Matrices A such that $A^{s+1}R = RA^*$ with $R^k = I$

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Abstract

We study matrices $A \in \mathbb{C}^{n\times n}$ such that $A^{s+1}R = RA^*$ where $R^k = I_n$, and s, k are nonnegative integers with $k \geq 2$; such matrices are called $\{R, s+1, k, *\}$-potent matrices. The $s = 0$ case corresponds to matrices such that $A = RA^*R^{-1}$ with $R^k = I_n$, and is studied using spectral properties of the matrix R. For $s \geq 1$, various characterizations of the class of $\{R, s+1, k, *\}$-potent matrices and relationships between these matrices and other classes of matrices are presented.

Keywords: $\{R, s+1, k, *\}$-potent matrix; k-involutory.
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1 Introduction

The set of $n \times n$ complex matrices is denoted by $\mathbb{C}^{n\times n}$. The symbols A^* and $A^\dagger$ denote the conjugate transpose and the Moore-Penrose inverse, respectively, of $A \in \mathbb{C}^{n\times n}$. The set of distinct eigenvalues of A (the spectrum of A) is denoted by $\sigma(A)$. The symbol I_n denotes the identity matrix of $\mathbb{C}^{n\times n}$.

Throughout this paper we will use matrices $R \in \mathbb{C}^{n\times n}$ such that $R^k = I_n$ where $k \in \{2, 3, 4, \ldots\}$. These matrices R are called *k-involutory* [29, 30, 32], and are a generalization of the well-studied *involutory matrices* (the $k = 2$ case). We say that k is *minimal* with respect to $R^k = I_n$ if k is the smallest integer over all $h \in \{2, 3, 4, \ldots\}$ such that $R^h = I_n$. Note that the definition given in [29, 30] differs from that in [32]; in this paper we adopt the definition given in [32], namely that R is k-involutory does not require that k be minimal with respect to $R^k = I_n$.

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For a k-involutory matrix $R \in \mathbb{C}^{n\times n}$ and $s \in \{0, 1, 2, 3, \dots\}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{R, s+1, k\}$*-potent* if it satisfies

$$A^{s+1}R = RA \tag{1}$$

[17, 8]. These matrices generalize the *centrosymmetric matrices* (matrices $A \in \mathbb{C}^{n\times n}$ such that $A = JAJ$ where J is the $n \times n$ antidiagonal matrix [31]), the matrices $A \in \mathbb{C}^{n\times n}$ such that $AP = PA$ where P is an $n\times n$ permutation matrix [25], and $\{K, s+1\}$*-potent matrices* (matrices $A \in \mathbb{C}^{n\times n}$ for which $KAK = A^{s+1}$ where $K^2 = I_n$ [18, 19, 20]).

Characterizations of $\{R, s+1, k\}$-potent matrices were presented in [17], and properties of a matrix group constructed from an $\{R, s+1, k\}$-potent matrix were studied in [8]. Motivated by the results found in these papers and their connections to important known classes of matrices, we investigate matrices $A \in \mathbb{C}^{n\times n}$ where A in the right-hand side of (1) is replaced by $f(A)$ for some function f. In (1), $f(A) = A$ and in this paper $f(A) = A^*$. We introduce and study this further class of matrices related to the $\{R, s+1, k\}$-potent matrices.

Definition 1. *Let $A \in \mathbb{C}^{n\times n}$, $R \in \mathbb{C}^{n\times n}$ be k-involutory (that is, $R^k = I_n$ for some integer $k \geq 2$), and $s \in \{0, 1, 2, 3, \dots\}$. The matrix A is called* $\{R, s+1, k, *\}$-potent *if it satisfies*

$$A^{s+1}R = RA^*. \tag{2}$$

The set of all $\{R, s+1, k, *\}$-potent matrices will be denoted by $\mathcal{P}_{R,s,k,*}$.

If $A \in \mathcal{P}_{R,s,k,*}$ and $A = A^*$, then A is an $\{R, s+1, k\}$-potent matrix. Hence, we are interested in non-Hermitian $\{R, s+1, k, *\}$-potent matrices. In this case, A^{s+1} and A have the same spectrum up to conjugation.

The $s = 0$ case corresponds to matrices such that $A = RA^*R^{-1}$. This class has been investigated when R is either a permutation matrix or an involution, and will be further addressed in Section 2. Matrices in $\mathcal{P}_{R,s,k,*}$ generalize the *perhermitian matrices* (matrices $A \in \mathbb{C}^{n\times n}$ such that $A = JA^*J$ where J is the $n \times n$ antidiagonal matrix [24]) and the κ*-Hermitian matrices* (matrices $A \in \mathbb{C}^{n\times n}$ such that $A = KA^*K$ where K is any $n \times n$ involutory permutation matrix [13]).

A Toeplitz matrix $T = [t_{ij}] \in \mathbb{C}^{n\times n}$ satisfies $t_{ij} = t_{j-i}$ for some given sequence $t_{-n}, \dots, t_n$, while a Hankel matrix $H = [h_{ij}] \in \mathbb{C}^{n\times n}$ satisfies $h_{ij} = h_{i+j-2}$ for some given sequence $h_0, \dots, h_{2n}$; note that if J is the $n \times n$ antidiagonal matrix, then JT is Hankel and HJ is Toeplitz [14]. Every real Toeplitz matrix T can be written as $T^t = J^{-1}TJ$, similarly $H^t = J^{-1}HJ$ for any Hankel matrix H with real entries (here B^t denotes the transpose of B); these matrices provide interesting examples of $\{R, s+1, k, *\}$-potent matrices ($R = J$, $s = 0$, and $k = 2$).

The well-known Sylvester equations $BX + X^*C = E$, for arbitrary matrices B, C and E of conformable sizes, are widely studied in the literature, and the homogeneous case ($E = 0$) has recently attracted the attention of several researchers due to its relationship with palindromic eigenvalue problems [27].

$\{R, s+1, k, *\}$-potent matrices can be seen as solutions to a subclass of the homogeneous Sylvester equations ($B = R^{-1}$, $C = -R^{-1}$ where R is k-involutory, and $s = 0$) since the resulting equations relate a matrix with its conjugate transpose by a k-involutory similarity.

It is known that any $n \times n$ matrix over any field is congruent to its transpose by an involutory congruence, i.e, for any $n \times n$ matrix A, there is an X with $X^2 = I_n$ such that $XAX^t = A^t$ [15, 10]. In [9], it was shown that any projector (idempotent matrix) is unitarily similar to its conjugate transpose.

The concepts of generalized and hypergeneralized projectors were introduced by Groß and Trenkler [12]: For $A \in \mathbb{C}^{n\times n}$, A is called a *generalized projector* if $A^2 = A^*$, and A is called a *hypergeneralized projector* if $A^2 = A^\dagger$. Benítez and Thome [6] have extended the definition of generalized projectors to *k-generalized projectors* and in this paper we define *k-hypergeneralized projectors*, for any integer k greater than or equal to 2. Results concerning generalized and hypergeneralized projectors and their extensions can be found in [2, 3, 4, 6, 12, 26, 28]. Matrices $A \in \mathbb{C}^{n\times n}$ satisfying $(A - pI_n)(A - qI_n) = O$ for some $p, q \in \mathbb{C}$ are called *quadratic matrices* [1]; such matrices were generalized and studied in [11]. We extend the definition in [1] to what we will call $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$-potent matrices.

Except in Section 2, we will assume $s \in \mathbb{N}$. The $s = 0$ case is discussed in Section 2. In Section 3, we derive properties of $\{R, s+1, k, *\}$-potent matrices and give various characterizations. In [8] it was proved that an $\{R, s+1, k\}$-potent matrix is always diagonalizable but this is not always true for matrices in $\mathcal{P}_{R,s,k,*}$. We impose conditions on R or on the matrix A to recover some of the properties obtained for the former class of matrices. In Section 4, we study the relationship between $\{R, s+1, k, *\}$-potent matrices and other classes of matrices such as the $\{s+1\}$-generalized projectors, the $\{s+1\}$-hypergeneralized projectors, and the $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$-potent matrices. We summarize these relationships in a diagram provided in Figure 1.

2 $AR = RA^*$ when $R^k = I_n$

In this section, we analyze the case $s = 0$. The techniques used for this case are different from those for the case $s \geq 1$, which will be discussed separately in the next section. We begin with the following lemma regarding k-involutory matrices. The expression $\gcd(a_1, \dots, a_n)$ is the greatest common divisor of integers $a_1, \dots, a_n$.

Lemma 2. *Let $R \in \mathbb{C}^{n\times n}$ with $R^k = I_n$ for some positive integer $k \geq 2$. Then $\sigma(R) \subseteq \{\omega, \omega^2, \omega^3, \dots, \omega^k = 1\}$ where $\omega = \exp\left(\frac{2\pi i}{k}\right)$. Further, there exists an invertible $S \in \mathbb{C}^{n\times n}$ such that $R = SDS^{-1}$ with*

$$D = \omega^{\alpha_1} I_{n_1} \oplus \omega^{\alpha_2} I_{n_2} \oplus \cdots \oplus \omega^{\alpha_p} I_{n_p}$$

where p is the number of distinct eigenvalues in $\sigma(R)$, where the α_j are positive integers with $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq k$, and where the dimension of the *eigenspace of R for ω^{α_j} is n_j for each j. The minimality of k with respect to $R^k = I_n$ is equivalent to* $\gcd(\alpha_1, \alpha_2, \ldots, \alpha_p, k) = 1$.

Proof. Since $R^k - I_n = O$, the minimum polynomial of R must divide $x^k - 1$, which has no repeated roots, and hence, all eigenvalues of R are k^{th} roots of unity, and all Jordan blocks for R are 1×1. Let $g = \gcd(\alpha_1, \alpha_2, \ldots, \alpha_p, k)$. Then there are positive integers $\beta_1, \beta_2, \ldots, \beta_p$ so that $\alpha_j = g\beta_j$ for each j, and a positive integer h so that $k = gh$. Then, for each j,

$$\omega^{\alpha_j} = \exp\left(\frac{2\pi i}{k}\alpha_j\right) = \exp\left(\frac{2\pi i}{k}g\beta_j\right) = \exp\left(\frac{2\pi i}{h}\beta_j\right)$$

so that ω^{α_j} is actually an h^{th} root of unity where $h = k/g$. Then

$$D^h = \bigoplus_{j=1}^{p} \left(\omega^{\alpha_j}\right)^h I_{n_j} = I_n.$$

Since $R^h = I_n$ if and only if $D^h = I_n$, the minimality of k is equivalent to $g = 1$. □

One would hope that $AR = RA^*$ would imply that $D = S^{-1}RS$ and $B = S^{-1}AS$ would satisfy $BD = DB^*$. However, this requires that

$$BD = \left(S^{-1}AS\right)\left(S^{-1}RS\right) = S^{-1}(AR)S = S^{-1}(RA^*)S$$

and

$$DB^* = \left(S^{-1}RS\right)\left(S^{-1}AS\right)^* = S^{-1}R(SS^*)A^*(S^{-1})^*$$

are the same, which need not be true. What is needed is that $S^{-1} = S^*$, which is to say, what is needed is that R is unitarily diagonalizable. While requiring that $R = R^*$ suffices, so does the weaker condition, $RR^* = R^*R$. (The matrix R is called a normal matrix when the weaker condition holds, and this condition is equivalent to unitary diagonalizability.)

Consequently, we assume that R is a normal matrix. We examine what the condition $BD = DB^*$ implies about the matrix B. Begin by imposing the block partitioning of D on B. Observe that under Hermitian transpose, the block $(B^*)_{ij}$ is the block $(B_{ji})^*$ for $1 \le i, j \le p$. Then $BD = DB^*$ is equivalent to the conditions

$$B_{ij}\omega^{\alpha_j} I_{n_j} = \omega^{\alpha_i} I_{n_i} (B^*)_{ij} \quad \text{for} \quad 1 \le i, j \le p.$$

Equivalently,

$$B_{ij} = \omega^{\alpha_i - \alpha_j}(B^*)_{ij} \quad \text{for} \quad 1 \le i, j \le p. \tag{3}$$

Observe that when $i = j$, it follows that $B_{ii} = (B^*)_{ii} = (B_{ii})^*$. Hence, each diagonal block of B must be Hermitian.

Now suppose that $i \neq j$. Note that (3) gives

$$B_{ij} = \omega^{\alpha_i - \alpha_j}(B^*)_{ij} = \omega^{\alpha_i - \alpha_j}(B_{ji})^*;$$

and it also gives $B_{ji} = \omega^{\alpha_j - \alpha_i} (B_{ij})^*$. The latter implies $(B_{ji})^* = \omega^{\alpha_i - \alpha_j} B_{ij}$. Combining these results, we see that when $i \neq j$,

$$B_{ij} = \omega^{\alpha_i - \alpha_j} (B_{ji})^* = \omega^{\alpha_i - \alpha_j} \omega^{\alpha_i - \alpha_j} B_{ij} = \omega^{2(\alpha_i - \alpha_j)} B_{ij}.$$

When $2(\alpha_i - \alpha_j) \not\equiv 0 \bmod k$, $B_{ij} = 0_{n_i \times n_j}$. Note that $2(\alpha_i - \alpha_j) \not\equiv 0 \bmod k$ can be restated as $2\alpha_i \not\equiv 2\alpha_j \bmod k$. Also, when $2\alpha_i \equiv 2\alpha_j \bmod k$, no restrictions are imposed on B_{ij}.

When is $2\alpha_i \equiv 2\alpha_j \bmod k$, and how does this depend on k?

When k is odd, 2 is invertible $\bmod k$, and consequently, $2\alpha_i \equiv 2\alpha_j \bmod k$ if and only if $\alpha_i \equiv \alpha_j \bmod k$. Since α_i and α_j are distinct integers in $\{1, 2, \ldots, k\}$, $2(\alpha_i - \alpha_j) \not\equiv 0 \bmod k$. Thus, when k is odd, B must be a direct sum of Hermitian matrices.

What about when $k = 2m$ for some positive integer m? Note that $\omega^m = \exp\left(\frac{2\pi i}{k} m\right) = \exp(\pi i) = -1$. Since α_i and α_j are distinct integers in $\{1, 2, \ldots, k\}$, $0 < |\alpha_i - \alpha_j| < k$, and consequently, $2(\alpha_i - \alpha_j) \equiv 0 \bmod k$ if and only if $2|\alpha_i - \alpha_j| = k$, or equivalently, if and only if $|\alpha_i - \alpha_j| = m$. That is, when $\alpha_i < \alpha_j$, this means $\alpha_j = \alpha_i + m$, and when $\alpha_i > \alpha_j$, this means $\alpha_i = \alpha_j + m$. Thus, if $k = 2m$, and if whenever ω^{α_i} is in $\sigma(R)$, $\omega^{\alpha_i + m} = -\omega^{\alpha_i} \notin \sigma(R)$, then B must be a direct sum of Hermitian matrices.

The interesting case is when $k = 2m$ and for at least one i, $\{\omega^{\alpha_i}, -\omega^{\alpha_i}\} \subseteq \sigma(R)$. In this case, the diagonal blocks of B are all Hermitian, and for B_{ij} where $\alpha_j \equiv \alpha_i + m \bmod k$, $B_{ji} = \omega^{\alpha_j - \alpha_i} (B_{ij})^* = \omega^m (B_{ij})^* = -(B_{ij})^*$. Apparently, in this case, there will be some nontrivial off-diagonal blocks, which are connected by a skew-Hermitian relationship to other off-diagonal blocks.

The preceding arguments lead to the main result of this section.

Theorem 3. *Suppose n, k are positive integers, and $A, R \in \mathbb{C}^{n \times n}$ where R is normal and k-involutory, with k minimal with respect to $R^k = I_n$. Let $S, D \in \mathbb{C}^{n \times n}$ be the unitary and diagonal matrices, respectively, given in Lemma 2 such that $R = SDS^*$. Then, $AR = RA^*$ holds if and only if $BD = DB^*$ where $B = S^*AS$. Further,*

1. *When k is odd, $BD = DB^*$ if and only if $B = \bigoplus_{j=1}^{p} B_{jj}$ where each B_{jj} is an arbitrary $n_j \times n_j$ Hermitian matrix.*

2. *When $k = 2m$ for some positive integer m, partition B into blocks using the natural partition of D. The following are equivalent:*

   (a) $BD = DB^*$

   (b) For $1 \leq j \leq p$, B_{jj} is an arbitrary $n_j \times n_j$ Hermitian matrix. $B_{ij} = 0_{n_i \times n_j}$ whenever $|\alpha_i - \alpha_j| \neq m$. If $\alpha_j = \alpha_i \pm m$ (equivalently, $\omega^{\alpha_j} = -\omega^{\alpha_i}$) for some α_i with $1 \leq \alpha_i \leq m$ and some α_j, then B_{ij} is an arbitrary $n_i \times n_j$ complex matrix such that $B_{ji} = -(B_{ij})^*$.

Corollary 4. *Suppose* $A, R \in \mathbb{C}^{n\times n}$, $R = R^*$, *and* $R^k = I_n$ *for some minimal positive integer* k. *Then* $k \in \{1, 2\}$. *If* $R = \pm I_n$, *then* $AR = RA^*$ *if and only if* $A = A^*$. *If* $R \neq \pm I_n$, *then* $\sigma(R) = \{-1, 1\}$, $k = 2$, *and there exists a unitary* $S \in \mathbb{C}^{n\times n}$ *such that* $R = S\left(I_{n_1} \oplus (-1)I_{n_2}\right)S^*$ *where* $n_1 > 0$ *is the multiplicity of* 1 *in* $\sigma(R)$ *and* $n_2 > 0$ *is the multiplicity of* -1 *in* $\sigma(R)$. *Let* $B = S^*AS$. *Then* $AR = RA^*$ *if and only if*

$$B = \begin{bmatrix} B_{11} & B_{12} \\ -\left(B_{12}\right)^* & B_{22} \end{bmatrix}$$

where $B_{11} \in \mathbb{C}^{n_1\times n_1}$ *and* $B_{22} \in \mathbb{C}^{n_2\times n_2}$ *are Hermitian, and* $B_{12} \in \mathbb{C}^{n_1\times n_2}$ *is arbitrary.*

Proof. If $R = R^*$, then $\sigma(R)$ must be real, so $\sigma(R) \subseteq \{-1, 1\}$, and hence, $k \in \{1, 2\}$ by the minimality condition. If $\sigma(R) = \{1\}$, then $k = 1$ and $R = I_n$. If $\sigma(R) = \{-1\}$, then $k = 2$ and $R = -I_n$. If $\sigma(R) = \{-1, 1\}$, then use the preceding theorem with $k = 2$ and $p = 2$. □

The next corollary follows by using a similar argument.

Corollary 5. *Suppose* $A, R \in \mathbb{C}^{n\times n}$, $R^* = -R$, *and* $R^k = I_n$ *for some minimal positive integer* k. *Then* $k = 4$. *If* $R = \pm iI_n$, *then* $AR = RA^*$ *if and only if* $A = A^*$. *If* $R \neq \pm iI_n$, *then* $\sigma(R) = \{-i, i\}$ *and there exists a unitary* $S \in \mathbb{C}^{n\times n}$ *such that* $R = S\left(iI_{n_1} \oplus (-i)I_{n_2}\right)S^*$ *where* $n_1 > 0$ *is the multiplicity of* i *in* $\sigma(R)$ *and* $n_2 > 0$ *is the multiplicity of* $-i$ *in* $\sigma(R)$. *Let* $B = S^*AS$. *Then* $AR = RA^*$ *if and only if*

$$B = \begin{bmatrix} B_{11} & O \\ O & B_{22} \end{bmatrix}$$

where $B_{11} \in \mathbb{C}^{n_1\times n_1}$ *and* $B_{22} \in \mathbb{C}^{n_2\times n_2}$ *are Hermitian.*

The following example illustrates the second case in Theorem 3.

Example 6. *Suppose that* $k = 4$ *and* $\sigma(R) = \{i, -1, -i\}$. *Here* $\omega = i$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $n_1 = 4$ *and* $n_2 = n_3 = 1$. *Then* $k = 2m$ *where* $m = 2$; ω^{α_1} *and* $\omega^{\alpha_3} = -\omega^{\alpha_1}$ *are in* $\sigma(R)$; *and* ω^{α_2} *is in* $\sigma(R)$ *but* $\omega^{\alpha_2+m} = -\omega^{\alpha_2}$ *is not. Suppose that* $S = I_6$ *so* $R = D$. *If* $A \in \mathbb{C}^{6\times 6}$ *satisfies* $AR = RA^*$, *then* A_{11}, A_{22} *and* A_{33} *must be arbitrary Hermitian matrices;* A_{12}, A_{21}, A_{23} *and* A_{32} *must be zero matrices;* A_{13} *must be arbitrary, and* $A_{31} = -\left(A_{13}\right)^*$. *That is,* $AR = RA^*$ *holds if and only if* A *satisfies*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & a_{16} \\ a_{12}^* & a_{22} & a_{23} & a_{24} & 0 & a_{26} \\ a_{13}^* & a_{23}^* & a_{33} & a_{34} & 0 & a_{36} \\ a_{14}^* & a_{24}^* & a_{34}^* & a_{44} & 0 & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ -a_{16}^* & -a_{26}^* & -a_{36}^* & -a_{46}^* & 0 & a_{66} \end{bmatrix}$$

where each diagonal entry of A *is real.*

3 Characterizations of $\{R, s+1, k, *\}$-potent matrices

For a matrix $A \in \mathbb{C}^{n\times n}$, the *group inverse*, if it exists, is the unique matrix $A^{\#}$ satisfying the matrix equations $AA^{\#}A = A$, $A^{\#}AA^{\#} = A^{\#}$, and $AA^{\#} = A^{\#}A$; it is well known that $A^{\#}$ exists if and only if $\operatorname{rank} A^2 = \operatorname{rank} A$ [5].

Throughout this section, we assume that s, k are integers with $s \geq 1$ and $k \geq 2$. First, we list some properties of $\{R, s+1, k, *\}$-potent matrices.

Lemma 7. *Suppose that $A \in \mathcal{P}_{R,s,k,*}$. Then the following statements hold.*

a. $A^{\#}$ *exists.*

b. $A^{\#} \in \mathcal{P}_{R,s,k,*}$.

c. $AA^{\#} \in \mathcal{P}_{R,s,k,*}$.

d. $\sigma(A) \subseteq \{0\} \cup \left\{\exp\left(\frac{2\pi t}{s+2} i\right),\ t \in \{0, 1, \ldots, s+1\}\right\}$.

Proof. (a) Since $s \geq 1$, $\operatorname{rank}(A) = \operatorname{rank}(A^*) = \operatorname{rank}(R^{-1}A^{s+1}R) = \operatorname{rank}(A^{s+1}) \leq \operatorname{rank}(A^2) \leq \operatorname{rank}(A)$. Thus, $\operatorname{rank}(A^2) = \operatorname{rank}(A)$.
(b) Using the relation $(A^*)^{\#} = (A^{\#})^*$, we obtain $(A^*)^{\#} = (R^{-1}A^{s+1}R)^{\#} = R^{-1}(A^{s+1})^{\#}R = R^{-1}(A^{\#})^{s+1}R = (A^{\#})^*$.
(c) Since $A, A^{\#} \in \mathcal{P}_{R,s,k,*}$, $(AA^{\#})^{s+1} = A^{s+1}(A^{\#})^{s+1} = RA^*R^{-1}R(A^{\#})^*R^{-1} = RA^*(A^{\#})^*R^{-1} = R(A^{\#}A)^*R^{-1} = R(AA^{\#})^*R^{-1}$.
(d) From $RA^*R^{-1} = A^{s+1}$, we have $[\sigma(A)]^{s+1} = \sigma(A^{s+1}) = \sigma(RA^*R^{-1}) = \sigma(A^*) = \overline{\sigma(A)}$, where $\overline{\sigma(A)}$ means the set of the conjugate of the eigenvalues of A. Thus, $\lambda \in \sigma(A)$ if and only if $\lambda^{s+1} = \overline{\lambda}$, which becomes $r^{s+1}\exp((s+1)i\theta) = r\exp(-i\theta)$ where we assume that $\lambda = r\exp(i\theta)$. Now, taking modulus the two possibilities are $r = 0$ which implies $\lambda = 0$, or $\lambda = \exp\left(\frac{2\pi t}{s+2} i\right)$, $t \in \{0, 1, \ldots, s+1\}$. □

Some results related to Lemma 7 were given in [16].

The next result presents a characterization of matrices in $\mathcal{P}_{R,s,k,*}$.

Theorem 8. *Let $A, R \in \mathbb{C}^{n\times n}$ be such that $R^k = I_n$ and $r = \operatorname{rank}(A)$. Then A is an $\{R, s+1, k, *\}$-potent matrix if and only if there exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ such that*

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1} \qquad \text{and} \qquad R = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^*, \tag{4}$$

for $X \in \mathbb{C}^{r\times r}$ satisfying $XC^* = C^{s+1}X$ with X nonsingular and for any nonsingular $T \in \mathbb{C}^{(n-r)\times(n-r)}$.

Proof. By Lemma 7, A has index at most 1. So, the core-nilpotent representation gives

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}$$

for some nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$. Substituting in $A^{s+1} = RA^*R^{-1}$ we get

$$P^{-1}R(P^{-1})^* \begin{bmatrix} C^* & O \\ O & O \end{bmatrix} P^*R^{-1}P = \begin{bmatrix} C^{s+1} & O \\ O & O \end{bmatrix}.$$

Denoting $Z = P^{-1}R(P^{-1})^*$ and partitioning Z as

$$Z = \begin{bmatrix} X & Y \\ V & T \end{bmatrix}$$

of adequate sizes, we arrive at

$$\begin{bmatrix} X & Y \\ V & T \end{bmatrix}\begin{bmatrix} C^* & O \\ O & O \end{bmatrix} = \begin{bmatrix} C^{s+1} & O \\ O & O \end{bmatrix}\begin{bmatrix} X & Y \\ V & T \end{bmatrix},$$

from where we obtain $XC^* = C^{s+1}X$, $Y = O$, and $V = O$. Since R is nonsingular, X and T are nonsingular as well. Substituting in the expression $R = PZP^*$, we get the representation (4). □

From Theorem 8, it follows that if A is an $\{R, s+1, k, *\}$-potent matrix with A as in (4) then

$$A^\# = P \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} P^{-1}.$$

Observe that in Theorem 8 we obtain the condition $XC^* = C^{s+1}X$ but, in general, we cannot conclude that C is an $\{X, s+1, k, *\}$-potent matrix. Moreover, while A is similar to a block diagonal matrix via the matrix P, the corresponding relation for R using the same P is a congruence to a block diagonal matrix. The concept of EP matrices allows us to improve the form in (4) by giving (unitary) similarity in R as well.

Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is called EP if $AA^\dagger = A^\dagger A$ [7], or equivalently, if there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C \in \mathbb{C}^{r\times r}$ such that

$$A = U \begin{bmatrix} C & O \\ O & O \end{bmatrix} U^*.$$

Theorem 9. *Let $A, R \in \mathbb{C}^{n\times n}$ be such that $R^k = I_n$ and $r = \mathrm{rank}(A)$. Consider the following three conditions:*

a. A is an EP matrix.

b. A is an $\{R, s+1, k, *\}$-potent matrix.

c. There exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C \in \mathbb{C}^{r\times r}$ such that

$$A = U \begin{bmatrix} C & O \\ O & O \end{bmatrix} U^* \qquad \text{and} \qquad R = U \begin{bmatrix} X & O \\ O & T \end{bmatrix} U^*$$

where C is a $\{X, s+1, k, *\}$-potent matrix for $X \in \mathbb{C}^{r\times r}$ and any $T \in \mathbb{C}^{(n-r)\times(n-r)}$ satisfying $T^k = I_{n-r}$.

Then any two of these conditions (a)-(c) imply the third one.

Proof. $(a) + (b) \Longrightarrow (c)$: Assume that

$$A = U \begin{bmatrix} C & O \\ O & O \end{bmatrix} U^*$$

for some unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C \in \mathbb{C}^{r\times r}$. Now, a similar proof as that of Theorem 8 gives (c). $(a) + (c) \Longrightarrow (b)$: This can be directly derived from Theorem 8. $(b) + (c) \Longrightarrow (a)$: This direction is trivial. $\square$

The findings in the next result relate to some facts about the diagonalization of a matrix in $\mathcal{P}_{R,s,k,*}$.

Theorem 10. *Let $A, R \in \mathbb{C}^{n\times n}$ be such that $R^k = I_n$ and A is an $\{R, s+1, k, *\}$-potent matrix. Then*

a. $A^{(s+1)^{2j}} = (R(R^{-1})^*)^j A(R^*R^{-1})^j, \;\; j = 1, \ldots, k.$

b. *If R is normal, then $A^{(s+1)^{2k}} = A$. In this case, $A^\# = A^{(s+1)^{2k}-2}$.*

c. *If R is Hermitian, then $A^{(s+1)^2} = A$. In this case, $A^\# = A^{(s+1)^2-2}$.*

d. *If R is normal, then A is diagonalizable.*

Proof. (a) The definition $A^{s+1} = RA^*R^{-1}$ implies $A^{(s+1)^2} = (A^{s+1})^{s+1} = R(A^{s+1})^*R^{-1} = R(R^{-1})^*AR^*R^{-1}$. Similarly,

$$A^{(s+1)^3} = (A^{(s+1)^2})^{s+1} = R(R^{-1})^*RA^*R^{-1}R^*R^{-1}$$

and $A^{(s+1)^4} = (R(R^{-1})^*)^2A^*(R^{-1}R^*)^2$. The result follows by induction. (b) If R is normal, then $(R^*)^{-1}R = R(R^*)^{-1}$ and inductively,

$$(R(R^{-1})^*)^k = R^k((R^{-1})^*)^k = R^k((R^k)^*)^{-1} = I_n$$

and

$$(R^*R^{-1})^k = (R^*)^k(R^{-1})^k = (R^k)^*(R^k)^{-1} = I_n,$$

since $R^k = I_n$. Now, the result follows from (a). (c) If $R^* = R$ and $R^k = I_n$, then $R^2 = I_n$ because R is (unitarily) diagonalizable and

$$\sigma(R) \subseteq \mathbb{R} \cap \left\{ \exp\left(\frac{2\pi q}{k} i\right), q \in \{0, 1, \ldots, q-1\} \right\} \subseteq \{-1, 1\}.$$

Hence, $R^{-1} = R = R^*$. Now, again the result follows from (a). (d) This follows from (b) and by taking into account that all the roots of the polynomial $p(z) = z^{(s+1)^{2k}} - z$ are simple. In order to compute the group inverses of A in parts (b) and (c) the following general fact is used: $A^\# = A^\ell$ if and only if $A^{\ell+2} = A$ for some given integer $\ell \geq 1$. $\square$

While in [8] it was proved that an $\{R, s+1, k\}$-potent matrix is always diagonalizable, this property is not always true for matrices in $\mathcal{P}_{R,s,k,*}$. The next example illustrates this fact.

Example 11. *Let ω be a primitive root of unity of order $2m$,*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad R_\omega = \begin{bmatrix} 0 & \sqrt{s+1} & 0 \\ \frac{1}{\sqrt{s+1}} & 0 & 0 \\ 0 & 0 & \omega \end{bmatrix}.$$

Then $R_\omega^{2m} = I_3$ and the matrix

$$X = \begin{bmatrix} 0 & \sqrt{s+1} \\ \frac{1}{\sqrt{s+1}} & 0 \end{bmatrix}$$

satisfies $XC^* = C^{s+1}X$ and $X^2 = I_2$ where $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Hence, A is a $\{R_\omega, s+1, 2m, *\}$-potent matrix. It is clear that A is not diagonalizable.

Recall that for a pair of matrices $A, B \in \mathbb{C}^{n\times n}$, the *commutator* $[A, B]$ is defined as $[A, B] = AB - BA$.

Lemma 12. *Let $R \in \mathbb{C}^{n\times n}$ be such that $R^k = I_n$. The set*

$$G = \{A \in \mathcal{P}_{R,s,k,*} : [A, B] = O, \forall B \in \mathcal{P}_{R,s,k,*}\}$$

is a semigroup under matrix multiplication.

Proof. Let $A_1, A_2 \in G$. Then, $A_1, A_2 \in \mathcal{P}_{R,s,k,*}$, and for $i = 1, 2$ we have $A_iB = BA_i$ for all $B \in \mathcal{P}_{R,s,k,*}$. In particular, $A_1A_2 = A_2A_1$. Since $RA_i^*R^{-1} = A_i^{s+1}$ for $i = 1, 2$, we get

$$(A_1A_2)^{s+1} = A_1^{s+1}A_2^{s+1} = RA_1^*A_2^*R^{-1} = R(A_2A_1)^*R^{-1} = R(A_1A_2)^*R^{-1},$$

that is $A_1A_2 \in \mathcal{P}_{R,s,k,*}$. Moreover, $(A_1A_2)B = A_1BA_2 = B(A_1A_2)$ for all $B \in \mathcal{P}_{R,s,k,*}$. Hence, $A_1A_2 \in G$. $\square$

Remark 13. *If $A, B \in \mathcal{P}_{R,s,k,*}$ satisfy $AB = BA$, then $AB \in \mathcal{P}_{R,s,k,*}$.*

4 Relationship between $\mathcal{P}_{R,s,k,*}$ and other classes of matrices

First, we present a general result whose proof will be useful in this section.

Lemma 14. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index 1 and $\operatorname{rank}(A) = r > 0$. Then A is a normal matrix if and only if there exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ be such that*

$$A = P\begin{bmatrix} C & O \\ O & O \end{bmatrix}P^{-1} \qquad \text{and} \qquad P^*P = \begin{bmatrix} M & O \\ O & N \end{bmatrix}$$

where $M \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ are both positive definite matrices and C^* commutes with MCM^{-1}.

Proof. It is well known that any matrix of index 1 has the form

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}$$

for some nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$. Substituting in $AA^* = A^*A$ and reordering factors yield

$$P^*P \begin{bmatrix} C & O \\ O & O \end{bmatrix} (P^*P)^{-1} \begin{bmatrix} C^* & O \\ O & O \end{bmatrix} P^*P = \begin{bmatrix} C^* & O \\ O & O \end{bmatrix} P^*P \begin{bmatrix} C & O \\ O & O \end{bmatrix}. \quad (5)$$

Partitioning P^*P with adequate sizes to the partition considered for A we obtain

$$P^*P = \begin{bmatrix} M & Q \\ Q^* & N \end{bmatrix},$$

with M and N Hermitian. Since P is nonsingular, by using the positive definiteness of P^*P it is easy to see that M and N are positive definite. The inversion formula of Banachiewicz-Schur ensures the nonsingularity of the Schur complement $W = (P^*P)/M = N - Q^*M^{-1}Q$ and gives

$$(P^*P)^{-1} = \begin{bmatrix} M^{-1} + M^{-1}QW^{-1}Q^*M^{-1} & -M^{-1}QW^{-1} \\ -W^{-1}Q^*M^{-1} & W^{-1} \end{bmatrix}.$$

Substituting in (5) and making the block products we get

$$\begin{bmatrix} MLM & MLQ \\ Q^*LM & Q^*LQ \end{bmatrix} = \begin{bmatrix} C^*MC & O \\ O & O \end{bmatrix}$$

where $L = C(M^{-1} + M^{-1}QW^{-1}Q^*M^{-1})C^*$. Thus, $MLM = C^*MC$, $MLQ = O$, $Q^*LM = O$, and $Q^*LQ = O$. By the nonsingularity of M and N we get $LQ = O$ and $Q^*L = O$, that is

$$O = C(M^{-1} + M^{-1}QW^{-1}Q^*M^{-1})C^*Q = CM^{-1}(I_r + QW^{-1}Q^*M^{-1})C^*Q.$$

This last expression gives $(I_r + QW^{-1}Q^*M^{-1})C^*Q = O$. Similarly, from $O = Q^*L = Q^*C(I_r + M^{-1}QW^{-1}Q^*)M^{-1}C^*$ we get $Q^*(I_r + M^{-1}QW^{-1}Q^*) = O$. Now, substituting the expression of L in $MLM = C^*MC$ we arrive at $MCM^{-1}(I_r + QW^{-1}Q^*M^{-1})C^*M = C^*MC$ which implies

$$O = MCM^{-1}(I_r + QW^{-1}Q^*M^{-1})C^*Q = C^*MCM^{-1}Q,$$

from where $Q = O$ due to the nonsingularity of C and M. Hence,

$$P^*P = \begin{bmatrix} M & O \\ O & N \end{bmatrix},$$

with $MCM^{-1}C^* = C^*MCM^{-1}$ since $L = CM^{-1}C^*$. The converse is evident. $\square$

In Lemma 7 we proved that the projector $AA^{\#} \in \mathcal{P}_{R,s,k,*}$ provided that $A \in \mathcal{P}_{R,s,k,*}$. The next result characterizes all projectors that belong to $\mathcal{P}_{R,s,k,*}$.

Theorem 15. *Let $A \in \mathbb{C}^{n\times n}$ be a projector, i.e., $A^2 = A$. Then the following conditions are equivalent:*

a. A is $\{R, s+1, k, *\}$-potent.

b. $AR = RA^*$.

c. There exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$A = P \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} P^{-1} \qquad \text{and} \qquad R = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^*$$

where $X \in \mathbb{C}^{r\times r}$ and $T \in \mathbb{C}^{(n-r)\times(n-r)}$ are nonsingular matrices.

Proof. Since $A^2 = A$, we get $A^{s+1} = A$ for all s and

$$A = P \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} P^{-1}. \tag{6}$$

(a) $\Longleftrightarrow$ (b) This follows directly from the definitions. (b) $\Longleftrightarrow$ (c) The form of R can be found by substituting (6) into $AR = RA^*$ and partitioning

$$P^{-1}R(P^{-1})^* = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}.$$

□

Remark 16. *Note that in the above theorem the value used for s was not relevant.*

In Theorem 9 we have characterized all $\{R, s+1, k, *\}$-potent matrices that are EP. Next, we characterize $\{R, s+1, k, *\}$-potent matrices that are normal.

Theorem 17. *Let $A \in \mathbb{C}^{n\times n}$ be a nonzero $\{R, s+1, k, *\}$-potent matrix. Then A is normal if and only if there exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ such that*

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1} \qquad \text{and} \qquad R = P \begin{bmatrix} XM & O \\ O & TN \end{bmatrix} P^{-1}$$

where $M \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ are both positive definite matrices and $X \in \mathbb{C}^{r\times r}$ and $T \in \mathbb{C}^{(n-r)\times(n-r)}$ are nonsingular matrices such that $XC^* = C^{s+1}X$.

Proof. By Theorem 8 there exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ such that

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1} \qquad \text{and} \qquad R = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^*,$$

for $X \in \mathbb{C}^{r\times r}$ satisfying $XC^* = C^{s+1}X$ with X nonsingular and for any nonsingular $T \in \mathbb{C}^{(n-r)\times(n-r)}$. Assume that A is normal. Then, a similar proof to that of Lemma 14 yields

$$P^* = \begin{bmatrix} M & O \\ O & N \end{bmatrix} P^{-1}$$

where $M \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ are both positive definite matrices. Thus, we can deduce that

$$R = P \begin{bmatrix} XM & O \\ O & TN \end{bmatrix} P^{-1}.$$

The converse is evident. □

In [12], Groß and Trenkler defined *generalized projectors* as matrices $A \in \mathbb{C}^{n\times n}$ that satisfy $A^2 = A^*$ and denoted this class of matrices by GP. In [6], Benítez and Thome introduced $\{s+1\}$-generalized projectors (for $s \geq 1$) and for ease we call these matrices $\{s+1\}$-GP matrices [19]. A matrix $A \in \mathbb{C}^{n\times n}$ is called an $\{s+1\}$-*GP matrix* if $A^* = A^{s+1}$; we denote the set of all $n \times n$ $\{s+1\}$-GP matrices by GP_{s+1}. The matrices in GP_{s+1} are characterized as follows [6]:

$$A \in GP_{s+1} \iff A \text{ is normal and } \sigma(A) \subseteq \{0\} \cap \Omega_{s+2} \iff A \text{ is normal and } A^{s+3} = A$$

where Ω_{s+2} denotes the roots of unity of order $s+2$. We next give another characterization.

Lemma 18. *Let $A \in \mathbb{C}^{n\times n}$. Then A is a $\{s+1\}$-GP matrix if and only if there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D = [d_{ij}] \in \mathbb{C}^{r\times r}$ such that*

$$A = U \begin{bmatrix} D & O \\ O & O \end{bmatrix} U^*,$$

with $d_{jj} \in \Omega_{s+2}$.

Proof. This is a straightforward extension of [6, Corollary 2.2]. □

Now, we characterize $\{R, s+1, k, *\}$-potent matrices that are in GP_{s+1}.

Theorem 19. *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k, *\}$-potent matrix. Then, the following statements are equivalent:*

a. A is a $\{s+1\}$-GP.

b. $A^*R = RA^*$.

c. There exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D = [d_{ij}] \in \mathbb{C}^{r\times r}$ such that

$$A = U \begin{bmatrix} D & O \\ O & O \end{bmatrix} U^* \quad \text{and} \quad R = U \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} U^*$$

where $d_{jj} \in \Omega_{s+2}$ with $R_1 \in \mathbb{C}^{r\times r}$ satisfying $R_1^* D = DR_1$ and $R_2 \in \mathbb{C}^{(n-r)\times(n-r)}$.

Proof. From the definition $A^{s+1} = RA^*R^{-1}$, it is easy to see that $A^{s+1} = A^*$ and $A^*R = RA^*$ are equivalent; thus $(a) \iff (b)$. Now, suppose that A is a $\{s+1\}$-GP matrix. By Lemma 18

$$A = U \begin{bmatrix} D & O \\ O & O \end{bmatrix} U^*,$$

under the conditions indicated there. Consider the partition

$$U^*RU = \begin{bmatrix} R_1 & R_3 \\ R_4 & R_2 \end{bmatrix},$$

according to the sizes of the partition of U^*AU. Equating blocks, we obtain that the expression $A^*R = RA^*$ is equivalent to $D^*R_1 = R_1D^*$, $R_3 = O$, and $R_4 = O$, since D is nonsingular; thus $(a) \iff (c)$. □

Now, we relate the class of $\{R, s+1, k, *\}$-potent matrices to the class of $\{s+1\}$-HGP matrices. In [12], Groß and Trenkler defined *hypergeneralized projectors* as matrices $A \in \mathbb{C}^{n\times n}$ that satisfy $A^2 = A^\dagger$ and denoted this class of matrices by HGP. We call a matrix $A \in \mathbb{C}^{n\times n}$ an $\{s+1\}$-*hypergeneralized potent (or* $\{s+1\}$-HGP) *matrix* if $A^{s+1} = A^\dagger$ and we denote the set of all $n \times n$ $\{s+1\}$-HGP matrices by HGP_{s+1}. The matrices in HGP_{s+1} are characterized as follows:

$$A \in HGP_{s+1} \iff A \text{ is } EP \text{ and } A^{s+3} = A.$$

Theorem 20. *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k, *\}$-potent matrix. Then, the following statements are equivalent:*

a. A is an $\{s+1\}$-HGP matrix.

b. $A^\dagger R = RA^*$.

c. There exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C \in \mathbb{C}^{r\times r}$ such that

$$A = U \begin{bmatrix} C & O \\ O & O \end{bmatrix} U^* \quad \text{and} \quad R = U \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} U^*$$

where $C^{-1}R_1 = R_1C^*$ with $R_1 \in \mathbb{C}^{r\times r}$ and $R_2 \in \mathbb{C}^{(n-r)\times(n-r)}$ satisfying $R_1^k = I_r$ and $R_2^k = I_{n-r}$.

Proof. The equivalence (a) $\iff$ (b) follows directly from the definitions. Suppose that A is a $\{s+1\}$-HGP. Then A is EP, so there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C \in \mathbb{C}^{r\times r}$ such that

$$A = U \begin{bmatrix} C & O \\ O & O \end{bmatrix} U^*.$$

It is clear that

$$A^\dagger = U \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} U^*.$$

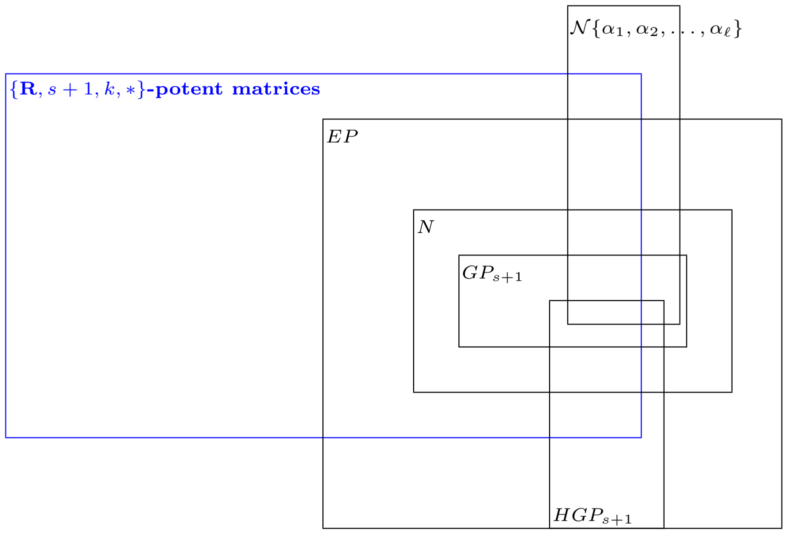

Figure 1: Relation between $\{R, s+1, k+*\}$-potent matrices and other classes

Now we consider the partition

$$U^*RU = \begin{bmatrix} R_1 & R_3 \\ R_4 & R_2 \end{bmatrix},$$

according to the sizes of the partition of U^*AU. Substituting in $RA^* = A^\dagger R$ and equating blocks we obtain $R_1C^* = C^{-1}R_1$, $R_3 = O$, and $R_4 = O$. Thus, the conditions on R have been obtained. Observe that $R^k = I_n$ implies $R_1^k = I_r$ and $R_2^k = I_{n-r}$. Hence (a) $\Longrightarrow$ (c). Finally, (c) $\Longrightarrow$ (b) is straightforward. $\square$

We summarize all the information studied in this section in Figure 1.

A matrix $A \in \mathbb{C}^{m\times n}$ is a *partial isometry* if $A^\dagger = A^*$, or equivalently, $AA^*A = A$ [22]. The relation between $\mathcal{P}_{R,s,k,*}$ and partial isometries is presented in the next result.

Theorem 21. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix in $\mathcal{P}_{R,s,k,*}$. As in Theorem 8, let*

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1} \qquad \text{and} \qquad R = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^*,$$

and partition P^*P as

$$P^*P = \begin{bmatrix} M & L \\ L^* & N \end{bmatrix}.$$

Then A is a partial isometry if and only if $I_r + L(N - L^*M^{-1}L)^{-1}L^*M^{-1} = MC^{-1}M^{-1}(C^{-1})^*$.

Proof. The result is obtained by substituting in $AA^*A = A$ the expression of A given in the statement and by using the Banachiewicz-Schur formula for the inverse of P^*P. $\square$

Finally, we present the relationship between $\mathcal{P}_{R,s,k,*}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$-potent matrices. The latter is an extension of the $\{\alpha_1, \alpha_2\}$-quadratic matrices [23].

Definition 22. *A matrix $A \in \mathbb{C}^{n\times n}$ is called an $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$-potent matrix if*

$$(A - \alpha_1 I_n)(A - \alpha_2 I_n)\ldots(A - \alpha_\ell I_n) = 0$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathbb{C}$ are pairwise distinct.
The set of all $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$-potent matrices will be denoted by $\mathcal{N}\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$.

If $\ell = 2$, matrices in $\mathcal{N}\{\alpha_1, \alpha_2\}$ are called $\{\alpha_1, \alpha_2\}$-quadratic [1, 11]. Allowing equalities between $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, the choice $\alpha_1 = \alpha_2 = \cdots = \alpha_\ell = 0$ leads to nilpotent matrices.

Lemma 23. *A nonzero $\{R, s+1, k, *\}$-potent matrix is not nilpotent.*

Proof. Suppose that $A \in \mathbb{C}^{n\times n}$ is an $\{R, s+1, k, *\}$-potent matrix that is not the zero matrix. By Theorem 8,

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}$$

for some nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$. If we assume that $A^m = O$ for some positive integer m then $C^m = O$, which is impossible. □

Theorem 24. *Let $A \in \mathcal{P}_{R,s,k,*}$, and let $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathbb{C}$ be pairwise distinct. Then $A \in \mathcal{N}\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ if and only if $\alpha_1 = 0$,*

$$A = L \begin{bmatrix} D & O \\ O & O \end{bmatrix} L^{-1} \qquad \text{and} \qquad R = L \begin{bmatrix} Y & O \\ O & T \end{bmatrix} L^*$$

for some nonsingular matrix $L \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D = [d_{ij}] \in \mathbb{C}^{r\times r}$ where $d_{jj} \in \{\alpha_2, \ldots, \alpha_\ell\} \cap \left\{\exp\left(\frac{2\pi t}{s+2} i\right),\ t \in \{0, 1, \ldots, s+1\}\right\}$ for $j = 1, 2, \ldots, r$ and some nonsingular matrices $Y \in \mathbb{C}^{r\times r}$, $T \in \mathbb{C}^{(n-r)\times(n-r)}$ such that $YD^* = D^{s+1}Y$.

Proof. Since $A \in \mathcal{P}_{R,s,k,*}$, by Theorem 8 we have

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1} \qquad \text{and} \qquad R = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^*$$

for some nonsingular matrices $P \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{r\times r}$, and $T \in \mathbb{C}^{(n-r)\times(n-r)}$ such that $XC^* = C^{s+1}X$. Suppose that $(A - \alpha_1 I_n)(A - \alpha_2 I_n)\ldots(A - \alpha_\ell I_n) = 0$ where $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \mathbb{C}$ are pairwise distinct. Then

$$P \begin{bmatrix} \Pi_{j=1}^{\ell}(C - \alpha_j I_r) & O \\ O & (-1)^{\ell}\Pi_{j=1}^{\ell}\alpha_j I_{n-r} \end{bmatrix} P^{-1} = O.$$

So, $\Pi_{j=1}^{\ell}(C-\alpha_j I_r)=O$ and $\Pi_{j=1}^{\ell}\alpha_j=0$. It is clear that there is at least one $j\in\{1,2,\ldots,\ell\}$ such that $\alpha_j=0$ (since $\alpha_i\neq\alpha_q$ if $i\neq q$). Without loss of generality, we can assume that $\alpha_1=0$ (consequently, $\alpha_j\neq 0$ for all $j\in\{2,\ldots,\ell\}$). Now, $\Pi_{j=2}^{\ell}(C-\alpha_j I_r)=O$ because C is nonsingular, and $p(x)=(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_\ell)$ is a (monic) annihilator polinomial of C with all its factors linear. Since all $\alpha_j\in\mathbb{C}$ and $\mathbb{C}$ is algebraically closed, C must be diagonalizable. Let $C=QDQ^{-1}$ with D diagonal. Then

$$A=P\left[\begin{array}{cc} QDQ^{-1} & O \\ O & O \end{array}\right]P^{-1}=L\left[\begin{array}{cc} D & O \\ O & O \end{array}\right]L^{-1}$$

where $L=P\left[\begin{array}{cc} Q & O \\ O & I_{n-r} \end{array}\right]$. Hence, A is diagonalizable. Substituting now, $C=QDQ^{-1}$ in $\Pi_{j=2}^{\ell}(C-\alpha_j I_r)=O$ we get $\Pi_{j=2}^{\ell}(D-\alpha_j I_r)=O$, that is for every $i=1,2,\ldots,\ell$, $\Pi_{j=2}^{\ell}(d_{jj}-\alpha_j)=0$, thus, $d_{jj}\in\{\alpha_2,\ldots,\alpha_\ell\}$ for all $j=1,2,\ldots,r$. From Lemma 7, $d_{jj}\in\left\{\exp\left(\frac{2\pi t}{s+2}i\right),\, t\in\{0,1,\ldots,s+1\}\right\}$ for all $j\in\{1,2,\ldots,r\}$. By using $XC^*=C^{s+1}X$ and $C=QDQ^{-1}$, we can denote $Y=Q^{-1}X(Q^*)^{-1}$ to arrive at

$$R=P\left[\begin{array}{cc} X & O \\ O & T \end{array}\right]P^*=L\left[\begin{array}{cc} Y & O \\ O & T \end{array}\right]L^*,$$

with $YD^*=D^{s+1}Y$ and Y nonsingular. □

Remark 25. *Notice that, if either $\alpha_j\neq 0$ for all $j\in\{1,2,\ldots,\ell\}$ or if $\alpha_j\notin\left\{\exp\left(\frac{2\pi t}{s+2}i\right),\, t\in\{0,1,\ldots,s+1\}\right\}$ for some $j\in\{2,\ldots,\ell\}$ then $\mathcal{N}\{\alpha_1,\alpha_2,\ldots,\alpha_\ell\}\cap\mathcal{P}_{R,s,k,*}=\emptyset$.*

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