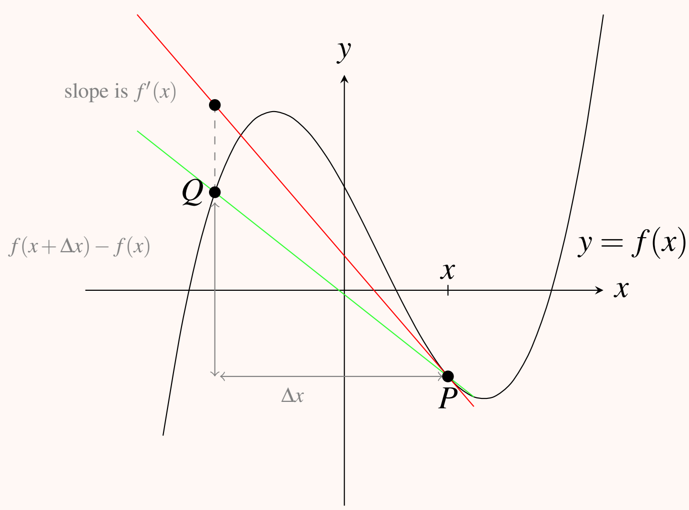
In the particular case of a circle, there's a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining $(0,0)$ to $(7,24)$ has slope 24/7. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{625-x^2})$ has slope $\sqrt{625-x^2}/x$, so the slope of the tangent line is $-x/\sqrt{625-x^2}$, as before. It is **NOT** always true that a tangent line is perpendicular to a line from the origin—don't use this shortcut in any other circumstance.

We now summarize our findings.

From Tangent Line Slope to Derivative

Given a function f and a point x we can compute the derivative of $f(x)$ at x as follows:

1. Form the difference quotient $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, which is the slope of a general secant line of the curve f throught the points $P = (x, f(x))$ and $Q = (x+\Delta x, f(x+\Delta x))$.
2. Take the limits as Δx goes to zero: $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$, which is the slope of the tangent line of the curve f at the point $P = (x, f(x))$.
3. If this limit exists, then the derivative exists and is equal to this limit.



In other words,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x},$$

provided the limit exists.

Applications

We started this section by saying, "It is often useful to know how sensitive the value of y is to small changes in x." We have seen one purely mathematical example of this, involving the function $f(x) = \sqrt{625 - x^2}$. Here are some more applied examples.

With careful measurement it might be possible to discover that the height of a dropped ball t seconds after it is released is $h(t) = h_0 - kt^2$. (Here h_0 is the initial height of the ball, when $t = 0$, and k is some number determined by the experiment.) A natural question is then, "How fast is the ball going at time t?" We can certainly get a pretty good idea with a little simple arithmetic.

Example 4.4: Analyzing Velocity

Suppose that the height h in metres of a dropped ball t seconds after it is released is given by

$$h(t) = h_0 - kt^2$$

with $h_0 = 100$ m and $k = 4.9$. We will answer the question "How fast is the ball travelling at time $t = 2$?" by exploring average speed near time $t = 2$ and discussing the difference between speed and velocity.

Solution. We know that when $t = 2$ the height is $100 - 4 \cdot 4.9 = 80.4$ metres. A second later, at $t = 3$, the height is $100 - 9 \cdot 4.9 = 55.9$ metres. The change in height during that second is $55.9 - 80.4 = -24.5$ metres. The negative sign means the height has decreased, as we expect for a falling ball, and the number 24.5 is the average speed of the ball during the time interval, in metres per second.

We might guess that 24.5 metres per second is not a terrible estimate of the speed at $t = 2$, but certainly we can do better. At $t = 2.5$ the height is $100 - 4.9(2.5)^2 = 69.375$ metres. During the half second from $t = 2$ to $t = 2.5$, the change in height is $69.375 - 80.4 = -11.025$ metres giving an average speed of $11.025/(1/2) = 22.05$ metres per second. This should be a better estimate of the speed at $t = 2$. So it's clear now how to get better and better approximations: compute average speeds over shorter and shorter time intervals. Between $t = 2$ and $t = 2.01$, for example, the ball drops 0.19649 metres in one hundredth of a second, at an average speed of 19.649 metres per second.

We still might reasonably ask for the precise speed at $t = 2$ (the *instantaneous* speed) rather than just an approximation to it. For this, once again, we need a limit. Let's calculate the average speed during the time interval from $t = 2$ to $t = 2 + \Delta t$ without specifying a particular value for Δt. The change in height during the time interval from $t = 2$ to $t = 2 + \Delta t$ is

$$\begin{aligned} h(2+\Delta t) - h(2) &= (100 - 4.9(2+\Delta t)^2) - 80.4 \\ &= 100 - 4.9(4 + 4\Delta t + \Delta t^2) - 80.4 \\ &= 100 - 19.6 - 19.6\Delta t - 4.9\Delta t^2 - 80.4 \\ &= -19.6\Delta t - 4.9\Delta t^2 \\ &= -\Delta t(19.6 + 4.9\Delta t) \end{aligned}$$

The average speed during this time interval is then

$$\frac{\Delta t(19.6 + 4.9\Delta t)}{\Delta t} = 19.6 + 4.9\Delta t.$$

When Δt is very small, this is very close to 19.6. Indeed, $\lim_{\Delta t \to 0}(19.6 + 4.9\Delta t) = 19.6$. So the exact speed at $t = 2$ is 19.6 metres per second.

At this stage we need to make a distinction between *speed* and *velocity*. Velocity is signed speed, that is, speed with a direction indicated by a sign (positive or negative). Our algebra above actually told us that the instantaneous velocity of the ball at $t = 2$ is -19.6 metres per second. The number 19.6 is the speed and the negative sign indicates that the motion is directed downwards (the direction of decreasing height).

In the language of the previous section, we might have started with $f(x) = 100 - 4.9x^2$ and asked for the slope of the tangent line at $x = 2$. We would have answered that question by computing

$$\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = \lim_{\Delta x \to 0}(-19.6 - 4.9\Delta x) = -19.6$$

The algebra is the same. Thus, the velocity of the ball is the value of the derivative of a certain function, namely, of the function that gives the position of the ball. ♣

The upshot is that this problem, finding the velocity of the ball, is *exactly* the same problem mathematically as finding the slope of a curve. This may already be enough evidence to convince you that whenever some quantity is changing (the height of a curve or the height of a ball or the size of the economy or the distance of a space probe from earth or the population of the world) the *rate* at which the quantity is changing can, in principle, be computed in exactly the same way, by finding a derivative.

Example 4.5: Demand for Sweaters

A clothing manufacturer has determined that the weekly demand function of their sweaters is given by

$$p = f(q) = 144 - q^2$$

where p is measured in dollars and q is measured in units of a thousand. Find the average rate of change in the unit price of a sweater if the quantity demanded is between 5000 and 6000 sweaters, between 5000 and 5100 sweaters, and between 5000 and 5010 sweaters. What is the instantaneous rate of change of the unit price when the quantity demanded is 5000 units?

Solution. The average rate of change of the unit price of a sweater if the quantity demanded is between q and $q + \Delta q$ is

$$\begin{aligned}\frac{f(q+\Delta q) - f(q)}{\Delta q} &= \frac{\left(144 - (q+\Delta q)^2\right) - \left(144 - q^2\right)}{\Delta q} \\ &= \frac{144 - q^2 - 2q\Delta q - \Delta q^2 - 144 + q^2}{\Delta q} \\ &= -2q - \Delta q\end{aligned}$$

To find the average rate of change of the unit price of a sweater when the quantity demanded is between 5000 and 6000 sweaters (that is, over the interval $[5, 6]$), we take $q = 5$ and $\Delta q = 1$, obtaining

$$-2(5) - 1 = -11$$

or -\$11 per 1000 sweaters. Similarly, taking $\Delta q = 0.1$ and $\Delta q = 0.01$ with $q = 5$, we find that the average rates of change of the unit price when the quantities demanded are between 5000 and 5100 and between

5000 and 5010 are -\$10.10 and -\$10.01 per 1000 sweaters, respectively.
The instantaneous rate of change of the unit price of a sweater when the quantity demanded is q units is given by

$$\begin{aligned}\lim_{\Delta q\to 0}\frac{f(q+\Delta q)-f(q)}{\Delta q} &= \lim_{\Delta q\to 0}\frac{\left(144-(q+\Delta q)^2\right)-\left(144-q^2\right)}{\Delta q}\\ &= \lim_{\Delta q\to 0}(-2q-\Delta q)\\ &= -2q\end{aligned}$$

In particular, the instantaneous rate of change when the quantity demanded is 5000 sweaters is

$$-2(5) = -10$$

or -\$10 per 1000 sweaters. ♣

Exercises for Section 4.1

Exercise 4.1.1 *Draw the graph of the function* $y = f(x) = \sqrt{169-x^2}$ *between* $x = 0$ *and* $x = 13$. *Find the slope* $\Delta y/\Delta x$ *of the secant line between the points of the circle lying over (a)* $x = 12$ *and* $x = 13$, *(b)* $x = 12$ *and* $x = 12.1$, *(c)* $x = 12$ *and* $x = 12.01$, *(d)* $x = 12$ *and* $x = 12.001$. *Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative* $f'(12)$. *Your answers to (a)–(d) should be getting closer and closer to your answer to (e).*

Exercise 4.1.2 *Use geometry to find the derivative* $f'(x)$ *of the function* $f(x) = \sqrt{625-x^2}$ *in the text for each of the following x: (a) 20, (b) 24, (c)* -7, *(d)* -15. *Draw a graph of the upper semicircle, and draw the tangent line at each of these four points.*

Exercise 4.1.3 *Draw the graph of the function* $y = f(x) = 1/x$ *between* $x = 1/2$ *and* $x = 4$. *Find the slope of the secant line between (a)* $x = 3$ *and* $x = 3.1$, *(b)* $x = 3$ *and* $x = 3.01$, *(c)* $x = 3$ *and* $x = 3.001$. *Now use algebra to find a simple formula for the slope of the secant line between* $(3, f(3))$ *and* $(3+\Delta x, f(3+\Delta x))$. *Determine what happens when* Δx *approaches 0. In your graph of* $y = 1/x$, *draw the straight line through the point* $(3, 1/3)$ *whose slope is this limiting value of the difference quotient as* Δx *approaches 0.*

Exercise 4.1.4 *Find an algebraic expression for the difference quotient* $\big(f(1+\Delta x) - f(1)\big)/\Delta x$ *when* $f(x) = x^2 - (1/x)$. *Simplify the expression as much as possible. Then determine what happens as* Δx *approaches 0. That value is* $f'(1)$.

Exercise 4.1.5 *Draw the graph of* $y = f(x) = x^3$ *between* $x = 0$ *and* $x = 1.5$. *Find the slope of the secant line between (a)* $x = 1$ *and* $x = 1.1$, *(b)* $x = 1$ *and* $x = 1.001$, *(c)* $x = 1$ *and* $x = 1.00001$. *Then use algebra to find a simple formula for the slope of the secant line between* 1 *and* $1+\Delta x$. *(Use the expansion* $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.*) Determine what happens as* Δx *approaches 0, and in your graph of* $y = x^3$ *draw the straight line through the point* $(1,1)$ *whose slope is equal to the value you just found.*

Exercise 4.1.6 *Find an algebraic expression for the difference quotient* $(f(x+\Delta x)-f(x))/\Delta x$ *when* $f(x) = mx+b$*. Simplify the expression as much as possible. Then determine what happens as* Δx *approaches 0. That value is* $f'(x)$*.*

Exercise 4.1.7 *Sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle* θ*? Why? Hint: think in terms of ratios of sides of triangles.*

Exercise 4.1.8 *Sketch the parabola* $y = x^2$*. For what values of x on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?*

Exercise 4.1.9 *An object is traveling in a straight line so that its position (that is, distance from some fixed point) is given by this table:*

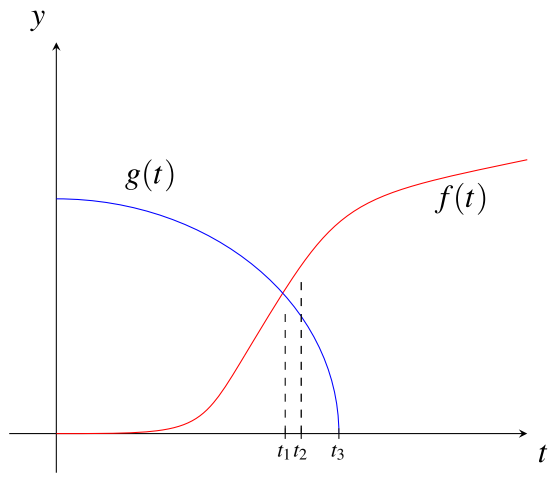
time (seconds)	0	1	2	3
distance (metres)	0	10	25	60

Find the average speed of the object during the following time intervals: $[0,1]$, $[0,2]$, $[0,3]$, $[1,2]$, $[1,3]$, $[2,3]$. *If you had to guess the speed at* $t = 2$ *just on the basis of these, what would you guess?*

Exercise 4.1.10 *Let* $y = f(t) = t^2$*, where t is the time in seconds and y is the distance in metres that an object falls on a certain airless planet. Draw a graph of this function between* $t = 0$ *and* $t = 3$*. Make a table of the average speed of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average speed between time* 2 *and time* $2+\Delta t$*. (If you substitute* $\Delta t = 1, 0.1, 0.01, 0.001$ *in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average speed (which should be in simplified form) determine what happens as* Δt *approaches zero. This is the instantaneous speed. Finally, in your graph of* $y = t^2$ *draw the straight line through the point* $(2,4)$ *whose slope is the instantaneous velocity you just computed; it should of course be the tangent line.*

Exercise 4.1.11 *If an object is dropped from an 80-metre high window, its height y above the ground at time t seconds is given by the formula* $y = f(t) = 80-4.9t^2$*. (Here we are neglecting air resistance; the graph of this function was shown in Figure 1.1.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and* $1+\Delta t$ *sec. Determine what happens to this average velocity as* Δt *approaches 0. That is the instantaneous velocity at time* $t = 1$ *second (it will be negative, because the object is falling).*

Exercise 4.1.12 *The following figure shows the devastating effect the opening of a new chain coffee shop had on a 3-generation rural café in a small town. The revenue of the chain coffee shop at time t (in months) is given by* $f(t)$ *million dollars, whereas the revenue of the rural café at time t is given by* $g(t)$ *million dollars. Answer the following questions by giving the value of t at which the specifying event took place.*



(a) *The revenue of the rural café is decreasing at the slowest rate.*

(b) *The revenue of the rural café is decreasing at the fastest rate.*

(c) *The revenue of the chain coffe shop first overtakes that of the rural café.*

(d) *The revenue of the chain coffee shop is increasing at the fastest rate.*

Exercise 4.1.13 *The demand function for tires is given by*

$$p = f(q) = -0.1q^2 - q + 125$$

where p is measured in dollars and q is measured in units of a thousand.

(a) *Find the average rate of change in the unit price of a tire if the quantity demanded is between* 5000 *and* 5050 *tires; between* 5000 *and* 5010 *tires.*

(b) *What is the rate of change of the unit price if the quantity demanded is* 5000*?*

4.2 The Derivative Function

In Section 4.1, we have seen how to create, or derive, a new function $f'(x)$ from a function $f(x)$, and that this new function carries important information. In one example we saw that $f'(x)$ indicates how steep the graph of $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ represents the position of the object at time x. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "rate" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

We begin by formally defining the derivative.

Definition 4.6: Definition of Derivative

The derivative of a function $y = f(x)$ with respect to x is the function f' defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

provided this limit exists.
Some textbooks use h in place of Δx in the definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists.

The process of finding the derivative is called **differentiation**. There are several different notations in use for the derivative.

Derivative Notations

Given a function $y = f(x)$, its derivative function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

provided it exists, can be written in the following ways:

$f'(x)$	$= y'$	read "y prime" or "the derivative of y"
	$= \dfrac{dy}{dx}$	read "$dy\ dx$" or "the derivative of y w.r.t x"
	$= \dfrac{df}{dx}$	read "$df\ dx$" or "the derivative of f w.r.t x"
	$= \dfrac{df(x)}{dx}$	read "df of x dx" or "the derivative of f w.r.t x"
	$= \dfrac{d}{dx}f(x)$	read "$d\ dx$ of f of x" or "the derivative of f w.r.t x"

The symbol d/dx is called a **differential operator** which means to take the derivative of the function $f(x)$ with respect to the variable x.

The notation, dy/dx, and its derivations remind us that the derivative is related to an actual slope between two points. This notation is called **Leibniz notation**, after Gottfried Leibniz, who developed the fundamentals of calculus independently at about the same time that Isaac Newton did. This notation has the added benefit that it indicates what we are differentiating with respect to, which is important in applications such as *related rates* (see Section 5.2), or *multi-variable calculus* (see Chapter 7).

Let us come back to our familiar function $y = \sqrt{625 - x^2}$ to provide an example of how Leibniz notation is used. If the function $f(x)$ is written out in full we often write the derivative as something like this

$$f'(x) = \frac{d}{dx}\sqrt{625 - x^2}$$

with the function written to the side, instead of trying to fit it into the numerator.

Example 4.7: Derivative of $y = t^2$ Using Δt Notation

Find the derivative of $y = f(t) = t^2$.

Solution. We compute

$$\begin{aligned} y' &= \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{(t + \Delta t)^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} (2t + \Delta t) = 2t. \end{aligned}$$

♣

Remember that Δt is a single quantity, not a "Δ" times a "t", and so Δt^2 is $(\Delta t)^2$ not $\Delta(t^2)$. Doing the same example using the second formula for the derivative with h in place of Δt gives the following. Note that we compute $f(t+h)$ by substituting $t+h$ in place of t everywhere we see t in the expression $f(t)$, *while making no other changes* (at least initially). For example, if $f(t) = t + \sqrt{(t+3)^2 - t}$ then $f(t+h) = (t+h) + \sqrt{((t+h)+3)^2 - (t+h)} = t + h + \sqrt{(t+h+3)^2 - t - h}$.

Example 4.8: Derivative of $y = t^2$ Using h Notation

Find the derivative of $y = f(t) = t^2$.

Solution. We compute

$$\begin{aligned} f'(t) &= \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} \\ &= \lim_{h\to 0}\frac{(t+h)^2-t^2}{h} \\ &= \lim_{h\to 0}\frac{t^2+2th+h^2-t^2}{h} \\ &= \lim_{h\to 0}\frac{2th+h^2}{h} \\ &= \lim_{h\to 0}(2t+h) = 2t. \end{aligned}$$

♣

Example 4.9: Derivative

Find the derivative of $y = f(x) = 1/x$.

Solution. We compute

$$\begin{aligned} y' &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\frac{x}{x(x+\Delta x)}-\frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{x-x-\Delta x}{x(x+\Delta x)\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{-\Delta x}{x(x+\Delta x)\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)} = \frac{-1}{x^2} \end{aligned}$$

♣

Note: If you happen to know some "derivative formulas" from an earlier course, for the time being you should pretend that you do not know them. In examples like the ones above and the exercises below, you are required to know how to find the derivative function using the definition of the derivative, i.e from basic principles. We will later develop some formulas so that we do not always need to do such computations, but we will continue to need to know how to do the more involved computations.

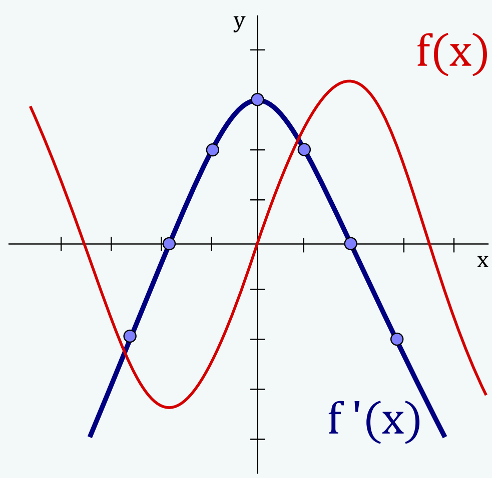
To recap, given any function f and any number x in the domain of f, we define $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ wherever this limit exists, and we call the number $f'(x)$ the derivative of f at x. Geometrically, $f'(x)$ is the slope of the tangent line to the graph of f at the point $(x, f(x))$. The following symbols also represent the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x).$$

As above, and as you might expect, for different values of x we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of f, or the slope of its tangent line, is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line to a line, but if it makes sense at all the tangent line must be the line itself. It is not hard to see that the derivative of $f(x) = mx + b$ is $f'(x) = m$. In the next example we emphasize the geometrical interpretation of derivative.

Example 4.10: Geometrical Interpretation of Derivative

Consider the function $f(x)$ given by the graph below. Verify that the graph of $f'(x)$ is indeed the derivative of $f(x)$ by analyzing slopes of tangent lines to the graph at different points.



Solution. We must think about the tangent lines to the graph of f, because the slopes of these lines are the values of $f'(x)$.

We start by checking the graph of f for horizontal tangent lines, since horizontal lines have a slope of 0. We find that the tangent line is horizontal at the points where x has the values -1.9 and 1.8 (approximately). At each of these values of x, we must have $f'(x) = 0$, which means that the graph of f' has an x-intercept (a point where the graph intersects the x-axis).

Note that horizontal tangent lines have a slope of zero and these occur approximately at the points $(-1.9, -3.2)$ and $(1.8, 3.2)$ of the graph. Therefore $f'(x)$ will cross the x-axis when $x = -1.9$ and $x = 1.8$.

Analyzing the slope of the tangent line of $f(x)$ at $x = 0$ gives approximately 3.0, thus, $f'(0) = 3.0$. Similarly, analyzing the slope of the tangent lines of $f(x)$ at $x = 1$ and $x = -1$ give approximately 2.0 for both, thus, $f'(1) = f'(-1) = 2.0$. ♣

In the next example we verify that the slope of a straight line is m.

Example 4.11: Derivative of a Linear Function

Let m,b be any two real numbers. Determine $f'(x)$ if $f(x) = mx+b$.

Solution. By the definition of derivative (using h in place of Δx) we have,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(m(x+h)+b)-(mx+b)}{h}$$

$$= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m.$$

This is not surprising. We know that $f'(x)$ always represents the slope of a tangent line to the graph of f. In this example, since the graph of f is a straight line $y = mx+b$ already, every tangent line is the same line $y = mx+b$. Since this line has a slope of m, we must have $f'(x) = m$. ♣

4.2.1. Differentiable

Now that we have introduced the derivative of a function at a point, we can begin to use the adjective **differentiable**.

Definition 4.12: Differentiable at a Point

A function f is **differentiable at point** *a if $f'(a)$ exists.*

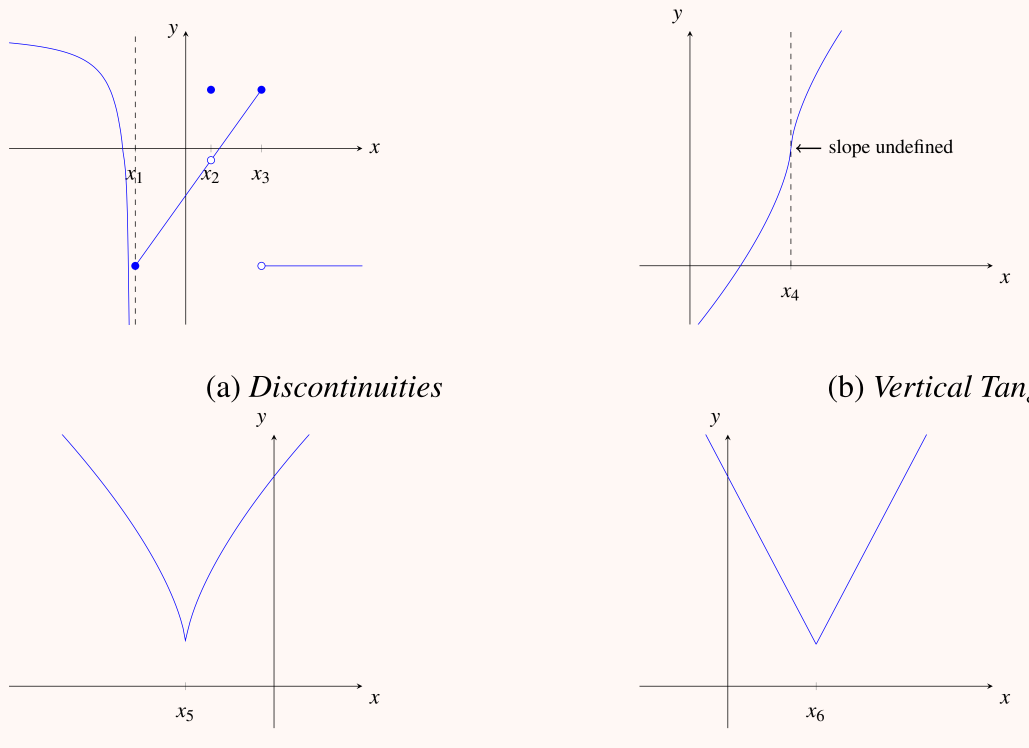
Definition 4.13: Differentiable on an Interval

A function f is **differentiable on an open interval** *(a,b) if it is differentiable at every point in the interval.*

Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is **no derivative**, because there is no tangent line. In order for the notion of the tangent line at a point to make sense, the curve must be "smooth" at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are four types of situations you should be aware of—discontinuities, vertical tangent lines, corners and cusps—where the slope takes on two equal values at the same x-value, or the slope is undefined at x, and hence no derivative can exist at x.

Non-Existence of Derivative

Given a function $y = f(x)$, the derivative does not xists at x if there is a discontinuity (x_1, x_2, x_3), vertical tangent (x_4), cusp (x_5) or corner (x_6) as shown. We say that f is **not differentiable** at x.



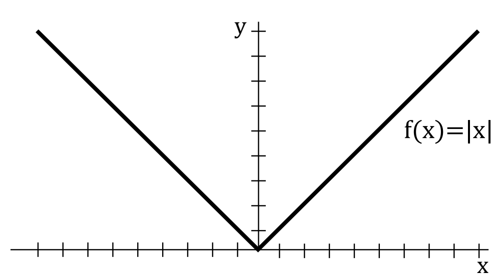
(a) *Discontinuities* (b) *Vertical Tangent*

(c) *Cusp* (d) *Corner*

Example 4.14: Derivative of the Absolute Value

Discuss the derivative of the absolute value function $y = f(x) = |x|$.

Solution. If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve—but there are *two* directions of the curve that come together at the origin.



We can summarize this as

$$y' = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ \text{undefined}, & \text{if } x = 0. \end{cases}$$

In particular, the absolute value function $f(x) = |x|$ is *not* differentiable at $x = 0$. ♣

We note that the following theorem can be proved using limits.

Theorem 4.15: Differentiability implies Continuity

If f is differentiable at a, then f is continuous at a.

Proof. Suppose that f is differentiable at a. That is,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. At this stage, we find it convenient to write this limit in an alternative form so that its connection with continuity can become more easily seen. If we let $x = a + h$, then $h = x - a$. Furthermore, $h \to 0$ is equivalent to $x \to a$. So,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

(This alternative formulation of the derivative is also standard. We will use it whenever we find it convenient to do so. You should get familiar with it.) Continuity at a can now be proved as follows:

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) + \lim_{x \to a} f(a) \\ &= f'(a) \cdot (a - a) + f(a) \\ &= f(a). \end{aligned}$$

♣

Note: If f is continuous at a it is *not* necessarily true that f is differentiable at a. For example, it was shown that $f(x) = |x|$ is not differentiable at $x = 0$ in the previous example, however, one can observe that $f(x) = |x|$ is continuous everywhere.

Example 4.16: Derivative of $y = x^{2/3}$

Discuss the derivative of the function $y = x^{2/3}$, shown in Figure 4.2.

Solution. We will later see how to compute this derivative; for now we use the fact that $y' = (2/3)x^{-1/3}$. Visually this looks much like the absolute value function, but it technically has a cusp, not a corner. The absolute value function has no tangent line at 0 because there are (at least) two obvious contenders—the tangent line of the left side of the curve and the tangent line of the right side. The function $y = x^{2/3}$ does not have a tangent line at 0, but unlike the absolute value function it can be said to have a single direction: as we approach 0 from either side the tangent line becomes closer and closer to a vertical line; the curve is vertical at 0. But as before, if you imagine traveling along the curve, an abrupt change in direction is required at 0: a full 180 degree turn. ♣

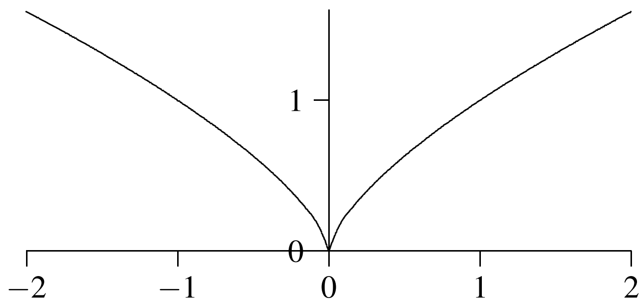


Figure 4.2: A cusp on $x^{2/3}$.

In practice we won't worry much about the distinction between these examples; in both cases the function has a "sharp point" where there is no tangent line and no derivative.

4.2.2. Tangent Line Equation

Recall from Section 1.2 that the point-slope form of a straight line passing through the point (x_1, y_1) with slope m is given by $y - y_1 = m(x - x_1)$. We are now in a position to provide the equation of a tangent line to a curve described by $y = f(x)$ at $x = a$, provided the derivative exists.

Definition 4.17: Tangent Line Equation

Given a function $y = f(x)$, the tangent line equation at $x = a$ is given by

$$y - f(a) = f'(a)(x - a),$$

provided $f'(a)$ exists.

Example 4.18: Horizontal Tangent Line

Let $f(x) = x^2 - 4x$.

(a) *Compute $f'(x)$.*

(b) *Find the point on the graph of f where the tangent line to the curve is horizontal.*

(c) *Sketch the graph of f and the tangent line to the curve at the point found above.*

(d) *What is the rate of change of f at this point?*

Solution.

(a) To find $f'(x)$, we compute

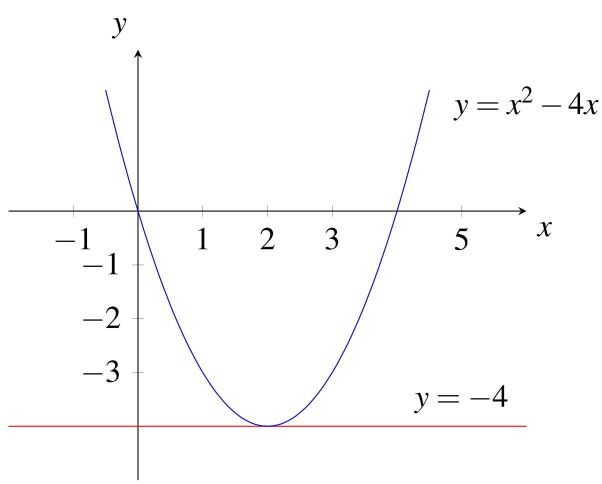
$$\begin{aligned}
f(x+h)-f(x) &= (x+h)^2-4(x+h) \\
&= x^2+2xh+h^2-4x-4h-(x^2-4x) \\
&= 2xh+h^2-4h \\
&= h(2x+h-4) \\
\frac{f(x+h)-f(x)}{h} &= 2x+h-4
\end{aligned}$$

Therefore,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(2x+h-4)=2x-4.$$

(b) At a point on the graph of f where the tangent line to the curve is horizontal and hence has zero slope, the derivative f' of f is zero. Accordingly, to find such points, we set $f'(x)=0$, which gives $2x-4=0$, or $x=2$. The corresponding value of y is given by $y=f(2)=-4$, and so the desired point is $(2,-4)$.

(c) The graph of f and the tangent line are shown below:



(d) The rate of change of f at $x=2$ is zero.

♣

4.2.3. Second and Higher Derivatives

If f is a differentiable function then its derivative f' is also a function and so we can take the derivative of f'. The new function, denoted by f'', read "f double prime", is called the **second derivative** of f, since it is the derivative of the derivative of f.

The following symbols represent the second derivative:

$$f''(x) = y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

We can continue this process to get the third derivative of f. In general, the ***n*-th derivative** of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, then we write:

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

We have establised that the derivative of a function f at a point x measures the rate of change of the function f at that point, and so the second derivative of f (the derivative of f') measures the rate of change of the derivative f' of the function f. Similarly, the third derivative of the function f, f''', measures the rate of change of f'', and so on.

In Chapter 5 we will discuss applications such as *curve sketching* involving the geometric interpretation of the second derivative of a function. The following examples provide an interpretation of both the first and the second derivative in familiar roles.

4.2.4. Applications

In economics, the derivative $y = E'(q)$ of a certain function $y = E(q)$ representing either cost, average cost, revenue, or profit is referred to by yet another name, namely **marginal cost function**, **marginal average cost function**, **marginal revenue function**, **marginal profit function** respectively. The study of the rate of change of economic functions is referred to as **marginal analysis**. As before, the marginal function $y = E'(q)$ indicates the rate of change of the function $y = E(q)$ with respect to the number q of units produced. For example: We are given a revenue function $R(q) = pq = qf(q)$, where p is the unit selling price of the commodity, q is the quantity of the commodity demanded, and f is the demand function. Then the **marginal revenue function** $R'(q)$ yields the actual revenue realized from the sale of an additional unit of the commodity given that sales are already at a certain level.

Example 4.19: Marginal Cost Function

A certain manufacturer produces tactical flashlights with a daily total manufacturing cost in dollars of

$$C(q) = 0.0001q^3 - 0.08q^2 + 40q + 5000$$

where q stands for the number of tactical flashlights produced.

(a) *Determine the marginal cost function.*

(b) *Calculate the marginal cost when $q = 200, 300, 400,$ and 600.*

(c) *What can you deduce from your results?*

Solution.

(a) The marginal cost C' is given by the derivative of the total cost function C. Thus,

$$\begin{aligned} C'(q) &= \lim_{h\to 0} \frac{C(q+h)-C(q)}{h} \\ &= \lim_{h\to 0} \frac{0.0001(q+h)^3 - 0.08(q+h)^2 + 40(q+h) + 5000 - 0.001q^3 + 0.08q^2 - 40q - 5000}{h} \end{aligned}$$

We expand, combine like-terms, and simplify by h to find

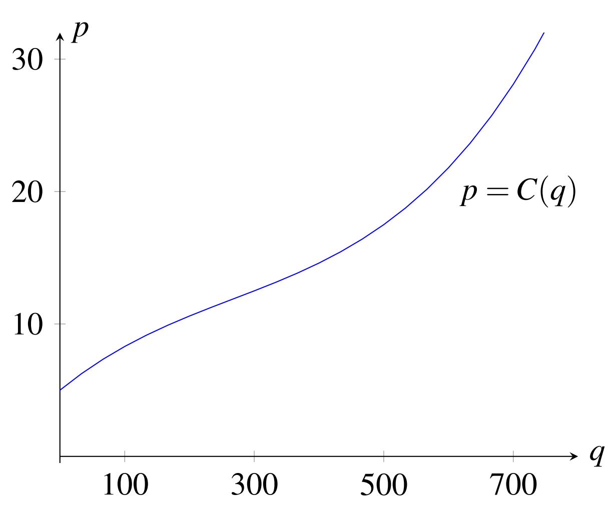
$$C'(q) = 0.003q^2 - 0.16q + 40.$$

(b) The marginal cost when $q = 200, 300, 400$, and 600 is given by

$$\begin{aligned} C'(200) &= 0.003(200)^2 - 0.16(200) + 40 = 20 \\ C'(300) &= 0.003(300)^2 - 0.16(300) + 40 = 19 \\ C'(400) &= 0.003(400)^2 - 0.16(400) + 40 = 24 \\ C'(600) &= 0.003(600)^2 - 0.16(600) + 40 = 52 \end{aligned}$$

or \$20, \$19, \$24 and \$52, respectively.

(c) We deduce from part (b) that the manufacturer's actual cost for producing the 201st tactical flashlight is approximated by \$20. Similarly, the approximated cost is \$19 for the 301st tactical flashlight, and so on. We also notice that when the level of production is already 600 units, the actual cost of producing one additional unit is approximately \$52. This increase in cost may be the result of several factors, among them excessive costs incurred because of higher maintenance, overtime to keep up with demand, production breakdown due to greater stress and strain on the equipment, and so on. Below, the graph of the total cost function is shown with $C(q)$ measured in thousands of dollars.



♣

Example 4.20: Marginal Revenue Function

The unit price p in dollars and the quantity demanded q of a certain product are related by the equation

$$p = -0.02q + 400 \quad 0 \leq q \leq 20,000$$

(a) Determine the revenue function R.

(b) Determine the marginal revenue function R'.

(c) Calculate $R'(2000)$. What you can deduce from your result?

Solution.

(a) We determine the revenue function R to be

$$\begin{aligned} R(q) &= pq \\ &= q(-0.02q + 400) \\ &= -0.02q^2 + 400q \quad 0 \leq q \leq 20,000 \end{aligned}$$

(b) We determine the marginal revenue function R' to be

$$R'(q) = \lim_{h \to 0} \frac{R(q+h) - R(q)}{h} = \lim_{h \to 0} \frac{-0.02(q+h)^2 + 400(q+h) + 0.02q^2 - 400q}{h}$$

We expand and simplify:

$$\begin{aligned} R'(q) &= \lim_{h \to 0} \frac{-0.02(q^2 + 2qh + h^2) + 400q + 400h + 0.02q^2 - 400q}{h} \\ &= \lim_{h \to 0} \frac{-0.04qh - 0.02h^2 + 400h}{h} \\ &= \lim_{h \to 0} \left(-0.04q - 0.02h^2 + 400\right) \\ &= -0.04q + 400 \end{aligned}$$

(c) $R'(2000) = -0.04(2000) + 400 = 320$. Therefore, the actual revenue from selling the 2001st product is approximated by \$320.

♣

Example 4.21: Marginal Profit Function

Refer to example 4.20. Suppose the cost of producing q units of this product is

$$C(q) = 100q + 200,000$$

dollars.

(a) Determine the profit function P.

(b) Determine the marginal profit function P'.

(c) Compute $P'(2000)$. What can you deduce from your result?

(d) Sketch the graph of the profit function P.

Solution.

(a) From part (a) of example 4.20, we have

$$R(q) = -0.02q^2 + 400q.$$

Hence, we determine the profit function P to be

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= \left(-0.02q^2 + 400q\right) - (100q + 200,000) \\ &= -0.02q^2 + 300q - 200,000 \end{aligned}$$

(b) The marginal profit function P' is given by

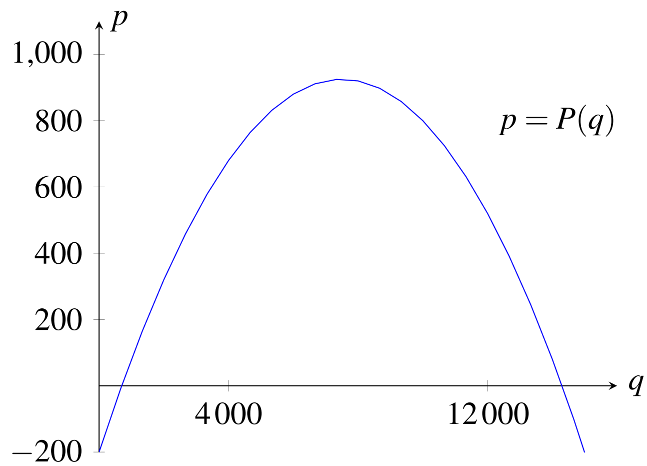
$$\begin{aligned} P'(q) &= \lim_{h\to 0} \frac{P(q+h) - P(q)}{h} \\ &= \lim_{h\to 0} \frac{-0.02(q+h)^2 + 300(q+h) - 200,000 + 0.02q^2 - 300q + 200,000}{h} \end{aligned}$$

As above, we expand and simplify to find

$$P'(q) = -0.04q + 300.$$

(c) $P'(2000) = -0.04(2000) + 300 = 220$. Therefore, the actual profit from selling the 2001st product is approximated by \$220

(d) The graph of the profit function P in thousands of dollars is shown below.



♣

The **Consumer Price Index** (CPI) provides a broad measure of some countries living cost by periodically measuring changes in the price level of market basket of consumer goods and services purchased by households. Let $I(t)$ with $a \leq t \leq b$ describe the CPI of an economy between the years a and b as shown in Figure 4.3. Then we can determine the rate of change of I for some value c with $a < c < b$ by calculating the derivative $I'(c)$. The *relative rate of change* of $I(t)$ with respect to t at $t = c$ is given by the quantity

$$\frac{I'(c)}{I(c)}$$

and measures the **inflation rate** of the economy at $t = c$. We can also determine the rate of change of $I'(c)$ for some value c with $a < c < b$ by calculating the second derivative $I''(c)$. When $I'(t)$ is positive and $I''(t)$ is negative at $t = c$ as determined in Example 4.22, then we deduce that the economy is experiencing inflation (the CPI is increasing) at $t = c$, but the rate at which inflation is growing is in fact slowing down. This is often used by a politician to claim that because inflation is slowing, the prices of goods and services are about to drop. A claim that is often false!

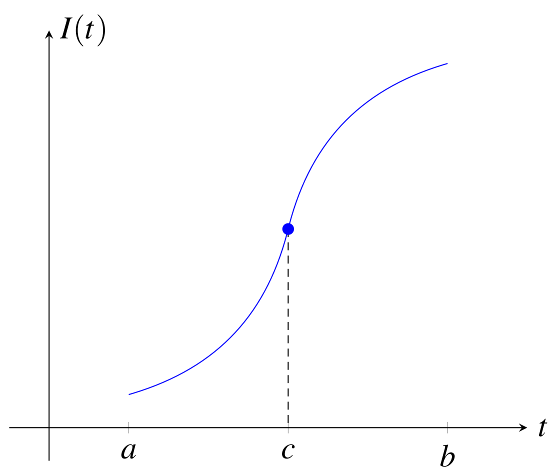


Figure 4.3: The CPI of a certain economy, $I(t)$, shown from year a to year b.

Example 4.22: Working with CPI

Suppose the function

$$I(t) = -0.2t^3 + 3t^2 + 100 \quad (0 \leq t \leq 9)$$

gives the CPI of an economy, where $t = 0$ *corresponds to the beginning of* 1995.

(a) *Determine* $I'(t)$ *using the definition of the derivative.*

(b) *Determine the inflation rate at the beginning of* 2001 *(*$t = 6$*).*

(c) *Determine* $I''(t)$ *from* $I'(t)$ *using the definition of the derivative.*

(d) *Calculate* $I''(6)$. *What you can deduce from your result?*

Solution.

(a) From the definition of the derivative, we have

$$I'(t) = \lim_{h \to 0} \frac{I(t+h) - I(t)}{h} = \lim_{h \to 0} \frac{-0.2(t+h)^3 + 3(t+h)^2 + 100 + 0.2t^3 - 3t^2 - 100}{h}.$$

We expand, group like-terms, and simplify by h to find

$$I'(t) = \lim_{h \to 0} \left(-0.6t^2 - 0.2h^2 - 0.6ht + 6t + 3h\right) = -0.6t^2 + 6t.$$

(b) We compute

$$I'(6) = -0.6(6)^2 + 6)6) = 14.4 \text{ and } I(6) = -0.2(6)^3 + 3(6)^2 + 100 = 164.8$$

From which we see that the inflation rate is

$$\frac{I'(6)}{I(6)} = \frac{14.4}{164.8} \simeq 0.0874$$

or approximately 8.7 percent.

(c) We must now compute

$$I''(t) = \lim_{h \to 0} \frac{I'(t+h) - I'(t)}{h}.$$

Using the above result, we get

$$I''(t) = \lim_{h \to 0} \frac{(0.6(t+h)^2 + 6(t+h)) - (-0.6t^2 + 6t)}{h}.$$

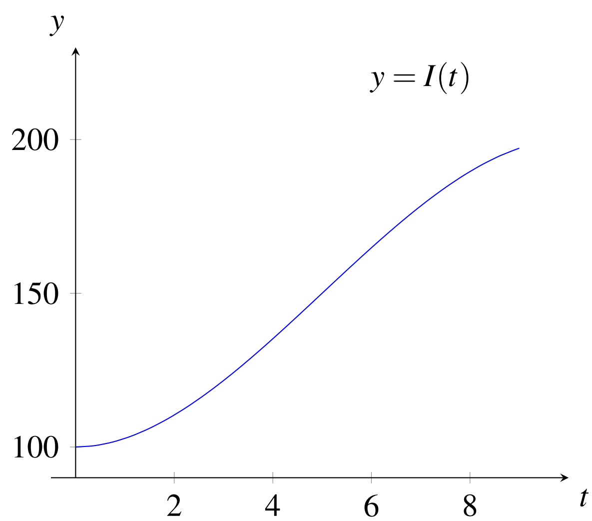
We again expand, simplify and take the limit to find

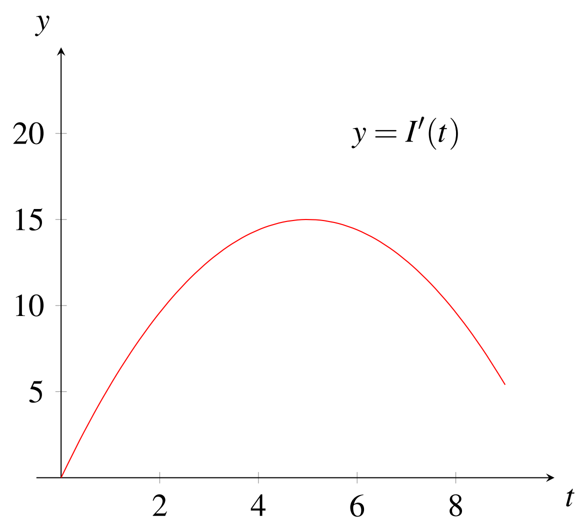
$$I''(t) = -1.2t + 6.$$

(d) Since

$$I''(6) = -1.2(6)+6 = -1.2,$$

we see that I' is indeed decreasing at $t = 6$ and conclude that inflation was levelling off at the time (see figures below).





♣

Suppose $f(t)$ is a position function of an object, representing the displacement of the object from the origin at time t. In terms of derivatives, the **velocity of an object is:**

$$v(t) = f'(t).$$

The change of velocity with respect to time is called the **acceleration** and can be found as follows:

$$a(t) = v'(t) = f''(t).$$

Acceleration is the derivative of the velocity function and the second derivative of the position function.

Example 4.23: Position, Velocity and Acceleration

Suppose the position function of an object is $f(t) = t^2$ metres at t seconds. Find the velocity and acceleration of the object at time $t = 1s$.

Solution. By the definition of velocity and acceleration we need to compute $f'(t)$ and $f''(t)$. Using the definition of derivative, we have,

$$f'(t) = \lim_{h\to 0}\frac{(t+h)^2 - t^2}{h} = \lim_{h\to 0}\frac{2th+h^2}{h} = \lim_{h\to 0}(2t+h) = 2t.$$

Therefore, $v(t) = f'(t) = 2t$. Thus, the velocity at time $t = 1$ is $v(1) = 2\ m/s$. We now have that the acceleration at time t is:

$$a(t) = f''(t) = \lim_{h\to 0}\frac{2(t+h)-2t}{h} = \lim_{h\to 0}\frac{2h}{h} = 2.$$

Therefore, $a(t) = 2$. Substituting $t = 1$ into the function $a(t)$ gives $a(1) = 2\ m/s^2$. ♣

Exercises for Section 4.2

Exercise 4.2.1 *Find the derivatives of the following functions.*

(a) $f(x) = \sqrt{169 - x^2}$

(b) $h(t) = 80 - 4.9t^2$

(c) $g(x) = x^2 - (1/x)$

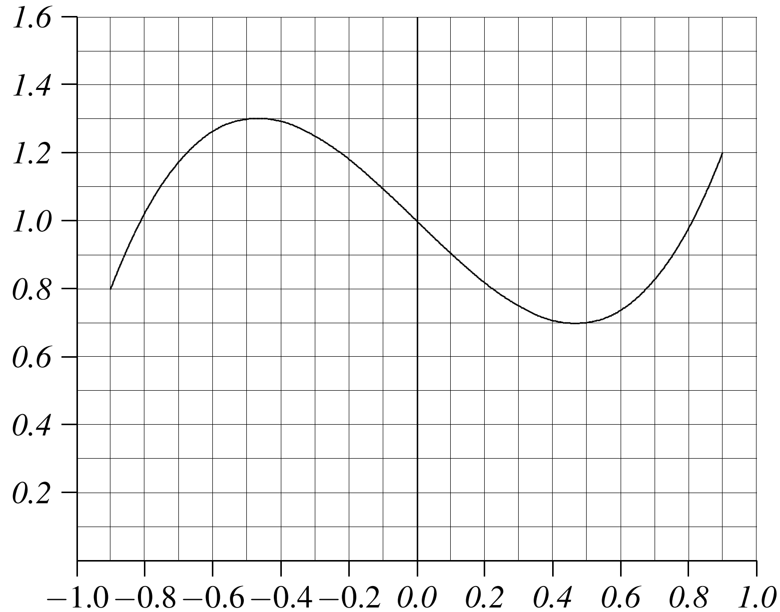
(d) $f(s) = as^2 + bs + c$, *where a, b, and c are constants.*

(e) $h(x) = x^3$

(f) $f(t) = 2/\sqrt{2t+1}$

(g) $g(t) = (2t-1)/(t+2)$

Exercise 4.2.2 *Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.*



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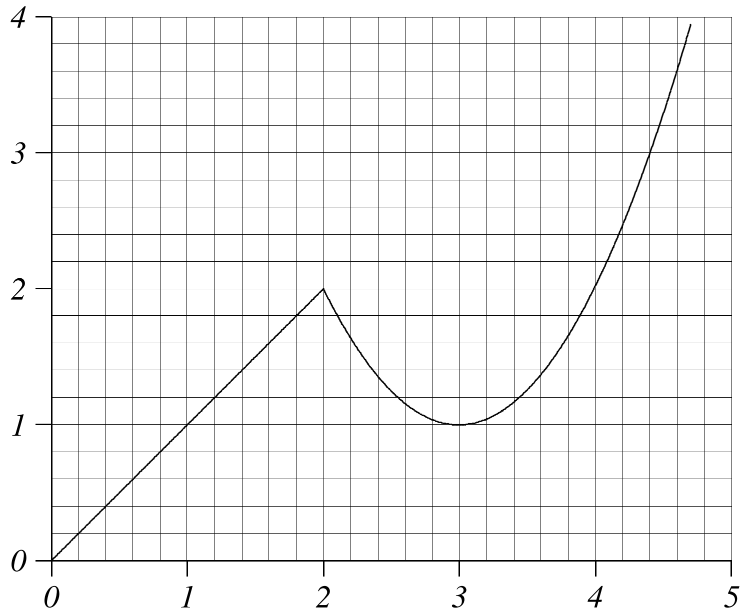


Exercise 4.2.4 *Find an equation for the tangent line to the graph of* $f(x) = 5 - x - 3x^2$ *at the point* $x = 2$

Exercise 4.2.5 *Find a value for a so that the graph of* $f(x) = x^2 + ax - 3$ *has a horizontal tangent line at* $x = 4$.

Exercise 4.2.6 *Find the slope of the tangent line to the graph of the given function at any point.*

(a) $f(x) = 7$

(b) $f(x) = x + 5$

(c) $f(x) = 2x^2$

(d) $f(x) = \sqrt{3x-1}$

Exercise 4.2.7 *Find the slope of the tangent line to the graph of each function at the given point and determine an equation of the tangent line.*

(a) $f(x) = 2x + 7$ *at* $(1, 1)$

(b) $f(x) = -\dfrac{1}{x}$ *at* $(2, -1)$

Exercise 4.2.8 *Let* $f(x) = x^2 - 2x + 1$.

(a) *Determine the derivative* f' *of* f.

(b) *Determine the point on the graph of f where the tangent line to the curve is horizontal.*

(c) *Sketch the graph of f and the tangent line to the curve at the point found in part (b).*

(d) *Determine the rate of change of f at the point found in part(b).*

Exercise 4.2.9 *Let $y = h(t) = t^2 + t$.*

(a) *Find the average rate of change of y with respect to t in the interval from $t = 2$ to $t = 3$, from $t = 2$ to $t = 2.5$, and from $t = 2$ to $t = 2.1$.*

(b) *Find the (instantaneous) rate of change of y at $t = 2$.*

(c) *Compare the results obtained in part (a) with that of part (b).*

Exercise 4.2.10 *The total cost $C(q)$ (in dollars) incurred by a certain manufacturer in producing q units a day is given by*

$$C(q) = -10q^2 + 300q + 130 \quad (0 \leq q \leq 15)$$

(a) *Find $C'(q)$.*

(b) *What is the rate of change of the total cost when the level of production is ten units?*

(c) *What is the average cost the manufacturer incurs when the level of production is ten units?*

Exercise 4.2.11 *A certain manufacturer produces beach umbrellas with a daily cost $C(q)$ of*

$$C(q) = 0.000002q^3 + 5q + 400.$$

Calculate

$$\frac{C(100+h) - C(100)}{h}$$

for $h = 1, 0.1, 0.001$, and 0.0001. Use your results to estimate the rate of change of the total cost function when the level of production is 100 units/day.

Exercise 4.2.12 *The gross domestic product (GDP) of a certain country over 8 years is approximated by*

$$G(t) = -0.2t^3 + 2.4t^2 + 60 \quad (0 \leq t \leq 8),$$

billions of dollars, where $t = 0$ corresponds to 1992. The derivative of $G(t)$ is given by

$$G'(t) = -0.6t^2 + 4.8t$$

(a) *Compute $G'(0), G'(1), ..., G'(8)$.*

(b) *Compute $G''(0), G''(1), ..., G''(8)$.*

(c) *What can you deduce from your results?*

Exercise 4.2.13 *It is estimated that the number $N(t)$ of individuals infected with a certain contagious disease is*

$$N(t) = -0.1t^3 + 1.5t^2 + 100 \quad (0 \leq t \leq 7)$$

where t is in months and $t = 0$ corresponds to the initial outbreak. The derivative of $N(t)$ is given by

$$N'(t) = -0.3t^2 + 3t.$$

After 4 months, a drug which reduces the infectiousness of the disease is developped.

(a) *Verify that the number of infected individuals was increasing for 7 months. (Hint: Compute $N'(0), N'(1), \ldots, N'(7)$*

(b) *Show that the drug was working by computing $N''(4), N''(5), N''(6)$ and $N''(7)$.*

Exercise 4.2.14 *A certain manufacturer determines that the amount of defective products coming out of their new Ontario plant t days after it opened can be estimated by*

$$A(t) = -0.00006t^5 + 0.00468t^4 - 0.1316t^3 + 1.915t^2 - 17.63t + 100$$

percent of the total number of units produced. The first and second derivatives of $A(t)$ are given:

$$A'(t) = -0.0003t^4 + 0.01872t^3 - 0.3948t^2 + 3.83t - 17.63$$

$$A''(t) = -0.0012t^3 + 0.05616t^2 - 0.7896t + 3.83.$$

Compute $A'(10)$ and $A''(10)$ and interpret your results.

Exercise 4.2.15 *Between the years* 1980 *and* 2001*, it is estimated that the percentage of young families who own their own home can be approximated by the function*

$$P(t) = 33.55(t+5)^{0.205} \quad (0 \leq t \leq 21)$$

where t is measured in years, with $t = 0$ corresponding to the beginning of 1980*. The second derivative of $P(t)$ is given by*

$$-5.46781(t+5)^{-1.795}.$$

Compute $P''(20)$ and interpret your results.

Exercise 4.2.16 *A certain manufacturer estimates that the total weekly cost in producing q units is*

$$C(q) = 2000 + 2q - 0.0001q^2 \quad 0 \leq q \leq 6000,$$

dollars.

(a) *What is the actual cost incurred in producing the* 1001*st and the* 2001*st unit?*

(b) *What is the marginal cost when $q = 1000$ and* 2000*?*

Exercise 4.2.17 *A movie theatre determines that their monthly revenue can be estimated by*

$$R(q) = 8000q - 100q^2$$

dollars when the price per movie ticket is q dollars.

(a) *Find the marginal revenue* R'.

(b) *Compute* $R'(39)$, $R'(40)$, *and* $R'(41)$.

(c) *Based on the results above, what price should the theatre charge in order to maximize their revenue?*

Exercise 4.2.18 *The demand function for a certain product is given by*

$$p = -0.04q + 800 \quad 0 \leq q \leq 20,000$$

where p *denotes the unit price in dollars and* q *denotes the quantity demanded.*

(a) *Determine the revenue function* R.

(b) *Determine the marginal revenue function* R'.

(c) *Compute* $R'(5000)$. *What can you deduce from your results?*

(d) *If the total cost in producing* q *units is given by*

$$C(q) = 200q + 300,000$$

determine the profit function $P(q)$.

(e) *Find the marginal profit function* P'.

(f) *Compute* $P'(5000)$ *and* $P'(8000)$.

(g) *Sketch the graph of the profit function. What can you deduce from your results?*

4.3 Derivative Rules

Using the definition of the derivative of a function is quite tedious. In this section we introduce a number of different shortcuts that can be used to compute the derivative. Recall that the *definition of derivative* is:

Given any number x for which the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, we assign to x the number $f'(x)$.

Next, we give some basic Derivative Rules for finding derivatives without having to use the limit definition directly.

Theorem 4.24: Derivative of a Constant Function

Let c *be a constant, then* $\frac{d}{dx}(c) = 0$.

Proof. Let $f(x) = c$ be a constant function. By the definition of derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

♣

We can also see the above theorem from a geometric point of view. Recall that the graph of a constant function is a horizontal straight line in the standard Cartesian coordinate system (see Figure 4.4). Now the tangent line to a horizontal straight line at any point on this line coincides with the line itself. But a horizontal line has slope zero, and therefore the slope of the tangent line must be zero as well.

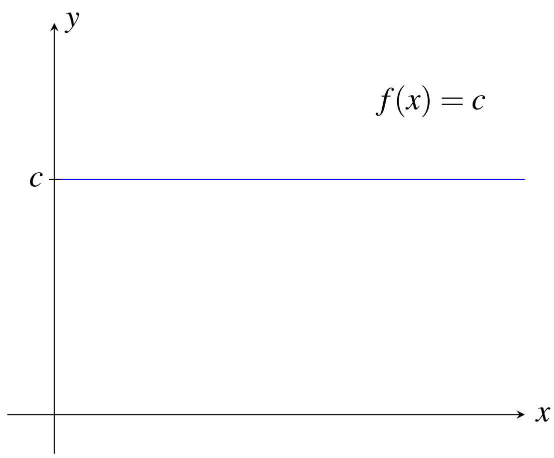


Figure 4.4: A constant function in the Cartesian coordinate system.

Example 4.25: Derivative of a Constant Function

The derivative of $f(x) = 17$ *is* $f'(x) = 0$ *since the derivative of a constant is* 0.

Theorem 4.26: Power Rule

If n is a positive integer, then $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

Proof. We use the formula:

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

which can be verified by multiplying out the right side. Let $f(x) = x^n$ be a power function for some positive integer n. Then at any number a we have:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) = na^{n-1}.$$

♣

It turns out that the Power Rule holds for any real number n; however, the proof of the Power Rule for the general case is a bit more difficult to prove and will be omitted.

Theorem 4.27: Power Rule (General)

If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Example 4.28: Derivative of a Power Function

By the Power Rule, the derivative of $g(x) = x^4$ is $g'(x) = 4x^3$.

Theorem 4.29: Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Proof. For convenience let $g(x) = cf(x)$. Then:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x),$$

where c can be moved in front of the limit by the Limit Rules. ♣

Example 4.30: Derivative of a Multiple of a Function

Find the derivative of

$$f(x) = 5x^3.$$

Solution. Using the Constant Multiple Rule, we obtain

$$f'(x) = \frac{d}{dx}\left(5x^3\right) = 5\frac{d}{dx}\left(x^3\right).$$

From the Power Rule, we have

$$\frac{d}{dx}\left(x^3\right) = 3x^2.$$

Putting together, we get that

$$\frac{d}{dx}\left(5x^3\right) = 5\left(3x^2\right) = 15x^2.$$

♣

Example 4.31: Derivative of a Multiple of a Function

Find the derivative of

$$f(x) = \frac{3}{\sqrt{x}}.$$

Solution. Rewrite $\frac{1}{\sqrt{x}}$ as $x^{-1/2}$. Now, we can again use the Constant Multiple and Power Rules:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(3x^{-1/2}\right) \\ &= 3\frac{d}{dx}\left(x^{-1/2}\right) \\ &= 3\left(-\tfrac{1}{2}x^{-3/2}\right) \\ &= -\frac{3}{2x^{3/2}} \end{aligned}$$

♣

Theorem 4.32: Sum and Difference Rules

If f and g are both differentiable functions, then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x),$$

and

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Proof. We demonstrate the proof for the Sum Rule and leave the proof for the Difference Rule to the reader.

For convenience, let $r(x) = f(x) + g(x)$. Then by the Limit Rules:

$$\begin{aligned} r'(x) &= \lim_{h\to 0}\frac{r(x+h)-r(x)}{h} \\ &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

♣

Example 4.33: Derivative of a Sum and Difference of Functions

Find the derivative of

$$f(x) = 4x^5 + 3x^4 - 8x^2 + x + 3.$$

Solution.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(4x^5 + 3x^4 - 8x^2 + x + 3\right) \\ &= \frac{d}{dx}(4x^5) + \frac{d}{dx}(3x^4) - \frac{d}{dx}(8x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(3) \\ &= 4\frac{d}{dx}(x^5) + 3\frac{d}{dx}(x^4) - 8\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(3) \\ &= 20x^4 + 12x^3 - 16x + 1 \end{aligned}$$

♣

Example 4.34: Tangent Line

Find the slope and an equation of the tangent line to the graph of $f(x) = 2x + \frac{1}{\sqrt{x}}$ *at the point* $(1,3)$.

Solution. The slope of the tangent line at any point on the graph of f is given by

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x + \frac{1}{\sqrt{x}}\right) \\ &= \frac{d}{dx}\left(2x + x^{-1/2}\right) \\ &= \frac{d}{dx}(2x) + \frac{d}{dx}(x^{-1/2}) \\ &= 2 - \tfrac{1}{2}x^{-3/2} \\ &= 2 - \frac{1}{2x^{3/2}} \end{aligned}$$

In particular, the slope of the tangent line to the graph of f at $(1,3)$ (where $x = 1$) is

$$f'(1) = 2 - \frac{1}{2(1)^{3/2}} = 2 - \frac{1}{2} = \frac{3}{2}.$$

Using the point-slope form of the equation of a line with slope $m = \frac{3}{2}$ and the point $(x_1, y_1) = (1,3)$, we see that an equation of the tangent line is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= \tfrac{3}{2}(x - 1), \end{aligned}$$

or, upon simplification,

$$y = \frac{3}{2}x + \frac{3}{2}.$$

♣

Example 4.35: Rate of Change Application

The demand function for a certain product is given by

$$p(x) = \frac{\sqrt{x}}{2} - \frac{x}{40} + 2000,$$

where p is the price measured in dollars and the quantity x is measured in units.

(a) Find the rate of change of price p per thousand products with respect to quantity x.

(b) How fast is the price changing with respect to x when $x = 25$ and $x = 400$? Interpret your result.

Solution.

(a) The rate of change of the price with respect to quantity is given by

$$\begin{aligned} p'(x) &= \frac{d}{dx}\left(\frac{\sqrt{x}}{2} - \frac{x}{40} + 2000\right) \\ &= \frac{d}{dx}\left(\frac{1}{2}x^{1/2} - \frac{1}{40}x + 2000\right) \\ &= \frac{1}{2}\frac{d}{dx}(x^{1/2}) - \frac{1}{40}\frac{d}{dx}(x) \\ &= \frac{1}{2}\left(\frac{1}{2}x^{-1/2}\right) - \frac{1}{40} \\ &= \frac{1}{4}x^{-1/2} - \frac{1}{40}. \end{aligned}$$

(b) When $x = 25$, we have

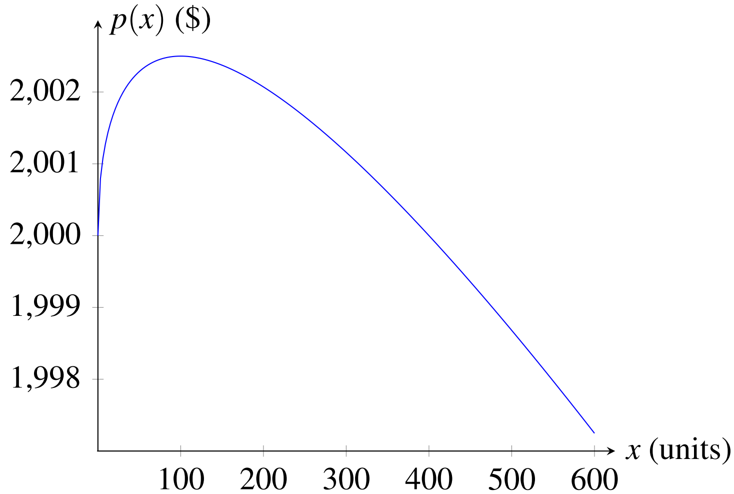
$$p'(25) = \frac{1}{4}(25)^{-1/2} - \frac{1}{40} = \frac{1}{20} - \frac{1}{40} = \frac{1}{40} = 0.025.$$

This means that when 25 products are demanded, one additional product demanded by consumers increases the price by \$0.025.

When $x = 400$, we have

$$p'(400) = \frac{1}{4}(400)^{-1/2} - \frac{1}{40} = \frac{1}{80} - \frac{1}{40} = -\frac{1}{80} = -0.0125.$$

This means that when 400 products are demanded, one additional product demanded by consumers decreases the price by \$0.0125. The graph of p is shown below. Notice that although the price is decreasing for large numbers of products demanded, the decrease is minimal.



♣

Example 4.36: Second Derivative

Find the second derivative of $f(x) = 5x^3 + 3x^2$.

Solution. We must differentiate $f(x)$ twice:

$$f'(x) = 15x^2 + 6x,$$

$$f''(x) = 30x + 6.$$

♣

Theorem 4.37: Product Rule

If f and g are both differentiable functions, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof. For convenience let $r(x) = f(x) \cdot g(x)$. Then:

$$\begin{aligned} r'(x) &= \lim_{h \to 0} \frac{r(x+h) - r(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \end{aligned}$$

As in the previous proof, we want to separate the functions f and g. The trick is to add and subtract $f(x+h)g(x)$ in the numerator. Then by Limit Rules:

$$\begin{aligned} r'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} \right] \\ &= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

♣

Note: we have just proven that the derivative of the product of two functions is *not* given by the product of the derivatives of the functions. In other words,

$$\frac{d}{dx}[f(x)g(x)] \neq \frac{d}{dx}f(x)\frac{d}{dx}g(x).$$

Example 4.38: Derivative of a Product of Functions

Find the derivative of $h(x) = (3x-1)(2x+3)$.

Solution. One way to do this question is to expand the expression. Alternatively, we use the Product Rule with $f(x) = 3x-1$ and $g(x) = 2x+3$. Note that $f'(x) = 3$ and $g'(x) = 2$, so,

$$h'(x) = (3)\cdot(2x+3) + (3x-1)\cdot(2) = 6x+9+6x-2 = 12x+7.$$

♣

Theorem 4.39: Quotient Rule

If f and g are both differentiable functions, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Proof. The proof is similar to the previous proof but the trick is to add and subtract the term $f(x)g(x)$ in the numerator. We omit the details. ♣

Example 4.40: Derivative of a Quotient of Functions

Find the derivative of $h(x) = \dfrac{3x-1}{2x+3}$.

Solution. By the Quotient Rule (using $f(x) = 3x-1$ and $g(x) = 2x+3$) we have:

$$h'(x) = \frac{\frac{d}{dx}(3x-1)\cdot(2x+3)-(3x-1)\cdot\frac{d}{dx}(2x+3)}{(2x+3)^2}$$
$$= \frac{3(2x+3)-(3x-1)(2)}{(2x+3)^2} = \frac{11}{(2x+3)^2}.$$

♣

Example 4.41: Rate of Change Application

The sales (in millions of dollars) of a DVD recording of a hit movie t years from the date of release is given by

$$S(t) = \frac{5t}{t^2+1}.$$

(a) Find the rate at which the sales are changing at time t.

(b) How fast are the sales changing at the time the DVDs are relased ($t = 0$)? Two years from the date of release?

Solution.

(a) The rate at which the sales are changing at time t is given by $S'(t)$. Using the Quotient Rule, we obtain

$$S'(t) = \frac{d}{dt}\left(\frac{5t}{t^2+1}\right) = 5\frac{d}{dt}\left(\frac{t}{t^2+1}\right)$$
$$= 5\left(\frac{(t^2+1)(1)-t(2t)}{(t^2+1)^2}\right)$$
$$= 5\left(\frac{t^2+1-2t^2}{(t^2+1)^2}\right)$$
$$= \frac{5(1-t^2)}{(t^2+1)^2}.$$

(b) The rate at which the sales are changing at the time the DVDs are released is given by

$$S'(0) = \frac{5(1-0)}{(0+1)^2} = 5.$$

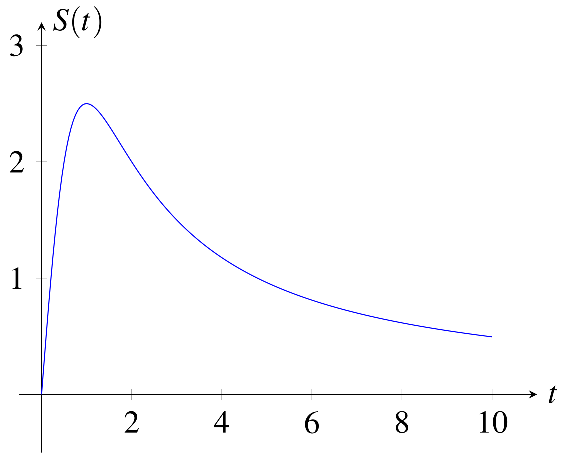
That is, they are increasing at the rate of \$5 million per year.

Two years from the date of release, the sales are changing at the rate of

$$S'(2) = \frac{5(1-4)}{(4+1)^2} = -0.6.$$

That is, they are decreasing at the rate of \$600,000 per year.

The graph of the function S is shown below where $S(t)$ is in millions of dollars and t is in years. Notice that after a spectacular rise, sales of the DVD begin to taper off.



♣

Exercises for Section 4.3

Exercise 4.3.1 *Find the derivatives of the following functions.*

(a) $f(x) = x^{100}$

(b) $f(t) = t^{-100}$

(c) $g(x) = \dfrac{1}{x^5}$

(d) $f(x) = x^{\pi}$

(e) $h(x) = x^{3/4}$

(f) $g(s) = s^{-9/7}$

(g) $f(x) = 5x^3 + 12x^2 - 15$

(h) $f(s) = -4s^5 + 3s^2 - 5/s^2$

(i) $f(x) = 5(-3x^2 + 5x + 1)$

(j) $f(t) = (t+1)(t^2 + 2t - 3)$

(k) $h(x) = (x+1)(x^2 + 2x - 3)^{-1}$

(l) $g(x) = x^3(x^3 - 5x + 10)$

(m) $g(s) = (s^2 + 5s - 3)(s^5)$

(n) $f(x) = (x^2 + 5x - 3)(x^{-5})$

(o) $h(x) = (5x^3 + 12x^2 - 15)^{-1}$

Exercise 4.3.2 *Find an equation for the tangent line to* $f(x) = x^3/4 - 1/x$ *at* $x = -2$.

Exercise 4.3.3 *Find an equation for the tangent line to* $f(x) = 3x^2 - \pi^3$ *at* $x = 4$.

Exercise 4.3.4 *Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t.*

Exercise 4.3.5 *Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.*

Exercise 4.3.6 *The general polynomial P of degree n in the variable x has the form* $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. *What is the derivative (with respect to x) of P?*

Exercise 4.3.7 *Find a cubic polynomial whose graph has horizontal tangents at* $(-2,5)$ *and* $(2,3)$.

Exercise 4.3.8 *Prove that* $\frac{d}{dx}(cf(x)) = cf'(x)$ *using the definition of the derivative.*

Exercise 4.3.9 *Suppose that f and g are differentiable at x. Show that* $f - g$ *is differentiable at x using the two linearity properties from this section.*

Exercise 4.3.10 *Use the Product Rule to compute the derivative of* $f(x) = (2x-3)^2$. *Sketch the function. Find an equation of the tangent line to the curve at* $x = 2$. *Sketch the tangent line at* $x = 2$.

Exercise 4.3.11 *Suppose that f, g, and h are differentiable functions. Show that* $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

Exercise 4.3.12 *Compute the derivative of* $\frac{x^3}{x^3 - 5x + 10}$.

Exercise 4.3.13 *Compute the derivative of* $\frac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1}$.

Exercise 4.3.14 *Compute the derivative of* $\frac{x}{\sqrt{x - 625}}$.

Exercise 4.3.15 *Compute the derivative of* $\frac{\sqrt{x-5}}{x^{20}}$.

Exercise 4.3.16 *Find an equation for the tangent line to* $f(x) = (x^2 - 4)/(5 - x)$ *at* $x = 3$.

Exercise 4.3.17 *Find an equation for the tangent line to* $f(x) = (x-2)/(x^3 + 4x - 1)$ *at* $x = 1$.

Exercise 4.3.18 *If* $f'(4) = 5$, $g'(4) = 12$, $(fg)(4) = f(4)g(4) = 2$, *and* $g(4) = 6$, *compute* $f(4)$ *and* $\frac{d}{dx}\frac{f}{g}$ *at 4.*

Exercise 4.3.19 *Let* $f(x) = x^3$. *Find the point on the graph of f where the tangent line is horizontal. Sketch the graph of f and draw the horizontal tangent line.*

Exercise 4.3.20 *Let* $f(x) = x^3 + 1$.

(a) Find the point(s) on the graph of f where the slope of the tangent line is equal to 12.

(b) Find the equation(s) of the tangent line(s) of part (a).

(c) Sketch the graph of f showing the tangent line(s).

Exercise 4.3.21 *Let* $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 6$. *Find the values of x for which*

(a) $f'(x) = -12$

(b) $f'(x) = 0$

(c) $f'(x) = 12$

Exercise 4.3.22 *An economy's consumer price index (CPI) is described by the function*

$$I(t) = -0.2t^3 + 3t^2 + 100 \quad 0 \leq t \leq 10$$

where $t = 0$ *corresponds to* 1994.

(a) *At what rate was the CPI changing in* 1999? *In* 2001? *In* 2004?

(b) *What was the average rate of increase in the CPI over the period from* 1999 *to* 2004?

Exercise 4.3.23 *The demand function for the Luminar desk lamp is given by*

$$p = f(x) = -0.1x^2 - 0.4x + 35$$

where x is the quantity demanded (measured in thousands) and p is the unit price in dollars.

(a) *Find* $f'(x)$.

(b) *What is the rate of change of the unit price when the quantity demanded is* 10,000 *units* ($x = 10$)? *What is the unit price at that level of demand?*

Exercise 4.3.24 *The supply function for a certain make of transistor radio is given by*

$$p = f(x) = 0.00001x^{5/4} + 10$$

where x is the quantity supplied and p us the unit price in dollars.

(a) *Find* $f'(x)$.

(b) *What is the rate of change of the unit price if the quantity supplied is* 10,000 *radios?*

Exercise 4.3.25 *Despite efforts at cost containment, the cost of medical care is increasing. Two major reasons for this increase are an aging population and extensive use by physicians of new technologies. Health-care spending through the year* 2000 *may be approximated by*

$$S(t) = 0.02836t^3 - 0.05167t^2 + 9.60881t + 41.9 \quad 0 \leq t \leq 35$$

where $S(t)$ *is the spending in millions of dollars and t is measured in years, with* $t = 0$ *corresponding to the beginning of* 1965.

(a) *Find an expression for the rate of change of health-care spending at any time t.*

(b) *How fast was health-care spending changing at the beginning of* 1980*? At the beginning of* 2000*?*

(c) *What was the amount of health-care spending at the beginning of* 1980*? At the beginning of* 2000*?*

Exercise 4.3.26 *Find the point(s) on the graph of the function*

$$f(x) = (x^2+6)(x-5)$$

where the slope of the tangent line is equal to -2.

Exercise 4.3.27 *A straight line perpendicular to and passing through the point of tangency of the tangent line is called the normal to the curve. Find the equation of the tangent line and the normal to the curve $y = \frac{1}{1+x^2}$ at the point $(1, \frac{1}{2})$.*

Exercise 4.3.28 *The total revenue in dollars for a video game is given by*

$$R(x) = \tfrac{1}{100}(x+2000)(1600-x) - 36,000$$

where x is the number of units sold. What is the rate of change of revenue with respect to x when 600 *units are sold? Interpret your result.*

Exercise 4.3.29 *A city's main well was recently found to be contaminated with trichloroethylene, a cancer-causing chemical, as a result of an abandoned chemical dump leaching chemicals into the water. A proposal submitted to the city's council members indicates that the cost, measured in millions of dollars, of removing x% of the toxic pollutant is given by*

$$C(x) = \frac{0.5x}{100-x}.$$

Find $C'(80), C'(90), C'(95), C'(99)$. What do your results tell you about the cost of removing all of the pollutant?

Exercise 4.3.30 *The total worldwide box-office receipts for a long-running movie are approximated by the function*

$$T(x) = \frac{120x^2}{x^2+4}$$

where $T(x)$ is measured in millions of dollars and x is the number of years since the movie's release. How fast are the total receipts changing 1, 3 *and* 5 *years after its release?*

Exercise 4.3.31 *Grand & Toy makes a line of executive desks. It is estimated that the total cost for making x units of their Senior Executive model is*

$$C(x) = 100x + 200,000$$

dollars/year

(a) *Find the average cost function,* $\overline{C}$.

(b) *Find the marginal average cost function,* $\overline{C}'$.

(c) *What happens to* $\overline{C}(x)$ *when* x *is very large? Interpret your results.*

Exercise 4.3.32 *The quantity of Sicard wristwatches demanded each month is related to the unit price by the equation*

$$p = \frac{50}{0.001x^2+1} \quad 0 \leq x \leq 20$$

where p *is measured in dollars and* x *in units of a thousand.*

(a) *Find the revenue function* R.

(b) *Find the marginal revenue function* R'.

(c) *Compute* $R'(2)$ *and interpret your result.*

4.4 The Chain Rule

Let $h(x) = \sqrt{625-x^2}$. The rules stated previously do not allow us to find $h'(x)$. However, $h(x)$ is a composition of two functions. Let $f(x) = \sqrt{x}$ and $g(x) = 625 - x^2$. Then we see that

$$h(x) = (f \circ g)(x).$$

From our rules we know that $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = -2x$, thus it would be convenient to have a rule which allows us to differentiate $f \circ g$ in terms of f' and g'. This gives rise to the **Chain Rule**.

Theorem 4.42: Chain Rule

If g *is differentiable at* x *and* f *is differentiable at* $g(x)$, *then the composite function* $h = f \circ g$ *[recall* $f \circ g$ *is defined as* $f(g(x))$*] is differentiable at* x *and* $h'(x)$ *is given by:*

$$h'(x) = f'(g(x)) \cdot g'(x).$$

The Chain Rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity $f'(g(x))$ is the derivative of f with x replaced by g; this can be written df/dg. As usual, $g'(x) = dg/dx$. Then the Chain Rule becomes

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

This looks like trivial arithmetic, but it is not: dg/dx is not a fraction, that is, not literal division, but a single symbol that means $g'(x)$. Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the Chain Rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

Example 4.43: Chain Rule

Compute the derivative of $\sqrt{625-x^2}$.

Solution. We already know that the answer is $-x/\sqrt{625-x^2}$, computed directly from the limit. In the context of the Chain Rule, we have $f(x) = \sqrt{x}$, $g(x) = 625-x^2$. We know that $f'(x) = (1/2)x^{-1/2}$, so $f'(g(x)) = (1/2)(625-x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x) = -2x$ we have

$$f'(g(x))g'(x) = \frac{1}{2\sqrt{625-x^2}}(-2x) = \frac{-x}{\sqrt{625-x^2}}.$$

♣

Example 4.44: Chain Rule

Compute the derivative of $1/\sqrt{625-x^2}$.

Solution. This is a quotient with a constant numerator, so we could use the Quotient Rule, but it is simpler to use the Chain Rule. The function is $(625-x^2)^{-1/2}$, the composition of $f(x) = x^{-1/2}$ and $g(x) = 625-x^2$. We compute $f'(x) = (-1/2)x^{-3/2}$ using the Power Rule, and then

$$f'(g(x))g'(x) = \frac{-1}{2(625-x^2)^{3/2}}(-2x) = \frac{x}{(625-x^2)^{3/2}}.$$

♣

Suppose the only data available to you are some points on the graphs of a function $y = f(x)$ and its derivative $f'(x)$.

Example 4.45: Chain Rule and Data Points

Given

$$f(2) = -1\ ,\ f(-1) = 3\ ,\ f'(2) = 4\ ,\ f'(-1) = -5 \text{ and,}$$

$$g(2) = 2\ ,\ g(-1) = -2\ ,\ g'(-1) = 0$$

If possible, find the following derivatives:

(a) $(f \circ g)'(2)$

(b) $(f \circ f)'(2)$

(c) $(g \circ f)'(-1)$

Solution.

(a) $(f \circ g)'(2) = f'(g(2)) \cdot g'(2) = f'(2) \cdot 7 = 4 \cdot 7 = 28$

(b) $(f \circ f)'(2) = f'(f(2)) \cdot f'(2) = f'(-1) \cdot 4 = (-5) \cdot 4 = -20$

(c) $(g \circ f)'(-1) = g'(f(-1)) \cdot f'(-1) = g'(3) \cdot (-5)$, which cannot be found since we do not know what $g'(-3)$ evaluates to.

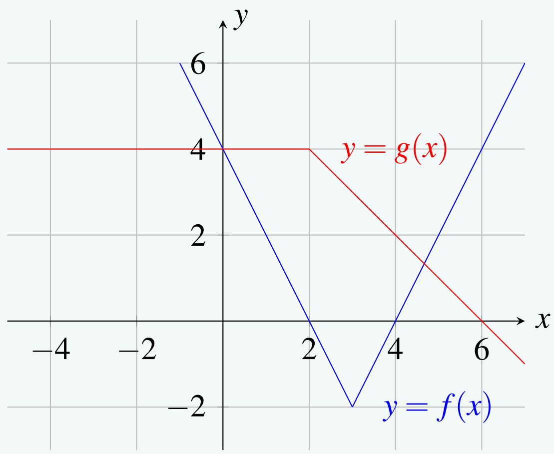
♣

Example 4.46: Investigating $f(g(a))$ and $g(f(a))$

The graphs of the functions $y = f(x)$ and $y = g(x)$ are given below. Suppose $h(x) = f(g(x))$ and $k(x) = g(f(x))$.

(a) Find $h(4)$ and $k(4)$. Are those values the same?

(b) Find $h'(4)$ and $k'(4)$. Are those values the same?



Solution.

(a) Since $h(x) = f(g(x))$, we have
$$h(4) = f(g(4)).$$
From the graph of g (in red), we read off that $g(4) = 2$, therefore
$$h(4) = f(g(4)) = f(2).$$
From the graph of f (in blue), we read off that $f(2) = 0$, therefore
$$h(4) = f(2) = 0.$$
Since $k(x) = g(f(x))$, we similarly read off from the graphs of f and g respectively to get
$$k(4) = g(f(4)) = g(0) = 4.$$
Comparing values, we see that $h(4) \neq k(4)$.

(b) By the Chain Rule, we have

$$h'(x) = f'(g(x)) \cdot g'(x) \text{ and,}$$
$$k'(x) = g'(f(x)) \cdot f'(x).$$

Therefore,

$$h'(4) = f'(g(4)) \cdot g'(4).$$

We already know from (1) that $g(4) = 2$, and so

$$h'(4) = f'(2) \cdot g'(4).$$

Now, the slope of the graph of f at $x = 2$ is read off as $-2/1 = -2$, and the slope of the graph of g at $x = 4$ is read off as $-1/1 = -1$. We compute

$$h'(4) = f'(2) \cdot g'(4) = (-2)(-1) = 2.$$

Similarly, we compute

$$k'(4) = g'(f(4)) \cdot f'(4) = g'(0) \cdot f'(4) = (0)(-2/1) = 0.$$

Comparing values, we conclude that $h'(4) \neq k'(4)$.

♣

Note: In general,

$$f(g(x)) \neq g(f(x)) \quad \text{and,}$$
$$\frac{d}{dx}(f(g(x))) \neq \frac{d}{dx}(g(f(x)))$$

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

Example 4.47: Derivative of Quotient

Compute the derivative of

$$f(x) = \frac{x^2 - 1}{x\sqrt{x^2 + 1}}.$$

Solution. The "last" operation here is division, so to get started we need to use the Quotient Rule first. This gives

$$\begin{aligned} f'(x) &= \frac{(x^2-1)'x\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)} \\ &= \frac{2x^2\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \end{aligned}$$

Now we need to compute the derivative of $x\sqrt{x^2+1}$. This is a product, so we use the Product Rule:

$$\frac{d}{dx}x\sqrt{x^2+1} = x\frac{d}{dx}\sqrt{x^2+1} + \sqrt{x^2+1}.$$

Finally, we use the Chain Rule:

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(x^2+1)^{1/2} = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

And putting it all together:

$$\begin{aligned} f'(x) &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \\ &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)\left(x\frac{x}{\sqrt{x^2+1}}+\sqrt{x^2+1}\right)}{x^2(x^2+1)}. \end{aligned}$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left. ♣

Using the Chain Rule, the Power Rule, and the Product Rule, it is possible to avoid using the Quotient Rule entirely.

Example 4.48: Derivative of Quotient without Quotient Rule

Compute the derivative of $f(x) = \dfrac{x^3}{x^2+1}$.

Solution. Write $f(x) = x^3(x^2+1)^{-1}$, then

$$\begin{aligned} f'(x) &= x^3\frac{d}{dx}(x^2+1)^{-1} + 3x^2(x^2+1)^{-1} \\ &= x^3(-1)(x^2+1)^{-2}(2x) + 3x^2(x^2+1)^{-1} \\ &= -2x^4(x^2+1)^{-2} + 3x^2(x^2+1)^{-1} \\ &= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2}{x^2+1} \\ &= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2(x^2+1)}{(x^2+1)^2} \\ &= \frac{-2x^4+3x^4+3x^2}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} \end{aligned}$$

Note that we already had the derivative on the second line; all the rest is simplification. It is easier to get to this answer by using the Quotient Rule, so there's a trade off: more work for fewer memorized formulas. ♣

Example 4.49: Chain Rule and Tangent Line

Find the slope of the tangent line to the graph of the function

$$f(x) = \left(\frac{2x+1}{3x+2}\right)^3$$

at the point $\left(0, \frac{1}{8}\right)$.

Solution. The slope of the tangent line to the graph of f at any point is given by $f'(x)$. To compute $f'(x)$, we use the general Power Rule followed by the Quotient Rule, obtaining,

$$\begin{aligned} f'(x) &= 3\left(\frac{2x+1}{3x+2}\right)^2 \frac{d}{dx}\left(\frac{2x+1}{3x+2}\right) \\ &= 3\left(\frac{2x+1}{3x+2}\right)^2 \left[\frac{(3x+2)(2)-(2x+1)(3)}{(3x+2)^2}\right] \\ &= 3\left(\frac{2x+1}{3x+2}\right)^2 \left[\frac{6x+4-6x-3}{(3x+2)^2}\right] \\ &= \frac{3(2x+1)^2}{(3x+2)^4} \end{aligned}$$

In particular, the slope of the tangent line to the graph at $\left(0, \frac{1}{8}\right)$ is given by

$$f'(0) = \frac{3(0+1)^2}{(0+2)^4} = \frac{3}{16}.$$

♣

Example 4.50: Chain of Composition

Compute the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$.

Solution. Here we have a more complicated chain of compositions, so we use the Chain Rule three times. At the outermost "layer" we have the function $g(x) = 1+\sqrt{1+\sqrt{x}}$ plugged into $f(x) = \sqrt{x}$, so applying the Chain Rule once gives

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2} \frac{d}{dx}\left(1+\sqrt{1+\sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1+\sqrt{x}}$. Using the Chain Rule again:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\begin{aligned}\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} &= \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}. \\ &= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}\end{aligned}$$

♣

To see the full power of the Chain Rule, we shall consider more complex compositions of functions.

Example 4.51: Complex Chain of Composition

Suppose we are given the functions $f(x)=\sqrt[5]{x}$, $g(x)=\left(x^2+3x\right)^{24}$, *and,* $h(x)=\frac{1}{x+2}$. *Find the first derivative of the following compositions:*

(a) $(f\circ h\circ g)(x)$

(b) $(g\circ f\circ h)(x)$

Solution. We will first make some general observations. By applying the Chain Rule twice, we find that the derivative of $(f\circ g\circ h)(x)$ is

$$\begin{aligned}\frac{d}{dx}(f\circ g\circ h)(x) &= \frac{d}{dx}(f(g(h(x)))) \\ &= f'(g(h(x))\cdot\frac{d}{dx}(g(h(x))) \\ &= f'(g(h(x))\cdot g'(h(x))\cdot h'(x).\end{aligned}$$

The derivative of f is

$$f'(x)=\frac{d}{dx}x^{1/5}=\frac{1}{5}x^{-4/5}.$$

The derivative of g is

$$\begin{aligned}g'(x) &= \frac{d}{dx}\left(x^2+3x\right)^{24} \\ &= 24\left(x^2+3x\right)^{23}\cdot\frac{d}{dx}\left(x^2+3x\right) \\ &= 24\left(x^2+3x\right)^{23}\cdot(2x+3) \\ &= 24(2x+3)\left(x^2+3x\right)^{23}.\end{aligned}$$

The derivative of h is

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left(\frac{1}{x+2}\right) \\
&= \frac{(x+2)\frac{d}{dx}(1) - 1\frac{d}{dx}(x+2)}{(x+2)^2} \\
&= \frac{0(x+2) - 1(1)}{(x+2)^2} \\
&= -\frac{1}{(x+2)^2}.
\end{aligned}$$

(a) The derivative of $(f \circ h \circ g)(x)$ is therefore given by

$$\frac{d}{dx}(f \circ h \circ g)(x) = f'(h(g(x)) \cdot h'(g(x)) \cdot g'(x).$$

We can now use the derivatives of f, g and h we calculated above and evaluate them at the appropriate inner functions.

$$\begin{aligned}
f'(h(g(x)) &= f'\left(h(x^2+3x)^{24})\right) \\
&= f'\left(\frac{1}{(x^2+3x)^{24}+2}\right) \\
&= \frac{1}{5}\left(\frac{1}{(x^2+3x)^{24}+2}\right)^{-4/5} \\
&= \frac{1}{5}\left((x^2+3x)^{24}+2\right)^{4/5},
\end{aligned}$$

and

$$h'(g(x)) = h'\left((x^2+3x)^{24}\right) = -\frac{1}{\left((x^2+3x)^{24}+2\right)^2}.$$

We can finally put together the derivative we are seeking,

$$\begin{aligned}
\frac{d}{dx}((f \circ h \circ g)(x)) &= \frac{1}{5}\left((x^2+3x)^{24}+2\right)^{4/5} \cdot \left(-\frac{1}{\left((x^2+3x)^{24}+2\right)^2}\right) \cdot 24\,(2x+3)\left(x^2+3x\right)^{23} \\
&= -\frac{24(2x+3)(x^2+3x)^{23}((x^2+3x)^{24}+2)^{4/5}}{5((x^2+3x)^{24}+2)^2}.
\end{aligned}$$

(b) In a similar fashion, we find the first derivative of $(g \circ f \circ h)(x)$ to be

$$\begin{aligned}
\frac{d}{dx}(g \circ f \circ h)(x) &= 24\left(2\sqrt[5]{\frac{1}{x+2}}+3\right)\left(\sqrt[5]{\frac{1}{x+2}}^{2}+3\sqrt[5]{\frac{1}{x+2}}\right)^{23}\cdot\frac{1}{5}\left(\frac{1}{x+2}\right)^{-4/5}\cdot\left(-\frac{1}{(x+2)^2}\right)\\
&= 24\left(2\sqrt[5]{\frac{1}{x+2}}+3\right)\left(\sqrt[5]{\frac{1}{x+2}}^{2}+3\sqrt[5]{\frac{1}{x+2}}\right)^{23}\cdot\frac{1}{5}(x+2)^{4/5}\cdot\left(-\frac{1}{(x+2)^2}\right)\\
&= \frac{24\left(2\sqrt[5]{\frac{1}{x+2}}+3\right)\left(\sqrt[5]{\frac{1}{x+2}}^{2}+3\sqrt[5]{\frac{1}{x+2}}\right)^{23}(x+2)^{4/5}}{5(x+2)^2}
\end{aligned}$$

♣

Example 4.52: Rate of Change Application

The membership of The Fitness Centre, which opened a few years ago, is approximated by the function

$$N(t) = 100(64+4t)^{2/3} \quad 0 \le t \le 52$$

where $N(t)$ gives the number of members at the beginning of week t.

(a) *Find $N'(t)$.*

(b) *How fast was the centre's membership increasing initially ($t = 0$)?*

(c) *How fast was the membership increasing at the beginning of the 40th week?*

(d) *What was the membership when the centre first opened? At the beginning of the 40th week?*

Solution.

(a) Using the general Power Rule, we obtain

$$\begin{aligned}
N'(t) &= \frac{d}{dt}\left[100(64+4t)^{2/3}\right]\\
&= 100\frac{d}{dt}(64+4t)^{2/3}\\
&= \frac{200}{3}(64+4t)^{-1/3}\frac{d}{dt}(64+4t)\\
&= \frac{200}{3}(64+4t)^{-1/3}(4)\\
&= \frac{800}{3(64+4t)^{1/3}}
\end{aligned}$$

(b) The rate at which the membership was increasing when the centre first opened is given by

$$N'(0) = \frac{800}{3(64)^{1/3}} \approx 66.7,$$

or approximately 67 people per week.

(c) The rate at which the membership was increasing at the beginning of the 40th week is given by

$$N'(40) = \frac{800}{3(64+160)^{1/3}} \approx 43.9,$$

or approximately 44 people per week.

(d) The membership when the centre first opened is given by

$$N(0) = 100(64)^{2/3} = 100(16),$$

or approximately 1600 people. The membership at the beginning of the 40th week is given by

$$N(40) = 100(64+160)^{2/3} \approx 3688.3,$$

or approximately 3688 people.

♣

Exercises for Section 4.4

Exercise 4.4.1 *Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.*

(a) $f(x) = x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$

(b) $f(x) = x^3 - 2x^2 + 4\sqrt{x}$

(c) $g(t) = (t^2+1)^3$

(d) $h(x) = x\sqrt{169-x^2}$

(e) $f(t) = (t^2-4t+5)\sqrt{25-t^2}$

(f) $f(x) = \sqrt{r^2-x^2}$, *r is a constant*

(g) $h(s) = \sqrt{1+s^4}$

(h) $g(x) = \dfrac{1}{\sqrt{5-\sqrt{x}}}$

(i) $f(x) = (1+3x)^2$

(j) $f(x) = \dfrac{(x^2+x+1)}{(1-x)}$

(k) $h(x) = \dfrac{\sqrt{25-x^2}}{x}$

(l) $f(t) = \sqrt{\dfrac{169}{t} - t}$

(m) $f(x) = \sqrt{x^3 - x^2 - (1/x)}$

(n) $f(x) = 100/(100-x^2)^{3/2}$

(o) $g(t) = \sqrt[3]{t+t^3}$

(p) $h(x) = \sqrt{(x^2+1)^2 + \sqrt{1+(x^2+1)^2}}$

(q) $f(x) = (x+8)^5$

Exercise 4.4.2 *Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.*

(a) $f(t) = (4-t)^3$

(b) $f(x) = (x^2+5)^3$

(c) $f(x) = (6-2x^2)^3$

(d) $g(x) = (1-4x^3)^{-2}$

(e) $f(x) = 5(x+1-1/x)$

(f) $f(t) = 4(2t^2-t+3)^{-2}$

(g) $h(s) = \dfrac{1}{1+1/s}$

(h) $f(x) = \dfrac{-3}{4x^2-2x+1}$

(i) $f(x) = (x^2+1)(5-2x)/2$

(j) $g(t) = (3t^2+1)(2t-4)^3$

(k) $f(x) = \dfrac{x+1}{x-1}$

(l) $h(s) = \dfrac{s^2-1}{s^2+1}$

(m) $h(t) = \dfrac{(t-1)(t-2)}{t-3}$

(n) $g(x) = \dfrac{2x^{-1}-x^{-2}}{3x^{-1}-4x^{-2}}$

(o) $f(x) = 3(x^2+1)(2x^2-1)(2x+3)$

(p) $f(x) = \dfrac{1}{(2x+1)(x-3)}$

(q) $f(s) = ((2s+1)^{-1}+3)^{-1}$

(r) $h(x) = (2x+1)^3(x^2+1)^2$

Exercise 4.4.3 *Find an equation for the tangent line to* $f(x) = (x-2)^{1/3}/(x^3+4x-1)^2$ *at* $x = 1$.

Exercise 4.4.4 *Find an equation for the tangent line to* $y = 9x^{-2}$ *at* $(3,1)$.

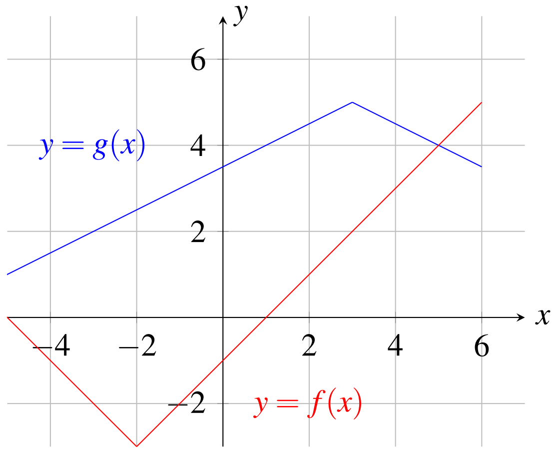
Exercise 4.4.5 *Find an equation for the tangent line to* $(x^2-4x+5)\sqrt{25-x^2}$ *at* $(3,8)$.

Exercise 4.4.6 *Find an equation for the tangent line to* $\dfrac{(x^2+x+1)}{(1-x)}$ *at* $(2,-7)$.

Exercise 4.4.7 *Find an equation for the tangent line to* $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ *at* $(1,\sqrt{4+\sqrt{5}})$.

Exercise 4.4.8 *Let* $y = f(x)$ *and* $x = g(t)$. *If* $g(1) = 2$, $f(2) = 3$, $g'(1) = 4$ *and* $f'(2) = 5$, *find the derivative of* $f \circ g$ *at 1.*

Exercise 4.4.9 *Use the graphical information of the functions f and g to find the derivative of the composition function and the given x-value.*



(a) $(f\circ g)'(2)$

(b) $(f\circ f)'(1)$

(c) $(g\circ f)'(3)$

(d) $(f\circ g)'(3)$

(e) $(g\circ f)'(-2)$

(f) $(g\circ f)'(-4)$

Exercise 4.4.10 *Express the derivative of $g(x)=x^2f(x^2)$ in terms of f and the derivative of f.*

Exercise 4.4.11 *Let $F(x)=f(f(x))$. Does it follow that $F'(x)=[f'(x)]^2$? Hint: Let $f(x)=x^2$.*

Exercise 4.4.12 *Suppose $h=g\circ f$. Does it follow that $h'=g'\circ f'$? Hint: Let $f(x)=x$ and $g(x)=x^2$.*

Exercise 4.4.13 *The number of viewers of a television series introduced several years ago is approximated by the function*

$$N(x)=(60+2x)^{2/3}\quad 1\leq x\leq 26$$

where $N(x)$ (measured in thousands) denotes the number of weekly viewers of the series in the x-th week. Find the rate of increase of the weekly audience at the end of week 2 and at the end of week 12. How many viewers were there in week 2? In week 24?

Exercise 4.4.14 *The registrar of a prominent city in Montreal estimates that the total student enrollment in the Continuing Education division will be given by*

$$N(t)=-\frac{20,000}{\sqrt{1+0.2t}}+21,000$$

where $N(t)$ denotes the number of students enrolled in the division t yr from now. Find an expression for $N'(t)$. How fast is the student enrollment increasing currently? How fast will it be increasing 5 yr from now?

Exercise 4.4.15 *The president of a major housing construction firm claims that the number of construction jobs created is given by*

$$N(x)=1.42x$$

where x denotes the number of housing starts. Suppose the number of housing starts in the next t mo is expected to be

$$x(t) = \frac{7t^2 + 140t + 700}{3t^2 + 80t + 550}$$

million units/year. Find an expression that gives the rate of change at which the number of construction jobs will be created t mo from now. At what rate will construction jobs be created 1 yr from now?

Exercise 4.4.16 *Government economists of a developing country determined that the purchase of imported perfume is related to a proposed "luxury tax" by the formula*

$$N(x) = \sqrt{10{,}000 - 40x - 0.02x^2} \quad 0 \le x \le 200$$

where $N(x)$ measured the percentage of normal consumption of perfume when a "luxury tax" of x% is imposed on it. Find the rate of change of $N(x)$ for taxes of 10, 100, and 150 %

Exercise 4.4.17 *The quantity demanded per month, x, of a certain make of personal computer (PC) is related to the average unit price,p (in dollars), of PCs by the equation*

$$x = f(p) = \frac{100}{9}\sqrt{810{,}000 - p^2}.$$

It is estimated that t mo from now, the average price of a PC will be given by

$$p(t) = \frac{400}{1 + \frac{\sqrt{t}}{8}} + 200 \quad 0 \le t \le 60$$

dollars. Find the rate at which the quantity demanded per month of the PCs will be changed 16 mo from now.

4.5 Derivative Rules for Trigonometric Functions

We next look at the derivative of the sine function. In order to prove the derivative formula for sine, we recall two limit computations from earlier:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0,$$

and the double angle formula

$$\sin(A + B) = \sin A \cos B + \sin B \cos A.$$

Theorem 4.53: Derivative of Sine Function

$(\sin x)' = \cos x$

Proof. Let $f(x) = \sin x$. Using the definition of derivative we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x \end{aligned}$$

since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$$

♣

A formula for the derivative of the *cosine function* can be found in a similar fashion:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Using the Quotient Rule we get formulas for the remaining trigonometric ratios. To summarize, here are the derivatives of the six trigonometric functions:

Theorem 4.54: Derivatives of Basic Trigonometric Functions

$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$

Example 4.55: Derivative of Product of Trigonometric Functions

Find the derivative of $f(x) = \sin x \tan x$.

Solution. Using the Product Rule we obtain

$$f'(x) = \cos x \tan x + \sin x \sec^2 x.$$

♣

Example 4.56: Chain Rule with Trigonometric Functions

Differentiate each of the following functions:

(a) $h(x) = (x + \sin(x^2))^{10}$

(b) $k(x) = \sin(\cos^2 x)$

Solution.

(a) We use the Chain Rule and the general Power Rule to find

$$h'(x) = 10(x + \sin x^2)^9 \frac{d}{dx}(x + \sin x^2)$$

We apply the Chain Rule once more using the derivatives of trigonometric functions:

$$\begin{aligned} h'(x) &= 10(x + \sin x^2)^9 \left[1 + \cos(x^2)\frac{d}{dx}(x^2)\right] \\ &= 10(x + \sin x^2)^9(1 + 2x\cos(x^2)) \end{aligned}$$

(b) We first rewrite $k(x)$ as $k(x) = \sin((\cos x)^2)$. Then

$$\begin{aligned} k'(x) &= \cos(\cos^2 x)\frac{d}{dx}((\cos x)^2) \\ &= \cos(\cos^2 x) 2\cos x \frac{d}{dx}(\cos x) \\ &= \cos(\cos^2 x) 2\cos x(-\sin x) \\ &= -2(\sin x)(\cos x)(\cos(\cos^2 x)) \end{aligned}$$

♣

Example 4.57: Tangent Line

Find an equation of the tangent line to the graph of the function $f(x) = \tan 2x$ *at the point* $\left(\frac{\pi}{8}, 1\right)$.

Solution. The slope of the tangent line at any point on the graph of f is given by

$$f'(x) = 2\sec^2(2x).$$

In particular, the slope of the tangent line at the point $\left(\frac{\pi}{8}, 1\right)$ is given by

$$f'\left(\frac{\pi}{8}\right) = 2\sec^2\left(\frac{\pi}{4}\right) = 2\left(\frac{2}{\sqrt{2}}\right)^2 = 4.$$

Therefore, a required equation is given by

$$y-1=4\left(x-\frac{\pi}{8}\right),$$

or,

$$y=4x+\left(1-\frac{\pi}{2}\right).$$

♣

Exercises for Section 4.5

Exercise 4.5.1 *Find the derivatives of the following functions.*

(a) $f(x)=\sin x\cos x$

(b) $f(x)=\cot x$

(c) $f(x)=\csc x - x\tan x$

(d) $h(x)=2\cos\pi x$

(e) $f(t)=\sin(t^2+1)$

(f) $f(x)=\tan 2x^2$

(g) $g(x)=x\sin x$

(h) $f(x)=2\sin 3x+3\cos 2x$

(i) $f(s)=2\cot 2s+\sec 3s$

(j) $f(x)=x^2\cos 2x$

(k) $h(s)=\sin\sqrt{s^2-1}$

(l) $f(x)=x\cos\frac{1}{x}$

(m) $f(x)=\frac{x-\sin x}{1+\cos x}$

(n) $g(t)=\sqrt{\tan t}$

Exercise 4.5.2 *Find the points on the curve $y=x+2\cos x$ that have a horizontal tangent line.*

Exercise 4.5.3 *The revenue of McMenamy's Fish Shanty located at a popular summer resort is approximately*

$$R(t)=2\left(5-4\cos\frac{\pi}{6}t\right)\quad 0\le t\le 12$$

during the t-th week ($t=1$ corresponds to the first week of June), where R is measured in thousands of dollars. On what week is the marginal revenue zero?

Exercise 4.5.4 *The revenue in thousands of dollars received from the sale of electronic fans is seasonal, with maximum revenue in the summer. Let the revenue received from the sales of fans be approximated by*

$$R(x)=-100\cos(2\pi x)+120$$

where x is time in years, measured from January 1. Calculate the marginal revenue for September 1. (Take that one month represents $1/12$ of the years).

Exercise 4.5.5 *Sales of large kitchen appliances such as ovens and fridges are usually subject to seasonal fluctuations. Everything Kitchen's sales of fridge models from the beginning of 2001 to the end of 2002 can be approximated by*

$$S(x)=\frac{1}{10}\sin\left(\frac{\pi}{2}(x+1)\right)+\frac{1}{2}$$

where x is time in quarters, $x=1$ corresponds to the end of the first quarter of 2001, and S is measured in millions of dollars. Find the rate of change of sales for the end of the third quarter of 2002.

4.6 Derivatives of Exponential & Logarithmic Functions

As with the sine function, we don't know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:

$$\begin{aligned}
\frac{d}{dx}a^x &= \lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0} a^x\frac{a^{\Delta x}-1}{\Delta x}\\
&= a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}
\end{aligned}$$

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves Δx but not x, which means that if $\lim_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ exists, then it is a constant number. This means that a^x has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim_{x\to 0}\sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ even exists—does this fraction really get closer and closer to some fixed value? Yes it does, but we will prove this property at the end of this section.

We can look at some examples. Consider $(2^x-1)/x$ for some small values of x: 1, 0.828427124, 0.756828460, 0.724061864, 0.70838051, 0.70070877 when x is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(3^x-1)/x$: 2, 1.464101616, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of x. It turns out to be true that in the limit this is about 1.1. Two examples don't establish a pattern, but if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1; the value at which this happens is called e, so that

$$\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=1.$$

As you might guess from our two examples, e is closer to 3 than to 2, and in fact $e\approx 2.718$.

Now we see that the function e^x has a truly remarkable property:

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{\Delta x\to 0}\frac{e^{x+\Delta x}-e^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{e^x e^{\Delta x}-e^x}{\Delta x}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x
\end{aligned}
$$

That is, e^x is its own derivative, or in other words the slope of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (z, e^z) and has slope e^z there, no matter what z is. It is sometimes convenient to express the function e^x without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1+x^2)$ instead of e^{1+x^2}.

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so is the logarithm easier to do now that we know the derivative of the exponential function. Let's start with $\log_e x$, which as you probably know is often abbreviated $\ln x$ and called the "natural logarithm" function.

Consider the relationship between the two functions, namely, that they are inverses, that one "undoes" the other. Graphically this means that they have the same graph except that one is "flipped" or "reflected" through the line $y = x$ as shown in Figure 4.5.

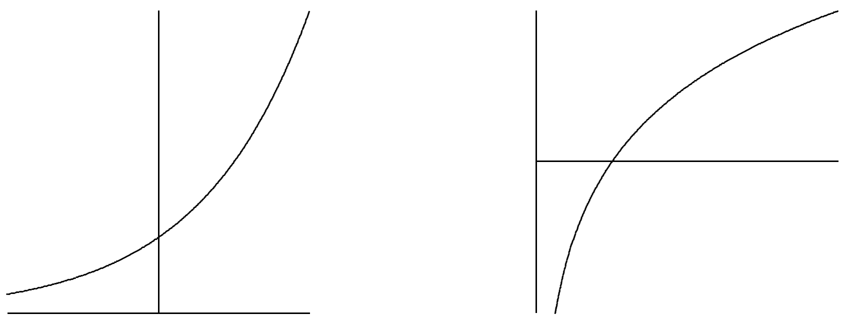


Figure 4.5: The exponential and logarithmic functions.

This means that the slopes of these two functions are closely related as well: For example, the slope of e^x is e at $x = 1$; at the corresponding point on the $\ln(x)$ curve, the slope must be $1/e$, because the "rise" and the "run" have been interchanged. Since the slope of e^x is e at the point $(1, e)$, the slope of $\ln(x)$ is $1/e$ at the point $(e, 1)$.

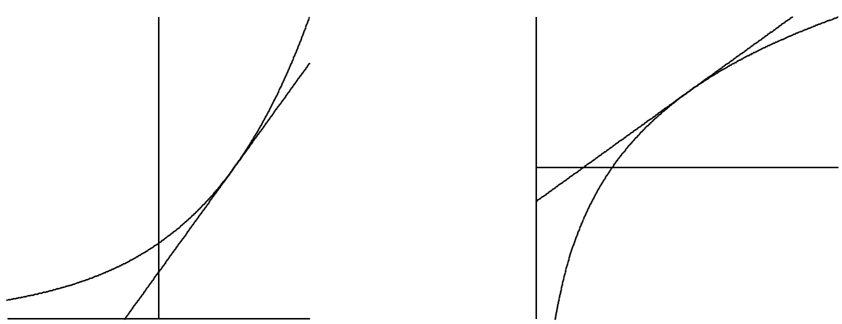


Figure 4.6: The exponential and logarithmic functions and their tangent lines at (e^z, z) and (z, e^z), respectively.

More generally, we know that the slope of e^x is e^z at the point (z, e^z), so the slope of $\ln(x)$ is $1/e^z$ at (e^z, z), as shown in Figure 4.6. In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x, \ln x)$ is $1/x$. The upshot is:

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln|x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus whether x is positive or negative, the derivative is the same.

Theorem 4.58: Derivative Formulas for e^x and $\ln_x$

$$\frac{d}{dx}e^x = e^x \quad \text{and} \quad \frac{d}{dx}\ln x = \frac{1}{x}.$$

What about the functions a^x and $\log_a x$? We know that the derivative of a^x is some constant times a^x itself, but what constant? Remember that "the logarithm is the exponent" and you will see that $a = e^{\ln a}$. Then

$$a^x = (e^{\ln a})^x = e^{x\ln a},$$

and we can compute the derivative using the Chain Rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}e^{x\ln a} = (\ln a)e^{x\ln a} = (\ln a)a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since

$$x = e^{\ln x}$$

we can take the logarithm base a of both sides to get

$$\log_a(x) = \log_a(e^{\ln x}) = \ln x \log_a e.$$

Then

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$\begin{aligned} a &= e^{\ln a} \\ \log_a(a) &= \log_a(e^{\ln a}) = \ln a \log_a e \\ 1 &= \ln a \log_a e \end{aligned}$$

$$\frac{1}{\ln a} = \log_a e,$$

we can replace $\log_a e$ to get

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}.$$

You may if you wish memorize the formulas.

Theorem 4.59: Derivative Formulas for a^x and $\log_a x$

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}.$$

Because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

Example 4.60: Derivative of Exponential Function

Compute the derivative of $f(x) = 2^x$.

Solution.

$$\begin{aligned} \frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\ &= \frac{d}{dx}e^{x\ln 2} \\ &= \left(\frac{d}{dx}x\ln 2\right)e^{x\ln 2} \\ &= (\ln 2)e^{x\ln 2} = 2^x \ln 2 \end{aligned}$$

♣

Example 4.61: Derivative of Exponential Function

Compute the derivative of $f(x) = 2^{x^2} = 2^{(x^2)}$.

Solution.

$$\begin{aligned} \frac{d}{dx}2^{x^2} &= \frac{d}{dx}e^{x^2\ln 2} \\ &= \left(\frac{d}{dx}x^2\ln 2\right)e^{x^2\ln 2} \end{aligned}$$

$$= (2\ln 2)xe^{x^2\ln 2}$$

$$= (2\ln 2)x2^{x^2}$$

♣

Example 4.62: Proving the General Power Rule

Recall that we have not justified the Power Rule except when the exponent is a positive or negative integer. We can now show that

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

where n is any real number.

Solution. We can use the exponential function to take care of other exponents.

$$\begin{aligned}\frac{d}{dx}x^n &= \frac{d}{dx}e^{n\ln x} \\ &= \left(\frac{d}{dx}n\ln x\right)e^{n\ln x} \\ &= \left(n\frac{1}{x}\right)x^n \\ &= nx^{n-1}\end{aligned}$$

♣

Example 4.63: Derivative of Exponential Function with Chain Rule

Find the derivative of

$$g(t) = e^{2t^2+t}$$

Solution. $g'(t) = e^{2t^2+t}\frac{d}{dt}(2t^2+t) = (4t+1)e^{2t^2+t}$. ♣

Example 4.64: Derivative of Exponential Function with Quotient Rule

Find the derivative of

$$R(q) = \frac{q}{10^q+5}$$

Solution. Using the Quotient Rule, we calculate

$$R'(q) = \frac{\frac{d}{dx}(q)(10^q+5) - q\frac{d}{dx}(10^q+5)}{(10^q+5)^2}$$

Now, using the derivative formula for the exponential function with base 10, we find

$$\begin{aligned}R'(q) &= \frac{(1)(10^q+5)-q(\ln 10)(10^q)}{(10^q+5)^2} \\ &= \frac{10^q+5-(\ln 10)(q10^q)}{(10^q+5)^2}.\end{aligned}$$

♣

Example 4.65: Derivative of Logarithmic Function with Chain Rule

Find the derivative of

$$y = \log_2(\sqrt{x-5})$$

Solution. We use the Chain Rule and the formula for the derivative of a logarithmic function of base 2 to find

$$\begin{aligned}y' &= \frac{1}{(\ln 2)\sqrt{x-5}}\frac{d}{dx}\left(\sqrt{x-5}\right) \\ &= \frac{1}{(\ln 2)\sqrt{x-5}}\frac{1}{2\sqrt{x-5}} \\ &= \frac{1}{2(\ln 2)(\sqrt{x-5})^2}.\end{aligned}$$

Note that we must have $x > 5$. Therefore, we need to be careful when we want to simplify the expression on the right side of the equation, and state the restriction as well:

$$y' = \frac{1}{2(\ln 2)(x-5)} \quad x > 5.$$

♣

Example 4.66: Derivative of Logarithmic Function with Chain Rule

Find the derivative of

$$g(t) = \ln\left(t^2 e^{-t^2}\right)$$

Solution. To save a lot of work, we first simplify the given expression using properties of logarithms. We have

$$\begin{aligned}g(t) &= \ln\left(t^2 e^{-t^2}\right) \\ &= \ln t^2 + \ln e^{-t^2} \\ &= 2\ln t - t^2.\end{aligned}$$

Therefore,

$$g'(t) = \frac{2}{t} - 2t = \frac{2(1-t^2)}{t}.$$

♣

The Fundamental Limit of Calculus

In Section 2.3 we defined e to be the base so that the slope of the tangent line to the function $y = a^x$ at the point $x = 0$ was equal to 1, i.e. $\frac{d(e^x)}{dx}\Big|_{x=0} = 1$. We will now see how we can write e **in terms of a limit**.

Theorem 4.67: The Fundamental Limit of Calculus

$$e = \lim_{h\to 0}(1+h)^{1/h} \qquad \textit{and} \qquad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Proof. Let $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ and so $f'(1) = 1$. We now write $f'(1)$ using the definition of derivative:

$$1 = f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\ln(1+h)^{1/h}$$

Taking e of both sides and using continuity of the exponential function gives:

$$e^1 = e^{\lim\limits_{h\to 0}\ln(1+h)^{1/h}} \iff e = \lim_{h\to 0} e^{\ln(1+h)^{1/h}} = \lim_{h\to 0}(1+h)^{1/h}$$

For the second limit, we let $n = \frac{1}{h}$. Then $h \to 0 \implies n \to \infty$ and $h = \frac{1}{n}$:

$$e = \lim_{h\to 0}(1+h)^{1/h} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

♣

Exercises for Section 4.6

Exercise 4.6.1 *Find the derivatives of the functions.*

(a) $f(x) = 3^{x^2}$

(b) $f(x) = \dfrac{\sin x}{e^x}$

(c) $f(t) = (e^t)^2$

(d) $f(x) = \sin(e^x)$

(e) $f(x) = e^{\sin x}$

(f) $h(x) = x^{\sin x}$

(g) $f(x) = x^3 e^x$

(h) $g(t) = t + 2^t$

(i) $g(x) = (1/3)^{x^2}$

(j) $f(s) = e^{4s}/s$

(k) $h(x) = \ln(x^3 + 3x)$

(l) $f(x) = \ln(\cos(x))$

(m) $f(t) = \sqrt{\ln(t^2)/t}$

(n) $f(x) = \ln(\sec(x) + \tan(x))$

(o) $g(x) = x^{\cos(x)}$

(p) $h(s) = s\ln s$

(q) $f(x) = \ln(\ln(3x))$

(r) $f(t) = \dfrac{1+\ln(3t^2)}{1+\ln(4t)}$

Exercise 4.6.2 *Find the second derivative of the function.*

(a) $f(t) = 3e^{-2t} - 5e^{-t}$

(b) $f(x) = x^2 e^{-2x}$

(c) $f(t) = (t-5)3^t$

(d) $y = 6^{\sqrt{x}}$

Exercise 4.6.3 *Find an equation of the tangent line to the graph of* $y = e^{2x-3}$ *at the point* $\left(\frac{3}{2}, 1\right)$.

Exercise 4.6.4 *Find an equation of the tangent line to the graph of* $y = e^{-x^2}$ *at the point* $\left(1, \frac{1}{e}\right)$.

Exercise 4.6.5 *Find the value of a so that the tangent line to* $y = \ln(x)$ *at* $x = a$ *is a line through the origin. Sketch the resulting situation.*

Exercise 4.6.6 *If* $f(x) = \ln(x^3 + 2)$ *compute* $f'(e^{1/3})$.

Exercise 4.6.7 *The unit selling price p (in dollars) and the quantity x demanded (in pairs) of a certain brand of women's gloves is given by the demand equation*

$$p = 100e^{-0.0001x} \quad 0 \le x \le 20,000$$

(a) *Find the revenue function R (Hint:* $R(x) = px$*).*

(b) *Find the marginal revenue function* R'.

(c) *What is the marginal revenue when* $x = 10$*?*

Exercise 4.6.8 *The monthly demand for a certain brand of table wine is given by the demand equation*

$$p = 240\left(1 - \frac{3}{3 + e^{-0.0005x}}\right)$$

where p denotes the wholesale price per case (in dollars) and x denotes the number of cases demanded.

(a) *Find the rate of change of the price per case when* $x = 1000$.

(b) *What is the price per case when* $x = 1000$*?*

Exercise 4.6.9 *The price of a certain commodity in dollars per unit time (measured in weeks) is given by*

$$p = 8 + 4e^{-2t} + te^{-2t}$$

(a) *What is the price of the commodity at* $t = 0$*?*

(b) *How fast is the price of the commodity changing at* $t = 0$*?*

(c) *Find the equilibrium price of the commodity (Hint: It's given by* $\lim_{t\to\infty} p$*. Also use the fact that* $\lim_{t\to\infty} te^{-2t} = 0$.

Exercise 4.6.10 *The percent of households using online banking may be approximated by the formula*

$$f(t) = 1.5e^{0.78t} \quad 0 \le t \le 4$$

where t is measured in years, with $t = 0$ corresponding to the beginning of 2000*.*

(a) *What is the projected percent of households using online banking at the beginning of* 2003*?*

(b) *How fast will the projected percent of households using online banking be changing at the beginning of* 2003*?*

(c) *How fast will the rate of the projected percent of households using online banking be changing at the beginning of* 2003*? (Hint: we want $f''(3)$. Why?)*

Exercise 4.6.11 *The average energy consumption of the typical refrigerator/freezer manufactured by York Industries is approximately*

$$C(t) = 1486e^{-0.073t} + 500 \quad 0 \le t \le 20$$

killowatt-hours (kWh) per year, where t is measured in years, with $t = 0$ corresponding to 1972*.*

(a) *What was the average energy consumption of the York refrigerator/freezer at the beginning of* 1972*?*

(b) *What is the rate of change of the average energy consumption?*

(c) *All refrigerator/freezers manufactured as of Janurary* 1, 1990*, must meet a* 950*-kWh/yr maximum energy-consumption standard. Show that the York refigerator/freezer satisfies this requirement.*

4.7 Implicit and Logarithmic Differentiation

4.7.1. Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of e^x and $\ln x$ because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

We will begin by illustrating the technique to find what we already know, the derivative of $\ln x$. Let's write $y = \ln x$ and then $x = e^{\ln x} = e^y$, that is, $x = e^y$. We say that this equation defines the function $y = \ln x$ implicitly because while it is not an explicit expression $y = \ldots$, it is true that if $x = e^y$ then y is in fact the natural logarithm function. Now, for the time being, pretend that all we know of y is that $x = e^y$; what can we say about derivatives? We can take the derivative of both sides of the equation:

$$\frac{d}{dx}x = \frac{d}{dx}e^y.$$

Then using the Chain Rule on the right hand side:

$$1 = \left(\frac{d}{dx}y\right)e^y = \frac{dy}{dx}e^y.$$

Then we can solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

There is one little difficulty here. To use the Chain Rule to compute $d/dx(e^y) = \dfrac{dy}{dx}e^y$ we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $1/x$. When using this method we will always have to assume that the desired derivative exists, but fortunately this is a safe assumption for most such problems.

The example $y = \ln x$ involved an inverse function defined implicitly, but other functions can be defined implicitly, and sometimes a single equation can be used to implicitly define more than one function.

Guideline for Implicit Differentiation

Given an implicitly defined relation $f(x,y) = k$ for some constant k, the following steps outline the implicit differentiation process for finding dy/dx:

1. Apply the differentiation operator d/dx to both sides of the equation $f(x,y) = k$.
2. Follow through with the differentiation by keeping in mind that y is a function of x, and so the Chain Rule applies.
3. Solve for dy/dx.

Here's a familiar example.

Example 4.68: Derivative of Circle Equation

Given $r^2 = x^2 + y^2$, find the derivative $\frac{dy}{dx}$.

Solution. The equation $r^2 = x^2 + y^2$ describes a circle of radius r. The circle is not a function $y = f(x)$ because for some values of x there are two corresponding values of y. If we want to work with a function, we can break the circle into two pieces, the upper and lower semicircles, each of which is a function. Let's call these $y = U(x)$ and $y = L(x)$; in fact this is a fairly simple example, and it's possible to give explicit expressions for these: $U(x) = \sqrt{r^2 - x^2}$ and $L(x) = -\sqrt{r^2 - x^2}$. But it's somewhat easier, and quite useful, to view both functions as given implicitly by $r^2 = x^2 + y^2$: both $r^2 = x^2 + U(x)^2$ and $r^2 = x^2 + L(x)^2$ are true, and we can think of $r^2 = x^2 + y^2$ as defining both $U(x)$ and $L(x)$.

Now we can take the derivative of both sides as before, remembering that y is not simply a variable but a function—in this case, y is either $U(x)$ or $L(x)$ but we're not yet specifying which one. When we take the derivative we just have to remember to apply the Chain Rule where y appears.

$$\frac{d}{dx}r^2 = \frac{d}{dx}(x^2 + y^2)$$

$$\begin{aligned} 0 &= 2x + 2y\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y} \end{aligned}$$

Now we have an expression for $\frac{dy}{dx}$, but it contains y as well as x. This means that if we want to compute $\frac{dy}{dx}$ for some particular value of x we'll have to know or compute y at that value of x as well. It is at this point that we will need to know whether y is $U(x)$ or $L(x)$. Occasionally it will turn out that we can avoid explicit use of $U(x)$ or $L(x)$ by the nature of the problem. ♣

Example 4.69: Slope of the Circle

Find the slope of the circle $4 = x^2 + y^2$ *at the point* $(1, -\sqrt{3})$.

Solution. Since we know both the x- and y-coordinates of the point of interest, we do not need to explicitly recognize that this point is on $L(x)$, and we do not need to use $L(x)$ to compute y – but we could. Using the calculation of $\frac{dy}{dx}$ from above,

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

It is instructive to compare this approach to others.

We might have recognized at the start that $(1, -\sqrt{3})$ is on the function $y = L(x) = -\sqrt{4 - x^2}$. We could then take the derivative of $L(x)$, using the Power Rule and the Chain Rule, to get

$$L'(x) = -\frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{4 - x^2}}.$$

Then we could compute $L'(1) = 1/\sqrt{3}$ by substituting $x = 1$.

Alternately, we could realize that the point is on $L(x)$, but use the fact that $\frac{dy}{dx} = -x/y$. Since the point is on $L(x)$ we can replace y by $L(x)$ to get

$$\frac{dy}{dx} = -\frac{x}{L(x)} = -\frac{x}{\sqrt{4 - x^2}},$$

without computing the derivative of $L(x)$ explicitly. Then we substitute $x = 1$ and get the same answer as before. ♣

In the case of the circle it is possible to find the functions $U(x)$ and $L(x)$ explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for y and implicit differentiation is the only way to find the derivative.

Example 4.70: Derivative of Function defined Implicitly

Find the derivative of any function defined implicitly by $yx^2 + y^2 = x$.

Solution. We treat y as an unspecified function and use the Chain Rule:

$$\frac{d}{dx}(yx^2 + y^2) = \frac{d}{dx}x$$

$$\begin{aligned}(y\cdot 2x+\frac{dy}{dx}\cdot x^2)+2y\frac{dy}{dx} &= 1\\ \frac{dy}{dx}\cdot x^2+2y\frac{dy}{dx} &= -y\cdot 2x\\ \frac{dy}{dx} &= \frac{-2xy}{x^2+2y}\end{aligned}$$

♣

Example 4.71: Derivative of Function defined Implicitly

Find the derivative of any function defined implicitly by $yx^2+e^y=x$.

Solution. We treat y as an unspecified function and use the Chain Rule:

$$\begin{aligned}\frac{d}{dx}(yx^2+e^y) &= \frac{d}{dx}x\\ (y\cdot 2x+\frac{dy}{dx}\cdot x^2)+\frac{dy}{dx}e^y &= 1\\ \frac{dy}{dx}x^2+\frac{dy}{dx}e^y &= 1-2xy\\ \frac{dy}{dx}(x^2+e^y) &= 1-2xy\\ \frac{dy}{dx} &= \frac{1-2xy}{x^2+e^y}\end{aligned}$$

♣

You might think that the step in which we solve for $\frac{dy}{dx}$ could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation $yx^2+e^y=x$ for y, so maybe after taking the derivative we get something that is hard to solve for $\frac{dy}{dx}$. In fact, *this never happens.* All occurrences $\frac{dy}{dx}$ come from applying the Chain Rule, and whenever the Chain Rule is used it deposits a single $\frac{dy}{dx}$ multiplied by some other expression. So it will always be possible to group the terms containing $\frac{dy}{dx}$ together and factor out the $\frac{dy}{dx}$, just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

Example 4.72: Derivative of Function defined Implicitly

Find $\frac{dy}{dx}$ by implicit differentiation if

$$2x^3+x^2y-y^9=3x+4.$$

Solution. Differentiating both sides with respect to x gives:

$$\begin{aligned} 6x^2 + \left(2xy + x^2\frac{dy}{dx}\right) - 9y^8\frac{dy}{dx} &= 3, \\ x^2\frac{dy}{dx} - 9y^8\frac{dy}{dx} &= 3 - 6x^2 - 2xy \\ \left(x^2 - 9y^8\right)\frac{dy}{dx} &= 3 - 6x^2 - 2xy \\ \frac{dy}{dx} &= \frac{3 - 6x^2 - 2xy}{x^2 - 9y^8}. \end{aligned}$$

♣

Example 4.73: Derivative of Function defined Implicitly

Suppose that s and t are related by the equation $s^2 + te^{st} = 2$. Find $\frac{ds}{dt}$.

Solution. We assume that s is a function of t, $s(t)$. Differentiate both sides of the equation defining the curve and group the terms involving $\frac{ds}{dt}$ obtaining,

$$\begin{aligned} \frac{d}{dt}\left(s^2 + te^{st}\right) &= \frac{d}{dt}2 \\ 2s\frac{ds}{dt} + e^{st} + t\left(s + t\frac{ds}{dt}\right) &= 0 \\ \left(2s + ste^{st}\right)\frac{ds}{dt} + ste^{st} &= 0. \end{aligned}$$

We used the Product Rule and the Chain Rule to carry out the differentiation. Solving for $\frac{ds}{dt}$ gives

$$\frac{ds}{dt} = \frac{-ste^{st}}{2s + ste^{st}}.$$

♣

In the previous examples we had functions involving x and y, and we thought of y as a function of x. In these problems we differentiated with respect to x. So when faced with x's in the function we differentiated as usual, but when faced with y's we differentiated as usual except we multiplied by a $\frac{dy}{dx}$ for that term because we were using Chain Rule.

4.7.2. Differentiating x and y as Functions of t

In the following example we will assume that both x and y are functions of t and want to differentiate the equation with respect to t. This means that every time we differentiate an x we will be using the Chain Rule, so we must multiply by $\frac{dx}{dt}$, and whenever we differentiate a y we multiply by $\frac{dy}{dt}$.

Example 4.74: Derivative of Function of an Additional Variable

Thinking of x and y as functions of t, i.e. $x(t)$ and $y(t)$, differentiate the following equation with respect to t:

$$x^2 + y^2 = 100.$$

Solution. Using the Chain Rule we have:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

♣

Example 4.75: Derivative of Function of an Additional Variable

If $y = x^3 + 5x$ and $\frac{dx}{dt} = 7$, find $\frac{dy}{dt}$ when $x = 1$.

Solution. Differentiating each side of the equation $y = x^3 + 5x$ with respect to t gives:

$$\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 5\frac{dx}{dt}.$$

When $x = 1$ and $\frac{dx}{dt} = 7$ we have:

$$\frac{dy}{dt} = 3(1^2)(7) + 5(7) = 21 + 35 = 56.$$

♣

Example 4.76: Differentiation with Parametric Functions

Find $\frac{dy}{dx}$ when $x = \cos t$ and $y = \sin t$.

Solution. We differentiate both x and y with respect to the parameter t:

$$\frac{dx}{dt} = -\sin t \quad \text{and} \quad \frac{dy}{dt} = \cos t$$

From the Chain Rule, we know that

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dx}{dt}$$

so that, by rearrangement

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

provided $\frac{dx}{dt} \neq 0$. So, in this case

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dx}{dt} = \frac{\cos t}{\sin t} = -\cot t.$$

♣

4.7.3. Logarithmic Differentiation

Previously we've seen how to take the derivative of a number to a function $(a^{f(x)})'$ – exponential differentiation, and also a function to a number $[(f(x))^n]'$ – Power Rule. But what about the derivative of a function to a function $[(f(x))^{g(x)}]'$?

In this case, we use a procedure known as **logarithmic differentiation**.

Guideline for Logarithmic Differentiation

Given a function $y = f(x)$, the following steps outline the logarithmic differentiation process:

1. Take ln of both sides of $y = f(x)$ to get $\ln y = \ln f(x)$ and simplify using logarithm properties.
2. Differentiate implicitly with respect to x and solve for $\frac{dy}{dx}$.
3. Replace y with its function of x (i.e., $f(x)$).

Example 4.77: Logarithmic Differentiation

Differentiate $y = x^x$.

Solution. *Method 1:* We take ln of both sides:

$$\ln y = \ln x^x.$$

Using log properties we have:

$$\ln y = x \ln x.$$

Differentiating implicitly gives:

$$\frac{y'}{y} = (1)\ln x + x\frac{1}{x},$$

$$\frac{y'}{y} = \ln x + 1.$$

Solving for y' gives:

$$y' = y(1 + \ln x).$$

Replace $y = x^x$ gives:

$$y' = x^x(1 + \ln x).$$

Method 2: Another method to find this derivative is as follows:

$$\begin{aligned} \frac{d}{dx}x^x &= \frac{d}{dx}e^{x\ln x} \\ &= \left(\frac{d}{dx}x\ln x\right)e^{x\ln x} \end{aligned}$$

$$= \left(x\frac{1}{x} + \ln x\right) x^x$$
$$= (1 + \ln x) x^x$$

♣

In fact, logarithmic differentiation can be used on more complicated products and quotients (not just when dealing with functions to the power of functions).

Example 4.78: Logarithmic Differentiation

Differentiate (assuming $x > 0$):

$$y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}.$$

Solution. Using Product and Quotient Rules for this problem is a complete nightmare! Let's apply logarithmic differentiation instead. Take ln of both sides:

$$\ln y = \ln\left(\frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\right).$$

Applying log properties:

$$\ln y = \ln\left((x+2)^3(2x+1)^9\right) - \ln\left(x^8(3x+1)^4\right)$$

$$\ln y = \ln\left((x+2)^3\right) + \ln\left((2x+1)^9\right) - \left[\ln\left(x^8\right) + \ln\left((3x+1)^4\right)\right]$$
$$\ln y = 3\ln(x+2) + 9\ln(2x+1) - 8\ln x - 4\ln(3x+1).$$

Now, differentiating implicitly with respect to x gives:

$$\frac{y'}{y} = \frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}.$$

Solving for y' gives:

$$y' = y\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

Replace $y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}$ gives:

$$y' = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

♣

Example 4.79: Logarithmic Differentiation

Differentiate $y = x^2(x-1)(x^2+4)^3$.

Solution. Taking the natural logarithm on both side of the given equation and using the laws of logarithms, we obtain

$$\begin{aligned}\ln y &= \ln\left(x^2(x-1)(x^2+4)^3\right)\\ &= \ln x^2 + \ln(x-1) + \ln(x^2+4)^3\\ &= 2\ln x + \ln(x-1) + 3\ln(x^2+4)\end{aligned}$$

Differentiating both sides of the equation with respect to x, we have

$$\frac{d}{dx}\ln y = \frac{y'}{y} = \frac{2}{x} + \frac{1}{x-1} + 3\frac{2x}{x^2+4}.$$

Finally, solving for y', we have

$$\begin{aligned}y' &= y\left(\frac{2}{x} + \frac{1}{x-1} + 3\frac{2x}{x^2+4}\right)\\ &= \left(x^2(x-1)(x^2+4)^3\right)\left(\frac{2}{x} + \frac{1}{x-1} + 3\frac{2x}{x^2+4}\right)\end{aligned}$$

♣

Exercises for Section 4.7

Exercise 4.7.1 *Find a formula for the derivative y' at the point (x,y):*

(a) $y^2 = 1 + x^2$

(b) $x^2 + xy + y^2 = 7$

(c) $x^3 + xy^2 = y^3 + yx^2$

(d) $4\cos x \sin y = 1$

(e) $\sqrt{x} + \sqrt{y} = 9$

(f) $\tan(x/y) = x + y$

(g) $\sin(x+y) = xy$

(h) $\frac{1}{x} + \frac{1}{y} = 7$

(i) $\cos(xy) - \sin x = 1$

(j) $x\sec(y) = \ln(\sin x)$

(k) $\sin(x+y) - \sin^{-1} y = 0$

(l) $(x^2 - y^2)\tan(y) = \sqrt{y}$

Exercise 4.7.2 *Use logarithmic differentiation to find the derivative of y.*

(a) $y = (x+1)^2(x+2)^3$

(b) $y = (3x+2)^4(5x-1)^2$

(c) $y = (x-1)^2(x+1)^3(x+3)^4$

(d) $y = \sqrt{3x+5}(2x-3)^4$

(e) $y = \frac{(2x^2-1)^5}{\sqrt{x+1}}$

(f) $y = \frac{\sqrt{4+3x^2}}{\sqrt[3]{x^2+1}}$

(g) $y = 3^x$

(h) $y = x^{x+2}$

(i) $y = (x^2+1)^x$

(j) $y = x^{\ln x}$

Exercise 4.7.3 *A hyperbola passing through* $(8,6)$ *consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at* $(8,6)$.

Exercise 4.7.4 *The graph of the equation* $x^2 - xy + y^2 = 9$ *is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel.*

Exercise 4.7.5 *Repeat the previous problem for the points at which the ellipse intersects the y-axis.*

Exercise 4.7.6 *If* $y = \log_a x$ *then* $a^y = x$. *Use implicit differentiation to find* y'.

Exercise 4.7.7 *Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.*

Exercise 4.7.8 *Find an equation for the tangent line to* $x^4 = y^2 + x^2$ *at* $(2, \sqrt{12})$. *(This curve is the* **kampyle of Eudoxus**.*)*

Exercise 4.7.9 *Find an equation for the tangent line to* $x^{2/3} + y^{2/3} = a^{2/3}$ *at a point* (x_1, y_1) *on the curve, with* $x_1 \neq 0$ *and* $y_1 \neq 0$. *(This curve is an* **a***stroid.)*

Exercise 4.7.10 *Find an equation for the tangent line to* $(x^2 + y^2)^2 = x^2 - y^2$ *at a point* (x_1, y_1) *on the curve, with* $x_1 \neq 0, -1, 1$. *(This curve is a* **lemniscate**.*)*

Exercise 4.7.11 *Two curves are* **orthogonal** *if at each point of intersection, the angle between their tangent lines is* $\pi/2$. *Two families of curves,* $\mathscr{A}$ *and* $\mathscr{B}$, *are* **orthogonal trajectories** *of each other if given any curve C in* $\mathscr{A}$ *and any curve D in* $\mathscr{B}$ *the curves C and D are orthogonal. For example, the family of horizontal lines in the plane is orthogonal to the family of vertical lines in the plane.*

(a) *Show that* $x^2 - y^2 = 5$ *is orthogonal to* $4x^2 + 9y^2 = 72$. *(Hint: You need to find the intersection points of the two curves and then show that the product of the derivatives at each intersection point is* -1.*)*

(b) *Show that* $x^2 + y^2 = r^2$ *is orthogonal to* $y = mx$. *Conclude that the family of circles centered at the origin is an orthogonal trajectory of the family of lines that pass through the origin.*

Note that there is a technical issue when $m = 0$. *The circles fail to be differentiable when they cross the x-axis. However, the circles are orthogonal to the x-axis. Explain why. Likewise, the vertical line through the origin requires a separate argument.*

(c) *For* $k \neq 0$ *and* $c \neq 0$ *show that* $y^2 - x^2 = k$ *is orthogonal to* $yx = c$. *In the case where k and c are both zero, the curves intersect at the origin. Are the curves* $y^2 - x^2 = 0$ *and* $yx = 0$ *orthogonal to each other?*

(d) *Suppose that* $m \neq 0$. *Show that the family of curves* $\{y = mx + b \mid b \in \mathbb{R}\}$ *is orthogonal to the family of curves* $\{y = -(x/m) + c \mid c \in \mathbb{R}\}$.

Exercise 4.7.12 *Differentiate the function* $y = \dfrac{(x-1)^8 (x-23)^{1/2}}{27x^6 (4x-6)^8}$

Exercise 4.7.13 *Differentiate the function* $f(x) = (x+1)^{\sin x}$.

Exercise 4.7.14 *Differentiate the function* $g(x) = \dfrac{e^x(\cos x+2)^3}{\sqrt{x^2+4}}$.

4.8 Derivatives of Inverse Functions

Suppose we wanted to find the *derivative of the inverse*, but do not have an actual formula for the inverse function? Then we can use the following derivative formula for the inverse evaluated at a.

Theorem 4.80: Derivative of Inverse Functions

Given an invertible function $f(x)$*, the derivative of its inverse function* $f^{-1}(x)$ *evaluated at* $x = a$ *is:*

$$\left[f^{-1}\right]'(a) = \frac{1}{f'\left[f^{-1}(a)\right]}$$

To see why this is true, start with the function $y = f^{-1}(x)$. Write this as $x = f(y)$ and differentiate both sides implicitly with respect to x using the Chain Rule:

$$1 = f'(y) \cdot \frac{dy}{dx}.$$

Thus,

$$\frac{dy}{dx} = \frac{1}{f'(y)},$$

but $y = f^{-1}(x)$, thus,

$$\left[f^{-1}\right]'(x) = \frac{1}{f'\left[f^{-1}(x)\right]}.$$

At the point $x = a$ this becomes:

$$\left[f^{-1}\right]'(a) = \frac{1}{f'\left[f^{-1}(a)\right]}$$

Example 4.81: Derivatives of Inverse Functions

Suppose $f(x) = x^5 + 2x^3 + 7x + 1$*. Find* $\left[f^{-1}\right]'(1)$.

Solution. First we should show that f^{-1} exists (i.e. that f is one-to-one). In this case the derivative $f'(x) = 5x^4 + 6x^2 + 7$ is strictly greater than 0 for all x, so f is strictly increasing and thus one-to-one.

It's difficult to find the inverse of $f(x)$ (and then take the derivative). Thus, we use the above formula evaluated at 1:

$$\left[f^{-1}\right]'(1) = \frac{1}{f'\left[f^{-1}(1)\right]}.$$

Note that to use this formula we need to know what $f^{-1}(1)$ is, and the derivative $f'(x)$. To find $f^{-1}(1)$ we make a table of values (plugging in $x = -3, -2, -1, 0, 1, 2, 3$ into $f(x)$) and see what value of x gives 1. We omit the table and simply observe that $f(0) = 1$. Thus,

$$f^{-1}(1) = 0.$$

Now we have:

$$\left[f^{-1}\right]'(1) = \frac{1}{f'(0)}.$$

And so, $f'(0) = 7$. Therefore,

$$\left[f^{-1}\right]'(1) = \frac{1}{7}.$$

♣

Example 4.82: Tangent Line of Inverse Functions

Find the equation of the tangent line to the inverse of

$$f(x) = \frac{e^{-3x}}{x^2+1}$$

at $(-1, 0)$.

Solution. First, we differentiate

$$\begin{aligned} f'(x) &= \frac{(x^2+1)(3e^{-3x})(2x)}{(x^2+1)^2} \\ &= \frac{3x^2e^{-3x} + 2xe^{-3x} + 3e^{-3x}}{x^4+2x^2+1} \end{aligned}$$

The derivative of f^{-1} is:

$$\begin{aligned} \frac{d}{dx}f^{-1}(x) &= \frac{1}{\frac{3y^2e^{-3y}+2ye^{-3y}+3e^{-3y}}{y^4+2y^2+1}} \\ &= \frac{y^4+2y^2+1}{3y^2e^{-3y}+2ye^{-3y}+3e^{-3y}} \\ \frac{d}{dx}f^{-1}(-1) &= 0^4+0^2+10^2e^0+2ye^0+3e^0 \\ &= \frac{1}{3} \end{aligned}$$

An equation of the tangent line at $(-1, 0)$ is then

$$y = \frac{1}{3}(x+1).$$

♣

4.8.1. Derivatives of Inverse Trigonometric Functions

We can apply the technique used to find the derivative of f^{-1} above to find the derivatives of the inverse trigonometric functions.

In the following examples we will derive the formulae for the derivative of the inverse sine, inverse cosine and inverse tangent. The other three inverse trigonometric functions have been left as exercises at the end of this section.

Example 4.83: Derivative of Inverse Sine

Find the derivative of $\sin^{-1}(x)$.

Solution. As above, we write $y = \sin^{-1}(x)$, so $x = \sin(y)$ and $-\pi/2 \le y \le \pi/2$, and differentiate both sides with respect to x using the Chain Rule.

$$\begin{aligned}
\frac{d}{dx}x &= \frac{d}{dx}\sin(y) \\
1 &= \cos(y)\frac{dy}{dx} \\
1 &= \cos\left(\sin^{-1}(x)\right)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{\cos\left(\sin^{-1}(x)\right)}
\end{aligned}$$

Although correct, this formula is cumbersome to use, and can be simplified significantly with a bit of trigonometry. Let $\theta = \sin^{-1}(x)$, so $\sin(\theta) = x$, and construct a right angle triangle with angle θ, opposite side length x and hypotenuse 1. The Pythagorean Theorem gives an adjacent side length of $\sqrt{1-x^2}$, so $\cos\left(\sin^{-1}(x)\right) = \cos(\theta) = \sqrt{1-x^2}$. Note that we choose the non-negative square root $\sqrt{1-x^2}$ since $\cos(\theta) \ge 0$ when $-\pi/2 \le \theta \le \pi/2$.

Finally, the derivative of inverse sine is

$$\left(\sin^{-1}(x)\right)' = \frac{1}{\sqrt{1-x^2}}$$

♣

Example 4.84: Derivative of Inverse Cosine

Find the derivative of $\cos^{-1}(x)$.

Solution. Let $y = \cos^{-1}(x)$, so $\cos(y) = x$ and $0 \le y \le \pi$. Next we differentiate implicitly:

$$\begin{aligned}
\frac{d}{dx}(\cos y) &= \frac{d}{dx}(x) \\
-\sin y \cdot \frac{dy}{dx} &= 1
\end{aligned}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

This time, just for variety, we will leave the derivative in terms of y and apply some trigonometry. Since $\cos y = x$, we construct a triangle with angle y, adjacent side length x and hypotenuse 1. Solving for the opposite side length using Pythagorean Theorem we obtain $\sqrt{1-x^2}$. Using this triangle we can see that $\sin y = \sqrt{1-x^2}$ $(0 \le y \le \pi)$. Substituting this into the equation for dy/dx, we find that

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

. ♣

In the following example we explore an alternate method of finding the derivative.

Example 4.85: Derivative of Inverse Tangent

Find the derivative of $\tan^{-1}(x)$.

Solution. We begin with $\tan\left(\tan^{-1}(x)\right) = x$. Taking the derivative using the Chain Rule we obtain

$$\sec^2\left(\tan^{-1}(x)\right) \cdot \frac{d}{dx}\left(\tan^{-1}(x)\right) = 1,$$

which we rearrange to obtain

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sec^2\left(\tan^{-1}(x)\right)}.$$

Let $\tan^{-1}(x) = \theta$, then $\tan(\theta) = x$. We construct a triangle with angle θ, adjacent side 1 and opposite side x. The hypotenuse is $\sqrt{1+x^2}$ using Pythagorean Rheorem. Then $\sec^2\left(\tan^{-1}(x)\right) = \sec^2(\theta) = (\sec(\theta))^2 = \left(\sqrt{1+x^2}\right)^2 = 1+x^2$. Recall that $\sec(x) = 1/\cos(x)$. Finally, the derivative is

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}.$$

♣

The derivatives of the three remaining inverse trigonometric functions can be found in a similar manner. The table below provides a summary of the derivatives of all six inverse trigonometric functions and their domains.

Theorem 4.86: Inverse Trig Derivatives

$$\begin{aligned}
\frac{d}{dx}\sin^{-1}x &= \frac{1}{\sqrt{1-x^2}}, && x\in(-1,1)\\
\frac{d}{dx}\cos^{-1}x &= -\frac{1}{\sqrt{1-x^2}}, && x\in(-1,1)\\
\frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2}, && x\in(-\infty,\infty)\\
\frac{d}{dx}\sec^{-1}x &= \frac{1}{|x|\sqrt{x^2-1}}, && x\in(-\infty,-1)\cup(1,\infty)\\
\frac{d}{dx}\csc^{-1}x &= -\frac{1}{|x|\sqrt{x^2-1}}, && x\in(-\infty,-1)\cup(1,\infty)\\
\frac{d}{dx}\cot^{-1}x &= -\frac{1}{1+x^2}, && x\in(-\infty,\infty)
\end{aligned}$$

Example 4.87: Inverse Functions and Chain Rule

Find the derivative of $f(x)=\log_2\left(\sin^{-1}(x^2-3x)\right)$.

Solution.

$$\begin{aligned}
f'(x) &= \frac{1}{(\ln 2)\sin^{-1}(x^2-3x)}\frac{d}{dx}\left(\sin^{-1}(x^2-3x)\right)\\
&= \frac{1}{(\ln 2)\sin^{-1}(x^2-3x)}\left(\frac{1}{\sqrt{1-(x^2-3x)^2}}\frac{d}{dx}(x^2-3x)\right)\\
&= \frac{1}{(\ln 2)\sin^{-1}(x^2-3x)}\frac{1}{\sqrt{1-(x^2-3x)^2}}(2x-3)\\
&= \frac{2x-3}{(\ln 2)\sin^{-1}(x^2-3x)\sqrt{1-(x^2-3x)^2}}
\end{aligned}$$

♣

Example 4.88: Inverse Functions and Implicit Differentiation

Find $\frac{dy}{dx}$ *for* $\cos^{-1}(xy)=x^2$.

Solution.

$$
\begin{aligned}
\frac{d}{dx}\left(cos^{-1}(xy)\right) &= \frac{d}{dx}(x^2)\\
-\frac{1}{\sqrt{1-(xy)^2}}\frac{d}{dx}(xy) &= 2x\\
\frac{y+x\frac{dy}{dx}}{\sqrt{1-(xy)^2}} &= 2x\\
-\frac{x}{\sqrt{1-(xy)^2}}\frac{dy}{dx} &= 2x+\frac{y}{\sqrt{1-(xy)^2}}\\
\frac{dy}{dx} &= -\frac{\sqrt{1-(xy)^2}}{x}\left(2x+\frac{y}{\sqrt{1-(xy)^2}}\right)
\end{aligned}
$$

♣

Exercises for Section 4.8

Exercise 4.8.1 *Find the derivative of the function.*

(a) $f(x) = \csc^{-1}(5x^2+1)$

(b) $f(x) = (\tan^{-1}(2x))^3$

(c) $g(x) = \sqrt{e^{cos^{-1}(x)}}$

(d) $f(t) = \ln(\sin^{-1} t)$

(e) $f(x) = \sec^{-1}(x^{3/2})$

(f) $h(s) = \cos^{-1}(\log_2 s)$

(g) $f(x) = (\cot^{-1} x)^{1/3}$

(h) $g(t) = \sin^{-1}(3^t)$

Exercise 4.8.2 *Find* $\frac{dy}{dx}$ *by implicit differentiation.*

(a) $\sin^{-1}(xy) + xy = x$

(b) $\tan^{-1}(x-y) = xy$

Exercise 4.8.3 *Given* $f(x) = 1+\ln(x-2)$, *first show that* f^{-1} *exists, then compute* $\left[f^{-1}\right]'(1)$.

Exercise 4.8.4 *The* **inverse cotangent function**, *denoted by* $\cot^{-1}(x)$, *is defined to be the inverse of the restricted cotangent function:* $\cot(x)$, $0 < x < \pi$. *Find the derivative of* $\cot^{-1}(x)$.

Exercise 4.8.5 *The* **inverse secant function**, *denoted by* $\sec^{-1}(x)$, *is defined to be the inverse of the restricted secant function:* $\sec(x)$, $x \in [0, \pi/2) \cup [\pi, 3\pi/2)$. *Find the derivative of* $\sec^{-1}(x)$.

Exercise 4.8.6 *The* **inverse cosecant function**, *denoted by* $\csc^{-1}(x)$, *is defined to be the inverse of the restricted cosecant function:* $\csc(x)$, $x \in (0, \pi/2] \cup (\pi, 3\pi/2]$. *Find the derivative of* $\csc^{-1}(x)$.

Exercise 4.8.7 *Suppose* $f(x) = x^3 + 4x + 2$. *Find the slope of the tangent line to the graph of* $g(x) = xf^{-1}(x)$ *at the point where* $x = 7$.

Exercise 4.8.8 *Find the derivatives of* $\sin^{-1}(x) + \cos^{-1}(x)$ *and* $(x^2 + 1)\tan^{-1}(x)$.

Exercise 4.8.9 *Differentiate* $y = \sin^{-1}(x^2)$ *and* $y = \tan^{-1}(3x)$.

4.9 Additional Exercises

Exercise 4.9.1 *Find the derivatives of the following functions from definition.*

(a) $f(x) = (2x+3)^2$

(b) $g(x) = x^{3/2}$

Exercise 4.9.2 *Let* $f(x) = \begin{cases} x^3 & \text{if } x \leq 1 \\ 5x - x^2 & \text{if } x > 1 \end{cases}$. *Use the definition of the derivative to find* $f'(1)$.

Exercise 4.9.3 *Differentiate the following functions.*

(a) $y = 7x^4 - 7\pi^4 + \dfrac{1}{\pi \sqrt[3]{x}}$

(b) $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

(c) $f(x) = |x-1| + |x+2|$

(d) $f(x) = x^2 \sin x \cos x$

(e) $y = \dfrac{x \sin x}{1 + \sin x}$

(f) $g(x) = \sqrt{2 + \dfrac{3}{\sqrt{x}}}$

(g) $y = \sqrt[3]{x^4 + x^2 + 1} + \dfrac{1}{(x^3 - x + 4)^5}$

(h) $y = \sin^3 x - \sin(x^3)$

(i) $F(x) = \sec^4 x + \tan^4 x$

(j) $y = \cos^2\left(\dfrac{1-x}{1+x}\right)$

(k) $y = \tan(\sin(x^2 + \sec^2 x))$

(l) $y = \dfrac{1}{2+\sin\frac{\pi}{x}}$

Exercise 4.9.4 *Differentiate the following functions.*

(a) $y = e^{3x} + e^{-x} + e^2$

(b) $y = e^{2x}\cos 3x$

(c) $f(x) = \tan(x + e^x)$

(d) $g(x) = \dfrac{e^x}{e^x+2}$

(e) $y = \ln(2+\sin x) - \sin(2+\ln x)$

(f) $f(x) = e^{x^\pi} + x^{\pi^e} + \pi^{e^x}$

(g) $y = \log_a(b^x) + b^{\log_a x}$, *where a and b are positive real numbers and* $a \neq 1$

(h) $y = (x^2+1)^{x^3+1}$

(i) $y = (x^2+e^x)^{1/\ln x}$

(j) $y = \dfrac{x\sqrt{x^2+x+1}}{(2+\sin x)^4(3x+5)^7}$

Exercise 4.9.5 *Find* $\dfrac{dy}{dx}$ *if y is a differentiable function that satisfy the given equation.*

(a) $x^2 + xy + y^2 = 7$

(b) $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$

(c) $x^2 \sin y + y^3 = \cos x$

(d) $x^2 + xe^y = 2y + e^x$

Exercise 4.9.6 *Differentiate the following functions.*

(a) $y = x\sin^{-1} x$

(b) $f(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$

(c) $g(x) = \tan^{-1}\left(\dfrac{x}{a}\right)$, *where* $a > 0$

(d) $y = x\tan^{-1} x - \dfrac{1}{2}\ln(x^2+1)$

5. Applications of Derivatives

In this chapter we explore how to use derivative and differentiation to solve a variety of problems, some mathematical and some practical. We explore some applications which motivated and were formalized in the definition of the derivative, and look at a few clever uses of the tangent line (which has immediate geometric ties to the definition of the derivative).

5.1 Elasticity of Demand

We begin by analyzing a real example from the air travel industry, and have a detailed look at how the cost of air plane tickets impact the revenue of tickets sold. A simplistic view may lead one to believe that a decrease in the cost for an airplane ticket would cause the revenue to increase and vice versa. In economics, this particular relationship between unit price and revenue is referred to as *elastic demand* as we will learn later. Particularly in Canada, start-up airlines can collapse more readily under this condition. The Department of Finance in Canada studied the aforementioned relationship and published the research results in

$$Air\ Travel\ Demand\ Elasticities:\ Concepts,\ Issues\ and\ Measurement:1$$

by differentiating between six types of air travel that are associated pairwise: business and leisure, long-haul and short-haul, and international long-haul and North American long-haul air travel. The results of the study corroborate that the demand for business air travel is less elastic than that for leisure air travel. This finding does not come as a surprise, since even a costly booked vacation can be more readily moved to different dates than business travels. The other two results of the study are that the demand for long-haul flights is less elastic than that for short-haul flights, and similarly, the demand for international flights is less elastic than that for North American flights. This make sense, because the further the destination, the less likely it is that an alternative mode of transport can be found as a substitute for an expensive flight.

We now derive the mathematical model that helps us to analyze the relationship between unit price and revenue, and determines the *elasticity of demand* of a particular economic situation when the demand function is given.

In order to aid our analysis, it will be more convenient to write the demand function f in the form $q = f(p)$. In other words, we will think of the quantity demanded q of a certain product as a function of its unit price p. As is shown in Figure 5.1, the function f is usually a decreasing function of p, because the quantity demanded of a product typically decreases as the associated unit price increases.

Note: During problem solving, it is often easier to use the inverse function of f, namely $p = g(q)$ than f itself.

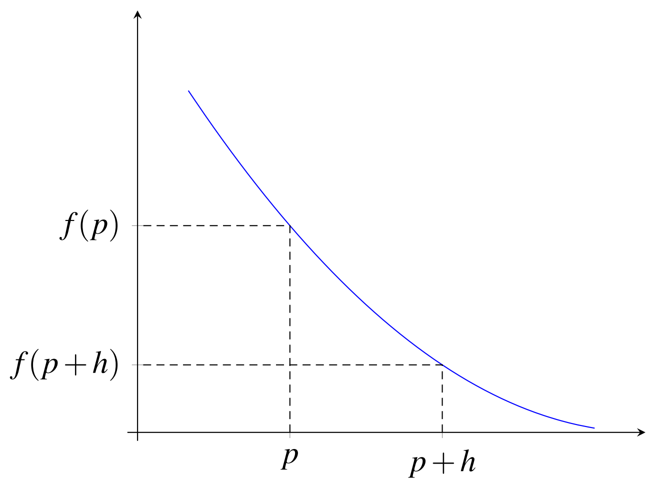


Figure 5.1: The demand function $q = f(p)$ and the effects on this demand from an increase in price by h dollars.

We now take a similar approach as in our analysis of the derivative in Chapter 4. Figure 5.1 shows an increase of h dollars in the unit price p for some product to a unit price of $p+h$ dollars. Therefore, the associated quantity demanded changes from $f(p)$ units to $f(p+h)$ units with an overall decrease of $f(p+h)-f(p)$ units. We can now calculate the percentage change in the unit price to be

$$\frac{\text{Change in unit price}}{\text{Price } p} \times 100 = \frac{h}{p}(100),$$

and the corresponding percentage change in the quantity demanded to be

$$\frac{\text{Change in quantity demanded}}{\text{Quantity demanded at price } p} \times 100 = \frac{f(p+h)-f(p)}{f(p)}(100).$$

By calculating the ratio of the percentage change in the quantity demanded to the percentage change in price, we can determine the effect the latter has on the former:

$$\begin{aligned}\frac{\text{Percentage change in the quantity demanded}}{\text{Percentage change in the unit price}} &= \frac{100\dfrac{f(p+h)-f(p)}{f(p)}}{100\dfrac{h}{p}} \\ &= \frac{\dfrac{f(p+h)-f(p)}{f(p)}}{\dfrac{f(p)}{p}}\end{aligned}$$

We now recognize the difference quotient in this fraction. So, if f is differentiable at p, we can deduce for small h that

$$\frac{f(p+h)-f(p)}{h} \approx f'(p).$$

In other words,

$$\frac{\text{Percentage change in the quantity demanded}}{\text{Percentage change in the unit price}} = \frac{f'(p)}{\dfrac{f(p)}{p}} = \frac{pf'(p)}{f(p)},$$

when h is small.

Note: In Section 5.7 it will be shown that for a decreasing function $q = f(p)$ on a certain interval I, its derivative $f'(p) < 0$ for all $p \in I$. But this means that the value of the quantity $pf'(p)/f(p)$ is negative. Since it is preferred to work with positive values, economists define the elasticity of demand $E(p)$ to be the negative of the quantity $pf'(p)/f(p)$.

Definition 5.1: Elasticity of Demand

Suppose that the demand function $q = f(p)$ is differentiable. Then the **elasticity of demand**, *E, at price p is defined by*

$$E(p) = -\frac{pf'(p)}{f(p)}$$

Example 5.2: Elasticity of Demand

The unit price p in dollars and the quantity demanded q of a certain product are related by the equation

$$p = -0.02q + 400 \quad 0 \leq q \leq 20,000$$

(a) Determine the elasticity of demand $E(p)$.

(b) Calculate $E(100)$. What can you determine from your result?

(c) Calculate $E(300)$. What can you determine from your result?

Solution.

(a) Writing q in terms of p, we have

$$q = f(p) = -50p + 20,000$$

and so $f'(p) = -50$. The elasticity of demand is thus

$$E(p) = -\frac{pf'(p)}{f(p)} = \frac{50p}{-50p + 20,000} = \frac{p}{400 - p}$$

(b)

$$E(100) = \frac{100}{400 - 100} = \frac{1}{3}.$$

Therefore, when the unit price p is \$100 per unit, a small increase in p will lead to a decrease of approximately 0.33% in the quantity demanded q.

(c)

$$E(300) = \frac{300}{400 - 300} = 3.$$

Here, we see that a small increase in p from \$300 per unit will lead to a decrease of approximately 3% in the quantity demanded q.

♣

The following economic terminology is useful when describing demand in terms of elasticity.

Definition 5.3: Elastic, Unitary and Inelastic Demand

1. *The demand is* **elastic** *if $E(p) > 1$. That is to say, the demand is elastic if the percentage change in demand is greater than than the percentage change in price.*
2. *The demand is* **unitary** *if $E(p) = 1$. That is to say, the demand is unitary if the percentage change in demand and price are relatively equal.*
3. *The demand is* **inelastic** *if $E(p) < 1$. That is to say, the demand is inelastic if the percentage change in demand is less than the percentage change in price.*

In Example 5.2, we determined that the demand for the given product is elastic when $p = 300$ and inelastic when $p = 100$. These calculations illustrate that a small percentage change in the unit price will result in a greater percentage change in the quantity demanded, i.e. when the demand is elastic; and a small percentage change in the unit price will cause a smaller percentage change in the quantity demanded, i.e. when the demand is inelastic; and lastly, a small percentage change in the unit price will result in the same percentage change in the quantity demanded, i.e. when the demand is unitary.

5.1.1. Elasticity and Revenue

In the previous section, we developed the notion of elasticity of demand by analyzing the relationship between quantity demanded and unit price in terms of percentage change. Of course this change influences revenue, and so we now have a closer look at the effects of elasticity on revenue. Again we assume that $q = f(p)$ relates the quantity q demanded of a certain product to its unit price p in dollars. When q units of the product are sold at the price p, then the revenue is given by

$$R(p) = pq = pf(p).$$

We now calculate the marginal revenue with respect to p and obtain

$$\begin{aligned} R'(p) &= f(p) + pf'(p) \\ &= f(p)\left[1 + \frac{pf'(p)}{f(p)}\right] \\ &= f(p)\,[1 - E(p).] \end{aligned}$$

This last equation tells us that elasticity influences revenue. In order to determine what the effects are, we analyze the sign of the marginal revenue. We first note that $f(p)$ is positive for all values of p and consider three cases:

1. Suppose the demand is elastic when the unit price is set at p dollars. Then

$$E(p) > 1 \implies 1 - E(p) < 0,$$

and so

$$R'(p) = f(p)\,[1 - E(p)] < 0,$$

which means that revenue R is decreasing at p. In other words, a small increase/decrease in the unit price results in a decrease/increase respectively in the revenue. This is illustrated on the revenue curve of the white region in Figure 5.2.

2. Suppose the demand is unitary when the unit price is set at p dollars. Then

$$E(p) = 1 \implies 1 - E(p) = 0,$$

and so

$$R'(p) = f(p)\,[1 - E(p)] = 0,$$

which causes revenue R to be stationary at p, i.e. neither increasing nor decreasing. This means that a small increase/decrease in the unit price does not affect a change in the revenue. This is visualized on the revenue curve in Figure 5.2 where arrows point to.

3. Lastly, suppose the demand is inelastic when the unit price is set at p dollars.

$$E(p) < 1 \implies 1 - E(p) > 0,$$

and so

$$R'(p) = f(p)\,[1 - E(p)] > 0,$$

which necessitates that revenue R is increasing at p. This implies that a small increase/decrease in the unit price results in an increase/decrease respectively in the revenue. This is visualized on the revenue curve of the grey region in Figure 5.2.

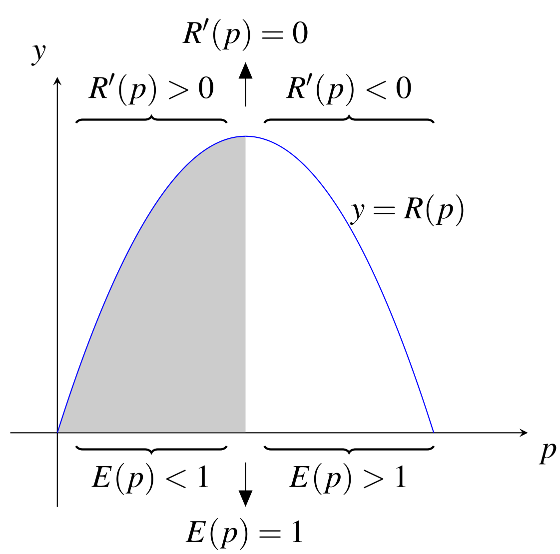


Figure 5.2: Inelastic demand corresponds to an increase in revenue (see grey), elastic demand corresponds to a decrease in revenue (see white), and at unitary demand revenue is stationary (see arrows)

The results of this analysis are summarized below:

Effects of Unit Price Changes to Revenue

1. If the demand is *elastic* at p, i.e. $E(p) > 1$, then a small increase/decrease in the unit price results in a decrease/increase respectively in the revenue.
2. If the demand is *unitary* at p, i.e. $E(p) = 1$, then a small increase/decrease in the unit price does not affect a change in the revenue. revenue
3. If the demand is *inelastic* at p, i.e. $E(p) < 1$, then a small increase/decrease in the unit price results in an increase/decrease respectively in the revenue.

Note: By noticing the following relationships between the unit price and the revenue, you may better be able to remember the effects on a unit price change on the revenue.

1. When the demand is elastic, then the change in unit price and the change in revenue move in *opposite* direction.
2. When the demand is inelastic, then the change in unit price and the change in revenue move in the *same* direction.

Example 5.4: Elasticity of Demand

Refer to example 5.2.

(a) *For $p = 100$ and $p = 300$, calculate whether the demand is elastic, unitary or inelastic.*

(b) *What can you deduce from your results when $p = 100$?*

Solution.

(a) From part (b) of Example 5.2, we see that $E(100) = \frac{1}{3} < 1$. Therefore, the demand is inelastic. From part (c) of Example 5.2, we see that $E(300) = 3 > 1$, and so the demand is elastic.

(b) Since the demand is inelastic when $p = 100$, a slight raise in the unit price will lead to an increase in revenue.

♣

Example 5.5: Elasticity of Demand

The demand equation for a certain product is given by

$$p = -0.02q + 300 \quad 0 \leq q \leq 15,000$$

where p denotes the unit price in dollars and q denotes the quantity demanded. The weekly total cost function associated with this product is

$$C(q) = 0.000003q^3 - 0.04q^2 + 200q + 70,000$$

dollars.

(a) Determine the revenue function R and the profit function P.

(b) Determine the marginal cost function C', the marginal revenue function R', and the marginal profit function P'.

(c) Determine the marginal average cost function $\overline{C}'$.

(d) Calculate $C'(3000)$, $R'(3000)$ and $P'(3000)$. What can you deduce from your results?

(e) Determine whether the demand is elastic, unitary, or inelastic when $p = 100$ and when $p = 200$.

Solution.

(a) $$\begin{aligned} R(q) &= pq \\ &= q(-0.02q^2 + 300q) \\ &= -0.02q^2 + 300q \end{aligned}$$

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= -0.02q^2 + 300q - (0.000003q^3 - 0.04q^2 + 200q + 70,000) \\ &= -0.000003q^3 + 0.02q^2 + 100q - 70,000 \quad (0 \leq q \leq 15,000) \end{aligned}$$

(b) $C'(q) = 0.000009q^2 - 0.08q + 200$
$R'(q) = -0.04q + 300$
$P'(q) = -0.000009q^2 + 0.04q + 100.$

(c) The average cost function is

$$\begin{aligned} \overline{C}(q) &= \frac{C(q)}{q} \\ &= \frac{0.000003q^3 - 0.04q^2 + 200q + 70,000}{q} \\ &= 0.000003q^2 - 0.04q + 200 + \frac{70,000}{q} \end{aligned}$$

Therefore, the marginal average cost function is

$$\overline{C}'(q) = 0.000006q - 0.04 - \frac{70,000}{q^2}.$$

(d) Using the above results, we find

$$C'(3000) = 0.000009(3000)^2 - 0.08(3000) + 200 = 41$$

That is, when the level of production is already 3000 units, the actual cost of producing one additional unit is approximately \$41.

$$R'(3000) = -0.04(3000) + 300 = 180$$

That is, the actual revenue to be realized from selling the 3001st unit is approximately \$180.

$$P'(3000) = -0.000009(3000)^2 + 0.04(3000) + 100 = 139$$

That is, the actual profit realized from selling the 3001st unit is approximately \$139.

(e) We first solve the given demand equation for q in terms of p, obtaining

$$q = f(p) = -50p + 15{,}000$$
$$\implies f'(p) = -50$$

Therefore,

$$\begin{aligned} E(p) &= \frac{pf'(p)}{f(p)} \\ &= \frac{p}{-50p + 15{,}000} \\ &= \frac{p}{300 - p} \quad 0 \le p < 300 \end{aligned}$$

Next, we compute

$$E(100) = \frac{100}{300 - 100} = \frac{1}{2} < 1$$

and we conclude that the demand is inelastic when $p = 100$. Next,

$$E(200) = \frac{200}{300 - 200} = 2 > 1$$

and we conclude that the demand is elastic when $p = 200$.

♣

Exercises for Section 5.1

Exercise 5.1.1 *For each demand equation, compute the elasticity of demand and determine whether or not the demand is elastic, unitary, or inelastic at the indicated price, p.*

(a) $q = -\frac{1}{2}p + 10,\ p = 10.$

(b) $q = -\frac{3}{2}p + 9,\ p = 1.$

(c) $q + \frac{1}{3}p - 24 = 0,\ p = 3.$

(d) $0.4q + p = 20,\ p = 12.$

(e) $p = 16 - 2q^2,\ p = 4.$

(f) $2p = 144 - q^2,\ p = 48.$

Exercise 5.1.2 *It is determined that the demand equation for a certain product is*

$$q = \frac{1}{5}(225 - p^2) \quad 0 \leq p \leq 15$$

where q is the quantity demanded in units of hundreds and p is the unit price in dollars.

(a) For $p = 18$ *and* $p = 10$*, determine whether the demand elastic or inelastic.*

(b) Determine the value of p for which the demand is unitary.

(c) If the unit price is lowered slightly from \$10, will the revenue increase or decrease?

(d) If the unit price is increased slightly from \$8, will the revenue increase or decrease?

Exercise 5.1.3 *It is estimated that the quantity q of fair tickets purchased is related to the ticket price p by the demand equation*

$$q = \frac{2}{3}\sqrt{36 - p^2} \quad 0 \leq p \leq 6.$$

Currently, the price is set at \$2 each.

(a) Is the demand elastic or inelastic at this price?

(b) If the ticket price is increased, will the revenue increase or decrease?

Exercise 5.1.4 *The demand function for a certain product is*

$$p = \sqrt{9 - 0.02q} \quad 0 \leq q \leq 450$$

where p is the unit price in hundreds of dollars and q is the quantity demanded per week.

(a) Calculate the elasticity of demand.

(b) Determine the values of p for which the demand if inelastic, unitary and elastic.

5.2 Related Rates

When defining the derivative $f'(x)$, we define it to be exactly the rate of change of $f(x)$ with respect to x. Consequently, any question about rates of change can be rephrased as a question about derivatives. **When we calculate derivatives, we are calculating rates of change**. Results and answers we obtain for derivatives translate directly into results and answers about rates of change. Let us look at some examples where more than one variable is involved, and where our job is to analyze and exploit relations between the rates of change of these variables. As an aside, this class of problems is known as **related rates problems**. The mathematical step of relating the rates of change turns out to be largely an exercise in differentiation using the Chain Rule or implicit differentiation. This explains why some textbooks place this section shortly after the sections on the Chain Rule and implicit differentiation.

Let's say we are interested in the relationship between the rate of change of a mortgage rate and the rate of change of the number of houses sold over time. If x represents the mortgage rate and y the number of houses sold at any time t, then x and y are each functions of this third variable t. Suppose furthermore that the mortgage rate x is related to the number of houses sold y, i.e. we also have an equation relating x to y:

$$f(x) = g(y).$$

Then we can differentiate both sides of this equation implicitly with respect to t, and get

$$f'(x)\frac{dx}{dt} = g'(y)\frac{dy}{dt}.$$

In other words, we now have an equation that relates dx/dt to dy/dt. In terms of our problem, this means that the rate of change of the mortgage rate and the rate of change of the number of houses sold are related as a function of time. And so, as dx/dt changes determines how dy/dt changes, i.e. the rate of change of mortgage w.r.t. time controls the rate of change of houses at that instant of time.

Example 5.6: Speed at which a Coordinate is Changing

Suppose an object is moving along a path described by $y = x^2$, that is, it is moving on a parabolic path. At a particular time, say $t = 5$, the x-coordinate is 6 and we measure the speed at which the x-coordinate of the object is changing and find that $dx/dt = 3$.
At the same time, how fast is the y-coordinate changing?

Solution. Using the Chain Rule,

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

At $t = 5$ we know that $x = 6$ and $dx/dt = 3$, so $dy/dt = (2)(6)(3) = 36$. ♣

In many cases, particularly interesting ones, x and y will be related in some other way, for example $x = f(y)$, or $F(x,y) = k$, or perhaps $F(x,y) = G(x,y)$, where $F(x,y)$ and $G(x,y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in all the known values (namely, x, y, and dx/dt), and then solving for dy/dt.

To summarize, here are the steps in doing a related rates problem.

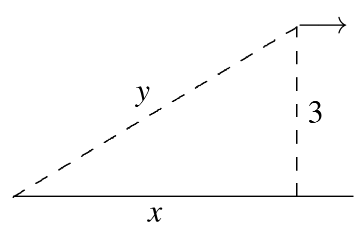
Steps for Solving Related Rates Problems

1. Read the problem at least twice.
2. Sketch and label a diagram of the problem if applicable.
3. Identify the independent variable (often, but not always, *time*).
4. Unless already introduced, use a let statement to introduce dependent variables.
5. State the known and unknown rate(s) and value(s) using your variable name(s).
6. Find an equation relating the independent and dependent variables.
7. Differentiate the equation implicitly w.r.t the independent variable.
8. Use substitution of known values to solve the new equation.
9. Critically evaluate if your answer makes sense.

Example 5.7: Receding Airplanes

A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Solution. To see what's going on, we first draw a schematic representation of the situation, as shown below.



Because the plane is in level flight directly away from you, the rate at which x changes is the speed of the plane, $dx/dt = 500$ mph. The distance between you and the plane is y; it is dy/dt that we wish to know. By the Pythagorean Theorem we know that $x^2 + 9 = y^2$. Taking the derivative with respect to the independent variable t, we obtain

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}y.$$

We are interested in the time at which $x = 4$; at this time we know that $4^2 + 9 = y^2$, so $y = 5$. Putting together all the information we get

$$2(4)(500) = 2(5)\frac{dy}{dt}.$$

Thus, $\frac{dy}{dt} = 400$ mph. ♣

Example 5.8: Rate of Change of Housing Starts

It is estimated that the number of housing starts, $N(t)$ (in units of a million), over the next 5 years is related to the mortgage rate $r(t)$ (percent per year) by the equation

$$8N^2 + r = 36.$$

What is the rate of change of the number of housing starts with respect to time when the mortage rate is 4% per year and is increasing at the rate of 0.25% per year?

Solution. We want to find dN/dt when

$$r = 4 \quad \text{and} \quad \frac{dr}{dt} = 0.25.$$

We are given the relationship

$$8N^2 + r = 36,$$

so we can find the relationship between the rate of change of N and the rate of change of r by differentiating this equation implicitly with respect to time. This gives

$$\begin{aligned} \frac{d}{dt}\left(8N^2 + r\right) &= \frac{d}{dt}(36) \\ 2(8)N\frac{dN}{dt} + \frac{dr}{dt} &= 0 \\ 16N\frac{dN}{dt} &= -\frac{dr}{dt} \\ \frac{dN}{dt} &= \frac{-1}{16N}\frac{dr}{dt} \end{aligned}$$

At the instant in time we are considering, the number of housing starts N is unknown. However, we know that N satisfies

$$8N^2 + r = 36,$$

so when $r = 4$, we must have

$$\begin{aligned} 8N^2 + 4 &= 36 \\ N^2 &= 4 \\ N &= 2, \end{aligned}$$

where we have rejected the negative root. Therefore, when $r = 4$ and $dr/dt = 0.25$, the rate of change of housing starts is

$$\frac{dN}{dt} = \frac{-1}{16(2)}(0.25) = -0.0078125,$$

that is, the number of housing starts is decreasing by approximately 7,813 units. ♣

Example 5.9: Supply-Demand

It is found that a certain manufacturer produces q thousand units per week when the unit price is \$$p$. Suppose the relationship between q and p is

$$q^2 - 3qp + p^2 = 5.$$

What is the rate of change of the supply when the quantity produced is 4000 units and the unit price is \$11, increasing at a rate of \$0.10 per week?

Solution. We differentiate the supply equation on both sides with respect to t, obtaining

$$\begin{aligned} \frac{d}{dt}\left(q^2\right) - \frac{d}{dt}(3qp) + \frac{d}{dt}\left(p^2\right) &= \frac{d}{dt}(5) \\ 2q\frac{dq}{dt} - 3\left(p\frac{dq}{dt} + q\frac{dp}{dt}\right) + 2p\frac{dp}{dt} &= 0, \end{aligned}$$

where we used the Product Rule on the second term. So when $p = 11$, $dp/dt = 0.1$ and $q = 4000$, we have

$$\begin{aligned} 2(4)\frac{dq}{dt} - 3\left((11)\frac{dq}{dt} + 4(0.1)\right) + 2(11)(0.1) &= 0 \\ 8\frac{dq}{dt} - 33\frac{dq}{dt} - 1.2 + 2.2 &= 0 \\ 25\frac{dq}{dt} &= 1 \\ \frac{dq}{dt} &= 0.04. \end{aligned}$$

Thus, at the instant of time under consideration, the supply is increasing at the rate of $(0.04)(1000)$, or 40, units per week. ♣

Example 5.10: Spherical Balloon

You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing when the radius is 4 cm?

Solution. Here the independent variable is time t and the dependent variables are the radius r and the volume V. We know dV/dt, and we want dr/dt. The two variables are related by the equation $V = 4\pi r^3/3$. Taking the derivative with respect to the independent variable t, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We now substitute the values we know at the instant in question:

$$7 = 4\pi 4^2 \frac{dr}{dt},$$

so $dr/dt = 7/(64\pi)$ cm/sec. ♣

Example 5.11: Conical Container

Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm; see Figure 5.3. How fast is the water level rising when the water is 4 cm deep (at its deepest point)?

Solution. The water forms a conical shape within the big cone; its height and base radius and volume are all increasing as water is poured into the container. This means that we actually have three things varying with time: the water level h (the height of the cone of water), the radius r of the circular top surface of water (the base radius of the cone of water), and the volume of water V. The volume of a cone is given by

$$V = \pi r^2 h/3.$$

Again, the independent variable is time t. We know dV/dt, and we want dh/dt. At first something seems to be wrong: we have a third variable, r, whose rate we don't know.

However, the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles,

$$\frac{r}{h} = \frac{10}{30},$$

so $r = h/3$. Now we can eliminate r from the problem entirely:

$$V = \pi(h/3)^2 h/3 = \pi h^3/27.$$

We take the derivative of both sides and plug in $h = 4$ and $dV/dt = 10$, obtaining

$$10 = 3\pi(\frac{4^2}{27})\frac{dh}{dt}.$$

Thus, $dh/dt = 90/(16\pi)$ cm/sec. ♣

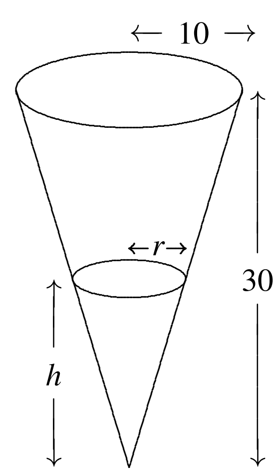


Figure 5.3: Conical water tank.

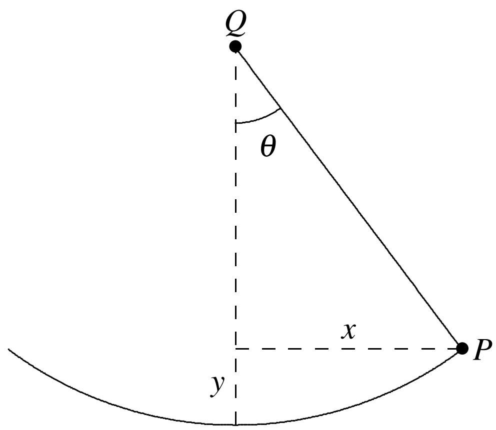
Example 5.12: Swing Set

A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. Find

(a) how fast the swing is rising after 1s;

(b) the angular speed of the rope in deg/sec after 1s.

Solution. We start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we're being asked about? Again, the independent variable is time t. Note that the person pushing the swing is moving horizontally at a rate we know. In other words, the horizontal coordinate of P is increasing at 6 ft/sec. In the x-y-plane let us make the convenient choice of putting the origin at the location of P at time $t = 0$, i.e., a distance 10 directly below the point of attachment as shown below. Then the rate we know is dx/dt, and in part (a) the rate we want is dy/dt (the rate at which P is rising). In part (b) the rate we want is $d\theta/dt$, where θ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $d\theta/dt$ from rad/sec by multiplying by $180/\pi$.)



(a) From the diagram we see that we have a right triangle whose legs are x and $10-y$, and whose hypotenuse is 10. Hence

$$x^2 + (10-y)^2 = 100.$$

Taking the derivative of both sides with respect to t we obtain

$$2x\frac{dx}{dt} + 2(10-y)(0-\frac{dy}{dt}) = 0.$$

We now look at what we know after 1 second, namely $x = 6$ (because x started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), thus $y = 2$ (because we get $10-y = 8$ from the Pythagorean Theorem applied to the triangle with hypotenuse 10 and leg 6), and $dx/dt = 6$. Putting in these values gives us

$$2 \cdot 6 \cdot 6 - 2 \cdot 8\frac{dy}{dt} = 0,$$

from which we can easily solve for dy/dt: $dy/dt = 4.5$ ft/sec.

(b) Here our two variables are x and θ, so we want to use the same right triangle as in part (a), but this time relate θ to x. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin\theta = x/10$. Taking derivatives we obtain

$$(\cos\theta)\frac{d\theta}{dt} = 0.1\frac{dx}{dt}.$$

At the instant in question ($t = 1$ sec), when we have a right triangle with sides 6–8–10, $\cos\theta = 8/10$ and $dx/dt = 6$. Thus $(8/10)d\theta/dt = 6/10$, i.e., $d\theta/dt = 6/8 = 3/4$ rad/sec, or approximately 43 deg/sec.

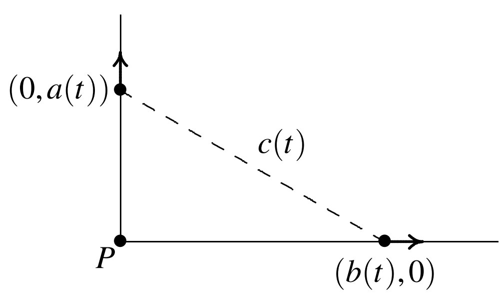
♣

We have seen that sometimes there are apparently more than two variables that change with time, but in reality there are just two, as the others can be expressed in terms of just two. However sometimes there really are several variables that change with time; as long as you know the rates of change of all but one of them you can find the rate of change of the remaining one. As in the case when there are just two variables, take the derivative of both sides of the equation relating all of the variables, and then substitute all of the known values and solve for the unknown rate.

Example 5.13: Distance Changing Rate

A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing accurate to one decimal place?

Solution. Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and let $c(t)$ be the distance from car A to car B at time t as shown below.



By the Pythagorean Theorem, $c(t)^2 = a(t)^2 + b(t)^2$. Taking derivatives we get

$$2c\frac{dc}{dt} = 2a(t)\frac{da}{dt} + 2b\frac{db}{dt},$$

so

$$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{\sqrt{a^2+b^2}}.$$

Substituting known values we get:

$$\frac{dc}{dt} = \frac{10 \cdot 80 + 15 \cdot 100}{\sqrt{10^2 + 15^2}} = \frac{460}{\sqrt{13}} \approx 127.6 \text{ km/hr}$$

at the time of interest. ♣

Notice how this problem differs from Example 5.7. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in Example 5.7 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this Example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

Exercises for Section 5.2

Exercise 5.2.1 *Suppose the quantity demanded weekly of a product is related to its unit price by the equation*

$$p + q^2 = 144$$

where p is measured in dollars and q is measured in units of a thousand. What is the rate of change of the quantity demanded when $q = 9$, $p = 63$, and the unit price is increasing at the rate of \$2/week?

Exercise 5.2.2 *The demand equation for a certain product is*

$$100q^2 + 9p^2 = 3600$$

where q is the number (in thousands) of units demanded each week when the unit price is \$$p$. What is the rate of change of the quantity demanded when the unit price is \$14 and the selling price is dropping at the rate of \$.15/unit/week?

Exercise 5.2.3 *Suppose the price p (in dollars/unit) of a product is related to the weekly supply q (in units of a thousand) by the equation*

$$625p^2 - q^2 = 100.$$

If 25,000 units are produced and the supply is falling at the rate of 1000 units/week, at what rate is the price changing?

Exercise 5.2.4 *The demand function for a certain product is*

$$p = -0.01q^2 - 0.1q + 6$$

where p is the unit price in dollars and q is the quantity demanded each week (in units of a thousand). Compute the elasticity of demand and determine whether the demand is inelastic, unitary, or elastic when $q = 10$.

Exercise 5.2.5 *Air is being pumped into a spherical balloon at a constant rate of 3 cm^3/s. How fast is the radius of the balloon increasing when the radius reaches 5cm?*

Exercise 5.2.6 *A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec?*

Exercise 5.2.7 *A cylindrical tank standing upright (with one circular base on the ground) has radius 1 meter. How fast does the water level in the tank drop when the water is being drained at 3 liters per second?*

Exercise 5.2.8 *A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?*

Exercise 5.2.9 *A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at* 0.1 *meters per second. How fast is the foot of the ladder approaching the wall when the foot of the ladder is 5 m from the wall?*

Exercise 5.2.10 *A rotating beacon is located 2 miles out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles along the shore from the point A?*

Exercise 5.2.11 *A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base?*

Exercise 5.2.12 *Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?*

Exercise 5.2.13 *A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out?*

Exercise 5.2.14 *A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later?*

Exercise 5.2.15 *A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m $\times$ 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m^3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is* $(1/3)(\textit{height})(\textit{area of base})$.

Exercise 5.2.16 *A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening?*

Exercise 5.2.17 *A man 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow shortening?*

Exercise 5.2.18 *A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car.*

Exercise 5.2.19 *A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car.*

Exercise 5.2.20 *A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time t seconds is* $h(t) = 20 - 9.8t^2/2$. *How fast is the object's shadow moving on the ground one second later?*

5.3 Linear and Higher Order Approximations

When we define the derivative $f'(x)$ as the rate of change of $f(x)$ with respect to x, we notice that in relation to the graph of f, the derivative is the slope of the tangent line, which (loosely speaking) is the line that just grazes the graph. But what precisely do we mean by this? In short, **the tangent line approximates the graph near the point of contact**. The definition of the derivative $f'(a)$ guarantees this when it exists: By taking x sufficiently close to a but not equal to a,

$$\frac{f(x) - f(a)}{x - a} \approx f'(a),$$

and consequently,

$$f(x) \approx f'(a)(x - a) + f(a).$$

The left hand side gives us the y-value of the function $y = f(x)$ and the right hand side gives us the y-value $y = f'(a)(x - a) + f(a)$ for the tangent line to the graph of f at the point $(a, f(a))$.

In this section we will explore how to apply this idea to approximate some values of f, some changes in the values of f, and also the roots of f.

5.3.1. Linear Approximations

We begin by the first derivative as an application of the tangent line to approximate f.

Recall that the tangent line to $f(x)$ at a point $(a, f(a))$ is given by

$$\begin{aligned} y - f(a) &= f'(a)(x - a) \\ y &= f'(a)(x - a) + f(a) \end{aligned}$$

provided that f is differentiable at $x = a$. As mentioned earlier, notice that the expression

$$f'(a)(x-a)+f(a)$$

is linear in x. Therefore, the above equation is also called the **linear approximation** of f at a. The function defined by

$$L(x) = f'(a)(x-a)+f(a)$$

is called the **linearization** of f at a.

If f is differentiable at a then L is a good approximation of f so long as x is "not too far" from a. Put another way, if f is differentiable at a then under a microscope f will look very much like a straight line, and thus will look very much like L; since $L(x)$ is often much easier to compute than $f(x)$, then it makes sense to use L as an approximation. Figure 5.4 shows a tangent line to $y = x^2$ at three different magnifications.

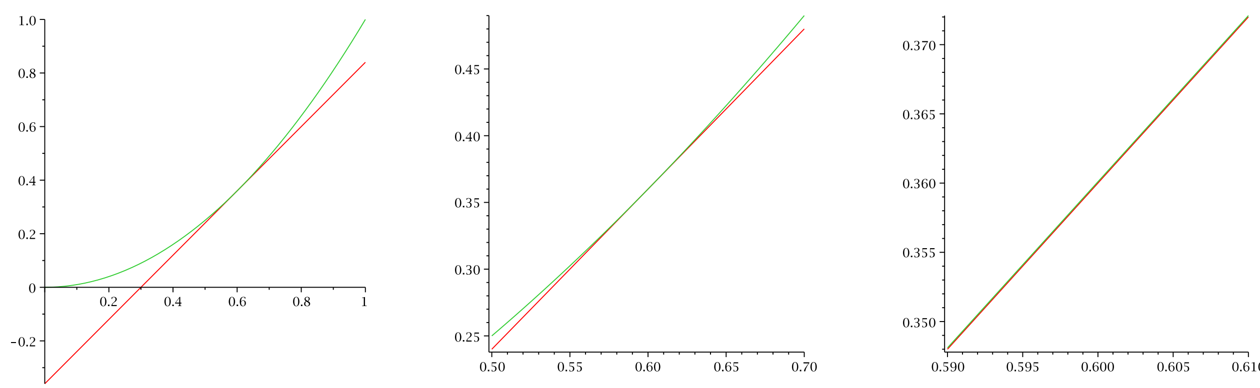


Figure 5.4: The linear approximation to $y = x^2$.

Definition 5.14: Linear Approximation

Suppose we are given a function $y = f(x)$.

1. *The* **linearization** *of f at $x = a$ is given by*

$$L(x) = f(a)+f'(a)(x-a)$$

provided that f is differentiable at $x = a$.

2. *The* **linear approximation** *of f at $x = a$ is given by*

$$f(x) \approx L(x) = f(a)+f'(a)(x-a)$$

provided that f is differentiable at $x = a$.

Thus in practice if we want to approximate a difficult value of $f(b)$, then we may be able to approximate this value using a linear approximation, provided that we can compute the tangent line at some point a close to b. Here are some examples.

Example 5.15: Linear Approximation

Let $f(x) = \sqrt{x+4}$, what is $f(6)$?

Solution. We are asked to calculate $f(6) = \sqrt{6+4} = \sqrt{10}$ which is not easy to do without a calculator. However 9 is (relatively) close to 10 and of course $f(5) = \sqrt{9}$ is easy to compute, and we use this to approximate $\sqrt{10}$.

To do so we have $f'(x) = 1/(2\sqrt{x+4})$, and thus the linear approximation to f at $x = 5$ is

$$L(x) = \left(\frac{1}{2\sqrt{5+4}}\right)(x-5) + \sqrt{5+4} = \frac{x-5}{6} + 3.$$

Notice that we did not create a common denominator to add teh two terms. This is because calculations are often easier in this form. Now to estimate $\sqrt{10}$, we substitute 6 into the linear approximation $L(x)$ instead of $f(x)$, to obtain

$$\sqrt{6+4} \approx \frac{6-5}{6} + 3 = \frac{19}{6} = 3^1/_6 = 3.1\bar{6} \approx 3.17$$

It turns out the exact value of $\sqrt{10}$ is actually 3.16227766... but our estimate of 3.17 was very easy to obtain and is relatively accurate. This estimate is only accurate to two decimal places. ♣

With modern calculators and computing software it may not appear necessary to use linear approximations, but in fact they are quite useful. For example in cases requiring an explicit numerical approximation, they allow us to get a quick estimate which can be used as a "reality check" on a more complex calculation. Further in some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible without serious loss of accuracy.

Example 5.16: Linear Approximation of Sine

Find the linear approximation of $\sin x$ *at* $x = 0$, *and use it to compute small values of* $\sin x$.

Solution. If $f(x) = \sin x$, then $f'(x) = \cos x$, and thus the linear approximation of $\sin x$ at $x = 0$ is:

$$L(x) = \cos(0)(x-0) + \sin(0) = x.$$

Thus when x is small this is quite a good approximation and is used frequently by engineers and scientists to simplify some calculations.

For example you can use your calculator (in radian mode since the derivative of $\sin x$ is $\cos x$ only in radian) to see that

$$\sin(0.1) = 0.099833416\ldots$$

and thus $L(0.1) = 0.1$ is a very good and quick approximation without any calculator! ♣

Exercises for Section 5.3.1

Exercise 5.3.1 *Determine the linear approximation $L(x)$ at a of each function below. Then use $L(x)$ to approximate the value of each function at the given x-value.*

(a) $f(x) = \sqrt{x},\ a = 4,\ x = 3$

(b) $f(x) = \sqrt[3]{x},\ a = 8,\ x = 9$

(c) $f(x) = \frac{1}{x},\ a = 5,\ x = 5.3$

(d) $f(x) = \frac{1}{x^2},\ a = 3,\ x = 2.8$

(e) $f(x) = x^2 + 3,\ a = 2,\ x = 2.2$

(f) $f(x) = (x-2)^3,\ a = 3,\ x = 3.1$

Exercise 5.3.2 *Find the linearization $L(x)$ of $f(x) = \ln(1+x)$ at $a = 0$. Use this linearization to approximate $f(0.1)$.*

Exercise 5.3.3 *Use linear approximation to estimate $(1.9)^3$.*

Exercise 5.3.4 *Show in detail that the linear approximation of $\sin x$ at $x = 0$ is $L(x) = x$ and the linear approximation of $\cos x$ at $x = 0$ is $L(x) = 1$.*

Exercise 5.3.5 *Use $f(x) = \sqrt[3]{x+1}$ to approximate $\sqrt[3]{9}$ by choosing an appropriate point $x = a$. Are we over- or under-estimating the value of $\sqrt[3]{9}$? Explain.*

5.3.2. Differentials

Very much related to linear approximations are the *differentials* dx and dy, used not to approximate values of f, but instead the change (or rise) in the values of f.

Definition 5.17: Differentials dx and dy

Let $y = f(x)$ be a differentiable function. We define a new independent variable dx, and a new dependent variable $dy = f'(x)\,dx$. Notice that dy is a function both of x (since $f'(x)$ is a function of x) and of dx. We call both dx and dy **differentials**.

Now fix a point a and let $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. If x is near a then Δx is clearly small. If we set $dx = \Delta x$ then we obtain

$$dy = f'(a)\,dx \approx \frac{\Delta y}{\Delta x}\Delta x = \Delta y.$$

Thus, dy can be used to approximate Δy, the actual change in the function f between a and x. This is exactly the approximation given by the tangent line:

$$dy = f'(a)(x-a) = f'(a)(x-a) + f(a) - f(a) = L(x) - f(a).$$

While $L(x)$ approximates $f(x)$, dy approximates how $f(x)$ has changed from $f(a)$. Figure 5.5 illustrates the relationships.

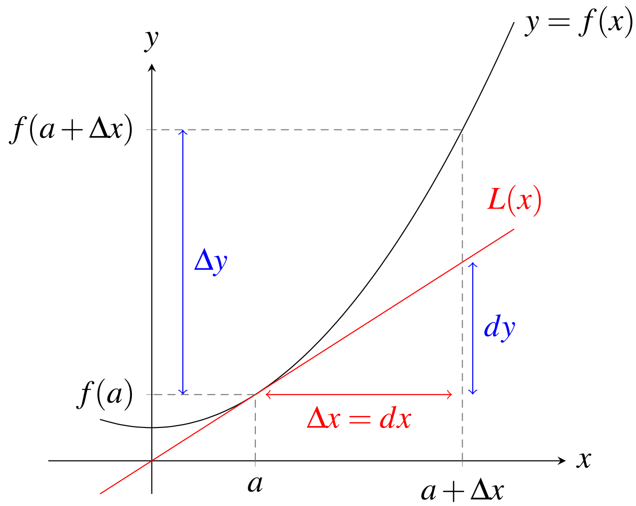


Figure 5.5: Differentials.

Note:

1. x, dx and Δx are independent variables.
2. Both dx and Δx measure the change as x changes from x to $x+\Delta x$.
3. y is a dependent variable of x, and Δy is a dependent variable of both x and Δx.
4. Δy measures the *actual* change in y as x changes from x to $x+\Delta x$.
5. dy measures the *approximate* change in y as x changes from x to $x+\Delta x$.

Since differentials are used to estimate the change in the dependent variable corresponding to a small change in the independent variable, they are a useful concept to analyze change in cost, revenue, and profit functions, which is summarized below.

Definition 5.18: Differentials in Marginal Analysis

1. *Suppose we are given the cost function $p = C(q)$. If C is differentiable, then*

$$dC = C'(q)dq \text{ and } \Delta C \approx C'(q)\Delta q.$$

2. *Suppose we are given the revenue function $p = R(q)$. If R is differentiable, then*

$$dR = R'(q)dq \text{ and } \Delta R \approx R'(q)\Delta q.$$

3. *Suppose we are given the profit function $p = P(q)$. If P is differentiable, then*

$$dP = P'(q)dq \text{ and } \Delta P \approx P'(q)\Delta q.$$

Example 5.19: Actual and Approximate Changes in y

Let $y = x^4$.

(a) Calculate Δx and Δy when x changes from 2 to 2.1, and from 2 to 1.9.

(b) Calculate the differential dy of y. Use dy to approximate Δy when x changes from 2 to 2.1, and from 2 to 1.9.

(c) Compare the results of part (b) with those of part (a).

Solution. Let $f(x) = x^3$.

(a) When x changes from 2 to 2.1, $\Delta x = 2.1 - 2 = 0.1$. Next,

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x) = f(2.1) - f(2) \\ &= (2.1)^3 - 2^3 = 9.261 - 8 = 1.261.\end{aligned}$$

Similarly, when x changes from 2 to 1.9, we have $\Delta x = 1.9 - 2 = -0.1$, and

$$\begin{aligned}\Delta y &= f(x+\Delta x) - f(x) = f(1.9) - f(2) \\ &= (1.9)^3 - 2^3 = 6.859 - 8 = -1.141.\end{aligned}$$

(b) $dy = f'(x)dx = 3x^2dx$. First, take $x = 2$ and $dx = 2.1 - 2 = 0.1$. Then,

$$dy = 3x^2dx = 3(2)^2(0.1) = 1.2.$$

Next, we have $x = 2$ and $dx = 1.9 - 2 = -0.1$. Therefore,

$$dy = 3x^2dx = 3(2)^2(-0.1) = -1.2.$$

(c) Comparing the actual changes from part (a) to the approximate changes from part (b), we notice that the approximation $dy = 1.2$ is quite close to the actual change $\Delta y = 1.20601$, but the approximation $dy = -1.41$ is not very close to the actual change $\Delta y = -1.2$.

♣

Example 5.20: Rise of Natural Logarithm

Approximate the rise of $f(x) = \ln x$ from $x = 1$ to $x = 1.1$, using differentials.

Solution. Note that $\ln(1.1)$ is not readily calculated (without a calculator) hence why we wish to use linear approximation to approximate $f(1.1) - f(1)$.

We fix $a = 1$ since we know that $f(1) = \ln(1) = 0$, and so

$$\Delta x = dx = x - 1 \quad \text{and} \quad \Delta y = f(x) - f(1) = \ln(x) - \ln(1) = \ln(x).$$

Since

$$f'(x) = \frac{1}{x},$$

we also have that

$$dy = f'(1)dx = \left(\frac{1}{1}\right)(x-1) = x-1.$$

Then the actual change in f as x changes from 1 to 1.1 is approximated as follows:

$$\begin{aligned} \Delta y &\approx dy \\ f(1.1) - f(1) &\approx f'(1)(1.1-1) \\ \ln(1.1) &\approx 0.1 \end{aligned}$$

The correct value of $\ln(1.1)$ is $0.0953\ldots$ and thus we were relatively close. ♣

Example 5.21: Approximating Function Value

Use differentials to approximate the value of $\sqrt{24.5}$. Compare your result using a calculator.

Solution. Consider the function $y = f(x) = \sqrt{x}$. We want to pick a number x close to 24.5 for which we know the value of $\sqrt{x}$. Appropriately, we take $x = 25$. Then the change in y, Δy, as x changes from $x = 25$ to $x = 24.5$ is

$$\begin{aligned} \Delta y \approx dy &= f'(x)\Delta x \\ &= \left(\frac{1}{2\sqrt{x}}\right)\bigg|_{x=25} \cdot (-1.5) \\ &= \left(\frac{1}{10}\right)(-1.5) = -0.15 \end{aligned}$$

Therefore,

$$\begin{aligned} \sqrt{24.5} - \sqrt{25} &= \Delta y \approx -0.15 \\ \sqrt{24.5} &\approx \sqrt{25} - 0.15 = 4.75. \end{aligned}$$

A calculator tells us that $\sqrt{24.5} \simeq 4.94975$, and so the error in our approximation is about 0.2. ♣

Example 5.22: Approximating Operating Cost

Suppose that the total operating cost of relocating a car 500 km at an average speed of v km/h, is

$$C(v) = 150 + v + \frac{6000}{v}$$

dollars. Find the approximate change in cost when the average speed is increased from 80 km/h to 85 km/h.

Solution. We use $v = 80$ to approximate $C(85)$.

$$\begin{aligned}\Delta C \approx dC &= C'(v)dv \\ &= \left(1 - \frac{6000}{v^2}\right)\bigg|_{v=80} \cdot (5) \\ &= \left(1 - \frac{6000}{6400}\right)(5) = 0.0625,\end{aligned}$$

that is, the total cost would increase by approximately \$0.06. ♣

Example 5.23: Approximating Sales

A company determines that the relationship between the amount of money q spent on advertising and total sales $S(q)$ is

$$S(q) = -0.02q^3 + q^2 + 2q + 100 \quad 0 \le q \le 60$$

where q is measured in thousands of dollars. Estimate the change in the company's total sales if the amount spent on advertising is increased from \$50,000 ($q = 50$) to \$55,000 ($q = 55$).

Solution. We approximate

$$\begin{aligned}\Delta S \approx dS &= S'(55)dq \\ &= \left(-0.06q^2 + 2q + 2\right)\big|_{x=50} \cdot (55 - 50) \\ &= (-30 + 100 + 2)(5) = 360.\end{aligned}$$

That is, total sales will increase by approximately \$360,000. ♣

Example 5.24: Approximating Drop in Price

Suppose the demand for a certain product is given by

$$p = f(q) = \frac{100}{q^2 + 2}$$

where p is expressed in dollars/unit and q is the quantity demanded each year. The manufacturer predicts they will be able to produce 6 billion units for the year. If the actual production is 6.2 billion units instead, what would happen to the predicted price, p?

Solution. The differential is given by

$$dp = -\frac{100q}{(q^2 + 2)^2} dq.$$

So when $q = 6$ and $dq = 0.2$,

$$\Delta p \approx dp = -\frac{100(6)}{(36 + 2)^2}(0.2) = -0.0831,$$

that is, the price will drop by approximately \$0.08. ♣

Exercises for Section 5.3.2

Exercise 5.3.6 *Find the differential of the given function.*

(a) $f(x) = 2x^2$

(b) $g(t) = t^3 - t$

(c) $f(t) = \sqrt{t+1}$

(d) $p(q) = 2q^{3/2} + q^{1/2}$

(e) $h(s) = s + \dfrac{2}{s}$

(f) $p(q) = \dfrac{q-1}{q^2+1}$

(g) $f(x) = \sqrt{3x^2 - x}$

Exercise 5.3.7 *For the following functions $f(x)$, determine Δy and dy at the given values of a and Δx.*

(a) $f(x) = x^4$, $a = 1$, $dx = \Delta x = 1/2$

(b) $f(x) = \sqrt{x}$. *If* $a = 1$ *and* $\Delta x = 1/10$

(c) $f(x) = \sin(2x)$. *If* $a = \pi$ *and* $\Delta x = \pi/100$

Exercise 5.3.8 *For the functions (i) $f(x) = x^2 - 1$ with x changing from 1 to 0.9 and (ii) $f(x) = \frac{1}{x}$ with x changing from -1 to -1.01, do the following:*

(a) *Calculate the differential of f.*

(b) *Use your results from part (a) to find the approximate change in y for the given change in x.*

(c) *Calculate the actual change in y for the given change in x and compare your results with that obtained in part (b).*

Exercise 5.3.9 *Use differentials to approximate the given quantity.*

(a) $\sqrt{48.5}$

(b) $\sqrt[3]{8.2}$

(c) $\sqrt{4.05} + \dfrac{1}{\sqrt{4.05}}$. *Hint: Let $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ and compute dy with $x = 4$ and $dx = 0.05$.*

Exercise 5.3.10 *Approximate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 40 meters. You may use the fact that the volume of a sphere of radius r is $V = (4/3)\pi r^3$, where in this example, $dr = 0.02$.*

Exercise 5.3.11 *It is determined that a certain country's gross domestic product (GDP) can be approximated by*

$$f(x) = 350x^{1/4}$$

where $f(x)$ is measured in millions of dollars and x is the capital expenditure in billions of dollars. Approximate the change in GDP if the country's capital expenditure changes from \$200 million to \$210 million.

Exercise 5.3.12 *A major supermarket determines that their yearly profit $P(q)$ is related to the amount q spent on advertising by*

$$P(q) = -\frac{1}{6}q^2 + 12q + 15 \quad 0 \le q \le 73$$

where both $P(q)$ and q are measured in thousands of dollars. Approximate the change in profits when advertising expenditure is increased from \$30,000 to \$32,000.

Exercise 5.3.13 *A bank determines that the number $N(t)$ of loans issued over the course of one year is related to the interest rate r by*

$$N(t) = \frac{8}{1+0.02r^2}$$

where N is measured in millions. Approximate the change in the number of loans the bank issues when the interest rate is increased from 10% to 10.5%.

Exercise 5.3.14 *The supply equation for a certain product is given by*

$$p = s(q) = 0.5\sqrt{q} + 8$$

where q is the quantity supplied and p is the unit price in dollars. Approximate the change in price when the quantity supplied is increased from 10,000 to 10,200 units.

5.3.3. Error Approximation

When working with differentials, we approximate function values, and therefore an error is introduced compared to the actual function values. Suppose we are given a function $y = f(x)$ with a measured quantity as input. If a is the exact value of the measured quantity, but $a + dx$ is the measured value, then $dx = \Delta x$ represents the so-called **measurement error**. Furthermore, this measurement error causes an error in the calculation of $f(x)$, which is known as **propagation error** $\Delta y = f(a+dx) - f(a)$. Both types of errors are known as **absolute errors**.

If $y = f(a)$ is calculated with the absolute error Δy, then the **relative error** in the calculation of y is given by the quantity $\frac{\Delta y}{y}$, while the **percentage error** is given by the quantity $\frac{\Delta y}{y} \times 100\%$. Since Δy is approximated by dy, the relative error is approximated by $\frac{dy}{y}$ and the percentage error by $\frac{dy}{y} \times 100\%$.

	True Value	Approximate Value
Absolute Error	Δy	dy
Relative Error	$\dfrac{\Delta y}{y}$	$\dfrac{dy}{y}$
Percentage Error	$\dfrac{\Delta y}{y} \times 100\%$	$\dfrac{dy}{y} \times 100\%$

Table 5.1: Types of Error when Working with Differentials.

Example 5.25: Approximating Errors in Measurement

We are given that the radius of a spherical object is measured to be 0.4 *m to within an error of* ± 0.001 *m. What are the relative and percentage errors?*

Solution. The relative error in r is

$$\frac{dr}{r} = \pm \frac{0.001}{0.4} = \pm 0.0004.$$

The percentage error is then

$$\frac{dr}{r} \times 100\% = 0.0004 \times 100\% = \pm 0.04\%.$$

♣

Example 5.26: Approximating Errors in Measurement

The sides of a cubical object are measured with an absolute percentage error of 3%. Approximate the maximum percentage error in the calculated volume of the cube using differentials.

Solution. Let x be the side-length of the cube. Then its volume is

$$V = x^3,$$

and

$$dV = 3x^2 dx.$$

Therefore,

$$\frac{dV}{V} = \frac{3x^2 dx}{x^3} = 3\frac{dx}{x}.$$

Hence,

$$\left|\frac{dV}{V}\right| = 3\left|\frac{dx}{x}\right| \leq 3(0.03) = 0.09,$$

where we used the fact that $\left|\frac{dx}{x}\right| \leq 0.03$.

Thus, the maximum percentage error in the measurement of the volume of the cube is 9%. ♣

Exercises for Section 5.3.3

Exercise 5.3.15 *The edges of a cube are measured to be 12 cm in length, with a maximum possible error of 0.02 cm. What is the maximum possible error that could occur when calculating the volume of the cube?*

Exercise 5.3.16 *A wooden box with a lid is lacquered to an even thickness of 0.04 cm. If the edges of the box measure 0.5 m, then calculate the approximate amount of lacquer required.*

Exercise 5.3.17 *A dome of radius 20 m is to be coated with a layer of paint. What is the approximate amount of paint needed if the coat is to be 0.05 cm thick? Note that the volume of a dome of radius r is* $V = \frac{2}{3}\pi r^3$.

Exercise 5.3.18 *True or false: If* $A = f(x)$*, then the percentage change in A is*

$$\frac{100f'(x)}{f(x)}dx.$$

Explain your answer.

Exercise 5.3.19 *The quarterly profit of a certain manufacturer is given by*

$$P(q) = -0.000032q^3 + 6q - 300$$

million dollars, where q is measured in tens of thousands of units. The expected number of units sold over the next quarter is 320,000, with a maximum error of 18%. Determine the maximum error in the expected profit.

Exercise 5.3.20 *The demand equation for a certain product is given by*

$$p = f(q) = \frac{25}{3q^2+2}$$

where p is the unit price in dollars and q is the quantity demanded each year, measured in thousands of units. It is expected that the demand will be 2000 units for the year, with a maximum error of 10%. What is the maximum error in the predicted price?

Exercise 5.3.21 *Suppose a monthly mortgage payment P, in dollars, is computed using the formula*

$$P = \frac{10,000r}{1-\left(1+\frac{r}{12}\right)^{-360}}$$

where r is the interest rate per year.

(a) *Find the differential of P.*

(b) *Approximately how much more will the monthly mortgage payments be if the interest rate increases from the present rate of 3% per year to 3.2 % per year? From 3% to 3.3% per year? To 3.4% per year? To 3.5% per year?*

Exercise 5.3.22 *Suppose $10,000 is deposited into an account that pays interest at the rate r/year compounded monthly. Then the account balance at the end of 10 years is given by*

$$A = 10{,}000\left(1+\frac{r}{12}\right)^{120}.$$

(a) *Find the differential of A.*

(b) *Approximately how much more would the account be worth at the end of 10 years with an interest rate of 1.1% per year instead of 1%? 1.2% per year instead of 1%? 1.3% per year instead of 1%?*

Exercise 5.3.23 *Suppose $2000 per month is deposited into an account that pays interest at the rate r/year compounded monthly. Then the account balance at the end of 25 years is given by*

$$S = \frac{24{,}000\left(\left(1+\frac{r}{12}\right)^{300}-1\right)}{r}$$

dollars.

(a) *Find the differential of S.*

(b) *Approximately how much more would the account be worth at the end of 25 years with an interest rate of 1.6%/year instead of 1.5%? 1.7%/year instead of 1.5%? 1.8%/year instead of 1.5%?*

5.3.4. Newton's Method

A well-known numerical method is *Newton's Method* (also sometimes referred to as *Newton-Raphson's Method*), named after Isaac Newton and Joseph Raphson. This method is used to find roots, or x-intercepts, of a function. While we may be able to find the roots of a polynomial which we can easily factor, we saw in the previous chapter on **Limits**, that for example the function $e^x + x = 0$ has a solution (*i.e.* root, or x-intercept) at $x \approx -0.56714$. By the Intermediate Value Theorem we know that the function $e^x + x = 0$ does have a solution. We cannot here simply solve for such a root algebraically, but we can use a numerical method such as $Newton's$. Such a process is typically classified as an $iterative$ method, a name given to a technique which involves repeating similar steps until the desired accuracy is obtained. Many computer algorithms are coded with a for-loop, repeating an iterative step to converge to a solution.

The idea is to start with an initial value x_0 (approximating the root), and use linear approximation to create values $x_1, x_2, \cdots$ getting closer and closer to a root.

The first value x_1 corresponds to the intercept of the tangent line of $f(x_0)$ with the x-axis, which is:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

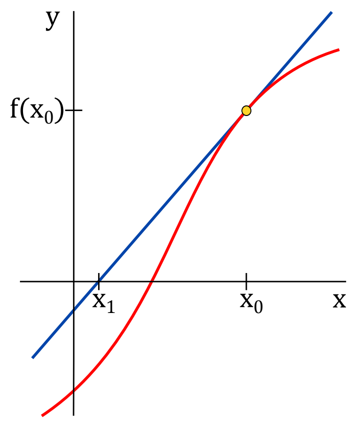


Figure 5.6: First iteration of Newton's Method.

We can see in Figure 5.6, that if we compare the point $(x_0, 0)$ to $(x_1, 0)$, we would likely come to the conclusion that $(x_1, 0)$ is closer to the actual root of $f(x)$ than our original guess, $(x_0, 0)$. As will be discussed, the choice of x_0 must be done correctly, and it may occur that x_1 does not yield a better estimate of the root.

Newton's method is simply to repeat this process again and again in an effort to obtain a more accurate solution. Thus at the next step we obtain:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

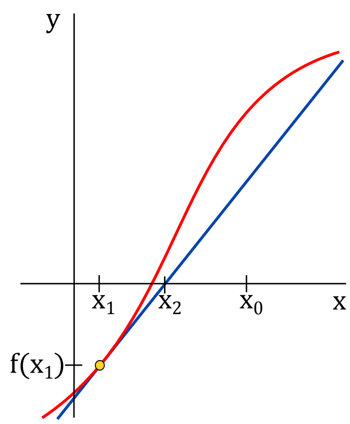


Figure 5.7: Second iteration of Newton's Method.

We can now clearly see how $(x_2, 0)$ is a better estimate of the root of $f(x)$, rather than any of the previous points. Moving forward, we will get:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Rest assured, $(x_3, 0)$ will be an even better estimate of the root! We express the general iterative step as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The idea is to iterate these steps to obtain the desired accuracy.

Newton's Method

1. Choose an initial estimate x_0 of the root r.
2. Calculate the next estimate using the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, \ldots)$$

3. Calculate $|x_n - x_{n+1}|$, which determines the number of accurate digits that have been achieved in the estimation of the root r.
4. Either repeat as of step 2 or terminate the algorithm with $r \approx x_{n+1}$ unless Newton's Method failed (see *Key Points* at the end of this section).

Here is an example.

Example 5.27: Newton's Method to Approximate a Root

Approximate the roots of $f(x) = x^3 - x + 1$ by Newton's Method, accurate to six decimal places.

Solution. Since the function is a cubic, solving the equation algebraically is difficult. We therefore use Newton's Method to compute an approximate root.

Our function f has only one real root as a sketch confirms (see Section 5.7 on how to perform *curve sketching*. We note that $f(-1) = -5$ and $f(0) = 1$. We apply the Intermediate Value Theorem to determine that f has a root between these two values. We choose to start with the initial value $x_0 = -1$. We encourage you to try Newton's Method with a different initial value such as -0.5 or -0.7 or any other value between -1 and 0.

We calculate the derivative to be $f'(x) = 3x^2 - 1$. Therefore, Newton's formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x + n - \frac{x_n^3 - x_n + 1}{3x_n^2 - 1}.$$

We compute the following approximations of the root. We also encourage you to verify that these are indeed the correct iterations through Newton's algorithm.

$$\begin{aligned}
x_0 &= -1 \\
x_1 &= -1.5000 \\
x_2 &= -1.347826... \\
x_3 &= -1.325200... \\
x_4 &= -1.324718... \\
x_5 &= -1.324717... \\
x_6 &= -1.324717... \\
&\cdots
\end{aligned}$$

Hence, the root we are seeking is approximately -1.324717. ♣

Applications

Example 5.28: Market Equilibrium

The monthly demand q (in units of a thousand) for a certain product is related to the unit price p (in dollars) by the demand equation

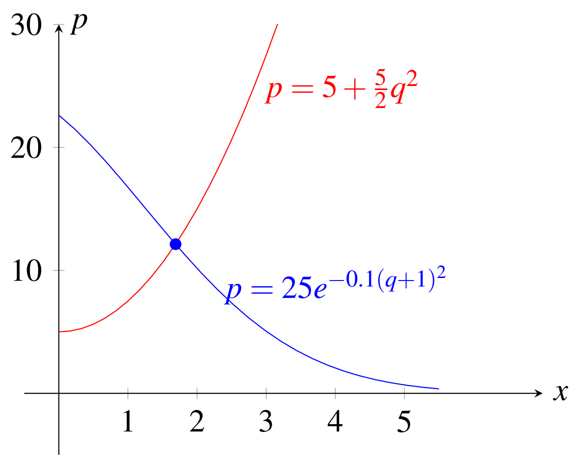
$$p = 25e^{-0.1(q+1)^2}.$$

The monthly supply for the same product is given by the supply equation

$$p = 5 + \frac{5}{2}q^2.$$

Estimate the equilibrium point of this system.

Solution. We determine the equilibrium point by finding the point of intersection of the demand curve and the supply curve.



To find the equilibrium point, we equate both equations, giving

$$5 + \frac{5}{2}q^2 = 25e^{-0.1(q+1)^2}$$

$$5 + \frac{5}{2}q^2 - 25e^{-0.1(q+1)^2} = 0$$

Instead of attempting to solve this equation exactly using algebraic techniques, we instead employ Newton's Method to find an approximate solution. First, write

$$10 + 5q^2 - 50e^{-0.1(q+1)^2} = 0,$$

since it is easier to work without the fraction. Then,

$$\begin{aligned} f(q) &= 10+5q^2-50e^{-0.1(q+1)^2} \\ &= 5\left(2+q^2-10e^{-0.1(q+1)^2}\right), \quad \text{and} \\ f'(q) &= 10q-50e^{-0.1(q+1)^2}\cdot(-0.2(q+1)) \\ &= 10\left(q+(q+1)e^{-0.1(q+1)^2}\right). \end{aligned}$$

And so we construct the required iterative formula as

$$q_{n+1} = q_n - \frac{2+q_n^2-10e^{-0.1(q_n+1)^2}}{2\left(q_n+(q_n+1)e^{-0.1(q_n+1)^2}\right)}$$

From the above sketch, we see that a reasonable estimate of the intersection point is $q = 2$. We now carry out Newton's Method using an initial guess of $q_0 = 2$.

$$q_1 = 2 - \frac{2+2^2-10e^{-0.1(3)^2}}{2\left(2+(3)e^{-0.1(3)^2}\right)} \approx 1.69962$$

$$q_2 = 1.69962 - \frac{2+(1.69962)^2-10e^{-0.1(2.69962)^2}}{2\left(2+(2.69962)e^{-0.1(2.69962)^2}\right)} \approx 1.68899$$

$$q_3 = 1.68899 - \frac{2+(1.68899)^2-10e^{-0.1(2.68899)^2}}{2\left(2+(2.68899)e^{-0.1(2.68899)^2}\right)} \approx 1.68898$$

Therefore, the equilibrium quantity is approximately 1.689 units, and the equilibrium price is correspondingly

$$p = 5+\frac{5}{2}(1.689)^2 \approx 12.1316$$

or approximately \$12.1316 per unit. The equilibrium point is thus $(1.689, 12.1316)$. ♣

Another application of Newton's Method is to the **internal rate of return** on an investment. Suppose an investment yields returns of $R_1, R_2, \ldots, R_n$ dollars at the end of the first, second,$\ldots$,n-th periods, respectively with an initial payment of C dollars. Then this investment has a net present value of

$$\frac{R_1}{1+r}+\frac{R_2}{(1+r)^2}+\frac{R_3}{(1+r)^3}+\cdots+\frac{R_n}{(1+r)^n}-C=0.$$

By multiplying both sides of the above equation with $(1+r)^n$, we obtain

$$C(1+r)^n-R_1(1+r)^{n-1}-R_2(1+r)^{n-2}-R_3(1+r)^{n-3}-\cdots-R_n=0.$$

Typically, a company's executives use the internal rate of return to determine whether an investment is profitable or not.

Example 5.29: Internal Rate of Return

A company is deciding on whether or not to purchase new equipment. The upfront cost of the equipment is \$50,000, but the company predicts that they will save \$15,000-$1000(m-1)$ *per year after* m *years for up to* 4 *years, after which the equipment will be useless. Approximate the internal rate of return on this investment.*

Solution.

This investment would yield returns of $R_1 = 15,000-1000(1-1) = 15,000$ after the first year, of $R_2 = 15,000-1000(2-1) = 14,000$ after the second year, $R_3 = 15,000-1000(3-1) = 13,000$ after the third year, and $R_4 = 15,000-1000(4-1) = 12,000$ after the fourth year. We also have that the initial investment is $C = 50,000$. Therefore, we wish to solve

$$50,000(1+r)^4-15,000(1+r)^3-14,000(1+r)^2-13,000(1+r)-12,000=0$$

for r. Let $x = 1+r$ for simplicity. Then,

$$f(x) = 50,000x^4-15,000x^3-14,000x^2-13,000x-12,000,$$

where we are looking to solve $f(x) = 0$. We can approximate the root of f using Newton's Method. Since

$$f'(x) = 200,000x^3-45,000x^2-28,000x-13,000$$

the required iterative formula is

$$x_{n+1} = x_n - \frac{50,000x_n^4-15,000x_n^3-14,000x_n^2-13,00x_n-12,000}{200,000x_n^3-45,000x_n^2-28,000x_n-13,000}.$$

Choose $x_0 = 1.0$. Then our iterates are

$$\begin{aligned} x_1 &= 1.0 - \frac{50,000(1.0)^4-15,000(1.0)^3-14,000(1.0)^2-13,000(1.0)-12,000}{200,000(1.0)^3-45,000(1.0)^2-28,00(1.0)-13,000} \\ &\approx 1.03509 \\ x_2 &\approx 1.03277 \\ x_3 &\approx 1.03276 \end{aligned}$$

So if $x \approx 1.033$, then $r \approx 1-x = 0.033$. Therefore, we find the rate of return on the investment to be about 3.3%. ♣

As with any numerical method, we need to be aware of the weaknesses of any technique we are using.

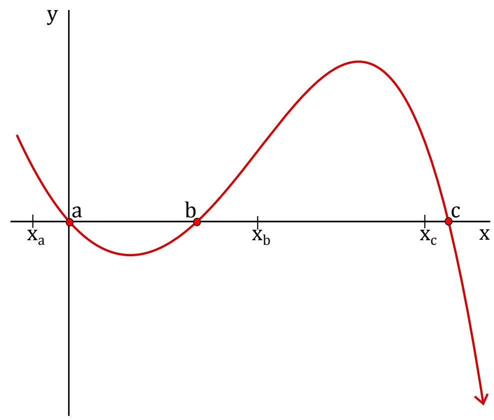


Figure 5.8: Function with three distinct solutions.

If we know our root is somewhere near a, we would make our guess $x_0 = a$. Generally speaking, a good practice is to make our guess as close to the actual root as possible. In some cases we may have no idea where the root is, so it would be prudent to perform the algorithm several times on several different initial guesses and analyze the results.

For example we can see in Figure 5.8 that $f(x)$ in fact has three roots, and depending on our initial guess, we may get the algorithm to converge to different roots. If we did not know where the roots were, we would try the technique several times. In one instance, if our initial guess was x_a, we'd likely converge to $(a,0)$. Then if we were to choose another guess, x_b, then we'd likely converge to $(b,0)$. Eventually, using various initial guesses we'd get one of three roots: a, b, or c. Under these circumstances we can clearly see the effectiveness of this numeric method.

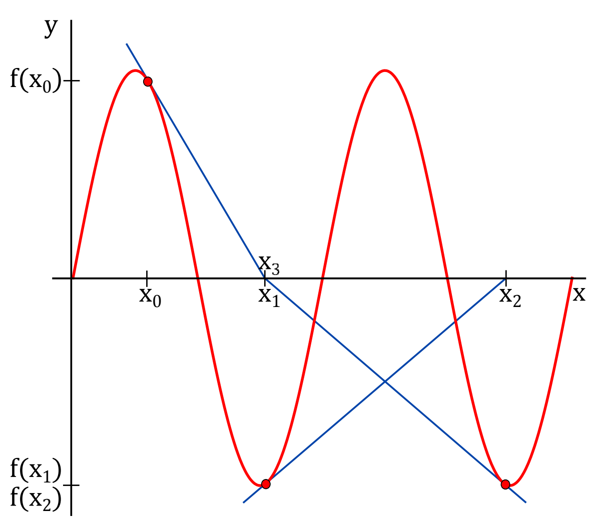


Figure 5.9: Newton's Method applied to $\sin x$ **with unstable point** x_0**.**

As another example if we attempt to use $Newton's\ Method$ on $f(x) = \sin x$ using $x_0 = \pi/2$, then $f'(x_0) = 0$ so x_1 is undefined and we cannot proceed. Even in general x_{n+1} is typically nowhere near x_n, and in general not converging to the root nearest to our initial guess of x_0. In effect, the algorithm keeps "bouncing around". An example of which is depicted in Figure 5.9. Based on our initial guess for such a function, the algorithm may or may not converge to a root, or it may or may not converge to the root **closest** to the initial guess. This gives rise to the more common issue: Selection of the initial guess, x_0.

Here is a summary.

Key Points in Using Newton's Method

1. We attempt to choose x_0 as close as possible to the root we wish to find.
2. A guess for x_0 which makes the algorithm 'bounce around' is considered **unstable**.
3. Even the smallest changes to x_0 can have drastic effects: We may converge to another root (see Figure 5.8), we may converge very slowly requiring many more iterations, or we may not converge at all due to an unstable point (see Figure 5.9).
4. We may encounter a **stationary point** if we choose x_0 such that $f'(x_0) = 0$ (*i.e.* x_0 is a critical point, see Definition 5.5.1!) in which case the algorithm fails (see Figure 5.10).

This is all to say that your initial guess for x_0 can be extremely important.

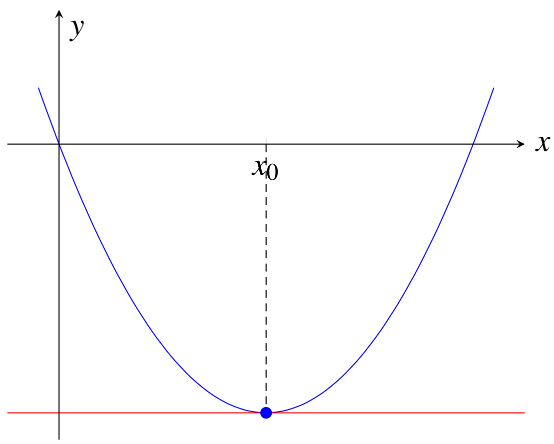


Figure 5.10: $f'(x_0) = 0$

Exercises for Section 5.3.4

Exercise 5.3.24 *Apply Newton's Method using the following steps: First, determine the function given the radical. Second, approximate the given radical with three iterations based on the given initial guess.*

(a) $\sqrt{3}$, $x_0 = 1.5$

(b) $\sqrt{7}$, $x_0 = 2.5$

(c) $\sqrt[3]{14}$, $x_0 = 2.5$

Exercise 5.3.25 *Apply Newton's Method to approximate the root of each function within the given interval, accurate to three decimal places.*

(a) $f(x) = e^{-x} - x$, $(0,1)$.

(b) $f(x) = e^{x} - \frac{1}{x}$, $(0,1)$.

(c) $f(x) = \ln(x^2)$, $(6,7)$.

Exercise 5.3.26 *Sketch the graphs of f and g on the same Cartesian coordinate system. Based on the sketch, choose an initial value to approximate the x-coordinate of the intersection point(s) of the two graphs. Then apply Newton's Method to determine the x-coordinate of the intersection point accurate to within two decimal places.*

(a) $f(x) = \sqrt{x}$, $g(x) = -0.5x + 2$

(b) $f(x) = e^{-x^2}$, $g(x) = x^2$

(c) $f(x) = \ln x$, $g(x) = 2 - x$

Exercise 5.3.27 *For each of the given functions $f(x)$, show that $f(x) = 0$ has a root between the given x-values. Use Newton's Method to find the zero(s). Hint: use the Intermediate Value Theorem.*

(a) $f(x) = 3x^2 - 9x - 11$ *between* $x = -1$ *and* $x = 0$; *and between* $x = 3$ *and* $x = 4$.

(b) $f(x) = x^3 - x - 1$ *between* $x = 1$ *and* $x = 2$.

(c) $f(x) = x^4 - 4x^3 + 10$ *between* $x = 1$ *and* $x = 2$.

Exercise 5.3.28 *Consider* $f(x) = x^3 - x^2 + x - 1$.

(a) *Using initial approximation* $x_0 = 2$, *find* x_4.

(b) *What is the exact value of the root of f? How does this compare to our approximation x_4 in part (a)?*

(c) *What would happen if we chose $x_0 = 0$ as our initial approximation?*

Exercise 5.3.29 *Consider* $f(x) = \sin x$. *What happens when we choose* $x_0 = \pi/2$? *Explain.*

Exercise 5.3.30 *Suppose a company purchases \$6000 worth of new equipment which will be used for the next 3 years. The investment is expected to yield returns of \$2,000 at the end of the first year, \$3500 at the end of the second year and \$1000 at the end of the third year. What is the internal rate of return on this investment?*

Exercise 5.3.31 *Executives of a certain company are contemplating the purchase of \$100,000 worth of equipment, which would be in use over the next 4 years. The investment is expected to yield returns of \$20,000 at the end of the first year, \$30,000 at the end of the second year, \$45,000 at the end of the third year and \$15,000 at the end of the fourth year. What is the internal rate of return on this investment?*

Exercise 5.3.32 *A first time home buyer borrows $250,000 from a bank to finance the purchase of a house. Interest is computed at the end of each month at a rate of 12r per year on the unpaid balance. If the home buyer repays the loan in equal monthly instalments of $2226.10 over the next 10 years, what is the rate of interest charged by the bank?*

Exercise 5.3.33 *A down payment of 10% is made towards a purchase of $10,000. Financing for this purchase is available with monthly payments of $255.50 over 4 years. What is the rate of interest charged?*

Exercise 5.3.34 *Suppose the demand equation for a certain product is given by*

$$p = d(q) = \frac{100}{0.02q^2 + 2} \quad 1 \leq q \leq 20$$

where p is the unit price in dollars and q is the quantity demanded in units of hundreds. The corresponding supply equation is given by

$$p = s(q) = 0.1q + 30$$

dollars. What is the equilibrium point?

Exercise 5.3.35 *A certain company determines that the demand equation for a product is*

$$p = -3q + 900 \quad 0 \leq q \leq 300$$

where p is the unit price in dollars and q is the quantity demanded in units of a thousand. If the weekly total cost function associated with the production of this product is

$$C(q) = q^2 + 2q + 700,$$

what is the break-even level(s) of operation for the company?

5.4 Indeterminate Form & L'Hôpital's Rule

5.4.1. Indeterminate Forms

Before we embark on introducing one more limit rule, we need to recall a concept from algebra. In your work with functions (see Chapter 2) and limits (see Chapter 4) we sometimes encountered expressions that were **undefined**, because they either lead to a contradiction or to numbers that are not in the set of numbers we started out with. Let us look at an example for either scenario to investigate the concept "undefined" more deeply.

Example 5.30: Undefined because it Leads to Contradiction

Suppose that

$$f(x) = \frac{1}{x}.$$

What happens when $x = 0$*? Then* $f(0) = 1/0$*, but* $1/0$ *is undefined. Why is that? Let's assume this value is defined. This means that* $1/0$ *is equal to some number, call it* n*. Then*

$$\begin{aligned} \frac{1}{0} &= n \\ 1 \div 0 &= n \\ 1 &= n \times 0 \\ 1 &= 0 \end{aligned}$$

Clearly, 1 is not equal to 0, and so this statement is a contradiction. In fact, if we analyze the satament

$$1 = n \times 0,$$

we notice that there is no number for n *that will satisfy this equation. Therefore,* $1/0$ *could not have been a number, and hence we say* $1/0$ *is undefined. This is the reason why we write that the domain of* f *is given by*

$$\mathscr{D}_f = \left\{ x \in \mathbb{R} \middle| x \neq 0 \right\}.$$

Example 5.31: Different Number Set

Suppose that $f(x) = \sqrt{x-1}$ *and that we are working over the real numbers. What happens when* $x = 0$*? Then*

$$f(0) = \sqrt{-1},$$

but $\sqrt{-1}$ *is undefined over the real numbers. Why is that? Let's assume this value is defined. Then by the definition of square root, there is a real number* n *such that*

$$-1 = n^2.$$

Clearly, the square of a real number cannot produce a negative real number because positive $\times$ *positive and negative* $\times$ *negative are both positive real numbers. In fact,* $\sqrt{-1}$ *is the imaginary number* i*, which belongs to the set of complex numbers.*

When we work out limit problems algebraically, we will often get as an initial answer something that is undefined. This is because the places where a function is undefined are the "interesting" places to look for limits. For example, if

$$g(x) = \frac{x^2 - 9}{x - 3},$$

then

$$g(3) = \frac{3^2 - 9}{3 - 3} = \frac{0}{0},$$

but

$$\begin{aligned}\lim_{x\to 3} g(x) &= \lim_{x\to 3}\frac{x^2-9}{x-3}\\ &= \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}\\ &= \lim_{x\to 3}(x+3) = 6.\end{aligned}$$

The function g is a line with a hole at $x = 3$ and the limit showed us that this hole can be removed with the y-value 6 at $x = 3$ (see Fig 5.11).

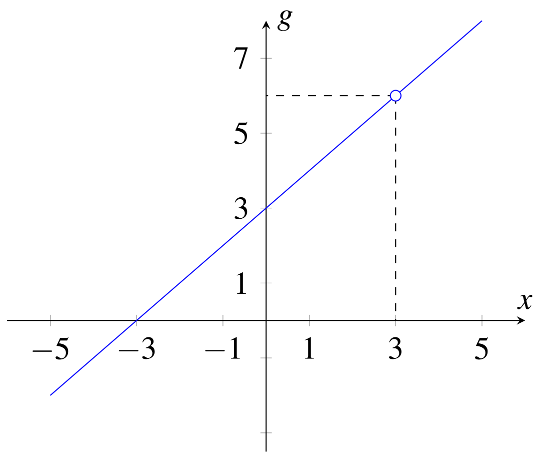


Figure 5.11: The function $g(x) = \frac{x^2-9}{x-3}$ **is undefined at** $x = 3$**.**

However, we must remember that when we are calculating the limit of $f(x)$ as $x \to a$ we are not interested in the behavior of $f(x)$ *at* a, but we want to know the behavior of $f(x)$ *around* a. It is therefore important for us to identify an undefined value a of a function, and furthermore, to investigate whether the type of undefined value can tell us something about the behavior of the function around a.

Before we continue, we need to draw attention to a notation that we have been using when calculating limits. When we write $f(x) \to 0$ as $x \to a$, we actually mean that $f(x)$ gets arbitrarily close to zero as x gets closer and closer to a. However, the function value never reaches zero. Similarly, when we write $f(x) \to \infty$ as $x \to a$, we actually mean that $f(x)$ grows ever larger, without bound as x gets closer and closer to a. However, the function value never reaches infinity, since infinity is not even a number.

Limit Behaviour

When calculating limits,

1. 0 represents a number arbitrarily close to zero;
2. $+\infty$ represents an arbitrarily large positive number; and
3. $-\infty$ represents an arbitrarily large negative number.

Therefore, $f(x) \to \frac{0}{0}$ as $x \to a$ means that $f(x)$ is a fraction for which both the numerator and the denominator get arbitrarily close to zero as x gets closer and closer to a, and $f(x) \to \frac{\infty}{\infty}$ as $x \to a$ means that $f(x)$ is a fraction for which both the numerator and the denominator grow ever larger, without bound as x gets closer and closer to a. We also know from experience that some limits that demonstrate $\frac{0}{0}$ or $\frac{\infty}{\infty}$ behaviour work out to be real numbers, i.e. the limit exists, while others do not, as the following four examples remind us:

Example 5.32: Limit exists when 0/0

$$\lim_{x\to 3}\frac{x^2-9}{x-3} \overset{\frac{0}{0}}{=} \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6$$

Example 5.33: Limit does not exist when 0/0

$$\lim_{x\to 0^+}\frac{\sqrt{x+1}-1}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0^+}\frac{\sqrt{x+1}-1}{x^2}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0^+}\frac{1}{x\left(\sqrt{x+1}+1\right)} \overset{\frac{1}{0^+}}{=} \infty$$

Example 5.34: Limit exists when ∞/∞

$$\lim_{x\to\infty}\frac{1-x}{2x} \overset{\frac{-\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}-1}{2} = -\frac{1}{2}$$

Example 5.35: Limit does not exist when ∞/∞

$$\lim_{x\to\infty}\frac{1-x^2}{2x} \overset{\frac{-\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}-x}{2} = -\infty$$

Upon closer inspection of the undefined expressions 0/0 and ∞/∞, we should realize that both terms are based on the division operation and ask ourselves whether there are other undefined expressions that we may encounter when taking limits. We therefore investigate arithmetic ($a+b$,$a-b$,ab,a/b) and exponentiating (a^b) operations where a and b are values that approach 0, 1, some arbitrary number $n \neq 0, 1$ or ∞. We leave it up to the reader to perform an exhaustive listing of all combinations, and instead limit ourselves to the combinations that are of interest as shown in Table 5.2.

$0+0$	$\infty+\infty$	$0\cdot\infty$	$n\cdot\infty$	0^0	0^∞
$0-0$	$\infty-\infty$	$\frac{0}{\infty}$	$\frac{n}{\infty}$	1^0	1^∞
$0\cdot 0$	$\pm\infty\cdot\pm\infty$	$\frac{\infty}{0}$	$\frac{\infty}{n}$	n^0	n^∞
$\frac{0}{0}$	$\frac{\pm\infty}{\pm\infty}$			∞^0	∞^∞

Table 5.2: Arithmetic and Exponentiating Combinations.

We now encourage the reader to investigate each one of the terms shown in Table 5.2 and decide whether the undefined expression resolves to give a single number value or infinity (**determinate form**), or whether this cannot be determined (**indeterminate form**), all the while keeping in mind our earlier discussion on limit behaviour around $x = a$. We formally define this new terminology before we explore some terms together.

Definition 5.36: Determinate and Indeterminate Forms

An undefined expression involving some operation between two quantities is called a **determinate form** *if it evaluates to a single number value or infinity.*
An undefined expression involving some operation between two quantities is called an **indeterminate form** *if it does not evaluate to a single number value or infinity.*

We will inspect multiplication more closely. Consider 0×0. Clearly, a number that is getting arbitrarily close to zero that is multiplied by another number that is getting arbitrarily close to zero gets even closer to zero, i.e. $0 \times 0 \to 0$. Now consider $\infty \times \infty$. Here, multiplying two values that are growing large without bound simply means that their product grows large without bound, i.e. $\infty \times \infty \to \infty$. Similarly, $(-\infty) \times \infty$ means that the magnitude of the product grows large without bound and that $(-\infty) \times \infty \to -\infty$. What about $n \times \infty$, when $n \neq 0$? Here we need to differentiate between negative and positive values of n: If $n > 0$, then $n \times \infty \to \infty$, and if $n < 0$, then $n \times \infty \to -\infty$. So far, we have only encountered determinate forms involving multiplication. Lastly, consider $0 \times \infty$. Here, we have a number that is getting arbitrarily close to zero being multiplied with a value that is growing large without bounds. This is like two ends of a rope being tugged and we do not know which side is going to win. Therefore, $0 \times \infty$ is an expression that cannot be determined.

We leave the remaining terms up to the reader to investigate and simply present the determinate and indeterminate forms of the expressions from Table 5.2 in Table 5.3.

Determinate Forms	**Indeterminate Forms**
$0+0$	$\infty-\infty$
$0-0$	$\dfrac{0}{0}$
$0\cdot 0$	$\dfrac{\pm\infty}{\pm\infty}$
$\pm\infty\cdot\pm\infty$	$0\cdot\infty$
$\dfrac{0}{\infty},\dfrac{n}{\infty}$	0^0
$\dfrac{\infty}{0},\dfrac{\infty}{n}$	∞^0
$n\cdot\infty \quad {\scriptstyle n\neq 0}$	1^∞
0^∞	
$n^\infty \quad {\scriptstyle n\neq 1}$	
∞^∞	

Table 5.3: Determinate and Indeterminate Forms

5.4.2. L'Hôpital's Rule for Finding Limits

We are now in a position to introduce one more technique for trying to evaluate a limit.

Definition 5.37: Limits of the Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

A limit of a quotient $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is said to be an ***indeterminate form of the type*** $\frac{0}{0}$ *if both $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$. Likewise, it is said to be an* ***indeterminate form of the type*** $\frac{\infty}{\infty}$ *if both $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$ (Here, the two $\pm$ signs are independent of each other).*

Theorem 5.38: L'Hôpital's Rule

For a limit $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists or equals ∞ or $-\infty$.

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here.

Note:

1. There may be instances where we would need to apply L'Hôpital's Rule multiple times, but we must confirm that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ is still indeterminate before we attempt to apply L'Hôpital's Rule again.
2. L'Hôpital's Rule is also valid for one-sided limits and limits at infinity.

Notation when Applying L'Hôpital's Rule

We use the symbol $\overset{H}{=}$ to denote we are using l'Hôpital's Rule in that step.

Example 5.39: L'Hôpital's Rule and Indeterminate Form 0/0

Compute $\lim_{x\to\pi}\frac{x^2-\pi^2}{\sin x}$.

Solution. We use L'Hôpital's Rule: Since the numerator and denominator both approach zero,

$$\lim_{x\to\pi}\frac{x^2-\pi^2}{\sin x}\overset{H}{=}\lim_{x\to\pi}\frac{2x}{\cos x},$$

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches -1, so

$$\lim_{x\to\pi}\frac{x^2-\pi^2}{\sin x}=\frac{2\pi}{-1}=-2\pi.$$

♣

Example 5.40: L'Hôpital's Rule and Indeterminate Form ∞/∞

Compute $\lim_{x\to\infty}\frac{2x^2-3x+7}{x^2+47x+1}$.

Solution. As x goes to infinity, both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{2x^2-3x+7}{x^2+47x+1}\overset{H}{=}\lim_{x\to\infty}\frac{4x-3}{2x+47}.$$

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

$$\lim_{x\to\infty}\frac{4x-3}{2x+47}\overset{H}{=}\lim_{x\to\infty}\frac{4}{2}=2.$$

So the original limit is 2 as well. ♣

Example 5.41: L'Hôpital's Rule and Indeterminate Form 0/0

Compute $\lim_{x\to 0} \frac{\sec x - 1}{\sin x}$.

Solution. Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sec x - 1}{\sin x} \stackrel{H}{=} \lim_{x\to 0} \frac{\sec x \tan x}{\cos x} = \frac{1\cdot 0}{1} = 0.$$

♣

Example 5.42: L'Hôpital's Rule and Indeterminate Form ∞/∞

Compute $\lim_{x\to 0^+} \frac{\frac{1}{x^2}}{\ln x}$.

Solution. As x aproaches zero from the right, the numerator approaches $+\infty$ and the denominator approaches $-\infty$. We may therefore apply L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 0^+} \frac{\frac{1}{x^2}}{\ln x} &\stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{-2}{x^3}}{\frac{1}{x}} \\ &\stackrel{H}{=} \lim_{x\to 0^+} \frac{-2}{x^2} \\ &= -\infty\end{aligned}$$

♣

Note: In order to decide which of two functions f and g grows faster as the independent variable, say x, becomes larger, we can apply the limit as x goes to infinity to the ratio f/g of these two functions. If the function f in the numerator grows faster, then the limit approaches infinity. If the function g in the denominator grows faster, then the limit approaches zero. If the functions have similar growth rates, then then the limit approaches a constant. This type of limit is readily computed using L'Hôpital's Rule, and so L'Hôpital's Rule is a useful tool to know.

We now exemplify this idea of growth rate. Let us have a closer look at the two functions $f(x) = 5x^3$ and $g(x) = x^3$. Then the function $f(x) = 5x^3$ grows exactly five times as fast as the function $g(x) = x^3$. However, the ratio of the two functions

$$\frac{f(x)}{g(x)} = \frac{5x^3}{x^3} = 5, \quad x \neq 0,$$

is a constant, and so both functions have fundamentally the same growth rate.

5.4.3. Informally Extending L'Hôpital's Rule

L'Hôpital's Rule concerns limits of a quotient that are indeterminate forms. But not all functions are given in the form of a quotient. But all the same, nothing prevents us from re-writing a given function in the form of a quotient. Indeed, some functions whose given form involve either a product $f(x)g(x)$ or a power $f(x)^{g(x)}$ carry indeterminacies such as $0 \cdot (\pm\infty)$ or $1^{\pm\infty}$. Something small times something numerically large (positive or negative) could be anything. It depends on how small and how large each piece turns out to be. A number close to 1 raised to a numerically large (positive or negative) power could be anything. It depends on how close to 1 the base is, whether the base is larger than or smaller than 1, and how large the exponent is (and its sign). We can use suitable algebraic manipulations to relate them to indeterminate quotients. We will illustrate with three examples, a product, a power and a difference.

Example 5.43: L'Hôpital's Rule and Indeterminate Form $0 \times \infty$

Compute $\lim_{x\to 0^+} x\ln x$.

Solution. This doesn't appear to be suitable for L'Hôpital's Rule, but it also is not "obvious". As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like:

$$(\text{something very small}) \cdot (\text{something very large and negative}).$$

This could be anything: it depends on *how small* and *how large* each piece of the function turns out to be. As defined earlier, this is a type of $\pm$"$0 \cdot \infty$", which is indeterminate. So we can in fact apply L'Hôpital's Rule after re-writing it in the form $\frac{\infty}{\infty}$:

$$x\ln x = \frac{\ln x}{1/x} = \frac{\ln x}{x^{-1}}.$$

Now as x approaches zero, both the numerator and denominator approach infinity (one $-\infty$ and one $+\infty$, but only the size is important). Using L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \stackrel{H}{=} \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} \frac{1}{x}(-x^2) = \lim_{x\to 0^+} -x = 0.$$

One way to interpret this is that since $\lim_{x\to 0^+} x\ln x = 0$, the x approaches zero much faster than the $\ln x$ approaches $-\infty$. ♣

Finally, we illustrate how a limit of the type "1^∞" can be indeterminate.

Example 5.44: L'Hôpital's Rule and Indeterminate Form 1^∞

Compute $\lim_{x\to 1^+} x^{1/(x-1)}$.

Solution. Plugging in $x = 1$ (from the right) gives a limit of the type "1^∞". To deal with this type of limit we will use logarithms. Let

$$L = \lim_{x\to 1^+} x^{1/(x-1)}.$$

Now, take the natural log of both sides:

$$\ln L = \lim_{x\to 1^+} \ln\left(x^{1/(x-1)}\right).$$

Using log properties we have:

$$\ln L = \lim_{x\to 1^+} \frac{\ln x}{x-1}.$$

The right side limit is now of the type $0/0$, therefore, we can apply L'Hôpital's Rule:

$$\ln L = \lim_{x\to 1^+} \frac{\ln x}{x-1} \stackrel{H}{=} \lim_{x\to 1^+} \frac{1/x}{1} = 1$$

Thus, $\ln L = 1$ and hence, our original limit (denoted by L) is: $L = e^1 = e$. That is,

$$L = \lim_{x\to 1^+} x^{1/(x-1)} = e.$$

In this case, even though our limit had a type of "1^∞", it actually had a value of e. ♣

Example 5.45: L'Hôpital's Rule and Indeterminate Form $\infty - \infty$

Compute $\lim_{x\to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution. As x approaches zero from the right,

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

This is not a form on which we know we can use L'Hôpital's Rule, however, if we combine the fractions, the problem becomes

$$\lim_{x\to 0^+} \frac{x-\sin x}{x\sin x},$$

which gives us the indeterminate form $0/0$. We can now apply L'Hôpital's Rule twice:

$$\begin{aligned}\lim_{x\to 0^+} \frac{x-\sin x}{x\sin x} &\stackrel{H}{=} \lim_{x\to 0^+} \frac{1-\cos x}{\sin x + x\cos x} \\ &\stackrel{H}{=} \lim_{x\to 0^+} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2-0} = 0.\end{aligned}$$

♣

Exercises for Section 5.4

Exercise 5.4.1 *Compute the following limits.*

(a) $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{\sin x}$

(b) $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^3}$

(c) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$

(d) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

(e) $\displaystyle\lim_{x\to 0}\frac{\sqrt{9+x}-3}{x}$

(f) $\displaystyle\lim_{x\to 2}\frac{2-\sqrt{x+2}}{4-x^2}$

(g) $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[1/3]{x}-1}$

(h) $\displaystyle\lim_{x\to 0}\frac{x^2}{\sqrt{2x+1}-1}$

(i) $\displaystyle\lim_{u\to 1}\frac{(u-1)^3}{(1/u)-u^2+3/u-3}$

(j) $\displaystyle\lim_{x\to 0}\frac{2+(1/x)}{3-(2/x)}$

(k) $\displaystyle\lim_{x\to 0^+}\frac{1+5/\sqrt{x}}{2+1/\sqrt{x}}$

(l) $\displaystyle\lim_{x\to\pi/2}\frac{\cos x}{(\pi/2)-x}$

(m) $\displaystyle\lim_{x\to 0}\frac{x^2}{e^x-x-1}$

(n) $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$

(o) $\displaystyle\lim_{x\to 0}\frac{\ln(x^2+1)}{x}$

(p) $\displaystyle\lim_{x\to 1}\frac{x\ln x}{x^2-1}$

(q) $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{\ln(x+1)}$

(r) $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

(s) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

(t) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1}$

Exercise 5.4.2 *Compute the following limits.*

(a) $\displaystyle\lim_{x\to 0^+}\sqrt{x}\ln x$ *[Hint: Let $t=1/x$]*

(b) $\displaystyle\lim_{x\to 0}\frac{(1-x)^{1/4}-1}{x}$

(c) $\displaystyle\lim_{t\to 0}\left(t+\frac{1}{t}\right)((4-t)^{3/2}-8)$

(d) $\displaystyle\lim_{t\to 0^+}\left(\frac{1}{t}+\frac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$

(e) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}$

(f) $\displaystyle\lim_{x\to 1}\frac{x^{1/4}-1}{x}$

(g) $\displaystyle\lim_{x\to 0}\frac{3x^2+x+2}{x-4}$

(h) $\displaystyle\lim_{x\to 0^+}\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$

(i) $\displaystyle\lim_{x\to 1}(x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(j) $\displaystyle\lim_{x\to 2}\frac{x^3-6x-2}{x^3+4}$

Exercise 5.4.3 *Discuss what happens if we try to use L'Hôpital's Rule to find the limit* $\displaystyle\lim_{x\to\infty}\frac{x+\sin x}{x+1}$.

5.5 Extrema of a Function

In calculus, there is much emphasis placed on analyzing the behaviour of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? How does the interval I impact our discussion of extrema?

5.5.1. Relative Extrema

A **relative maximum** point on a function is a point (x,y) on the graph of the function whose y-coordinate is larger than all other y-coordinates on the graph at points "close to" (x,y). More precisely, $(x, f(x))$ is a relative maximum if there is an interval (a,b) with $a < x < b$ and $f(x) \geq f(z)$ for every z in (a,b). Similarly, (x,y) is a **relative minimum** point if it has locally the smallest y-coordinate. Again being more precise: $(x, f(x))$ is a relative minimum if there is an interval (a,b) with $a < x < b$ and $f(x) \leq f(z)$ for every z in (a,b). A **relative extremum** is either a relative minimum or a relative maximum.

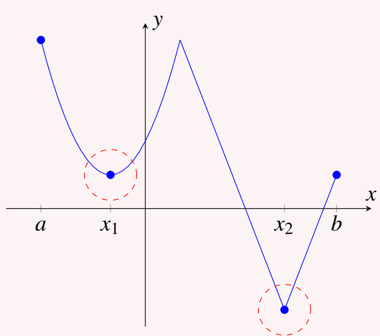
Note:

1. The plural of *extremum* is *extrema* and similarly for *maximum* and *minimum*.
2. Because a relative extremum is "extreme" locally by looking at points "close to" it, it is also referred to as a **local extremum**.

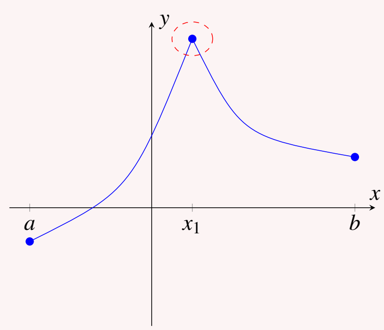
Definition 5.46: Relative Maxima and Minima

A real-valued function f has a **relative maximum** *at x_0 if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0.*

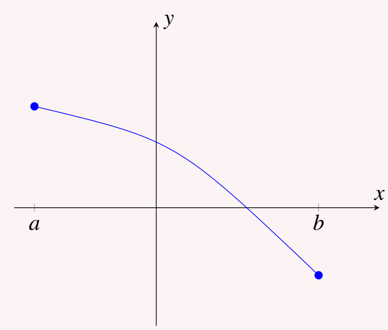
A real-valued function f has a **relative minimum** *at x_0 if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0.*



(a) *relative minima at $x = x_1$, x_2*



(b) *relative maximum at $x = x_1$*



(e) *no relative extrema in (a,b)*

Relative maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of relative maximum and minimum points are shown in Figure 5.12.

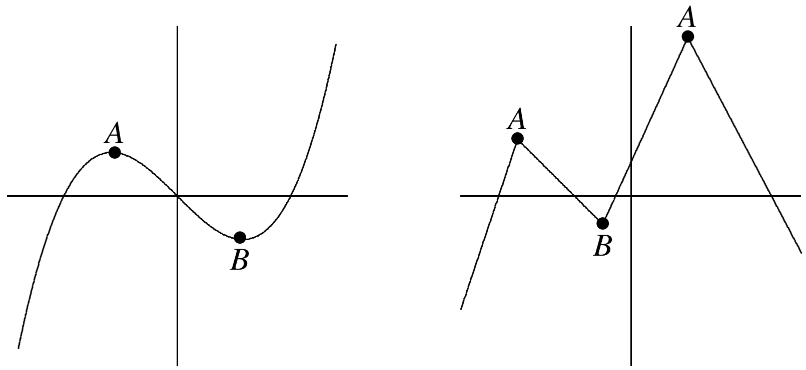


Figure 5.12: Some relative maximum points (A) and minimum points (B).

If $(x, f(x))$ is a point where $f(x)$ reaches a relative maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line *must* be horizontal. This is important enough to state as a theorem.

The proof is simple enough and we include it here, but you may accept Fermat's Theorem based on its strong intuitive appeal and come back to its proof at a later time.

Theorem 5.47: Fermat's Theorem

If $f(x)$ has a relative extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$, provided that $f'(a)$ exists.

Proof. We shall give the proof for the case where $f(x)$ has a relative maximum at $x = a$. The proof for the relative minimum case is similar.

Since $f(x)$ has a relative maximum at $x = a$, there is an open interval (c, d) with $c < a < d$ and $f(x) \leq f(a)$ for every x in (c, d). So, $f(x) - f(a) \leq 0$ for all such x. Let us now look at the sign of the difference quotient $\dfrac{f(x) - f(a)}{x - a}$. We consider two cases according as $x > a$ or $x < a$.

If $x > a$, then $x - a > 0$ and so, $\dfrac{f(x) - f(a)}{x - a} \leq 0$. Taking limit as x approach a from the right, we get

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

On the other hand, if $x < a$, then $x - a < 0$ and so, $\dfrac{f(x) - f(a)}{x - a} \geq 0$. Taking limit as x approach a from the left, we get

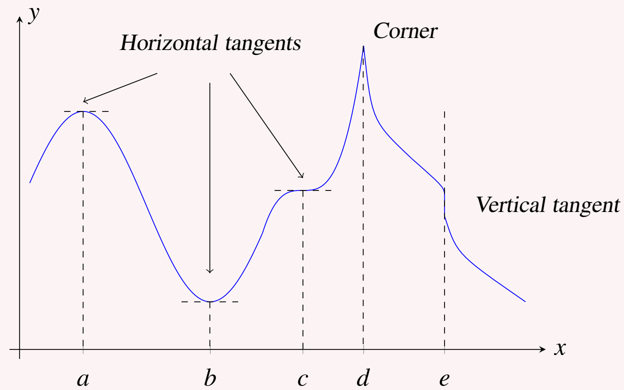
$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0.$$

Since f is differentiable at a, $f'(a) = \displaystyle\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$. Therefore, we have both $f'(a) \leq 0$ and $f'(a) \geq 0$. So, $f'(a) = 0$. ♣

Thus, the only points at which a function can have a relative maximum or minimum are points at which the derivative is zero, as in the left-hand graph in Figure 5.12, or the derivative is undefined, as in the right-hand graph. This leads us to define these special points.

Definition 5.48: Critical Point

Any value of x in the domain of f for which $f'(x)$ is zero or undefined is called a **critical point** *of f.*



The x-values a, b and c above are places for which $f'(x)$ is zero, and the x-values d and e above are places for which $f'(x)$ is undefined.

Note: When looking for relative maximum and minimum points, you are likely to make two sorts of mistakes.

1. You may forget that a maximum or minimum can occur where the derivative does not exist. You should therefore check whether the derivative exists everywhere.

2. You might also assume that any place that the derivative is zero is a relative maximum or minimum point, but this is not true. A portion of the graph of $f(x) = x^3$ is shown in Figure 5.13. The derivative of f is $f'(x) = 3x^2$, and $f'(0) = 0$, but there is neither a maximum nor minimum at $(0,0)$. In other words, the converse of Fermat's Theorem – if $f'(a) = 0$ at some point $x = a$, then f must have a relative extremum at that point – is *not* true.

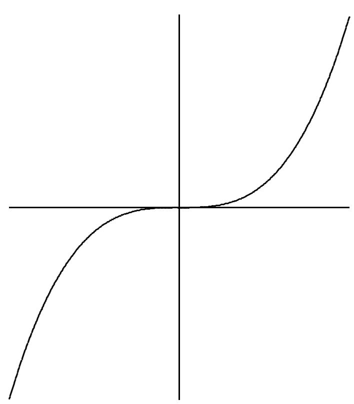


Figure 5.13: No relative extrema even though the derivative is zero at $x = 0$.

Since the derivative is zero or undefined at both relative maximum and relative minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach, but one that is often tedious or difficult, is to test directly whether the y-coordinates "near" the potential maximum or minimum are above or below the y-coordinate at the point of interest. Of course, there are too many points "near" the point to test, but a little thought shows we need only test two provided we know that f is continuous (recall that this means that the graph of f has no jumps or gaps).

Suppose, for example, that we have identified three points at which f' is zero or nonexistent: (x_1,y_1), (x_2,y_2), (x_3,y_3), and $x_1 < x_2 < x_3$ (see Figure 5.14). Suppose that we compute the value of $f(a)$ for $x_1 < a < x_2$, and that $f(a) < f(x_2)$. What can we say about the graph between a and x_2? Could there be a point $(b, f(b))$, $a < b < x_2$ with $f(b) > f(x_2)$? No: if there were, the graph would go up from $(a, f(a))$ to $(b, f(b))$ then down to $(x_2, f(x_2))$ and somewhere in between would have a relative maximum point. (This is not obvious; it is a result of the Extreme Value Theorem stated in the next section.) But at that relative maximum point the derivative of f would be zero or nonexistent, yet we already know that the derivative is zero or nonexistent only at x_1, x_2, and x_3. The upshot is that one computation tells us that $(x_2, f(x_2))$ has the largest y-coordinate of any point on the graph near x_2 and to the left of x_2. We can perform the same test on the right. If we find that on both sides of x_2 the values are smaller, then there must be a relative maximum at $(x_2, f(x_2))$; if we find that on both sides of x_2 the values are larger, then there must be a relative minimum at $(x_2, f(x_2))$; if we find one of each, then there is neither a relative maximum or minimum at x_2.

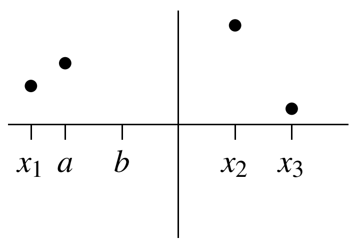


Figure 5.14: Testing for a maximum or minimum.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

Example 5.49: Testing for Relative Extrema in Cubic Function

Find all relative maximum and minimum points for the function $f(x) = x^3 - x$.

Solution. The derivative is $f'(x) = 3x^2 - 1$. This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that $f(\sqrt{3}/3) = -2\sqrt{3}/9$. Now we test two points on either side of $x = \sqrt{3}/3$, choosing one point in the interval $(-\sqrt{3}/3, \sqrt{3}/3)$ and one point in the interval $(\sqrt{3}/3, \infty)$. Since $f(0) = 0 > -2\sqrt{3}/9$ and $f(1) = 0 > -2\sqrt{3}/9$, there must be a relative minimum at $x = \sqrt{3}/3$. For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a relative maximum at $x = -\sqrt{3}/3$. ♣

Of course this example is made very simple by our choice of points to test, namely $x = -1$, 0, 1. We could have used other values, say $-5/4$, $1/3$, and $3/4$, but this would have made the calculations considerably more tedious, and we should always choose very simple points to test if we can.

Example 5.50: Testing for Relative Extrema in Trigonometric Function

Find all relative maximum and minimum points for $f(x) = \sin x + \cos x$.

Solution. The derivative is $f'(x) = \cos x - \sin x$. This is always defined and is zero whenever $\cos x = \sin x$. Recalling that the $\cos x$ and $\sin x$ are the x- and y-coordinates of points on a unit circle, we see that $\cos x = \sin x$ when x is $\pi/4$, $\pi/4 \pm \pi$, $\pi/4 \pm 2\pi$, $\pi/4 \pm 3\pi$, etc. Since both sine and cosine have a period of 2π, we need only determine the status of $x = \pi/4$ and $x = 5\pi/4$. We can use 0 and $\pi/2$ to test the critical value $x = \pi/4$. We find that $f(\pi/4) = \sqrt{2}$, $f(0) = 1 < \sqrt{2}$ and $f(\pi/2) = 1$, so there is a relative maximum when $x = \pi/4$ and also when $x = \pi/4 \pm 2\pi$, $\pi/4 \pm 4\pi$, etc. We can summarize this more neatly by saying that there are relative maxima at $\pi/4 \pm 2k\pi$ for every integer k.

We use π and $\frac{3\pi}{2}$ to test the critical value $x = 5\pi/4$. The relevant values are $f(5\pi/4) = -\sqrt{2}$, $f(\pi) = -1 > -\sqrt{2}$, $f(3\pi/2) = 1 > -\sqrt{2}$, so there is a relative minimum at $x = 5\pi/4$, $5\pi/4 \pm 2\pi$, $5\pi/4 \pm 4\pi$, etc. More succinctly, there are relative minima at $5\pi/4 \pm 2k\pi$ for every integer k. ♣

Example 5.51: Testing for Relative Extrema in Power Function

Find all relative maximum and minimum points for $g(x) = x^{2/3}$.

Solution. The derivative is $g'(x) = \frac{2}{3}x^{-1/3}$. This is undefined when $x = 0$ and is not equal to zero for any x in the domain of g'. Now we test two points on either side of $x = 0$. We use $x = -1$ and $x = 1$. Since $g(0) = 0$, $g(-1) = 1 > 0$ and $g(1) = 1 > 0$, there must be a relative minimum at $x = 0$. ♣

Exercises for Section 5.5.1

Exercise 5.5.1 *Find all relative maximum and minimum points* (x, y) *by the method of this section.*

(a) $y = x^2 - x$

(b) $y = 2 + 3x - x^3$

(c) $y = x^3 - 9x^2 + 24x$

(d) $y = x^4 - 2x^2 + 3$

(e) $y = 3x^4 - 4x^3$

(f) $y = (x^2 - 1)/x$

(g) $y = 3x^2 - (1/x^2)$

(h) $y = \cos(2x) - x$

(i) $f(x) = \begin{cases} x - 1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$

(j) $f(x) = \begin{cases} x - 3 & x < 3 \\ x^3 & 3 \leq x \leq 5 \\ 1/x & x > 5 \end{cases}$

(k) $f(x) = x^2 - 98x + 4$

(l) $f(x) = \begin{cases} -2 & x = 0 \\ 1/x^2 & x \neq 0 \end{cases}$

Exercise 5.5.2 *For any real number x there is a unique integer n such that $n \leq x < n+1$, and the greatest integer function is defined as $\lfloor x \rfloor = n$. Where are the critical values of the greatest integer function? Which are relative maxima and which are relative minima?*

Exercise 5.5.3 *Explain why the function $f(x) = 1/x$ has no relative maxima or minima.*

Exercise 5.5.4 *How many critical points can a quadratic polynomial function have?*

Exercise 5.5.5 *Show that a cubic polynomial can have at most two critical points. Give examples to show that a cubic polynomial can have zero, one, or two critical points.*

Exercise 5.5.6 *Explore the family of functions $f(x) = x^3 + cx + 1$ where c is a constant. How many and what types of relative extremes are there? Your answer should depend on the value of c, that is, different values of c will give different answers.*

Exercise 5.5.7 *We generalize the preceding two questions. Let n be a positive integer and let f be a polynomial of degree n. How many critical points can f have? (Hint: Recall the* **Fundamental Theorem of Algebra**, *which says that a polynomial of degree n has at most n roots.)*

5.5.2. Absolute Extrema

Unlike a relative extremum, which is only "extreme" relative to points "close to" it, an absolute extremum is "extreme" compared to *all* other points in the interval under consideration. Some examples of absolute maximum and minimum points are shown in Figure 5.15. This leads us to the following definitions.

Definition 5.52: Absolute Maxima and Minima

A real-valued function f has an **absolute maximum** *at x_0 if $f(x_0) \geq f(x)$ for all x in the domain of f.*

A real-valued function f has an **absolute minimum** *at x_0 if $f(x_0) \leq f(x)$ for all x in the domain of f.*

Note:

1. Notice that the definition of absolute extrema entails that an absolute extremum, unlike a relative extremum, can fall on an endpoint as shown in Figure 5.15.

2. Because of the "global" nature of an absolute extremum it is also often referred to as a **global extremum**.

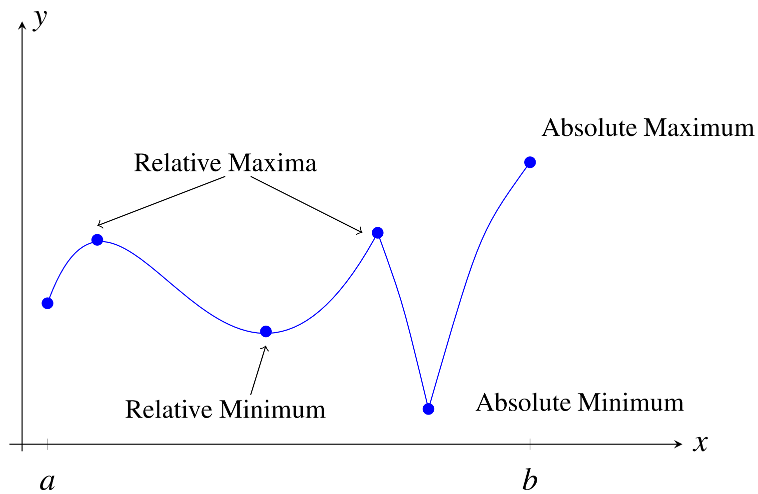


Figure 5.15: Classification of Extrema

Example 5.53: Absolute Extrema Using a Graph

Find the absolute extrema of the following functions using their graphs.

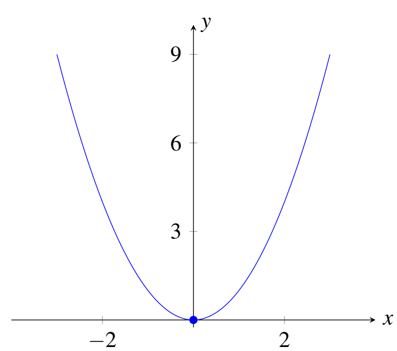
(a) $f(x) = x^2$ *on the interval* $(-\infty, \infty)$.

(b) $f(x) = |x|$ *on the interval* $[-1, 2]$.

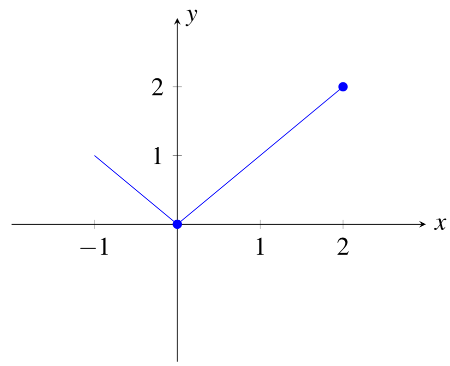
(c) $f(x) = \cos x$ *on the interval* $[0, \pi]$.

Solution.

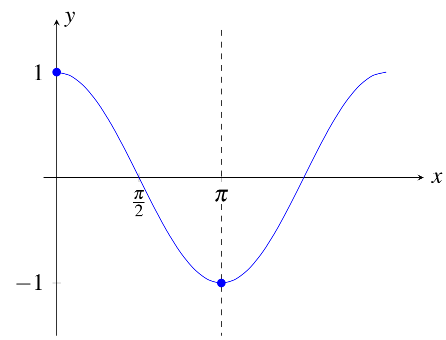
(a) This parabola has an absolute minimum at $x = 0$. However, it does not have an absolute maximum.



(b) This graph looks like a check mark. It has an absolute minimum at $x = 0$ and an absolute maximum at $x = 2$.



(c) $f(x) = cos(x)$ has an absolute minimum at $x = \pi$ and an absolute maximum at $x = 0$ on the interval $[0, \pi]$.



♣

Like Fermat's Theorem, the following theorem has an intuitive appeal. However, unlike Fermat's Theorem, the proof relies on a more advanced concept called **compactness**, which will only be covered in a course typically entitled Analysis. So, we will be content with understanding the statement of the theorem.

Theorem 5.54: Extreme Value Theorem

If a function f is continuous on a closed interval $[a,b]$, then f has both an absolute maximum and an absolute minimum on $[a,b]$.

Although this theorem tells us that an absolute extremum exists, it does not tell us what it is or how to find it.

Note that if an absolute extremum is inside the interval (i.e. not an endpoint), then it must also be a relative extremum. This immediately tells us that to find the absolute extrema of a function on an interval, we need only examine the relative extrema inside the interval, and the endpoints of the interval. We can devise a method for finding absolute extrema for a function f on a closed interval $[a,b]$.

Guideline for Finding Absolute Extrema Given Continuity of f and Closed Interval

1. Verify the function is continuous on $[a,b]$.
2. Find the derivative and determine all critical values of f that are in (a,b).
3. Evaluate the function at the critical values found in Step 2 and the endpoints $x=a$ and $x=b$ of the interval.
4. The absolute maximum value and absolute minimum value of f correspond to the largest and smallest y-values respectively found in Step 3.

Why must a function be continuous on a closed interval in order to use this theorem? Consider the following example.

Example 5.55: Absolute Extrema and Continuity

Find any absolute extrema for $f(x)=1/x$ on the interval $[-1,1]$.

Solution. The function f is not continuous at $x=0$. Since $0\in[-1,1]$, f is not continuous on the closed interval:

$$\lim_{x\to 0^+} f(x) = +\infty$$
$$\lim_{x\to 0^-} f(x) = -\infty,$$

so we are *unable* to apply the Extreme Value Theorem. Therefore, $f(x)=1/x$ does not have an absolute maximum or an absolute minimum on $[-1,1]$. ♣

However, if we consider the same function on an interval where it is continuous, the theorem will apply. This is illustrated in the following example.

Example 5.56: Absolute Extrema and Continuity

Find any absolute extrema for $f(x)=1/x$ on the interval $[1,2]$.

Solution. The function f is continous on the interval, so we can apply the Extreme Value Theorem. We begin with taking the derivative to be $f'(x)=-1/x^2$ which has a critical value at $x=0$, but since this critical value is not in $[1,2]$ we ignore it. The only points where an extrema can occur are the endpoints of the interval. To find the maximum or minimum we can simply evaluate the function: $f(1)=1$ and $f(2)=1/2$, so the absolute maximum is at $x=1$ and the absolute minimum is at $x=2$. ♣

Why must an interval be closed in order to use the above theorem? Recall the difference between open and closed intervals. Consider a function f on the open interval $(0,1)$. If we choose successive values of x moving closer and closer to 1, what happens? Since 1 is not included in the interval we will not attain exactly the value of 1. Suppose we reach a value of 0.9999 — is it possible to get closer to 1? Yes: There are infinitely many real numbers between 0.9999 and 1. In fact, any conceivable real number close to 1 will have infinitely many real numbers between itself and 1. Now, suppose f is decreasing on $(0,1)$: As we approach 1, f will continue to decrease, even if the difference between successive values of f is slight. Similarly if f is increasing on $(0,1)$.

Consider a few more examples:

Example 5.57: Determining Absolute Extrema

Determine the absolute extrema of $f(x) = x^3 - x^2 + 1$ *on the interval* $[-1,2]$.

Solution. First, notice f is continuous on the closed interval $[-1,2]$, so we're able to use Theorem 5.54 to determine the absolute extrema. The derivative is $f'(x) = 3x^2 - 2x$, and the critical values are $x = 0, 2/3$ which are both in the interval $[-1,2]$. In order to find the absolute extrema, we must consider all critical values that lie within the interval (that is, in $(-1,2)$) *and* the endpoints of the interval.

$$\begin{aligned} f(-1) &= (-1)^3 - (-1)^2 + 1 = -1 \\ f(0) &= (0)^3 - (0)^2 + 1 = 1 \\ f(2/3) &= (2/3)^3 - (2/3)^2 + 1 = 23/27 \\ f(2) &= (2)^3 - (2)^2 + 1 = 5 \end{aligned}$$

The absolute maximum is at (2,5) and the absolute minimum is at (-1,-1). ♣

Example 5.58: Determining Absolute Extrema

Determine the absolute extrema of $f(x) = -9/x - x + 10$ *on the interval* $[2,6]$.

Solution. First, notice f is continuous on the closed interval $[2,6]$, so we're able to use Theorem 5.54 to determine the absolute extrema. The function is not continuous at $x = 0$, but we can ignore this fact since 0 is not in $[2,6]$. The derivative is $f'(x) = 9/x^2 - 1$, and the critical values are $x = \pm 3$, but only $x = +3$ is in the interval. In order to find the absolute extrema, we must consider all critical values that lie within the interval *and* the endpoints of the interval.

$$\begin{aligned} f(2) &= -9/(2) - (2) + 10 = 7/2 = 3.5 \\ f(3) &= -9/(3) - (3) + 10 = 4 \\ f(6) &= -9/(6) - (6) + 10 = 5/2 = 2.5 \end{aligned}$$

The absolute maximum is at (3,4) and the absolute minimum is at (6,2.5). ♣

When we are trying to find the absolute extrema of a function on an open interval, we cannot use the Extreme Value Theorem. However, if the function is continuous on the interval, many of the same ideas apply. In particular, if an absolute extremum exists, it must also be a relative extremum. In addition to checking values at the relative extrema, we must check the behaviour of the function as it approaches the ends of the interval.

Some examples to illustrate this method.

Example 5.59: Determining Absolute Extrema

Find the extrema of $y = \sec(x)$ *on* $(-\pi/2, \pi/2)$.

Solution. Notice $\sec(x)$ is continuous on $(-\pi/2, \pi/2)$ and has one relative minimum at 0. Also

$$\lim_{x\to(-\pi/2)^+} \sec(x) = \lim_{x\to(\pi/2)^-} \sec(x) = +\infty,$$

so $\sec(x)$ has no absolute maximum, but the point $(0,1)$ is the absolute minimum. ♣

A similar approach can be used for infinite intervals.

Example 5.60: Determining Absolute Extrema

Find the extrema of $y = \dfrac{x^2}{x^2+1}$ *on* $(-\infty, \infty)$.

Solution. Since $x^2+1 \neq 0$ for all x in $(-\infty,\infty)$ the function is continuous on this interval. This function has only one critical value at $x=0$, which is the relative minimum and also the absolute minimum. Now, $\lim_{x\to\pm\infty} \frac{x^2}{x^2+1} = 1$, so the function does not have an absolute maximum: It continues to increase towards 1, but does not attain this exact value. ♣

Exercises for Section 5.5.2

Exercise 5.5.8 *Find the absolute extrema for the following functions over the given interval.*

(a) $f(x) = -\frac{x+4}{x-4}$ *on* $[0,3]$

(b) $f(x) = -\frac{x+4}{x-4}$ *on* $[0,3]$

(c) $f(x) = \csc(x)$ *on* $[0,\pi]$

(d) $f(x) = \ln(x)/x^2$ *on* $[1,4]$

(e) $f(x) = x\sqrt{1-x^2}$ *on* $[-1,1]$

(f) $f(x) = xe^{-x^2/32}$ *on* $[0,2]$

(g) $f(x) = x - \tan^{-1}(2x)$ *on* $[0,2]$

(h) $f(x) = \frac{x}{x^2+1}$

Exercise 5.5.9 *For each of the following, sketch a potential graph of a continuous function on the closed interval* $[0,4]$ *with the given properties.*

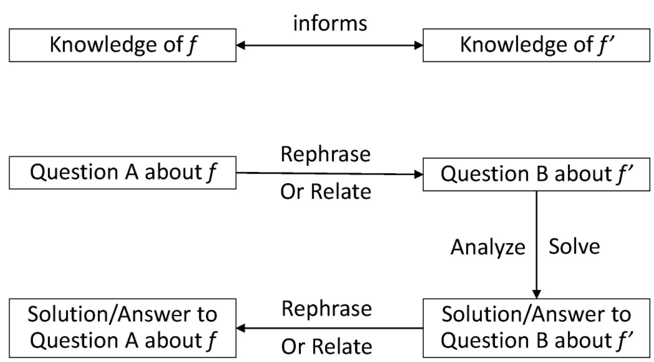
(a) *Absolute minimum at 0, absolute maximum at 2, relative minimum at 3.*

(b) *Absolute maximum at 1, absolute minimum at 2, relative maximum at 3.*

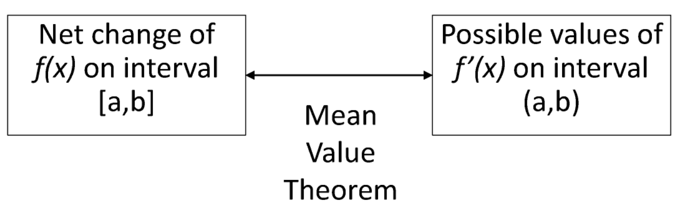
(c) *Absolute minimum at 4, absolute maximum at 1, relative minimum at 2, relative maxima at 1 and 3.*

5.6 The Mean Value Theorem

There are numerous applications of the derivative through its **definition** as rate of change and as the slope of the tangent line. In this section we shall look at some deeper reasons why the derivative turns out to be so useful. The simple answer is that **the derivative of a function tells us a lot about the function**. More important, "hard" questions about a function can sometimes be answered by solving a relatively simple problem about the derivative of the function.



The Mean Value Theorem tells us that there is an intimate connection between the net change of the value of any "sufficiently nice" function over an interval and the possible values of its derivative on that interval. Because of this connection, we can draw conclusions about the possible values of the derivative based on information about the values of the function, and conversely, we can draw conclusions about the values of the function based on information about the values of its derivative.



Let us illustrate the idea through the following two interesting questions involving derivatives:

1. Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?

2. Suppose you drive a car from toll booth on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit?

While these sound very different, it turns out that the two problems are very closely related. We know that "speed" is really the derivative by a different name; let's start by translating the second question into something that may be easier to visualize. Suppose that the function $f(t)$ gives the position of your car on the toll road at time t. Your change in position between one toll booth and the next is given by $f(t_1) - f(t_0)$, assuming that at time t_0 you were at the first booth and at time t_1 you arrived at the second booth. Your average speed for the trip is $(f(t_1) - f(t_0))/(t_1 - t_0)$. If we think about the graph of $f(t)$, the average speed is the slope of the line that connects the two points $(t_0, f(t_0))$ and $(t_1, f(t_1))$. Your speed at any particular time t between t_0 and t_1 is $f'(t)$, the slope of the curve. Now question (2) becomes a question about slope. In particular, if the slope between endpoints is 70, what can be said of the slopes at points between the endpoints?

As a general rule, when faced with a new problem it is often a good idea to examine one or more simplified versions of the problem, in the hope that this will lead to an understanding of the original problem. In this case, the problem in its "slope" form is somewhat easier to simplify than the original, but equivalent, problem.

Here is a special instance of the problem. Suppose that $f(t_0) = f(t_1)$. Then the two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn't take much experimentation before you are convinced of the truth of this statement: Somewhere between t_0 and t_1 the slope is exactly zero, that is, somewhere between t_0 and t_1 the slope is equal to the slope of the line between the endpoints. This suggests that perhaps the same is true even if the endpoints are at different heights, and again a bit of experimentation will probably convince you that this is so. But we can do better than "experimentation"—we can prove that this is so.

We start with the simplified version:

Theorem 5.61: Rolle's Theorem

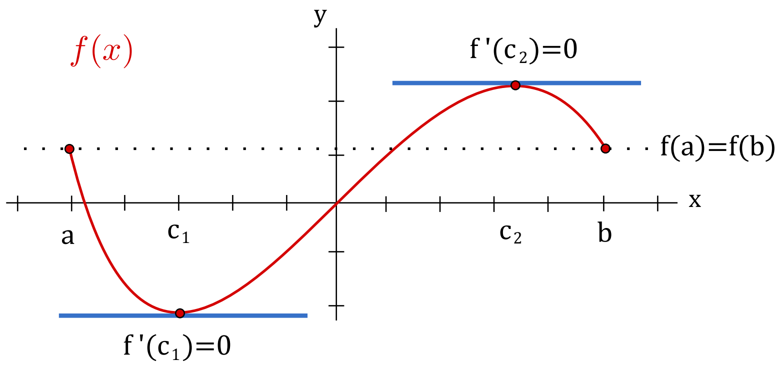
Suppose that $f(x)$ has a derivative on the interval (a,b), is continuous on the interval $[a,b]$, and $f(a) = f(b)$. Then at some value $c \in (a,b)$, $f'(c) = 0$.

Proof. We know that $f(x)$ has a maximum and minimum value on $[a,b]$ (because it is continuous), and we also know that the maximum and minimum must occur at an endpoint, at a point at which the derivative is zero, or at a point where the derivative is undefined. Since the derivative is never undefined, that possibility is removed.

If the maximum or minimum occurs at a point c, other than an endpoint, where $f'(c) = 0$, then we have found the point we seek. Otherwise, the maximum and minimum both occur at an endpoint, and since the endpoints have the same height, the maximum and minimum are the same. This means that $f(x) = f(a) = f(b)$ at every $x \in [a,b]$, so the function is a horizontal line, and it has derivative zero everywhere in (a,b). Then we may choose any c at all to get $f'(c) = 0$. ♣

Rolle's Theorem is illustrated below for a function $f(x)$ where $f'(x) = 0$ holds for two values of $x = c_1$

and $x = c_2$:



Perhaps remarkably, this special case is all we need to prove the more general one as well.

Theorem 5.62: Mean Value Theorem

Suppose that $f(x)$ has a derivative on the interval (a,b) and is continuous on the interval $[a,b]$. Then at some value $c \in (a,b)$, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof. Let $m = \dfrac{f(b)-f(a)}{b-a}$, and consider a new function $g(x) = f(x) - m(x-a) - f(a)$. We know that $g(x)$ has a derivative everywhere, since $g'(x) = f'(x) - m$. We can compute $g(a) = f(a) - m(a-a) - f(a) = 0$ and

$$g(b) = f(b) - m(b-a) - f(a) \quad = f(b) - \tfrac{f(b)-f(a)}{b-a}(b-a) - f(a)$$
$$= f(b) - (f(b) - f(a)) - f(a) = 0.$$

So the height of $g(x)$ is the same at both endpoints. This means, by Rolle's Theorem, that at some c, $g'(c) = 0$. But we know that $g'(c) = f'(c) - m$, so

$$0 = f'(c) - m = f'(c) - \frac{f(b)-f(a)}{b-a},$$

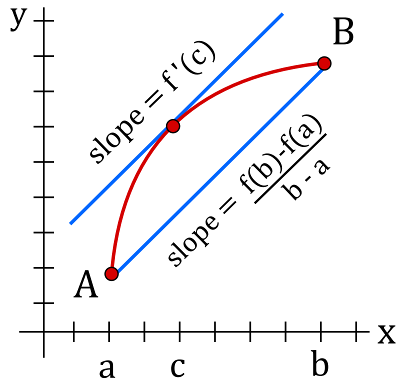
which turns into

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

exactly what we want. ♣

The Mean Value Theorem is illustrated below showing the existence of a point $x = c$ for a function $f(x)$ where the tangent line at $x = c$ (with slope $f'(c)$) is parallel to the secant line connecting $A(a, f(a))$

and $B(b, f(b))$ (with slope $\frac{f(b)-f(a)}{b-a}$):



Returning to the original formulation of question (2), we see that if $f(t)$ gives the position of your car at time t, then the Mean Value Theorem says that at some time c, $f'(c) = 70$, that is, at some time you must have been traveling at exactly your average speed for the trip, and that indeed you exceeded the speed limit.

Now let's return to question (1). Suppose, for example, that two functions are known to have derivative equal to 5 everywhere, $f'(x) = g'(x) = 5$. It is easy to find such functions: $5x$, $5x+47$, $5x-132$, etc. Are there other, more complicated, examples? No—the only functions that work are the "obvious" ones, namely, $5x$ plus some constant. How can we see that this is true?

Although "5" is a very simple derivative, let's look at an even simpler one. Suppose that $f'(x) = g'(x) = 0$. Again we can find examples: $f(x) = 0$, $f(x) = 47$, $f(x) = -511$ all have $f'(x) = 0$. Are there non-constant functions f with derivative 0? No, and here's why: Suppose that $f(x)$ is not a constant function. This means that there are two points on the function with different heights, say $f(a) \neq f(b)$. The Mean Value Theorem tells us that at some point c, $f'(c) = (f(b) - f(a))/(b-a) \neq 0$. So any non-constant function does not have a derivative that is zero everywhere; this is the same as saying that the only functions with zero derivative are the constant functions.

Let's go back to the slightly less easy example: suppose that $f'(x) = g'(x) = 5$. Then $(f(x) - g(x))' = f'(x) - g'(x) = 5 - 5 = 0$. So using what we discovered in the previous paragraph, we know that $f(x) - g(x) = k$, for some constant k. So any two functions with derivative 5 must differ by a constant; since $5x$ is known to work, the only other examples must look like $5x + k$.

Now we can extend this to more complicated functions, without any extra work. Suppose that $f'(x) = g'(x)$. Then as before $(f(x) - g(x))' = f'(x) - g'(x) = 0$, so $f(x) - g(x) = k$. Again this means that if we find just a single function $g(x)$ with a certain derivative, then every other function with the same derivative must be of the form $g(x) + k$.

Example 5.63: Given Derivative

Describe all functions that have derivative $5x - 3$.

Solution. It's easy to find one: $g(x) = (5/2)x^2 - 3x$ has $g'(x) = 5x - 3$. The only other functions with the same derivative are therefore of the form $f(x) = (5/2)x^2 - 3x + k$.

Alternately, though not obviously, you might have first noticed that $g(x) = (5/2)x^2 - 3x + 47$ has $g'(x) = 5x - 3$. Then every other function with the same derivative must have the form $f(x) = (5/2)x^2 -$

$3x+47+k$. This looks different, but it really isn't. The functions of the form $f(x) = (5/2)x^2 - 3x + k$ are exactly the same as the ones of the form $f(x) = (5/2)x^2 - 3x + 47 + k$. For example, $(5/2)x^2 - 3x + 10$ is the same as $(5/2)x^2 - 3x + 47 + (-37)$, and the first is of the first form while the second has the second form. ♣

This is worth calling a theorem:

Theorem 5.64: Functions with the Same Derivative

If $f'(x) = g'(x)$ for every $x \in (a,b)$, then for some constant k, $f(x) = g(x) + k$ on the interval (a,b).

Example 5.65: Same Derivative

Describe all functions with derivative $\sin x + e^x$. One such function is $-\cos x + e^x$, so all such functions have the form $-\cos x + e^x + k$.

Theorem 5.64 and the above example illustrate what the Mean Value Theorem allows us to say about $f(x)$ when we have perfect information about $f'(x)$. Specifically, $f(x)$ is determined up to a constant. Our next example illustrates almost the opposite extreme situation, one where we have much less information about $f'(x)$ beyond the fact that $f'(x)$ exists. Specifically, assuming that we know an upper bound on the values of $f'(x)$, what can we say about the values of $f(x)$?

Example 5.66: Conclusion Regarding Function Value Based on Derivative Information

Suppose that f is a differentiable function such that $f'(x) \leq 2$ for all x. What is the largest possible value of $f(7)$ if $f(3) = 5$?

Solution. We are interested in the values of $f(x)$ at $x = 3$ and $x = 7$. It makes sense to focus our attention on the interval between 3 and 7. It is given that $f(x)$ is differentiable for all x. So, $f(x)$ is also continuous at all x. In particular, $f(x)$ is continuous on the interval $[3,7]$ and differentiable on the interval $(3,7)$. By the Mean Value Theorem, we know that there is some c in $(3,7)$ such that

$$f'(c) = \frac{f(7) - f(3)}{7-3}.$$

Simplifying and using the given information $f(3) = 5$, we get

$$f'(c) = \frac{f(7) - 5}{4},$$

or, after re-arranging the terms,

$$f(7) = 4f'(c) + 5.$$

We do not know the exact value of c, but we do know that $f'(x) \leq 2$ for all x. This implies that $f'(c) \leq 2$. Therefore,

$$f(7) \leq 4 \cdot 2 + 5 = 13.$$

That is, the value of $f(7)$ cannot exceed 13. To convince ourselves that 13 (as opposed to some smaller number) is the largest possible value of $f(7)$, we still need to show that it is possible for the value of $f(7)$ to reach 13. If we review our proof, we notice that the inequality will be an equality if $f'(c) = 2$. One way to guarantee this without knowing anything about c is to require $f'(x) = 2$ for all x. This means that $f(x) = 2x + k$ for some constant k. From the condition $f(3) = 5$, we see that $k = -1$. We can easily verify that indeed $f(x) = 2x - 1$ meets all our requirements and $f(7) = 13$. ♣

Exercises for Section 5.6

Exercise 5.6.1 *Let $f(x) = x^2$. Find a value $c \in (-1,2)$ so that $f'(c)$ equals the slope between the endpoints of $f(x)$ on $[-1,2]$.*

Exercise 5.6.2 *Verify that $f(x) = x/(x+2)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1,4]$ and then find all of the values, c, that satisfy the conclusion of the theorem.*

Exercise 5.6.3 *Verify that $f(x) = 3x/(x+7)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2,6]$ and then find all of the values, c, that satisfy the conclusion of the theorem.*

Exercise 5.6.4 *Let $f(x) = \tan x$. Show that $f(\pi) = f(2\pi) = 0$ but there is no number $c \in (\pi, 2\pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?*

Exercise 5.6.5 *Let $f(x) = (x-3)^{-2}$. Show that there is no value $c \in (1,4)$ such that $f'(c) = (f(4) - f(1))/(4-1)$. Why is this not a contradiction of the Mean Value Theorem?*

Exercise 5.6.6 *Describe all functions with derivative $x^2 + 47x - 5$.*

Exercise 5.6.7 *Describe all functions with derivative $\frac{1}{1+x^2}$.*

Exercise 5.6.8 *Describe all functions with derivative $x^3 - \frac{1}{x}$.*

Exercise 5.6.9 *Describe all functions with derivative $\sin(2x)$.*

Exercise 5.6.10 *Find $f(x)$ if $f'(x) = e^{-x}$ and $f(0) = 2$.*

Exercise 5.6.11 *Suppose that f is a differentiable function such that $f'(x) \geq -3$ for all x. What is the smallest possible value of $f(4)$ if $f(-1) = 2$?*

Exercise 5.6.12 *Show that the equation $6x^4 - 7x + 1 = 0$ does not have more than two distinct real roots.*

Exercise 5.6.13 *Let f be differentiable on $\mathbb{R}$. Suppose that $f'(x) \neq 0$ for every x. Prove that f has at most one real root.*

Exercise 5.6.14 *Prove that for all real x and y* $|\cos x - \cos y| \leq |x - y|$. *State and prove an analogous result involving sine.*

Exercise 5.6.15 *Show that* $\sqrt{1+x} \leq 1 + (x/2)$ *if* $-1 < x < 1$.

Exercise 5.6.16 *Suppose that* $f(a) = g(a)$ *and that* $f'(x) \leq g'(x)$ *for all* $x \geq a$.

(a) *Prove that* $f(x) \leq g(x)$ *for all* $x \geq a$.

(b) *Use part (a) to prove that* $e^x \geq 1 + x$ *for all* $x \geq 0$.

(c) *Use parts (a) and (b) to prove that* $e^x \geq 1 + x + \dfrac{x^2}{2}$ *for all* $x \geq 0$.

(d) *Can you generalize these results?*

5.7 Curve Sketching

In this section, we discuss how we can tell what the graph of a function looks like by performing simple tests on its derivatives.

5.7.1. The First Derivative Test and Intervals of Increase/Decrease

The method of Section 5.5.1 for deciding whether there is a relative maximum or minimum at a critical value is not always convenient. We can instead use information about the derivative $f'(x)$ to decide; since we have already had to compute the derivative to find the critical values, there is often relatively little extra work involved in this method.

How can the derivative tell us whether there is a maximum, minimum, or neither at a point? The following so-called **First Derivative Test** is a procedure for finding relative extrema of a continuous function based on critical points and analyzing behaviour around the critical points:

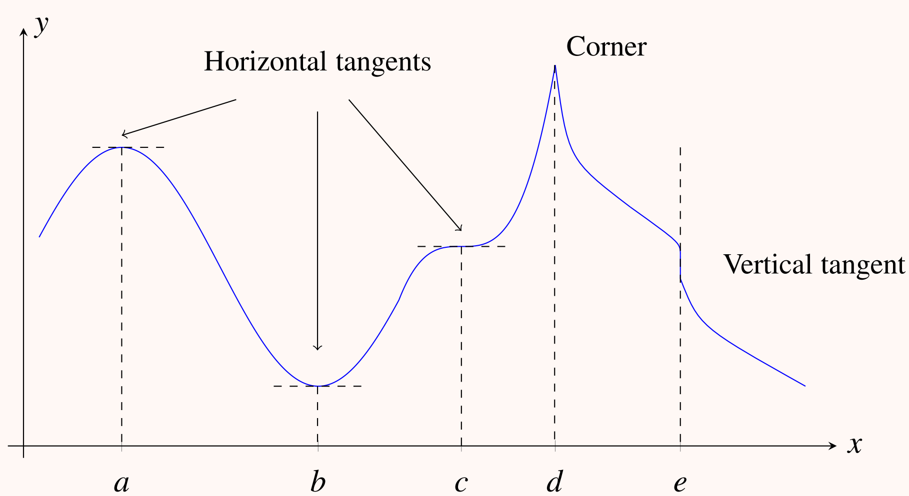
First Derivative Test

Let k be a critical point in the domain of a continuous function f and suppose that f is differentiable around $x = k$.

1. There is a relative maximum at $x = k$. This happens if $f'(x) > 0$ as we approach $x = k$ from the left (i.e. when x is in the vicinity of k, and $x < k$) and $f'(x) < 0$ as we move to the right of $x = a$ (i.e. when x is in the vicinity of k, and $x > k$). See $x = a, d$ in the graph below.

2. There is a relative minimum at $x = a$. This happens if $f'(x) < 0$ as we approach $x = k$ from the left (i.e. when x is in the vicinity of k, and $x < k$) and $f'(x) > 0$ as we move to the right of $x = k$ (i.e. when x is in the vicinity of k, and $x > k$). See $x = b$ in the graph below.

3. There is neither a relative maximum or relative minimum at $x = a$. This happens if $f'(x)$ does not change from negative to positive, or from positive to negative, as we move from the left of $x = a$ to the right of $x = k$ (that is, $f'(x)$ is positive on both sides of $x = k$, or negative on both sides of $x = k$). See $x = c, e$ in the graph below.



Example 5.67: Relative Extrema

Find all relative maximum and minimum points for $f(x) = \sin x + \cos x$ using the First Derivative Test.

Solution. The domain of f is $D_f = \mathbb{R}$. The derivative is $f'(x) = \cos x - \sin x$ and from Example 5.50 the critical values we need to consider are $\pi/4$ and $5\pi/4$.

To classify these critical points using the First Derivative Test, we need to determine the sign of $f'(x)$ as x approaches each point from the left and from the right.

First pick a test point in the interval $(-\infty, \pi/4)$, say $x = 0$. Evaluate

$$f'(0) = \cos(0) - \sin(0) = 1.$$

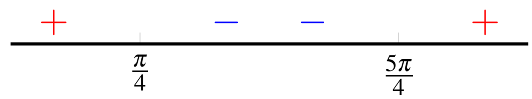
The next test point should be in the interval $(\pi/4, 5\pi/4)$, for example $x = \pi$. Here,

$$f'(\pi) = \cos(\pi) - \sin(\pi) = -1.$$

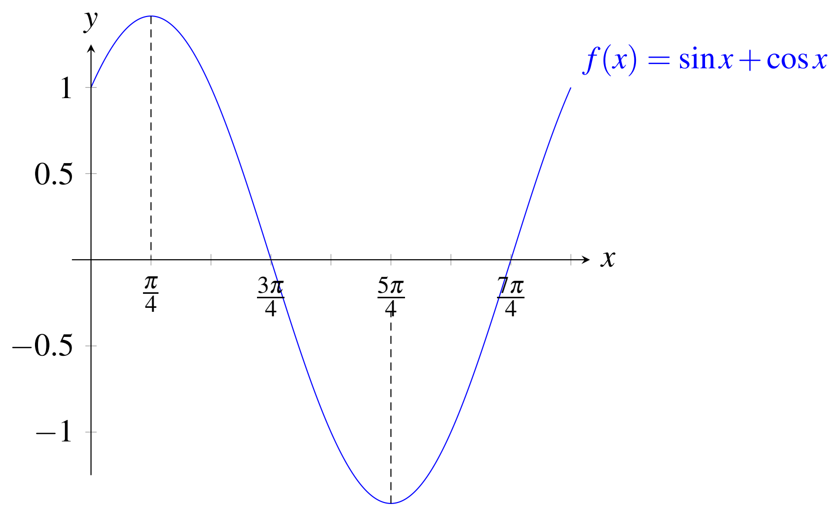
Lastly, we need a test point in the interval $(5\pi/4, \infty)$, such as $x = 2\pi$. Then

$$f'(2\pi) = \cos(2\pi) - \sin(2\pi) = 1.$$

From this information, we can construct the following sign diagram for $f'(x)$:



We conclude that $f(x)$ must be increasing on the entire interval $(-\infty, \pi/4)$, decreasing on $(\pi/4, 5\pi/4)$, and increasing on $(5\pi/4, \infty)$. Therefore, by the First Derivative Test, f has a relative minimum at $5\pi/4$ and a relative maximum at $\pi/4$, as shown in the graph below.



♣

Example 5.68: Relative Extrema

Find the relative maxima and minima of the function $f(x) = x^{2/3}$.

Solution. The derivative of f is

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.$$

The function f' is not defined at $x = 0$, so f' is discontinuous there. It is continuous everywhere else. Furthermore, f' is not equal to zero anywhere. Thus, $x = 0$ is the only critical point of the function f.

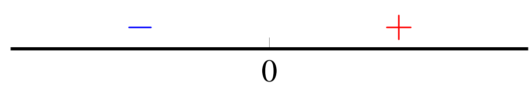
Pick a test point (say, $x = -1$) in the interval $(-\infty, 0)$ and compute

$$f'(-1) = -\frac{2}{3}.$$

Next, we pick a test point (say $x = 1$) in the interval $(0, \infty)$ and compute

$$f'(1) = \frac{2}{3}.$$

We now construct the sign diagram for $f'(x)$:



We see that the sign of $f'(x)$ changes from negative to positive as we move across $x = 0$ from left to right. Thus, an application of the First Derivative Test tells us that $f(0) = 0$ is a relative minimum of f. We confirm these results with the graph of f, shown in Figure 5.16. ♣

Perhaps you noticed in the graphs from this section that the critical points seem to divide the domain of a function into intervals, where the function either **increases** or **decreases** on this interval. We will now formally introduce this concept of *increase* and *decrease*.

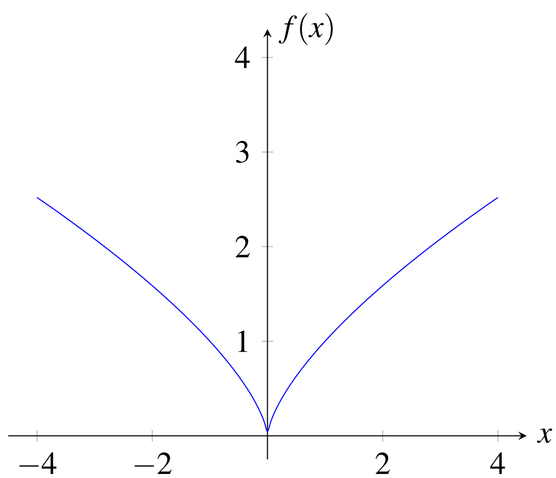
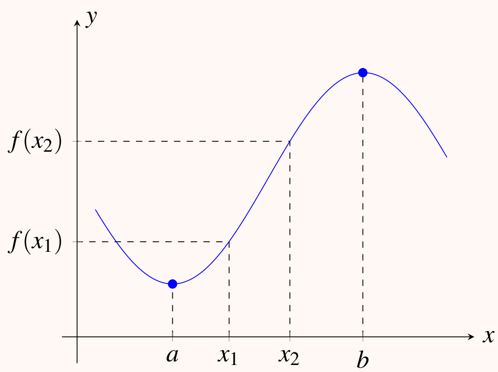


Figure 5.16: The graph of $f(x) = x^{2/3}$ decreases on $(-\infty, 0)$, and increases on $(0, \infty)$.

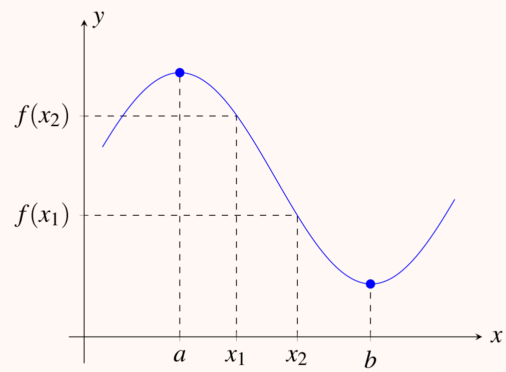
Increasing and Decreasing Functions

A function f is **increasing** on an interval (a,b) if for any two numbers x_1 and x_2 in (a,b), $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

A function f is **decreasing** on an interval (a,b) if for any two numbers x_1 and x_2 in (a,b), $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.



(a) *f is increasing on (a,b).*



(b) *f is decreasing on (a,b).*

We say that f is increasing at $x = c$ if $c \in (a,b)$ such that f is increasing on the interval (a,b), see Figure 5.17. Similarly, we say that f is decreasing at $x = c$ if $c \in (a,b)$ such that f is decreasing on the interval (a,b), see Figure 5.18. Increasing functions slope upward from left to right (Figure 5.17) and decreasing functions slope downward from left to right (Figure 5.18). But the slope of a function at a point $x = c$ is the rate of change of the function at $x = c$, which is given by the derivative of the function at that point. Therefore, the derivative lends itself naturally as the tool for determining intervals of increase and decrease, provided the function is differentiable on the intervals, see Figures 5.17 and 5.18. This leads us to the following theorem, which we can prove using the Mean Value Theorem from Section 5.6.

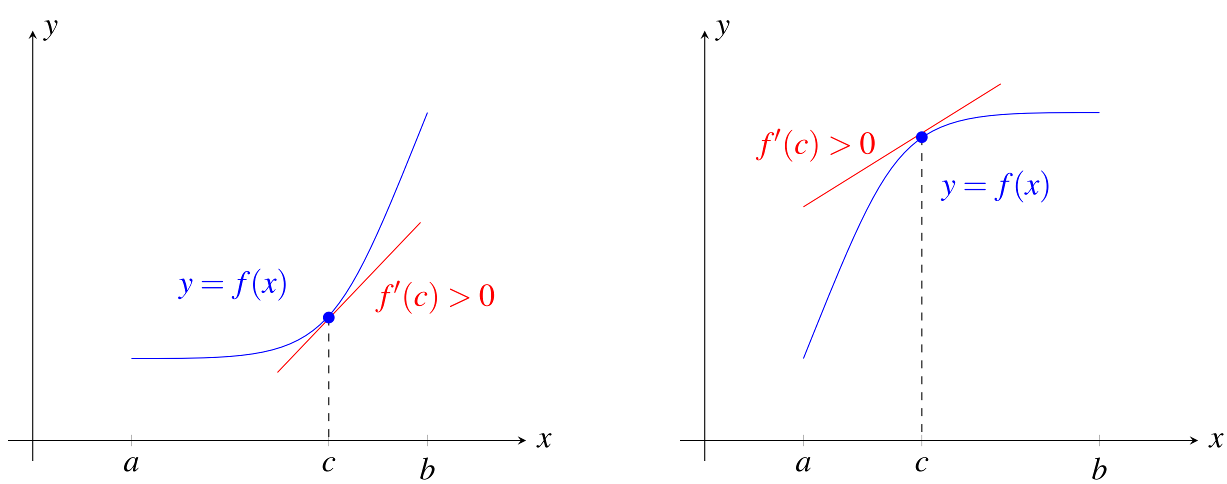


Figure 5.17: f is increasing at $x = c$.

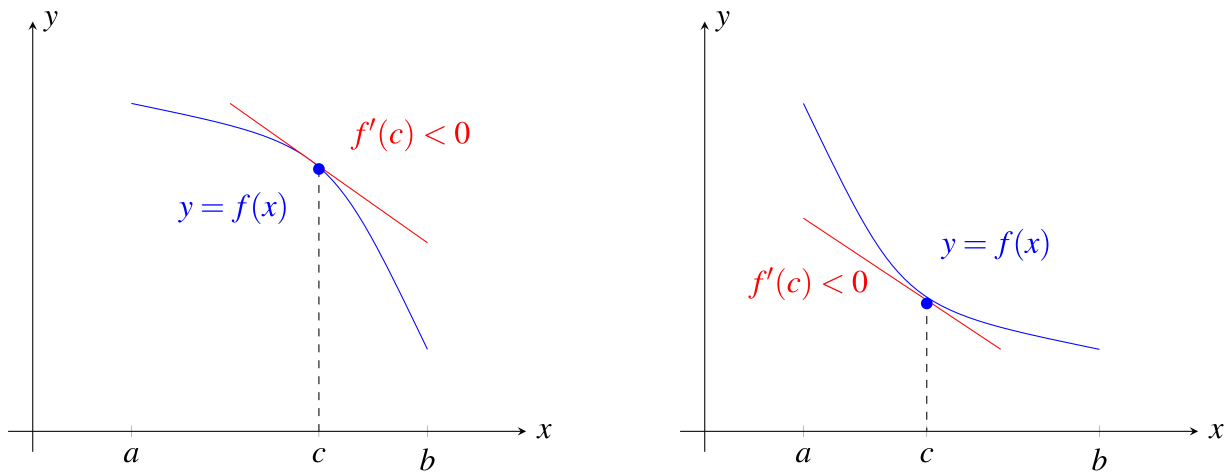


Figure 5.18: f **is decreasing at** $x = c$**.**

Theorem 5.69: Intervals of Increase and Decrease Test

1. *If* $f'(x) > 0$ *for every* x *in an interval, then* f *is increasing on that interval.*
2. *If* $f'(x) < 0$ *for every* x *in an interval, then* f *is decreasing on that interval.*
3. *If* $f'(x) = 0$ *for each value of* x *in an interval* (a,b)*, then* f *is constant on* (a,b)*.*

Proof. We will prove the increasing case. The proof of the decreasing case is similar. Suppose that $f'(x) > 0$ on an interval I. Then f is differentiable, and hence also, continuous on I. If x_1 and x_2 are any two numbers in I and $x_1 < x_2$, then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem, there is some c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But c must be in I, and thus, since $f'(x) > 0$ for every x in I, $f'(c) > 0$. Also, since $x_1 < x_2$, we have $x_2 - x_1 > 0$. Therefore, both the left hand side and the denominator of the right hand side are positive. It follows that the numerator of the right hand must be positive. That is, $f(x_2) - f(x_1) > 0$, or in other words, $f(x_1) < f(x_2)$. This shows that between x_1 and x_2 in I, the larger one, x_2, necessarily has the larger function value, $f(x_2)$, and the smaller one, x_1, necessarily have the smaller function value, $f(x_1)$. This means that f is increasing on I. ♣

Example 5.70: Intervals of Increase and Decrease for $f'(x) = 0$

Determine the intervals where the function $f(x) = x^3 - 3x^2 - 24x + 32$ *is increasing and where it is decreasing.*

Solution. The derivative of f is

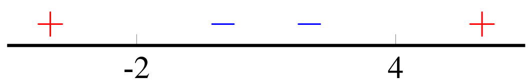
$$f'(x) = 3x^2 - 6x - 24 = 3(x+2)(x-4),$$

and it is continuous everywhere. The zeros of $f'(x)$ are $x=-2$ and $x=4$, and these points divide the real line into the intervals $(-\infty,-2)$,$(-2,4)$, and $(4,\infty)$.

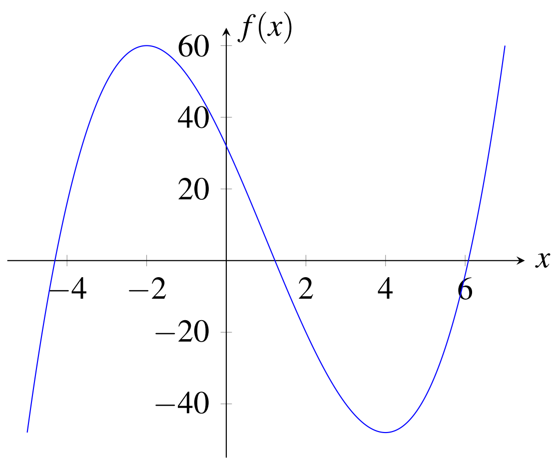
To determine the sign of $f'(x)$ in the above intervals, compute $f'(x)$ at a convenient test point within each interval. We find,

Interval	*Test Point*	$f'(c)$	*Sign of* $f'(x)$
$(-\infty,-2)$	-3	21	$+$
$(-2,4)$	0	-24	$-$
$(4,\infty)$	5	21	$+$

Using these results, we obtain the following sign diagram:



We conclude that f is increasing on the intervals $(-\infty,-2)$ and $(4,\infty)$ and is decreasing on the interval $(-2,4)$. The graph of f is shown below.



♣

Example 5.71: Intervals of Increase and Decrease for $f'(x)$ DNE

Find the interval where the function $f(x)=x^{2/3}$ is increasing and the interval where it is decreasing.

Solution. The derivative of f is

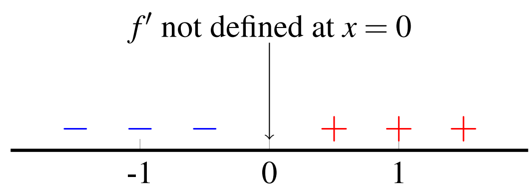
$$f'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3x^{1/3}}.$$

As noted in Example 5.7.1, f' is not defined at $x=0$, is continuous everywhere else, and is not equal to zero in its domain.

Since $f'(-1) < 0$, we see that $f'(x) < 0$ on $(-\infty.0)$. Next, we pick a test point (say $x = 1$) in the interval $(0,\infty)$ and compute

$$f'(1) = \frac{2}{3}.$$

Since $f'(1) > 0$, we see that $f'(x) > 0$ on $(0,\infty)$. From the consequent sign diagram,



we conclude that f is decreasing on the interval $(-\infty,0)$ and increasing on the interval $(0,\infty)$. The graph of f, shown in Figure 5.16 confirms these results. ♣

Example 5.72: Intervals of Increase and Decrease

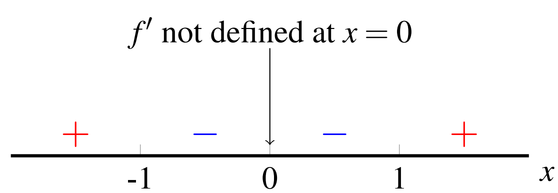
Find the intervals where the function $f(x) = x + \dfrac{1}{x}$ *is increasing and where it is decreasing.*

Solution. The derivative of f is

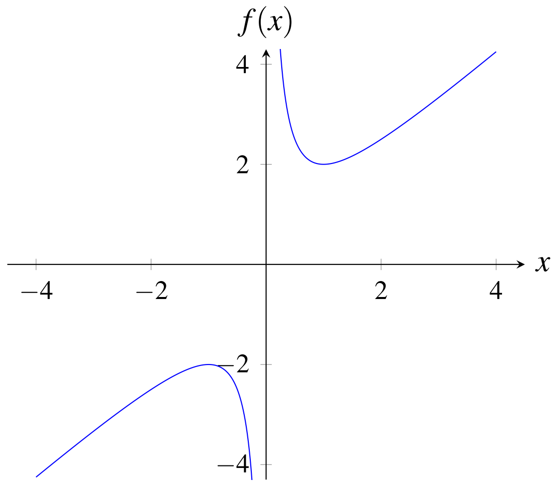
$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}.$$

Since f' is not defined at $x = 0$, it is discontinuous there. Furthermore, $f'(x)$ is equal to zero when $x^2 - 1 = 0$, or $x = \pm 1$. These values of x partition the domain of f' into the open intervals $(-\infty,-1)$,$(-1,0)$,$(0,1)$ and $(1,\infty)$, where the sign of f' is different from zero.

To determine the sign of f' in each of these intervals, we compute $f'(x)$ at the test points $x = -2, -\frac{1}{2}, \frac{1}{2}$ and 2,respectively, obtaining $f'(-2) = \frac{3}{4}$,$f'(-\frac{1}{2}) = -3$, $f'(\frac{1}{2} = -3$, and $f'(2) = \frac{3}{4}$.



From the above sign diagram for f', we conclude that f is increasing on $(-\infty,-1)$ and $(1,\infty)$ and decreasing on $(-1,0)$ and $(0,1)$. The graph of f is shown below. Note that f' does not change sign as we move accros the point of discontinuity, $x = 0$.



♣

Note: Example 5.7.1 reminds us that we must *not* automatically conclude that the derivative f' must change sign when we move across a discontinuity or zero of f'.

Example 5.73: Intervals of Increase and Decrease and Relative Extrema

Consider the function $f(x) = x^4 - 2x^2$. Find where f is increasing and where f is decreasing. Use this information to find the relative maximum and minimum points of f.

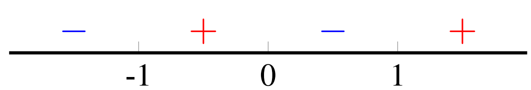
Solution. We first compute $f'(x)$ and analyze its sign.

$$f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x-1)(x+1).$$

Thus, $f'(x) = 0$ when $x = 0, 1$, and -1. $f'(x)$ is a continuous function, and so these are the only critical points. This splits the domain into the open intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$, where f is either increasing or decreasing. Picking appropriate test points, we find

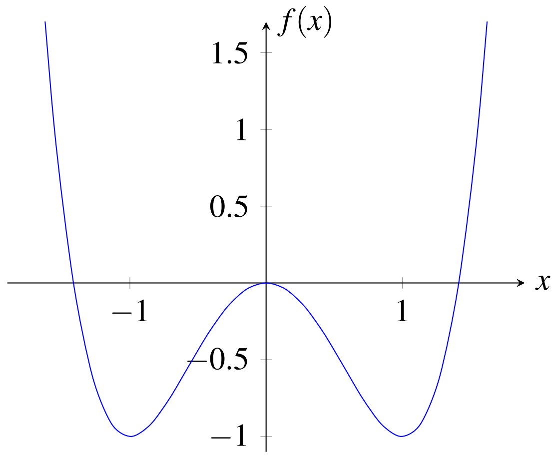
$$f'(-2) = -24, \;\; f'(-0.5) = 1.5, \;\; f'(0.5) = -1.5, \text{ and } \; f'(2) = 24.$$

This leads us to the sign diagram,



from which we see that f is increasing on the interval $(-1, 0)$ and on the interval $(1, \infty)$. Similarly, f is decreasing on the interval $(-\infty, -1)$ and on the interval $(0, 1)$.

Therefore, at the critical points -1, 0 and 1, respectively, f has a relative minimum, a relative maximum and a relative minimum, as shown below.



♣

Example 5.74: Intervals of Increase and Decrease in Profit Function

The profit function of Acrosonic is given by

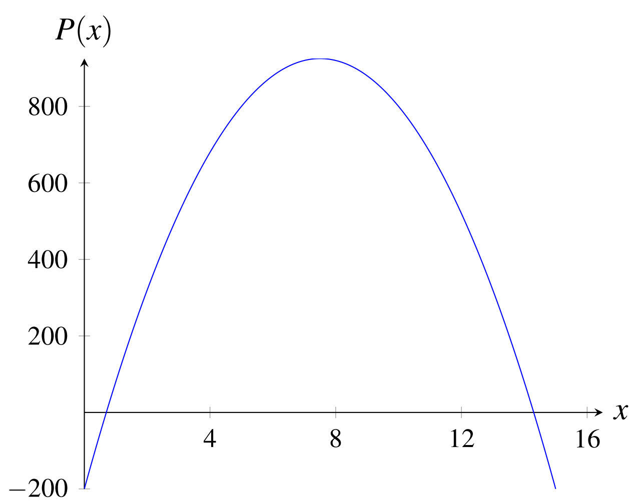
$$P(x) = -0.02x^2 + 300x - 200,000$$

dollars, where x is the number of Acrosonic model F loudspeakers systems produced. Find where the function P is increasing and where it is decreasing.

Solution. The derivative P' is

$$P'(x) = -0.04x + 300 = -0.04(x - 7500).$$

Thus, $P'(x) = 0$ when $x = 7500$. Furthermore, $P'(x) > 0$ for x in the interval $(0, 7500)$, and $P'(x) < 0$ for x in the interval $(7500, \infty)$. This means that the profit function P is increasing on $(0, 7500)$, and decreasing on $(7500, \infty)$ (see graph below, where both P and x are in units of a thousand).



♣

Exercises for Section 5.7.1

Exercise 5.7.1 *Find all critical points and identify them as relative maximum points, relative minimum points, or neither.*

(a) $y = x^2 - x$

(b) $y = 2 + 3x - x^3$

(c) $y = x^3 - 9x^2 + 24x$

(d) $y = x^4 - 2x^2 + 3$

(e) $y = 3x^4 - 4x^3$

(f) $y = (x^2 - 1)/x$

(g) $y = 3x^2 - (1/x^2)$

(h) $y = \cos(2x) - x$

(i) $f(x) = (5 - x)/(x + 2)$

(j) $f(x) = |x^2 - 121|$

(k) $f(x) = x^3/(x + 1)$

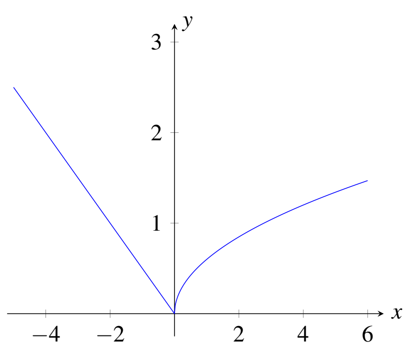
(l) $f(x) = \sin^2 x$

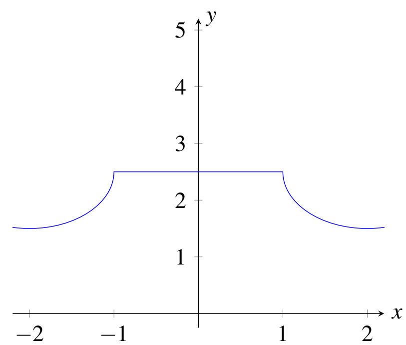
(m) $f(x) = \sec x$

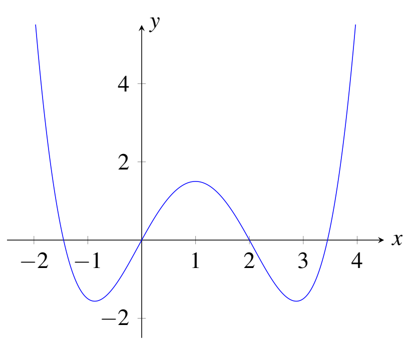
Exercise 5.7.2 *Let $f(\theta) = \cos^2(\theta) - 2\sin(\theta)$. Find the intervals where f is increasing and the intervals where f is decreasing in $[0, 2\pi]$. Use this information to classify the critical points of f as either relative maximums, relative minimums, or neither.*

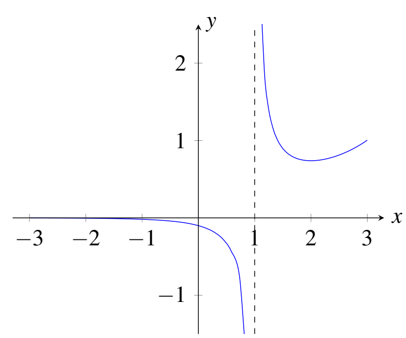
Exercise 5.7.3 *Let $r > 0$. Find the relative maxima and minima of the function $f(x) = \sqrt{r^2 - x^2}$ on its domain $[-r, r]$.*

Exercise 5.7.4 *Given the graph of a function f, determine the intervals where f is increasing, constant, and decreasing.*

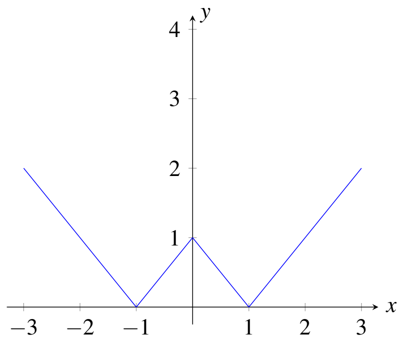
(a)


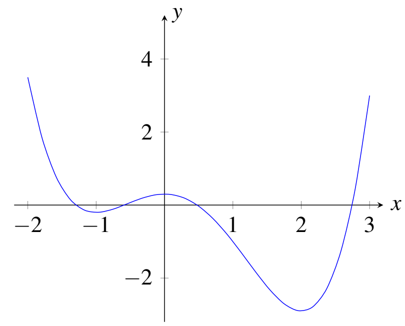
(c)


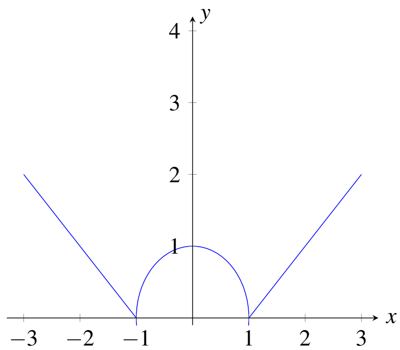
(b)


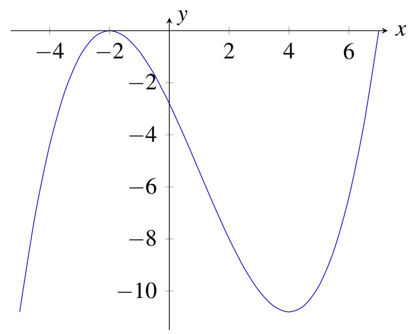
(d)


Exercise 5.7.5 *Given the graph of a function f, determine the relative maxima and relative minima, if any.*

(a)


(c)


(b)


(d)


Exercise 5.7.6 *A subsidiary of ThermoMaster manufactures an indoor-outdoor thermometer. Management estimates that the profit (in dollars) realizable by the company for the manufacture and sale of x units of thermometers each week is*

$$P(x) = -0.001x^2 + 8x - 5000.$$

Find the intervals where the profit function P is increasing and the intervals where P is decreasing.

Exercise 5.7.7 *Based on data from the Central Provident Fund of a certain country (a government agency similar to the Canada Pension Plan), the estimated cash in the fund in 1995 is given by*

$$A(t) = -96.6t^4 + 403.6t^3 + 660.9t^2 + 250 \quad 0 \leq t \leq 5$$

where $A(t)$ is measure in billions of dollars and t is measured in decades, with $t = 0$ corresponding to 1995. Find the interval where A is increasing and the interval where A is decreasing and interpret your results. Hint: Use the quadratic formula.

Exercise 5.7.8 *Sales in the Web-hosting industry are projected to grow in accordance with the function*

$$f(t) = -0.05t^3 + 0.56t^2 + 5.47t + 7.5 \quad 0 \leq t \leq 6$$

where $f(t)$ is measured in billions of dollars and t is measured in years, with $t = 0$ corresponding to 1999.

(a) Find the interval where f is increasing and the interval where f is decreasing.

(b) What does your result tell you about the sales in the Web-hosting industry from 1999 through 2005?

Exercise 5.7.9 *According to a study conducted in 1997, the number of subscribers (in thousands) to the Canadian cellular market in the next 6 years is approximated by the function*

$$N(t) = 0.09444t^3 - 1.44167t^2 + 10.65695t + 52 \quad 0 \leq t \leq 6$$

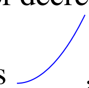
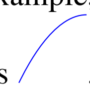
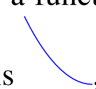
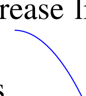
where t is measured in years, with $t = 0$ corresponding to 1997.

(a) *Find the interval where N is increasing and the interval where N is decreasing.*

(b) *What does your result tell you about the number of subscribers in the Canadian cellular market in the years under consideration?*

Exercise 5.7.10 *Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. Show that f has exactly one critical point using the First Derivative Test. Give conditions on a and b which guarantee that the critical point will be a maximum. It is possible to see this without using calculus at all; explain.*

5.7.2. Concavity and Inflection Points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, when $f'(x) > 0$, $f(x)$ is increasing. However, a function can increase like this , or like this . And similarly, a function can decrease like this , or like this . How can you determine which way it is? For example, as you can see in Figure 5.19, the function $y = x^3$ increases as we move from left to right, but the sections of the curve defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ increase in different ways. As we approach the origin from the left along the curve, the curve turns clockwise and stays below its tangents as shown in Figure 5.19. In other words, the slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away to the right of the origin along the curve, the curve turns counterclockwise and stays above its tangents as shown in Figure 5.19. In other words, the slopes of the tangents are increasing on the interval $(0, \infty)$. This manner of curving defines the **concavity** of the curve.

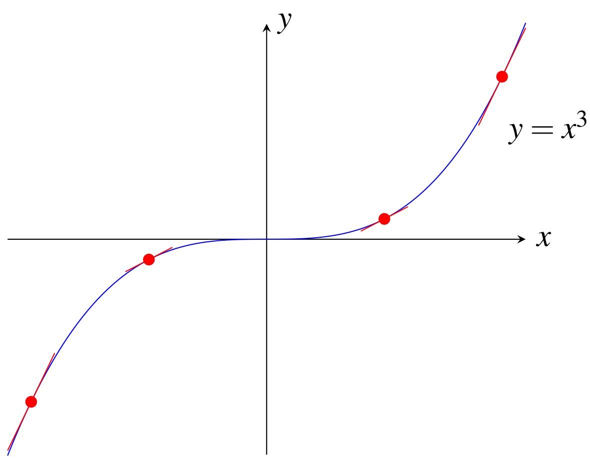


Figure 5.19: The graph of $y = x^3$ is concave down $(-\infty, 0)$, and concave up on $(0, \infty)$.

Definition 5.75: Concavity

Suppose that $y = f(x)$ is a differentiable function on the open interval (a,b), then f is

1. **concave up** *on (a,b) if f' is increasing on (a,b); and*
2. **concave down** *on (a,b) if f' is decreasing on (a,b).*

Suppose that a function is twice differentiable. This means that we can get information from the sign of f'' even when f' is not zero. Suppose that $f''(a) > 0$. This means that near $x = a$, f' is increasing. If $f'(a) > 0$, this means that f slopes up and is getting steeper; if $f'(a) < 0$, this means that f slopes down and is getting less steep. The two situations of concave up are shown in Figure 5.20.

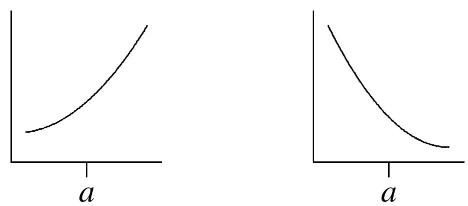


Figure 5.20: $f''(a) > 0$: $f'(a)$ positive and increasing, $f'(a)$ negative and increasing.

Now suppose that $f''(a) < 0$. This means that near $x = a$, f' is decreasing. If $f'(a) > 0$, this means that f slopes up and is getting less steep; if $f'(a) < 0$, this means that f slopes down and is getting steeper. The two situations of concave down are shown in Figure 5.21.

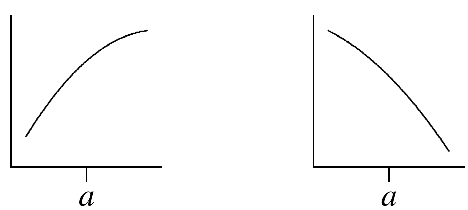


Figure 5.21: $f''(a) < 0$: $f'(a)$ positive and decreasing, $f'(a)$ negative and decreasing.

These observations lead us to the following theorem on concavity.

Theorem 5.76: Second Derivative Test for Concavity

Suppose that $y = f(x)$ is a twice differentiable function on the open interval (a,b).

1. *If $f''(x) > 0$ for all $x \in (a,b)$, then the graph of f is concave up on (a,b).*
2. *If $f''(x) < 0$ for all $x \in (a,b)$, then the graph of f is concave down on (a,b).*

We now have all the tools to determine the intervals, where the curve of a function is either concave up or concave down.

Guideline for Determining Intervals of Concavity

Suppose that $y = f(x)$ is a twice differentiable function on its domain.

1. Find all critical points of $y = f'(x)$ (note that these are not necessarily the same as for the function f) and all x-values where f' is undefined.

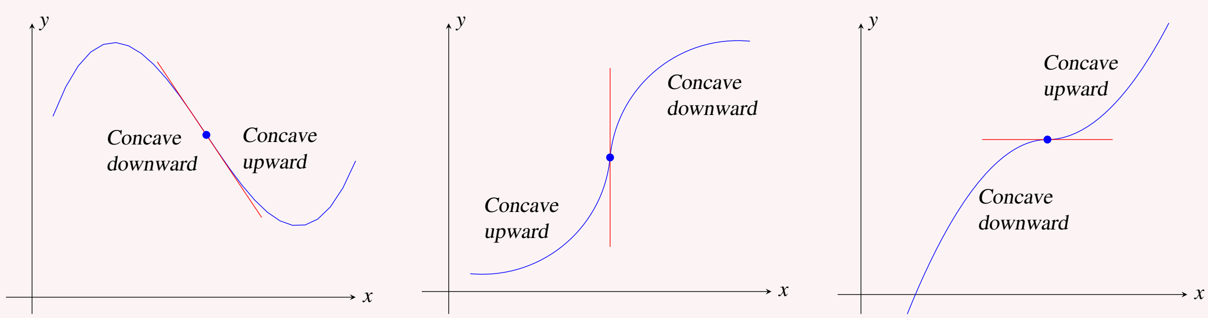
2. These x-values section the domain into open intervals.

3. Choose a number n in each interval. Then apply the Second Derivative Test for Concavity to these numbers n to decide if f is concave up or concave down on these intervals.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called **inflection points**.

Definition 5.77: Inflection Point

Any value of x in the domain of f where the tangent line exists and where the concavity changes is called an **inflection point**.



If the concavity changes from up to down at $x = a$, f'' changes from positive to the left of a to negative to the right of a, and usually $f''(a) = 0$. We can identify such points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points. Note that it is possible that $f''(a) = 0$ but the concavity is the same on both sides; $f(x) = x^4$ at $x = 0$ is an example. These observations lead us to the following guideline for determining inflection points.

Guideline for Determining Inflection Points

Suppose that $y = f(x)$ is a twice differentiable function on its domain.

1. Compute $f'(x)$ and $f''(x)$.

2. Find all critical points c of the function $y = f'(x)$ (note that these are not necessarily the same as for the function f).

3. Determine the sign of f'' to the left and to the right of c.

4. If there is a change in signs, then c is an inflection point.

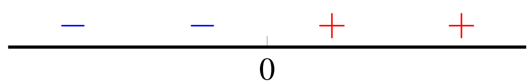
Example 5.78: Concavity and Inflection Point

Describe the concavity of $f(x) = x^3 - x$ using intervals, and determine if there are any inflection points.

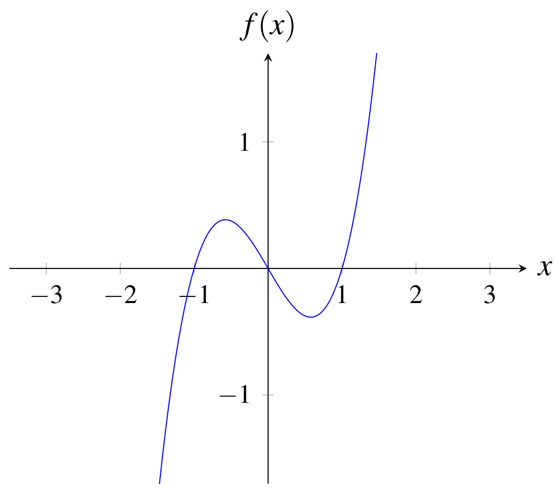
Solution. We find the first two derivatives of f to be

$$f'(x) = 3x^2 - 1 \text{ and } f''(x) = 6x.$$

Next, we create the sign diagram for f'':



From this sign diagram, we see that f is concave up for $x \in (-\infty, 0)$ and concave down for $x \in (0, \infty)$. The concavity changes as we move through $x = 0$ and so the point $(0, f(0)) = (0, 0)$ is an inflection point of f. The graph of f is shown below.



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Example 5.79: Concavity and Inflection Point

Describe the concavity of

$$f(x) = \frac{1}{x^2 + 1}$$

using intervals, and determine if there are any inflection points.

Solution. We find the first and second derivatives of f to be

$$f'(x) = \frac{d}{dx}(x^2 + 1)^{-1} = -2x(x^2 + 1)^{-2} = -\frac{2x}{(x^2 + 1)^2},$$

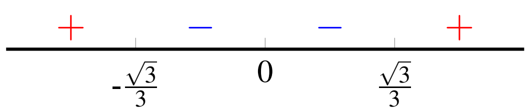
and

$$\begin{aligned}
f''(x) &= \frac{(x^2+1)^2(-2)+2(2x)(x^2+1)(2x)}{(x^2+1)^4} \\
&= \frac{(x^2+1)(-2(x^2+1)+8x^2)}{(x^2+1)^4} = \frac{(x^2+1)(6x^2-2)}{(x^2+1)^4} \\
&= \frac{2(3x^2-1)}{(x^2+1)^3}.
\end{aligned}$$

The critical points of f'' will occur when $f''=0$ since f'' is continuous everywhere. We then solve

$$\begin{aligned}
3x^2-1 &= 0 \\
x^2 &= \frac{1}{3} \\
x &= \pm\sqrt{3}.
\end{aligned}$$

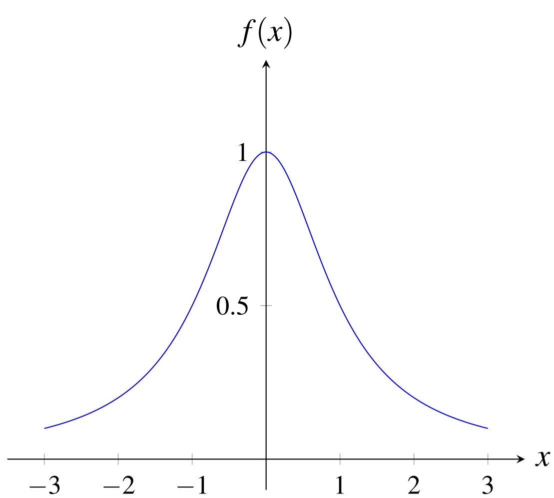
The sign diagram for f'' is shown below:



Therefore, f is concave upward on $(-\infty,\sqrt{3}/3)\cup(\sqrt{3}/3,\infty)$ and concave downward on $(-\sqrt{3}/3,\sqrt{3}/3)$. Furthermore,$f''(x)$ changes sign as we move across the points $x=\pm\sqrt{3}/3$.

$$f\left(\pm\frac{\sqrt{3}}{3}\right) = \frac{3}{4},$$

and so the points $(\pm\sqrt{3}/3, 3/4)$ are inflection points of f. The graph of f is shown below.



♣

Example 5.80: Concavity

Describe the concavity of

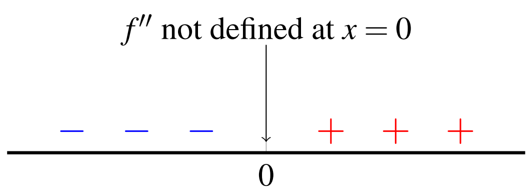
$$f(x) = x + \frac{1}{x}$$

using intervals.

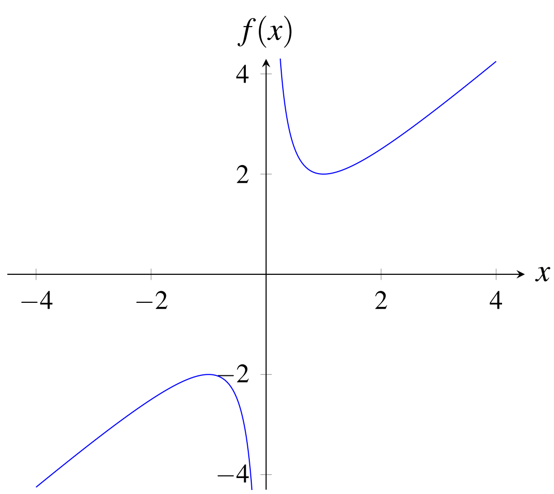
Solution. We compute

$$f'(x) = 1 - \frac{1}{x^2} \quad \text{and} \quad f''(x) = \frac{2}{x^3}.$$

From the sign diagram for f'',



we see that f is concave downward for $x \in (-\infty, 0)$ and concave upward for $x \in (0, \infty)$. The graph of f is shown below.



♣

Law of Diminishing Returns

The **law of diminishing returns** in economics is related to concavity. The graph of the function shown in Figure 5.22 gives the output y from a given input x. For example, if the input were advertising costs for some product, then the output might be the corresponding revenue from sales. The graph shows an inflection point at $(c, f(c))$. For $x < c$, the graph is concave up, which means that the rate of change of the slope is increasing. Therefore, the output y is increasing at a faster rate with each additional dollar spent.

For $x > c$, the graph is concave down, which means that the rate of change of the slope is decreasing. Therefore, the output y is decreasing at a faster rate with each additional dollar spent, i.e. producing a diminishing return. Therefore, the point $(c, f(c))$ is referred to as the **point of diminishing return**. Another example for diminishing returns comes from agriculture, where there is a fixed amount of land, machinery, etc.. Here, adding workers increases production a lot at first, then less and less with each additional worker.

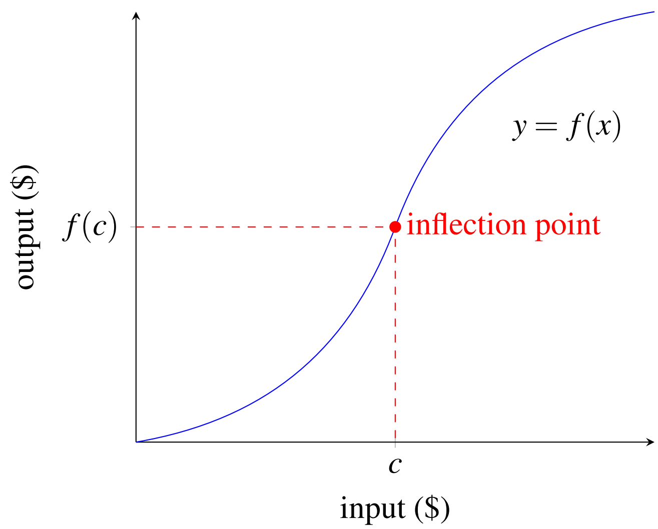


Figure 5.22: The function $y = f(x)$ is concave up on $(0,c)$, concave down on (c,∞) and has an inflection point at $(c, f(c))$.

Example 5.81: Consumer Price Index

Suppose a certain country's consumer price index (CPI) between the years 1990 and 1999 is approximated by

$$I(t) = -0.2t^3 + 3t^2 + 50 \quad 0 \leq t \leq 9.$$

Determine the inflection point of I.

Solution. We calculate the first and second derivatives of I:

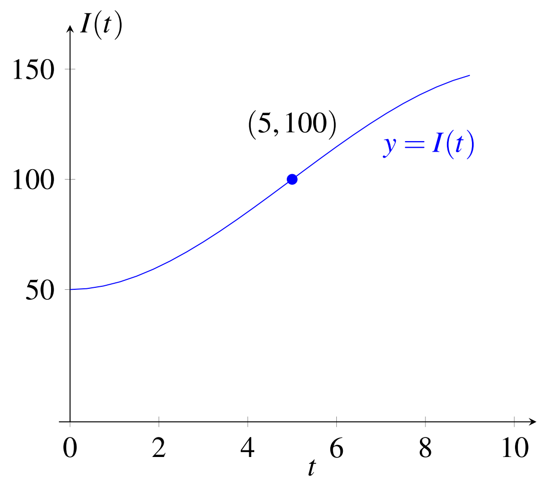
$$I'(t) = -0.6t^2 + 6, \text{ and}$$

$$I''(t) = -1.2t + 6 = -1.2(t-5).$$

So I'' is continuous everywhere and $I''(t) = 0 \implies t = 5$. This gives the only critical point of I''. Since

$$I''(t) > 0 \text{ for } t < 5, \text{ and } I''(t) < 0 \text{ for } t > 5,$$

the point $(5, I(5)) = (5, 100)$ is an inflection point of I. The graph of I is sketched below:



This result reveals that the inflation rate only started decelerating when $t = 5$, that is, in the year 1995. ♣

Exercises for Section 5.7.2

Exercise 5.7.11 *Describe the concavity of the functions below.*

(a) $y = x^2 - x$

(b) $y = 2 + 3x - x^3$

(c) $y = x^3 - 9x^2 + 24x$

(d) $y = x^4 - 2x^2 + 3$

(e) $y = 3x^4 - 4x^3$

(f) $y = (x^2 - 1)/x$

(g) $y = 3x^2 - (1/x^2)$

(h) $y = \sin x + \cos x$

(i) $y = 4x + \sqrt{1-x}$

(j) $y = (x+1)/\sqrt{5x^2+35}$

(k) $y = x^5 - x$

(l) $y = 6x + \sin 3x$

(m) $y = x + 1/x$

(n) $y = x^2 + 1/x$

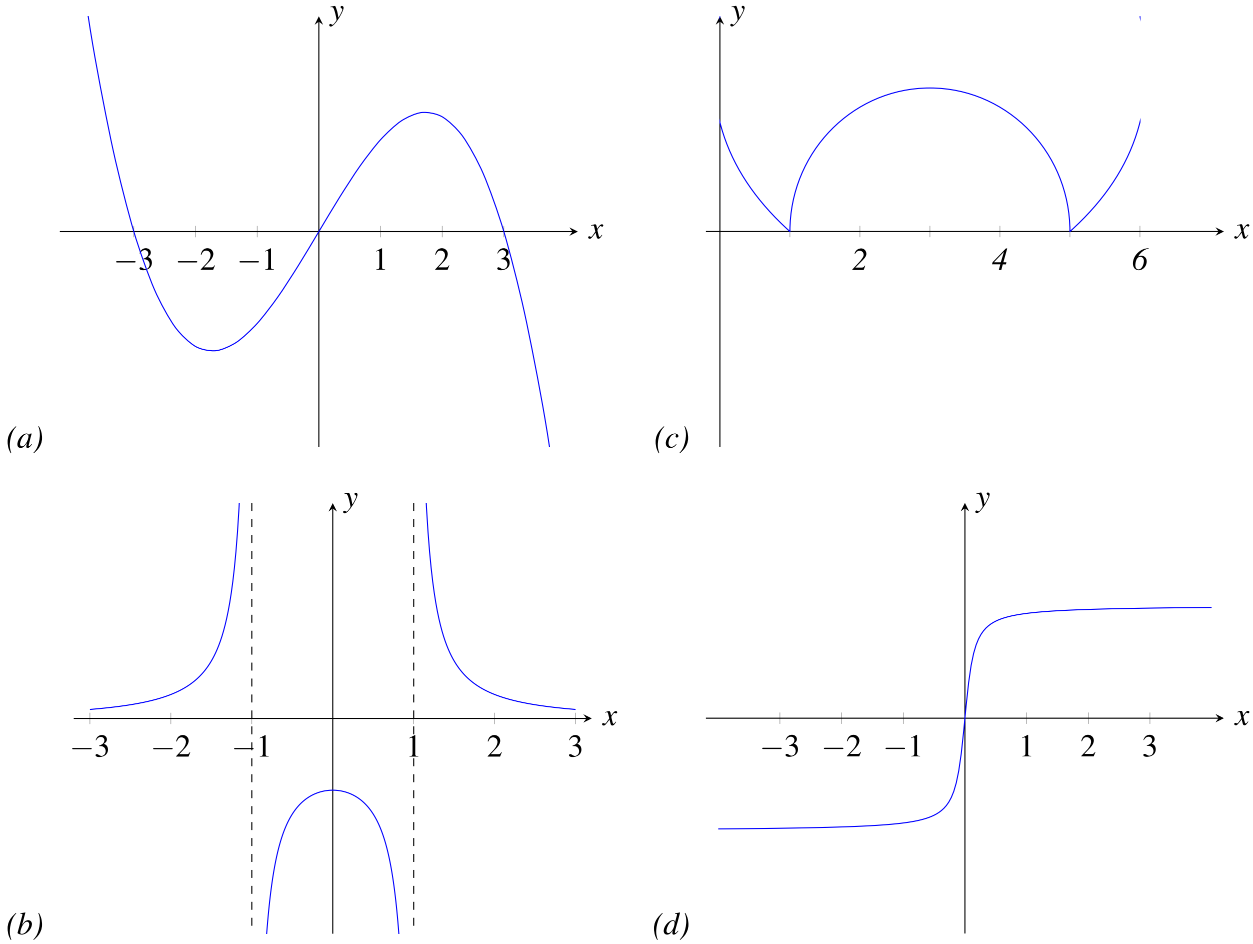
(o) $y = (x+5)^{1/4}$

(p) $y = \tan^2 x$

(q) $y = \cos^2 x - \sin^2 x$

(r) $y = \sin^3 x$

Exercise 5.7.12 *Describe the concavity of the graphs shown below using intervals, and determine if there are any inflection points.*



Exercise 5.7.13 *Determine if there are any inflection points.*

(a) $f(x) = x^3 + 4$

(b) $f(t) = 6t^3 - 18t^2 + 12t - 15$

(c) $g(x) = 3x^4 - 4x^3 + 1$

(d) $h(t) = \sqrt[3]{t} + 12$

(e) $p(q) = (q-1)^3 - 3$

(f) $h(s) = \frac{2}{1+s^2}$

(g) $p(q) = qe^{-4q}$

(h) $f(a) = \cos(3a) - \frac{1}{2}$

Exercise 5.7.14 *Identify the intervals on which the graph of the function* $f(x) = x^4 - 4x^3 + 10$ *is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.*

Exercise 5.7.15 *Describe the concavity of* $y = x^3 + bx^2 + cx + d$*. You will need to consider different cases, depending on the values of the coefficients.*

Exercise 5.7.16 *Let n be an integer greater than or equal to two, and suppose f is a polynomial of degree n. How many inflection points can f have? Hint: Use the Second Derivative Test and the* **Fundamental Theorem of Algebra**.

Exercise 5.7.17 *A retailer determines that their revenue R as a function of the amount q (in thousands of dollars) spent on advertising can be approximated by*

$$R(q) = -0.003q^3 + 1.35q^2 + 2q + 8000 \quad 0 \leq q \leq 400$$

thousands of dollars.

(a) *Determine the concavity of R using intervals and determine if there are any inflection points.*

(b) *If the retailer currently spends $140,000 on advertising, should they consider increasing this amount?*

Exercise 5.7.18 *A grocery store determines that their sales S as a function of the amount q (in thousands of dollars) spent on advertising can be approximated by*

$$S(q) = -0.002q^3 + 0.6q^2 + q + 500 \quad 0 \leq q \leq 200$$

thousands of dollars. Determine if there are any inflection points and, if there is, discuss its significance.

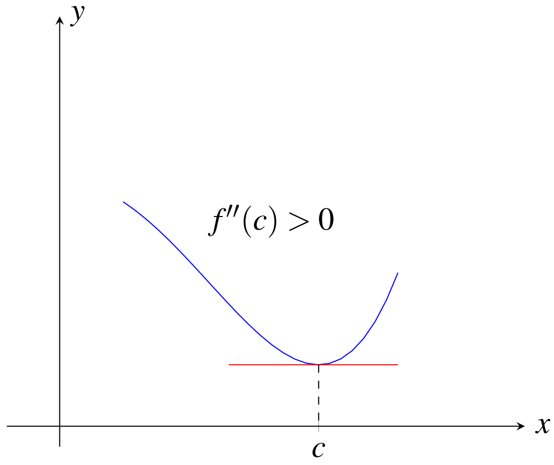
Exercise 5.7.19 *A company wants to determine whether or not their cost-cutting measures (implemented at time $t \equiv 0$) will be effective. Suppose the profit P (in hundreds of dollars) of the company over the next 8 years can be approximated by*

$$P(t) = t^3 - 9t^2 + 40t + 50 \quad 0 \leq t \leq 8.$$

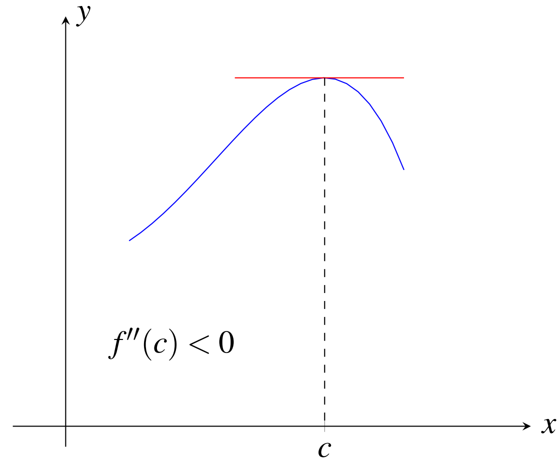
By studying the concavity of P, deduce whether or not these measures will be effective.

5.7.3. The Second Derivative Test for Relative Extrema

The basis of the First Derivative Test is that if the derivative changes from positive to negative at a critical point then there is a relative maximum at the point, and similarly for a relative minimum. However, we can also use our knowledge from concavity to test for relative extrema at a critical point. In Figure 5.23a, we observe that the graph of the function f shown here has a relative minimum at the critical point $x = c$ and that the graph is concave upward at that point. Similarly, in Figure 5.23b, we observe that the graph of the function f shown here has a relative maximum at the critical point $x = c$ and that the graph is concave downward at that point. Furthermore, in Section 5.7.2 we just learned that given a twice differentiable function f and a critical point $x = c$ of f the function f is concave upward at $x = c$ if $f''(c) > 0$ and concave downward at $x = c$ if $f''(c) < 0$. We therefore introduce the so-called **Second Derivative Test for relative extrema** as an alternative technique to the First Derivative Test for checking critical points for relative extrema, as long as the second derivative exists.



(a) *f has a relative minimum at $x = c$.*



(b) *f has a relative maximum at $x = c$.*

Figure 5.23

Theorem 5.82: Second Derivative Test for Relative Extrema

Suppose that $y = f(x)$ is a twice differentiable function on its domain.

1. *Compute $f'(x)$ and $f''(x)$.*
2. *Find all critical points c of the function $y = f(x)$ such that $f'(c) = 0$.*
3. *Compute $f''(c)$ for all such points c.*
   (a) *If $f''(c) < 0$, then the graph of f has a relative maximum at c.*
   (b) *If $f''(c) > 0$, then the graph of f has a relative minimum at c.*
   (c) *If $f''(c) = 0$, then the test is inconclusive and we must use some other method to learn about the behaviour of the curve f at c.*

Note: The above theorem indicates that the Second Derivative Test for relative extrema is inconclusive if $f''(c) = 0$ for a critical point c as required, but furthermore, this test also tells us nothing when $f''(c)$ does not exist and also when there are critical points such that $f'(c)$ does not exist. You may wonder why we should bother with the Second Derivative Test. Well, if the second derivative is easy to compute, then this test is an efficient method for finding relative extrema.

Example 5.83: Second Derivative Test for Relative Extrema

Consider again $f(x) = \sin x + \cos x$, with $f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$. Use the Second Derivative Test to determine which critical points are relative maxima or minima.

Solution. Since $f''(\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0$, we know there is a relative maximum at $\pi/4$. Since $f''(5\pi/4) = -(-\sqrt{2}/2) - (-\sqrt{2}/2) = \sqrt{2} > 0$, there is a relative minimum at $5\pi/4$. ♣

Example 5.84: Second Derivative Test for Relative Extrema

Let $f(x) = x^4$ and $g(x) = -x^4$. Classify the critical points of $f(x)$ and $g(x)$ as either maximum or minimum.

Solution. The derivatives for $f(x)$ are $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Zero is the only critical value, but $f''(0) = 0$, so the Second Derivative Test tells us nothing. However, $f(x)$ is positive everywhere except at zero, so clearly $f(x)$ has a relative minimum at zero.

On the other hand, for $g(x) = -x^4$, $g'(x) = -4x^3$ and $g''(x) = -12x^2$. So $g(x)$ also has zero as its only only critical value, and the second derivative is again zero, but $g(x)$ has a relative maximum at zero. ♣

We conclude this section with a summary of the graphical information gained from both the first and second derivatives of a function f. The first derivative tells us about the *monotonicity* of the graph of f, i.e. where the graph goes up (f is increasing) or down (f is decreasing). The second derivative tells us about the *concavity* of the graph of f, i.e. where the graph curves up (f is concave up) or down (f is concave down). Together, these properties tell us *how* the function f is increasing or decreasing as is shown in Table 5.4.

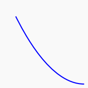
Signs of f' **and** f''	**Properties of** **the Graph of** f	**General Shape** **of the Graph of** f
$f'(x) > 0$ $f''(x) > 0$	f *increasing* f *concave upward*	
$f'(x) > 0$ $f''(x) < 0$	f *increasing* f *concave downward*	
$f'(x) < 0$ $f''(x) > 0$	f *decreasing* f *concave upward*	
$f'(x) < 0$ $f''(x) < 0$	f *decreasing* f *concave downward*	

Table 5.4

Exercises for Section 5.7.3

Exercise 5.7.20 *Find all relative maximum and minimum points by the Second Derivative Test.*

(a) $y = x^2 - x$

(b) $y = 2 + 3x - x^3$

(c) $y = x^3 - 9x^2 + 24x$

(d) $y = x^4 - 2x^2 + 3$

(e) $y = 3x^4 - 4x^3$

(f) $y = (x^2 - 1)/x$

(g) $y = 3x^2 - (1/x^2)$

(h) $y = \cos(2x) - x$

(i) $y = 4x + \sqrt{1-x}$

(j) $y = \dfrac{x+1}{\sqrt{5x^2+35}}$

(k) $y = x^5 - x$

(l) $y = 6x + \sin 3x$

(m) $y = x + 1/x$

(n) $y = x^2 + 1/x$

(o) $y = (x+5)^{1/4}$

(p) $y = \tan^2 x$

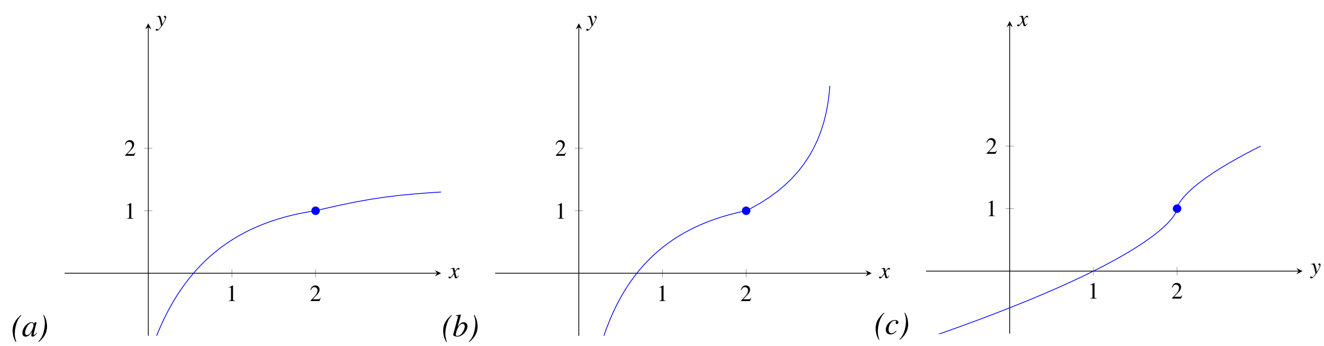
(q) $y = \cos^2 x - \sin^2 x$

(r) $y = \sin^3 x$

Exercise 5.7.21 *Suppose a function $y = f(x)$ has the following properties:*

$$f(2) = 1, \quad f'(2) > 0, \quad \text{and } f''(2) < 0.$$

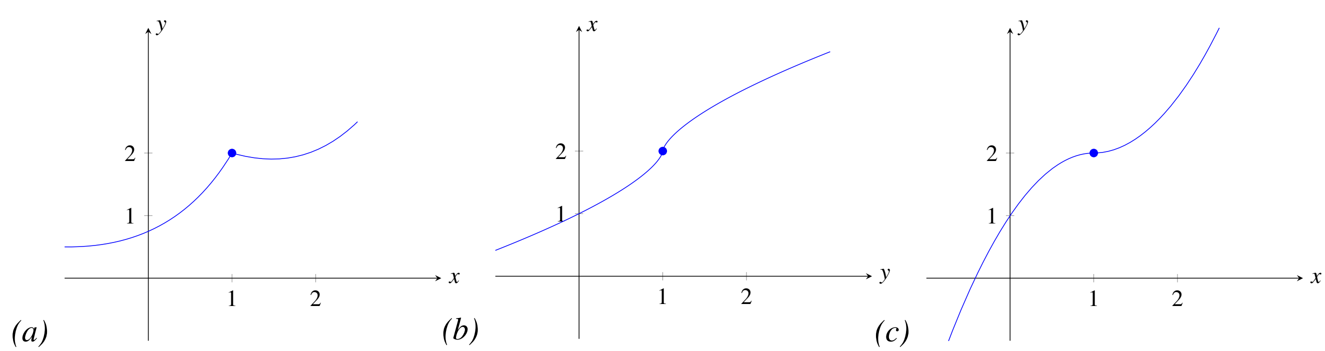
Decide which graph below corresponds to the function f.



Exercise 5.7.22 *Suppose a function $y = f(x)$ has the following properties:*

$$f(1) = 2, \quad f'(x) > 0 \quad \text{on} \quad (-\infty, 1) \cup (1, \infty), \quad \text{and} \quad f''(1) = 0.$$

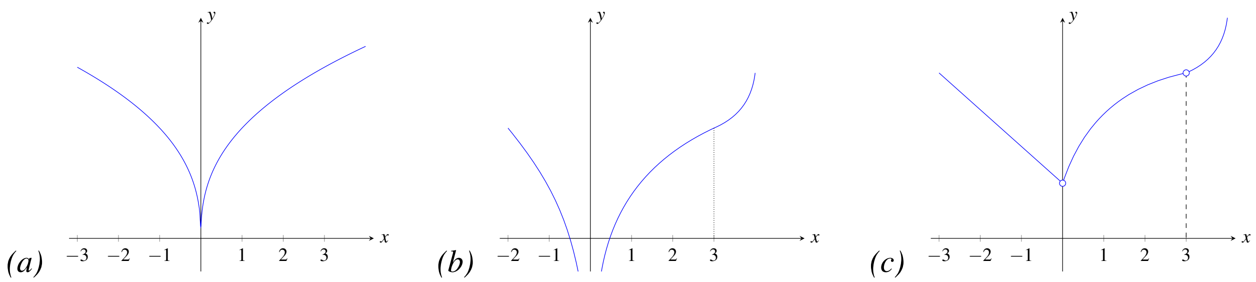
Decide which graph below corresponds to the function f.



Exercise 5.7.23 *Suppose a function $y = f(x)$ has the following properties:*

$$f'(0) \text{ undefined}, \quad f'(x) < 0 \quad \text{on} \quad (-\infty, 0), \quad f''(x) < 0 \quad \text{on} \quad (0, 3),$$

and f has an inflection point at $x = 3$. Decide which graph below corresponds to the function f.



Exercise 5.7.24 *A busy coffee shop determines that the number N of transactions processed t hours after opening at* 6 *am can be described by*

$$N(t) = -t^3 + 5t^2 + 25t \qquad 0 \leq t \leq 8.$$

What is the shop's busiest hour?

Exercise 5.7.25 *A manufacturer determines that the daily cost C (in dollars) of producing q units is given by*

$$C(q) = q^3 - 30q^2 + 300q + 50.$$

Determine if there are any inflection points and interpret your result.

Exercise 5.7.26 *A busy coffee shop determines that the number N of transactions processed t hours after opening at* 6 *am can be described by*

$$N(t) = -t^3 + 5t^2 + 25t \qquad 0 \leq t \leq 8.$$

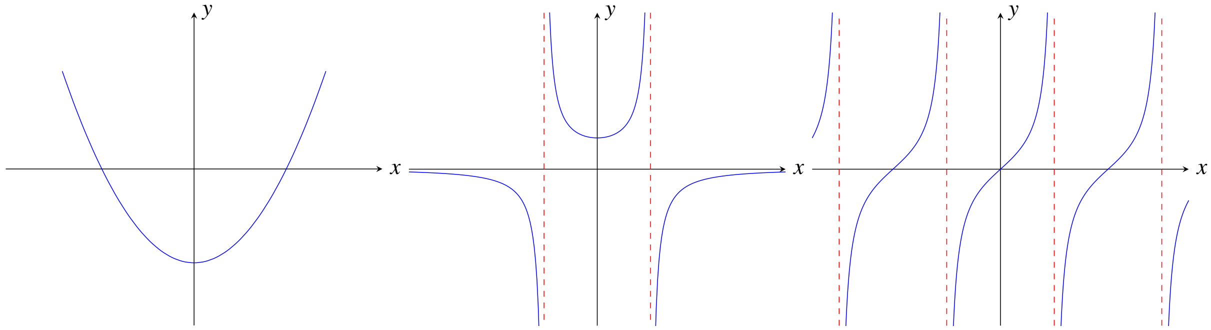
(a) *Describe the rate of change of the number of transactions between 6 am and 11 am, and between 11 am and 2 pm.*

(b) *When is the rate of change of the number of transactions maximal?*

5.7.4. Asymptotes and Other Things to Look For

Vertical Asymptotes Revisited

Vertical Asymptotes were introduced in Section 3.5.3. Since they play an important role in curve sketching, we provide a short summary as a reminder of the idea. A vertical asymptote is a place where the function approaches infinity, typically because the formula for the function has a denominator that becomes zero. For example, the reciprocal function $f(x) = 1/x$ has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.). Whenever the formula for a function contains a denominator it is worth looking for a vertical asymptote by checking to see if the denominator can ever be zero, and then checking the limit at such points. Note that there is not always a vertical asymptote where the derivative is zero: $f(x) = (\sin x)/x$ has a zero denominator at $x = 0$, but since $\lim\limits_{x\to 0}(\sin x)/x = 1$ there is no asymptote there. Note also that the graph of a function can have any number of vertical asymptotes from none (e.g., any polynomial function) to infinitely many (e.g., tangent function), see Figure 5.28 for some examples.

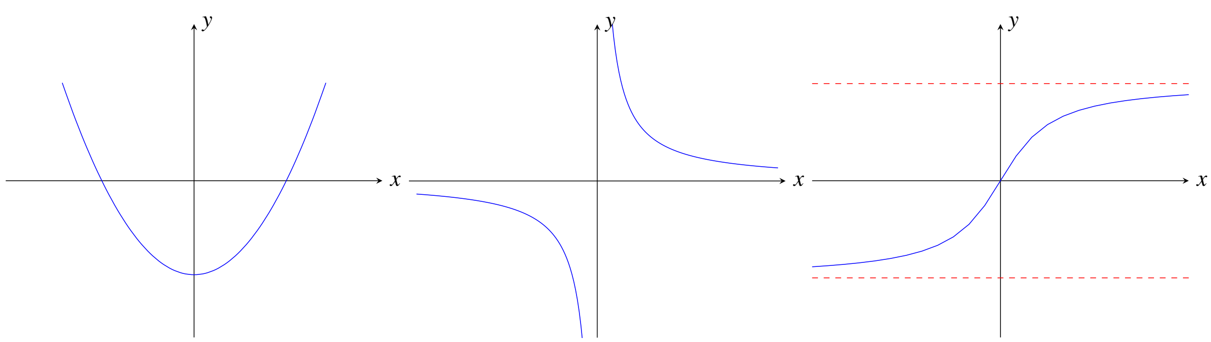


(a) *no vertical asymptote* (b) *two vertical asymptotes* (c) *infinitely many asymptotes*

Figure 5.24

Horizontal Asymptotes Revisited

In Section 3.5.4 we discussed *horizontal asymptotes*. These too are a fundamental feature when sketching the graph of a function, and so we offer a short summary about them. A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). Hence, the graph of a function can have at most two horizontal asymptotes. Horizontal asymptotes can be identified by computing the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For example, a polynomial function has no horizontal asymptotes as shown in Figure 5.25a. Since $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$, the line $y = 0$ (that is, the x-axis) is a horizontal asymptote in both directions; for example, see the reciprocal function in Figure 5.25b. The arctangent is an example of a function with two horizontal asymptotes (Figure 5.25c).



(a) *no horizontal asymptote* (b) *one horizontal asymptote* (c) *two horizontal asymptotes*

Figure 5.25

Slant Asymptotes and other Asymptotic Behaviour

Some functions have straight asymptotes that are neither horizontal nor vertical, but slanted.These too are a fundamental feature when sketching the graph of a function, and so we offer a short summary about *slant asymptotes* from our introduction in Section 3.5.5. A slant asymptote is again a line that $f(x)$ gets closer and closer to as x approaches ∞ (or as x approaches -∞). If $y = mx + b$ describes a slant asymptote, then $\lim_{x\to\infty}(f(x) - (mx+b)) = 0$ or $\lim_{x\to-\infty}(f(x) - (mx+b)) = 0$. In the case of rational functions, slant asymptotes occur when the degree of the polynomial in the numerator is one more than the degree of the polynomial in the denominator, see Figure 5.26.

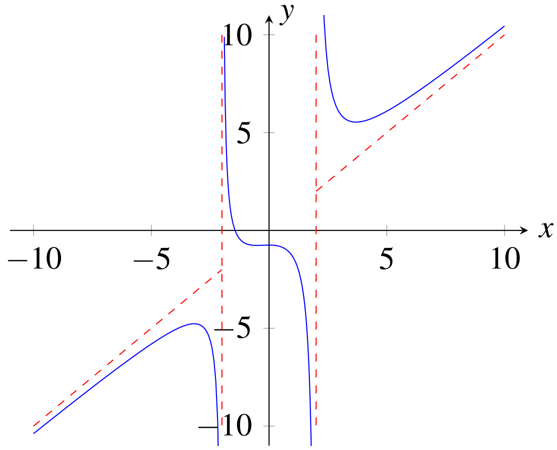


Figure 5.26: The function $f(x) = \frac{x^3+3}{x^2-4}$ has a slant asymptote given by $y = x$ and vertical asymptotes at $x = \pm 2$.

If the degree between the polynomials of the numerator and denominator are higher than one, say n, then the rational function exhibits asymptotic behaviour towards a polynomial with degree n. This polynomial can be found by long division and taking a similar limit approach as for slant asymptotes, but this is beyond the scope of this material. Other functions could also exhibit different asymptotic behaviour, but such asymptotes are somewhat more difficult to identify and we will ignore them.

Endpoints or Other Special Points of Domain

If the domain of the function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$ (see Figure 5.27). In this case we might also identify this behaviour because when $x = \pm r$ the denominator of the function is zero.

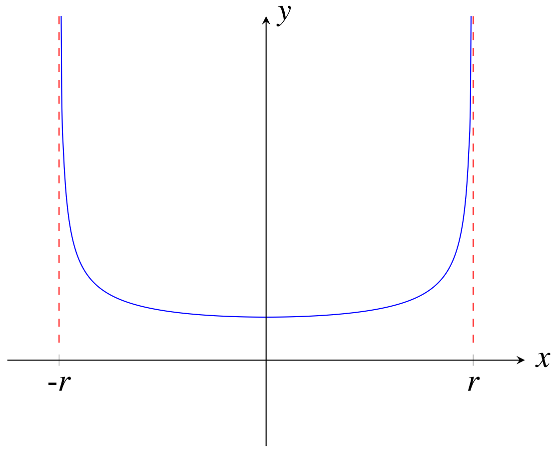
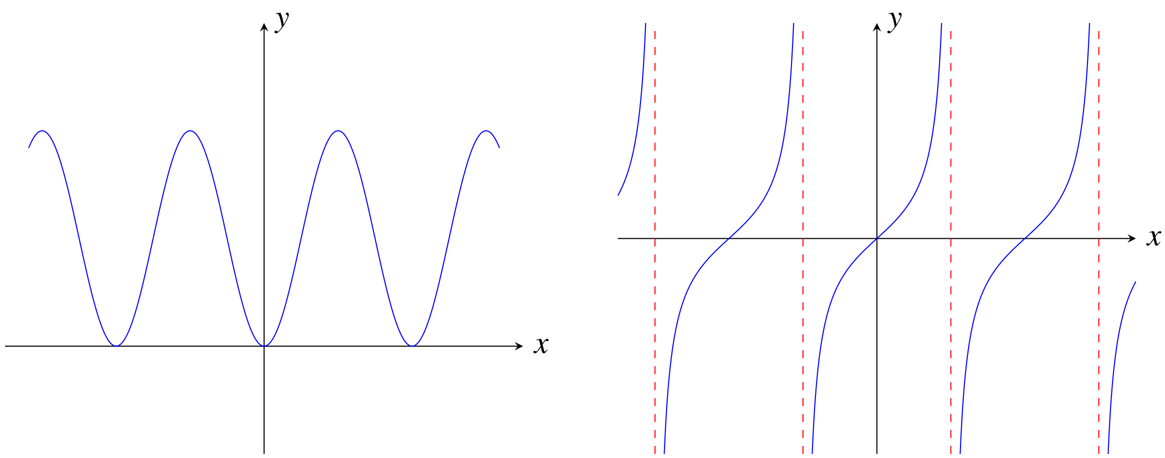


Figure 5.27: $y = f(x) = 1/\sqrt{r^2 - x^2}$ **with asymptotes at endpoints of domain.**

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

Function Symmetry

Finally, it is worthwhile to notice any **symmetry**. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an **even function**. Its graph is symmetric with respect to the y-axis. Some examples of even functions are: x^n when n is an even number, $\cos x$, and $\sin^2 x$, see Figure 5.28a. On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an **odd function**. Its graph is symmetric with respect to the origin. Some examples of odd functions are: x^n when n is an odd number, $\sin x$, and $\tan x$, see Figure 5.28b. Of course, most functions are neither even nor odd, and do not have any particular symmetry.



(a) $y = \sin^2 x$ *is an even function*

(b) $y = \tan x$ *is an odd function*

Figure 5.28

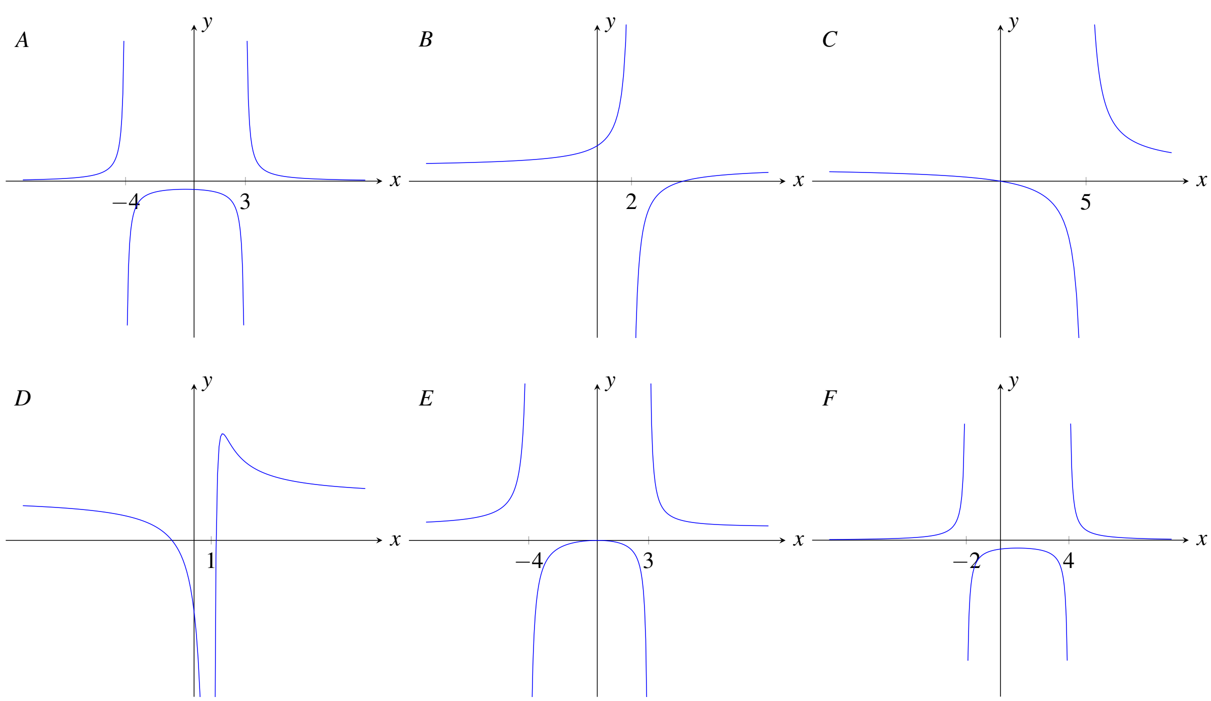
Exercises for Section 5.7.4

Exercise 5.7.27 *State the domain and determine the vertical and horizontal asymptotes, if any, of the following functions.*

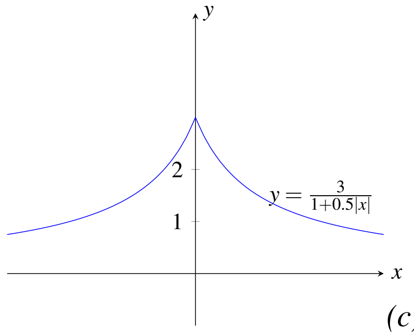
(a) $f(x) = \dfrac{9x-1}{x^2-9}$ (b) $f(x) = \dfrac{8x^2+9}{x^2-16}$ (c) $f(x) = \dfrac{6(x^2+10)}{x^2+2x-15}$

Exercise 5.7.28 *Match the graphs labelled A through F with one of the rational functions labelled a through f.*

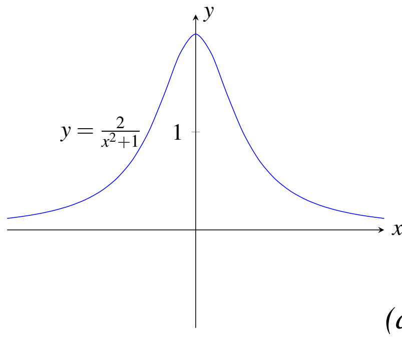
$$a(x) = \frac{x^2-25}{x^2+3x-10} \quad b(x) = \frac{5}{x^2-2x-8} \quad c(x) = \frac{3x^2-5}{x^2-2x+1}$$
$$d(x) = \frac{x^2}{x^2+x-12} \quad e(x) = \frac{7}{x^2+x-12} \quad f(x) = \frac{x^2-2x}{x^2-7x+10}$$



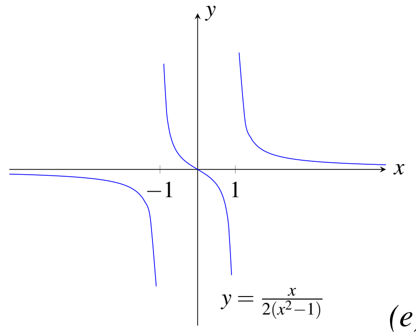
Exercise 5.7.29 *Draw and label the vertical and horizontal asymptotes, if any, directly on the graphs.*



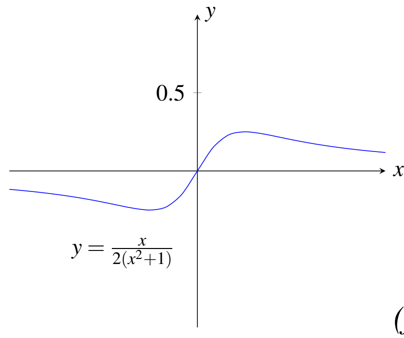
(a)



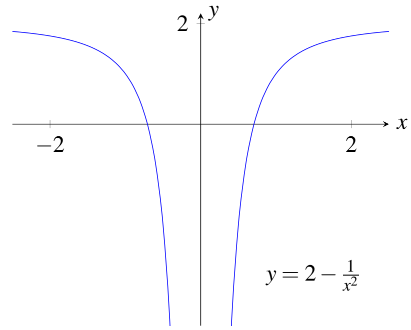
(b)



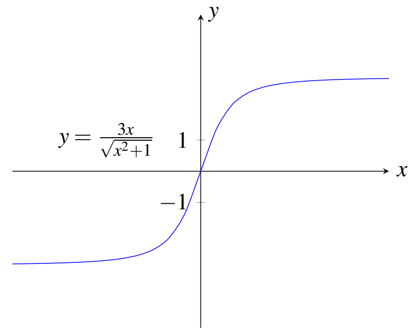
(c)



(d)



(e)



(f)

Exercise 5.7.30 *Determine the horizontal and vertical asymptotes of the following functions.*

(a) $f(x) = \dfrac{1}{x}$

(b) $f(q) = -\dfrac{2}{q^2}$

(c) $g(t) = \dfrac{t-1}{t+1}$

(d) $h(s) = s^3 - 3s^2 + s + 1$

(e) $f(t) = \dfrac{t^2}{t^2-9}$

(f) $f(x) = \dfrac{3x}{x^2-x-6}$

(g) $h(t) = 2 + \dfrac{5}{(t-2)^2}$

(h) $f(s) = \dfrac{s^2-2}{s^2-4}$

(i) $g(x) = \dfrac{x^3-x}{x(x+1)}$

Exercise 5.7.31 *Find the slant asymptote of the following functions:*

(a) $f(x) = \dfrac{5x^2-3x+1}{x+2}$ *(b)* $f(x) = \dfrac{x^2}{x-1}$ *(c)* $f(x) = \dfrac{x^2+3x+2}{x-1}$ *(d)* $f(x) = \dfrac{x^2-5x+4}{x-3}$

5.7.5. Summary of Curve Sketching

The following is a guideline for sketching a curve $y = f(x)$ by hand. Each item may not be relevant to the function in question, but utilizing this guideline will provide all information needed to make a detailed sketch of the function.

Guideline for Curve Sketching

Given a function $y = f(x)$, follow these steps to sketch the graph of f.

1. Determine the domain, and the x-values $u_1, u_2, ..., u_m$ where the function is undefined. Graph a set of coordinate axes that is suitable.
2. Find all intercepts. Graph them, if any.
3. Determine if f has any symmetry.
4. Determine asymptotes:
   a. Horizontal Asymptotes ($x \to \infty$ and $x \to -\infty$). Graph them, if any.
   b. Vertical Asymptotes ($x \to u_i$ for $i = 1, 2, ..., m$). Graph them, if any.
5. Calculate f':
   a. Determine the critical points $c_1, c_2, ..., c_n$ of f and use these numbers along with $u_1, u_2, ..., u_m$ to create a sign chart for f'.
   b. Determine the intervals of increase and decrease.
   c. Apply the First Derivative Test to determine relative extrema. Graph them, if any.
6. Calculate f'':
   a. Determine the critical points $c'_1, c'_2, ..., c'_n$ of f' and use these numbers along with $u_1, u_2, ..., u_m$ to create a sign chart for f''.
   b. Determine the intervals of concave up and concave down.
   c. Determine all inflection points. Graph them, if any.
7. Use both sign charts to complete the sketch of the graph and don't forget to label the graph with relevant information.

Example 5.85: Curve Sketching

Sketch the graph of the function $f(x) = \dfrac{2x^2}{x^2 - 1}$.

Solution.

1. The domain is $\{x : x^2 - 1 \neq 0\} = \{x : x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
2. There is an x-intercept at $x = 0$. The y intercept is $y = 0$.
3. $f(-x) = f(x)$, so f is an even function (symmetric about y-axis)
4. $\lim\limits_{x \to \pm\infty} \dfrac{2x^2}{x^2 - 1} = \lim\limits_{x \to \pm\infty} \dfrac{2}{1 - 1/x^2} = 2$, so $y = 2$ is a horizontal asymptote.

   Now the denominator is 0 at $x = \pm 1$, so we compute:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = +\infty, \quad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty, \quad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = +\infty.$$

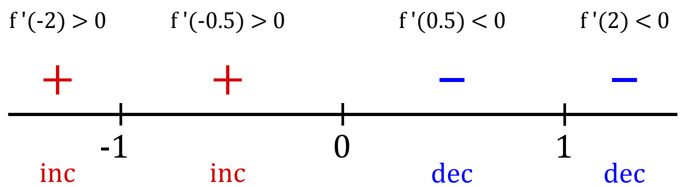
So the lines $x = 1$ and $x = -1$ are vertical asymptotes.

5. For critical values we take the derivative:

$$f'(x) = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

Note that $f'(x) = 0$ when $x = 0$ (the top is zero). Also, $f'(x) = DNE$ when $x = \pm 1$ (the bottom is zero). As $x = \pm 1$ is *not* in the domain of $f(x)$, the only critical point is $x = 0$ (recall that to be a critical point we need it to be in the domain of the original function).

Drawing a number line and including *all* of the split points of $f'(x)$ we have:



Thus f is increasing on $(-\infty, -1) \cup (-1, 0)$ and decreasing on $(0, 1) \cup (1, \infty)$.

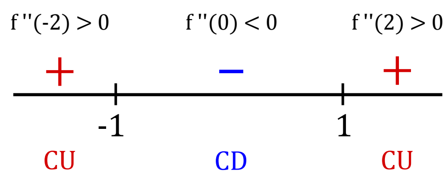
By the First Derivative Test, $x = 0$ is a relative max.

6. For possible inflection points we take the second derivative:

$$f''(x) = \frac{12x^2 + 4}{(x^2-1)^3}$$

The top is never zero. Also, the bottom is only zero when $x = \pm 1$ (neither of which are in the domain of $f(x)$). Thus, there are no possible inflection points to consider.

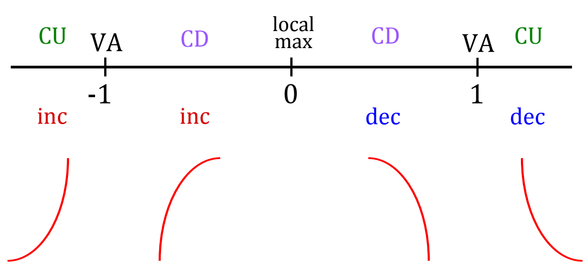
Drawing a number line and including *all* of the split points of $f''(x)$ we have:



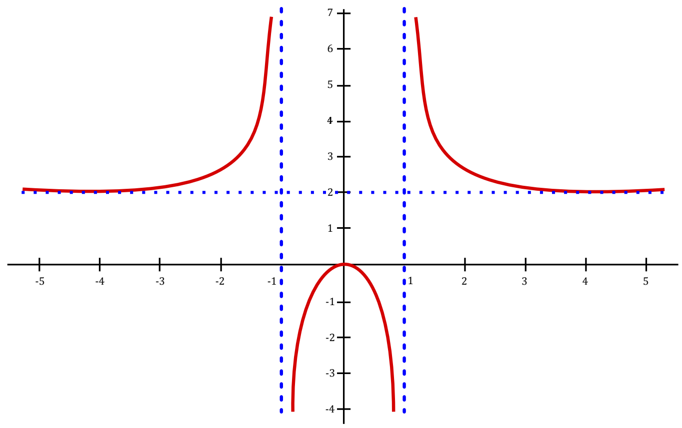
Hence f is concave up on $(-\infty, -1) \cup (1, \infty)$, concave down on $(-1, 1)$.

7. We put this information together and sketch the graph.

We combine some of this information on a single number line to see what *shape* the graph has on certain intervals:



Note that there is a horizontal asymptote at $y = 2$ and that the curve has x-int of $x = 0$ and y-int of $y = 0$. Therefore, a sketch of $f(x)$ is as follows:



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Example 5.86: Curve Sketching

Sketch the graph of the function

$$f(x) = x^3 - 6x^2 + 9x + 2.$$

Solution. Obtain the following information on the graph of f.

1. The domain of f is $(-\infty, \infty)$.

2. By setting $x = 0$, we find that the y-intercept is 2. The x-intercept is found by setting $y = 0$, which in this case leads to a cubic equation. Since the solution is not readily found, we will not use this information.

3. Since f is a cubic polynomial, we expect odd symmetry. This will become more obvious once we analyze f' and f''.

4. We now look for any asymptotes of f:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(x^3 - 6x^2 + 9x + 2\right) = \infty$$

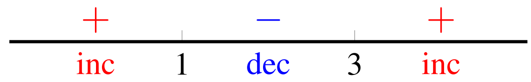
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(x^3 - 6x^2 + 9x + 2\right) = -\infty$$

   We see that f decreases without bound as x decreases and increases without bound as x increases. Therefore, f has no horizontal asymptotes. Since f is a polynomial,there are no vertical asymptotes.

5.

$$f'(x) = 3x^2 - 12x + 9 = 3(x-3)(x-1)$$

   Setting $f'(x) = 0$ gives $x = 1$ or $x = 3$ as our only critical points. The following sign diagram for f shows that f is increasing on the intervals $(-\infty, 1)$ and $(3, \infty)$ and decreasing on the interval $(1, 3)$.

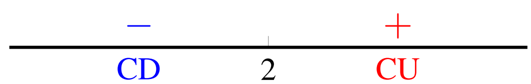


Since the sign of f' changes as we move across the critical point $x = 1$, a relative maximum occurs at $(1, f(1)) = (1,6)$. Similarly, a relative minimum of f occurs at $(3,2)$.

6. We find

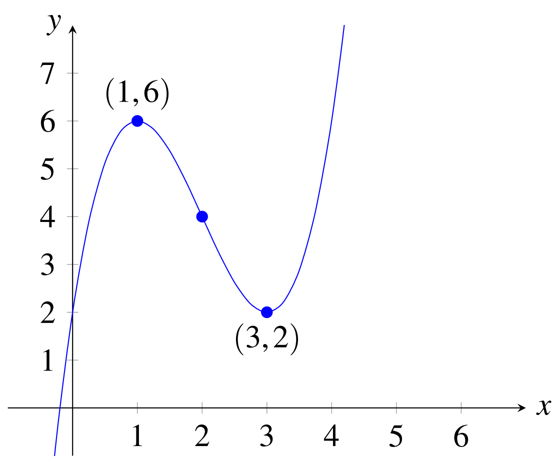
$$f''(x) = 6x - 12 = 6(x-2),$$

which is equal to zero when $x = 2$. The sign diagram for f'',



shows that f is concave downward on $(-\infty, 2)$ and concave upward on $(2, \infty)$. Since the sign of f'' changes sign as we move across $x = 2$, f must have an inflection point at $(2, f(2)) = (2,4)$. In fact, we can show that f exhibits odd symmetry about this point.

7. Putting all of the above information together, we arrive at the following graph of $f(x)$.



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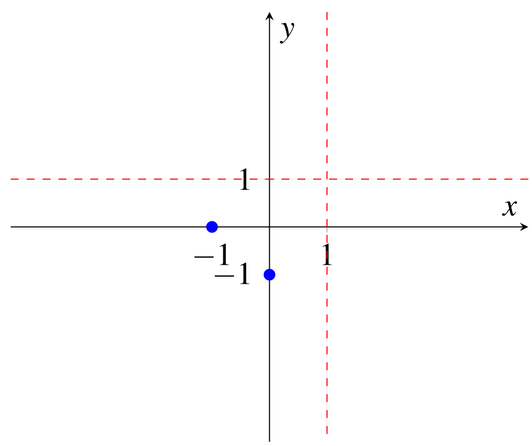
Suppose we are not given the definition of a function as in Examples **??** and **??**, but rather, we are being provided information about its domain and other behaviours. Sometimes, these pieces of information may still allow us to sketch the curve of such a function as shown in the next example.

Example 5.87: Curve Sketching with Pieces of Information

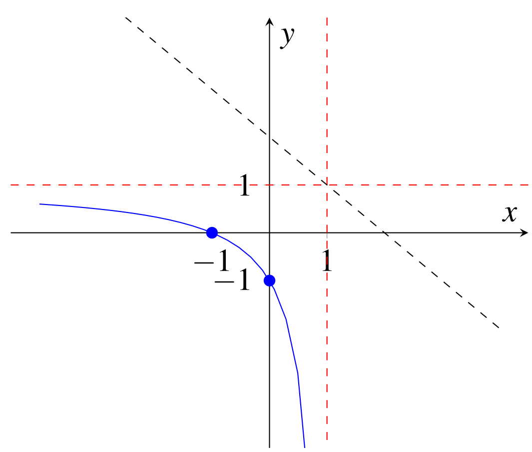
Suppose the following properties of a function f are given.

1. *Domain:* $(-\infty,1)\cup(1,\infty)$.
2. *Intercepts* $(0,-1)$ *and* $(-1,0)$.
3. f *is symmetric about the line* $y=-x+2$.
4. $x=1$ *is a vertical asymptote and* $y=1$ *is a horizontal asymptote.*
5. f *is decreasing on* $(-\infty,1)\cup(1,\infty)$.
6. f *has no relative extrema.*
7. f *is concave downward on* $(-\infty,1)$ *and concave upward on* $(1,\infty)$.
8. f *has no points of inflection.*

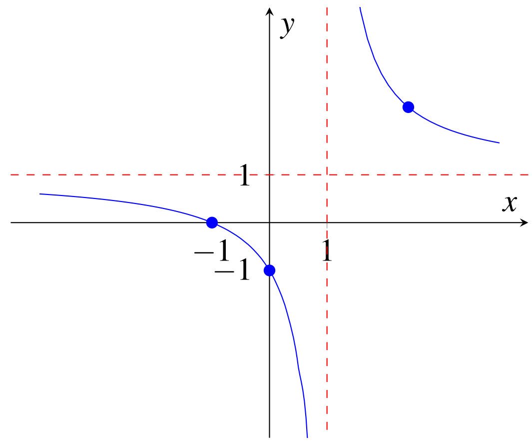
Solution. We initialize our graph with the x- and y- intercepts (item 2), along with the given horizontal and vertical asymptotes (item 4).



Let's first look at the bottom left-hand corner of the graph. We are given that, here, f is decreasing and concave downward (items 5 and 7). We can therefore connect the two intercepts like so:



The line of symmetry $y = -x + 2$ (item 3) is drawn in black. We reflect the curve across this line, ensuring that our final curve is concave up and decreasing on the interval $(1, \infty)$ (items 5 and 7). Our final result is below.



♣

Exercises for Section 5.7

Exercise 5.7.32 *Follow the Curve Sketching Guideline provided in this section to sketch the graphs of the following functions.*

(a) $y = x^5 - 5x^4 + 5x^3$

(b) $y = x^3 - 3x^2 - 9x + 5$

(c) $y = (x-1)^2(x+3)^{2/3}$

(d) $x^2 + x^2y^2 = a^2y^2$, $a > 0$.

(e) $y = x^5 - x$

(f) $y = x(x^2+1)$

(g) $y = x^3 + 6x^2 + 9x$

Exercise 5.7.33 *Follow the Curve Sketching Guideline provided in this section to sketch the graphs of the following functions.*

(a) $y = 4x + \sqrt{1-x}$

(b) $y = (x+1)/\sqrt{5x^2+35}$

(c) $y = x + 1/x$

(d) $y = x^2 + 1/x$

(e) $y = (x+5)^{1/4}$

(f) $y = x/(x^2-9)$

(g) $y = x^2/(x^2+9)$

(h) $y = 2\sqrt{x} - x$

(i) $y = (x-1)/(x^2)$

Exercise 5.7.34 *Follow the Curve Sketching Guideline provided in this section to sketch the graphs of the following functions.*

(a) $y = xe^x$

(b) $y = (e^x + e^{-x})/2$

(c) $y = e^{-x}\cos x$

(d) $y = e^x - \sin x$

(e) $y = e^x/x$

(f) $y = \tan^2 x$

(g) $y = \cos^2 x - \sin^2 x$

(h) $y = \sin^3 x$

(i) $y = 6x + \sin 3x$

(j) $y = 3\sin(x) - \sin^3(x)$, *for* $x \in [0, 2\pi]$

Exercise 5.7.35 *Sketch the graph of* $f(x) = x^3 - 6x^2 + 2$ *using the following information:*

Domain: $(-\infty, \infty)$
y-intercept: 2
Asmptotes: none
Increasing on: $(-\infty, 0) \cup (4, \infty)$
Decreasing on: $(0, 4)$
Relative Extrema: relative max at $(0, 2)$*, relative min at* $(4, -30)$
Concavity: Downward on $(-\infty, 4)$*, upward on* $(4, \infty)$
Inflection point: $(2, -14)$

Exercise 5.7.36 *Sketch the graph of* $f(x) = \frac{1}{5}\left(x^4 - 2x^3\right)$ *using the following information:*

Domain: $(-\infty, \infty)$
y-intercept: 0
x-intercepts: 0,2
Asmptotes: none
Increasing on: $(1.5, \infty)$
Decreasing on: $(-\infty, 0) \cup (0, 3/2)$
Relative Extrema: relative min at $(3/2, -27/80)$
Concavity: Downward on $(0, 1)$*, upward on* $(-\infty, 0) \cup (1, \infty)$
Inflection points: $(0, 0)$ *and* $(1, -1/5)$

Exercise 5.7.37 *Sketch the graph of* $f(x) = \frac{x-1}{x^2}$ *using the following information:*

Domain: $(-\infty, 0) \cup (0, \infty)$
x-intercept: 1
Asmptotes: x-axis and y-axis
Increasing on: $(0, 2)$
Decreasing on: $(-\infty, 0) \cup (2, \infty)$
Relative Extrema: relative max at $(2, 1/4)$
Concavity: Downward on $(-\infty, 0) \cup (0, 3)$*, upward on* $(3, \infty)$
Inflection point: $(3, 2/9)$

Exercise 5.7.38 *Sketch the graph of* $f(x) = x - 5x^{1/3}$ *using the following information:*

Domain: $(-\infty, \infty)$
x-intercepts: $5\sqrt{5}$*,0*
Asmptotes: none

Increasing on: $(-\infty, 0) \cup (2.15, \infty)$
Decreasing on: $(0, 2.15)$
Relative Extrema: relative max at $(0,0)$*, relative min at* $(2.15, -4.3)$
Concavity: Upward on $(-\infty, \infty)$

5.8 Optimization Problems

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form: Find the largest (or smallest) value of $f(x)$ when $a \leq x \leq b$. Sometimes a or b are infinite, but frequently the real world imposes some **constraint** on the values that x may have.

Such a problem differs in two ways from the relative maximum and minimum problems we encountered when graphing functions: We are interested only in the function between a and b, and we want to know the largest or smallest value that $f(x)$ takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a relative maximum or minimum but a *global* (or *absolute*) maximum or minimum.

Guideline for Solving Optimization Problems

1. Identify what is to be maximized or minimized and what the constraints are.
2. Draw a diagram (if appropriate) and label it.
3. Decide what the variables are and in what units their values are being measured in. For example, A for area in square metres, r for radius in inches, C for cost in Euros. In other words, if the problem does not introduce these variables, you need to do so.
4. Write a formula for the function that is to be maximized or minimized.
5. Use the given constraint to express the formula from Step 4 in terms of a single variable, namely something like $f(x)$ (or $A(x)$, $C(x)$,..., whatever name is appropriate). Then identify the domain of this function, which is typically $[a,b]$ or (a,b).
6. Find the critical points of f. Compare all critical values and endpoints (or perhaps $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ or curve sketching if the interval is open) to determine the absolute extrema of f.
7. Provide your solution meaningfully, which includes unit(s).

Example 5.88: Maximize your Profit

You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

Solution. The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function $P(x)$ representing the profit when the price per item is x. Profit is revenue minus costs, and revenue is number of items sold times the price per item, so we get $P = nx - 2000 - 0.50n$. The number of items sold is itself a function of x, $n = 5000 + 1000(1.5 - x)/0.10$, because $(1.5 - x)/0.10$ is the number of multiples of 10 cents that the price is below \$1.50. Now we substitute for n in the profit function:

$$\begin{aligned} P(x) &= (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.5(5000 + 1000(1.5 - x)/0.10) \\ &= -10000x^2 + 25000x - 12000 \end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is $P'(x) = -20000x + 25000$, which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a relative maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. (Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these.) Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items. ♣

Example 5.89: Minimize Average Cost

A manufacturer determines that the daily average cost of producing q units is

$$\overline{C}(q) = 0.0001q^2 - 0.08q + 65 + \frac{5000}{q} \quad q > 0$$

Determine the number of units produced per day which minimizes the average cost.

Solution. We first note that the domain of the function $\overline{C}$ is the open interval $(0, \infty)$. Calculate,

$$\overline{C}'(q) = 0.0002q - 0.08 - \frac{5000}{q^2}.$$

Then solving $C'(q) = 0$ gives $q = 500$, which is the only critical point of C. Next,

$$\overline{C}''(q) = 0.0002 + \frac{10,000}{q^3}.$$

And so

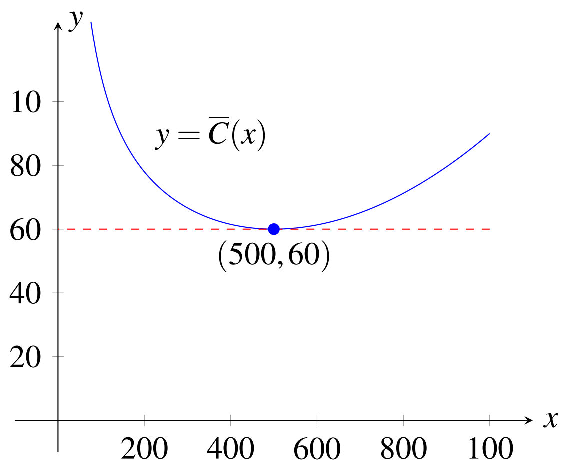
$$\overline{C}''(500) = 0.0002 + \frac{10,000}{(500)^3} > 0.$$

Therefore, by the Second Derivative Test, $q = 500$ is a relative minimum of $\overline{C}$.

Furthermore, $\overline{C}$ is concave upward everywhere, so the relative minimum of $\overline{C}$ must be the absolute maximum of $\overline{C}$. The minimum cost is thus

$$\overline{C}(500) = 0.0001(500)^2 - 0.08(500) + 65 + \frac{5000}{500} = 60$$

or \$60 per unit. The sketch of $\overline{C}$ follows.



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Example 5.90: Maximize Manufacturing Capacity

It is estimated that the operating rate of a major manufacturer's factories over a certain 365-day period is given by

$$f(t) = 100 + \frac{800t}{t^2 + 90,000} \quad 0 \leq t \leq 365$$

percent. Determine the day on which the operating rate is maximized.

Solution. We wish to find the absolute maximum of f on $[0, 365]$. We first calculate

$$\begin{aligned} f'(t) &= \frac{(t^2 + 90,000)(800) - 800t(2t)}{(t^2 + 90,000)^2} \\ &= \frac{-800(t^2 - 90,000)}{(t^2 + 90,000)^2} \end{aligned}$$

Therefore, $f'(t) = 0 \implies t = -300$ or 300. So the only critical point of f is $t = 300$ (since $t = -300$ is outside the domain of f). Evaluating $f(t)$ at this critical point and both endpoints, we see

$$f(0) = 100, \quad f(300) = 101.33, \quad f(365) = 101.308$$

Thus, the manufacturing capacity operating rate was at a maximum after 300 days. ♣

Example 5.91: Largest Rectangle

Find the largest rectangle (that is, the rectangle with largest area) that fits inside the graph of the parabola $y = x^2$ below the line $y = a$ (a is an unspecified constant value), with the top side of the rectangle on the horizontal line $y = a$; see Figure 5.29.)

Solution. We want to find the maximum value of some function $A(x)$ representing area. Perhaps the hardest part of this problem is deciding what x should represent. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. So we can let the x in $A(x)$ be the x of the parabola $f(x) = x^2$. Then the area is

$$A(x) = (2x)(a - x^2) = -2x^3 + 2ax.$$

We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. (You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area.)

Setting $0 = A'(x) = 6x^2 + 2a$ we get $x = \sqrt{a/3}$ as the only critical value. Testing this and the two endpoints, we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. The maximum area thus occurs when the rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$. ♣

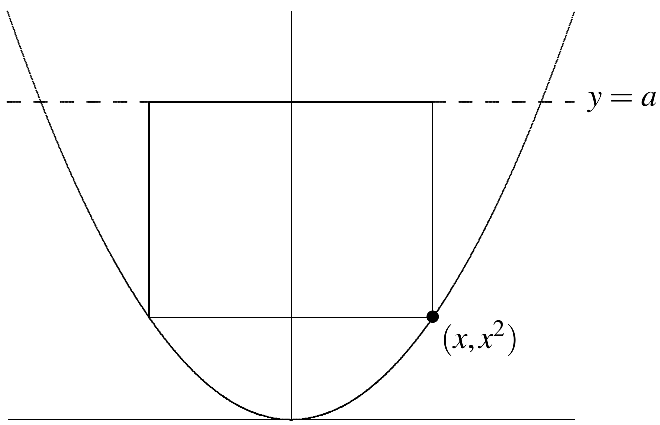
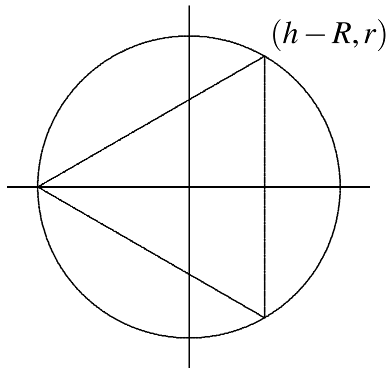


Figure 5.29: Rectangle in a parabola.

Example 5.92: Largest Cone

If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Solution. Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\pi r^2 h/3$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible—equal to R—by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the cross-section depicted in the figure shown below. We have situated the picture in a convenient way relative to the x- and y-axes, namely, with the centre of the sphere at the origin and the vertex of the cone at the far left on the x-axis.



Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. This is frequently the case, but often the two variables are related in some way so that "really" there is only one variable. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are $(h-R, r)$, must be on the circle of radius R. That is,

$$(h-R)^2 + r^2 = R^2.$$

We can solve for h in terms of r or for r in terms of h. Either involves taking a square root, but we notice that the volume function contains r^2, not r by itself, so it is easiest to solve for r^2 directly: $r^2 = R^2 - (h-R)^2$. Then we substitute the result into $\pi r^2 h/3$:

$$\begin{aligned} V(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V(h)$ when h is between 0 and $2R$ including the endpoints. Like in Example 5.8: Largest Rectangle, we argue that zero volume at the endpoints of the closed interval makes the problem easier to solve Now we solve $0 = f'(h) = -\pi h^2 + (4/3)\pi h R$, getting $h = 0$ or $h = 4R/3$. We compute $V(0) = V(2R) = 0$ and $V(4R/3) = (32/81)\pi R^3$. The maximum is the latter; since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

♣

Example 5.93: Containers of Given Volume

You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder.

Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Solution. Let us first choose letters to represent various things: h for the height, r for the base radius, V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder; V and c are constants, h and r are variables. Now we can write the cost of materials:

$$c(2\pi rh) + Nc(2\pi r^2).$$

Again we have two variables; the relationship is provided by the fixed volume of the cylinder: $V = \pi r^2 h$. We use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). The result is

$$f(r) = 2c\pi r \frac{V}{\pi r^2} + 2Nc\pi r^2 = \frac{2cV}{r} + 2Nc\pi r^2.$$

We want to know the minimum value of this function when r is in $(0,\infty)$. We now set $0 = f'(r) = -2cV/r^2 + 4Nc\pi r$, giving $r = \sqrt[3]{V/(2N\pi)}$. Since $f''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a relative minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{\pi(V/(2N\pi))} = 2N,$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius (or the height is equal to the diameter). ♣

Example 5.94: Rectangles of Given Area

Of all rectangles of area 100, which has the smallest perimeter?

Solution. First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$ (in order that the area be 100). So the function we want to minimize is

$$f(x) = 2x + 2\frac{100}{x}$$

since the perimeter is twice the length plus twice the width of the rectangle. Not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

We next find $f'(x)$ and set it equal to zero: $0 = f'(x) = 2 - 200/x^2$. Solving $f'(x) = 0$ for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $f'(x)$ is defined everywhere on the interval $(0,\infty)$, there are no more critical values, and there are no endpoints. Is there a relative maximum, minimum, or neither at $x = 10$? The second derivative is $f''(x) = 400/x^3$, and $f''(10) > 0$, so there is a relative minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square. ♣

Example 5.95: Minimize Travel Time

Suppose you want to reach a point A that is located across the sand from a nearby road (see Figure 5.30). Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Solution. Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB} = a - x$ and, by the Pythagorean Theorem, $\overline{BA} = \sqrt{x^2+b^2}$, the total time for the trip is

$$f(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$

We want to find the minimum value of f when x is between 0 and a. As usual we set $f'(x) = 0$ and solve for x:

$$0 = f'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}}$$

$$w\sqrt{x^2+b^2} = vx$$

$$w^2(x^2+b^2) = v^2x^2$$

$$w^2b^2 = (v^2-w^2)x^2$$

$$x = \frac{wb}{\sqrt{v^2-w^2}}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0,a]$, and $wb/\sqrt{v^2-w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$f''(x) = \frac{b^2}{(x^2+b^2)^{3/2}w}.$$

Since this is always positive there is a relative minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0,a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$f(0) \quad = \frac{a}{v} + \frac{b}{w}$$

$$f(a) = \frac{\sqrt{a^2+b^2}}{w}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $f''(x)$ is always positive, so the derivative $f'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $f(0) > f(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2-w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2-w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand. ♣

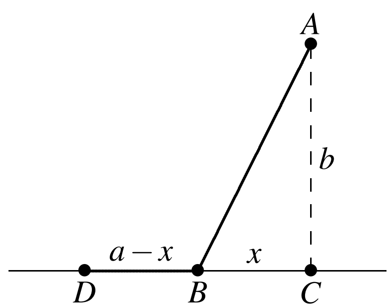


Figure 5.30: Minimizing travel time.

Exercises for Section 5.8

Exercise 5.8.1 *Suppose the monthly profit a manufacturer realizes from selling q units is given by*

$$P(q) = -5q^2 + 1300q - 15,000$$

dollars. What is the maximum monthly profit?

Exercise 5.8.2 *Suppose the daily profit a manufacturer realizes from selling q units is given by*

$$P(q) = -0.2q^3 - 2q^2 + 1000$$

What is the maximum daily profit?

Exercise 5.8.3 *A manufacturer determines that the total cost $C(q)$ of manufacturing q units per day is given by*

$$C(q) = 400 + 4q + 0.0001q^2$$

dollars. If each unit is sold at

$$p = 10 - 0.0004q$$

dollars, what is the daily level of production that maximizes the daily profit?

Exercise 5.8.4 *A certain company has a weekly fixed cost of \$9945, and a variable production cost of*

$$V(q) = q^2 - 3q + 80$$

dollars per unit. If the revenue from selling q units per week is

$$R(q) = q^2 - 6q + 20,$$

what is the level of production that will maximize the weekly profit?

Exercise 5.8.5 *Suppose the total monthly cost of manufacturing q units is given by*

$$C(q) = 0.5q^2 - 50q + 15,000$$

dollars.

(a) *Determine the average cost function* $\overline{C}$.

(b) *Determine the level of production that results in the smallest average production cost.*

(c) *Determine the level of production for which the average cost is equal to the marginal cost.*

(d) *What can you deduce from your results?*

Exercise 5.8.6 *Given the following demand equation,*

$$p = \sqrt{300 - 0.5q},$$

where p is unit price and q is the number of units manufactured per week, how many units should be manufactured and sold each week?

Exercise 5.8.7 *We define the average revenue as*

$$\overline{R}(q) = \frac{R(q)}{q} \quad q > 0.$$

Show that if $R(q)$ *is concave downward, then the maximum average revenue occurs when* $\overline{R}(q) = R'(q)$.

Exercise 5.8.8 *The gross domestic product (GDP) of a certain country following a national crisis (at* $t \equiv 0$*) is approximated by*

$$G(t) = -0.4t^3 + 4.8t^2 + 20 \quad 0 \leq t \leq 12$$

where $G(t)$ *is measured in billions of dollars. When during this time period is the GDP at its highest?*

Exercise 5.8.9 *Suppose the amount of money in an account is given by*

$$a(t) = -0.01t^4 + 0.5t^3 + 3.8t^2 + 12.6t + 1200 \quad 0 \leq t \leq 55$$

thousands of dollars over 55 years. Determine the year during which the value of the account is maximal.

Exercise 5.8.10 *Over a time period of 6 years, it is shown that the number N of independently owned bakeries is given by*

$$N(t) = 2 + 8.82x - 7.73x^2 + 2.08x^3 - 0.175x^4 \quad 0 \leq t \leq 6$$

(in millions of bakeries). Determine the absolute extrema of the function N on the interval $[0,6]$ *and interpret your results.*

Exercise 5.8.11 *Suppose*

$$C(q) = 0.02q^3 - 0.03q^2 + 20q + 300$$

is the daily cost function for a certain producer where q is measured in units of thousands. Determine the level of production that minimizes the daily average cost per unit.

Exercise 5.8.12 *Find the dimensions of the rectangle of largest area having fixed perimeter* 100.

Exercise 5.8.13 *Find the dimensions of the rectangle of largest area having fixed perimeter P.*

Exercise 5.8.14 *A box with square base and no top is to hold a volume* 100. *Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base.*

Exercise 5.8.15 *A box with square base is to hold a volume* 200. *The bottom and top are formed by folding in flaps from all four sides, so that the bottom and top consist of two layers of cardboard. Find the dimensions of the box that requires the least material. Also find the ratio of height to side of the base.*

Exercise 5.8.16 *A box with square base and no top is to hold a volume V. Find (in terms of V) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve V.)*

Exercise 5.8.17 *You have* 100 *feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area?*

Exercise 5.8.18 *You have l feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area?*

Exercise 5.8.19 *Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 that you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make?*

Exercise 5.8.20 *Find the area of the largest rectangle that fits inside a semicircle of radius* 10 *(one side of the rectangle is along the diameter of the semicircle).*

Exercise 5.8.21 *Find the area of the largest rectangle that fits inside a semicircle of radius r (one side of the rectangle is along the diameter of the semicircle).*

Exercise 5.8.22 *For a cylinder with surface area* 50*, including the top and the bottom, find the ratio of height to base radius that maximizes the volume.*

Exercise 5.8.23 *For a cylinder with given surface area S, including the top and the bottom, find the ratio of height to base radius that maximizes the volume.*

Exercise 5.8.24 *You want to make cylindrical containers to hold 1 liter using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side* $2r$*, so that* $2(2r)^2 = 8r^2$ *of material is needed (rather than* $2\pi r^2$*, which is the total area of the top and bottom). Find the dimensions of the container using the least amount of material, and also find the ratio of height to radius for this container.*

Exercise 5.8.25 *You want to make cylindrical containers of a given volume V using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side* $2r$*, so that* $2(2r)^2 = 8r^2$ *of material is needed (rather than* $2\pi r^2$*, which is the total area of the top and bottom). Find the optimal ratio of height to radius.*

Exercise 5.8.26 *Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.)*

Exercise 5.8.27 *A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side, and the container is made with no bottom; (b) the same as in (a), except that the container is made with a circular bottom, for which the cost per unit area is 1.5 times the cost per unit area of the side.*

Exercise 5.8.28 *A piece of cardboard is 1 meter by* $1/2$ *meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume?*

Exercise 5.8.29 *(a) A square piece of cardboard of side a is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides a and b?*

Exercise 5.8.30 *A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top; the colored glass transmits only* $1/2$ *as much light per unit area as the the clear* *glass. If the distance from top to bottom (across both the rectangle and the semicircle) is 2 meters and the window may be no more than 1.5 meters wide, find the dimensions of the rectangular portion of the window that lets through the most light.*

Exercise 5.8.31 *A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is between* 0 *and* 1*). If the distance from top to bottom (across both the rectangle and the semicircle) is a fixed distance H, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light.*

Exercise 5.8.32 *You are designing a poster to contain a fixed amount A of printing (measured in square centimeters) and have margins of a centimeters at the top and bottom and b centimeters at the sides. Find the ratio of vertical dimension to horizontal dimension of the printed area on the poster if you want to minimize the amount of posterboard needed.*

Exercise 5.8.33 *What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere?*

Exercise 5.8.34 *The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back.*

Exercise 5.8.35 *Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250* cm^3*) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side.*

Exercise 5.8.36 *A conical paper cup is to hold* $1/4$ *of a liter. Find the height and radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula* $\pi r\sqrt{r^2+h^2}$ *for the area of the side of a cone.*

Exercise 5.8.37 *A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula* $\pi r\sqrt{r^2+h^2}$ *for the area of the side of a cone, called the* **lateral area** *of the cone.*

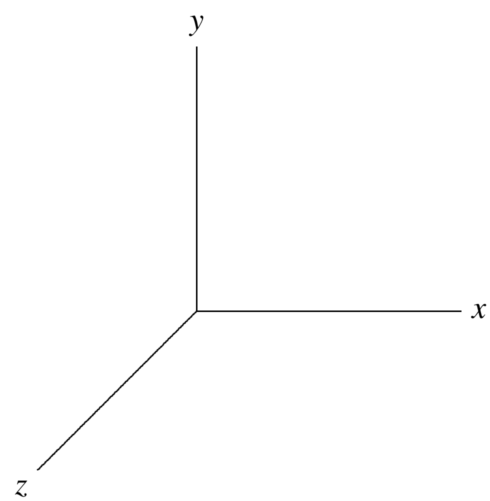
Exercise 5.8.38 *Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle.*

Exercise 5.8.39 *How are your answers to Problem 5.8.19 affected if the cost per item for the x items, instead of being simply \$2, decreases below \$2 in proportion to x (because of economy of scale and volume discounts) by 1 cent for each 25 items produced?*

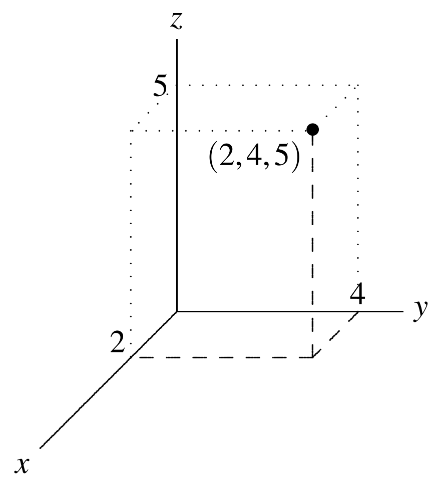
6. Three Dimensions

6.1 The Coordinate System

Throughout the text thus far we have focused investigating functions of the form $y = f(x)$, with one independent and one dependent variable. Such functions can be represented in two dimensions, using two numerical axes that allow us to identify every point in the plane with two numbers. We now shift our focus to three-dimensional space; to identify every point in three dimensions we require three numerical values. The obvious way to make this association is to add one new axis, perpendicular to the x- and y-axes we already understand. We could, for example, add a third axis, the z-axis, with the positive z-axis coming straight out of the page, and the negative z-axis going out the back of the page. This is difficult to work with on a printed page, so more often we draw a view of the three axes from an angle:



You must then imagine that the z-axis is perpendicular to the other two. Just as we have investigated functions of the form $y = f(x)$ in two dimensions, we will investigate three dimensions largely by considering functions; now the functions will (typically) have the form $z = f(x,y)$. Due to the fact that we are used to having the result of a function graphed in the vertical direction, it is somewhat easier to maintain that convention in three dimensions. To accomplish this, we normally rotate the axes so that z points up; the result is then:



Note that if you imagine looking down from above, along the z-axis, the positive z-axis will come straight toward you, the positive y-axis will point up, and the positive x-axis will point to your right, as usual. Any point in space is identified by providing the three coordinates of the point, as shown; naturally, we list the coordinates in the order (x,y,z). One useful way to think of this is to use the x and y coordinates to identify a point in the x-y-plane, then move straight up (or down) a distance given by the z coordinate.

It is now fairly simple to understand some "shapes" in three dimensions that correspond to simple conditions on the coordinates. In two dimensions the equation $x = 1$ describes the vertical line through $(1,0)$. In three dimensions, it still describes all points with x-coordinate 1, but this is now a plane, as in Figure 6.1.

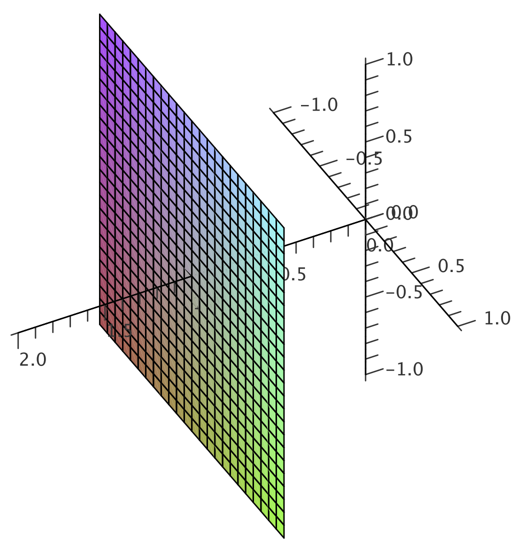


Figure 6.1: The plane $x = 1$.

Recall the very useful distance formula in two dimensions which comes directly from the Pythagorean Theorem: the distance between points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. What is the distance between two points (x_1,y_1,z_1) and (x_2,y_2,z_2) in three dimensions? Geometrically, we want the length of the long diagonal labelled c in the "box" in Figure 6.2. Since a, b, c form a right triangle, $a^2+b^2=c^2$. b is the vertical distance between (x_1,y_1,z_1) and (x_2,y_2,z_2), so $b=|z_1-z_2|$. The length a runs parallel to the x-y-plane, so it is simply the distance between (x_1, y_1) and (x_2, y_2), that is, $a^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Now we see that $c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

It is sometimes useful to give names to points, for example we might let $P_1 = (x_1, y_1, z_1)$, or more concisely we might refer to the point $P_1(x_1, y_1, z_1)$, and subsequently use just P_1. Distance between two points in either two or three dimensions is sometimes denoted by d, so for example the formula for the distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ might be expressed as

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Definition 6.1: Distance

The distance between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in two dimensions is

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The distance between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in three dimensions is

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

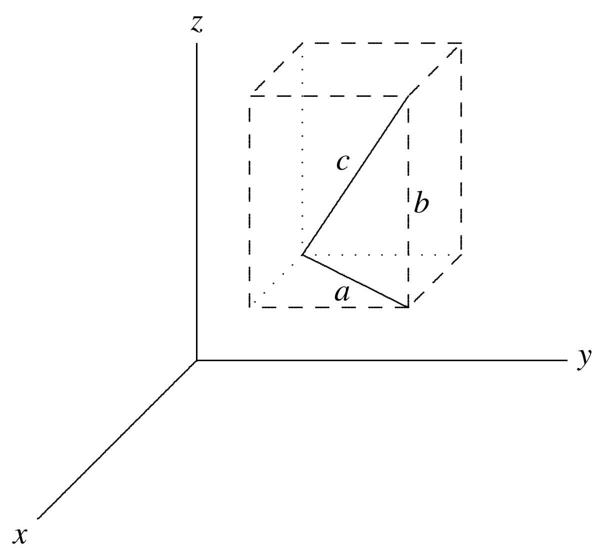


Figure 6.2: Distance in three dimensions.

In two dimensions, the distance formula immediately gives us the equation of a circle: the circle of radius r and center at (h, k) consists of all points (x, y) at distance r from (h, k), so the equation is $r = \sqrt{(x-h)^2 + (y-k)^2}$ or $r^2 = (x-h)^2 + (y-k)^2$. Now we can get the similar equation $r^2 = (x-h)^2 + (y-k)^2 + (z-l)^2$, which describes all points (x, y, z) at distance r from (h, k, l), namely, the sphere with radius r and center (h, k, l).

Definition 6.2: Equation of a Sphere

A sphere with centre (x_0, y_0, z_0) and radius r is described by

$$r^2 = (x - x_0)^2 + (y - y_0)^2 + (x - x_0)^2$$

Exercises for Section 6.1

Exercise 6.1.1 *Sketch the location of the points $(1,1,0)$, $(2,3,-1)$, and $(-1,2,3)$ on a single set of axes.*

Exercise 6.1.2 *Describe geometrically the set of points (x,y,z) that satisfy $z = 4$.*

Exercise 6.1.3 *Describe geometrically the set of points (x,y,z) that satisfy $y = -3$.*

Exercise 6.1.4 *Describe geometrically the set of points (x,y,z) that satisfy $x + y = 2$.*

Exercise 6.1.5 *The equation $x + y + z = 1$ describes some collection of points in $\mathbb{R}^3$. Describe and sketch the points that satisfy $x + y + z = 1$ and are in the x-y-plane, in the x-z-plane, and in the y-z-plane.*

Exercise 6.1.6 *Find the lengths of the sides of the triangle with vertices $(1,0,1)$, $(2,2,-1)$, and $(-3,2,-2)$.*

Exercise 6.1.7 *Find the lengths of the sides of the triangle with vertices $(2,2,3)$, $(8,6,5)$, and $(-1,0,2)$. Why do the results tell you that this isn't really a triangle?*

Exercise 6.1.8 *Find an equation of the sphere with center at $(1,1,1)$ and radius 2.*

Exercise 6.1.9 *Find an equation of the sphere with center at $(2,-1,3)$ and radius 5.*

Exercise 6.1.10 *Find an equation of the sphere with center $(3,-2,1)$ and that goes through the point $(4,2,5)$.*

Exercise 6.1.11 *Find an equation of the sphere with center at $(2,1,-1)$ and radius 4. Find an equation for the intersection of this sphere with the y-z-plane; describe this intersection geometrically.*

Exercise 6.1.12 *Consider the sphere of radius 5 centered at $(2,3,4)$. What is the intersection of this sphere with each of the coordinate planes?*

Exercise 6.1.13 *Show that for all values of θ and ϕ, the point $(a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$ lies on the sphere given by $x^2 + y^2 + z^2 = a^2$.*

Exercise 6.1.14 *Prove that the midpoint of the line segment connecting* (x_1,y_1,z_1) *to* (x_2,y_2,z_2) *is at* $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$.

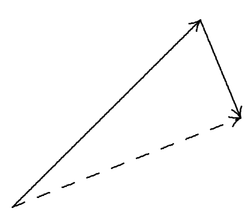
Exercise 6.1.15 *Any three points* $P_1(x_1,y_1,z_1)$*,* $P_2(x_2,y_2,z_2)$*,* $P_3(x_3,y_3,z_3)$*, lie in a plane and form a triangle. The* **triangle inequality** *says that* $d(P_1,P_3) \leq d(P_1,P_2)+d(P_2,P_3)$*. Prove the triangle inequality using either algebra (messy) or the law of cosines (less messy).*

Exercise 6.1.16 *Is it possible for a plane to intersect a sphere in exactly two points? Exactly one point? Explain.*

6.2 Vectors

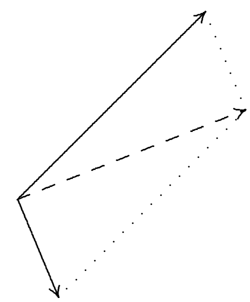
A **vector**, denoted **v**, is a quantity consisting of a non-negative magnitude and a direction. We could represent a vector in two dimensions as (m,θ), where m is the magnitude and θ is the direction, measured as an angle from some agreed upon direction. For example, we might think of the vector $(5,45^\circ)$ as representing "5 km toward the northeast"; that is, this vector might be a **displacement vector**, indicating, say, that your grandfather walked 5 kilometers toward the northeast to school in the snow. On the other hand, the same vector could represent a velocity, indicating that your grandfather walked at 5 km/hr toward the northeast. What the vector does not indicate is where this walk occurred: a vector represents a magnitude and a direction, but not a location. Pictorially it is useful to represent a vector as an arrow; the direction of the vector, naturally, is the direction in which the arrow points; the magnitude of the vector is reflected in the length of the arrow.

It turns out that many, many quantities behave as vectors, e.g., displacement, velocity, acceleration, force. Already we can get some idea of their usefulness using displacement vectors. Suppose that your grandfather walked 5 km NE and then 2 km SSE; if the terrain allows, and perhaps armed with a compass, how could your grandfather have walked directly to his destination? We can use vectors (and a bit of geometry) to answer this question. We begin by noting that since vectors do not include a specification of position, we can "place" them anywhere that is convenient. So we can picture your grandfather's journey as two displacement vectors drawn head to tail:



The displacement vector for the shortcut route is the vector drawn with a dashed line, from the tail of the first to the head of the second. With a little trigonometry, we can compute that the third vector has magnitude approximately 4.62 and direction 21.43°, so walking 4.62 km in the direction 21.43° north of east (approximately ENE) would get your grandfather to school. This sort of calculation is so common, we dignify it with a name: we say that the third vector is the **sum** of the other two vectors.

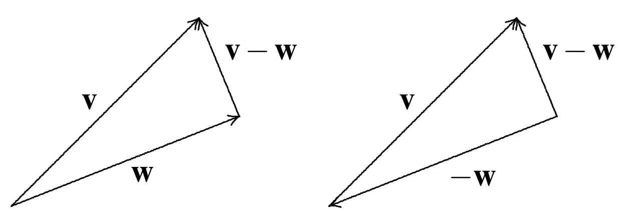
There is another common way to picture the sum of two vectors. Put the vectors tail to tail and then complete the parallelogram they indicate; the sum of the two vectors is the diagonal of the parallelogram:



This is a more natural representation in some circumstances. For example, if the two original vectors represent forces acting on an object, the sum of the two vectors is the net or effective force on the object, and it is convenient to draw all three with their tails at the location of the object.

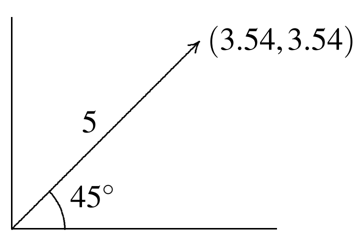
We also define **scalar multiplication** for vectors: if $\mathbf{v}$ is a vector (m,θ) and $a \geq 0$ is a real number, the vector $a\mathbf{v}$ is (am,θ), namely, it points in the same direction but has a times the magnitude. If $a < 0$, $a\mathbf{v}$ is $(|a|m,\theta+\pi)$, with $|a|$ times the magnitude and pointing in the opposite direction (unless we specify otherwise, angles are measured in radians).

Now we can understand subtraction of vectors: $\mathbf{v}-\mathbf{w} = \mathbf{v}+(-1)\mathbf{w}$:

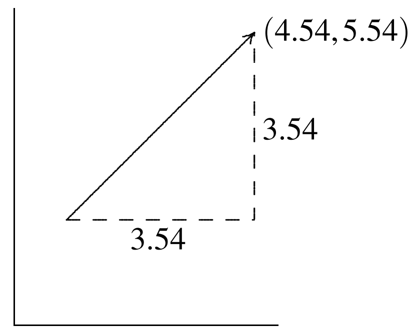


Note that as you would expect, $\mathbf{w}+(\mathbf{v}-\mathbf{w}) = \mathbf{v}$.

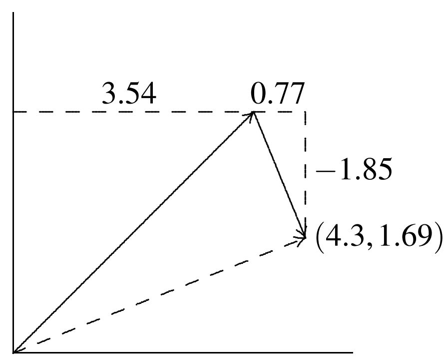
We can represent a vector in ways other than (m,θ), and in fact (m,θ) is not generally used at all. How else could we describe a particular vector? Consider again the vector $(5,45^\circ)$. Let's draw it again, but impose a coordinate system. If we put the tail of the arrow at the origin, the head of the arrow ends up at the point $(5/\sqrt{2},5/\sqrt{2}) \approx (3.54,3.54)$.



In this picture the coordinates $(3.54,3.54)$ identify the head of the arrow, provided we know that the tail of the arrow has been placed at $(0,0)$. Then in fact the vector can always be identified as $(3.54,3.54)$, no matter where it is placed; we just have to remember that the numbers 3.54 must be interpreted as a *change* from the position of the tail, not as the actual coordinates of the arrow head; to emphasize this we write $\langle 3.54, 3.54\rangle$ to mean the vector and $(3.54, 3.54)$ to mean the point. Then if the vector $\langle 3.54, 3.54\rangle$ is drawn with its tail at $(1,2)$ it looks like this:



Consider again the two part trip: 5 km NE and then 2 km SSE. The vector representing the first part of the trip is $\langle 5/\sqrt{2}, 5/\sqrt{2}\rangle$, and the second part of the trip is represented by $\langle 2\cos(-3\pi/8), 2\sin(-3\pi/8)\rangle \approx \langle 0.77, -1.85\rangle$. We can represent the sum of these with the usual head to tail picture:



It is clear from the picture that the coordinates of the destination point are $(5/\sqrt{2}+2\cos(-3\pi/8), 5/\sqrt{2}+2\sin(-3\pi/8))$ or approximately $(4.3, 1.69)$, so the sum of the two vectors is $\langle 5/\sqrt{2}+2\cos(-3\pi/8), 5/\sqrt{2}+2\sin(-3\pi/8)\rangle \approx \langle 4.3, 1.69\rangle$. Adding the two vectors is easier in this form than in the (m,θ) form, provided that we're willing to have the answer in this form as well.

It is easy to see that scalar multiplication and vector subtraction are also easy to compute in this form: $a\langle v,w\rangle = \langle av, aw\rangle$ and $\langle v_1, w_1\rangle - \langle v_2, w_2\rangle = \langle v_1 - v_2, w_1 - w_2\rangle$. What about the magnitude? The magnitude of the vector $\langle v,w\rangle$ is still the length of the corresponding arrow representation; this is the distance from the origin to the point (v,w), namely, the distance from the tail to the head of the arrow. Using the familiar distance formula the magnitude of the vector is simply $\sqrt{v^2+w^2}$, which we also denote with absolute value bars: $|\langle v,w\rangle| = \sqrt{v^2+w^2}$.

In three dimensions, vectors are still quantities consisting of a magnitude and a direction, but of course there are many more possible directions. It's not clear how we might represent the direction explicitly, but the coordinate version of vectors makes just as much sense in three dimensions as in two. By $\langle 1,2,3\rangle$ we mean the vector whose head is at $(1,2,3)$ if its tail is at the origin. As before, we can place the vector anywhere we want; if it has its tail at $(4,5,6)$ then its head is at $(5,7,9)$. It remains true that arithmetic is easy to do with vectors in this form.

Arithmetic of Vectors

Sum of vectors:

$$\langle v_1, v_2, v_3\rangle + \langle w_1, w_2, w_3\rangle = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle$$

Scalar Multiplication of vectors:

$$a\langle v_1, v_2, v_3\rangle = \langle av_1, av_2, av_3\rangle$$

Subtraction of vectors:

$$\langle v_1, v_2, v_3\rangle - \langle w_1, w_2, w_3\rangle = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3\rangle$$

The **magnitude** of the vector is the distance from the origin to the head of the vector, or $|\langle v_1, v_2, v_3\rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

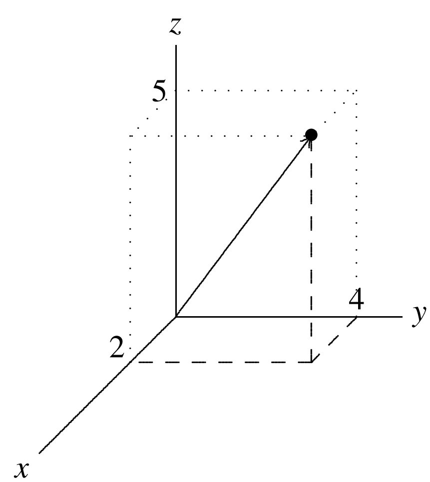


Figure 6.3: The vector $\langle 2,4,5\rangle$ with its tail at the origin.

Three particularly simple vectors turn out to be quite useful: $\mathbf{i} = \langle 1,0,0\rangle$, $\mathbf{j} = \langle 0,1,0\rangle$, and $\mathbf{k} = \langle 0,0,1\rangle$. These play much the same role for vectors that the axes play for points. In particular, notice that

$$\begin{aligned}\langle v_1, v_2, v_3\rangle &= \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle \\ &= v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\end{aligned}$$

Thus far, we have focused our discussion on vectors which begin at the origin and end at a point. However we frequently want to produce a vector that points from one point to another. That is, if P and Q are points, we seek the vector $\mathbf{x}$ such that when the tail of $\mathbf{x}$ is placed at P, its head is at Q; we refer to this vector as $\overrightarrow{PQ}$. If we know the coordinates of P and Q, the coordinates of the vector are easy to find.

Example 6.3:

Suppose $P = (1, -2, 4)$ and $Q = (-2, 1, 3)$. *The vector* $\overrightarrow{PQ}$ *is* $\langle -2-1, 1--2, 3-4\rangle = \langle -3, 3, -1\rangle$ and $\overrightarrow{QP} = \langle 3, -3, 1\rangle$.

Exercises for Section 6.2

Exercise 6.2.1 *Draw the vector* $\langle 3, -1\rangle$ *with its tail at the origin.*

Exercise 6.2.2 *Draw the vector* $\langle 3, -1, 2\rangle$ *with its tail at the origin.*

Exercise 6.2.3 *Let* $\mathbf{v}$ *be the vector with tail at the origin and head at* $(1,2)$*; let* $\mathbf{w}$ *be the vector with tail at the origin and head at* $(3,1)$. *Draw* $\mathbf{v}$ *and* $\mathbf{w}$ *and a vector* $\mathbf{u}$ *with tail at* $(1,2)$ *and head at* $(3,1)$. *Draw* $\mathbf{u}$ *with its tail at the origin.*

Exercise 6.2.4 *Let* $\mathbf{v}$ *be the vector with tail at the origin and head at* $(-1,2)$*; let* $\mathbf{w}$ *be the vector with tail at the origin and head at* $(3,3)$. *Draw* $\mathbf{v}$ *and* $\mathbf{w}$ *and a vector* $\mathbf{u}$ *with tail at* $(-1,2)$ *and head at* $(3,3)$. *Draw* $\mathbf{u}$ *with its tail at the origin.*

Exercise 6.2.5 *Let* $\mathbf{v}$ *be the vector with tail at the origin and head at* $(5,2)$*; let* $\mathbf{w}$ *be the vector with tail at the origin and head at* $(1,5)$. *Draw* $\mathbf{v}$ *and* $\mathbf{w}$ *and a vector* $\mathbf{u}$ *with tail at* $(5,2)$ *and head at* $(1,5)$. *Draw* $\mathbf{u}$ *with its tail at the origin.*

Exercise 6.2.6 *Find* $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ *and* $-2\mathbf{v}$ *for* $\mathbf{v} = \langle 1, 3\rangle$ *and* $\mathbf{w} = \langle -1, -5\rangle$.

Exercise 6.2.7 *Find* $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ *and* $-2\mathbf{v}$ *for* $\mathbf{v} = \langle 1, 2, 3\rangle$ *and* $\mathbf{w} = \langle -1, 2, -3\rangle$.

Exercise 6.2.8 *Find* $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ *and* $-2\mathbf{v}$ *for* $\mathbf{v} = \langle 1, 0, 1\rangle$ *and* $\mathbf{w} = \langle -1, -2, 2\rangle$.

Exercise 6.2.9 *Find* $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ *and* $-2\mathbf{v}$ *for* $\mathbf{v} = \langle 1, -1, 1\rangle$ *and* $\mathbf{w} = \langle 0, 0, 3\rangle$.

Exercise 6.2.10 *Find* $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ *and* $-2\mathbf{v}$ *for* $\mathbf{v} = \langle 3, 2, 1\rangle$ *and* $\mathbf{w} = \langle -1, -1, -1\rangle$.

Exercise 6.2.11 *Let* $P = (4, 5, 6)$, $Q = (1, 2, -5)$. *Find* $\overrightarrow{PQ}$. *Find a vector with the same direction as* $\overrightarrow{PQ}$ *but with length 1. Find a vector with the same direction as* $\overrightarrow{PQ}$ *but with length 4.*

Exercise 6.2.12 *If* A, B, *and* C *are three points, find* $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

Exercise 6.2.13 *Consider the 12 vectors that have their tails at the center of a clock and their respective heads at each of the 12 digits. What is the sum of these vectors? What if we remove the vector corresponding to 4 o'clock? What if, instead, all vectors have their tails at 12 o'clock, and their heads on the remaining digits?*

Exercise 6.2.14 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be nonzero vectors in two dimensions that are not parallel or anti-parallel. Show, algebraically, that if* $\mathbf{c}$ *is any two dimensional vector, there are scalars* s *and* t *such that* $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.

Exercise 6.2.15 *Does the statement in the previous exercise hold if the vectors* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$ *are three dimensional vectors? Explain.*

6.3 The Dot Product

The goal of this section is to answer the following question. Given two vectors, what is the angle between them?

Since vectors have no position, we are free to place vectors wherever we like. If the two vectors are placed tail-to-tail, there is now a reasonable interpretation of the question: we seek the measure of the smallest angle between the two vectors, in the plane in which they lie. Figure 6.4 illustrates the situation.

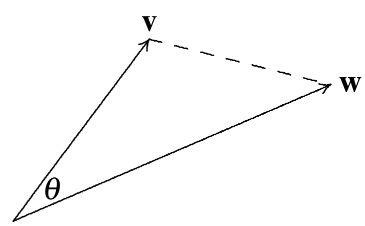


Figure 6.4: The angle between vectors v and w.

Since the angle θ lies in a triangle, we can compute it using a bit of trigonometry, namely, the law of cosines. Remember that the law of cosines states $c^2 = a^2 + b^2 - 2ab\cos C$.

The lengths of the sides of the triangle in Figure 6.4 are $|\mathbf{v}|$, $|\mathbf{w}|$, and $|\mathbf{v} - \mathbf{w}|$. Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$; then

$$\begin{aligned}
|\mathbf{v} - \mathbf{w}|^2 &= |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2|\mathbf{v}||\mathbf{w}|\cos\theta \\
2|\mathbf{v}||\mathbf{w}|\cos\theta &= |\mathbf{v}|^2 + |\mathbf{w}|^2 - |\mathbf{v} - \mathbf{w}|^2 \\
&= v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 - (v_1 - w_1)^2 - (v_2 - w_2)^2 - (v_3 - w_3)^2 \\
&= v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 \\
&\quad - (v_1^2 - 2v_1w_1 + w_1^2) - (v_2^2 - 2v_2w_2 + w_2^2) - (v_3^2 - 2v_3w_3 + w_3^2) \\
&= 2v_1w_1 + 2v_2w_2 + 2v_3w_3 \\
|\mathbf{v}||\mathbf{w}|\cos\theta &= v_1w_1 + v_2w_2 + v_3w_3 \\
\cos\theta &= (v_1w_1 + v_2w_2 + v_3w_3)/(|\mathbf{v}||\mathbf{w}|)
\end{aligned}$$

A bit of simple arithmetic with the coordinates of $\mathbf{v}$ and $\mathbf{w}$ allows us to compute the cosine of the angle between them. If necessary we can use the arccosine to get θ, but in many problems $\cos\theta$ turns out to be all we really need.

The numerator of the fraction that gives us $\cos\theta$ turns up a lot, so we give it a name and more compact notation: we call it the **dot product**, and write it as

$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$$

This is the same symbol we use for ordinary multiplication, but there should never be any confusion; you can tell from context whether we are "multiplying" vectors or numbers. (We might also use the dot for scalar multiplication: $a \cdot \mathbf{v} = a\mathbf{v}$; again, it is clear what is meant from context.)

Example 6.4:

Find the angle between the vectors $\mathbf{v} = \langle 1,2,1\rangle$ *and* $\mathbf{w} = \langle 3,1,-5\rangle$.

Solution. We know that $\cos\theta = \mathbf{v}\cdot\mathbf{w}/(|\mathbf{v}||\mathbf{w}|) = (1\cdot 3 + 2\cdot 1 + 1\cdot(-5))/(|\mathbf{v}||\mathbf{w}|) = 0$, so $\theta = \pi/2$, that is, the vectors are perpendicular. ♣

Example 6.5:

Find the angle between the vectors $\mathbf{v} = \langle 3,3,0\rangle$ *and* $\mathbf{w} = \langle 1,0,0\rangle$.

Solution. We compute

$$\begin{aligned}\cos\theta &= (3\cdot 1 + 3\cdot 0 + 0\cdot 0)/(\sqrt{9+9+0}\sqrt{1+0+0}) \\ &= 3/\sqrt{18} = 1/\sqrt{2}\end{aligned}$$

so $\theta = \pi/4$. ♣

The following are some special cases worth looking at.

Example 6.6:

Find the angles between:

1. $\mathbf{v}$ and $\mathbf{v}$
2. $\mathbf{v}$ and $-\mathbf{v}$
3. $\mathbf{v}$ and $\mathbf{0} = \langle 0,0,0\rangle$

Solution.

1. $\cos\theta = \mathbf{v}\cdot\mathbf{v}/(|\mathbf{v}||\mathbf{v}|) = (v_1^2+v_2^2+v_3^2)/(\sqrt{v_1^2+v_2^2+v_3^2}\sqrt{v_1^2+v_2^2+v_3^2}) = 1$, so the angle between $\mathbf{v}$ and itself is zero, which of course is correct.

2. $\cos\theta = \mathbf{v}\cdot -\mathbf{v}/(|\mathbf{v}||-\mathbf{v}|) = (-v_1^2-v_2^2-v_3^2)/(\sqrt{v_1^2+v_2^2+v_3^2}\sqrt{v_1^2+v_2^2+v_3^2}) = -1$, so the angle is π, that is, the vectors point in opposite directions, as of course we already knew.

3. $\cos\theta = \mathbf{v}\cdot\mathbf{0}/(|\mathbf{v}||\mathbf{0}|) = (0+0+0)/(\sqrt{v_1^2+v_2^2+v_3^2}\sqrt{0^2+0^2+0^2})$, which is undefined. On the other hand, note that since $\mathbf{v}\cdot\mathbf{0} = 0$ it looks at first as if $\cos\theta$ will be zero, which as we have seen means that vectors are perpendicular; only when we notice that the denominator is also zero do we run into trouble. One way to "fix" this is to adopt the convention that the zero vector $\mathbf{0}$ is perpendicular to all vectors; then we can say in general that if $\mathbf{v}\cdot\mathbf{w} = 0$, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

♣

Generalizing the examples, note the following useful facts:

- If $\mathbf{v}$ is parallel or anti-parallel to $\mathbf{w}$ then $\mathbf{v}\cdot\mathbf{w}/(|\mathbf{v}||\mathbf{w}|) = \pm 1$, and conversely, if $\mathbf{v}\cdot\mathbf{w}/(|\mathbf{v}||\mathbf{w}|) = 1$, $\mathbf{v}$ and $\mathbf{w}$ are parallel, while if $\mathbf{v}\cdot\mathbf{w}/(|\mathbf{v}||\mathbf{w}|) = -1$, $\mathbf{v}$ and $\mathbf{w}$ are anti-parallel. (Vectors are parallel if they point in the same direction, anti-parallel if they point in opposite directions.)
- If $\mathbf{v}$ is perpendicular to $\mathbf{w}$ then $\mathbf{v}\cdot\mathbf{w}/(|\mathbf{v}||\mathbf{w}|) = 0$, and conversely if $\mathbf{v}\cdot\mathbf{w}/(|\mathbf{v}||\mathbf{w}|) = 0$ then $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

Given two vectors, it is often useful to find the **projection** of one vector onto the other, because this turns out to have important meaning in many circumstances. More precisely, given $\mathbf{v}$ and $\mathbf{w}$, we seek a vector parallel to $\mathbf{w}$ but with length determined by $\mathbf{v}$ in a natural way, as shown in Figure 6.5. $\mathbf{p}$ is chosen so that the triangle formed by $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{v}-\mathbf{p}$ is a right triangle.

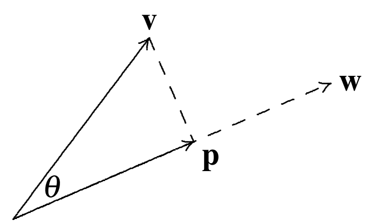


Figure 6.5: p is the projection of v onto w.

Using a little trigonometry, we see that

$$|\mathbf{p}| = |\mathbf{v}|\cos\theta = |\mathbf{v}|\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|};$$

this is sometimes called the **scalar projection of v onto w**. To get $\mathbf{p}$ itself, we multiply this length by a vector of length one parallel to $\mathbf{w}$:

$$\mathbf{p} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|}\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}.$$

Be sure that you understand why $\mathbf{w}/|\mathbf{w}|$ is a vector of length one (also called a **unit vector**) parallel to $\mathbf{w}$.

The discussion so far implicitly assumed that $0 \le \theta \le \pi/2$. If $\pi/2 < \theta \le \pi$, the picture is like Figure 6.6. In this case $\mathbf{v}\cdot\mathbf{w}$ is negative, so the vector

$$\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$

is anti-parallel to $\mathbf{w}$, and its length is

$$\left|\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|}\right|.$$

In general, the scalar projection of $\mathbf{v}$ onto $\mathbf{w}$ may be positive or negative. If it is negative, it means that the projection vector is anti-parallel to $\mathbf{w}$ and that the length of the projection vector is the absolute value of the scalar projection. Of course, you can also compute the length of the projection vector as usual, by applying the distance formula to the vector.

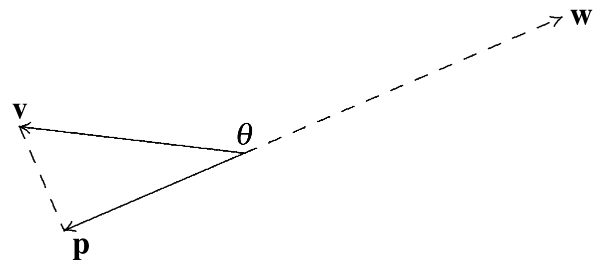


Figure 6.6: p is the projection of v onto w.

Note that the phrase "projection onto **w**" is a bit misleading if taken literally; all that **w** provides is a direction; the length of **w** has no impact on the final vector. In Figure 6.7, for example, **w** is shorter than the projection vector, but this is perfectly acceptable.

Figure 6.7: p is the projection of v onto w.

Physical force is a vector quantity. It is often necessary to compute the "component" of a force acting in a different direction than the force is being applied.

Example 6.7: Components of Force Vector

Suppose a ten pound weight is resting on an inclined plane—a pitched roof, for example. Gravity exerts a force of ten pounds on the object, directed straight down. It is useful to think of the component of this force directed down and parallel to the roof, and the component down and directly into the roof. These forces are the projections of the force vector onto vectors parallel and perpendicular to the roof. Suppose the roof is tilted at a 30° angle, as in Figure 6.8. Compute the component of the force directed down the roof and the component of the force directed into the roof.

Solution. A vector parallel to the roof is $\langle -\sqrt{3}, -1\rangle$, and a vector perpendicular to the roof is $\langle 1, -\sqrt{3}\rangle$. The force vector is $\mathbf{F} = \langle 0, -10\rangle$. The component of the force directed down the roof is then

$$\mathbf{F}_1 = \frac{\mathbf{F}\cdot\langle -\sqrt{3}, -1\rangle}{|\langle -\sqrt{3}, -1\rangle|^2}\langle -\sqrt{3}, -1\rangle = \frac{10}{2}\frac{\langle -\sqrt{3}, -1\rangle}{2} = \langle -5\sqrt{3}/2, -5/2\rangle$$

with length 5. The component of the force directed into the roof is

$$\mathbf{F}_2 = \frac{\mathbf{F}\cdot\langle 1, -\sqrt{3}\rangle}{|\langle 1, -\sqrt{3}\rangle|^2}\langle 1, -\sqrt{3}\rangle = \frac{10\sqrt{3}}{2}\frac{\langle 1, -\sqrt{3}\rangle}{2} = \langle 5\sqrt{3}/2, -15/2\rangle$$

with length $5\sqrt{3}$. Thus, a force of 5 pounds is pulling the object down the roof, while a force of $5\sqrt{3}$ pounds is pulling the object into the roof. ♣

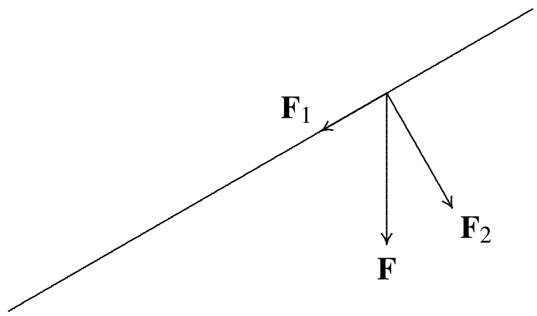


Figure 6.8: Components of a force.

The dot product has some familiar-looking properties that will be useful later, so we list them here. These may be proved by writing the vectors in coordinate form and then performing the indicated calculations; subsequently it can be easier to use the properties instead of calculating with coordinates.

Theorem 6.8: Dot Product Properties

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and a is a real number, then

1. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$
2. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
4. $(a\mathbf{u})\cdot\mathbf{v} = a(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(a\mathbf{v})$

Exercises for Section 6.3

Exercise 6.3.1 *Find* $\langle 1,1,1\rangle\cdot\langle 2,-3,4\rangle$.

Exercise 6.3.2 *Find* $\langle 1,2,0\rangle\cdot\langle 0,0,57\rangle$.

Exercise 6.3.3 *Find* $\langle 3,2,1\rangle\cdot\langle 0,1,0\rangle$.

Exercise 6.3.4 *Find* $\langle -1,-2,5\rangle\cdot\langle 1,0,-1\rangle$.

Exercise 6.3.5 *Find* $\langle 3,4,6\rangle\cdot\langle 2,3,4\rangle$.

Exercise 6.3.6 *Find the cosine of the angle between* $\langle 1,2,3\rangle$ *and* $\langle 1,1,1\rangle$*; use a calculator if necessary to find the angle.*

Exercise 6.3.7 *Find the cosine of the angle between* $\langle -1,-2,-3\rangle$ *and* $\langle 5,0,2\rangle$*; use a calculator if necessary to find the angle.*

Exercise 6.3.8 *Find the cosine of the angle between* $\langle 47,100,0\rangle$ *and* $\langle 0,0,5\rangle$*; use a calculator if necessary to find the angle.*

Exercise 6.3.9 *Find the cosine of the angle between* $\langle 1,0,1\rangle$ *and* $\langle 0,1,1\rangle$*; use a calculator if necessary to find the angle.*

Exercise 6.3.10 *Find the cosine of the angle between* $\langle 2,0,0\rangle$ *and* $\langle -1,1,-1\rangle$*; use a calculator if necessary to find the angle.*

Exercise 6.3.11 *Find the angle between the diagonal of a cube and one of the edges adjacent to the diagonal.*

Exercise 6.3.12 *Find the scalar and vector projections of* $\langle 1,2,3\rangle$ *onto* $\langle 1,2,0\rangle$.

Exercise 6.3.13 *Find the scalar and vector projections of* $\langle 1,1,1\rangle$ *onto* $\langle 3,2,1\rangle$.

Exercise 6.3.14 *A force of 10 pounds is applied to a wagon, directed at an angle of* $30°$. *Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground.*

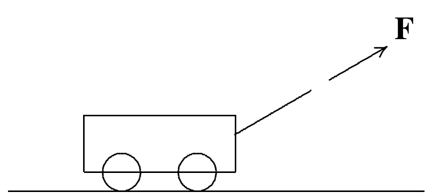


Figure 6.9: Pulling a wagon.

Exercise 6.3.15 *A force of 15 pounds is applied to a wagon, directed at an angle of* $45°$. *Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground.*

Exercise 6.3.16 *Use the dot product to find a non-zero vector* $\mathbf{w}$ *perpendicular to both* $\mathbf{u} = \langle 1,2,-3\rangle$ *and* $\mathbf{v} = \langle 2,0,1\rangle$.

Exercise 6.3.17 *Let* $\mathbf{x} = \langle 1,1,0\rangle$ *and* $\mathbf{y} = \langle 2,4,2\rangle$. *Find a unit vector that is perpendicular to both* $\mathbf{x}$ *and* $\mathbf{y}$.

Exercise 6.3.18 *Do the three points* $(1,2,0)$, $(-2,1,1)$, *and* $(0,3,-1)$ *form a right triangle?*

Exercise 6.3.19 *Do the three points* $(1,1,1)$*,* $(2,3,2)$*, and* $(5,0,-1)$ *form a right triangle?*

Exercise 6.3.20 *Show that* $|\boldsymbol{v}\cdot\boldsymbol{w}| \leq |\boldsymbol{v}||\boldsymbol{w}|$

Exercise 6.3.21 *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be perpendicular vectors. Use Theorem 6.8 to prove that* $|\boldsymbol{x}|^2+|\boldsymbol{y}|^2 = |\boldsymbol{x}+\boldsymbol{y}|^2$*. What is this result better known as?*

Exercise 6.3.22 *Prove that the diagonals of a rhombus intersect at right angles.*

Exercise 6.3.23 *Suppose that* $\boldsymbol{z} = |\boldsymbol{x}|\boldsymbol{y}+|\boldsymbol{y}|\boldsymbol{x}$ *where* $\boldsymbol{x}$*,* $\boldsymbol{y}$*, and* $\boldsymbol{z}$ *are all nonzero vectors. Prove that* $\boldsymbol{z}$ *bisects the angle between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$*.*

Exercise 6.3.24 *Prove Theorem 6.8.*

6.4 The Cross Product

Suppose we are given two vectors. In many cases it is useful to find a third vector perpendicular to the first two. There are of course an infinite number of such vectors of different lengths. Nevertheless, let us find one. Suppose $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$. We want to find a vector $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ with $\mathbf{c}\cdot\mathbf{v} = \mathbf{c}\cdot\ \mathbf{w} = 0$, or

$$\begin{aligned} v_1c_1 + v_2c_2 + v_3c_3 &= 0, \\ w_1c_1 + w_2c_2 + w_3c_3 &= 0. \end{aligned}$$

Multiply the first equation by w_3 and the second by v_3 and subtract to get

$$\begin{aligned} w_3v_1c_1 + w_3v_2c_2 + w_3v_3c_3 &= 0 \\ v_3w_1c_1 + v_3w_2c_2 + v_3w_3c_3 &= 0 \\ (v_1w_3 - w_1v_3)c_1 + (v_2w_3 - w_2v_3)c_2 &= 0 \end{aligned}$$

Of course, this equation in two variables has many solutions; a particularly easy one to see is $c_1 = v_2w_3 - w_2v_3$, $c_2 = w_1v_3 - v_1w_3$. Substituting back into either of the original equations and solving for c_3 gives $c_3 = v_1w_2 - w_1v_2$.

This particular answer to the problem turns out to have some nice properties, and it is dignified with a name: the **cross product**:

$$\mathbf{v}\times\mathbf{w} = \langle v_2w_3 - w_2v_3, w_1v_3 - v_1w_3, v_1w_2 - w_1v_2\rangle.$$

While there is a nice pattern to this vector, it can be a bit difficult to memorize; here is a convenient mnemonic. The determinant of a two by two matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

This is extended to the determinant of a three by three matrix:

$$\begin{aligned}\begin{vmatrix} x & y & z \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= x\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - y\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + z\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= x(v_2w_3 - w_2v_3) - y(v_1w_3 - w_1v_3) + z(v_1w_2 - w_1v_2) \\ &= x(v_2w_3 - w_2v_3) + y(w_1v_3 - v_1w_3) + z(v_1w_2 - w_1v_2).\end{aligned}$$

Each of the two by two matrices is formed by deleting the top row and one column of the three by three matrix; the subtraction of the middle term must also be memorized. This is not the place to extol the uses of the determinant; suffice it to say that determinants are extraordinarily useful and important. Here we want to use it merely as a mnemonic device. You will have noticed that the three expressions in parentheses on the last line are precisely the three coordinates of the cross product; replacing x, y, z by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ gives us

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= (v_2w_3 - w_2v_3)\mathbf{i} - (v_1w_3 - w_1v_3)\mathbf{j} + (v_1w_2 - w_1v_2)\mathbf{k} \\ &= (v_2w_3 - w_2v_3)\mathbf{i} + (w_1v_3 - v_1w_3)\mathbf{j} + (v_1w_2 - w_1v_2)\mathbf{k} \\ &= \langle v_2w_3 - w_2v_3, w_1v_3 - v_1w_3, v_1w_2 - w_1v_2 \rangle \\ &= \mathbf{v} \times \mathbf{w}.\end{aligned}$$

Given $\mathbf{v}$ and $\mathbf{w}$, there are typically two possible directions and an infinite number of magnitudes that will give a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. As we have picked a particular one, we should investigate the magnitude and direction.

We know how to compute the magnitude of $\mathbf{v} \times \mathbf{w}$; it's a bit messy but not difficult. It is somewhat easier to work initially with the square of the magnitude, so as to avoid the square root:

$$\begin{aligned}|\mathbf{v} \times \mathbf{w}|^2 &= (v_2w_3 - w_2v_3)^2 + (w_1v_3 - v_1w_3)^2 + (v_1w_2 - w_1v_2)^2 \\ &= v_2^2w_3^2 - 2v_2w_3w_2v_3 + w_2^2v_3^2 + w_1^2v_3^2 - 2w_1v_3v_1w_3 + v_1^2w_3^2 + v_1^2w_2^2 - 2v_1w_2w_1v_2 + w_1^2v_2^2\end{aligned}$$

While it is far from obvious, this nasty looking expression can be simplified:

$$\begin{aligned}|\mathbf{v} \times \mathbf{w}|^2 &= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) - (v_1w_1 + v_2w_2 + v_3w_3)^2 \\ &= |\mathbf{v}|^2|\mathbf{w}|^2 - (\mathbf{v} \cdot \mathbf{w})^2 \\ &= |\mathbf{v}|^2|\mathbf{w}|^2 - |\mathbf{v}|^2|\mathbf{w}|^2\cos^2\theta \\ &= |\mathbf{v}|^2|\mathbf{w}|^2(1 - \cos^2\theta) \\ &= |\mathbf{v}|^2|\mathbf{w}|^2\sin^2\theta \\ |\mathbf{v} \times \mathbf{w}| &= |\mathbf{v}||\mathbf{w}|\sin\theta\end{aligned}$$

The magnitude of $\mathbf{v} \times \mathbf{w}$ is thus very similar to the dot product. In particular, notice that if $\mathbf{v}$ is parallel to $\mathbf{w}$, the angle between them is zero, so $\sin\theta = 0$, so $|\mathbf{v} \times \mathbf{w}| = 0$, and likewise if they are anti-parallel, $\sin\theta = 0$, and $|\mathbf{v} \times \mathbf{w}| = 0$. Conversely, if $|\mathbf{v} \times \mathbf{w}| = 0$ and $|\mathbf{v}|$ and $|\mathbf{w}|$ are not zero, it must be that $\sin\theta = 0$, so $\mathbf{v}$ is parallel or anti-parallel to $\mathbf{w}$.

Here is a curious fact about this quantity that turns out to be quite useful later on: Given two vectors, we can put them tail to tail and form a parallelogram, as in Figure 6.10. The height of the parallelogram,

h, is $|\mathbf{v}|\sin\theta$, and the base is $|\mathbf{w}|$, so the area of the parallelogram is $|\mathbf{v}||\mathbf{w}|\sin\theta$, exactly the magnitude of $|\mathbf{v}\times\mathbf{w}|$.

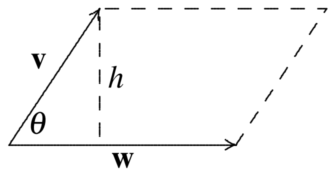


Figure 6.10: A parallelogram.

What about the direction of the cross product? Remarkably, there is a simple rule that describes the direction. Let's look at a simple example: Let $\mathbf{v} = \langle a,0,0\rangle$, $\mathbf{w} = \langle b,c,0\rangle$. If the vectors are placed with tails at the origin, $\mathbf{v}$ lies along the x-axis and $\mathbf{w}$ lies in the x-y-plane, so we know the cross product will point either up or down. The cross product is

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0 & 0 \\ b & c & 0 \end{vmatrix} = \langle 0,0,ac\rangle.$$

As predicted, this is a vector pointing up or down, depending on the sign of ac. Suppose that $a > 0$, so the sign depends only on c: if $c > 0$, $ac > 0$ and the vector points up; if $c < 0$, the vector points down. On the other hand, if $a < 0$ and $c > 0$, the vector points down, while if $a < 0$ and $c < 0$, the vector points up. Here is how to interpret these facts with a single rule: Imagine rotating vector $\mathbf{v}$ until it points in the same direction as $\mathbf{w}$; there are two ways to do this—use the rotation that goes through the smaller angle. If $a > 0$ and $c > 0$, or $a < 0$ and $c < 0$, the rotation will be counter-clockwise when viewed from above; in the other two cases, $\mathbf{v}$ must be rotated clockwise to reach $\mathbf{w}$. The rule is: counter-clockwise means up, clockwise means down. If $\mathbf{v}$ and $\mathbf{w}$ are any vectors in the x-y-plane, the same rule applies—$\mathbf{v}$ need not be parallel to the x-axis.

Although it is somewhat difficult computationally to see how this plays out for any two starting vectors, the rule is essentially the same. Place $\mathbf{v}$ and $\mathbf{w}$ tail to tail. The plane in which $\mathbf{v}$ and $\mathbf{w}$ lie may be viewed from two sides; view it from the side for which $\mathbf{v}$ must rotate counter-clockwise to reach $\mathbf{w}$; then the vector $\mathbf{v}\times\mathbf{w}$ points toward you.

This rule is usually called the **right hand rule**. Imagine placing the heel of your right hand at the point where the tails are joined, so that your slightly curled fingers indicate the direction of rotation from $\mathbf{v}$ to $\mathbf{w}$. Then your thumb points in the direction of the cross product $\mathbf{v}\times\mathbf{w}$.

One immediate consequence of these facts is that $\mathbf{v}\times\mathbf{w} \neq \mathbf{w}\times\mathbf{v}$, because the two cross products point in the opposite direction. On the other hand, since

$$|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}||\,\mathbf{w}|\sin\theta = |\mathbf{w}||\mathbf{v}|\sin\theta = |\mathbf{w}\times\mathbf{v}|,$$

the lengths of the two cross products are equal, so we know that $\mathbf{v}\times\mathbf{w} = -(\mathbf{w}\times\mathbf{v})$.

The cross product has some familiar-looking properties that will be useful later, so we list them here. As with the dot product, these can be proved by performing the appropriate calculations on coordinates, after which we may sometimes avoid such calculations by using the properties.

Theorem 6.9: Cross Product Properties

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors and* a *is a real number, then*

1. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$
2. $(\mathbf{v}+\mathbf{w})\times\mathbf{u} = \mathbf{v}\times\mathbf{u}+\mathbf{w}\times\mathbf{u}$
3. $(a\mathbf{u})\times\mathbf{v} = a(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\times(a\mathbf{v})$
4. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$
5. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

Exercises for Section 6.4

Exercise 6.4.1 *Find the cross product of* $\langle 1,1,1\rangle$ *and* $\langle 1,2,3\rangle$.

Exercise 6.4.2 *Find the cross product of* $\langle 1,0,2\rangle$ *and* $\langle -1,-2,4\rangle$.

Exercise 6.4.3 *Find the cross product of* $\langle -2,1,3\rangle$ *and* $\langle 5,2,-1\rangle$.

Exercise 6.4.4 *Find the cross product of* $\langle 1,0,0\rangle$ *and* $\langle 0,0,1\rangle$.

Exercise 6.4.5 *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are separated by an angle of* $\pi/6$, *and* $|\mathbf{u}|=2$ *and* $|\mathbf{v}|=3$. *Find* $|\mathbf{u}\times\mathbf{v}|$.

Exercise 6.4.6 *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are separated by an angle of* $\pi/4$, *and* $|\mathbf{u}|=3$ *and* $|\mathbf{v}|=7$. *Find* $|\mathbf{u}\times\mathbf{v}|$.

Exercise 6.4.7 *Find the area of the parallelogram with vertices* $(0,0)$, $(1,2)$, $(3,7)$, *and* $(2,5)$.

Exercise 6.4.8 *Find and explain the value of* $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}$ *and* $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j})$.

Exercise 6.4.9 *Prove that for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=0$.

Exercise 6.4.10 *Prove Theorem 6.9.*

Exercise 6.4.11 *Define the triple product of three vectors,* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$, *to be the scalar* $\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})$. *Show that three vectors lie in the same plane if and only if their triple product is zero. Verify that* $\langle 1,5,-2\rangle$, $\langle 4,3,0\rangle$ *and* $\langle 6,13,-4\rangle$ *all lie in the same plane.*

6.5 Lines and Planes

Lines and planes are perhaps the simplest of curves and surfaces in three dimensional space. They also will prove important as we seek to understand more complicated curves and surfaces.

You may recall that the equation of a line in two dimensions is $ax+by=c$; it is reasonable to expect that a line in three dimensions is given by $ax+by+cz=d$. However it turns out that this is the equation of a plane. We will turn our attention to a study of planes and return to consider lines later in this section.

A plane does not have an obvious "direction" as does a line. It is possible to associate a plane with a direction in a very useful way, however: there are exactly two directions perpendicular to a plane. Any vector with one of these two directions is called **normal** to the plane. While there are many normal vectors to a given plane, they are all parallel or anti-parallel to each other.

Suppose two points (v_1,v_2,v_3) and (w_1,w_2,w_3) are in a plane; then the vector $\langle w_1-v_1,w_2-v_2,w_3-v_3\rangle$ is parallel to the plane. In particular, if this vector is placed with its tail at (v_1,v_2,v_3) then its head is at (w_1,w_2,w_3) and it lies in the plane. As a result, any vector perpendicular to the plane is perpendicular to $\langle w_1-v_1,w_2-v_2,w_3-v_3\rangle$. In fact, it is easy to see that the plane consists of *precisely* those points (w_1,w_2,w_3) for which $\langle w_1-v_1,w_2-v_2,w_3-v_3\rangle$ is perpendicular to a normal to the plane, as indicated in Figure 6.11. Turning this around, suppose we know that $\langle a,b,c\rangle$ is normal to a plane containing the point (v_1,v_2,v_3). Then (x,y,z) is in the plane if and only if $\langle a,b,c\rangle$ is perpendicular to $\langle x-v_1,y-v_2,z-v_3\rangle$. In turn, we know that this is true precisely when $\langle a,b,c\rangle\cdot\langle x-v_1,y-v_2,z-v_3\rangle=0$. That is, (x,y,z) is in the plane if and only if

$$\begin{aligned}
\langle a,b,c\rangle\cdot\langle x-v_1,y-v_2,z-v_3\rangle &= 0\\
a(x-v_1)+b(y-v_2)+c(z-v_3) &= 0\\
ax+by+cz-av_1-bv_2-cv_3 &= 0\\
ax+by+cz &= av_1+bv_2+cv_3.
\end{aligned}$$

Working backwards, note that if (x,y,z) is a point satisfying $ax+by+cz=d$ then

$$\begin{aligned}
ax+by+cz &= d\\
ax+by+cz-d &= 0\\
a(x-d/a)+b(y-0)+c(z-0) &= 0\\
\langle a,b,c\rangle\cdot\langle x-d/a,y,z\rangle &= 0.
\end{aligned}$$

Namely, $\langle a,b,c\rangle$ is perpendicular to the vector with tail at $(d/a,0,0)$ and head at (x,y,z). This means that the points (x,y,z) that satisfy the equation $ax+by+cz=d$ form a plane perpendicular to $\langle a,b,c\rangle$. (This doesn't work if $a=0$, but in that case we can use b or c in the role of a. That is, either $a(x-0)+b(y-d/b)+c(z-0)=0$ or $a(x-0)+b(y-0)+c(z-d/c)=0$.)

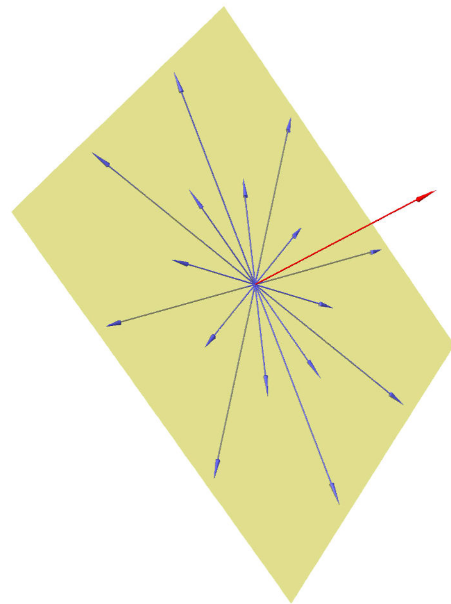


Figure 6.11: A plane defined via vectors perpendicular to a normal.

Thus, given a vector $\langle a,b,c\rangle$ we know that all planes perpendicular to this vector have the form $ax + by + cz = d$, and any surface of this form is a plane perpendicular to $\langle a,b,c\rangle$.

Definition 6.10: Scalar Equation of a Plane

Any plane can be written in the form

$$ax + by + cz = d$$

where a,b,c,d are constants and not all a,b,c are zero.
This plane is perpendicular to the vector $\langle a,b,c\rangle$.

Example 6.11: Perpendicular Plane

Find an equation for the plane perpendicular to $\langle 1,2,3\rangle$ and containing the point $(5,0,7)$.

Solution. Using the formula above, the plane is $1x + 2y + 3z = d$. To find d we may substitute the known point on the plane to get $5 + 2\cdot 0 + 3\cdot 7 = d$, so $d = 26$. ♣

Example 6.12: Normal Vector

Find a vector normal to the plane $2x - 3y + z = 15$.

Solution. One example is $\langle 2,-3,1\rangle$. Any vector parallel or anti-parallel to this works as well, so for example $-2\langle 2,-3,1\rangle = \langle -4,6,-2\rangle$ is also normal to the plane. ♣

We will frequently need to find an equation for a plane given certain information about the plane. While there may occasionally be slightly shorter ways to get to the desired result, it is always possible, and usually advisable, to use the given information to find a normal to the plane and a point on the plane, and then to find the equation as above.

Example 6.13: Plane Perpendicular

The planes $x-z=1$ and $y+2z=3$ intersect in a line. Find a third plane that contains this line and is perpendicular to the plane $x+y-2z=1$.

Solution. First, we note that two planes are perpendicular if and only if their normal vectors are perpendicular. Thus, we seek a vector $\langle a,b,c\rangle$ that is perpendicular to $\langle 1,1,-2\rangle$. In addition, since the desired plane is to contain a certain line, $\langle a,b,c\rangle$ must be perpendicular to any vector parallel to this line. Since $\langle a,b,c\rangle$ must be perpendicular to two vectors, we may find it by computing the cross product of the two.

Therefore we need a vector parallel to the line of intersection of the given planes. For this, it suffices to know two points on the line. To find two points on this line, we must find two points that are simultaneously on the two planes, $x-z=1$ and $y+2z=3$. Any point on both planes will satisfy $x-z=1$ and $y+2z=3$. It is easy to find values for x and z satisfying the first, such as $x=1, z=0$ and $x=2, z=1$. Then we can find corresponding values for y using the second equation, namely $y=3$ and $y=1$, so $(1,3,0)$ and $(2,1,1)$ are two such points. They are both on the line of intersection since they are contained in both planes.

Now $\langle 2-1, 1-3, 1-0\rangle = \langle 1,-2,1\rangle$ is parallel to the line. Finally, we may choose $\langle a,b,c\rangle = \langle 1,1,-2\rangle \times \langle 1,-2,1\rangle = \langle -3,-3,-3\rangle$. While this vector will do perfectly well, any vector parallel or anti-parallel to it will work as well. For example we might choose $\langle 1,1,1\rangle$ which is anti-parallel to it, and easier to work with.

Now we know that $\langle 1,1,1\rangle$ is normal to the desired plane and $(2,1,1)$ is a point on the plane. This gives an equation of $(1)x+(1)y+(1)z=d$. Substituting the value of the point into the equation gives $d=4$, and therefore an equation of the plane is $x+y+z=4$. As a quick check, since $(1,3,0)$ is also on the line, it should be on the plane; since $1+3+0=4$, we see that this is indeed the case.

Note that had we used $\langle -3,-3,-3\rangle$ as the normal, we would have discovered the equation $-3x-3y-3z=-12$. Then we might well have noticed that we could divide both sides by -3 to get the equivalent $x+y+z=4$. ♣

We will now turn our attention to a study of lines. Unfortunately, it turns out to be quite inconvenient to represent a typical line with a single equation; we need to approach lines in a different way.

Unlike a plane, a line in three dimensions does have an obvious direction, namely, the direction of any vector parallel to it. In fact a line can be defined and uniquely identified by providing one point on the line and a vector parallel to the line (in one of two possible directions). That is, the line consists of exactly those points we can reach by starting at the point and going for some distance in the direction of the vector. Let's see how we can translate this into more mathematical language.

Suppose a line contains the point (v_1,v_2,v_3) and is parallel to the vector $\langle a,b,c\rangle$. If we place the vector $\langle v_1,v_2,v_3\rangle$ with its tail at the origin and its head at (v_1,v_2,v_3), and if we place the vector $\langle a,b,c\rangle$ with its tail at (v_1,v_2,v_3), then the head of $\langle a,b,c\rangle$ is at a point on the line. We can get to *any* point on the line by doing the same thing, except using $t\langle a,b,c\rangle$ in place of $\langle a,b,c\rangle$, where t is some real number. Because of the way vector addition works, the point at the head of the vector $t\langle a,b,c\rangle$ is the point at the head of the vector $\langle v_1,v_2,v_3\rangle + t\langle a,b,c\rangle$, namely (v_1+ta, v_2+tb, v_3+tc); see Figure 6.12.

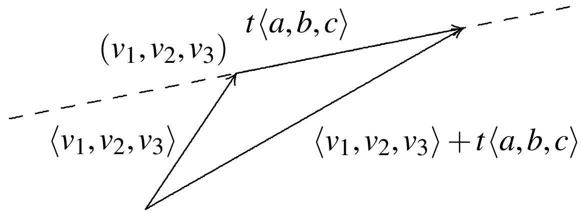


Figure 6.12: Vector form of a line.

In other words, as t runs through all possible real values, the vector $\langle v_1,v_2,v_3\rangle + t\langle a,b,c\rangle$ points to every point on the line when its tail is placed at the origin. It is occasionally useful to use this form of a line even in two dimensions; a vector form for a line in the x-y-plane is $\langle v_1,v_2\rangle + t\langle a,b\rangle$, which is the same as $\langle v_1,v_2,0\rangle + t\langle a,b,0\rangle$.

Definition 6.14: Vector Equation of a Line

An equation for a line passing through point (v_1,v_2,v_3) and parallel to the vector$\langle a,b,c\rangle$ is

$$\langle v_1,v_2,v_3\rangle + t\langle a,b,c\rangle$$

where the vector $\langle a,b,c\rangle$ is called the **direction vector** *for the line.*

Another common way to write this is as a set of **parametric equations**:

$$x = v_1 + ta \qquad y = v_2 + tb \qquad z = v_3 + tc.$$

Definition 6.15: Parametric Equations of a Line

A line in space can be described as

$$x = v_1 + ta$$
$$y = v_2 + tb$$
$$z = v_3 + tc$$

where (v_1,v_2,v_3) is a point on the line and $\langle a,b,c\rangle$ is parallel to the line.

Example 6.16: Vector Expression

Find a vector expression for the line through $(6,1,-3)$ and $(2,4,5)$.

Solution. To get a vector parallel to the line we subtract $\langle 6,1,-3\rangle - \langle 2,4,5\rangle = \langle 4,-3,-8\rangle$. The line is then given by $\langle 2,4,5\rangle + t\langle 4,-3,-8\rangle$; there are of course many other possibilities, such as $\langle 6,1,-3\rangle + t\langle 4,-3,-8\rangle$. ♣

Example 6.17: Intersecting Lines

Determine whether the lines $\langle 1,1,1\rangle + t\langle 1,2,-1\rangle$ *and* $\langle 3,2,1\rangle + t\langle -1,-5,3\rangle$ *are parallel, intersect, or neither.*

Solution. In two dimensions, two lines either intersect or are parallel; in three dimensions, lines that do not intersect might not be parallel. In this case, since the direction vectors for the lines are not parallel or anti-parallel we know the lines are not parallel. If they intersect, there must be two values a and b so that $\langle 1,1,1\rangle + a\langle 1,2,-1\rangle = \langle 3,2,1\rangle + b\langle -1,-5,3\rangle$. That is, the following must have a solution:

$$\begin{aligned} 1+a &= 3-b \\ 1+2a &= 2-5b \\ 1-a &= 1+3b \end{aligned}$$

This gives three equations in two unknowns, so there may or may not be a solution in general. In this case, it is easy to discover that $a=3$ and $b=-1$ satisfies all three equations. Substituting these values into $\langle 1,1,1\rangle + a\langle 1,2,-1\rangle = \langle 3,2,1\rangle + b\langle -1,-5,3\rangle$, the point of intersection is $(4,7,-2)$. ♣

Example 6.18: Distance from a Point to a Plane

Find the distance from the point $(1,2,3)$ *to the plane* $2x-y+3z=5$.

Solution. The distance from a point P to a plane is the shortest distance from P to any point on the plane; this is the distance measured from P perpendicular to the plane; see Figure 6.13. This distance is the absolute value of the scalar projection of $\overrightarrow{QP}$ onto a normal vector $\mathbf{n}$, where Q is any point on the plane. It is easy to find a point on the plane, say $(1,0,1)$. Thus the distance is

$$\frac{\langle 0,2,2\rangle \cdot \langle 2,-1,3\rangle}{|\langle 2,-1,3\rangle|} = \frac{4}{\sqrt{14}}.$$

♣

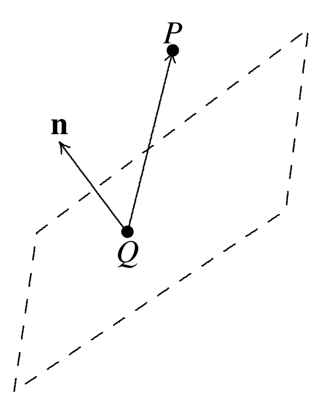


Figure 6.13: Distance from a point to a plane.

Example 6.19: Distance from a Point to a Line

Find the distance from the point $(-1,2,1)$ to the line $\langle 1,1,1\rangle + t\langle 2,3,-1\rangle$.

Solution. Again we want the distance measured perpendicular to the line, as indicated in Figure 6.14. The desired distance is

$$|\overrightarrow{QP}|\sin\theta = \frac{|\overrightarrow{QP}\times \mathbf{v}|}{|\mathbf{v}|},$$

where $\mathbf{v}$ is any vector parallel to the line. From the equation of the line, we can use $Q=(1,1,1)$ and $\mathbf{v}=\langle 2,3,-1\rangle$ along with $P=(-1,2,1)$, so the distance is

$$\frac{|\langle -2,1,0\rangle \times \langle 2,3,-1\rangle|}{\sqrt{14}} = \frac{|\langle -1,-2,-8\rangle|}{\sqrt{14}} = \frac{\sqrt{69}}{\sqrt{14}}.$$

♣

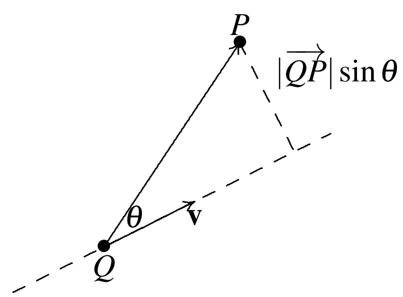


Figure 6.14: Distance from a point to a line.

Exercises for Section 6.5

Exercise 6.5.1 *Find an equation of the plane which satisfies the given conditions.*

(a) *contains* $(6,2,1)$, *perpendicular to* $\langle 1,1,1\rangle$

(b) *contains* $(-1,2,-3)$, *perpendicular to* $\langle 4,5,-1\rangle$

(c) *contains* $(1,2,-3)$, $(0,1,-2)$, $(1,2,-2)$

(d) *contains* $(1,0,0)$, $(4,2,0)$, $(3,2,1)$

(e) *contains* $(1,0,0)$, $(4,2,0)$, $(3,2,1)$

(f) *contains* $(1,0,0)$ *and the line* $\langle 1,0,2\rangle + t\langle 3,2,1\rangle$

(g) *contains the line of intersection of* $x+y+z=1$ *and* $x-y+2z=2$, *perpendicular to the x-y plane*

Exercise 6.5.2 *Find an equation of the line which satisfies the given conditions.*

(a) *passes through* $(1,0,3)$, $(1,2,4)$

(b) *passes through* $(1,0,3)$, *perpendicular to the plane* $x+2y-z=1$

(c) *passes through the origin, perpendicular to the plane* $x+y-z=2$

Exercise 6.5.3 *Find a and c so that* $(a,1,c)$ *is on the line through* $(0,2,3)$ *and* $(2,7,5)$.

Exercise 6.5.4 *Explain how to discover the solution in Example 6.17.*

Exercise 6.5.5 *Determine whether the given lines are parallel, intersect, or neither.*

(a) $\langle 1,3,-1\rangle + t\langle 1,1,0\rangle$ *and* $\langle 0,0,0\rangle + t\langle 1,4,5\rangle$

(b) $\langle 1,0,2\rangle + t\langle -1,-1,2\rangle$ *and* $\langle 4,4,2\rangle + t\langle 2,2,-4\rangle$

(c) $\langle 1,2,-1\rangle + t\langle 1,2,3\rangle$ *and* $\langle 1,0,1\rangle + t\langle 2/3,2,4/3\rangle$

(d) $\langle 1,1,2\rangle + t\langle 1,2,-3\rangle$ *and* $\langle 2,3,-1\rangle + t\langle 2,4,-6\rangle$

Exercise 6.5.6 *Find a unit normal vector to each of the coordinate planes.*

Exercise 6.5.7 *Show that* $\langle 2,1,3\rangle + t\langle 1,1,2\rangle$ *and* $\langle 3,2,5\rangle + s\langle 2,2,4\rangle$ *are the same line.*

Exercise 6.5.8 *Give a prose description for each of the following processes:*

(a) *Given two distinct points, find the line that goes through them.*

(b) *Given three points (not all on the same line), find the plane that goes through them. Why do we need the caveat that not all points be on the same line?*

(c) *Given a line and a point not on the line, find the plane that contains them both.*

(d) *Given a plane and a point not on the plane, find the line that is perpendicular to the plane through the given point.*

Exercise 6.5.9 *Find the distance:*

(a) *from* $(2,2,2)$ *to* $x+y+z=-1$

(b) *from* $(2,-1,-1)$ *to* $2x-3y+z=2$

(c) *from* $(2,-1,1)$ *to* $\langle 2,2,0\rangle + t\langle 1,2,3\rangle$

(d) *from* $(1,0,1)$ *to* $\langle 3,2,1\rangle + t\langle 2,-1,-2\rangle$

Exercise 6.5.10 *Find the cosine of the angle between the following planes:*

(a) $x+y+z=2$ *and* $x+2y+3z=8$

(b) $x-y+2z=2$ *and* $3x-2y+z=5$

6.6 Other Coordinate Systems

Coordinate systems are tools that let us use algebraic methods to understand geometry. While the **rectangular** (also called **Cartesian**) coordinates that we have been discussing are the most common, some problems are easier to analyze in alternate coordinate systems.

A coordinate system is a scheme that allows us to identify any point in the plane or in three-dimensional space by a set of numbers. In rectangular coordinates these numbers are interpreted, roughly speaking, as the lengths of the sides of a rectangular "box."

In two dimensions you may already be familiar with an alternative, called **polar coordinates**. In this system, each point in the plane is identified by a pair of numbers (r,θ). The number θ measures the counter-clockwise angle between the positive x-axis and a vector with tail at the origin and head at the point, as shown in Figure 6.15; the number r measures the distance from the origin to the point. Either of these may be negative; a negative θ indicates the angle is measured clockwise from the positive x-axis instead of counter-clockwise, and a negative r indicates the point at distance $|r|$ in the opposite of the direction given by θ.

The relationship between polar and rectangular coordinates is given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

and

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

Example 6.20: Rectangular to Polar Coordinates

Convert the point $(x,y) = (1,\sqrt{3})$ into polar coordinates.

Solution. First calculate r:

$$r = \sqrt{x^2+y^2} = \sqrt{1+3} = 2$$

Now find θ such that $\tan\theta = \frac{\sqrt{3}}{1}$. The required θ is $\frac{\pi}{3}$.

The polar coordinates are $(2,\frac{\pi}{3})$. ♣

Figure 6.15 also shows the point with rectangular coordinates $(1,\sqrt{3})$ and polar coordinates $(2,\pi/3)$, 2 units from the origin and $\pi/3$ radians from the positive x-axis.

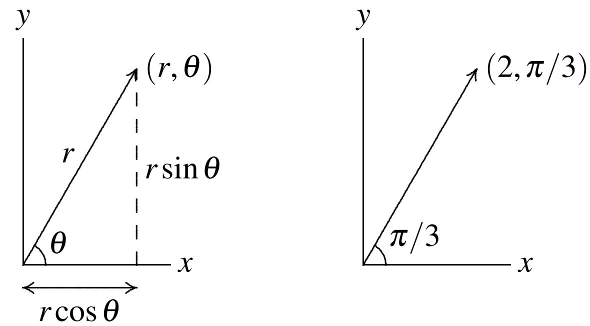


Figure 6.15: Polar coordinates: the general case and the point with rectangular coordinates $(1,\sqrt{3})$.

We can extend polar coordinates to three dimensions simply by adding a z coordinate; this is called **cylindrical coordinates**. Each point in three-dimensional space is represented by three coordinates (r,θ,z) in the obvious way: this point is z units above or below the point (r,θ) in the x-y-plane, as shown in Figure 6.16. The point with rectangular coordinates $(1,\sqrt{3},3)$ and cylindrical coordinates $(2,\pi/3,3)$ is also indicated in Figure 6.16.

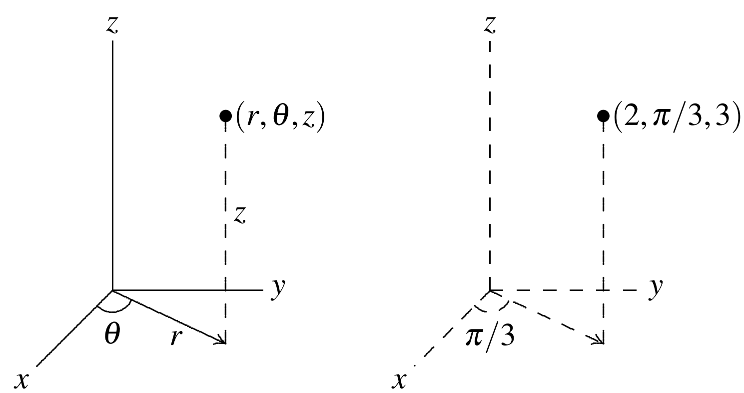


Figure 6.16: Cylindrical coordinates: the general case and the point with rectangular coordinates $(1,\sqrt{3},3)$.

Some figures with relatively complicated equations in rectangular coordinates will be represented by simpler equations in cylindrical coordinates. For example, the cylinder in Figure 6.17 has equation $x^2+y^2=4$ in rectangular coordinates, but equation $r=2$ in cylindrical coordinates.

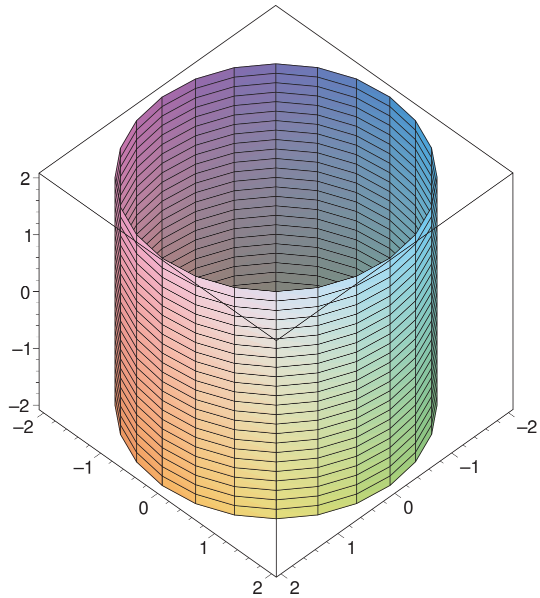


Figure 6.17: The cylinder $r = 2$.

Given a point (r, θ) in polar coordinates, it is easy to see (as in Figure 6.15) that the rectangular coordinates of the same point are $(r\cos\theta, r\sin\theta)$, and so the point (r, θ, z) in cylindrical coordinates is $(r\cos\theta, r\sin\theta, z)$ in rectangular coordinates. This means it is usually easy to convert any equation from rectangular to cylindrical coordinates: simply substitute

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

and leave z alone. For example, starting with $x^2 + y^2 = 4$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ gives

$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta &= 4 \\ r^2(\cos^2\theta + \sin^2\theta) &= 4 \\ r^2 &= 4 \\ r &= 2. \end{aligned}$$

Of course, it's easy to see directly that this defines a cylinder as mentioned above.

Cylindrical coordinates are an obvious extension of polar coordinates to three dimensions, but the use of the z coordinate means they are not as closely analogous to polar coordinates as another standard coordinate system. In polar coordinates, we identify a point by a direction and distance from the origin; in three dimensions we can do the same thing, in a variety of ways. The question is: how do we represent a direction? One way is to give the angle of rotation, θ, from the positive x-axis, just as in cylindrical coordinates, and also an angle of rotation, ϕ, from the positive z-axis. Roughly speaking, θ is like longitude and ϕ is like latitude. (Earth longitude is measured as a positive or negative angle from the prime meridian, and is always between 0 and 180 degrees, east or west; θ can be any positive or negative angle, and we use radians except in informal circumstances. Earth latitude is measured north or south from the equator; ϕ is measured from the north pole down.) This system is called **spherical coordinates**; the coordinates are listed in the order (ρ, θ, ϕ), where ρ is the distance from the origin, and like r in polar and cylindrical coordinates it may be negative. The general case and an example are pictured in Figure 6.18; the length marked r is the r of cylindrical coordinates.

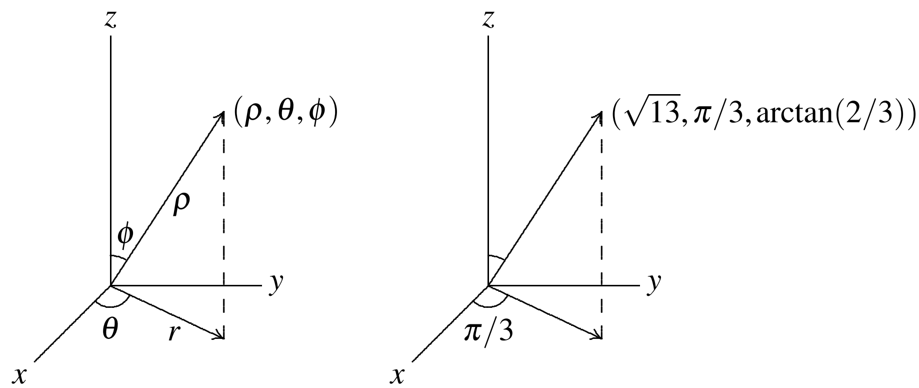


Figure 6.18: Spherical coordinates: the general case and the point with rectangular coordinates $(1,\sqrt{3},3)$.

As with cylindrical coordinates, we can easily convert equations in rectangular coordinates to the equivalent in spherical coordinates, though it is a bit more difficult to discover the proper substitutions. Figure 6.19 shows the typical point in spherical coordinates from Figure 6.18, viewed now so that the arrow marked r in the original graph appears as the horizontal "axis" in the left hand graph. From this diagram it is easy to see that the z coordinate is $\rho\cos\phi$, and that $r=\rho\sin\phi$, as shown. Thus, in converting from rectangular to spherical coordinates we will replace z by $\rho\cos\phi$. To see the substitutions for x and y we now view the same point from above, as shown in the right hand graph. The hypotenuse of the triangle in the right hand graph is $r=\rho\sin\phi$, so the sides of the triangle, as shown, are $x=r\cos\theta=\rho\sin\phi\cos\theta$ and $y=r\sin\theta=\rho\sin\phi\sin\theta$. Therefore to convert from rectangular to spherical coordinates, we make these substitutions:

$$\begin{aligned}x&=\rho\sin\phi\cos\theta\\y&=\rho\sin\phi\sin\theta\\z&=\rho\cos\phi.\end{aligned}$$

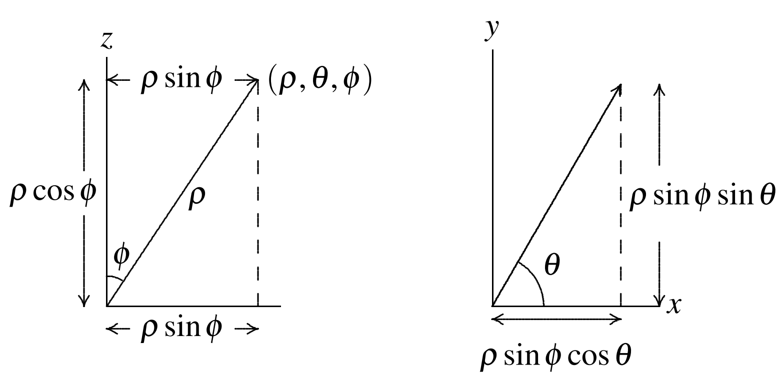


Figure 6.19: Converting from rectangular to spherical coordinates.

As the cylinder had a simple equation in cylindrical coordinates, so does the sphere in spherical coordinates.

Example 6.21:

Find an equation for the sphere of radius 2 in spherical coordinates.

Solution. If we start with the Cartesian equation of the sphere and substitute, we get the spherical equation:

$$\begin{aligned}
x^2+y^2+z^2 &= 2^2 \\
\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi &= 2^2 \\
\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)+\rho^2\cos^2\phi &= 2^2 \\
\rho^2\sin^2\phi+\rho^2\cos^2\phi &= 2^2 \\
\rho^2(\sin^2\phi+\cos^2\phi) &= 2^2 \\
\rho^2 &= 2^2 \\
\rho &= 2
\end{aligned}$$

Therefore, in spherical coordinates, a sphere of radius 2 is expressed $\rho = 2$. ♣

Although not as simple as with cylindrical coordinates, we can use spherical coordinates to describe the equation of a cylinder.

Example 6.22: Cylinder Equation in Spherical Coordinates

Find an equation for the cylinder $x^2+y^2=4$ in spherical coordinates.

Solution. Proceeding as in the previous example:

$$\begin{aligned}
x^2+y^2 &= 4 \\
\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta &= 4 \\
\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) &= 4 \\
\rho^2\sin^2\phi &= 4 \\
\rho\sin\phi &= 2 \\
\rho &= \frac{2}{\sin\phi}
\end{aligned}$$

♣

Exercises for Section 6.6

Exercise 6.6.1 *Convert the following points in rectangular coordinates to cylindrical and spherical coordinates:*

(a) $(1,1,1)$

(b) $(7,-7,5)$

(c) $(\cos(1),\sin(1),1)$

(d) $(0,0,-\pi)$

Exercise 6.6.2 *Find an equation for the sphere* $x^2+y^2+z^2=4$ *in cylindrical coordinates.*

Exercise 6.6.3 *Find an equation for the y-z-plane in cylindrical coordinates.*

Exercise 6.6.4 *Find an equation equivalent to* $x^2+y^2+2z^2+2z-5=0$ *in cylindrical coordinates.*

Exercise 6.6.5 *Suppose the curve* $z=e^{-x^2}$ *in the x-z-plane is rotated around the z-axis. Find an equation for the resulting surface in cylindrical coordinates.*

Exercise 6.6.6 *Suppose the curve* $z=x$ *in the x-z-plane is rotated around the z-axis. Find an equation for the resulting surface in cylindrical coordinates.*

Exercise 6.6.7 *Find an equation for the plane* $y=0$ *in spherical coordinates.*

Exercise 6.6.8 *Find an equation for the plane* $z=1$ *in spherical coordinates.*

Exercise 6.6.9 *Find an equation for the sphere with radius 1 and center at* $(0,1,0)$ *in spherical coordinates.*

Exercise 6.6.10 *Find an equation for the cylinder* $x^2+y^2=4$ *in spherical coordinates.*

Exercise 6.6.11 *Suppose the curve* $z=x$ *in the x-z-plane is rotated around the z-axis. Find an equation for the resulting surface in spherical coordinates.*

Exercise 6.6.12 *Plot the polar equations* $r=\sin(\theta)$ *and* $r=\cos(\theta)$ *and comment on their similarities. (If you get stuck on how to plot these, you can multiply both sides of each equation by r and convert back to rectangular coordinates).*

Exercise 6.6.13 *Extend Exercises 6.6.6 and 6.6.11 by rotating the curve* $z=mx$ *around the z-axis and converting to both cylindrical and spherical coordinates.*

Exercise 6.6.14 *Convert the spherical formula* $\rho=\sin\theta\sin\phi$ *to rectangular coordinates and describe the surface defined by the formula (Hint: Multiply both sides by* ρ*.)*

Exercise 6.6.15 *We can describe points in the first octant by* $x>0$*,* $y>0$ *and* $z>0$*. Give similar inequalities for the first octant in cylindrical and spherical coordinates.*

7. Multi-Variable Calculus

7.1 Functions of Several Variables

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ maps a single real value x to a single real value y. Such a function is referred to as a **single-variable** function and can be readily visualized in a two-dimensional coordinate system: above (or below) each point x on the x-axis we graph the point y, where of course $y = f(x)$. By now, you have seen the graphs of many such functions. We now extend this visualization process to **multi-variable** functions, also referred to as functions of several variables.

In single-variable calculus we were concerned with functions that map the real numbers $\mathbb{R}$ to $\mathbb{R}$, sometimes called "real functions of one variable", meaning the "input" is a single real number and the "output" is likewise a single real number. Now we turn to functions of several variables, where several input variables are mapped to one value: functions $f\colon \mathbb{R}^n \to \mathbb{R}$. We will deal primarily with $n = 2$ and to a lesser extent $n = 3$; in fact many of the techniques we discuss can be applied to larger values of n as well.

A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ maps a pair of values (x,y) to a single real number. The three-dimensional coordinate system is a convenient way to visualize such functions: above (or below) each point (x,y) in the x-y-plane we graph the point (x,y,z), where then $z = f(x,y)$. In other words, in interpreting the graph of a function $f(x,y)$, one often thinks of the value $z = f(x,y)$ of the function at the point (x,y) as the *height* of the point (x,y,z) on the graph of f. If $f(x,y) > 0$, then the point (x,y,z) is $f(x,y)$ units above the x-y-plane; if $f(x,y) < 0$, then the point (x,y,z) is $|f(x,y)|$ units below the x-y-plane.

In general, it is quite difficult to draw the graph of a function of two variables. But techniques have been developed that enable us to generate such graphs with minimum effort, using a computer. Still, it is valuable to be able to visualize relatively simple surfaces without such aids. It is often a good idea to examine the function on restricted subsets of the plane, especially lines. It can also be useful to identify those points (x,y) that share a common z-value.

Before we introduce some special and some general three-dimensional graphs and their equations, let us first consider the domain of functions in two variables.

The variables x and y are called *independent* variables, and the variable z, which is dependent on the values of x and y, is referred to as a *dependent* variable. As indicated already, the number $z = f(x,y)$ is called the *value* of f at the point (x,y). Unless specified otherwise, the domain of the function f will be taken to be the largest possible set for which the rule defining f is meaningful. We will demonstrate this with several examples.

Example 7.1: Domain of Two-Variable Functions

Find the domain of each of the following functions.

(a) $f(x,y) = \sqrt{1-x^2-y^2}$.

(b) $g(x,y) = \dfrac{1}{\sqrt{1-x^2-y^2}}$.

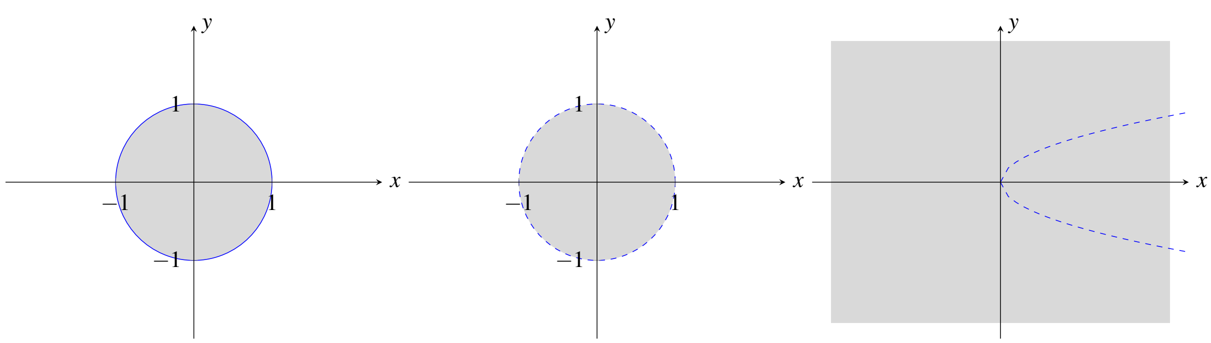
(c) $h(x,y) = \dfrac{1}{y-x^2}$.

Solution.

(a) The domain of f is the set of points (x,y) such that $1-x^2-y^2 \geq 0$. We recognize $x^2+y^2=1$ as the equation of a circle of radius 1 centred at the origin, and so the domain of f consists of all points which lie *on* or *inside* this circle (see below).

(b) We now require that $1-x^2-y^2 > 0$. The domain of g therefore contains all points (x,y) such that $x^2+y^2 < 1$; that is, all points which lie strictly *inside* the unit circle (see below).

(c) We see that $h(x,y)$ is undefined for $x = y^2$. The domain of h therefore consists of all points in the x-y-plane except those which satisfy $y = \pm\sqrt{x}$ (see below).



(a) *Domain of f shown in grey.* (b) *Domain of g shown in grey.* (c) *Domain of h shown in grey.*

♣

Example 7.2: Domain of Two-Variable Functions

A manufacturer produces a model X and a model Y, and determines that the unit prices of these two products are related. Let q_x be the weekly quantity demanded of model X, and let q_y be the weekly quantity demanded of model Y. The unit price of model X is found to be

$$p_x = 500 - q_x - \frac{1}{3}q_y,$$

and the unit price of model Y is found to be

$$p_y = 200 - \frac{1}{3}q_x - \frac{1}{5}q_y.$$

(a) *Determine the revenue function $R(q_x, q_y)$.*

(b) *Sketch the domain of R.*

Solution.

(a) Selling q_x units of model X yields a revenue of $q_x p_x$ dollars per week, and selling q_y units of model Y yields a revenue of $q_y p_y$ dollars per week. We therefore construct the revenue function R as

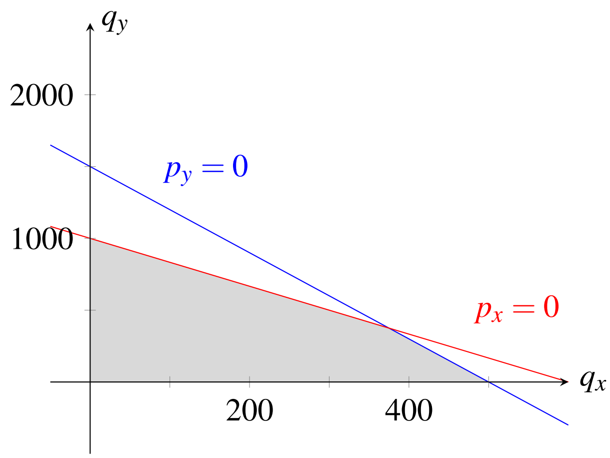
$$\begin{aligned} R(q_x, q_y) &= q_x p_x + q_y p_y \\ &= q_x\left(500 - q_x - \frac{1}{3}q_y\right) + q_y\left(200 - \frac{1}{3}q_x - \frac{1}{5}q_y\right) \\ &= \frac{1}{15}\left(-15q_x^2 - 10q_xq_y + 7500q_x - 3q_y^2 + 3000q_y\right), \end{aligned}$$

dollars per week.

(b) The domain of R is all points (q_x, q_y) in the plane such that q_x,q_y,p_x and $p_y \geq 0$. That is, where

$$\begin{aligned} q_x &\geq 0, \\ q_y &\geq 0, \\ 500 - q_x - \tfrac{1}{3}q_y \geq 0 &\iff q_y \leq 1500 - 3q_x \\ 200 - \tfrac{1}{3}q_x - \tfrac{1}{5}q_y \geq 0 &\iff q_y \leq 1000 - \tfrac{5}{3}q_x. \end{aligned}$$

We find the domain by graphing the two lines given by $p_x = 0$ and $p_y = 0$ in the first quadrant of the q_x-q_y-plane. The domain of R is the region which is bounded by the lines $p_x = 0$, $p_y = 0$, $q_x = 0$ and $q_y = 0$ for which the desired inequality holds. The final solution is given by the shaded region (including the boundary) in the figure below.



♣

Let us now consider some functions that we can describe and graph with the knowledge we have attained so far.

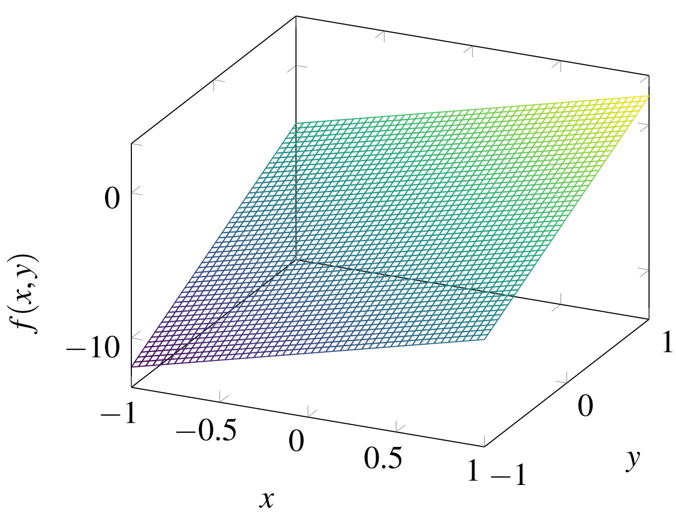
Example 7.3: Plane

Describe and graph the function $f(x,y) = 3x + 4y - 5$.

Solution.

We first write this as $z = 3x + 4y - 5$ and then $3x + 4y - z = 5$, which is the equation of a plane. However, we can also determine that this equation represents a plane from the following analysis: If we hold x constant, then the equation simplifies to that of a straight line in the y-z-plane. Generally speaking, if we choose any point on the graph of f, for example from $f(0,0) = -5$, this function grows linearly in every direction in the same way that a linear function behaves as can be seen below.

To graph the plane, we alternately let x, then y, then z be equal to zero, which leads to a linear equation in the y-z-, x-z-, and x-y-planes respectively. These lines represent the intersection of the functions' graph with each of the coordinate planes in the three-dimensional coordinate system and form part of the plane surface as shown below.

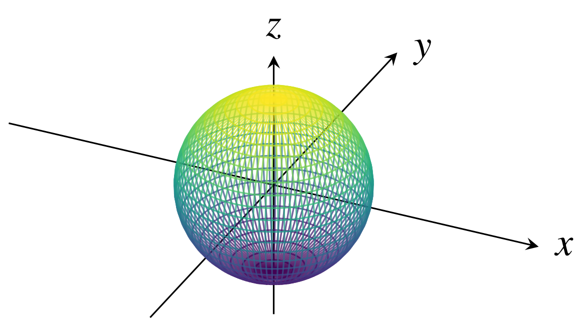


♣

Example 7.4: Sphere

Describe and graph the equation $x^2+y^2+z^2=4$.

Solution. The equation $x^2+y^2+z^2=4$ represents a sphere of radius 2 and centre $(0,0,0)$ as shown below. We cannot write this in the form $z=f(x,y)$, since for each x and y in the disk $x^2+y^2<4$ there are two corresponding points on the sphere, namely one above and one below this point (x,y). As with the equation of a circle, we can resolve this equation into two functions, $f_1(x,y)=\sqrt{4-x^2-y^2}$ and $f_2(x,y)=-\sqrt{4-x^2-y^2}$, representing the upper and lower hemispheres, respectively. Each of these is an example of a function with a restricted domain: only certain values of x and y make sense (namely, those for which $x^2+y^2\leq 4$) and the graphs of these functions are limited to a small region of the plane.

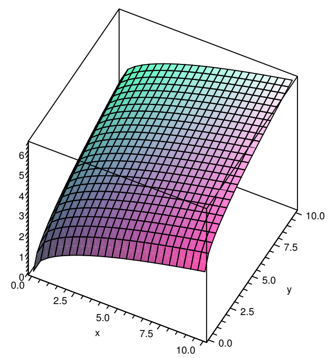


♣

Example 7.5: Square Root

Describe and graph the function $f(x,y)=\sqrt{x}+\sqrt{y}$.

Solution. This function is defined only when both x and y are non-negative. When $y=0$ we get $f(x,y)=\sqrt{x}$, the familiar square root function in the x-z-plane, and when $x=0$ we get the same curve in the y-z-plane. Generally speaking, we see that starting from $f(0,0)=0$ this function gets larger in every direction in roughly the same way that the square root function gets larger. For example, if we restrict attention to the line $x=y$, we get $f(x,y)=2\sqrt{x}$ and along the line $y=2x$ we have $f(x,y)=\sqrt{x}+\sqrt{2x}=(1+\sqrt{2})\sqrt{x}$ (see graph below).



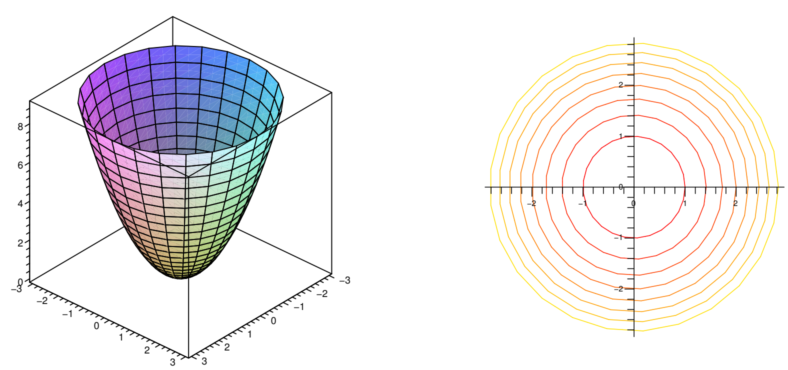
♣

Example 7.6: Elliptic Paraboloid

Describe and graph the function $f(x,y) = x^2 + y^2$.

Solution. When $x = 0$ this becomes $f(y) = y^2$, a parabola in the y-z-plane; when $y = 0$ we get the "same" parabola $f(x) = x^2$ in the x-z-plane.

Finally, picking a value $z = k$, at what points does $f(x,y) = k$? This means $x^2 + y^2 = k$, which we recognize as the equation of a circle of radius $\sqrt{k}$, as seen in the graph below to the right. So the graph of $f(x,y)$ has parabolic cross-sections as shown in the graph below to the left, and the same height everywhere on concentric circles with centre at the origin. This fits with what we have already discovered.



♣

As in this example, the points (x,y) such that $f(x,y) = k$ form a curve, called a **level curve** of the function. A graph of some level curves can give a good idea of the shape of the surface; it looks much like a topographic map of the surface. By drawing the level curves corresponding to several admissible values of k, we obtain a **contour map**. In the graph of Example 7.6 both the surface and its associated level curves are shown. Note that, as with a topographic map, the heights corresponding to the level curves are evenly spaced, so that where curves are closer together the surface is steeper.

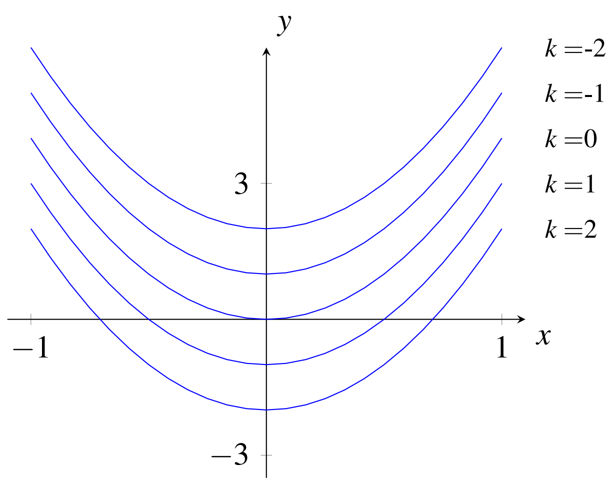
Example 7.7: Level Curves

Sketch the level curves of the function $z = f(x,y) = 4x^2 - y$ corresponding to $z = -2, -1, 0, 1, 2$.

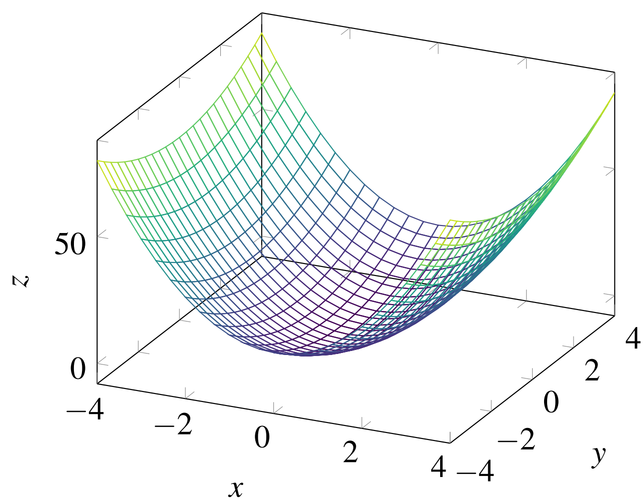
Solution. We find the level curves of f by setting $f(x,y) = z$ to be a constant. For $z = -2, -1, 0, 1, 2$, we find the equations

$$y = 4x^2 + 2$$
$$y = 4x^2 + 1$$
$$y = 4x^2$$
$$y = 4x^2 - 1$$
$$y = 4x^2 - 2$$

and so each level curve is a parabola in the x-y-plane, where only the y-intercept changes. These level curves are shown in the graph below to the left, and the graph of $f(x,y)$ is shown to the right for reference.



(a) *Level curves of f*



(b) *The graph of $z = f(x,y)$*

♣

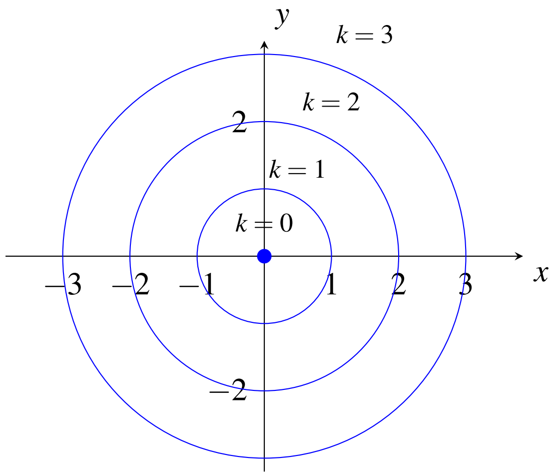
Example 7.8: Level Curves

Sketch the level curves of the function $z = f(x,y) = \sqrt{x^2 + y^2}$ corresponding to $z = 0, 1, 2, 3$.

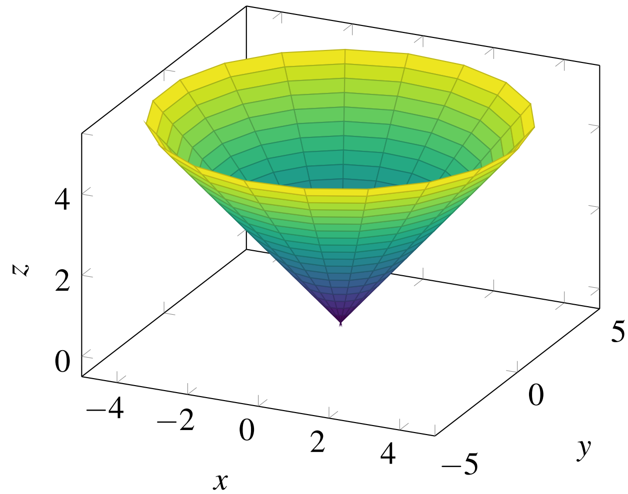
Solution. For each level curve corresponding to $z = c$, we have

$$\sqrt{x^2 + y^2} = c \implies x^2 + y^2 = c^2,$$

which is the equation of a circle centred at $(0,0)$ of radius c. These level curves are drawn below in the graph to the left. The graph of $z = f(x,y)$ is provided to the right.



(a) *Level curves of f*



(b) *The graph of $z = f(x,y)$*

♣

Example 7.9: Home Mortgage Payments

The monthly payment for a condo that amortizes a loan of A dollars in t years when the interest rate is r per year is given by

$$P = f(A,r,t) = \frac{Ar}{12\left[1-\left(1+\frac{r}{12}\right)^{-12t}\right]}$$

Find the monthly payment for a home mortgage of $240,000 to be amortized over 25 years when the interest rate is 4% per year.

Solution. P is a function of three variables: A, r and t. To find the required monthly payments, we evaluate P at the given values,

$$\begin{aligned} P &= f(240000, 0.04, 25) \\ &= \frac{240000(0.04)}{12\left[1-\left(1+\frac{0.04}{12}\right)^{-12(25)}\right]} \\ &\approx 1266.81. \end{aligned}$$

Thus, the monthly payments would be about $1266.81. ♣

Functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for $n > 2$ behave much like functions of two variables. The principal difficulty with such functions is visualizing them, as they do not "fit" in the three dimensions we are familiar with. For $n = 3$ variables there are various ways to interpret functions that make them easier to understand. For example, $f(x,y,z)$ could represent the temperature at the point (x,y,z), or the strength of a magnetic field. Similar to considering level curves with two-variable functions, it is useful to consider those points at which $f(x_1, x_2, \ldots, x_n) = k$, where k is some constant for n-variable functions. This collection of points is called a **level set**. For three variables, a level set is a surface, called a **level surface**.

Example 7.10: Level Surfaces

Suppose the temperature at (x,y,z) is $T(x,y,z) = e^{-(x^2+y^2+z^2)}$. Describe the level surfaces.

Solution. This function has a maximum value of 1 at the origin, and tends to 0 in all directions. If k is positive and at most 1, the set of points for which $T(x,y,z) = k$ is those points satisfying $x^2+y^2+z^2 = -\ln k$, a sphere centred at the origin. The level surfaces are the concentric spheres centred at the origin. ♣

Exercises for Section 7.1

Exercise 7.1.1 *Find the equations and describe the shapes of the cross-sections when $x = 0$, $y = 0$ and $x = y$. Plot the surface with a three-dimensional graphing tool.*

(a) $f(x,y) = (x-y)^2$

(b) $f(x,y) = |x|+|y|$

(c) $f(x,y) = e^{-(x^2+y^2)}\sin(x^2+y^2)$

(d) $f(x,y) = \sin(x-y)$

(e) $f(x,y) = (x^2-y^2)^2$

Exercise 7.1.2 *Find the domain of each of the following functions:*

(a) $f(x,y) = \sqrt{9-x^2}+\sqrt{y^2-4}$

(b) $h(u,v) = \sqrt{2-u^2-v^2}$

(c) $f(x,y) = \sqrt{16-x^2-4y^2}$

(d) $f(x,y) = x+4y$

(e) $g(x,y,z) = x^2+y^2+z^2$

(f) $f(x,y) = \arcsin(x^2+y^2-2)$

(g) $h(u,v) = \dfrac{uv}{u-v}$

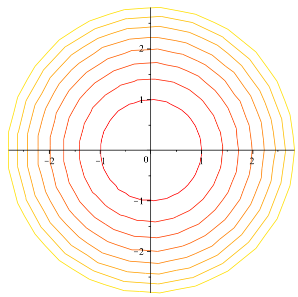
(h) $f(s,t) = \sqrt{s^2+t^2}$

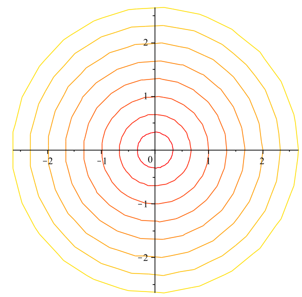
(i) $g(r,s) = \sqrt{rs}$

(j) $f(x,y) = e^{-xy}$

(k) $h(x,y) = \ln(x+y-7)$

Exercise 7.1.3 *Below are two sets of level curves at a sequence of equally-spaced heights z. One is for a* *cone, one is for a paraboloid. Which is which? Explain.*





Exercise 7.1.4 *Sketch the level curves of the function corresponding to the given values of z.*

(a) $f(x,y) = 3x + 2y$, $z = -2, -1, 0, 1, 2$

(b) $f(x,y) = x^2 - y$, $z = -2, -1, 0, 1, 2$

(c) $f(x,y) = 2y + x^2$, $z = -2, -1, 0, 1, 2$

(d) $f(x,y) = 2xy$, $z = -4, -2, 2, 4$

(e) $f(x,y) = \sqrt{25 - x^2 - y^2}$, $z = 0, 1, 3, 5$

(f) $f(x,y) = e^y - x^2$, $z = 1, 2, 3, 4$

Exercise 7.1.5 *A clothing retailer sells casual and business jackets. Let q_c denote the quantity of casual jackets demanded monthly and q_b denote the quantity of business jackets demanded monthly. It is determined that the unit price of casual jackets is approximately*

$$p_c = 100 - \frac{1}{4}q_c - \frac{1}{7}q_b,$$

and that the unit price of business jackets is approximately

$$p_b = 150 - \frac{1}{10}q_c - \frac{1}{3}q_b$$

dollars, respectively.

(a) *Determine the monthly revenue, $R(q_c, q_b)$.*

(b) *Determine the domain of R.*

Exercise 7.1.6 *A certain manufacturer produces basic and enhanced versions of their product. Let q_b denote the quantity of basic units daily demanded and q_e denote the quantity of enhanced units demanded daily. It is determined that the unit price of the basic units is approximately*

$$p_b = 10 - 0.1q_b - 0.5q_e,$$

and that the unit price of the enhanced units is approximately

$$p_e = 30 - 0.4q_b - q_e$$

dollars, respectively.

(a) *Determine the daily revenue, $R(q_b, q_e)$.*

(b) Determine the domain of R.

Exercise 7.1.7 *We can calculate the outstanding principal on a loan at the end of i months by the formula*

$$B(A,r,t,i) = A\left(\frac{\left(1+\frac{r}{12}\right)^{i}-1}{\left(1+\frac{r}{12}\right)^{12t}-1}\right) \quad 0 \le i \le 12t$$

where A is the principal loan, r is the annual interest rate, and t is the amortization period in years. Suppose the original amount borrowed is \$100,000, and the interest rate charged by the bank is 3%. What is the amount owed to the bank after 2 years if the loan is to be repaid in equal instalments over 25 years? What is the amount owed after 20 years?

Exercise 7.1.8 *In economics, the given optimal quantity Q of goods for a store to order is given by the Wilson lot-size formula:*

$$Q(C,N,H) = \sqrt{\frac{2CN}{H}}$$

where C is the cost of placing an order, N is the number of items the store sells per day, and H is the daily holding cost for each item. What is the most economical quantity of winter tires to order if the store pays \$25 for placing an order, pays \$2 for holding a tire per day, and expects to sell 60 tires a day?

7.2 Limits and Continuity

To develop calculus for functions of one variable, we needed to make sense of the concept of a limit, which was used in the definition of a continuous function and the derivative of a function. Limits involving functions of two variables can be considerably more difficult to deal with; fortunately, most of the functions we encounter are fairly easy to understand.

The potential difficulty is largely due to the fact that there are many ways to "approach" a point in the x-y-plane. If we want to say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we need to capture the idea that as (x,y) gets close to (a,b) then $f(x,y)$ gets close to L. For functions of one variable, $f(x)$, there are only two ways that x can approach a: from the left or right. But there are an infinite number of ways to approach (a,b): along any one of an infinite number of straight lines, or even along a curved path in the x-y-plane. We might hope that it's really not so bad—suppose, for example, that along every possible line through (a,b) the value of $f(x,y)$ gets close to L; surely this means that "$f(x,y)$ approaches L as (x,y) approaches (a,b)". Sadly, no.

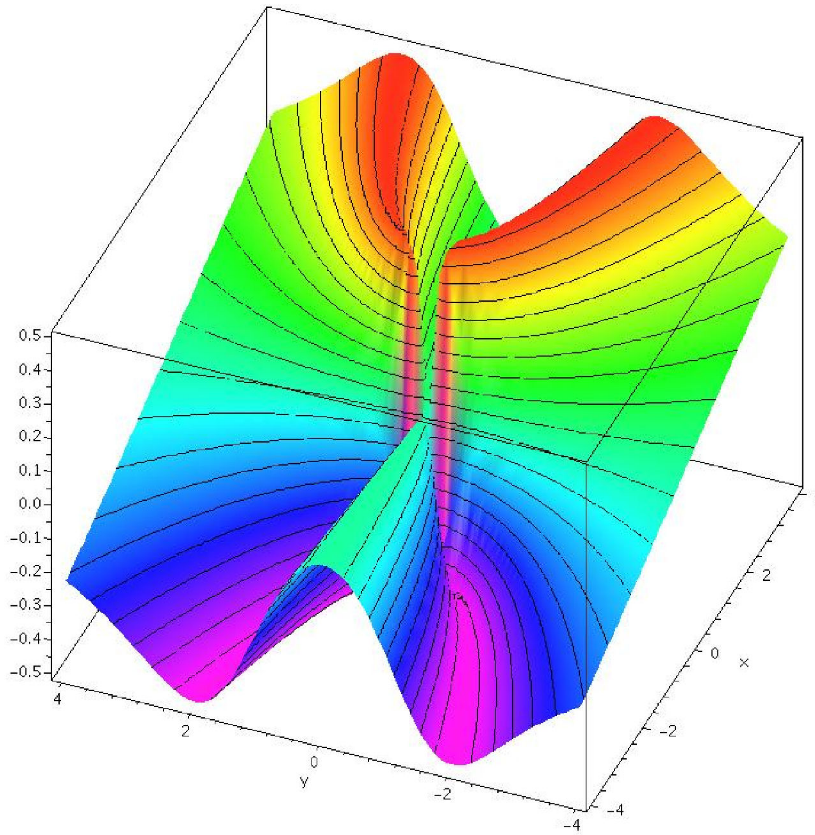


Figure 7.1: $f(x,y) = \dfrac{xy^2}{x^2+y^4}$

Example 7.11: Weird Limit

Analyze $f(x,y) = xy^2/(x^2+y^4)$.

Solution. When $x = 0$ or $y = 0$, $f(x,y)$ is 0, so the limit of $f(x,y)$ approaching the origin along either the x or y axis is 0. Moreover, along the line $y = mx$, $f(x,y) = m^2x^3/(x^2+m^4x^4)$. As x approaches 0 this expression approaches 0 as well. So along every line through the origin $f(x,y)$ approaches 0. Now suppose we approach the origin along $x = y^2$. Then

$$f(x,y) = \frac{y^2y^2}{y^4+y^4} = \frac{y^4}{2y^4} = \frac{1}{2},$$

so the limit is $1/2$. Looking at Figure 7.1, it is apparent that there is a ridge above $x = y^2$. Approaching the origin along a straight line, we go over the ridge and then drop down toward 0, but approaching along the ridge the height is a constant $1/2$. ♣

Fortunately, we can define the concept of limit without needing to specify how a particular point is approached—indeed, in Definition 3.4, we didn't need the concept of "approach." Roughly, that definition says that when x is close to a then $f(x)$ is close to L; there is no mention of "how" we get close to a. We can adapt that definition to two variables quite easily:

Definition 7.12: Limit of a Multivariate Function

Suppose $f(x,y)$ is a two-variable function. We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, $|f(x,y)-L| < \varepsilon$.

This says that we can make $|f(x,y)-L| < \varepsilon$, no matter how small ε is, by making the distance from (x,y) to (a,b) "small enough".

Example 7.13: Multivariate Limit

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

Solution. Suppose $\varepsilon > 0$. Then

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}3|y|.$$

Note that $x^2/(x^2+y^2) \le 1$ and $|y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \delta$. So

$$\frac{x^2}{x^2+y^2}3|y| < 1\cdot 3\cdot \delta.$$

We want to force this to be less than ε by picking δ "small enough." If we choose $\delta = \varepsilon/3$ then

$$\left|\frac{3x^2y}{x^2+y^2}\right| < 1\cdot 3\cdot \frac{\varepsilon}{3} = \varepsilon.$$

♣

Recall that a function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$. We can say exactly the same thing about a function of two variables: $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

The function $f(x,y) = 3x^2y/(x^2+y^2)$ is not continuous at $(0,0)$, because $f(0,0)$ is not defined. However, we know that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, so we can make a continuous function, by extending the definition of f so that $f(0,0) = 0$. This surface is shown in Figure 7.2.

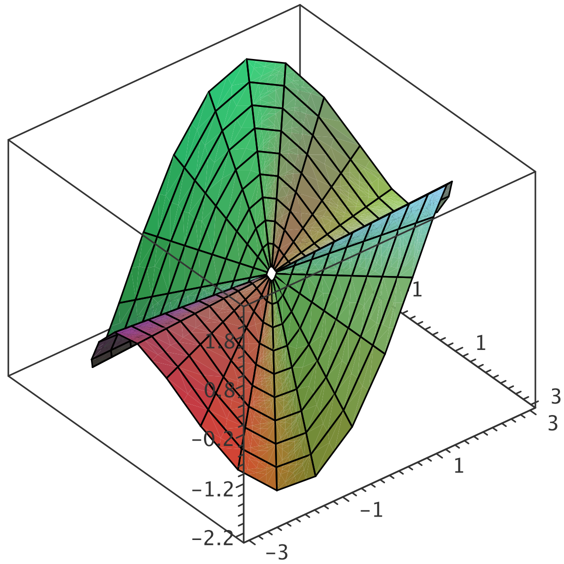


Figure 7.2: $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

Note that we cannot extend the definition of the function in Example 7.11 to create a continuous function, since the limit does not exist as we approach $(0,0)$.

Fortunately, the functions we will be working with will usually be continuous almost everywhere. As with single variable functions, two classes of common functions are particularly useful and easy to describe. A polynomial in two variables is a sum of terms of the form ax^my^n, where a is a real number and m and n are non-negative integers. A rational function is a quotient of polynomials.

Theorem 7.14: Continuity of Functions

Polynomials are continuous everywhere. Rational functions are continuous everywhere they are defined.

Exercises for Section 7.2

Exercise 7.2.1 *Determine whether each limit exists. If it does, find the limit and prove that it is the limit; if it does not, explain how you know.*

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

(e) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

(f) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{2x^2+y^2}}$

(g) $\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$

(h) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$

(i) $\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$

(j) $\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}$

(k) $\lim_{(x,y)\to(1,-1)} 3x+4y$

(l) $\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2}$

Exercise 7.2.2 *Does the function* $f(x,y) = \frac{x-y}{1+x+y}$ *have any discontinuities? What about* $f(x,y) = \frac{x-y}{1+x^2+y^2}$*? Explain.*

7.3 Partial Differentiation

The derivative of a function of a single variable tells us how quickly the value of the function changes as the value of the independent variable changes. Intuitively, it tells us how "steep" the graph of the function is. We might wonder if there is a similar idea for graphs of functions of two variables, that is, surfaces. It is not clear that this has a simple answer, nor how we might proceed. We will start with what seem to be very small steps toward the goal. Surprisingly, it turns out that these simple ideas hold the keys to a more general understanding.

7.3.1. First-Order Partial Derivatives

The derivative of a single-variable function $f(x)$ tells us how much $f(x)$ changes as x changes. There is no ambiguity when we speak about the rate of change of $f(x)$ with respect to x since x must be constrained to move along the x-axis. The situation becomes more complicated, however, when we study the rate of change of a function of two or more variables. The obvious analogue for a function of two variables $g(x,y)$ would be something that tells us how quickly $g(x,y)$ increases as x and y increase. However, in most cases this will depend on how quickly x and y are changing relative to each other.

Example 7.15: Analyzing a Simple Rate of Change in One Variable

Given $f(x,y) = y^2$, analyze the rate of change of f when one variable changes and the other variable is kept constant.

Solution. If we look at a point (x,y,y^2) on this surface, the value of a function does not change at all if we fix y and let x increase, but increases like y^2 if we fix x and let y increase. ♣

Example 7.16: Analyzing Rates of Change in One Variable

Given $f(x,y) = x^2 + y^2$, analyze the rate of change of f when one variable changes and the other variable is kept constant.

Solution. Now let us consider what happens to $f(x,y)$ when both x and y are increasing, perhaps at different rates. We can think of this as being a movement in a certain direction of a point in the x-y-plane. A point and a direction defines a line in the x-y-plane, and so we are asking how the function changes as we move along this line.

Introducing cross-sections: Let us then imagine a plane perpendicular to the x-y-plane that intersects the x-y-plane along this line. This plane will intersect the surface of f in a curve, so we can just look at the behaviour of this curve in the given plane.

Figure 7.3 shows the plane $x + y = 1$, which is the plane perpendicular to the line $x + y = 1$ in the x-y-plane. Observe that its intersection with the surface of f is a curve, in fact, a parabola. We will refer to such a curve as the cross-section of the surface above the line in the x-y-plane.

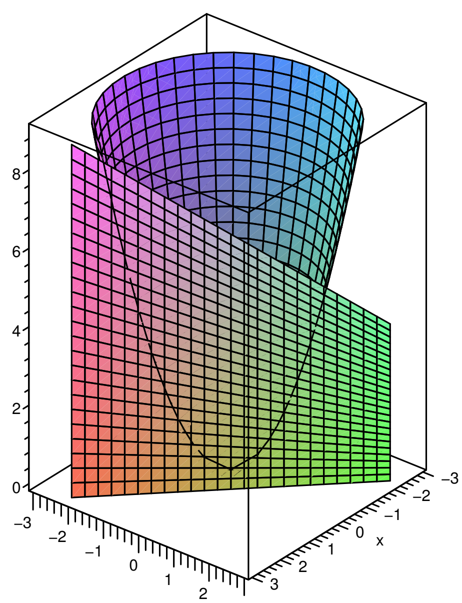


Figure 7.3: The intersection of a plane $x + y = 1$ and the surface $f(x,y) = x^2 + y^2$

How to think of a rate of change: We can now look at the rate of change (or slope) of f in a particular direction by looking at the slope of a curve in a plane — something we already have experience with.

Simple case - lines parallel to the x- or y- axis: Let's start by looking at some particularly easy lines: Those parallel to the x- or y- axis. Suppose we are interested in the cross-section of $f(x,y)$ above the line $y = b$. If we substitute b for y in $f(x,y)$, we get a function in one variable, describing the height of the cross-section as a function of x. Because $y = b$ is parallel to the x-axis, if we view it from a vantage point on the negative y-axis, we will see what appears to be simply an ordinary curve in the x-z-plane.

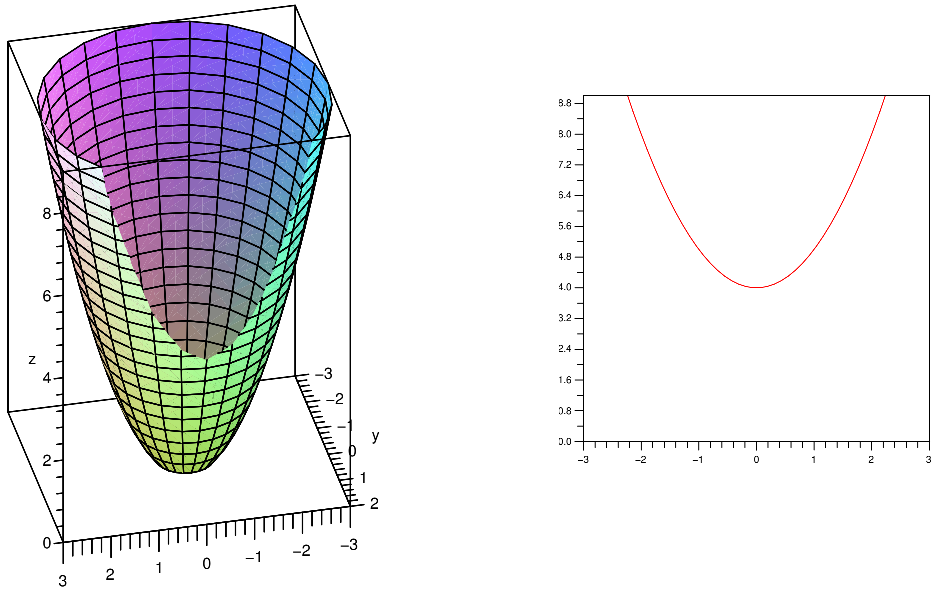


Figure 7.4: $f(x,y) = x^2 + y^2$**, cut by the plane** $y = 2$

We now consider a particular cross-section. The cross-section above the line $y = 2$ consists of all points $(x, 2, x^2 + 4)$. Looking at this cross-section we see what appears to be just the curve $f(x) = x^2 + 4$. At any point on the cross-section, $(a, 2, a^2 + 4)$, the slope of the surface *in the direction of the line* $y = 2$ is simply the slope of the curve $f(x) = x^2 + 4$, namely $2x$. Figure 7.4 shows the same parabolic surface as before, but now cut by the plane $y = 2$. The left graph shows the cut-off surface, the right shows just the cross-section.

If, for example, we're interested in the point $(-1, 2, 5)$ on the surface, then the slope in the direction of the line $y = 2$ is $2x = 2(-1) = -2$. This means that starting at $(-1, 2, 5)$ and moving on the surface, above the line $y = 2$, in the direction of increasing x-values, the surface goes down; of course moving in the opposite direction, toward decreasing x-values, the surface will rise.

Generalizing our findings: If we're interested in some other line $y = k$, there is really no change in the computation. The equation of the cross-section above $y = k$ is $x^2 + k^2$ with derivative $2x$. We can save ourselves the effort, small as it is, of substituting k for y: all we are in effect doing is temporarily assuming that y is some constant. With this assumption, the derivative $\frac{d}{dx}(x^2 + y^2) = 2x$. To emphasize that we are only temporarily assuming y is constant, we use a slightly different notation: $\frac{\partial}{\partial x}(x^2 + y^2) = 2x$; the "$\partial$" reminds us that there are more variables than x, but that only x is being treated as a variable. We read the equation as "the partial derivative of $(x^2 + y^2)$ with respect to x is $2x$." ♣

Based on this discussion, we are now in a position to formally introduce the **first-order partial derivatives of** f. Of course, we can do the same sort of calculation for lines parallel to the y-axis. We temporarily hold x constant, which gives us the equation of the cross-section above a line $x = k$. We can then compute the derivative with respect to y; this will measure the slope of the curve in the y direction.

Definition 7.17: First-Order Partial Derivatives of $f(x,y)$

Suppose $f(x,y)$ is a two-variable function.
Then, the **first partial derivative of f with respect to x** *at (x,y) is given by*

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

provided the limit exists,
and the **first partial derivative of f with respect to y** *at (x,y) is given by*

$$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

provided the limit exists.

Note: Convenient alternate notations for the partial derivatives of $z = f(x,y)$ are given by

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x,y)) = z_x = \frac{\partial z}{\partial x} = \partial_x f,$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f(x,y)) = z_y = \frac{\partial z}{\partial y} = \partial_y f.$$

Example 7.18: Partial Derivative with respect to y

Find the partial derivative with respect to y of $f(x,y) = \sin(xy) + 3xy$.

Solution. The partial derivative with respect to y of $f(x,y) = \sin(xy) + 3xy$ is

$$f_y(x,y) = \frac{\partial}{\partial y}\sin(xy) + 3xy = \cos(xy)\frac{\partial}{\partial y}[(xy)+3x] = x\cos(xy) + 3x.$$

♣

Example 7.19: Partial Derivatives

Given the two-variable function $f(x,y) = x^2 - xy + y^5$,

(a) *Find the first-order partial derivatives f_x and f_y.*

(b) *Calculate $f_x(2,1)$ and interpret your result.*

(c) *Calculate $f_y(2,1)$ and interpret your result.*

Solution.

(a) To find the first-order partial derivative of f with respect to x, we treat the variable y as a constant and then differentiate as usual. That is,

$$f_x(x,y) = \frac{\partial}{\partial x}\left(x^2 - xy + y^5\right) = \frac{\partial}{\partial x}(x^2) - y\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y^5) = 2x - y.$$

Next, to compute the first-order partial derivative of f with respect to y, we treat the variable x as a constant. Then,

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^2 - xy + y^5\right) = \frac{\partial}{\partial y}(x^2) - x\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}(y^5) = -x + 5y^4.$$

(b) Letting $x = 2$ and $y = 1$, we find

$$f_x(2,1) = 2(2) - 1 = 3.$$

This means that f is increasing at a rate of 3 in the x-direction at the point $(2,1)$.

(c) We again let $x = 2$ and $y = 1$. Then,

$$f_y(2,1) = -2 + 5(1) = 3,$$

which gives the rate at which f is changing in the y-direction at the point $(2,1)$.

♣

Example 7.20: First-Order Partial Derivatives

Find the first-order partial derivatives of the following functions.

(a) $f(x,y) = \dfrac{xy}{x+y^3}$.

(b) $g(s,t) = \left(s + st^2 - t^3\right)^2$.

(c) $p(x,y) = x\ln(x+y^3)$.

Solution.

(a) We first compute f_x: to do this, we need to treat y as a constant. Then

$$f_x(x,y) = y\frac{\partial}{\partial x}\frac{x}{x+y^3} = y\frac{(x+y^3) - x}{(x+y^3)^2} = \frac{y^4}{(x+y^3)^2}.$$

Now to compute f_y, we consider x to be constant, which gives

$$f_y(x,y) = x\frac{\partial}{\partial y}\frac{y}{x+y^3} = x\frac{(x+y^3) - 3y^2}{(x+y^3)^2} = \frac{x(x-2y^3)}{(x+y^3)^2}.$$

(b) Treating the variable t as a constant and differentiating with respect to s gives

$$g_s(s,t) = 2\frac{\partial}{\partial s}\left(s+st^2-t^3\right) = 2(t^2+1)(st^2+s-t^3).$$

And so if we treat s as a constant and differentiate in the t-direction, we obtain

$$g_t(s,t) = 2\frac{\partial}{\partial t}\left(s+st^2-t^3\right) = 2t(2s-3t)(st^2+s-t^3).$$

(c) Differentiating in the x-direction gives

$$p_x(x,y) = \ln(x+y^3) + x\frac{\partial}{\partial x}\ln(x+y^3) = \ln(x+y^3) + \frac{x}{x+y^3},$$

and differentiating in the y-direction gives

$$p_y(x,y) = x\frac{\partial}{\partial x}\ln(x+y^3) = \frac{3xy^2}{x+y^3}.$$

♣

Example 7.21: First-Order Partial Derivatives

Find the first-order partial derivatives of the function

$$w = f(x,y,z) = xyz - xe^x + x\sin y.$$

Solution. Since f is a three-variable function, it will have three first-order partial derivatives:

$$f_x, f_y, \text{ and } f_z.$$

So to compute the derivative in the x-direction, we need to consider both y and z as constants and then differentiate with respect to x, and similarly for the derivatives in the y- and z-direction. We then obtain

$$\begin{aligned} f_x(x,y) &= \tfrac{\partial}{\partial x}(xyz - xe^x + x\sin y) = yz - (e^x + xe^x) + \sin y \\ f_y(x,y) &= \tfrac{\partial}{\partial y}(xyz - xe^x + x\sin y) = xz + x\cos y \\ f_z(x,y) &= \tfrac{\partial}{\partial z}(xyz - xe^x + x\sin y) = xy \end{aligned}$$

♣

7.3.2. Applications

In economics, the **Cobb-Douglas production function** is a specific form of **production function** that yields the amount of output produced by the amounts of two or more inputs, which are in technological relationship among each other. Here, we only consider one type of example, namely the connection between physical capital and labour, which are defined as follows.

Definition 7.22: Cobb-Douglas Production Function

The Cobb-Douglas production function is defined by

$$Y(L,K) = aL^b K^{1-b}$$

to represent the amount of output, Y, produced by two inputs that are in technological relationship to each other, where specifically
L is the total number of person-hours worked per year;
K is the real value of all physical capital such as machinery, equipment, and buildings;
a is the total factor productivity; and
$b \in (0,1)$ and $1-b$ are the output elasticities of labour and capital respectively.

The partial derivative Y_L measures the rate of change of production with respect to the amount of money expended for labour, when the level of capital expenditure is held fixed. Therefore, Y_L is called the **marginal productivity of labour**. Similarly, the partial derivative Y_K measures the rate of change of production with respect to the amount of money expended for capital, when the level of labour expenditure is held constant. Therefore, Y_K is called the **marginal productivity of capital**.

We showcase these concepts with the following example.

Example 7.23: Marginal Productivity

Suppose a country's production can be described by the function

$$Y(L,K) = 15L^{1/3}K^{2/3}$$

units, where L is units of labour and K is units of capital.

(a) *Calculate the marginal productivity of labour and the marginal productivity of capital.*

(b) *Evaluate Y_L and Y_K when $L = 100$ and $K = 55$.*

(c) *Interpret your result.*

Solution.

(a) The marginal productivity of labour is calculated to be

$$Y_L = \frac{\partial}{\partial L} 15L^{1/3}K^{2/3} = 5\left(\frac{L}{K}\right)^{2/3},$$

and the marginal productivity of capital is

$$Y_K = \frac{\partial}{\partial K} 15L^{1/3}K^{2/3} = 10\left(\frac{L}{K}\right)^{1/3}.$$

(b) When $L = 100$ and $K = 55$, we have

$$Y_L(100,55) = 55\left(\frac{55}{100}\right)^{2/3} \approx 3.36,$$

and

$$Y_K(100,55) = 10\left(\frac{55}{100}\right)^{1/3} \approx 12.2.$$

(c) We see that the marginal productivity of labour is less than the marginal productivity of capital when 100 units is spent on labour and 55 units is spent on capital. Therefore, if we increase the amount spent on capital while keeping the amount spent on labour constant, the resulting productivity will be higher than if we were to increase the amount spent on labour instead.

♣

A further application in economics of partial derivatives is to two-variable functions that represent the **relative demands** of two commodities that are either *competitive* or *complementary*.

The relative demands of two commodities are defined as follows:

Definition 7.24: Relative Demand Equations of Two Commodities

Let A and B be two commodities, then the **relative demand equations** *are given by*

$$q_A = f(p_A, p_B) \quad \text{and} \quad q_B = g(p_A, p_B),$$

where q_A is the quantity demanded for commodity A;
q_B is the quantity demanded for commodity B;
p_A is the unit price for commodity A; and
p_B is the unit price for commodity B.

Definition 7.25: Competitive Commodities

The relative demands of two commodities are termed **competitive commodities***, if they have an inverse relationship to each other, i.e. an increase in price (or decrease in demand) of one commodity results in an increase in demand of the other commodity.*

Examples of competitive commodities are bread and cereal, beer and wine, or cherries and strawberries.

Definition 7.26: Complementary Commodities

The relative demands of two commodities are termed **complementary commodities***, if they have a direct relationship to each other, i.e. an increase in demand of one commodity results in an increase in demand of the other commodity and vice versa.*

Examples of complementary commodities are flat screen TVs and blue-ray players, hamburgers and hamburger buns, or apple pie and vanilla ice cream.

Using partial derivatives, we can readily test whether two relative demands are competitive or complementary.

Definition 7.27: Partial Derivatives of Competitive and Complementary Commodities

1. *If*

$$\frac{\partial q_A}{\partial p_B} > 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} > 0,$$

then the commodities A and B are competitive.

2. *If*

$$\frac{\partial q_A}{\partial p_B} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} < 0,$$

then the commodities A and B are complementary.

Example 7.28: Competitive and Complementary Commodities

Let q_A denote the quantity of apples demanded and q_B denote the quantity of blueberries demanded in a supermarket. If the demand equations for apples and blueberries are given by

$$q_A = 500 - p_A^2 + 0.5p_B^2 \quad \text{and} \quad q_B = 850 + 0.3p_A^2 - 0.8p_B^2,$$

respectively, determine whether the products are competitive, complementary or neither.

Solution. We calculate

$$\frac{\partial q_A}{\partial p_B} = p_B \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} = 0.6p_A$$

Since p_A and p_B must be positive, we determine that

$$\frac{\partial q_A}{\partial p_B} > 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} > 0.$$

We thus conclude that the two products are competitive. ♣

Example 7.29: Competitive and Complementary Commodities

A retailer sells both raincoats and umbrellas. Let q_A denote the quantity of umbrellas demanded and q_B denote the quantity of raincoats demanded. The demand equations for raincoats and umbrellas are, respectively,

$$q_A = \frac{2p_B}{3+5p_A^2} \quad \text{and} \quad q_B = \frac{p_A}{3+p_B^{1/3}}.$$

Determine whether these two commodities are competitive, complementary, or neither.

Solution. We calculate

$$\frac{\partial q_A}{\partial p_B} = \frac{2}{3+5p_A^2} \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} = \frac{1}{3+p_B^{1/3}}.$$

As always, p_A and p_B must be positive. We thus determine that

$$\frac{\partial q_A}{\partial p_B} > 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} > 0,$$

and that the two products are competitive. ♣

7.3.3. Tangent Plane

So far, using no new techniques, we have succeeded in measuring the slope of a surface in two quite special directions. For functions of one variable, the derivative is closely linked to the notion of tangent line. For surfaces, the analogous idea is the tangent plane—a plane that just touches a surface at a point, and has the same slope as the surface in all directions. Even though we haven't yet figured out how to compute the slope in all directions, we have enough information to find tangent planes. Suppose we want the plane tangent to a surface at a particular point (a,b,c). If we compute the two partial derivatives of the function for that point, we get enough information to determine two lines tangent to the surface, both through (a,b,c) and both tangent to the surface in their respective directions. These two lines determine a plane, that is, there is exactly one plane containing the two lines: the tangent plane. Figure 7.5 shows (part of) two tangent lines at a point, and the tangent plane containing them.

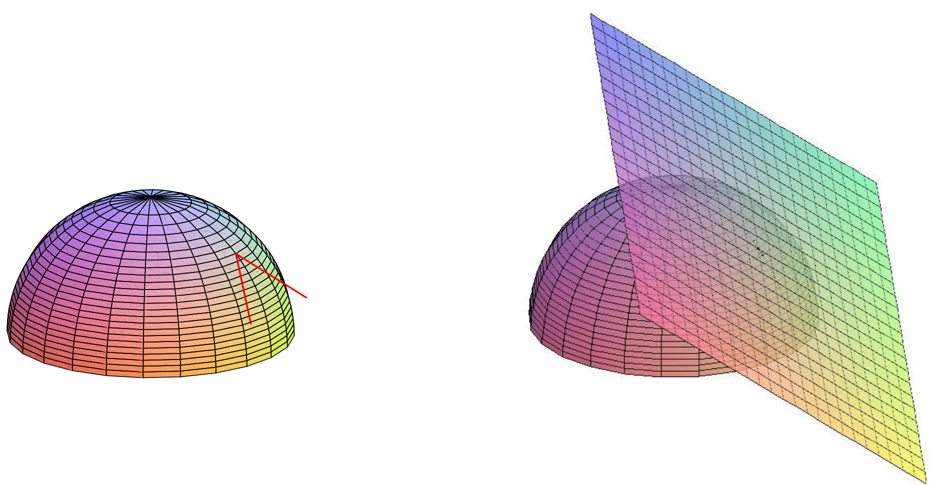


Figure 7.5: Tangent vectors and tangent plane.

How can we discover an equation for this tangent plane? We know a point on the plane, (a,b,c); we need a vector normal to the plane. If we can find two vectors, one parallel to each of the tangent lines we know how to find, then the cross product of these vectors will give the desired normal vector.

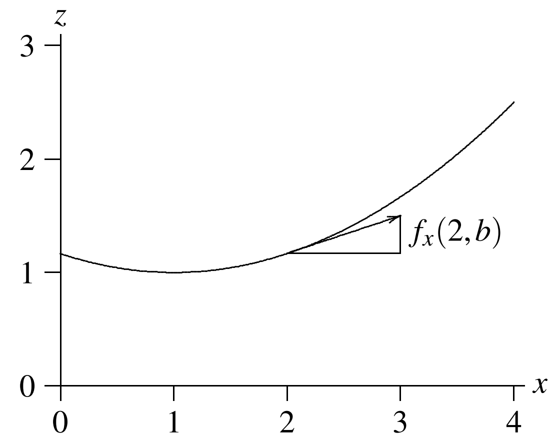


Figure 7.6: A tangent vector.

How can we find vectors parallel to the tangent lines? Consider first the line tangent to the surface above the line $y = b$. A vector $\langle u, v, w\rangle$ parallel to this tangent line must have y component $v = 0$, and we may as well take the x component to be $u = 1$. The ratio of the z component to the x component is the slope of the tangent line, precisely what we know how to compute. The slope of the tangent line is $f_x(a,b)$, so

$$f_x(a,b) = \frac{w}{u} = \frac{w}{1} = w.$$

In other words, a vector parallel to this tangent line is $\langle 1, 0, f_x(a,b)\rangle$, as shown in Figure 7.6. If we repeat the reasoning for the tangent line above $x = a$, we get the vector $\langle 0, 1, f_y(a,b)\rangle$.

Now to find the desired normal vector we compute the cross product, $\langle 0,1,f_y\rangle \times \langle 1,0,f_x\rangle = \langle f_x, f_y, -1\rangle$. From our earlier discussion of planes, we can write down the equation we seek: $f_x(a,b)x + f_y(a,b)y - z = k$, and k as usual can be computed by substituting a known point: $f_x(a,b)(a) + f_y(a,b)(b) - c = k$. There are various more-or-less nice ways to write the result:

$$\begin{aligned}
&f_x(a,b)x + f_y(a,b)y - z = f_x(a,b)a + f_y(a,b)b - c\\
&f_x(a,b)x + f_y(a,b)y - f_x(a,b)a - f_y(a,b)b + c = z\\
&f_x(a,b)(x-a) + f_y(a,b)(y-b) + c = z\\
&f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) = z
\end{aligned}$$

Example 7.30: Tangent Plane to a Sphere

Find the plane tangent to $x^2 + y^2 + z^2 = 4$ *at* $(1, 1, \sqrt{2})$.

Solution. The point $(1,1,\sqrt{2})$ is on the upper hemisphere, so we use $f(x,y) = \sqrt{4 - x^2 - y^2}$. Then $f_x(x,y) = -x(4 - x^2 - y^2)^{-1/2}$ and $f_y(x,y) = -y(4 - x^2 - y^2)^{-1/2}$, so $f_x(1,1) = f_y(1,1) = -1/\sqrt{2}$ and the equation of the plane is

$$z = -\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}.$$

The hemisphere and this tangent plane are pictured in Figure 7.5. ♣

So it appears that to find a tangent plane, we need only find two quite simple ordinary derivatives, namely f_x and f_y. This is true *if the tangent plane exists*. It is, unfortunately, not always the case that if f_x and f_y exist there is a tangent plane. Consider the function $xy^2/(x^2+y^4)$ with $f(0,0)$ defined to be 0, pictured in Figure 7.1. This function has value 0 when $x=0$ or $y=0$. Now it's clear that $f_x(0,0) = f_y(0,0) = 0$, because in the x and y directions the surface is simply a horizontal line. But it's also clear from the picture that this surface does not have anything that deserves to be called a tangent plane at the origin, certainly not the x-y-plane containing these two tangent lines.

When does a surface have a tangent plane at a particular point? What we really want from a tangent plane, as from a tangent line, is that the plane be a "good" approximation of the surface near the point. Here is how we can make this precise:

Definition 7.31: Tangent Plane

Let $\Delta x = x - x_0$, $\Delta y = y - y_0$, and $\Delta z = z - z_0$ where $z_0 = f(x_0,y_0)$. The function $z = f(x,y)$ is differentiable at (x_0,y_0) if

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where both ε_1 and ε_2 approach 0 as (x,y) approaches (x_0,y_0).

This definition takes a bit of absorbing. Let's rewrite the central equation a bit:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y. \tag{7.1}$$

The first three terms on the right are the equation of the tangent plane, that is,

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$

is the z-value of the point on the plane above (x,y). Equation 7.1 says that the z-value of a point on the surface is equal to the z-value of a point on the plane plus a "little bit," namely $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$. As (x,y) approaches (x_0,y_0), both Δx and Δy approach 0, so this little bit $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ also approaches 0, and the z-values on the surface and the plane get close to each other. But that by itself is not very interesting: since the surface and the plane both contain the point (x_0,y_0,z_0), the z values will approach z_0 and hence get close to each other whether the tangent plane is *tangent* to the surface or not. The extra condition in the definition says that as (x,y) approaches (x_0,y_0), the ε values approach 0—this means that $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ approaches 0 much, much faster, because $\varepsilon_1 \Delta x$ is much smaller than either ε_1 or Δx. It is this extra condition that makes the plane a tangent plane.

We can see that the extra condition on ε_1 and ε_2 is just what is needed if we look at partial derivatives. Suppose we temporarily fix $y = y_0$, so $\Delta y = 0$. Then the equation from the definition becomes

$$\Delta z = f_x(x_0,y_0)\Delta x + \varepsilon_1 \Delta x$$

or

$$\frac{\Delta z}{\Delta x} = f_x(x_0,y_0) + \varepsilon_1.$$

Now taking the limit of the two sides as Δx approaches 0, the left side turns into the partial derivative of z with respect to x at (x_0,y_0), or in other words $f_x(x_0,y_0)$, and the right side does the same, because as (x,y) approaches (x_0,y_0), ε_1 approaches 0. Essentially the same calculation works for f_y.

7.3.4. Second-Order Partial Derivatives

The first-order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of a two-variable function $f(x,y)$ are again functions of the two variables x and y. Therefore, we can compute the derivative of each function $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ to obtain the **second-order partial derivatives** of f.

Definition 7.32: Second-Order Partial Derivatives of $f(x,y)$

Suppose $f(x,y)$ is a two-variable function. Then we obtain the four **second-order partial derivatives** *by differentiating the functions f_x and f_y with respect to x and y:*

1. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$
2. $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$
3. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$
4. $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$

Note:

1. The notation for the second-order partial derivative is analogous to the notation for the ordinary second derivative in single-variable calculus:

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

2. When multiple distinct independent variables are involved in second-order partial derivatives, then they are also referred to as **mixed partial derivatives**. For example, f_{xy} and f_{yx}.

3. In general, we have that $f_{xy} \neq f_{yx}$. However, if the second-order partial derivatives are continuous around a point (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$. This is a theorem that is variously referenced to as Schwarz's theorem or Clairaut's theorem, but we would need real analysis in order to prove it, so we state it here without proof. Most functions we will encounter in this course satisfy Clairaut's theorem.

Figure 7.7 is a schematic way of demonstrating how the four second-order partial derivatives of f are calculated.

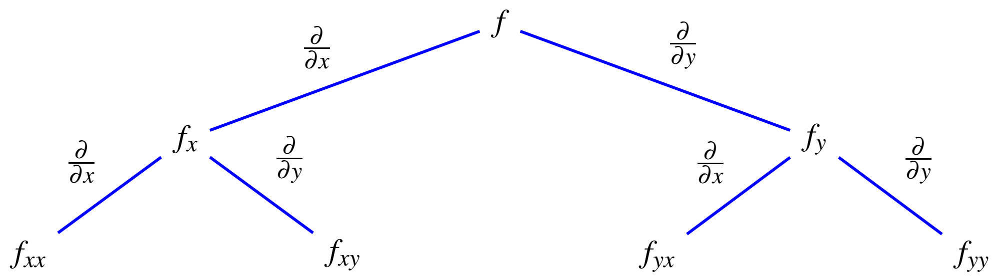


Figure 7.7: A schematic of the four second-order partial derivatives of f.

Theorem 7.33: Clairaut's Theorem

If the mixed partial derivatives are continuous, they are equal.

Example 7.34: Second-Order Partial Derivatives

Find the second-order partial derivatives of the function

(a) $f(x,y) = x^2 - 2x^2y + xy^2 + y^3$.

(b) $g(x,y) = e^{x^2y}$.

Solution.

(a) We will first calculate f_x and f_y.

$$f_x = 2x - 4xy + y^2, \quad \text{and} \quad f_y = -2x^2 + 2xy + 2y^2.$$

Hence, the four second-order partial derivatives of f are

$$\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}\left(2x - 4xy + y^2\right) = 2 - 4y \\
f_{xy} &= \frac{\partial}{\partial y}\left(2x - 4xy + y^2\right) = -4x + 2y \\
f_{yy} &= \frac{\partial}{\partial y}\left(-2x^2 + 2xy + 2y^2\right) = 2x + 4y \\
f_{yx} &= \frac{\partial}{\partial x}\left(-2x^2 + 2xy + 2y^2\right) = -4x + 2 = f_{xy}.
\end{aligned}$$

(b) The first-order partial derivatives of g are

$$g_x = 2xye^{x^2y}, \quad \text{and} \quad g_y = x^2e^{x^2y}.$$

Thus, the second-order partial derivatives of g are given by

$$\begin{aligned}
g_{xx} &= \frac{\partial}{\partial x}\left(2xye^{x^2y}\right) = 2ye^{x^2y} + 4x^2y^2e^{x^2y} \\
g_{xy} &= \frac{\partial}{\partial y}\left(2xye^{x^2y}\right) = 2x^3ye^{x^2y} + 2xe^{x^2y} \\
g_{yy} &= \frac{\partial}{\partial y}\left(x^2e^{x^2y}\right) = x^4e^{x^2y} \\
g_{yx} &= \frac{\partial}{\partial x}\left(x^2e^{x^2y}\right) = 2x^3ye^{x^2y} + 2xe^{x^2y} = g_{xy}.
\end{aligned}$$

♣

Exercises for Section 7.3

Exercise 7.3.1 *Find f_x and f_y of the following functions $f(x,y)$.*

(a) $f(x,y) = \cos(x^2y) + y^3$

(b) $f(x,y) = \dfrac{xy}{x^2+y}$

(c) $f(x,y) = e^{x^2+y^2}$

(d) $f(x,y) = xy\ln(xy)$

(e) $f(x,y) = \sqrt{1-x^2-y^2}$

(f) $f(x,y) = x\tan(y)$

(g) $f(x,y) = \dfrac{1}{xy}$

Exercise 7.3.2 *Evaluate the first partial derivatives of the function at the given point.*

(a) $f(x,y) = x^2y + xy^2$ *at* $(1,3)$

(b) $g(x,y) = x\sqrt{y}$ *at* $(4,4)$

(c) $f(s,t) = s/t$ *at* $(-1,2)$

(d) $p(x,y) = e^{xy}$ *at* $(1,-1)$

(e) $h(a,b,c) = a^2bc^3$ *at* $(1,1,2)$

(f) $f(p,q) = \sin(pq)$ *at* $(\pi/2,1)$

(g) $h(a,b) = e^{\sin(ab)}$ *at* $(\pi,1)$

(h) $g(x,y) = \frac{\sin(x)\cos(y)}{x+y}$ *at* $(\pi/3,\pi/3)$

Exercise 7.3.3 *Suppose the productivity of a country can be approximated by*

$$Y(L,K) = 15L^{3/5}K^{2/5}$$

where L is units of labour and K is units of capital.

(a) *Calculate* Y_K *and* Y_L.

(b) *Calculate the marginal productivity of labour and the marginal productivity of capital when* $L = 100$ *and* $K = 55$. *Interpret your results.*

Exercise 7.3.4 *Suppose the productivity of a country can be approximated by*

$$Y(L,K) = 5L^{1/4}K^{3/4}$$

where L is units of labour and K is units of capital.

(a) *Calculate* Y_K *and* Y_L.

(b) *Calculate the marginal productivity of labour and the marginal productivity of capital when* $L = 33$ *and* $K = 100$. *Interpret your results.*

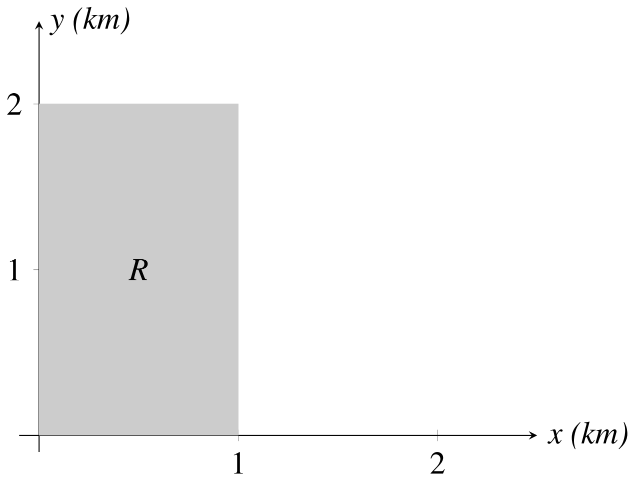
Exercise 7.3.5 *Suppose the density of houses in a city district is given by*

$$\rho(x,y) = 100 - 10(x-3) - 5(y-1)^2$$

where x and y are measured in kilometres and lie in the region described by the graph below.

(a) *Calculate* ρ_x *and* ρ_y.

(b) What can you say about the density of housing around the point $(1,0)$*?*



Exercise 7.3.6 *An office supply store sells* q_N *notebooks and* q_P *pencils. Suppose the demand equation for notebooks and pencils are, respectively*

$$q_N = f(p_N, p_P) = 8000 + 8p_N - 0.5p_P^2 \quad \text{and} \quad q_P = g(p_N, p_P) = 1000 - 0.8p_N^2 + 20p_P$$

(where p_N *and* p_P *are the respective unit prices). Are these two products are competitive, complementary, or neither?*

Exercise 7.3.7 *A network TV station sells online and cable subscriptions. Suppose the demand equation for online and cable subscriptions are, respectively*

$$q_O = f(p_O, p_C) = 7000 + 2p_O + p_C^2 \quad \text{and} \quad q_C = g(p_O, p_C) = 1500 + 0.8p_O^2 + 0.2p_C$$

(where p_O *and* p_C *are the respective unit prices). Are these two products are competitive, complementary, or neither?*

Exercise 7.3.8 *Find an equation for the plane tangent to the following functions* $f(x,y) = z$ *at the given point.*

(a) $2x^2 + 3y^2 - z^2 = 4$ *at* $(1,1,-1)$.

(b) $f(x,y) = \sin(xy)$ *at* $(\pi, 1/2, 1)$

(c) $f(x,y) = x^2 + y^3$ *at* $(3,1,10)$

(d) $f(x,y) = x\ln(xy)$ *at* $(2,1/2,0)$

Exercise 7.3.9 *Find an equation for the line normal to* $x^2 + 4y^2 = 2z$ *at* $(2,1,4)$.

Exercise 7.3.10 *Explain in your own words why, when taking a partial derivative of a function of multiple variables, we can treat the variables not being differentiated as constants.*

Exercise 7.3.11 *Consider a differentiable function,* $f(x,y)$*. Give physical interpretations of the meanings of* $f_x(a,b)$ *and* $f_y(a,b)$ *as they relate to the graph of* f*.*

Exercise 7.3.12 *In much the same way that we used the tangent line to approximate the value of a function from single variable calculus, we can use the tangent plane to approximate a function from multivariable calculus. Consider the tangent plane found in Exercise* **??**. *Use this plane to approximate* $f(1.98, 0.4)$.

Exercise 7.3.13 *The volume of a cylinder is given by* $V = \pi r^2 h$. *Suppose that the current values of r and h are* $r = 7$ *cm and* $h = 3$ *cm. Is the volume more sensitive to a small change in radius or the same amount of change in height? Why?*

Exercise 7.3.14 *Suppose that one of your colleagues has calculated the partial derivatives of a given function, and reported to you that* $f_x(x,y) = 2x + 3y$ *and that* $f_y(x,y) = 4x + 6y$. *Do you believe them? Why or why not? If not, what answer might you have accepted for* f_y*?*

Exercise 7.3.15 *Suppose* $f(t)$ *and* $g(t)$ *are single variable differentiable functions. Find* $\partial z/\partial x$ *and* $\partial z/\partial y$ *for each of the following two variable functions.*

(a) $z = f(x)g(y)$

(b) $z = f(xy)$

(c) $z = f(x/y)$

Exercise 7.3.16 *Let* $Q(p,q) = pq/(p^2 + q^2)$*; compute* Q_{pp}, Q_{qp}, *and* Q_{qq}.

Exercise 7.3.17 *Determine all first and second partial derivatives of the following functions.*

(a) $f(x,y) = x^2y^2 + y^3$

(b) $f(x,y) = 2x^2 + xy + 3$

(c) $f(x,y) = y\sin x$

(d) $f(x,y) = \sin(2y)\cos(x)$

(e) $f(x,y) = e^{x^2-y}$

(f) $f(x,y) = \ln\sqrt{x^2+y^2}$

Exercise 7.3.18 *Prove that the function* $u(x,t) = e^{-\alpha^2 k^2 t}\sin(kx)$ *is a solution to the heat equation*

$$u_t = \alpha^2 u_{xx},$$

where α *and k are constants.*

Exercise 7.3.19 *Prove that* $u = \sin(x - at) + \ln(x + at)$ *is a solution to the wave equation*

$$u_{tt} = a^2 u_{xx},$$

where a is a constant.

Exercise 7.3.20 *How many distinct third-order derivatives does a function of 2 variables have?*

Exercise 7.3.21 *How many distinct n-th-order derivatives does a function of 2 variables have?*

7.4 The Chain Rule

Consider the surface $z = x^2y + xy^2$, and suppose that $x = 2 + t^4$ and $y = 1 - t^3$. We can think of the latter two equations as describing how x and y change relative to, say, time. Then

$$z = x^2y + xy^2 = (2+t^4)^2(1-t^3) + (2+t^4)(1-t^3)^2$$

tells us explicitly how the z coordinate of the corresponding point on the surface depends on t. If we want to know dz/dt we can compute it more or less directly, but it's actually a bit simpler to use Product and Chain Rules:

$$\begin{aligned}\frac{dz}{dt} &= x^2y' + 2xx'y + x2yy' + x'y^2 \\ &= (2xy + y^2)x' + (x^2 + 2xy)y' \\ &= (2(2+t^4)(1-t^3) + (1-t^3)^2)(4t^3) + ((2+t^4)^2 + 2(2+t^4)(1-t^3))(-3t^2)\end{aligned}$$

If we look carefully at the middle step, $dz/dt = (2xy + y^2)x' + (x^2 + 2xy)y'$, we notice that $2xy + y^2$ is $\partial z/\partial x$, and $x^2 + 2xy$ is $\partial z/\partial y$. This turns out to be true in general, and gives us a new Chain Rule:

Theorem 7.35: Multivariate Chain Rule

Suppose that $z = f(x,y)$, f is differentiable, $x = g(t)$, and $y = h(t)$. Assuming that the relevant derivatives exist,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Proof. If f is differentiable, then

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y,$$

where ε_1 and ε_2 approach 0 as (x,y) approaches (x_0,y_0). Then

$$\frac{\Delta z}{\Delta t} = f_x\frac{\Delta x}{\Delta t} + f_y\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}. \tag{7.2}$$

As Δt approaches 0, (x,y) approaches (x_0,y_0) and so

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} &= \frac{dz}{dt} \\ \lim_{\Delta t\to 0}\varepsilon_1\frac{\Delta x}{\Delta t} &= 0\cdot\frac{dx}{dt} \\ \lim_{\Delta t\to 0}\varepsilon_2\frac{\Delta y}{\Delta t} &= 0\cdot\frac{dy}{dt}\end{aligned}$$

and so taking the limit of (7.2) as Δt goes to 0 gives

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt},$$

as desired. ♣

We can write the Chain Rule in a way that is somewhat closer to the single variable Chain Rule:

$$\frac{df}{dt} = \langle f_x, f_y \rangle \cdot \langle x', y' \rangle,$$

or (roughly) the derivatives of the outside function "times" the derivatives of the inside functions. Not surprisingly, essentially the same Chain Rule works for functions of more than two variables, for example, given a function of three variables $f(x,y,z)$, where each of x, y and z is a function of t,

$$\frac{df}{dt} = \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle.$$

We can even extend the idea further. Suppose that $f(x,y)$ is a function and $x = g(s,t)$ and $y = h(s,t)$ are functions of two variables s and t. Then f is "really" a function of s and t as well, and

$$\frac{\partial f}{\partial s} = f_x g_s + f_y h_s \qquad \frac{\partial f}{\partial t} = f_x g_t + f_y h_t.$$

The natural extension of this to $f(x,y,z)$ works as well.

Recall that we used the ordinary Chain Rule to do implicit differentiation. We can do the same with the new Chain Rule.

Example 7.36: Equation of a Sphere

Find the partial derivative of $x^2 + y^2 + z^2 = 4$.

Solution. The equation $x^2 + y^2 + z^2 = 4$ defines a sphere, which is not a function of x and y, though it can be thought of as two functions, the top and bottom hemispheres. We can think of z as one of these two functions, so really $z = z(x,y)$, and we can think of x and y as particularly simple functions of x and y, and let $f(x,y,z) = x^2 + y^2 + z^2$. Since $f(x,y,z) = 4$, $\partial f/\partial x = 0$, but using the Chain Rule:

$$\begin{aligned} 0 = \frac{\partial f}{\partial x} &= f_x \frac{\partial x}{\partial x} + f_y \frac{\partial y}{\partial x} + f_z \frac{\partial z}{\partial x} \\ &= (2x)(1) + (2y)(0) + (2z)\frac{\partial z}{\partial x}, \end{aligned}$$

noting that since y is temporarily held constant its derivative $\partial y/\partial x = 0$. Now we can solve for $\partial z/\partial x$:

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z}.$$

In a similar manner we can compute $\partial z/\partial y$. ♣

Exercises for Section 7.4

Exercise 7.4.1 *Use the Chain Rule to compute the following.*

(a) dz/dt *for* $z = \sin(x^2 + y^2)$, $x = t^2 + 3$, $y = t^3$

(b) dz/dt *for* $z = x^2 y$, $x = \sin(t)$, $y = t^2 + 1$

(c) $\partial z/\partial s$ *and* $\partial z/\partial t$ *for* $z = x^2 y$, $x = \sin(st)$, $y = t^2 + s^2$

(d) $\partial z/\partial s$ *and* $\partial z/\partial t$ *for* $z = x^2 y^2$, $x = st$, $y = t^2 - s^2$

(e) $\partial z/\partial x$ *and* $\partial z/\partial y$ *for* $2x^2 + 3y^2 - 2z^2 = 9$

(f) $\partial z/\partial x$ *and* $\partial z/\partial y$ *for* $2x^2 + y^2 + z^2 = 9$

Exercise 7.4.2 *Chemistry students will recognize the* ideal gas law, *given by* $PV = nRT$ *which relates the Pressure, Volume, and Temperature of n moles of gas. (R is the ideal gas constant). Thus, we can view pressure, volume, and temperature as variables, each one dependent on the other two.*

(a) *If pressure of a gas is increasing at a rate of* $0.2 Pa/min$ *and temperature is increasing at a rate of* $1K/min$*, how fast is the volume changing?*

(b) *If the volume of a gas is decreasing at a rate of* $0.3L/min$ *and temperature is increasing at a rate of* $.5K/min$*, how fast is the pressure changing?*

(c) *If the pressure of a gas is decreasing at a rate of* $0.4 Pa/min$ *and the volume is increasing at a rate of* $3L/min$*, how fast is the temperature changing?*

Exercise 7.4.3 *Verify the following identity in the case of the ideal gas law:*

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

Exercise 7.4.4 *The previous exercise was a special case of the following fact, which you are to verify here: If* $F(x,y,z)$ *is a function of 3 variables, and the relation* $F(x,y,z) = 0$ *defines each of the variables in terms of the other two, namely* $x = f(y,z)$, $y = g(x,z)$ *and* $z = h(x,y)$*, then*

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

7.5 Directional Derivatives

We still have not answered one of our first questions about the steepness of a surface: starting at a point on a surface given by $f(x,y)$, and walking in a particular direction, how steep is the surface? We are now ready to answer the question.

We already know roughly what has to be done: as shown in Figure 7.3, we extend a line in the x-y-plane to a vertical plane, and we then compute the slope of the curve that is the cross-section of the surface in that plane. The major stumbling block is that what appears in this plane to be the horizontal axis, namely the line in the x-y-plane, is not an actual axis—we know nothing about the "units" along the axis. Our goal is to make this line into a t axis; then we need formulas to write x and y in terms of this new variable t; then we can write z in terms of t since we know z in terms of x and y; and finally we can simply take the derivative.

So we need to somehow "mark off" units on the line, and we need a convenient way to refer to the line in calculations. It turns out that we can accomplish both by using the vector form of a line. Suppose that $\mathbf{u}$ is a unit vector $\langle u_1, u_2\rangle$ in the direction of interest. A vector equation for the line through (x_0, y_0) in this direction is $\mathbf{v}(t) = \langle u_1 t + x_0, u_2 t + y_0\rangle$. The height of the surface above the point $(u_1 t + x_0, u_2 t + y_0)$ is $g(t) = f(u_1 t + x_0, u_2 t + y_0)$. Because $\mathbf{u}$ is a unit vector, the value of t is precisely the distance along the line from (x_0, y_0) to $(u_1 t + x_0, u_2 t + y_0)$; this means that the line is effectively a t axis, with origin at the point (x_0, y_0), so the slope we seek is

$$\begin{aligned} g'(0) &= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \langle u_1, u_2\rangle \\ &= \langle f_x, f_y\rangle \cdot \mathbf{u} \\ &= \nabla f \cdot \mathbf{u} \end{aligned}$$

Here we have used the Chain Rule and the derivatives $\frac{d}{dt}(u_1 t + x_0) = u_1$ and $\frac{d}{dt}(u_2 t + y_0) = u_2$. The vector $\langle f_x, f_y\rangle$ is very useful, so it has its own symbol, ∇f, pronounced "del f"; it is also called the **gradient** of f.

Example 7.37: Slope

Find the slope of $z = x^2 + y^2$ at $(1,2)$ in the direction of the vector $\langle 3,4\rangle$.

Solution. We first compute the gradient at $(1,2)$: $\nabla f = \langle 2x, 2y\rangle$, which is $\langle 2,4\rangle$ at $(1,2)$. A unit vector in the desired direction is $\langle 3/5, 4/5\rangle$, and the desired slope is then $\langle 2,4\rangle \cdot \langle 3/5,4/5\rangle = 6/5 + 16/5 = 22/5$. ♣

Example 7.38: Tangent Vector

Find a tangent vector to $z = x^2 + y^2$ at $(1,2)$ in the direction of the vector $\langle 3,4\rangle$ and show that it is parallel to the tangent plane at that point.

Solution. Since $\langle 3/5, 4/5\rangle$ is a unit vector in the desired direction, we can easily expand it to a tangent vector simply by adding the third coordinate computed in the previous example: $\langle 3/5, 4/5, 22/5\rangle$. To see that this vector is parallel to the tangent plane, we can compute its dot product with a normal to the plane. We know that a normal to the tangent plane is

$$\langle f_x(1,2), f_y(1,2), -1\rangle = \langle 2,4,-1\rangle,$$

and the dot product is $\langle 2,4,-1\rangle \cdot \langle 3/5,4/5,22/5\rangle = 6/5+16/5-22/5 = 0$, so the two vectors are perpendicular. (Note that the vector normal to the surface, namely $\langle f_x, f_y, -1\rangle$, is simply the gradient with a -1 tacked on as the third component.) ♣

The slope of a surface given by $z = f(x,y)$ in the direction of a (two-dimensional) vector $\mathbf{u}$ is called the **directional derivative** of f, written $D_{\mathbf{u}}f$. The directional derivative immediately provides us with some additional information. We know that

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

if $\mathbf{u}$ is a unit vector; θ is the angle between ∇f and $\mathbf{u}$. This tells us immediately that the largest value of $D_{\mathbf{u}}f$ occurs when $\cos\theta = 1$, namely, when $\theta = 0$, so ∇f is parallel to $\mathbf{u}$. In other words, the gradient ∇f points in the direction of steepest ascent of the surface, and $|\nabla f|$ is the slope in that direction. Likewise, the smallest value of $D_{\mathbf{u}}f$ occurs when $\cos\theta = -1$, namely, when $\theta = \pi$, so ∇f is anti-parallel to $\mathbf{u}$. In other words, $-\nabla f$ points in the direction of steepest descent of the surface, and $-|\nabla f|$ is the slope in that direction.

Example 7.39: Direction of Steepest Ascent and Descent

Investigate the direction of steepest ascent and descent for $z = x^2 + y^2$.

Solution. The gradient is $\langle 2x, 2y\rangle = 2\langle x,y\rangle$; this is a vector parallel to the vector $\langle x,y\rangle$, so the direction of steepest ascent is directly away from the origin, starting at the point (x,y). The direction of steepest descent is thus directly toward the origin from (x,y). Note that at $(0,0)$ the gradient vector is $\langle 0,0\rangle$, which has no direction, and it is clear from the plot of this surface that there is a minimum point at the origin, and tangent vectors in all directions are parallel to the x-y-plane. ♣

If ∇f is perpendicular to $\mathbf{u}$, $D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = 0$, since $\cos(\pi/2) = 0$. This means that in either of the two directions perpendicular to ∇f, the slope of the surface is 0; this implies that a vector in either of these directions is tangent to the level curve at that point. Starting with $\nabla f = \langle f_x, f_y\rangle$, it is easy to find a vector perpendicular to it: either $\langle f_y, -f_x\rangle$ or $\langle -f_y, f_x\rangle$ will work.

If $f(x,y,z)$ is a function of three variables, all the calculations proceed in essentially the same way. The rate at which f changes in a particular direction is $\nabla f \cdot \mathbf{u}$, where now $\nabla f = \langle f_x, f_y, f_z\rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ is a unit vector. Again ∇f points in the direction of maximum rate of increase, $-\nabla f$ points in the direction of maximum rate of decrease, and any vector perpendicular to ∇f is tangent to the level surface $f(x,y,z) = k$ at the point in question. Of course there are no longer just two such vectors; the vectors perpendicular to ∇f describe the tangent plane to the level surface, or in other words ∇f is a normal to the tangent plane.

Example 7.40: Gradient

Suppose the temperature at a point in space is given by $T(x,y,z) = T_0/(1+x^2+y^2+z^2)$; *at the origin the temperature in Kelvin is* $T_0 > 0$, *and it decreases in every direction from there. It might be, for example, that there is a source of heat at the origin, and as we get farther from the source, the temperature decreases. The gradient is*

$$\begin{aligned}\nabla T &= \langle \frac{-2T_0 x}{(1+x^2+y^2+z^2)^2} + \frac{-2T_0 x}{(1+x^2+y^2+z^2)^2} + \frac{-2T_0 x}{(1+x^2+y^2+z^2)^2} \rangle \\ &= \frac{-2T_0}{(1+x^2+y^2+z^2)^2}\langle x,y,z\rangle.\end{aligned}$$

The gradient points directly at the origin from the point (x,y,z)*—by moving directly toward the heat source, we increase the temperature as quickly as possible.*

Example 7.41: Tangent Plane

Find the points on the surface defined by $x^2+2y^2+3z^2 = 1$ *where the tangent plane is parallel to the plane defined by* $3x - y + 3z = 1$.

Solution. Two planes are parallel if their normals are parallel or anti-parallel, so we want to find the points on the surface with normal parallel or anti-parallel to $\langle 3,-1,3\rangle$. Let $f = x^2+2y^2+3z^2$; the gradient of f is normal to the level surface at every point, so we are looking for a gradient parallel or anti-parallel to $\langle 3,-1,3\rangle$. The gradient is $\langle 2x,4y,6z\rangle$; if it is parallel or anti-parallel to $\langle 3,-1,3\rangle$, then

$$\langle 2x,4y,6z\rangle = k\langle 3,-1,3\rangle$$

for some k. This means we need a solution to the equations

$$2x = 3k \qquad 4y = -k \qquad 6z = 3k$$

but this is three equations in four unknowns—we need another equation. What we haven't used so far is that the points we seek are on the surface $x^2+2y^2+3z^2 = 1$; this is the fourth equation. If we solve the first three equations for x, y, and z and substitute into the fourth equation we get

$$\begin{aligned}1 &= \left(\frac{3k}{2}\right)^2 + 2\left(\frac{-k}{4}\right)^2 + 3\left(\frac{3k}{6}\right)^2 \\ &= \left(\frac{9}{4}+\frac{2}{16}+\frac{3}{4}\right)k^2 \\ &= \frac{25}{8}k^2\end{aligned}$$

so $k = \pm\frac{2\sqrt{2}}{5}$. The desired points are $\left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{5}\right)$ and $\left(-\frac{3\sqrt{2}}{5}, \frac{\sqrt{2}}{10}, -\frac{\sqrt{2}}{5}\right)$. ♣

Exercises for Section 7.5

Exercise 7.5.1 *Find* $D_{\mathbf{u}} f$ *for* $f = x^2 + xy + y^2$ *in the direction of* $\mathbf{u} = \langle 2, 1\rangle$ *at the point* $(1, 1)$.

Exercise 7.5.2 *Find* $D_{\mathbf{u}} f$ *for* $f = \sin(xy)$ *in the direction of* $\mathbf{u} = \langle -1, 1\rangle$ *at the point* $(3, 1)$.

Exercise 7.5.3 *Find* $D_{\mathbf{u}} f$ *for* $f = e^x \cos(y)$ *in the direction 30 degrees from the positive x axis at the point* $(1, \pi/4)$.

Exercise 7.5.4 *The temperature of a thin plate in the x-y-plane is* $T = x^2 + y^2$. *How fast does temperature change at the point* $(1, 5)$ *moving in a direction 30 degrees from the positive x axis?*

Exercise 7.5.5 *Suppose the density of a thin plate at* (x, y) *is* $1/\sqrt{x^2 + y^2 + 1}$. *Find the rate of change of the density at* $(2, 1)$ *in a direction* $\pi/3$ *radians from the positive x axis.*

Exercise 7.5.6 *Suppose the electric potential at* (x, y) *is* $\ln\sqrt{x^2 + y^2}$. *Find the rate of change of the potential at* $(3, 4)$ *toward the origin and also in a direction at a right angle to the direction toward the origin.*

Exercise 7.5.7 *A plane perpendicular to the x-y-plane contains the point* $(2, 1, 8)$ *on the paraboloid* $z = x^2 + 4y^2$. *The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane.*

Exercise 7.5.8 *A plane perpendicular to the x-y-plane contains the point* $(3, 2, 2)$ *on the paraboloid* $36z = 4x^2 + 9y^2$. *The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane.*

Exercise 7.5.9 *Suppose the temperature at* (x, y, z) *is given by* $T = xy + \sin(yz)$. *In what direction should you go from the point* $(1, 1, 1)$ *to decrease the temperature as quickly as possible? What is the rate of change of temperature in this direction?*

Exercise 7.5.10 *Suppose the temperature at* (x, y, z) *is given by* $T = xyz$. *In what direction can you go from the point* $(1, 1, 1)$ *to maintain the same temperature?*

Exercise 7.5.11 *Find an equation for the plane tangent to* $x^2 - 3y^2 + z^2 = 7$ *at* $(1, 1, 3)$.

Exercise 7.5.12 *Find an equation for the plane tangent to* $xyz = 6$ *at* $(1, 2, 3)$.

Exercise 7.5.13 *Find an equation for the line normal to* $x^2 + 2y^2 + 4z^2 = 26$ *at* $(2, -3, -1)$.

Exercise 7.5.14 *Find an equation for the line normal to* $x^2 + y^2 + 9z^2 = 56$ *at* $(4, 2, -2)$.

Exercise 7.5.15 *Find an equation for the line normal to* $x^2 + 5y^2 - z^2 = 0$ *at* $(4, 2, 6)$.

Exercise 7.5.16 *Find the directions in which the directional derivative of* $f(x,y) = x^2 + \sin(xy)$ *at the point* $(1,0)$ *has the value 1.*

Exercise 7.5.17 *Show that the curve* $\mathbf{r}(t) = \langle \ln(t), t\ln(t), t \rangle$ *is tangent to the surface* $xz^2 - yz + \cos(xy) = 1$ *at the point* $(0,0,1)$.

Exercise 7.5.18 *A bug is crawling on the surface of a hot plate, the temperature of which at the point x units to the right of the lower left corner and y units up from the lower left corner is given by* $T(x,y) = 100 - x^2 - 3y^3$.

(a) *If the bug is at the point* $(2,1)$, *in what direction should it move to cool off the fastest? How fast will the temperature drop in this direction?*

(b) *If the bug is at the point* $(1,3)$, *in what direction should it move in order to maintain its temperature?*

Exercise 7.5.19 *The elevation on a portion of a hill is given by* $f(x,y) = 100 - 4x^2 - 2y$. *From the location above* $(2,1)$, *in which direction will water run?*

Exercise 7.5.20 *Suppose that* $g(x,y) = y - x^2$. *Find the gradient at the point* $(-1,3)$. *Sketch the level curve to the graph of g when* $g(x,y) = 2$, *and plot both the tangent line and the gradient vector at the point* $(-1,3)$. *(Make your sketch large). What do you notice, geometrically?*

Exercise 7.5.21 *The gradient* ∇f *is a vector valued function of two variables. Prove the following gradient rules. Assume* $f(x,y)$ *and* $g(x,y)$ *are differentiable functions.*

(a) $\nabla(fg) = f\nabla(g) + g\nabla(f)$

(b) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$

(c) $\nabla((f(x,y))^n) = nf(x,y)^{n-1}\nabla f$

7.6 Maxima and Minima

Now we want to deal with extrema of two-variable functions. Similar to single-variable functions, we distinguish between *relative* and *absolute* extrema in two-variable functions.

Definition 7.42: Relative Extrema of a Two-Variable Function

Let $z = f(x,y)$ *be a function defined on a region* $R \subseteq \mathbb{R}^2$ *containing the point* (x_0,y_0).

Then f has a **relative maximum** *at* (x_0,y_0) *if* $f(x,y) \leq f(x_0,y_0)$ *for all points* (x,y) *that are sufficiently close to* (x_0,y_0). *The number* $f(x_0,y_0)$ *is the relative maximum value.*

Then f has a **relative minimum** *at* (x_0,y_0) *if* $f(x,y) \geq f(x_0,y_0)$ *for all points* (x,y) *that are sufficiently close to* (x_0,y_0). *The number* $f(x_0,y_0)$ *is the relative minimum value.*

In other words, if the point $(x_0, y_0, f(x_0, y_0))$ is the highest point on the graph of f compared to nearby points, then f has a relative maximum at (x_0, y_0), and similarly for a relative minimum. On the other hand, if the inequality holds for all points in the region R, then we have an *absolute* minimum or maximum.

Definition 7.43: Absolute Extrema of a Two-Variable Function

Let $z = f(x,y)$ be a function defined on a region $R \subseteq \mathbb{R}^2$ containing the point (x_0, y_0).

Then f has a **absolute maximum** *at (x_0, y_0) if $f(x,y) \le f(x_0, y_0)$ for all points (x,y) in R. The number $f(x_0, y_0)$ is the absolute maximum value.*

Then f has a **absolute minimum** *at (x_0, y_0) if $f(x,y) \ge f(x_0, y_0)$ for all points (x,y) in R. The number $f(x_0, y_0)$ is the absolute minimum value.*

Suppose a surface given by $f(x,y)$ has a relative maximum at (x_0, y_0, z_0); geometrically, this point on the surface looks like the top of a hill. If we look at the cross-section in the plane $y = y_0$, we will see a relative maximum on the curve at (x_0, z_0), and we know from single-variable calculus that $\frac{\partial z}{\partial x} = 0$ at this point. Likewise, in the plane $x = x_0$, $\frac{\partial z}{\partial y} = 0$. So if there is a relative maximum at (x_0, y_0, z_0), both partial derivatives at the point must be zero, and likewise for a relative minimum. Thus, to find relative maximum and minimum points, we need only consider those points at which both partial derivatives are 0. As in the single-variable case, it is possible for the derivatives to be 0 at a point that is neither a maximum or a minimum, so we need to test these points further.

Definition 7.44: Critical Point of a Two-Variable Function

Let $z = f(x,y)$ be a function defined on a region $R \subseteq \mathbb{R}^2$ containing the point (x_0, y_0).

The point (x_0, y_0) in R is a **critical point** *of f if* **both** *$f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or* **at least one** *of the derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ does not exist.*

You will recall that in the single-variable case, we examined three methods to identify maximum and minimum points; the most useful is the Second Derivative Test, though it does not always work. For functions of two variables there is also a Second Derivative Test; again it is by far the most useful test, though it doesn't always work.

Theorem 7.45: Second Derivative Test for Two-Variable Functions

Suppose that the second partial derivatives of $f(x,y)$ *are continuous near* (x_0,y_0), *and* $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$. *We denote by* D *the* **discriminant**

$$D(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2.$$

1. *If* $D > 0$ *and* $f_{xx}(x_0,y_0) < 0$ *or* $f_{yy}(x_0,y_0) < 0$ *there is a relative maximum at* (x_0,y_0).
2. *If* $D > 0$ *and* $f_{xx}(x_0,y_0) > 0$ *or* $f_{yy}(x_0,y_0) > 0$ *there is a relative minimum at* (x_0,y_0).
3. *If* $D < 0$ *there is neither a maximum nor a minimum at* (x_0,y_0), *but a saddle.*
4. *If* $D = 0$, *the test fails.*

Note: We interpret the Second Derivative Test as follows.

1. If the discriminant is positive at the point (x_0,y_0) and $f_{xx} < 0$, then the surface curves downward in all directions giving rise to a relative maximum.
2. If the discriminant is positive at the point (x_0,y_0) and $f_{xx} > 00$, then the surface curves upward in all directions giving rise to a relative minimum.
3. If the discriminant is negative at the point (x_0,y_0), then the surface curves up in some directions and down in others. This behaviour creates a so-called **saddle point** at (x_0,y_0,z_0) (see Figure 7.8).

Definition 7.46: Saddle Point

Given the function $f(x,y)$, *the point* (x_0,y_0) *is called a* **saddle point** *of* f *if*

1. *there is at least one point* (x,y) *arbitrarily close to* (x_0,y_0) *for which* $f(x,y) > f(x_0,y_0)$, *and*
2. *there is at least one point* (x,y) *arbitrarily close to* (x_0,y_0) *for which* $f(x,y) < f(x_0,y_0)$.

Example 7.47: Extrema on an Elliptic Paraboloid

Verify that $f(x,y) = x^2 + y^2$ *has a minimum at* $(0,0)$.

Solution. First, we compute all the needed derivatives:

$$f_x = 2x \qquad f_y = 2y \qquad f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{xy} = 0.$$

The derivatives f_x and f_y are zero only at $(0,0)$. Applying the Second Derivative Test there:

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 2 \cdot 2 - 0 = 4 > 0,$$

so there is a relative minimum at $(0,0)$, and there are no other possibilities. ♣

Example 7.48: Saddle Point on a Hyperbolic Paraboloid

Find all relative maxima and minima for $f(x,y) = x^2 - y^2$.

Solution. The derivatives:

$$f_x = 2x \qquad f_y = -2y \qquad f_{xx} = 2 \qquad f_{yy} = -2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 2 \cdot -2 - 0 = -4 < 0,$$

so there is neither a maximum nor minimum at $(0,0,0)$, but a saddle point. The surface is shown in Figure 7.8. ♣

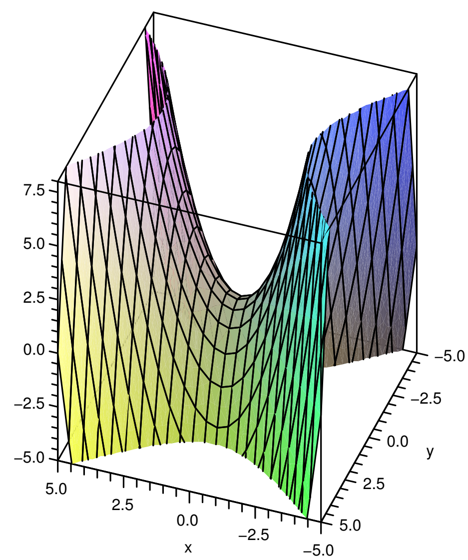


Figure 7.8: A saddle point at $(0,0,0)$ on the surface of $z = x^2 - y^2$.

Example 7.49: Finding Extrema

Find all relative maxima and minima for $f(x,y) = x^4 + y^4$.

Solution. The derivatives:

$$f_x = 4x^3 \qquad f_y = 4y^3 \qquad f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. However, in this case it is easy to see that there is a minimum at $(0,0)$, because $f(0,0) = 0$ and at all other points $f(x,y) > 0$. ♣

Example 7.50: Finding Extrema

Find all relative maxima and minima for $f(x,y) = x^3 + y^3$.

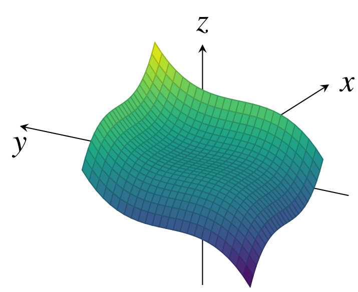
Solution. The derivatives:

$$f_x = 3x^2 \qquad f_y = 3y^2 \qquad f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. In this case, a little thought shows there is neither a maximum nor a minimum at $(0,0)$: when x and y are both positive, $f(x,y) > 0$, and when x and y are both negative, $f(x,y) < 0$, and there are points of both kinds arbitrarily close to $(0,0)$. Alternately, if we look at the cross-section when $y = 0$, we get $f(x,0) = x^3$, which does not have either a maximum or minimum at $x = 0$. In fact, $(0,0,0)$ is a saddle point.



♣

Example 7.51: Maximizing Profits

A certain company produces and sells entertainment units. The total weekly revenue and cost in dollars is given by

$$R(x,y) = -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 280x + 260y + 1000, \text{ and}$$

$$C(x,y) = 160x + 160y + 6000$$

respectively, where x denotes the number of fully assembled units and y the number of kits produced and sold each week. Find the company's maximum profit per week, and the number of assembled units and kits that must be produced and sold to achieve this profit.

Solution. The weekly profits is given by

$$\begin{aligned} P(x,y) &= R(x,y) - C(x,y) \\ &= \left(-\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 280x + 260y + 1000\right) - (160x + 160y + 6000) \\ &= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 120x + 110y - 5000, \end{aligned}$$

where $x \geq 0$ and $y \geq 0$.

To find the relative maximum profit per week, we calculate the critical points by setting the first partial derivatives equal to zero:

$$P_x = -\frac{1}{2}x - \frac{1}{4}y + 120 = 0$$

$$P_y = -\frac{3}{4}y - \frac{1}{4}x + 100 = 0$$

We solve for y by multiplying the second equation by 2 and subtracting it from the first equation, which yields

$$\frac{5}{4}y - 80 = 0 \implies y = 64.$$

Substituting this value of y into the first equation we obtain

$$-\frac{1}{2}x - \frac{1}{4}(64) + 120 = 0 \implies x + 32 - 240 = 0 \implies x = 208.$$

Therefore, the only critical point of P is $(208, 64)$.

We now apply the Second Derivative Test to confirm that the weekly maximum profit occurs at $(208, 64)$. For this we need the second partial derivatives of P:

$$P_{xx} = -\frac{1}{2}, \; P_{xy} = -\frac{1}{4}, \; P_{yy} = -\frac{3}{4}.$$

Therefore, the discriminant is given by

$$D(x,y) = \left(-\frac{1}{2}\right)\left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right)^2 = \frac{3}{8} - \frac{1}{16} = \frac{5}{16} > 0$$

for all (x,y) in the domain of P and, in particular, $D(208,64) > 0$. Since $P_{xx}(208,64) = -\frac{1}{2} < 0$, the point $(208,64)$ leads to a relative maximum of P. Hence, the maximizing number of assembled units is 208 units and the maximizing number of kits is 64 kits, whose production and sale leads to a weekly profit of

$$P(208,64) = -\frac{1}{4}(208)^2 - \frac{3}{8}(64)^2 - \frac{1}{4}(208)(64) + 120(208) + 110(64) - 5000 = \$10,680.$$

♣

Example 7.52: Optimizing Surface

A rectangular container with an open top needs to have a capacity of 10 m^3 and be constructed of a thin sheet of metal. What are the dimensions of the box if the amount of metal is minimized?

Solution. Let x, y and z be the length, width and height of the box respectively, measured in metres. Then the volume of the box is

$$xyz = 10 \implies z = \frac{10}{xy}.$$

Hence, the surface area of the box is

$$\begin{aligned} A(x,y,z) &= 2xz + 2yz + xy \\ A(x,y) &= 2x\left(\frac{10}{xy}\right) + 2y\left(\frac{10}{xy}\right) + xy \\ &= \frac{20}{y} + \frac{20}{x} + xy, \end{aligned}$$

with domain $\{(x,y) \in \mathbb{R}^2 | x > 0, y > 0\}$. We now find the critical points of A:

$$A_x = -\frac{20}{x^2} + y, \quad \text{and} \quad A_y = -\frac{20}{y^2} + x.$$

Therefore,

$$\begin{aligned} A_x = 0 \quad \text{and} \quad A_y = 0 &\implies y = \frac{20}{x^2} \quad \text{and} \quad x = \frac{20}{y^2} \\ &\implies y = \frac{20}{20/y^2} \implies 20y = y^4 \\ &\implies y(20 - y^3) = 0 \end{aligned}$$

Now, $y = 0 \notin D_A^y$, but

$$y = 20^{1/3} \in D_A^y \implies x = \frac{20}{(20^{1/3})^2} = 20^{1/3} \in D_A^x.$$

Hence, $(20^{1/3}, 20^{1/3})$ is the only critical point of A.
Since the domain of A is unbounded, we use the Second Derivative Test to classify this critical point. The second-order partial derivatives are

$$A_{xx} = \frac{40}{x^3}, \quad A_{yy} = \frac{40}{y^3}, \quad A_{xy} = 1,$$

and so the discriminant is

$$D\left(20^{1/3}, 20^{1/3}\right) = \frac{40}{\left(20^{1/3}\right)^3} \cdot \frac{40}{\left(20^{1/3}\right)^3} - 1 = 3 > 0.$$

Since

$$A_{xx}\left(20^{1/3}, 20^{1/3}\right) = \frac{40}{\left(20^{1/3}\right)^3} = 2 > 0,$$

the critical point yields a relative minimum with a value of

$$z = \frac{10}{(20^{1/3})(20^{1/3})} = \frac{20^{1/3}}{2}.$$

Is this an absolute minimum? It is, but it is difficult to see this analytically; physically and graphically it is clear that there is a minimum, in which case it must be at the single critical point.
Thus, the dimensions of the box which minimize the amount of metal are width $= 20^{1/3} \approx 2.7$ m, length $= 20^{1/3} \approx 2.7$ m and height $= \frac{20^{1/3}}{2} \approx 1.4$ m. ♣

Recall that when we did single-variable absolute maximum and minimum problems, the easiest cases were those for which the variable could be limited to a finite closed interval, for then we simply had to check all critical values and the endpoints. The previous example is difficult because there is no finite boundary to the domain of the problem—both w and l can be in $(0,\infty)$. As in the single-variable case, the problem is often simpler when there is a finite boundary.

Theorem 7.53: Absolute Extrema of a Two-Variable Function

If $f(x,y)$ is continuous on a closed and bounded subset of $\mathbb{R}^2$, then it has both an absolute maximum and an absolute minimum value.

As in the case of single-variable functions, this means that the absolute maximum and minimum values must occur at a critical point or on the boundary; however, in the two-variable case the boundary is a curve, not merely two endpoints.

Guideline for Finding Absolute Extrema

Suppose that $z = f(x,y)$ is a continuous function on a closed and bounded region $R \subseteq \mathbb{R}^2$.

1. List the *interior* points of R that are critical points of f, which may lead to relative extrema of f. Evaluate f at these points.

2. List the *boundary points* of R, which may lead to relative extrema of f. Evaluate f at these points.

3. Compare the maximum and minimum values from steps 1 and 2, the *largest* value is the absolute maximum and the *smallest* value is the absolute minimum.

We will demonstrate the above guideline with two examples.

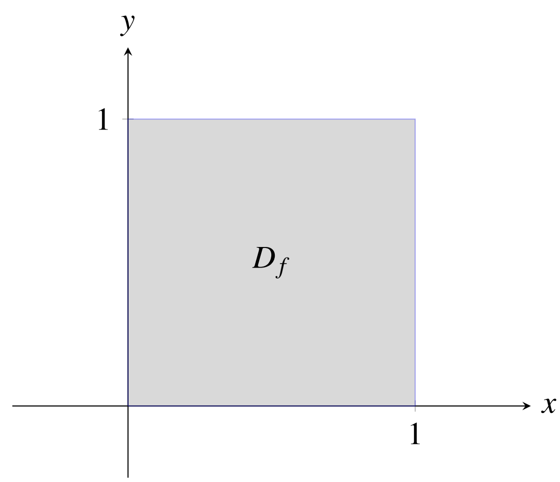
Example 7.54: Finding Absolute Extrema

Find the absolute extrema of

$$f(x,y) = xy - x^3y^2$$

on the region bounded by $0 \le x \le 1, \quad 0 \le y \le 1$.

Solution. We apply the guideline for finding absolute extrema.
We begin by determining the boundary points:
Notice that the constraint $0 \le x \le 1$, $0 \le y \le 1$ determines a square region bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 1$ with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$.



When $x=0$, then $f(0,y)=0$. Similarly, when $y=0$, then $f(x,0)=0$. On the edge $x=1, 0\le y\le 1$:

$$a(y)=f(1,y)=y-y^2 \implies a'(y)=1-2y=0 \implies y=0.5.$$

Hence, $(1,0.5)$ is the only critical point of a on this edge.
On the edge $y=1, 0\le x\le 1$:

$$b(x)=f(x,1)=x-x^3 \implies b'(x)=1-3x^2=0 \implies x=\frac{1}{\sqrt{3}}$$

Hence, $\left(\frac{1}{\sqrt{3}},1\right)$ is the only critical point of b on this edge.
Next, we determine the interior points:
To find the critical points of f, we begin by computing the first-order partial derivatives of f:

$$f_x=y-3x^2y^2, \quad f_y=x-2x^3y$$

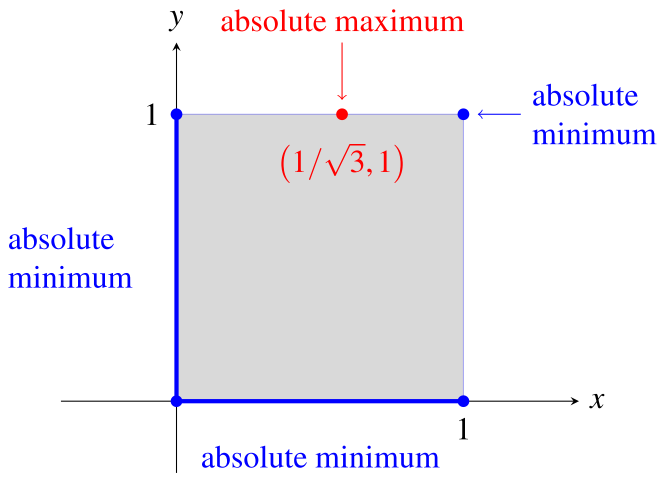
We set the derivatives equal to zero and solve for x and y:

$$\begin{aligned}f_x=y-3x^2y^2=0,&\quad f_y=x-2x^3y=0\\ \implies y(1-3x^2y)=0,&\quad x(1-2x^2y)=0\\ \implies y=0 \text{ or } 1-3x^2y=0,&\quad x=0 \text{ or } 1-2x^2y=0\end{aligned}$$

If $y=0$, we cannot have $2x^2y=1$, so we must have $x=0$, which yields the point $(0,0)$.
If $3x^2y=1$, we cannot have $x=0$, so we must have $2x^2y=1$ But then $x^2y=\frac{1}{3}=\frac{1}{2}$, which is impossible.
Hence, the only critical point on the square is $(0,0)$.
Lastly, we compare values of f at all interior and boundary points we have found:

$$\begin{aligned}&f(x,0)=0\\ &f(0,y)=0\\ &f(0,0)=f(0,1)=f(1,0)=0\\ &f(1,1)=1\cdot 1-1^3\cdot 1^2=0\\ &f\left(1,\frac{1}{2}\right)=1\cdot\frac{1}{2}-1^3\cdot\left(\frac{1}{2}\right)^2=\frac{1}{4}=0.25\\ &f\left(\frac{1}{\sqrt{3}},1\right)=\frac{1}{\sqrt{3}}\cdot 1-\left(\frac{1}{2}\right)^3\cdot 1^2=\frac{2}{3\sqrt{3}}\approx 0.385\end{aligned}$$

Thus, the absolute maximum occurs at $\left(\frac{1}{\sqrt{3}}, 1\right)$ with a value of $\frac{2}{3\sqrt{3}}$, and the absolute minimum occurs along the lines $x = 0$, $0 \leq y \leq 1$ and $y = 0$, $0 \leq x \leq 1$ as well as the point $(1, 1)$.



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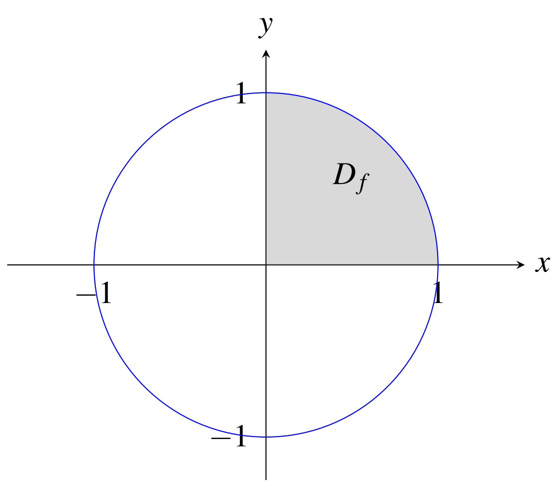
Example 7.55: Optimizing Volume of a Box

The length of the diagonal of a box is to be one metre; find the maximum possible volume.

Solution. If the box is placed with one corner at the origin, and sides along the axes, the length of the diagonal is $\sqrt{x^2 + y^2 + z^2}$, and the volume is

$$V = xyz = xy\sqrt{1 - x^2 - y^2}.$$

Clearly, $x^2 + y^2 \leq 1$, so the domain we are interested in is the quarter of the unit disk in the first quadrant as shown below.



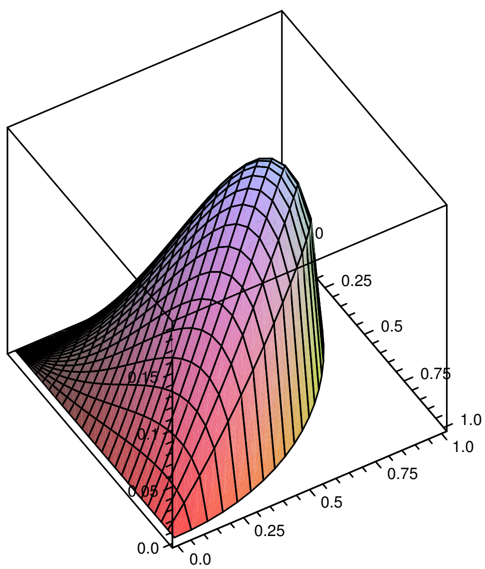
To find the interior points that lead to relative extrema, we calculate the derivatives:

$$V_x = \frac{y - 2yx^2 - y^3}{\sqrt{1 - x^2 - y^2}}$$
$$V_y = \frac{x - 2xy^2 - x^3}{\sqrt{1 - x^2 - y^2}}$$

If these are both 0, then $x = 0$ or $y = 0$, or $x = y = 1/\sqrt{3}$.

The boundary of the domain is composed of three curves: $x = 0$ for $y \in [0,1]$; $y = 0$ for $x \in [0,1]$; and $x^2 + y^2 = 1$, where $x \geq 0$ and $y \geq 0$. In all three cases, the volume $xy\sqrt{1 - x^2 - y^2}$ is 0.

Comparing the relative extrema, the maximum occurs at the only critical point $(1/\sqrt{3}, 1/\sqrt{3}, \sqrt{3}/\sqrt{3})$. The surface of f on its domain is shown below.



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Exercises for Section 7.6

Exercise 7.6.1 *Classify the critical points of the following functions:*

(a) $f(x,y) = x^2 + 4y^2 - 2x + 8y - 1$

(b) $f(x,y) = x^2 - y^2 + 6x - 10y + 2$

(c) $f(x,y) = xy$

(d) $f(x,y) = 9 + 4x - y - 2x^2 - 3y^2$

(e) $f(x,y) = x^2 + 4xy + y^2 - 6y + 1$

(f) $f(x,y) = x^2 - xy + 2y^2 - 5x + 6y - 9$

(g) $f(x,y)=xy(5x+y-15)$

(h) $f(x,y)=x^2y-xy^2+4xy-4x^2-4y^2$

(i) $f(x,y)=e^{-x^3/3+x-y^2}$

(j) $f(x,y)=2x^2+2xy+2y^2-6x$

(k) $f(x,y)=x^4+y^4-4xy$

(l) $f(x,y)=x^2y-2xy^2+3xy+4$

Exercise 7.6.2 *Find the absolute maximum and minimum points of $f=x^2+3y-3xy$ over the region bounded by $y=x$, $y=0$, and $x=2$.*

Exercise 7.6.3 *A six-sided rectangular box is to hold $1/2$ cubic meter; what shape should the box be to minimize surface area?*

Exercise 7.6.4 *A certain company produces and sells flat screen TVs. The total weekly revenue and cost in dollars is given by*

$$R(x,y)=-0.2x^2-0.25y^2-0.2xy+225x+135y+1500, \text{ and}$$

$$C(x,y)=125x+45y+5500$$

respectively, where x denotes the number of deluxe units and y the number of standard units produced and sold each week. Find the company's maximum profit per week, and the number of deluxe and standard units that must be produced and sold to achieve this profit.

Exercise 7.6.5 *A certain company produces and sells hand-made wooden living room tables. The total monthly revenue and cost in dollars is given by*

$$R(x,y)=-0.005x^2-0.003y^2-0.002xy+31x+22y, \text{ and}$$

$$C(x,y)=17x+10y+200$$

respectively, where x denotes the number of finished tables and y the number of unfinished tables produced and sold each month. Find the company's maximum profit per month, and the number of finished and unfinished tables that must be produced and sold to achieve this profit.

Exercise 7.6.6 *The post office will accept packages whose combined length and girth is at most 130 inches. (Girth is the maximum distance around the package perpendicular to the length; for a rectangular box, the length is the largest of the three dimensions.) What is the largest volume that can be sent in a rectangular box?*

Exercise 7.6.7 *The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost.*

Exercise 7.6.8 *Using the methods of this section, find the shortest distance from the origin to the plane $x+y+z=10$.*

Exercise 7.6.9 *Using the methods of this section, find the shortest distance from the point (x_0,y_0,z_0) to the plane $ax+by+cz=d$. You may assume that $c\neq 0$; use of Sage or similar software is recommended.*

Exercise 7.6.10 *A trough is to be formed by bending up two sides of a long metal rectangle so that the cross-section of the trough is an isosceles trapezoid. If the width of the metal sheet is 2 meters, how should it be bent to maximize the volume of the trough?*

Exercise 7.6.11 *Given the three points* $(1,4)$*,* $(5,2)$*, and* $(3,-2)$*,* $(x-1)^2+(y-4)^2+(x-5)^2+(y-2)^2+(x-3)^2+(y+2)^2$ *is the sum of the squares of the distances from point* (x,y) *to the three points. Find x and y so that this quantity is minimized.*

Exercise 7.6.12 *Suppose that* $f(x,y)=x^2+y^2+kxy$*. Find and classify the critical points, and discuss how they change when k takes on different values.*

Exercise 7.6.13 *Find the shortest distance from the point* $(0,b)$ *to the parabola* $y=x^2$*.*

Exercise 7.6.14 *Find the shortest distance from the point* $(0,0,b)$ *to the paraboloid* $z=x^2+y^2$*.*

Exercise 7.6.15 *Consider the function* $f(x,y)=x^3-3x^2y+y^3$*.*

(a) *Show that* $(0,0)$ *is the only critical point of* f*.*

(b) *Show that the Discriminant Test is inconclusive for* f*.*

(c) *Determine the cross-sections of* f *obtained by setting* $y=kx$ *for various values of k.*

(d) *What kind of critical point is* $(0,0)$*?*

Exercise 7.6.16 *Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid* $2x^2+72y^2+18z^2=288$*.*

7.7 Lagrange Multipliers

Many applied max/min problems take the following form: we want to find an extreme value of a function, like $V=xyz$, subject to a constraint, like $1=\sqrt{x^2+y^2+z^2}$. Often this can be done, as we have, by explicitly combining the equations and then finding critical points. There is another approach that is often convenient, the method of **Lagrange multipliers**.

It is somewhat easier to understand two variable problems, so we begin with one as an example. Suppose the perimeter of a rectangle is to be 100 units. Find the rectangle with largest area. This is a fairly straightforward problem from single variable calculus. We write down the two equations: $A=xy$, $P=100=2x+2y$, solve the second of these for y (or x), substitute into the first, and end up with a one-variable maximization problem. Let's now think of it differently: the equation $A=xy$ defines a surface, and the equation $100=2x+2y$ defines a curve (a line, in this case) in the x-y-plane. If we graph both of these in the three-dimensional coordinate system, we can phrase the problem like this: what is the highest point on the surface above the line? The solution we already understand effectively produces the equation of the cross-section of the surface above the line and then treats it as a single variable problem. Instead, imagine that we draw the level curves (the contour lines) for the surface in the x-y-plane, along with the line.

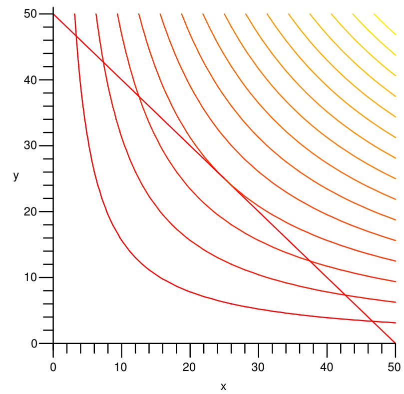


Figure 7.9: Constraint line with contour plot of the surface xy.

Imagine that the line represents a hiking trail and the contour lines are, as on a topographic map, the lines of constant altitude. How could you estimate, based on the graph, the high (or low) points on the path? As the path crosses contour lines, you know the path must be increasing or decreasing in elevation. At some point you will see the path just touch a contour line (tangent to it), and then begin to cross contours in the opposite order—that point of tangency must be a maximum or minimum point. If we can identify all such points, we can then check them to see which gives the maximum and which the minimum value. As usual, we also need to check boundary points; in this problem, we know that x and y are positive, so we are interested in just the portion of the line in the first quadrant, as shown. The endpoints of the path, the two points on the axes, are not points of tangency, but they are the two places that the function xy is a minimum in the first quadrant.

How can we actually make use of this? At the points of tangency that we seek, the constraint curve (in this case the line) and the level curve have the same slope—their tangent lines are parallel. This also means that the constraint curve is perpendicular to the gradient vector of the function; going a bit further, if we can express the constraint curve itself as a level curve, then we seek the points at which the two level curves have parallel gradients. The curve $100 = 2x + 2y$ can be thought of as a level curve of the function $2x + 2y$; Figure 7.10 shows both sets of level curves on a single graph. We are interested in those points where two level curves are tangent—but there are many such points, in fact an infinite number, as we've only shown a few of the level curves. All along the line $y = x$ are points at which two level curves are tangent. While this might seem to be a show-stopper, it is not.

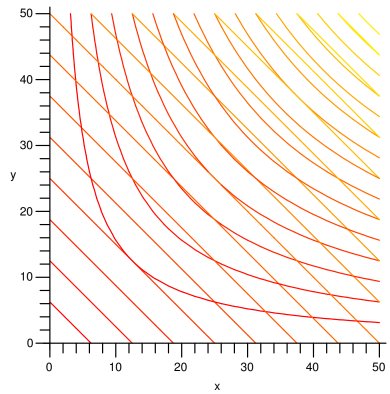


Figure 7.10: Contour plots for $2x+2y$ and xy.

The gradient of $2x+2y$ is $\langle 2,2\rangle$, and the gradient of xy is $\langle y,x\rangle$. They are parallel when $\langle 2,2\rangle = \lambda \langle y,x\rangle$, that is, when $2 = \lambda y$ and $2 = \lambda x$. We have two equations in three unknowns, which typically results in many solutions (as we expected). A third equation will reduce the number of solutions; the third equation is the original constraint, $100 = 2x+2y$. So we have the following system to solve:

$$2 = \lambda y \qquad 2 = \lambda x \qquad 100 = 2x+2y.$$

In the first two equations, λ can't be 0, so we may divide by it to get $x = y = 2/\lambda$. Substituting into the third equation we get

$$\begin{aligned} 2\frac{2}{\lambda} + 2\frac{2}{\lambda} &= 100 \\ \frac{8}{100} &= \lambda \end{aligned}$$

so $x = y = 25$. Note that we are not really interested in the value of λ—it is a clever tool, the Lagrange multiplier, introduced to solve the problem. In many cases, as here, it is easier to find λ than to find everything else without using λ.

The same method works for functions of three variables, except of course everything is one dimension higher: the function to be optimized is a function of three variables and the constraint represents a surface—for example, the function may represent temperature, and we may be interested in the maximum temperature on some surface, like a sphere. The points we seek are those at which the constraint surface is tangent to a level surface of the function. Once again, we consider the constraint surface to be a level surface of some function, and we look for points at which the two gradients are parallel, giving us three equations in four unknowns. The constraint provides a fourth equation.

Example 7.56: Optimization with Constraints

Maximize the function xyz *given the constraint* $1 = \sqrt{x^2+y^2+z^2}$.

Solution. The constraint is $1 = \sqrt{x^2+y^2+z^2}$, which is the same as $1 = x^2+y^2+z^2$. The function to maximize is xyz. The two gradient vectors are $\langle 2x, 2y, 2z\rangle$ and $\langle yz, xz, xy\rangle$, so the equations to be solved are

$$\begin{aligned} yz &= 2x\lambda \\ xz &= 2y\lambda \\ xy &= 2z\lambda \\ 1 &= x^2+y^2+z^2 \end{aligned}$$

If $\lambda = 0$ then at least two of x, y, z must be 0, giving a volume of 0, which will not be the maximum. If we multiply the first two equations by x and y respectively, we get

$$\begin{aligned} xyz &= 2x^2\lambda \\ xyz &= 2y^2\lambda \end{aligned}$$

so $2x^2\lambda = 2y^2\lambda$ or $x^2 = y^2$; in the same way we can show $x^2 = z^2$. Hence the fourth equation becomes $1 = x^2+x^2+x^2$ or $x = 1/\sqrt{3}$, and so $x = y = z = 1/\sqrt{3}$ gives the maximum volume. This is of course the same answer we obtained previously. ♣

Another possibility is that we have a function of three variables, and we want to find a maximum or minimum value not on a surface but on a curve; often the curve is the intersection of two surfaces, so that we really have two constraint equations, say $g(x,y,z) = c_1$ and $h(x,y,z) = c_2$. It turns out that at points on the intersection of the surfaces where f has a maximum or minimum value,

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

As before, this gives us three equations, one for each component of the vectors, but now in five unknowns, x, y, z, λ, and μ. Since there are two constraint functions, we have a total of five equations in five unknowns, and so can usually find the solutions we need.

Example 7.57: Intersection of a Plane with a Cylinder

The plane $x+y-z=1$ intersects the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

Solution. We want the extreme values of $f = \sqrt{x^2+y^2+z^2}$ subject to the constraints $g = x^2+y^2 = 1$ and $h = x+y-z = 1$. To simplify the algebra, we may use instead $f = x^2+y^2+z^2$, since this has a maximum or minimum value at exactly the points at which $\sqrt{x^2+y^2+z^2}$ does. The gradients are

$$\nabla f = \langle 2x, 2y, 2z\rangle \qquad \nabla g = \langle 2x, 2y, 0\rangle \qquad \nabla h = \langle 1, 1, -1\rangle,$$

so the equations we need to solve are

$$\begin{aligned} 2x &= \lambda 2x + \mu \\ 2y &= \lambda 2y + \mu \end{aligned}$$

$$\begin{aligned} 2z &= 0 - \mu \\ 1 &= x^2 + y^2 \\ 1 &= x + y - z. \end{aligned}$$

Subtracting the first two we get $2y - 2x = \lambda(2y - 2x)$, so either $\lambda = 1$ or $x = y$. If $\lambda = 1$ then $\mu = 0$, so $z = 0$ and the last two equations are

$$1 = x^2 + y^2 \qquad \text{and} \qquad 1 = x + y.$$

Solving these gives $x = 1$, $y = 0$, or $x = 0$, $y = 1$, so the points of interest are $(1,0,0)$ and $(0,1,0)$, which are both distance 1 from the origin. If $x = y$, the fourth equation is $2x^2 = 1$, giving $x = y = \pm 1/\sqrt{2}$, and from the fifth equation we get $z = -1 \pm \sqrt{2}$. The distance from the origin to $(1/\sqrt{2}, 1/\sqrt{2}, -1+\sqrt{2})$ is $\sqrt{4-2\sqrt{2}} \approx 1.08$ and the distance from the origin to $(-1/\sqrt{2}, -1/\sqrt{2}, -1-\sqrt{2})$ is $\sqrt{4+2\sqrt{2}} \approx 2.6$. Thus, the points $(1,0,0)$ and $(0,1,0)$ are closest to the origin and $(-1/\sqrt{2}, -1/\sqrt{2}, -1-\sqrt{2})$ is farthest from the origin. ♣

Exercises for Section 7.7

Exercise 7.7.1 *A six-sided rectangular box is to hold $1/2$ cubic meter; what shape should the box be to minimize surface area?*

Exercise 7.7.2 *The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box?*

Exercise 7.7.3 *The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost.*

Exercise 7.7.4 *Using Lagrange multipliers, find the shortest distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$.*

Exercise 7.7.5 *Find all points on the surface $xy - z^2 + 1 = 0$ that are closest to the origin.*

Exercise 7.7.6 *The material for the bottom of an aquarium costs half as much as the high strength glass for the four sides. Find the shape of the cheapest aquarium that holds a given volume V.*

Exercise 7.7.7 *The plane $x - y + z = 2$ intersects the cylinder $x^2 + y^2 = 4$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin.*

Exercise 7.7.8 *Find three positive numbers whose sum is 48 and whose product is as large as possible.*

Exercise 7.7.9 *Find all points on the plane $x+y+z=5$ in the first octant at which $f(x,y,z)=xy^2z^2$ has a maximum value.*

Exercise 7.7.10 *Find the points on the surface $x^2-yz=5$ that are closest to the origin.*

Exercise 7.7.11 *A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y-x)$ of the standard items and $45,000+500(x-2y)$ of the deluxe each year. How should the items be priced to maximize profit?*

Exercise 7.7.12 *A length of sheet metal is to be made into a water trough by bending up two sides as shown in Figure 7.11. Find x and ϕ so that the trapezoid–shaped cross section has maximum area, when the width of the metal sheet is 27 inches (that is, $2x+y=27$).*

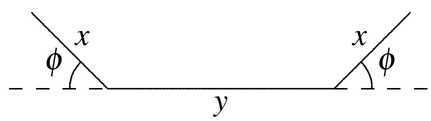


Figure 7.11: Cross-section of a trough.

Exercise 7.7.13 *Find the maximum and minimum values of $f(x,y,z)=6x+3y+2z$ subject to the constraint $g(x,y,z)=4x^2+2y^2+z^2-70=0$.*

Exercise 7.7.14 *Find the maximum and minimum values of $f(x,y)=e^{xy}$ subject to the constraint $g(x,y)=x^3+y^3-16=0$.*

Exercise 7.7.15 *Find the maximum and minimum values of $f(x,y)=xy+\sqrt{9-x^2-y^2}$ when $x^2+y^2\leq 9$.*

Exercise 7.7.16 *Find three real numbers whose sum is 9 and the sum of whose squares is a small as possible.*

Exercise 7.7.17 *Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.*

Selected Exercise Answers

1.1.1 (a) $\dfrac{x}{y}$

(b) $\sqrt{x}y$

(c) $\dfrac{2}{\sqrt[3]{x}}$

1.1.2 $a = 2$, $b = -\frac{5}{3}$, $c = \frac{3}{2}$.

1.1.3 $x = -4$ and $x = 6$.

1.1.5 (a) $(5/3, \infty)$

(b) $[1/7, 2/7]$

(c) $(-\infty, -3) \cup (-2, 1]$

(d) $(-\infty, \infty)$

(e) No solution

(f) $(-\infty, 1) \cup (1, \infty)$

(g) $(-2, 0) \cup (2, \infty)$

(h) $[4, \infty) \cup \{0\}$

(i) $(0, \frac{1}{2})$

(j) $(-2, -1] \cup (1, 4]$

1.1.6 $x = -\frac{1}{2}$ and $x = -\frac{1}{6}$.

1.1.7 (a) $(-\infty, -2] \cup [2, \infty)$

(b) $[2, 4]$

(c) $(-\infty, -9/2] \cup [-1/2, \infty)$

(d) $(4, \infty)$

(e) $(-\infty, \infty)$

(f) $(-9, -6) \cup (4, 7)$

1.2.1 (a) $(2/3)x + (1/3)$

(b) $y = -2x$

(c) $y = (-2/3)x + (1/3)$

(d) $y = -x/3 + 17/3$

(e) $y = -1/2x + 5/2$

1.2.2 (a) $y = 2x + 2$, 2, -1

(b) $y = -x + 6$, 6, 6

(c) $y = x/2 + 1/2$, $1/2$, -1

(d) $y = 3/2$, y-intercept: $3/2$, no x-intercept

(e) $y = (-2/3)x - 2$, -2, -3

1.2.3 Yes, the lines are parallel as they have the same slope of $-1/2$

1.2.4 $y = 0$, $y = -2x + 2$, $y = 2x + 2$

1.2.5 $y = (9/5)x + 32$, $(-40, -40)$

1.2.6 $y = 0.15x + 10$

1.2.7 $0.03x + 1.2$

1.2.8 (a) $P = -0.0001x + 2$
(b) $x = -10000P + 20000$

1.2.9 $(2/25)x - (16/5)$

1.2.10 (a) 2

(b) $\sqrt{2}$

(c) $\sqrt{2}$

1.2.12 (a) $x^2 + y^2 = 9$

(b) $(x-5)^2 + (y-6)^2 = 9$

(c) $(x+5)^2 + (y+6)^2 = 9$

1.2.14 (a) circle

(b) ellipse

(c) horizontal parabola

1.2.15 $(x + 2/7)^2 + (y - 41/7)^2 = 1300/49$

1.3.1 $2n\pi - \pi/2$, any integer n

1.3.2 $n\pi \pm \pi/6$, any integer n

1.3.4 $-\frac{5}{4}$

1.3.5 $\sin\theta = -x/\sqrt{x^2+1}$, $\cos\theta = -1/\sqrt{x^2+1}$.

1.3.6 $-\frac{2\pi}{7}$ is the unique answer.

1.3.7 $(\sqrt{2}+\sqrt{6})/4$

1.3.8 $-(1+\sqrt{3})/(1-\sqrt{3}) = 2+\sqrt{3}$

1.3.11 $t = \pi/2$

1.4.1 (a) $\dfrac{\sqrt{2}}{2}$

(b) $3(\sqrt{x+h+1}+\sqrt{x+1})$

1.4.2 (a) $-13/5$

(b) $-1/2, 3$

(c) $(1 \pm \sqrt{13})/2$

(d) No real solutions

(e) $\sqrt{3}$

1.4.3 Counter-examples may vary.

(a) $x = 3$

(b) $x = h = 1$

(c) $x = y = 1$

1.4.4 $x + 3y - 13 = 0$, or equivalents such as $y = -\frac{1}{3}x + \frac{13}{3}$

1.4.5 $(-\infty, 0] \cup (\frac{1}{3}, 3]$

1.4.6 It is impossible for both $x-2$ and $1-x$ to be non-negative for the same real number x.

1.4.7 $6x + 3h$

1.4.8 $-1/[(2x+2h-1)(2x-1)]$

1.4.9 -3

1.4.10 $\pi/6, 5\pi/6$

1.4.11 $2\pi/5$

1.4.12 It is equal to 2 for all x larger than 4.

1.4.13 $(x+2)^2+(y-3)^2=25$.

1.4.14 Centre is $(-3,2)$ and radius is 2.

1.4.15 y could be any real number greater than or equal to 6.

1.4.16 $x^6y^4/(36z^8)$

1.4.17 $x=(y-2)/(3+4y)$

1.4.18 $Q(x)=x+1$, $R=-7$

2.1.1 (a) $\{x \mid x \in \mathbb{R}\}$, i.e., all x

(b) $\{x \mid x \geq 3/2\}$

(c) $\{x \mid x \neq -1\}$

(d) $\{x \mid x \neq 1 \text{ and } x \neq -1\}$

(e) $\{x \mid x < 0\}$

(f) $\{x \mid x \in \mathbb{R}\}$, i.e., all x

(g) $\{x \mid h-r \leq x \leq h+r\}$

(h) $\{x \mid x \geq 0\}$

(i) $\{x \mid -1 \leq x \leq 1\}$

(j) $\{x \mid x \geq 1\}$

(k) $\{x \mid -1/3 < x < 1/3\}$

(l) $\{x \mid x \geq 0 \text{ and } x \neq 1\}$

(m) $\{x \mid x \geq 0 \text{ and } x \neq 1\}$

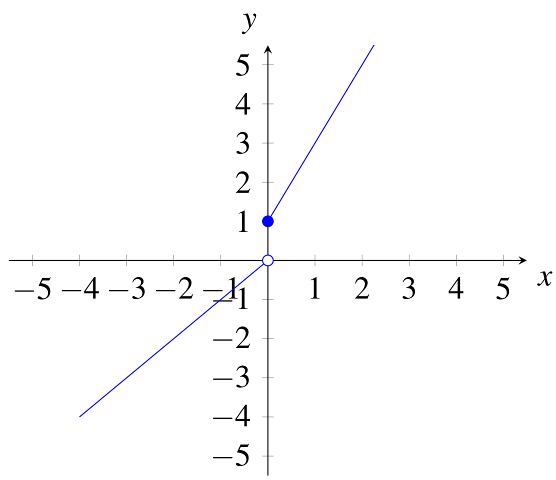
2.1.2 $A=x(500-2x)$, $\{x \mid 0 \leq x \leq 250\}$

2.1.3 $V=r(50-\pi r^2)$, $\{r \mid 0 < r \leq \sqrt{50/\pi}\}$

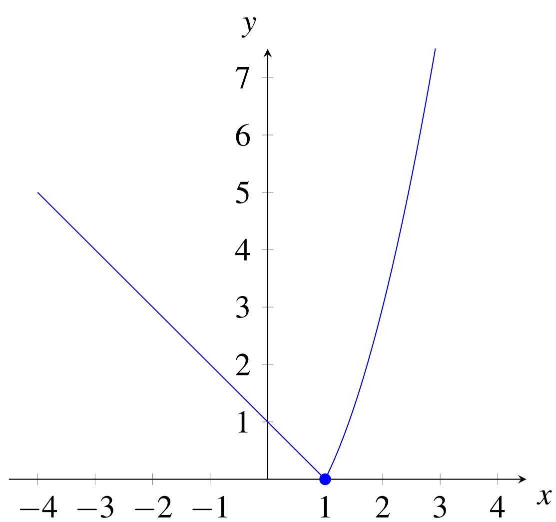
2.1.4 $A = 2\pi r^2 + 2000/r$, $\{r \mid 0 < r < \infty\}$

2.1.5 $\frac{5}{2}$,3,3,9.

2.1.6 (a) Domain is $(-\infty, \infty)$. Range is $(-\infty, 0) \cup [-\infty, \infty)$.

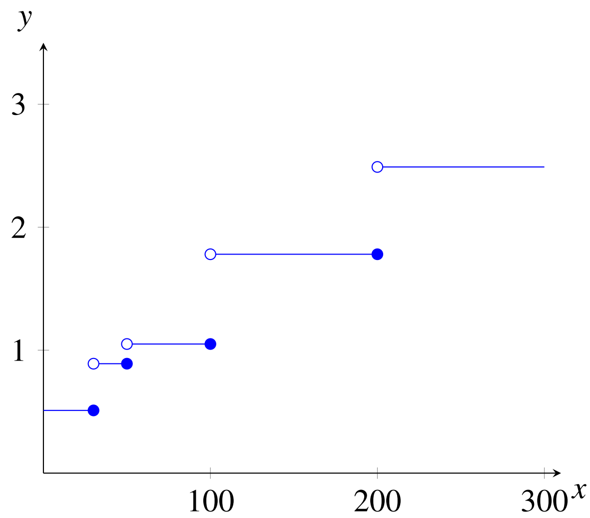


(b) Domain is $(-\infty, \infty)$. Range is $[0, \infty)$.



2.1.7

$$f(x) = \begin{cases} 0.51 & 0 < x \le 30 \\ 0.89 & 30 < x \le 50 \\ 1.05 & 50 < x \le 100 \\ 1.78 & 100 < x \le 200 \\ 2.49 & 200 < x \le 500 \end{cases}$$



$x \in [0, 500]$.

2.2.3 $\{x \mid x \geq 3\}$, $\{x \mid x \geq 0\}$

2.2.4 $4a - 2h - 1$

2.2.5 (a) $f(x) = \dfrac{2}{\sqrt{x^2+4x}}$, $g = \dfrac{x}{\sqrt{x^2+4x}}$

(b) $f(x) = 2 + x$, $g = \dfrac{1}{\sqrt{x^2+4x}}$

(c) $f(x) = \dfrac{x}{\sqrt{x^2-4}}$, $g = 2 + x$

2.3.1 $y = 2^x$

2.3.2 $y = 7$

2.3.3 $y = 2$

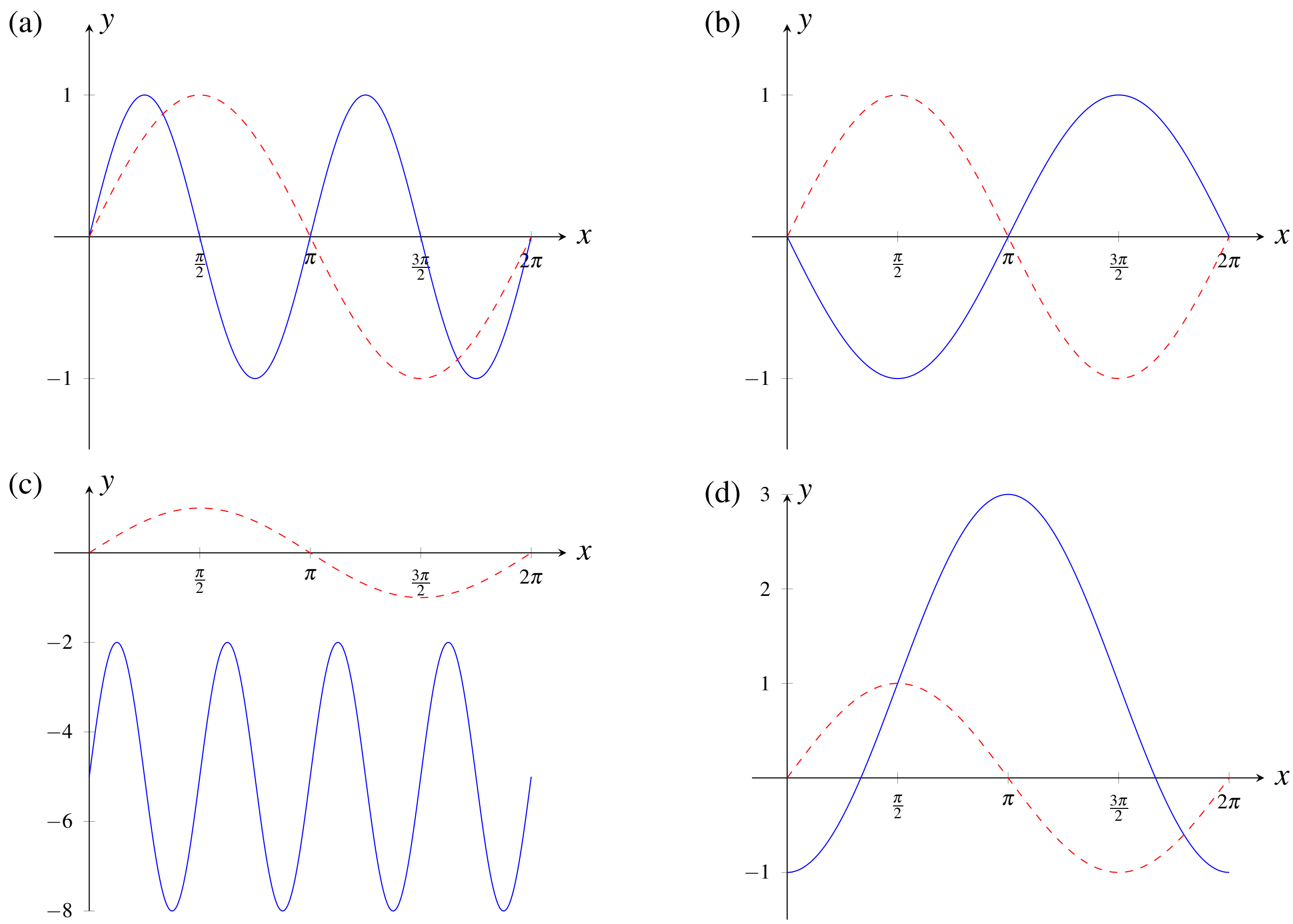
2.3.4 $x \neq 0$

2.4.4 (a) Inverses

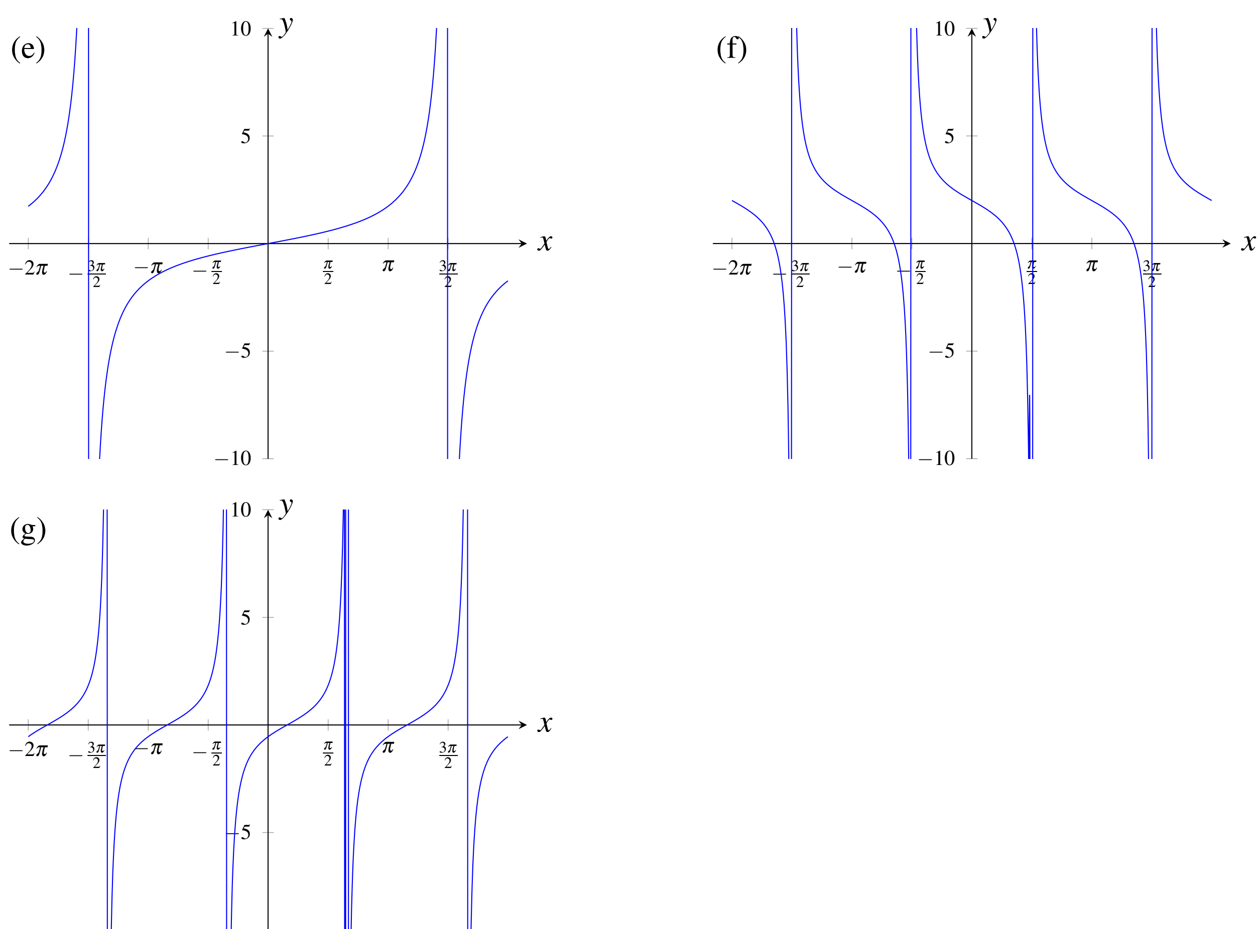
(b) Inverses

(c) Not inverses

2.5.7 $x = \dfrac{\log 2}{\log a}$



2.6.1



2.6.2 (a) \$600,000; \$400,000.

(b) $x = 4n + 2$, $(n = 0, 1, 2, ...)$.

2.6.3 (a) $\pi/3$

(b) $3\pi/4$

2.6.4 (a) $\pi/4$

(b) $-\pi/3$

(c) $1/3$

(d) $-3/4$

2.6.5 $\sqrt{1-x^2}/x$ with domain $[-1,0) \cup (0,1]$.

2.6.6 (a) $\frac{1}{\sqrt{2}}$

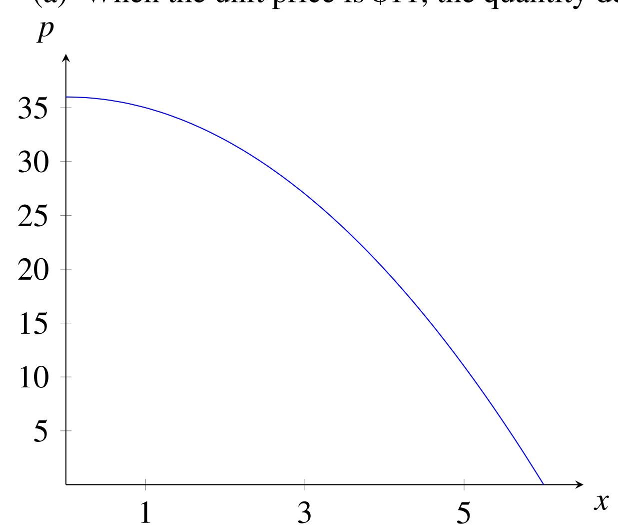
(b) $\frac{1}{\sqrt{2}}$

(c) $\sqrt{2}$

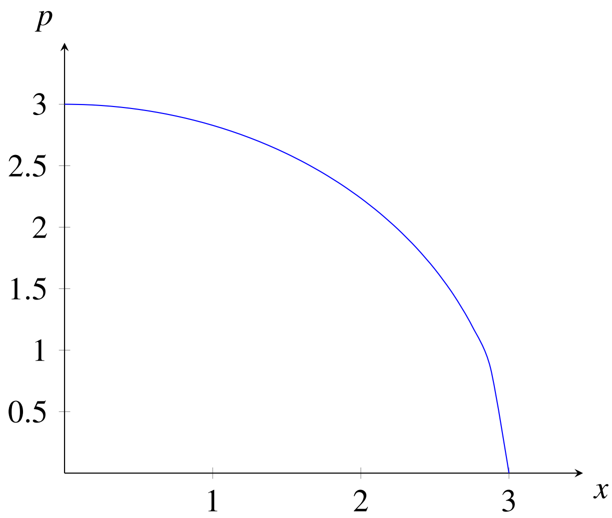
(d) 1

(e) 1

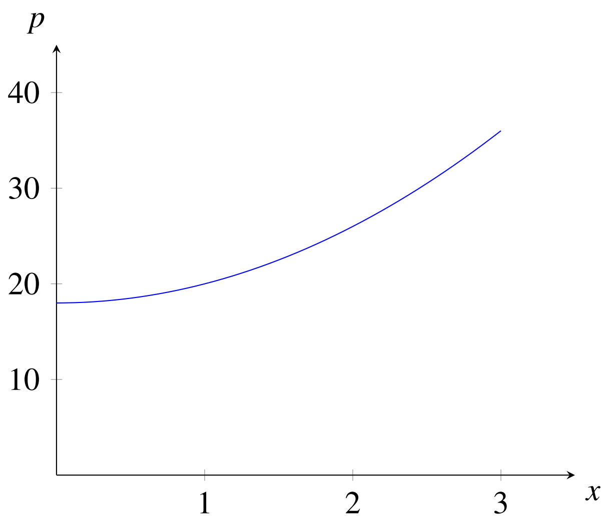
2.7.1 (a) When the unit price is \$11, the quantity demanded is 5000.



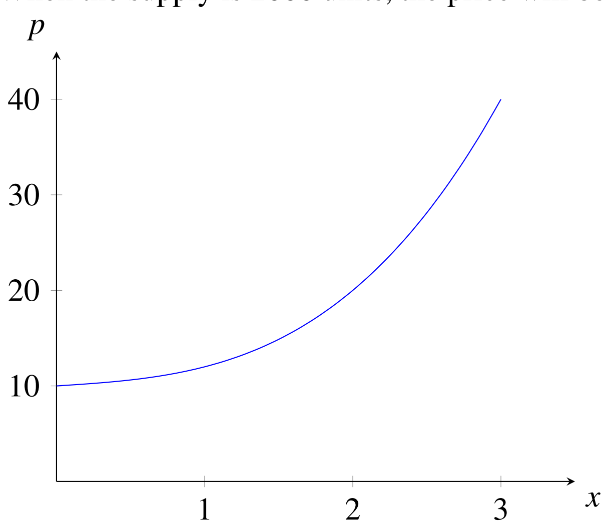
(b) When the unit price is \$2, the quantity demanded is approx. 2236.



2.7.2 (a) When the supply is 2000 units, the price will be 26.



(b) When the supply is 2000 units, the price will be 20.



2.7.3 (a) $(2500, 67.50)$

(b) $(11000, 3)$

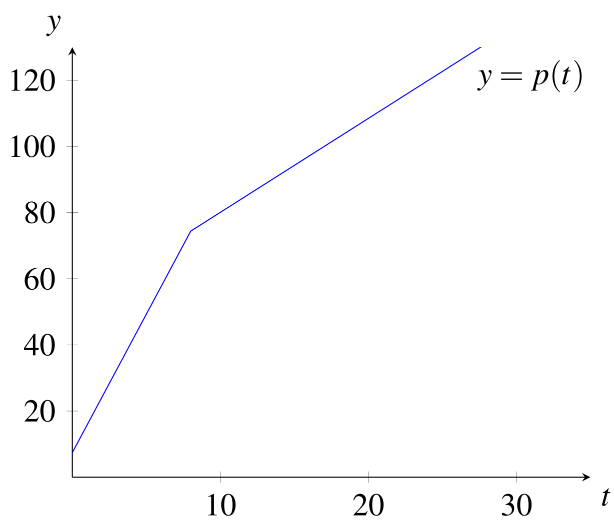
2.7.4 (a) $C(x) = 14x + 100,000$

(b) $R(x) = 12x$

(c) $P(x) = 6x - 100,000$

(d) Loss of \$28,000. Profit of \$20,000

2.7.5 (a)

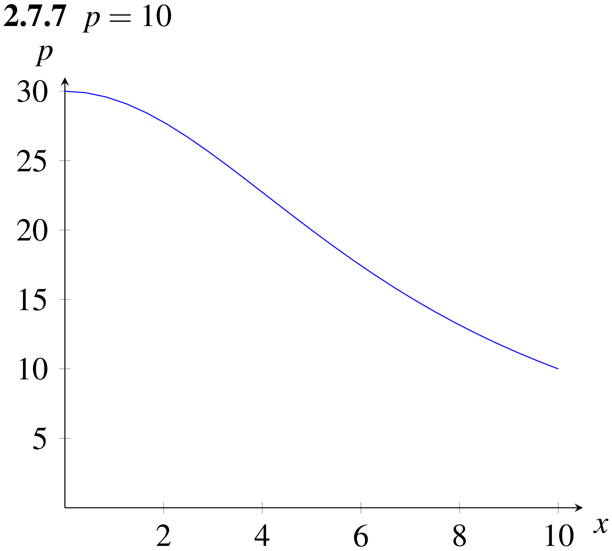


(b) \$7.44 per kg

(c) \$108.48 per kg

2.7.6 The slope of L_2 is greater than the slope of L_1. For each drop of a dollar in price, the quantity demanded for version X is greater than for version Y.

2.7.7 $p = 10$



2.7.8 $(8000, 80)$

2.7.9 (a) \$16

(b) \$11

(c) \$6

(d) 640 watches

(e) 480 watches

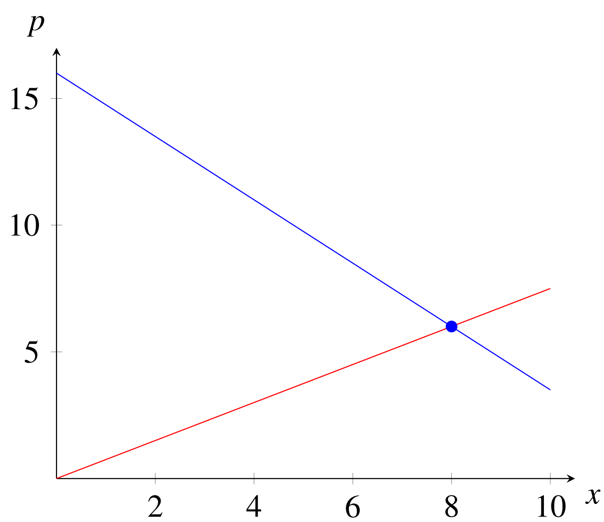
(f) 320 watches

(g) See below

(h) 0 watches

(i) Approximately 1333 watches

(j) Approximately 2667 watches

(k) 

(l) $(8, 6)$

2.7.10 (a) $C(x) = 3.50x + 90$

(b) 17 T-shirts

(c) 108 T-shirts

2.7.11 (a) Break-even quantity is about 41 units; produce. $P(x) = 145x - 6000$.

(b) Break-even quantity is -50 units; don't produce. Impossible to make a profit when $C(x) > R(x)$ for all $x > 0$. $P(x) = -100x - 5000$ (always a loss).

2.8.1 (d)

2.8.2 $3/[5(x+3)]$

2.8.3 $6+h$

2.8.4 (a) $[2,3)\cup(3,\infty)$

(b) $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

2.8.5 $\{x : x \neq 0\}$

2.8.6 $g(x)=(5x+26)/3$

2.8.7 (c)

2.8.8 $f^{-1}(x)=f(x)=\ln\left(\dfrac{e^x}{e^x-1}\right)$ and its domain is $(0,\infty)$.

2.8.9 (a) $2-\ln 3$

(b) 1,3

(c) $(e^2-1)^2$

(d) 3

2.8.10 $-\pi$

2.8.11 $2\pi/5$

2.8.12 1

3.3.1 (a) 8

(b) 6

(c) dne

(d) -2

(e) -1

(f) 8

(g) 7

(h) 6

(i) 3

(j) $-3/2$

(k) 6

(l) 2

3.4.1 (a) 7

(b) 5

(c) 0

(d) undefined

(e) $1/6$

(f) 0

(g) 3

(h) 172

(i) 0

(j) 2

(k) does not exist

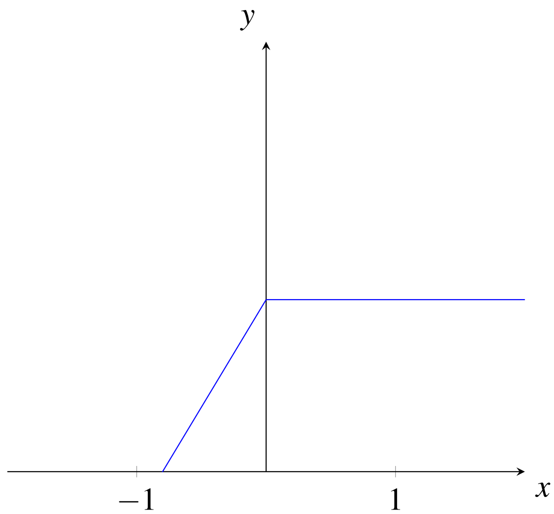
(l) $\sqrt{2}$

(m) $3a^2$

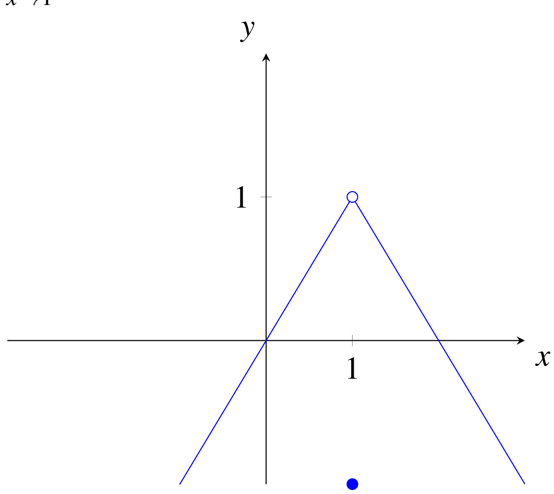
(n) 512

3.4.2 $L = 0$ and $M = 1$. No.

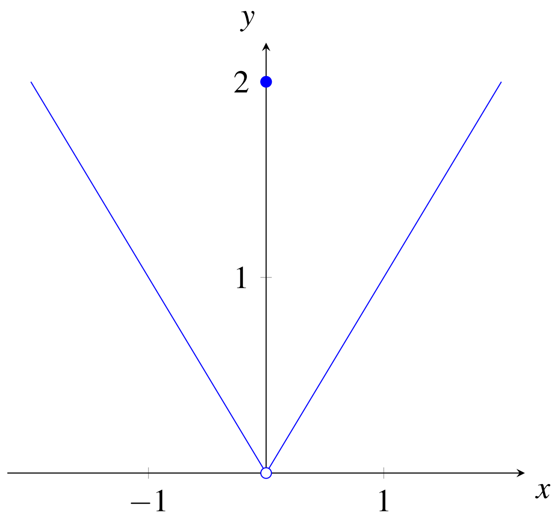
3.4.3 (a) $\lim_{x\to 0} f(x) = 1$



(b) $\lim_{x\to 1} f(x) = 1$



(c) $\lim_{x\to 0} f(x) = 0$



3.4.4 (a) -1

(b) 2

(c) $\frac{1}{6}$

3.4.5 (a) -1

(b) The limit does not exists.

(c) $\frac{2}{1}$

(d) $\frac{1}{3}$

(e) $\frac{3}{2}$

(f) 2

3.4.6 (a) 6

(b) $-\frac{1}{4}$

(c) The limit does not exist.

(d) -1

(e) The limit does not exist.

(f) 0;0

(g) 1;3

3.5.1 (a) 1

(b) 1

(c) $-\infty$

(d) $1/3$

(e) 0

(f) ∞

(g) ∞
(h) $2/7$
(i) 2
(j) $-\infty$
(k) ∞
(l) 0
(m) $1/2$
(n) 5
(o) $2\sqrt{2}$
(p) $3/2$
(q) ∞
(r) does not exist
(s) 3
(t) $-\infty$
(u) 0

3.5.2 $y = 1$ and $y = -1$

3.5.3 $x = 0$ and $x = 2$.

3.5.5 $y = x + 4$

3.5.6 (a) $-\infty$

(b) $\pi/2$

(c) $-\pi/2$

(d) ∞

(e) -5

(f) $\frac{1}{3}$

3.5.7 (a) \$500,000, \$750,000, \$1,166,667, \$2 million, \$4.5 million, \$9.5 million.

(b) The limit does not exist; as the percent of pollutant to be removes approaches 100, the cost becomes astronomical.

3.5.8 (a) \$24 million, \$60 million, \$83.1 million.

(b) \$120 million.

3.6.1 (a) 5

(b) $7/2$

(c) $3/4$

(d) 1

(e) $-\sqrt{2}/2$

3.6.2 7

3.6.3 2

3.6.5 3

3.7.1 (a) $x=0$. $\lim_{x\to 0}$ does not exist and is not equal to $f(0)$.

(b) No points of discontinuity. f is continuous everywhere.

(c) $x=0$. $\lim_{x\to 0}=2$, which is not equal to $f(0)=1$.

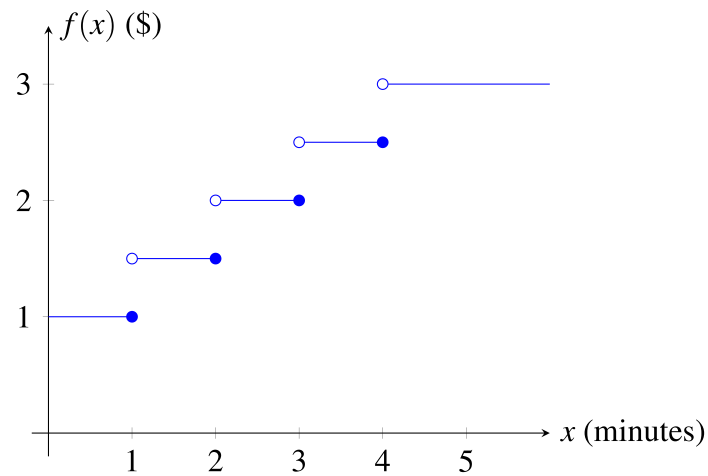
3.7.2 (a) $(-\infty,\infty)$

(b) $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$

(c) $(-\infty,\infty)$

3.7.4 $a=-1$

3.7.6 f is discontinous at $x=1,2,3,4,5$ minutes.



4.1.1 -5, -2.47106145, -2.4067927, -2.400676, -2.4

4.1.2 $-4/3$, $-24/7$, $7/24$, $3/4$

4.1.3 -0.107526881, -0.11074197, -0.1110741, $\dfrac{-1}{3(3+\Delta x)}\to\dfrac{-1}{9}$

4.1.4 $\dfrac{3+3\Delta x+\Delta x^2}{1+\Delta x}\to 3$

4.1.5 3.31, 3.003001, 3.0000,
$3+3\Delta x+\Delta x^2\to 3$

4.1.6 m

4.1.9 10, 25/2, 20, 15, 25, 35.

4.1.10 5, 4.1, 4.01, 4.001, $4+\Delta t \to 4$

4.1.11 -10.29, -9.849, -9.8049, $-9.8-4.9\Delta t \to -9.8$

4.1.12 (a) $t=0$

(b) $t=t_3$

(c) $t=t_1$

(d) $t=t_1$

4.1.13 (a) -\$2.00 per 1000 tires.

(b) -\$2.00 per 1000 tires.

4.2.1 (a) $-x/\sqrt{169-x^2}$

(b) $-9.8t$

(c) $2x+1/x^2$

(d) $2as+b$

(e) $3x^2$

(f) $-2/(2t+1)^{3/2}$

(g) $5/(t+2)^2$

4.2.4 $y=-13x+17$

4.2.5 -8

4.2.6 (a) 0

(b) 1

(c) $4x$

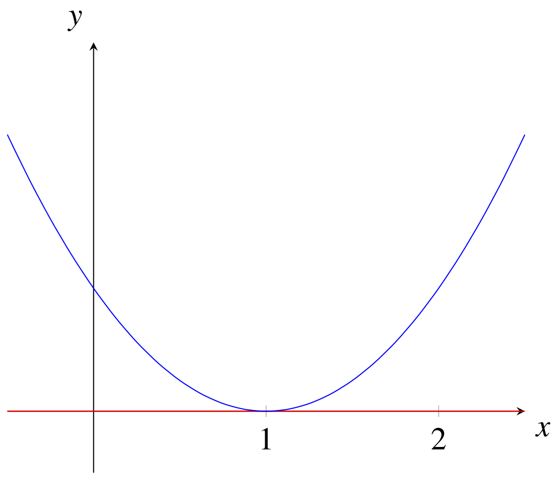
(d) $\dfrac{3}{2\sqrt{3x-1}}$

4.2.7 (a) 2; $y=2x-1$

(b) $\frac{1}{4}$; $y=\frac{1}{4}x-\frac{3}{2}$

4.2.8 (a) $f'(x)=2x-2$

(b) $(1,0)$

(c) 

(d) 0

4.2.9 (a) 6;5.5;5.1

(b) 5

(c) The computation in part(a) show that as h approaches 0, the average rate of change approaches the instantaneous rate of change.

4.2.10 (a) $C'(q) = -20q + 300$

(b) \$100 per unit

(c) \$200 per unit

4.2.11 \$5.06060, \$5.06006,\$5.060006,\$5.0600006. Estimate \$5.06.

4.2.12 (a) 0;4.2;7.2;9;9.6;9;7.2;4.2;0

(b) 4.8;3.6;2.4;1.2;0;-1.2;-2.4;-3.6;-4.8

(c) After a spectacular growth rate in the early years, the growth of the GDP cooled off.

4.2.13

t	0	1	2	3	4	5	6	7
$N'(t)$	0	2.7	4.8	6.3	7.2	7.5	7.2	6.3
$N''(t)$					0.6	0	-0.6	-1.2

4.2.14 $A'(10) = -3.09$; $A''(10) = 0.35$. 10 days after the plant opens, the amount of defective products is decreasing at a rate of 3 percent per day. The rate at which the amount of defective products is decreasing is increasing at the rate of 0.35 percent per day per day.

4.2.15 $P''(20) = -0.0169243$. The rate of change of young families owning their own homes is changing at a rate of approximately 0.02%/yr^2.

4.2.16 (a) \$1.80; \$1.60.

(b) \$1.80; \$1.60.

4.2.17 (a) $8000-200q$

(b) $200, 0, -200$.

(c) \$40.

4.2.18 (a) $R(q)=pq=-0.04q^2+800q$

(b) $R'(q)=-0.08q+800$

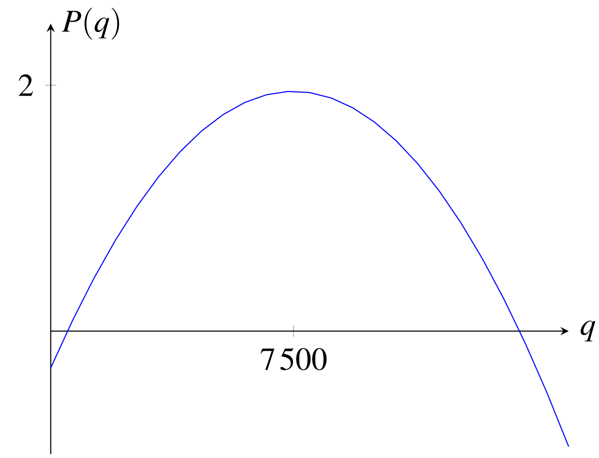
(c) $R'(5000)=400$

(d) $P(q)=-0.04q^2+600q-300,000$

(e) $P'(q)=-0.08q+600$

(f) $P'(5000)=200$, $P'(8000)=-40$

(g) The profit increases as production increases, peaking at 7500 units. Beyond this level, profit falls. $P(q)$ shown in millions of dollars.



4.3.1 (a) $100x^{99}$

(b) $-100t^{-101}$

(c) $-5x^{-6}$

(d) $\pi x^{\pi-1}$

(e) $(3/4)x^{-1/4}$

(f) $-(9/7)s^{-16/7}$

(g) $15x^2+24x$

(h) $-20s^4+6s+10/s^3$

(i) $-30x+25$

(j) $3t^2+6t-1$

(k) $-\dfrac{x^2+2x+5}{(x^2+2x-3)^2}$

(l) $3x^2(x^3-5x+10)+x^3(3x^2-5)$

(m) $s^4(7s^2+30s-15)$

(n) $\dfrac{-3x^2-20x+15}{x^6}$

(o) $-\dfrac{3x(5x+8)}{(5x^3+12x^2-15)^2}$

4.3.2 $y=13x/4+5$

4.3.3 $y=24x-48-\pi^3$

4.3.4 $-49t/5+5$, $-49/5$

4.3.6 $\sum_{k=1}^{n} k a_k x^{k-1}$

4.3.7 $x^3/16 - 3x/4 + 4$

4.3.10 $f' = 4(2x-3)$, $y = 4x - 7$

4.3.12 $\dfrac{3x^2}{x^3-5x+10} - \dfrac{x^3(3x^2-5)}{(x^3-5x+10)^2}$

4.3.13 $\dfrac{2x+5}{x^5-6x^3+3x^2-7x+1} - \dfrac{(x^2+5x-3)(5x^4-18x^2+6x-7)}{(x^5-6x^3+3x^2-7x+1)^2}$

4.3.14 $\dfrac{x-1250}{2(x-625)^{3/2}}$

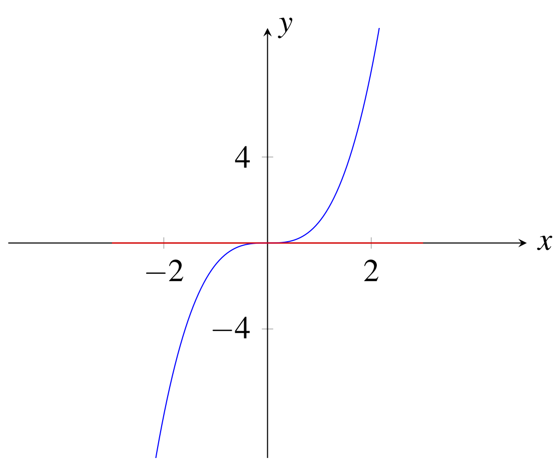
4.3.15 $\dfrac{200-39x}{2x^{21}\sqrt{x-5}}$

4.3.16 $y = 17x/4 - 41/4$

4.3.17 $y = 11x/16 - 15/16$

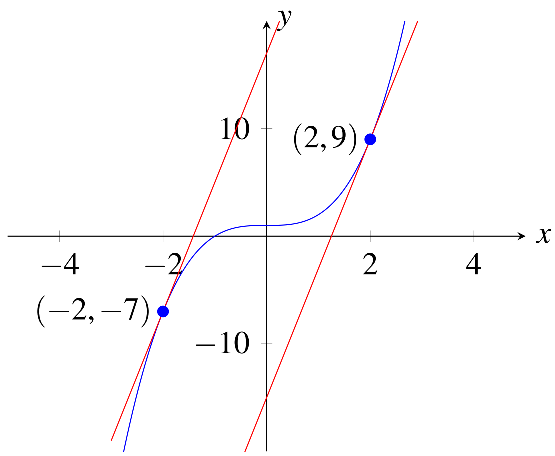
4.3.18 $13/18$

4.3.19 $(0,0)$



4.3.20 (a) $(-2,-7)$ and $(2,9)$.

(b) $y = 12x + 17$ and $y = 12x - 15$.



4.3.21 (a) $x = 0$ or $x = -1$.

(b) $x = -3$ or $x = 2$.

(c) $x = -4$ or $x = 3$.

4.3.22 (a) 12.6 pts/yr; 0 pts/yr.

(b) 10 pts/yr

4.3.23 (a) $f'(x) = -0.2x - 0.4$.

(b) \$2.40 per 1000 lamps; \$21

4.3.24 (a) $f'(x) = 0.000125x^{1/4}$.

(b) \$2.08million/yr.

4.3.25 (a) $S'(t) = 0.08508t^2 - 0.10334t + 9.60881$

(b) \$27.20171 million/yr; \$110.21491 million/yr

(c) \$270.1214 million; \$1530.8476 million

4.3.26 $(\frac{4}{3}, -\frac{770}{27})$ and $(2, -30)$.

4.3.27 $y = -\frac{1}{2}x + 1$ and $y = 2x - \frac{3}{2}$.

4.3.28 $R'(600) = -16$. The revenue is decreasing at a rate of \$16 per video game.

4.3.29 \$0.125,\$0.5,\$2,\$50 million per 1% more of the pollutant. It is too costly to remove *all* of the pollutant.

4.3.30 \$38.4, \$17.04, \$5.71 million per year.

4.3.31 (a) $\overline{C}(x) = 100 + \dfrac{200,000}{x}$

(b) $\overline{C}'(x) = -\dfrac{200,000}{x^2}$

(c) $\lim_{x\to\infty} \overline{C}(x) = 100$. The average cost approached \$100 per unit if the production level is very high.

4.3.32 (a) $R(x) = \frac{50x}{0.01x^2+1}$

(b) $R'(x) = \frac{50-0.5x^2}{(0.01x^2+1)^2}$

(c) $R'(2) \approx 44.379$. When the level of sales is at 2000 units, the revenue increases at the rate of approximately \$44,379 per 1000 units.

4.4.1 (a) $4x^3 - 9x^2 + x + 7$

(b) $3x^2 - 4x + 2/\sqrt{x}$

(c) $6t(t^2+1)^2$

(d) $\sqrt{169-x^2} - x^2/\sqrt{169-x^2}$

(e) $(2t-4)\sqrt{25-t^2} -$
$(t^2-4t+5)t/\sqrt{25-t^2}$

(f) $-x/\sqrt{r^2-x^2}$

(g) $2s^3/\sqrt{1+s^4}$

(h) $\dfrac{1}{4\sqrt{x}(5-\sqrt{x})^{3/2}}$

(i) $6 + 18x$

(j) $\dfrac{2x+1}{1-x} + \dfrac{x^2+x+1}{(1-x)^2}$

(k) $-1/\sqrt{25-x^2} - \sqrt{25-x^2}/x^2$

(l) $\dfrac{1}{2}\left(\dfrac{-169}{t^2} - 1\right) \Big/ \sqrt{\dfrac{169}{t} - t}$

(m) $\dfrac{3x^2 - 2x + 1/x^2}{2\sqrt{x^3 - x^2 - (1/x)}}$

(n) $\dfrac{300x}{(100-x^2)^{5/2}}$

(o) $\dfrac{1+3t^2}{3(t+t^3)^{2/3}}$

(p) $\left(4x(x^2+1) + \dfrac{4x^3+4x}{2\sqrt{1+(x^2+1)^2}}\right) \Big/ 2\sqrt{(x^2+1)^2 + \sqrt{1+(x^2+1)^2}}$

(q) $5(x+8)^4$

4.4.2 (a) $-3(4-t)^2$

(b) $6x(x^2+5)^2$

(c) $-12x(6-2x^2)^2$

(d) $24x^2(1-4x^3)^{-3}$

(e) $5 + 5/x^2$

(f) $-8(4t-1)(2t^2-t+3)^{-3}$

(g) $1/(s+1)^2$

(h) $3(8x-2)/(4x^2-2x+1)^2$

(i) $-3x^2 + 5x - 1$

(j) $6t(2t-4)^3 + 6(3t^2+1)(2t-4)^2$

(k) $-2/(x-1)^2$

(l) $4s/(s^2+1)^2$

(m) $(t^2-6t+7)/(t-3)^2$

(n) $-5/(3x-4)^2$

(o) $60x^4+72x^3+18x^2+18x-6$

(p) $(5-4x)/((2x+1)^2(x-3)^2)$

(q) $1/(2(2+3s)^2)$

(r) $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$

4.4.3 $y=23x/96-29/96$

4.4.4 $y=3-2x/3$

4.4.5 $y=13x/2-23/2$

4.4.6 $y=2x-11$

4.4.7 $y=\dfrac{20+2\sqrt{5}}{5\sqrt{4+\sqrt{5}}}x+\dfrac{3\sqrt{5}}{5\sqrt{4+\sqrt{5}}}$

4.4.8 $(f(g(1)))'=20$

4.4.9 (a) 0.5

(b) 1

(c) 0.5

(d) DNE

(e) DNE

(f) -0.5

4.4.10 $g'(x)=2x(f(x^2)+x^2f'(x^2))$

4.4.11 No.

4.4.12 No.

4.4.13 $\frac{1}{3}$ thousand/wk; 16 thousand; 22.7 thousand

4.4.14 $N'(t)=\frac{2000}{(1+0.2t)^{3/2}}$;2000 students; 707 students

4.4.15 $N'(t)=\frac{1.42(140t^2+3500t+21,000)}{(3t^2+80t+550)^2}$; 31,312 jobs/yr

4.4.16 -20.6%;-28.9%;-38.6%

4.4.17 Approximately 19 computer/mo.

4.5.1 (a) $f'(x)=\cos 2x$

(b) $f'(x)=-\csc^2 x$

(c) $f'(x)=-\tan x-x\sec^2 x-\cot x\csc x$

(d) $h'(x)=-2\pi\sin\pi x$

(e) $f'(t)=2t\cos(t^2+1)$

(f) $f'(x)=4x\sec^2 2x^2$

(g) $g'(x)=x\cos x+\sin x$

(h) $f'(x)=6(\cos 3x-\sin 2x)$

(i) $f'(s)=2s(\cos 2s-s\sin 2s)$

(j) $f'(x)=\dfrac{x\cos\sqrt{x^2-1}}{\sqrt{x^2-1}}$

(k) $h'(s) = \cos\frac{1}{s} + \frac{1}{s}\sin\frac{1}{s}$

(l) $f'(x) = \frac{x\sin x}{(1+\cos x)^2}$

(m) $g'(t) = \frac{\sec^2 t}{2\sqrt{\tan t}}$

4.5.2 $\pi/6 + 2n\pi$, $5\pi/6 + 2n\pi$, any integer n

4.5.3 $t = 0, 6$ and 12

4.5.4 -\$628.32 thousand dollars per year.

4.5.5 Rate of change of sales are increasing by \$157,079.63 per quarter.

4.6.1 (a) $2\ln(3)x3^{x^2}$

(b) $\frac{\cos x - \sin x}{e^x}$

(c) $2e^{2t}$

(d) $e^x\cos(e^x)$

(e) $\cos(x)e^{\sin x}$

(f) $x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$

(g) $3x^2e^x + x^3e^x$

(h) $1 + 2^t\ln(2)$

(i) $-2x\ln(3)(1/3)^{x^2}$

(j) $e^{4s}(4s-1)/s^2$

(k) $(3x^2+3)/(x^3+3x)$

(l) $-\tan(x)$

(m) $(1-\ln(t^2))/(t^2\sqrt{\ln(t^2)})$

(n) $\sec(x)$

(o) $x^{\cos(x)}(\cos(x)/x - \cos(x)\ln(x))$

4.6.2 (a) $f''(t) = 12e^{-2t} - 5e^{-t}$

(b) $f''(s) = 2(2x^2 - 4x + 1)e^{-2x}$

(c) $f''(t) = \left(2\ln 3 - 5(\ln 3)^2\right)3^t + t(\ln 3)^2 3^t$

(d) $y''(x) = \frac{(\ln 6)^2\sqrt{x}6^{\sqrt{x}} - (\ln 6)6^{\sqrt{x}}}{4(\sqrt{x})^3}$

4.6.3 $y = 2x - 2$

4.6.4 $y = -\frac{2x}{e} + \frac{3}{e}$

4.6.5 e

4.6.7 (a) $R(x) = 100xe^{-0.0001x}$

(b) $R'(x) = 100(1 - 0.0001x)e^{-0.0001x}$

(c) \$99.80 per thousand pairs.

4.6.8 (a) -\$0.0168 per case

(b) \$40.36 per case.

4.6.9 (a) \$12/unit

(b) -\$7/week

(c) \$8/unit

4.6.10 (a) 15.6%

(b) 12.1%/yr

(c) 9.5%/yr/yr

4.6.11 (a) 1986 kWh/yt

(b) $C'(t) = -108.48e^{-0.073t}$

(c) $C(18) = 899.35 < 950$

4.7.1 (a) x/y

(b) $-(2x+y)/(x+2y)$

(c) $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$

(d) $\sin(x)\sin(y)/(\cos(x)\cos(y))$

(e) $-\sqrt{y}/\sqrt{x}$

(f) $(y\sec^2(x/y)-y^2)/(x\sec^2(x/y)+y^2)$

(g) $(y-\cos(x+y))/(\cos(x+y)-x)$

(h) $-y^2/x^2$

(i) $-\dfrac{\cos x}{x\sin(xy)}-\dfrac{y}{x}$

(j) $\dfrac{\cot x-\sec y}{x\sec y\tan y}$

(k) $-\dfrac{\sqrt{1-y^2}\cos(x+y)}{\sqrt{1-y^2}\cos(x+y)-1}$

(l) $\dfrac{4x\sqrt{y}\tan(y)}{4y^{3/2}\tan(y)-2\sqrt{y}(x^2-y^2)\sec^2(y)+1}$

4.7.2 (a) $y' = (5x+7)(x+1)(x+2)^2$

(b) $y' = 2(3x+2)^3(5x-1)(45x+4)$

(c) $y' = (9x^2+14x-7)(x-1)(x+1)^2(x+3)^3$

(d) $y' = \dfrac{1}{2}(2x-3)^3(54x+71)(3x+5)^{-1/2}$

(e) $y' = \dfrac{(38x^2+40x+1)(2x^2-1)^4}{2(x+1)^{3/2}}$

(f) $y' = 3^x\ln 3$

(g) $y' = \dfrac{x\ln x+x+2)x^{x+2}}{x}$

(h) $y' = \dfrac{\left[(x^2+1)\ln(x^2+1)+2x^2\right](x^2+1)^x}{x^2+1}$

(i) $y' = 2(\ln x)x^{\ln x-1}$

4.7.3 1

4.7.4 $y = 2x \pm 6$

4.7.5 $y = x/2 \pm 3$

4.7.7 $(\sqrt{3}, 2\sqrt{3})$, $(-\sqrt{3}, -2\sqrt{3})$, $(2\sqrt{3}, \sqrt{3})$, $(-2\sqrt{3}, -\sqrt{3})$

4.7.8 $y = 7x/\sqrt{3} - 8/\sqrt{3}$

4.7.9 $y = (-y_1^{1/3}x + y_1^{1/3}x_1 + x_1^{1/3}y_1)/x_1^{1/3}$

4.7.10 $(y-y_1)/(x-x_1) = (2x_1^3+2x_1y_1^2-x_1)/(2y_1^3+2y_1x_1^2+y_1)$

4.8.1

(a) $f'(x) = -\dfrac{10x}{(5x^2+1)\sqrt{(5x^2+1)^2-1}}$

(b) $f'(x) = \dfrac{9(\tan^{-1}(3x))^2}{1+9x^2}$

(c) $g'(x) = -\dfrac{\sqrt{e^{\cos^{-1}x}}}{2\sqrt{1-x^2}}$

(d) $f'(t) = \dfrac{1}{\sqrt{1-t^2}\sin^{-1}t}$

(e) $f'(x) = \dfrac{2}{2x\sqrt{x^3-1}}$

(f) $h'(s) = -\dfrac{1}{(\ln 2)s\sqrt{1-(\log_2 s)^2}}$

(g) $f'(x) = -\dfrac{1}{3(1+x^2(\cot^{-1}x)^{2/3}}$

(h) $g'(t) = \dfrac{3^t(\ln 3)}{\sqrt{1-3^{2t}}}$

4.8.2 (a) $\dfrac{dy}{dx} = \left(1-y-\dfrac{y}{\sqrt{1-(xy)^2}}\right)\left(\dfrac{x}{\sqrt{1-(xy)^2}}+x\right)^{-1}$

(b) $\dfrac{dy}{dx} = \left(\dfrac{1}{1+(x-y)^2}-y\right)\left(\dfrac{1}{1+(x-y)^2}+x\right)^{-1}$

4.8.3 1

4.9.1 (a) $4(2x+3)$

(b) $\frac{3}{2}x^{1/2}$

4.9.2 3

4.9.3 (a) $28x^3 - \dfrac{1}{3\pi x^{4/3}}$

(b) $-\dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}$

(c) $f'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$

(d) $2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x$

(e) $\dfrac{(\sin x + x\cos x)(1+\sin x) - x\sin x\cos x}{(1+\sin x)^2}$

(f) $-\dfrac{3}{4x^{3/2}}\left(2+\dfrac{3}{\sqrt{x}}\right)^{-1/2}$

(g) $\frac{1}{3}(x^4+x^2+1)^{-2/3}(4x^3+2x) - \dfrac{5(3x^2-1)}{(x^3-x+4)^6}$

(h) $3\sin^2 x\cos x - 3x^2\cos(x^3)$

(i) $4\sec^4 x\tan x + 4\tan^3 x\sec^2 x$

(j) $\dfrac{4}{(1+x)^2}\cos\left(\dfrac{1-x}{1+x}\right)\sin\left(\dfrac{1-x}{1+x}\right)$

(k) $(2x+2\sec^2 x\tan x)\sec^2(\sin(x^2+\sec^2 x))\cos(x^2+\sec^2 x)$

(l) $-\dfrac{\pi\cos\frac{\pi}{x}}{x^2(2+\sin\frac{\pi}{x})^2}$

4.9.4 (a) $3e^{3x}-e^{-x}$

(b) $2e^{2x}\cos 3x-3e^{2x}\sin 3x$

(c) $(1+e^x)\sec^2(x+e^x)$

(d) $2e^x/(e^x+2)^2$

(e) $\dfrac{\cos x}{2+\sin x}-\dfrac{\cos(2+\ln x)}{x}$

(f) $e^{x^\pi}\cdot\pi x^{\pi-1}+\pi^e x^{\pi^e-1}+\pi^{e^x}\ln\pi\cdot e^x$

(g) $\log_a b+(\log_a b)x^{(\log_a b)-1}$

(h) $(x^2+1)^{x^3+1}\left(3x^2\ln(x^2+1)+\dfrac{2x(x^3+1)}{x^2+1}\right)$

(i) $\dfrac{(x^2+e^x)^{1/\ln x}}{(\ln x)^2}\left(\dfrac{2x+e^x}{x^2+e^x}\ln x-\dfrac{x^2+e^x}{x}\right)$

(j) $\dfrac{x\sqrt{x^2+x+1}}{(2+\sin x)^4(3x+5)^7}\left(\dfrac{1}{x}+\dfrac{2x+1}{2(x^2+x+1)}-\dfrac{4\cos x}{2+\sin x}-\dfrac{21}{3x+5}\right)$

4.9.5 (a) $-(2x+y)/(x+2y)$

(b) $\dfrac{x-(2x^2+2y^2-x)(4x-1)}{4y(2x^2+2y^2-x)-y}$

(c) $-\dfrac{\sin x+2x\sin y}{x^2\cos y+3y^2}$

(d) $\dfrac{2x+e^y-e^x}{2-xe^y}$

4.9.6 (a) $\sin^{-1}x+x/\sqrt{1-x^2}$

(b) $\dfrac{\cos^{-1}x+\sin^{-1}x}{(\cos^{-1}x)^2\sqrt{1-x^2}}$

(c) $a/(x^2+a^2)$

(d) $\tan^{-1}x$

5.1.1 (a) 1, unitary.

(b) $\frac{1}{5}$, inelastic.

(c) $\frac{1}{5}$, inelastic.

(d) $\frac{3}{2}$, elastic.

(e) $\frac{1}{4}$, inelastic.

(f) 1, unitary.

5.1.2 (a) Inelastic when $p = 8$. Elastic when $p = 10$.

(b) $p = 8.66$.

(c) Increase

(d) Increase

5.1.3 (a) Inelastic

(b) Increase

5.1.4 (a) $E(p) = \dfrac{2p^2}{9 - p^2}$.

(b) For $p < \sqrt{3}$, demand is inelastic. For $p = \sqrt{3}$, demand is unitary. And for $p > \sqrt{3}$, demand is elastic.

5.2.1 111 units/week

5.2.2 44 units/week

5.2.3 -$.37/unit

5.2.4 $E(p) = 0.37$; inelastic

5.2.6 $1/(16\pi)$ cm/s

5.2.7 $3/(1000\pi)$ meters/second

5.2.8 $1/4$ m/s

5.2.9 $-6/25$ m/s

5.2.10 80π mi/min

5.2.11 $3\sqrt{5}$ ft/s

5.2.12 $20/(3\pi)$ cm/s

5.2.13 $13/20$ ft/s

5.2.14 $5\sqrt{10}/2$ m/s

5.2.15 $75/64$ m/min

5.2.16 tip: 6 ft/s, length: $5/2$ ft/s

5.2.17 tip: $20/11$ m/s, length: $9/11$ m/s

5.2.18 $380/\sqrt{3}-150 \approx 69.4$ mph

5.2.19 $500/\sqrt{3}-200 \approx 88.7$ km/hr

5.2.20 $4000/49$ m/s

5.3.1 (a) $f(3) \approx L(3) = \frac{7}{4}$

(b) $f(9) \approx L(9) = \frac{25}{12}$

(c) $f(5.3) \approx L(5.3) = \frac{47}{250}$

(d) $f(2.8) \approx L(2.8) = \frac{17}{135}$

(e) $f(2.2) \approx L(2.2) = 7.8$

(f) $f(3.1) \approx L(3.1) = 4.1$

5.3.2 $L(x) = x$, $f(0.1) \approx L(0.1) = 0.1$

5.3.3 Choose $f(x) = x^3$ and $a = 2$, the closest integer to 1.9. The linearization of f at a is $L(x) = 12(x-2)+8$, and $(1.9)^3 = f(1.9) \approx L(1.9) = 12(1.9-2)+8 = 6.8$.

5.3.5 Choose $a = 7$ since $f(7) = \sqrt[3]{7+1} = \sqrt[3]{8} = 2$ is an integer close to $\sqrt[3]{9}$. The linearization of f at $a = 7$ is $L(x) = 1/12(x-7)+2$. Then $f(8) = \sqrt[3]{8+1} = \sqrt[3]{9} \approx L(8) = 1/12(8-7)+2 = 2.08\overline{3}$. We are over-estimating $\sqrt[3]{9}$ since $L(x) > f(x)$ for all x around $a = 7$.

5.3.6 (a) $dy = 4x\,dx$

(b) $dy = (3t^2-1)dt$

(c) $dy = \dfrac{1}{2\sqrt{t+1}}dt$

(d) $dy = \dfrac{6q+1}{2\sqrt{q}}dq$

(e) $dy = \dfrac{s^2-2}{s^2}ds$

(f) $dy = \dfrac{-q^2+2q+1}{(q^2+1)^2}dq$

(g) $dy = \dfrac{6x-1}{2\sqrt{3x^2-x}}dx$

5.3.7 (a) $\Delta y = 65/16$, $dy = 2$

(b) $\Delta y = \sqrt{11/10}-1$, $dy = 0.05$

(c) $\Delta y = \sin(\pi/50)$, $dy = \pi/50$

5.3.8 (a) (i) $dy = 2xdx$, and (ii) $dy = -\frac{1}{x^2}dx$

(b) (i) $dy \approx -0.02$, and (ii) $dy \approx -0.0098$

(c) (i) $\Delta y = -0.19$, and (ii) $\Delta y = -0.0099$.

5.3.9 (a) 6.9643

(b) 2.042

(c) 2.509

5.3.10 $dV = 8\pi/25$

5.3.11 \$16 million

5.3.12 \$4000

5.3.13 Approximately 40 fewer loans

5.3.14 \$0.5

5.3.15 ± 8.64 cm^3

5.3.16 300cm^3

5.3.17 1257 L

5.3.18 True. The percentage change in A is approximately

$$\frac{100[f(x+\Delta x)-f(x)]}{f(x)} \approx \frac{100f'(x)dx}{f(x)}$$

5.3.19 $\pm$ 1.06 million dollars

5.3.20 ± 0.36 %

5.3.21 (a) $dP = \dfrac{10,000\left(1-(1+\frac{r}{12})^{-360}-30r(1+\frac{r}{12})^{-361}\right)}{\left(1-\left(1+\frac{r}{12}\right)^{-360}\right)^2}dr$

(b) \$12.90,\$19.40,\$25.89,\$32.36

5.3.22 (a) $dA = 100,000\left(1+\frac{r}{12}\right)^{119}dr$

(b) \$110,\$220,\$331

5.3.23 (a) $dS = 24,000\left(\frac{300r(1+\frac{r}{12})^{299}(\frac{1}{12})-(1+\frac{r}{12})^{300}+1}{r^2}\right)dr$

(b) \$9617,\$19,235,\$28,852

5.3.24 (a) $f(x) = x^2 - 3$, 1.732051

(b) $f(x) = x^2 - 7$, 2.645751

(c) $f(x) = x^3 - 14$, 2.410142

5.3.25 (a) 0.567143

(b) 0.567143

(c) 6.98024

5.3.26 (a) 1.528

(b) 0.753

(c) 1.557

5.3.27 (a) Notice that $f(-2) = 19$, $f(0) = -11$, and $f(5) = 19$ and f is a continuous function. By the Intermediate Value Theorem there exists a root in $[-2,0]$ and $[0,5]$. Choose $x_0 = 0$, then $x_4 \approx -0.93242$. Choose $x_0 = 5$, then $x_4 \approx 3.93242$.

(b) Notice that $f(1) = -1$ and $f(2) = 5$ and f is a continuous function. By the IVT there exists a root in $[1,2]$. Required root is $x \approx 1.87939$.

(c) $f(1) = 7$ and $f(2) = -6$ and f continuous. Therefore there exists a root in the interval $[1,2]$. This root is $x \approx 1.61178$.

5.3.28 (a) $x_4 \approx 1.00022\ldots$

(b) $x = 1$ is the root of f. Our approximation in part (a) was correct to 3 decimal places.

(c) $x_1 = 1$. The root is found in one iteration of Newton's Method.

5.3.29 $\cos(\pi/2) = 0$, so x_1 is undefined.

5.3.30 4.5 %

5.3.31 3.91 %

5.3.32 1.33% per year

5.3.33 16.1% per year

5.3.34 $(789.88, 30.79)$.

5.3.35 224 units and 1 unit

5.4.1

(a) 0
(b) ∞
(c) 0
(d) 0
(e) $1/6$
(f) $1/16$
(g) $3/2$
(h) 0
(i) -1
(j) $-1/2$
(k) 5
(l) 1
(m) 2
(n) 1
(o) 0
(p) $1/2$
(q) 2
(r) $1/2$
(s) 2
(t) 0

5.4.2 (a) 0
(b) $-1/4$
(c) -3
(d) $1/2$
(e) 1
(f) 0
(g) $-1/2$
(h) ∞
(i) 5
(j) $-1/2$

5.5.1 (a) min at $x = 1/2$
(b) min at $x = -1$, max at $x = 1$
(c) max at $x = 2$, min at $x = 4$
(d) min at $x = \pm 1$, max at $x = 0$
(e) min at $x = 1$
(f) none
(g) none
(h) min at $x = 7\pi/12 + k\pi$, max at $x = -\pi/12 + k\pi$, for integer k
(i) none
(j) relative max at $x = 5$
(k) relative min at $x = 49$
(l) relative min at $x = 0$

5.5.4 one

5.5.8 (a) Absolute maximum $(3,7)$; Absolute minimum $(0,1)$
(b) Absolute maximum $(3,7)$; Absolute minimum $(0,1)$

(c) Absolute minimum $(\pi/2,1)$; No absolute maximum

(d) Absolute minimum $(1,0)$; Absolute maximum $(e^{1/2},\frac{1}{2e})$

(e) Absolute minimum $(1,0)$; Absolute maximum $(e^{1/2},\frac{1}{2e})$

(f) Absolute minimum $(0,0)$; Absolute maximum $(2,2e^{1/8})$

(g) Absolute minimum $(1/2,\frac{2-\pi}{4})$; Absolute maximum $(2,2-\tan^{-1}(4))$

(h) Absolute maximum $(1,1/2)$; Absolute minimum $(-1,-1/2)$

5.6.1 $c=1/2$

5.6.2 $c=\sqrt{18}-2$

5.6.6 $x^3/3+47x^2/2-5x+k$

5.6.7 $\arctan x+k$

5.6.8 $x^4/4-\ln x+k$

5.6.9 $-\cos(2x)/2+k$

5.7.1 (a) rel min at $x=1/2$

(b) rel min at $x=-1$, rel max at $x=1$

(c) rel max at $x=2$, rel min at $x=4$

(d) rel min at $x=\pm 1$, rel max at $x=0$

(e) rel min at $x=1$

(f) none

(g) none

(h) rel min at $x=7\pi/12+k\pi$, rel max at $x=-\pi/12+k\pi$, for integer k

(i) none

(j) rel max at $x=0$, rel min at $x=\pm 11$

(k) rel min at $x=-3/2$, neither at $x=0$

(l) rel min at $n\pi$, rel max at $\pi/2+n\pi$

(m) rel min at $2n\pi$, rel max at $(2n+1)\pi$, for integer n

5.7.2 min at $\pi/2+2n\pi$, max at $3\pi/2+2n\pi$

5.7.4 (a) decreasing on $(-\infty,0)$, increasing on $(0,\infty)$

(b) increasing on $(-1,1)\cup(3,\infty)$ and decreasing on $(-\infty,-1)\cup(1,3)$.

(c) increasing on $(-2,-1)$, decreasing on $(1,2)$, constant on $(-1,1)$

(d) decreasing on $(-\infty,1)\cup(1,2)$ and increasing on $(2,\infty)$.

5.7.5 (a) rel min at -1,1 and rel max at 0

(b) rel min at -1,2 and rel max at 0

(c) rel min at -1,1 and rel max at 0

(d) rel min at 4 and rel max at -2

5.7.6 Increasing on $(0,4000)$ and decreasing on $(4000,\infty)$.

5.7.7 Increasing from 1995 to 2035 and decreasing from 2035 to 2045.

5.7.8 (a) Increasing on $(0,6)$

(b) Increasing throughout the years from 1999 to 2005

5.7.9 (a) Increasing on $(0,6)$.

(b) Always increasing from 1997 to 2003.

5.7.11 (a) concave up everywhere

(b) concave up when $x<0$, concave down when $x>0$

(c) concave down when $x<3$, concave up when $x>3$

(d) concave up when $x<-1/\sqrt{3}$ or $x>1/\sqrt{3}$, concave down when $-1/\sqrt{3}<x<1/\sqrt{3}$

(e) concave up when $x<0$ or $x>2/3$, concave down when $0<x<2/3$

(f) concave up when $x<0$, concave down when $x>0$

(g) concave up when $x<-1$ or $x>1$, concave down when $-1<x<0$ or $0<x<1$

(h) concave down on $((8n-1)\pi/4,(8n+3)\pi/4)$, concave up on $((8n+3)\pi/4,(8n+7)\pi/4)$, for integer n

(i) concave down everywhere

(j) concave up on $(-\infty,(21-\sqrt{497})/4)$ and $(21+\sqrt{497})/4,\infty)$

(k) concave up on $(0,\infty)$

(l) concave down on $(2n\pi/3,(2n+1)\pi/3)$

(m) concave up on $(0,\infty)$

(n) concave up on $(-\infty,-1)$ and $(0,\infty)$

(o) concave down everywhere

(p) concave up everywhere

(q) concave up on $(\pi/4+n\pi,3\pi/4+n\pi)$ for n integer

(r) inflection points at $n\pi$, $\pm\arcsin(\sqrt{2/3})+n\pi$ for n integer

5.7.12 (a) CCU on $(-\infty,0)$, CCD on $(0,\infty)$. Inflection point at $(0,0)$.

(b) CCU on $(-\infty,1)\cup(5,\infty)$, CCD on $(1,5)$. Inflection points at $(1,0)$ and $(5,0)$.

(c) CCU on $(-infty,-1)\cup(1,\infty)$, CCD on $(-1,1)$. No inflection points.

(d) CCU on $(-\infty,0)$, CCD on $(0,\infty)$. Inflection point at $(0,0)$.

5.7.13 (a) $(0,4)$

(b) $(1,-15)$

(c) $(0,1)$ and $(2/3,11/27)$

(d) $(0,12)$

(e) $(1,-1)$

(f) $(-\sqrt{3}/3,3/2)$ and $(\sqrt{3}/3,3/2)$

(g) $(1/2,1/2e^{-2})$

(h) $(\pi/4+n\pi,-1/2)$ for n integer

5.7.14 up/incr: $(3,\infty)$, up/decr: $(-\infty,0)$, $(2,3)$, down/decr: $(0,2)$

5.7.17 (a) $(150,28850)$

(b) no

5.7.18 $(100,4600)$

5.7.19 After declining the first 3 years, the growth rate of the company's profit will once again rise.

5.7.20 (a) rel min at $x=1/2$

(b) rel min at $x=-1$, rel max at $x=1$

(c) rel max at $x=2$, rel min at $x=4$

(d) rel min at $x=\pm1$, rel max at $x=0$

(e) rel min at $x=1$

(f) none

(g) none

(h) rel min at $x=7\pi/12+n\pi$, rel max at $x=-\pi/12+n\pi$, for integer n

(i) rel max at $x=63/64$

(j) rel max at $x=7$

(k) rel max at $-5^{-1/4}$, rel min at $5^{-1/4}$

(l) none

(m) rel max at -1, rel min at 1

(n) rel min at $2^{-1/3}$

(o) none

(p) rel min at $n\pi$, for integer n

(q) rel max at $n\pi$, rel min at $\pi/2+n\pi$ for integer n

(r) rel max at $\pi/2+2n\pi$, rel min at $3\pi/2+2n\pi$ for integer n

5.7.21 (a)

5.7.22 (b)

5.7.23 (b)

5.7.24 11 am

5.7.25 $(10,1050)$

5.7.26 (a) The number of transactions is increasing from 6 to 11 am; the rate of transactions is increasing the fastest at around 8am, and then starts to decrease. From 11 am to 2 pm, the number of transactions is decreasing; the rate of transactions is decreasing until the shop closes.

(b) 11 am

5.7.27 (a) $D=(-\infty,-3)\cup(-3,3)\cup(3,\infty);x=\pm3;y=0.$

(b) $D=(-\infty,-4)\cup(-4,4)\cup(4,\infty);x=\pm3;y=8.$

(c) $D=(-\infty,-5)\cup(-5,3)\cup(3,\infty);x=-5,3;y=6.$

5.7.28 (a) B

(b) D

(c) C

(d) F

(e) A

(f) E

5.7.29 (a) HA at $y=0$

(b) HA at $y=0$

(c) VA at $x=\pm 1$, HA at $y=0$

(d) VA at $x=0$, HA at $y=2$

(e) HA at $y=\pm 3$

5.7.30 (a) HA at $y=0$; VA at $x=0$

(b) HA at $y=0$; VA at $x=0$

(c) HA at $y=1$; VA at $x=-1$

(d) none

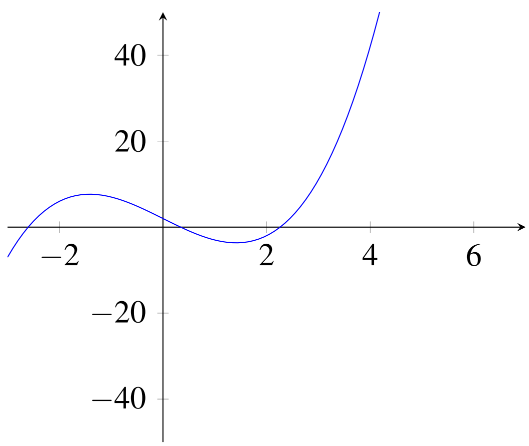
(e) HA at $y=1$; VA at $t=\pm 3$

(f) HA at $y=0$; VA at $x=-2,3$

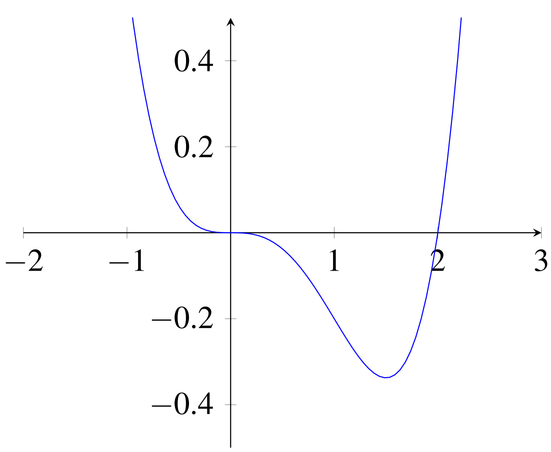
(g) HA at $y=2$; VA at $t=2$

(h) HA at $y=1$; VA at $x=\pm 2$

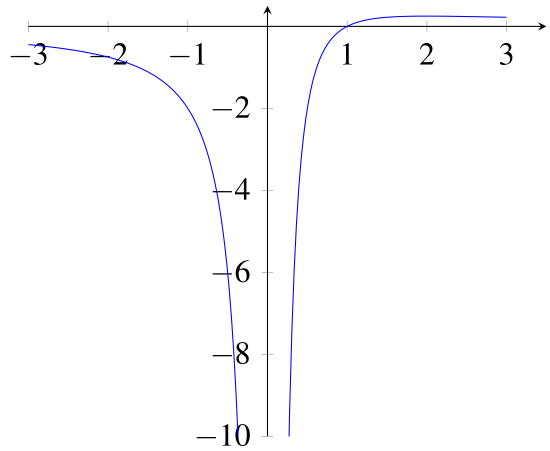
(i) none



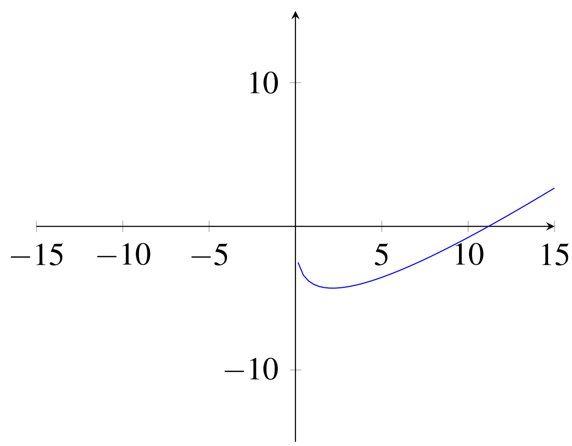
5.7.35



5.7.36



5.7.37



5.7.38

5.8.1 \$69,800 per month when $q = 130$.

5.8.2 \$214.16 per day

5.8.3 6000 units/day

5.8.4 100/3 units per week

5.8.5 (a) $\overline{C}(q) = 0.5q - 50 + \frac{10{,}000}{q}$

(b) $q = 13,750$

(c) $q = 13,750$

(d) The minimum average production occurs when average cost is equal to marginal cost.

5.8.6 300 units

5.8.7 By the Second Derivative Test, the critical point does give a maximum revenue.

5.8.8 G maximal at $t = 8$.

5.8.9 $a(t)$ maximal at $t = 42.181$.

5.8.10 N has an absolute minimum of 0.86 million bakeries at about $x = 3.1$ and an absolute maximum of 5.1 million bakeries at about $x = 0.8$.

5.8.11 About 19,000 units/day

5.8.12 25×25

5.8.13 $P/4 \times P/4$

5.8.14 $w = l = 2 \cdot 5^{2/3}$, $h = 5^{2/3}$, $h/w = 1/2$

5.8.15 $\sqrt[3]{100} \times \sqrt[3]{100} \times 2\sqrt[3]{100}$, $h/s = 2$

5.8.16 $w = l = 2^{1/3}V^{1/3}$, $h = V^{1/3}/2^{2/3}$, $h/w = 1/2$

5.8.17 1250 square feet

5.8.18 $l^2/8$ square feet

5.8.19 \$5000

5.8.20 100

5.8.21 r^2

5.8.22 $h/r = 2$

5.8.23 $h/r = 2$

5.8.24 $r = 5$, $h = 40/\pi$, $h/r = 8/\pi$

5.8.25 $8/\pi$

5.8.26 $4/27$

5.8.27 (a) 2, (b) $7/2$

5.8.28 $\dfrac{\sqrt{3}}{6} \times \dfrac{\sqrt{3}}{6} + \dfrac{1}{2} \times \dfrac{1}{4} - \dfrac{\sqrt{3}}{12}$

5.8.29 (a) $a/6$, (b) $(a + b - \sqrt{a^2 - ab + b^2})/6$

5.8.30 1.5 meters wide by 1.25 meters tall

5.8.31 If $k \le 2/\pi$ the ratio is $(2-k\pi)/4$; if $k \ge 2/\pi$, the ratio is zero: the window should be semicircular with no rectangular part.

5.8.32 a/b

5.8.33 $1/\sqrt{3} \approx 58\%$

5.8.34 $18 \times 18 \times 36$

5.8.35 $r = 5/(2\pi)^{1/3} \approx 2.7$ cm,
$h = 5 \cdot 2^{5/3}/\pi^{1/3} = 4r \approx 10.8$ cm

5.8.36 $h = \dfrac{750}{\pi}\left(\dfrac{2\pi^2}{750^2}\right)^{1/3}$, $r = \left(\dfrac{750^2}{2\pi^2}\right)^{1/6}$

5.8.37 $h/r = \sqrt{2}$

5.8.38 $1/2$

5.8.39 \$7000

6.1.6 3, $\sqrt{26}$, $\sqrt{29}$

6.1.7 $\sqrt{14}$, $2\sqrt{14}$, $3\sqrt{14}$.

6.1.8 $(x-1)^2+(y-1)^2+(z-1)^2 = 4$.

6.1.9 $(x-2)^2+(y+1)^2+(z-3)^2 = 25$.

6.1.11 $(x-2)^2+(y-1)^2+(z+1)^2 = 16$, $(y-1)^2+(z+1)^2 = 12$

6.2.6 $\sqrt{10}$, $\langle 0,-2\rangle$, $\langle 2,8\rangle$ 2, $2\sqrt{17}$, $\langle -2,-6\rangle$

6.2.7 $\sqrt{14}$, $\langle 0,4,0\rangle$, $\langle 2,0,6\rangle$ 4, $2\sqrt{10}$, $\langle -2,-4,-6\rangle$

6.2.8 $\sqrt{2}$, $\langle 0,-2,3\rangle$, $\langle 2,2,-1\rangle$ $\sqrt{13}$, 3, $\langle -2,0,-2\rangle$

6.2.9 $\sqrt{3}$, $\langle 1,-1,4\rangle$, $\langle 1,-1,-2\rangle$ $3\sqrt{2}$, $\sqrt{6}$, $\langle -2,2,-2\rangle$

6.2.10 $\sqrt{14}$, $\langle 2,1,0\rangle$, $\langle 4,3,2\rangle$ $\sqrt{5}$, $\sqrt{29}$, $\langle -6,-4,-2\rangle$

6.2.11 $\langle -3,-3,-11\rangle$, $\langle -3/\sqrt{139},-3/\sqrt{139},-11/\sqrt{139}\rangle$ $\langle -12/\sqrt{139},-12/\sqrt{139},-44/\sqrt{139}\rangle$

6.2.12 $\langle 0,0,0\rangle$

6.2.13 $\mathbf{0}$; $\langle -r\sqrt{3}/2, r/2\rangle$; $\langle 0, -12r\rangle$; where r is the radius of the clock

6.3.1 3

6.3.2 0

6.3.3 2

6.3.4 -6

6.3.5 42

6.3.6 $\sqrt{6}/\sqrt{7}$, ≈ 0.39

6.3.7 $-11\sqrt{14}\sqrt{29}/406$, ≈ 2.15

6.3.8 0, $\pi/2$

6.3.9 $1/2$, $\pi/3$

6.3.10 $-1/\sqrt{3}$, ≈ 2.19

6.3.11 $\arccos(1/\sqrt{3}) \approx 0.96$

6.3.12 $\sqrt{5}$, $\langle 1,2,0\rangle$.

6.3.13 $3\sqrt{14}/7$, $\langle 9/7, 6/7, 3/7\rangle$.

6.3.14 $\langle 0,5\rangle$, $\langle 5\sqrt{3}, 0\rangle$

6.3.15 $\langle 0, 15\sqrt{2}/2\rangle$,$\langle 15\sqrt{2}/2, 0\rangle$

6.3.16 Any vector of the form $\langle a, -7a/2, -2a\rangle$

6.3.17 $\langle 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\rangle$

6.3.18 No.

6.3.19 Yes.

6.4.1 $\langle 1,-2,1\rangle$

6.4.2 $\langle 4,-6,-2\rangle$

6.4.3 $\langle -7,13,-9\rangle$

6.4.4 $\langle 0,-1,0\rangle$

6.4.5 3

6.4.6 $21\sqrt{2}/2$

6.4.7 1

6.5.1 (a) $(x-6)+(y-2)+(z-1)=0$

(b) $4(x+1)+5(y-2)-(z+3)=0$

(c) $(x-1)-(y-2)=0$

(d) $-2(x-1)+3y-2z=0$

(e) $4(x-1)-6y=0$

(f) $x+3y=0$

6.5.2 (a) $\langle 1,0,3\rangle+t\langle 0,2,1\rangle$

(b) $\langle 1,0,3\rangle+t\langle 1,2,-1\rangle$

(c) $t\langle 1,1,-1\rangle$

6.5.3 $-2/5$, $13/5$

6.5.5 (a) neither

(b) parallel

(c) intersect at $(3,6,5)$

(d) same line

6.5.9 (a) $7/\sqrt{3}$

(b) $4/\sqrt{14}$

(c) $\sqrt{131}/\sqrt{14}$

(d) $\sqrt{68}/3$

6.5.10 (a) $\sqrt{42}/7$

(b) $\sqrt{21}/6$

6.6.1 (a) $(\sqrt{2},\pi/4,1)$, $(\sqrt{3},\pi/4,\arccos(1/\sqrt{3}))$

(b) $(7\sqrt{2},7\pi/4,5)$, $(\sqrt{123},7\pi/4,\arccos(5/\sqrt{123})$

(c) $(1,1,1)$, $(\sqrt{2},1,\pi/4)$

(d) $(0,0,-\pi)$, $(\pi,0,\pi)$

6.6.2 $r^2+z^2=4$

6.6.3 $r\cos\theta=0$

6.6.4 $r^2+2z^2+2z-5=0$

6.6.5 $z=e^{-r^2}$

6.6.6 $z=r$

6.6.7 $\sin\theta=0$

6.6.8 $1=\rho\cos\phi$

6.6.9 $\rho=2\sin\theta\sin\phi$.

6.6.10 $\rho\sin\phi=2$

6.6.11 $\cos\phi=1/\sqrt{2}$

6.6.13 $z=mr$; $\cot\phi=m$ if $m\neq 0$, $\phi=0$ if $m=0$

6.6.14 A sphere with radius $1/2$, center at $(0,1/2,0)$

6.6.15 $0<\theta<\pi/2$, $0<\phi<\pi/2$, $\rho>0$; $0<\theta<\pi/2$, $r>0$, $z>0$

7.1.1 (a) $z=y^2$, $z=x^2$, $z=0$, lines of slope 1

(b) $z=|y|$, $z=|x|$, $z=2|x|$, diamonds

(c) $z=e^{-y^2}\sin(y^2)$, $z=e^{-x^2}\sin(x^2)$, $z=e^{-2x^2}\sin(2x^2)$, circles

(d) $z=-\sin(y)$, $z=\sin(x)$, $z=0$, lines of slope 1

(e) $z=y^4$, $z=x^4$, $z=0$, hyperbolas

7.1.2 (a) $\{(x,y)\mid |x|\leq 3 \text{ and } |y|\geq 2\}$

(b) $\{(u,v)\mid u^2+v^2\leq 2\}$

(c) $\{(x,y)\mid x^2+4y^2\leq 16\}$

(d) All real x and y.

(e) All real x,y, and z

(f) $\{(x,y) \mid 1 \leq x^2+y^2 \leq 3\}$

(g) $\{(u,v) \mid u \neq v\}$

(h) All real s and t

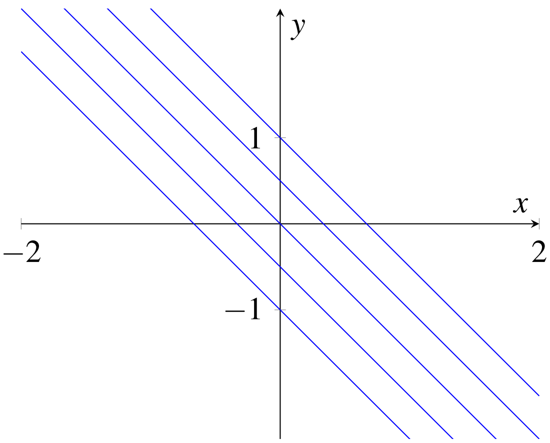
(i) $\{(r,s) \mid rs > 0\}$

(j) All real x and y

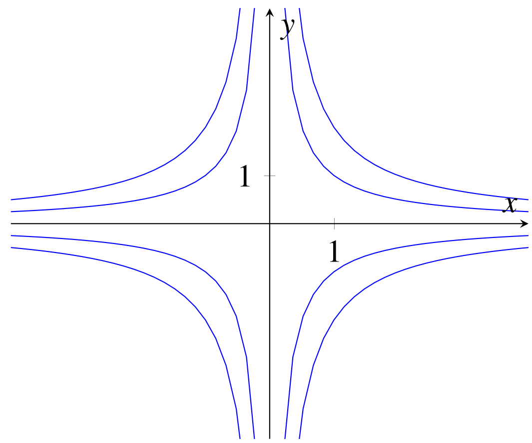
(k) $\{(x,y) \mid x+y > 7\}$

7.1.4

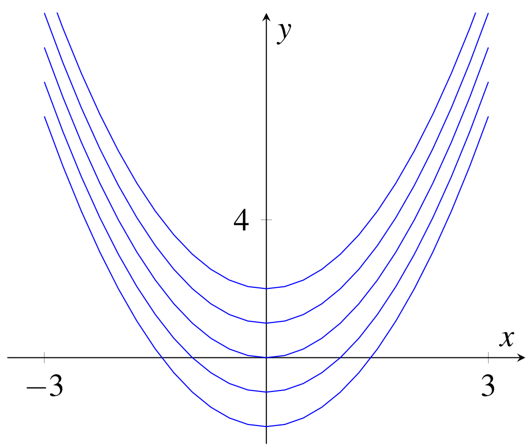
We draw the level curves of f:



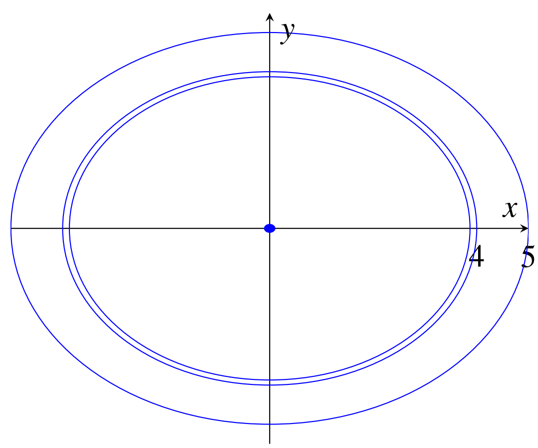
(a)



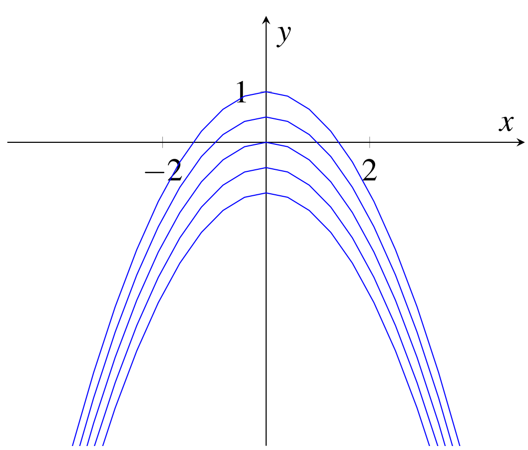
(d)



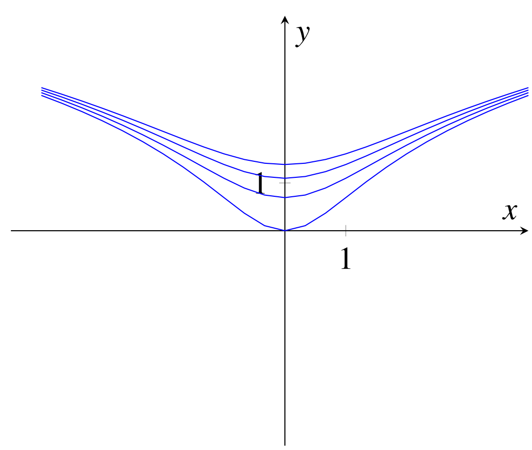
(b)



(e)



(c)



(f)

7.1.5 (a) $R(q_c, q_b) = \frac{1}{420}\left(-105q_c^2 - 102q_cq_b + 42000q_c - 140q_b^2 + 63000q_b\right)$

(b) The region where all of $p_c, p_b, q_c, q_b \geq 0$.

7.1.6 (a) $R(x, y) = -0.1q_b^2 - 0.9q_bq_c + 10q_b - q_c^2 + 30q_c$

(b) The region where all of $p_b, p_c, q_b, q_c \geq 0$.

7.1.7 \$50,148.53; \$28,661.05

7.1.8 About 39 tires.

7.2.1 (a) No limit; use $x = 0$ and $y = 0$

(b) No limit; use $x = 0$ and $x = y$

(c) No limit; use $x = 0$ and $x = y$

(d) Limit is 1

(e) Limit is zero

(f) Limit is zero

(g) Limit is -1

(h) Limit is zero

(i) No limit; use $x = 0$ and $y = 0$

(j) Limit is zero

(k) Limit is -1

(l) Limit is zero

7.3.1 (a) $f_x = -2xy\sin(x^2y)$, $f_y = -x^2\sin(x^2y) + 3y^2$

(b) $f_x = (y^2 - x^2y)/(x^2+y)^2$, $f_y = x^3/(x^2+y)^2$

(c) $f_x = 2xe^{x^2+y^2}$, $f_y = 2ye^{x^2+y^2}$

(d) $f_x = \ln(xy) + y$, $f_y = x\ln(xy) + x$

(e) $f_x = -x/\sqrt{1-x^2-y^2}$, $f_y = -y/\sqrt{1-x^2-y^2}$

(f) $f_x = \tan y$, $f_y = x\sec^2 y$

(g) $f_x = -1/(x^2y)$, $f_y = -1/(xy^2)$

7.3.2 (a) $f_x(1,3) = 15$, $f_y(1,3) = 7$

(b) $g_x(4,4) = 2$, $g_y(4,4) = 1$

(c) $f_s(-1,2) = 1/2$, $f_t(-1,2) = 1/4$

(d) $p_x(1,-1) = -1/e$, $p_y(1,-1) = 1/e$

(e) $h_a(1,1,2) = 16$, $h_b(1,1,2) = 8$, $h_c(1,1,2) = 12$

(f) $f_p(\pi/2,1) = 0$, $f_q(\pi/2,1) = 0$

(g) $h_a = -1$, $h_b = -\pi$

(h) $g_x = \frac{1}{1+\pi}$, $g_y = 0$

7.3.3 (a) $Y_K = 9(K/L)^{2/5}, Y_L = 6(K/L)^{3/5}$.

(b) 7.09,8.58

7.3.4 (a) $Y_L = \left(\frac{5K}{4L}\right)^{3/4}, Y_K = \left(\frac{15L}{4K}\right)^{1/4}$.

(b) 2.87,2.84.

7.3.5 (a) $\rho_x = -10$, $\rho_y = -10(y-1)$.

(b) $\rho_x(1,0) = -10$ and $\rho_y(1,0) = 10$. Therefore density is increasing in the y-direction and decreasing in the x-direction.

7.3.6 complementary

7.3.7 competitive

7.3.8 (a) $z = -2(x-1) - 3(y-1) - 1$

(b) $z = 1$

(c) $z = 6(x-3) + 3(y-1) + 10$

(d) $z = (x-2) + 4(y-1/2)$

7.3.9 $\mathbf{r}(t) = \langle 2,1,4\rangle + t\langle 2,4,-1\rangle$

7.3.13 height

7.3.16 $Q_{pp} = (2p^3q - 6pq^3)/(p^2+q^2)^3$, $Q_{qq} = (2pq^3 - 6p^3q)/(p^2+q^2)^3$

7.3.17 (a) $f_x = 2xy^2$, $f_y = y(2x^2+3y)$,$f_{xx} = 2y^2$, $fyy = 2(x^2+3y)$, $f_{xy} = 4xy = f_{yx}$.

(b) $f_x = 4x + y^3$, $f_y = 3xy^2$, $f_{xx} = 4$, $f_{yy} = 6xy$, $f_{xy} = 3y^2 = f_{yx}$.

(c) $f_x = y\cos(x)$, $f_y = \sin(x)$, $f_{xx} = -y\sin(x)$, $f_{yy} = 0$, $f_{xy} = \cos(x) = f_{yx}$.

(d) $f_x = -\sin(x)sin(2y)$, $f_y = 2\cos(x)\cos(2y)$, $f_{xx} = -\cos(x)\sin(2y)$, $f_{yy} = -4\cos(x)\sin(2y)$, $f_{xy} = -2\sin(x)\cos(2y) = f_{yx}$.

(e) $f_x = 2xe^{x^2-y}$, $f_y = -e^{x^2-y}$, $f_{xx} = 2(2x^2+1)e^{x^2-y}$, $f_{yy} = e^{x^2-y}$, $f_{xy} = -2xe^{x^2-y} = f_{yx}$.

(f) $f_x = \frac{x}{x^2+y^2}$, $f_y = \frac{y}{x^2+y^2}$, $f_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $f_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$ $f_{xy} = \frac{-2xy}{(x^2+y^2)^2} = f_{yx}$

7.4.1 (a) $4xt\cos(x^2+y^2) + 6yt^2\cos(x^2+y^2)$

(b) $2xy\cos t + 2x^2t$

(c) $2xyt\cos(st) + 2x^2s$, $2xys\cos(st) + 2x^2t$

(d) $2xy^2t - 4yx^2s$, $2xy^2s + 4yx^2t$

(e) x/z, $3y/(2z)$

(f) $-2x/z$, $-y/z$

7.4.2 (a) $V' = (nR - 0.2V)/P$

(b) $P' = (nR + 0.6P)/2V$

(c) $T' = (3P - 0.4V)/(nR)$

7.5.1 $9\sqrt{5}/5$

7.5.2 $\sqrt{2}\cos 3$

7.5.3 $e\sqrt{2}(\sqrt{3}-1)/4$

7.5.4 $\sqrt{3}+5$

7.5.5 $-\sqrt{6}(2+\sqrt{3})/72$

7.5.6 $-1/5, 0$

7.5.7 $4(x-2)+8(y-1)=0$

7.5.8 $2(x-3)+3(y-2)=0$

7.5.9 $\langle -1, -1-\cos 1, -\cos 1\rangle$, $-\sqrt{2+2\cos 1+2\cos^2 1}$

7.5.10 Any direction perpendicular to $\nabla T = \langle 1,1,1\rangle$, for example, $\langle -1,1,0\rangle$

7.5.11 $2(x-1)-6(y-1)+6(z-3)=0$

7.5.12 $6(x-1)+3(y-2)+2(z-3)=0$

7.5.13 $\langle 2+4t, -3-12t, -1-8t\rangle$

7.5.14 $\langle 4+8t, 2+4t, -2-36t\rangle$

7.5.15 $\langle 4+8t, 2+20t, 6-12t\rangle$

7.5.16 $\langle 0,1\rangle$, $\langle 4/5, -3/5\rangle$

7.5.18 (a) $\langle 4,9\rangle$

(b) $\langle -81,2\rangle$ or $\langle 81,-2\rangle$

7.5.19 in the direction of $\langle 8,1\rangle$

7.5.20 $\nabla g(-1,3) = \langle 2,1\rangle$

7.6.1

(a) minimum at $(1,-1)$

(b) none

(c) none

(d) maximum at $(1,-1/6)$

(e) none

(f) minimum at $(2,-1)$

(g) minimum at $(1,5)$

(h) maximum at $(0,0)$

(i) saddle at $(-1,0)$, maximum at $(1,0)$

(j) minimum at $(2,-1)$

(k) minimum at $(-1,-1)$, maximum at $(1,1)$, saddle at $(0,0)$

(l) minimum at $(-1,1/2)$, saddle at $(-3,0)$, $(0,0)$ and $(0,3/2)$.

7.6.2 $f(2,2)=-2$, $f(2,0)=4$

7.6.3 a cube $1/\sqrt[3]{2}$ on a side

7.6.4 200 deluxe units and 100 standard units; \$10,500

7.6.5 1071 finished units and 1643 unfinished units; \$17,157.14

7.6.6 $65/3\times 65/3\times 130/3$

7.6.7 It has a square base, and is one and one half times as tall as wide. If the volume is V the dimensions are $\sqrt[3]{2V/3}\times\sqrt[3]{2V/3}\times\sqrt[3]{9V/4}$.

7.6.8 $\sqrt{100/3}$

7.6.9 $|ax_0+by_0+cz_0-d|/\sqrt{a^2+b^2+c^2}$

7.6.10 The sides and bottom should all be $2/3$ meter, and the sides should be bent up at angle $\pi/3$.

7.6.11 $(3,4/3)$

7.6.13 $|b|$ if $b\le 1/2$, otherwise $\sqrt{b-1/4}$

7.6.14 $|b|$ if $b\le 1/2$, otherwise $\sqrt{b-1/4}$

7.6.16 $1024/\sqrt{3}$

7.7.1 a cube, $\sqrt[3]{1/2}\times\sqrt[3]{1/2}\times\sqrt[3]{1/2}$

7.7.2 $65/3\cdot 65/3\cdot 130/3=2\cdot 65^3/27$

7.7.3 It has a square base, and is one and one half times as tall as wide. If the volume is V the dimensions are $\sqrt[3]{2V/3}\times\sqrt[3]{2V/3}\times\sqrt[3]{9V/4}$.

7.7.4 $|ax_0 + by_0 + cz_0 - d|/\sqrt{a^2 + b^2 + c^2}$

7.7.5 $(0,0,1)$, $(0,0,-1)$

7.7.6 $\sqrt[3]{4V} \times \sqrt[3]{4V} \times \sqrt[3]{V/16}$

7.7.7 Farthest: $(-\sqrt{2}, \sqrt{2}, 2 + 2\sqrt{2})$; closest: $(2,0,0)$, $(0,-2,0)$

7.7.8 $x = y = z = 16$

7.7.9 $(1,2,2)$

7.7.10 $(\sqrt{5},0,0)$, $(-\sqrt{5},0,0)$

7.7.11 standard \$65, deluxe \$75

7.7.12 $x = 9$, $\phi = \pi/3$

7.7.13 35, -35

7.7.14 maximum e^4, no minimum

7.7.15 5, $-9/2$

7.7.16 3, 3, 3

7.7.17 a cube of side length $2/\sqrt{3}$

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