# A VARIATIONAL PRINCIPLE FOR FREE SEMIGROUP ACTIONS

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Abstract. In this paper we introduce a notion of measure theoretical entropy for a finitely generated free semigroup action and establish a variational principle when the semigroup is generated by continuous self maps on a compact metric space and has finite topological entropy. In the case of semigroups generated by Ruelle-expanding maps we prove the existence of equilibrium states and describe some of their properties. Of independent interest are the different ways we will present to compute the metric entropy and a characterization of the stationary measures.

## 1. Introduction

The concept of entropy, introduced into the realm of dynamical systems more than fifty years ago, has become an important ingredient in the characterization of the complexity of dynamical systems. The topological entropy reflects the complexity of the dynamical system and is an important invariant. The metric entropy of invariant measures turns out to be a surprisingly universal concept in ergodic theory since it appears in the study of different subjects, such as information theory, Poincaré recurrence or the analysis of either local or global dynamics. The classical variational principle for the topological entropy of continuous maps on compact metric spaces (see e.g. [27]) not only relates both concepts as allows one to describe the dynamics by means of a much richer combined review of both topological and ergodic aspects. An extension of the variational principle for more general group and semigroup actions face some nontrivial challenges. If, on the one hand, it runs into the difficulty to establish a suitable notion of topological complexity for the action, on the other hand, the existence of probability measures that are invariant by the (semi)group action is a rare event beyond the setting of finitely generated abelian groups. While the latter has become a major obstruction for the proof of variational principles for other group actions, leading contributions were established in the case of amenable and sofic groups [11, 16, 21].

We will consider random walks associated to finitely generated semigroup actions of continuous maps on a compact metric space $X$. More precisely, after fixing a finite set $\{g_1, g_2, \cdots, g_p\}$ of endomorphisms of $X$ and taking the unilateral shift $\sigma\colon \Sigma_p^+ \to \Sigma_p^+$ defined on the space of sequences with values in $\{1, 2, \cdots, p\}$, endowed with a Borel $\sigma-$invariant probability measure $\mathbb{P}$, we associate to each $\omega = \omega_1\,\omega_2\,\cdots \in \Sigma_p^+$ the sequence of compositions $(g_{\omega_1}\,g_{\omega_2}\,\cdots g_{\omega_n})_{n\in\mathbb{N}}$. Our aim is to carry on the analysis, started in [9, 24], of the ergodic and statistical properties of these random compositions, and to set up a thermodynamic formalism. In this context common invariant measures seldom exist. Moreover, as we will discuss later on, stationary measures are not enough to describe the thermodynamic formalism of these semigroup actions.

One way to study a finitely generated free semigroup action is through the corresponding skew product $\mathcal{F}_G\colon \Sigma_p^+ \times X \to \Sigma_p^+ \times X$ defined by $\mathcal{F}_G(\omega, x) = (\sigma(\omega), g_{\omega_1}(x))$. This is an advantageous approach because $\mathcal{F}_G$ is a true dynamical system and, moreover, if each generator is a Ruelle-expanding map, then so is $\mathcal{F}_G$ and for it a complete formalism is known [25]. Yet, we are looking for intrinsic dynamical and ergodic concepts, the less dependent on the skew product the better. In [20], the authors proposed a notion of measure theoretical entropy for free semigroup actions and probability measures invariant by all the generators of the semigroup, but only a partial variational principle

2000 *Mathematics Subject Classification.* Primary: 37B05, 37B40, 22F05. Secondary: 37D20, 37D35, 37C85.
*Key words and phrases.* Semigroup actions, expanding maps, skew-product, entropy.

was obtained there (as happened in [5] and [10]). The major obstruction to use this strategy, in order to extend the ergodic theory known for a single expanding map, is the fact that probability measures which are invariant by all the generators of a free semigroup action may fail to exist. On the other hand, although stationary measures describe some of the statistics of the iteration of random transformations and are suited to the relativised variational principle proved for the skew product in [18] (using a notion of metric entropy different from ours and without any reference to semigroup actions), its role is hard to uncover without a major intervention of properties of the skew product $\mathcal{F}_G$. Moreover, as we will see, the set of stationary measures is scarce to describe in its full extent the ergodic attributes of a free semigroup action (though adequate in the setting of group actions; see [6], concerning group actions of bimeasurable transformations and a notion of metric entropy similar to the one employed in [18]).

Following the idea of complexity presented in [8], a notion of topological entropy for a free semigroup action of Ruelle-expanding maps was introduced in [24], and its main properties have been described in [9] and [10] through a suitable transfer operator. In the former reference, the topological information has also been linked to ergodic measurements, both with respect to the relative metric entropy (cf. Proposition 8.8 of [9]) and the underlying Bernoulli random walk (cf. Proposition 8.6 of [9]). In this paper we work with a new notion of measure theoretical entropy of a free semigroup action. Inspired by the fact that, when the topological entropy of a continuous map on a compact metric space is finite, the pressure determines both its space of Borel invariant probability measures (cf. Theorem 9.11 of [27]) and the entropy function (cf. Theorem 9.12 of [27]), we define the metric entropy of a Borel probability measure on $X$ using the semigroup topological pressure, a concept presented in [9] that we will recall on Section 2. To make up for the absence of invariant measures, we summon the Gibbs plans explored in [23] in the context of a single dynamical system and a family of continuous observables, and thereby establish a variational principle for continuous potentials and discuss the existence of equilibrium states for expansive and mean expansive semigroup actions, as well as the existence and uniqueness of equilibrium states for Hölder potentials whenever the semigroup is generated by Ruelle-expanding maps. In particular, we show that, as happens with a single Ruelle-expanding transformation, the measure of maximal entropy of a finitely generated free semigroup action of Ruelle-expanding maps ties in with other dynamical behavior: it describes the distribution of periodic orbits; it is a fixed point of a natural transfer operator; and it may be computed as the weak$^*$ limit of the sequence of averages of Dirac measures supported on pre-images of any point by the semigroup action. Moreover, if the generators are $C^2-$expanding self maps of a compact connected Riemannian manifold $X$, the measure of maximal entropy of the semigroup action is the marginal on $X$ of the measure of maximal entropy of the skew product $\mathcal{F}_G$, whose projection on the shift $\Sigma_p^+$ is the Bernoulli measure determined by the probability vector $\left(\frac{\deg\, g_1}{\sum_{i=1}^p \deg g_i}, \frac{\deg\, g_2}{\sum_{i=1}^p \deg g_i}, \cdots, \frac{\deg\, g_p}{\sum_{i=1}^p \deg g_i}\right)$, where $\deg g_i$ stands for the degree of the map $g_i$.

This paper is organized as follows. Section 2 contains the main definitions and background information. In Section 3 we give precise statements of our results. Section 4 concerns the connection between the topological pressure of a semigroup action with respect to a continuous potential and the corresponding annealed pressure of the associated skew product, from which we deduce in Section 5 a generalization of Bufetov formula of [8]. Section 6 is devoted to a characterization of stationary measures in terms of the quenched pressure, resembling Theorem 9.11 of [27]. In Section 7 we introduce the notion of metric entropy of a finitely generated free semigroup action, justifying our choice with several relevant properties. The variational principle is proved in Sections 8 and 9. We finish the paper by reviewing the definition of metric entropy of a single dynamics proposed by Katok in [15] and extending its meaning to the context of semigroup actions.

## 2. Setting

2.1. **Finitely generated free semigroups.** Given $p \in \mathbb{N}$, a compact metric space $(X, D)$, a finite set of continuous maps $g_i : X \to X$, $i \in \{1, 2, \dots, p\}$, consider the finitely generated semigroup $(G, \circ)$ with the finite set of generators $G_1 = \{id, g_1, g_2, \dots, g_p\}$, where the semigroup operation $\circ$ is the composition of maps. Write $G = \bigcup_{n \in \mathbb{N}_0} G_n$, where $G_0 = \{id\}$ and $\underline{g} \in G_n$ if and only if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$, with $g_{i_j} \in G_1$ (for notational simplicity's sake we will use $g_j\, g_i$ instead of the composition $g_j \circ g_i$). We note that a semigroup may have multiple generating sets, and the dynamical or ergodic properties we will prove depend on the chosen generator set. In what follows, we will assume that the generator set $G_1$ is minimal, meaning that no function $g_j$, for $j = 1, \dots, p$, can be expressed as a composition of the remaining generators.

Set $G_1^* = G_1 \setminus \{id\}$ and, for every $n \geq 1$, let $G_n^*$ denote the space of concatenations of $n$ elements in $G_1^*$. To summon each element $\underline{g}$ of $G_n^*$, we will write $|\underline{g}| = n$ instead of $\underline{g} \in G_n^*$. In $G$, one consider the semigroup operation of concatenation defined as usual: if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$ and $\underline{h} = h_{i_m} \dots h_{i_2} h_{i_1}$, where $n = |\underline{g}|$ and $m = |\underline{h}|$, then

$$\underline{g}\, \underline{h} = g_{i_n} \dots g_{i_2} g_{i_1} h_{i_m} \dots h_{i_2} h_{i_1} \in G_{m+n}^*.$$

Each element $\underline{g}$ of $G_n$ may be seen as a word which originates from the concatenation of $n$ elements in $G_1$. Yet, different concatenations may generate the same element in $G$. Nevertheless, in most of the computations to be done, we shall consider different concatenations instead of the elements in $G$ they create. One way to interpret this statement is to consider the itinerary map

$$\begin{array}{cccc} \iota: & F_p & \to & G \\ & \underline{i} = i_n \dots i_1 & \mapsto & \underline{g}_{\underline{i}} := g_{i_n} \dots g_{i_1} \end{array}$$

where $\mathbb{F}_p$ is the free semigroup with $p$ generators, thus regarding concatenations on $G$ as images by $\iota$ of paths on $\mathbb{F}_p$.

2.2. **Free semigroup actions.** The finitely generated semigroup $G$ induces an *action* in $X$, say

$$\begin{array}{cccc} \mathbb{S}: & G \times X & \to & X \\ & (g, x) & \mapsto & g(x). \end{array}$$

We say that $\mathbb{S}$ is a *semigroup action* if, for any $\underline{g}$, $\underline{h} \in G$ and every $x \in X$, we have $\mathbb{S}(\underline{g}\,\underline{h}, x) = \mathbb{S}(\underline{g}, \mathbb{S}(\underline{h}, x))$. The action $\mathbb{S}$ is continuous if the map $\underline{g} : X \to X$ is continuous for any $\underline{g} \in G$.

We say that $x \in X$ is a *fixed point* of the action $\mathbb{S}$ if there exists $\underline{g} \in G_1^*$ such that $\underline{g}(x) = x$; the set of these fixed points will be denoted by $\mathrm{Fix}(\underline{g})$. More generally, $x \in X$ is said to be a *periodic point with period* $n \in \mathbb{N}$ of the action $\mathbb{S}$ if there exist $n \in \mathbb{N}$ and $\underline{g} \in G_n^*$ such that $\underline{g}(x) = x$. Write $\mathrm{Per}(G_n) = \bigcup_{|\underline{g}|=n} \mathrm{Fix}(\underline{g})$ for the set of all periodic points with period $n$. Accordingly, $\mathrm{Per}(G) = \bigcup_{n \geq 1} \mathrm{Per}(G_n)$ will stand for the set of periodic points of the whole semigroup action. We observe that, when $G_1^* = \{f\}$, these definitions coincide with the usual ones for the dynamical system $f$.

2.3. **Skew products.** The action of semigroups of dynamics has a strong connection with skew products which has been analyzed in order to obtain properties of semigroup actions by means of fibred and annealed quantities associated to the skew product dynamics (see for instance [9]). We recall that, if $X$ is a compact metric space and one considers a finite set of continuous maps $g_i : X \to X$, $i \in \{1, 2, \dots, p\}$, $p \geq 1$, we have defined a skew product dynamics

$$\begin{array}{cccc} \mathcal{F}_G: & \Sigma_p^+ \times X & \to & \Sigma_p^+ \times X \\ & (\omega, x) & \mapsto & (\sigma(\omega), g_{\omega_1}(x)) \end{array} \tag{2.1}$$

where $\omega = (\omega_1, \omega_2, \dots)$ is an element of the full unilateral space of sequences $\Sigma_p^+ = \{1, 2, \dots, p\}^{\mathbb{N}}$ and $\sigma$ denotes the shift map on $\Sigma_p^+$. For every $x \in X$ and $n \in \mathbb{N}$, we will write

$$\mathcal{F}_G^n(\omega, x) = (\sigma^n(\omega), g_\omega^n(x))$$

where

$$g^0_\omega(x) = x \qquad \text{and} \qquad g^n_\omega(x) = g_{\omega_n} \ldots g_{\omega_2}\, g_{\omega_1}(x).$$

We endow the space $\Sigma_p^+$ with the distance $d_\Sigma(\omega,\omega') := e^{-n}$, where $n = \inf\{k \geq 1 \colon \omega_k \neq \omega'_k\}$ if this set in nonempty; otherwise, $d_\Sigma(\omega,\omega') := 0$. In particular, $d_\Sigma(\omega,\omega') < \varepsilon$ if and only if $\omega_k = \omega'_k$ for every $1 \leq k \leq [-\log\varepsilon]$, and so the dynamical ball

$$B_{d_\Sigma}(\omega, m, \varepsilon) := \Big\{\omega' \in \Sigma_p^+ \colon d_\Sigma\left(\sigma^j(\omega), \sigma^j(\omega')\right) \leq \varepsilon,\ \forall\, 0 \leq j \leq m\Big\}$$

coincides with the cylinder $[\omega]_{m+[-\log\varepsilon]} := \{\omega' \in \Sigma_p^+ \colon \omega'_k = \omega_k,\ \forall\, 1 \leq k \leq m + [-\log\varepsilon]\}$.

2.4. **Random walks.** A *random walk on $G$* is determined by a Borel probability measure $\mathbb{P}$ on $\Sigma_p^+ = \{1,\ldots,p\}^{\mathbb{N}}$ which is invariant by the unilateral shift dynamics $\sigma : \Sigma_p^+ \to \Sigma_p^+$. More precisely, we take $\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots) \in \Sigma_p^+$ in the support of $\mathbb{P}$ and, along this path, we follow the dynamics of $(g^n_\omega(x))_{n\in\mathbb{N}_0}$ for every $x \in X$. For reasons that will be clearer later, we will be specially interested in Bernoulli random walks which are obtained as follows: we take a probability vector $\underline{a} = (a_1, a_2, \cdots, a_p)$, where $0 \leq a_i \leq 1$ for all $i \in \{1,\ldots,p\}$ and $\sum_{i=1}^p a_i = 1$ (which may be understood as a probability on the $\sigma-$algebra of all subsets of $\{1,2,\ldots,p\}$) and the corresponding Bernoulli probability measure $\eta_{\underline{a}} = \underline{a}^{\mathbb{N}}$ in the Borel subsets of $\Sigma_p^+$. For instance, if $\underline{a}(i) = \frac{1}{p}$ for any $i \in \{1,\ldots,p\}$, then $\eta_{\underline{p}}$ is the equally distributed symmetric Bernoulli probability measure on $\Sigma_p^+$.

2.5. **Ruelle-expanding maps.** We will focus on free semigroup actions of expanding maps, that is, finitely generated semigroups whose generators in $G_1^*$ are Ruelle-expanding maps, a notion we now recall.

*Definition* 2.1. Let $(X,D)$ be a compact metric space and $T : X \to X$ a continuous map. The dynamical system $(X,T)$ is *Ruelle-expanding* if $T$ is open and the following conditions are satisfied:

(1) There exists $c > 0$ such that, for all $x, y \in X$ with $x \neq y$, we have

$$T(x) = T(y) \quad \Rightarrow \quad D(x,y) > c.$$

(2) There are $r > 0$ and $0 < \rho < 1$ such that, for each $x \in X$ and all $a \in T^{-1}(\{x\})$ there exists a map $\varphi : B_r(x) \to X$, defined on the open ball $B_r(x)$ centered at $x$ with radius $r$, such that

$$\begin{aligned} \varphi(x) &= a \\ T \circ \varphi(z) &= z \\ D(\varphi(z), \varphi(w)) &\leq \rho\, D(z,w) \qquad \forall\, z,\, w \in B_r(x). \end{aligned}$$

Notice that, in particular, a Ruelle-expanding map is expansive. Thus, the corresponding entropy map (defined on the compact set of $T-$ invariant Borel probability measures endowed with the weak$^*$ topology and given by $\mu \mapsto h_\mu(T)$) is upper semi-continuous, which ensures the existence of an equilibrium state for every continuous potential (cf. [27, Theorem 9.13]).

In some instances of this work, we will assume that $G_1^*$ is either a subset of Ruelle-expanding maps defined on a compact metric space $X$ or is contained in the space of $C^2-$expanding endomorphisms acting on a compact connected Riemannian manifold $X$.

2.6. **Topological pressure of free semigroup actions.** Let $(X,D)$ be a compact metric space and denote by $C^0(X)$ the space of continuous maps $\varphi\colon X \to \mathbb{R}$.

Given $\varepsilon > 0$ and $\underline{g} := g_{i_n} \ldots g_{i_2}\, g_{i_1} \in G_n$, the dynamical ball $B(x, \underline{g}, \varepsilon)$ is the set

$$B(x,\underline{g},\varepsilon) := \Big\{y \in X : D(\underline{g}_j(y), \underline{g}_j(x)) \leq \varepsilon,\ \forall\, 0 \leq j \leq n\Big\}$$

where, for every $0 \leq j \leq n$, the notation $\underline{g}_j$ stands for the concatenation $g_{i_j} \ldots g_{i_2}\, g_{i_1} \in G_j$, and $\underline{g}_0 = id$. Observe that this is a classical ball with respect to the dynamical metric $D_{\underline{g}}$ defined by

$$D_{\underline{g}}(x,y) := \max_{0 \leq j \leq n} D(\underline{g}_j(x), \underline{g}_j(y)). \tag{2.2}$$

Notice that both the dynamical ball and the metric depend on the underlying concatenation of generators $g_{i_n} \dots g_{i_1}$ and not on the group element $\underline{g}$, since the latter may have distinct representations.

Given $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$, we say that a set $K \subset X$ is $(\underline{g}, n, \varepsilon)-$*separated* if $D_{\underline{g}}(x, y) > \varepsilon$ for any distinct $x, y \in K$. The maximal cardinality of a $(\underline{g}, \varepsilon, n)-$separated set on $X$ will be denoted by $s(\underline{g}, n, \varepsilon)$.

The *topological entropy of the semigroup action* $\mathbb{S}$ with respect to a fixed set of generators $G_1$ and a random walk $\mathbb{P}$ is given by

$$h_{\text{top}}(\mathbb{S}, \mathbb{P}) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{\Sigma_p^+} s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)\, d\mathbb{P}(\omega),$$

where $\omega = \omega_1 \omega_2 \cdots \omega_n \cdots$ and $s(g_{\omega_n} \dots g_{\omega_1}, n, \epsilon)$ is the maximum cardinality of $(\underline{g}, n, \varepsilon)-$separated sets. The topological entropy of a semigroup action estimates the average (with respect to a random walk $\mathbb{P}$) growth rate in $n$ of the number of orbits of length $n$, up to some small error $\varepsilon$.

*The topological entropy of the semigroup action* $\mathbb{S}$ is defined by

$$h_{\text{top}}(\mathbb{S}) = h_{\text{top}}(\mathbb{S}, \eta_{\underline{p}}).$$

More generally, given $\varphi \colon X \to \mathbb{R} \in C^0(X)$, the *topological pressure of the semigroup action* $\mathbb{S}$ with respect to a fixed set of generators $G_1$, a random walk $\mathbb{P}$ and the potential $\varphi$ is

$$P_{\text{top}}(\mathbb{S}, \varphi, \mathbb{P}) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{\Sigma_p^+} \sup_{E} \sum_{x \in E} e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j} \cdots g_{\omega_1}(x))}\, d\,\mathbb{P}(\omega)$$

where the supremum is taken over all sets $E = E_{\underline{g}, n, \varepsilon}$ that are $(\underline{g}, n, \varepsilon)-$separated. More information may be found in [9, 24].

2.7. **Annealed and quenched topological pressures.** Let $\pi_1 : \Sigma_p^+ \times X \to \Sigma_p^+$ and $\pi_2 : \Sigma_p^+ \times X \to X$ denote the natural projections on $\Sigma_p^+$ and $X$, respectively. Following [3], given a continuous potential $\psi : \Sigma_p^+ \times X \to \mathbb{R}$ and a nontrivial probability vector $\underline{a}$, the *annealed topological pressure of* $\mathcal{F}_G$ *with respect to* $\psi$ *and the Bernoulli probability measure* $\eta_{\underline{a}}$ is defined by

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) = \sup_{\{\mu \,:\, (\mathcal{F}_G)_*(\mu) = \mu\}} \Big\{ h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu)) + \int_{\Sigma_p^+ \times X} \psi \, d\mu \Big\} \tag{2.3}$$

where $\omega = (\omega_1, \omega_2, \dots) \in \Sigma_p^+$, $(\pi_1)_*(\mu)$ is the marginal of $\mu$ in $\Sigma_p^+$ (and is $\sigma-$invariant) and the *entropy per site* $h^{\underline{a}}((\pi_1)_*(\mu))$ with respect to $\eta_{\underline{a}}$ is given by

$$h^{\underline{a}}((\pi_1)_*(\mu)) = -\int_{\Sigma_p^+} \log \left( \mathcal{R}_{(\pi_1)_*(\mu)}(\omega) \right) \; d(\pi_1)_*(\mu)(\omega) \tag{2.4}$$

if $d(\pi_1)_*(\mu)(\omega_1, \omega_2, \dots) \ll d\underline{a}(\omega_1) \; d(\pi_1)_*(\mu)(\omega_2, \omega_3, \dots)$, where

$$\mathcal{R}_{(\pi_1)_*(\mu)} := \frac{d(\pi_1)_*(\mu)}{d\underline{a} \; d(\pi_1)_*(\mu) \circ \sigma}$$

denotes the Radon-Nykodin derivative of $(\pi_1)_*(\mu)$ with respect to $\underline{a} \times [(\pi_1)_*(\mu) \circ \sigma]$; otherwise, $h^{\underline{a}}((\pi_1)_*(\mu)) = -\infty$. Recall from [13] (or [3, page 676]) that

$$h^{\underline{a}}((\pi_1)_*(\mu)) = 0 \quad \Leftrightarrow \quad (\pi_1)_*(\mu) = \eta_{\underline{a}}.$$

According to [3, Equation (2.28)], the annealed pressure can also be evaluated by

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) = \sup_{\{\mu \,:\, (\mathcal{F}_G)_*(\mu) = \mu\}} \Big\{ h_\mu(\mathcal{F}_G) + \int_{\Sigma_p^+ \times X} \log \left( \underline{a}_{\,\omega_1} \, e^{\psi(\omega, x)} \right) d\mu(\omega, x) \Big\}. \tag{2.5}$$

The *quenched topological pressure of* $\mathcal{F}_G$ *with respect to* $\psi$ *and* $\underline{a}$ is defined by

$$P_{\text{top}}^{(q)}(\mathcal{F}_G, \psi, \underline{a}) = \sup_{\mu \,:\, (\mathcal{F}_G)_*(\mu) = \mu,\, (\pi_1)_*(\mu) = \eta_{\underline{a}}} \Big\{ h_\mu(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma) + \int_{\Sigma_p^+ \times X} \psi \, d\mu \Big\}. \tag{2.6}$$

Notice that $P^{(a)}_{\text{top}}(\mathcal{F}_G, \psi, \underline{a}) \geq P^{(q)}_{\text{top}}(\mathcal{F}_G, \psi, \underline{a})$ for every continuous potential $\psi$ and every $\underline{a}$.

An $\mathcal{F}_G-$invariant probability measure is said to be an *annealed equilibrium state for $\mathcal{F}_G$ with respect to $\psi$ and $\underline{a}$* if it attains the supremum in equation (2.3). Similar definition of *quenched equilibrium state for $\mathcal{F}_G$ with respect to $\psi$ and $\underline{a}$*, concerning the maximum at (2.6). It was shown in [3] applying functional analytic methods that, if each generator in $G_1^*$ is a $C^2-$expanding map on a Riemannian manifold $X$, then there are unique annealed and quenched equilibrium states for every Hölder potential $\psi : \Sigma_p^+ \times X \to \mathbb{R}$; the same results hold if each generator is a Ruelle-expanding map on a compact metric space (cf. [12]). Moreover, it was proved in [9, Section 8] that

$$h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) = P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a})$$

and, more generally, that, for any continuous potential $\varphi : X \to \mathbb{R}$, we have

$$P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) = P^{(a)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a}) \tag{2.7}$$

where $\psi_\varphi(\omega, x) = \varphi(x)$ for every $(\omega, x) \in \Sigma_p^+ \times X$. In Section 4 we will prove that (2.7) holds for a wider class of continuous finitely generated semigroup actions.

## 3. Main results

We start characterizing the stationary measures of a free finitely generated semigroup action, generalizing a similar description of invariant measures by a single dynamics given by Theorem 9.11 of [27].

**Theorem A.** *Assume that any element of $G_1^*$ is a continuous map acting freely on a compact metric space $X$, fix a random walk $\eta_{\underline{a}}$ in $\Sigma_p^+$, and assume $h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) < \infty$. Then a Borel probability measure $\nu$ on $X$ is $\eta_{\underline{a}}-$stationary if and only if*

$$P^{(q)}_{top}(\mathcal{F}_G, \psi_\varphi, \underline{a}) \geq \int_X \varphi \, d\nu \qquad \forall \varphi \in C^0(X).$$

Observe that, from (2.7), this result implies that for every $\eta_{\underline{a}}-$stationary Borel probability measure $\nu$ we have

$$P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) \geq \int_X \varphi \, d\nu \qquad \forall \varphi \in C^0(X).$$

This motivates the following definition. For a Borel probability measure $\nu$ in $X$ and a random walk $\eta_{\underline{a}}$ associated to some nontrivial probability vector $\underline{a}$, the *metric entropy of the semigroup action $\mathbb{S}$ with respect to $\nu$ and $\eta_{\underline{a}}$* is given by

$$h_\nu(\mathbb{S}, \eta_{\underline{a}}) = \inf_{\varphi \in C^0(X)} \left\{ P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) - \int_X \varphi \, d\nu \right\}. \tag{3.1}$$

Let $\Pi(\nu, \sigma)$ be the set of Borel $\mathcal{F}_G-$invariant probability measures $\mu$ on $\Sigma_p^+ \times X$ such that the marginal $(\pi_1)_*(\mu)$ of $\mu$ on $\Sigma_p^+$ is a $\sigma-$invariant probability measure and the marginal $(\pi_2)_*(\mu)$ of $\mu$ on $X$ is $\nu$. Notice that the existence of $\eta_{\underline{a}}-$stationary measures (cf. Subsection 6.2) guarantees that the set of measures $\nu$ such that $\Pi(\nu, \sigma) \neq \emptyset$ is nonempty.

We say that the semigroup action $\mathbb{S}$ is *expansive with respect to the random walk $\eta_{\underline{a}}$* if there exists a finite measurable partition $\beta$ on $X$ such that, for $\eta_{\underline{a}}-$almost every $\omega$, the diameter of the dynamical partition $\beta_0^{(n)}(\omega)$ tends to zero as $n \to \infty$ (see (7.1) for the definition of $\beta_0^{(n)}(\omega)$). Examples of expansive semigroup actions include, for instance, the ones whose generators in $G_1^*$ satisfy: (i) each such generator is Ruelle-expanding; (ii) $g_1$ is Ruelle-expanding and, for every $2 \leq i \leq p$, the map $g_i$ does not contract distances; (iii) all generators are everywhere expanding except for a finite number of indifferent fixed points.

**Theorem B.** *Assume that any element of $G_1^*$ is a continuous map acting on a compact metric space $X$, and fix a random walk $\eta_{\underline{a}}$ and a potential $\varphi \in C^0(X)$. If $h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) < \infty$ then*

$$P_{top}(\mathbb{S}, \varphi, \eta_{\underline{a}}) = \sup_{\{\nu \in \mathcal{M}(X)\,:\, \Pi(\nu,\sigma)\neq\emptyset\}} \Big\{ h_\nu(\mathbb{S}, \eta_{\underline{a}}) + \int \varphi \, d\nu \Big\}. \tag{3.2}$$

*In particular,*

$$h_{top}(\mathbb{S}, \eta_{\underline{a}}) = \sup_{\{\nu \in \mathcal{M}(X)\,:\, \Pi(\nu,\sigma)\neq\emptyset\}} h_\nu(\mathbb{S}, \eta_{\underline{a}}).$$

*If, in addition, any element of $G_1^*$ is Ruelle-expanding and $\underline{a} = \underline{p}$, then the previous supremum is only attained at the marginal on $X$ of the measure of maximal entropy of the skew product $\mathcal{F}_G$.*

Regarding the existence of equilibrium states, that is, measures where the supremum in (3.2) is attained, one could expect that, as in the classical setting, expansive and continuous semigroup actions admit equilibrium states. And, indeed, this is the case, as will be proved in Section 7. In particular, we find equilibrium states for the following class of smooth semigroup actions. We say that a semigroup action acting on a compact manifold $X$ is $C^1$ if its generators are $C^1$ maps. Given a random walk $\eta_{\underline{a}}$, a $C^1-$semigroup action is said to be *expanding on average* if

$$\int \log \|Dg_\omega^{-1}\|_\infty \, d\eta_{\underline{a}}(\omega) < 0. \tag{3.3}$$

The latter is a mean non-uniform expanding assumption for the semigroup action which, since $G_1$ is finite, defines an open condition with respect to the $C^1-$topology on the space of generators. Besides, it is not hard to check that (3.3) implies that the semigroup is expansive. Hence we conclude that every smooth, expanding on average action admits equilibrium states.

The variational principle just established may be formulated in terms of the annealed pressure of the skew-product $\mathcal{F}_G$, making explicit the intervention, in the concept of topological entropy of the semigroup action, of the complexity of both the skew-product and the shift, as well as the entropy per site with respect to the fixed random walk. Besides, the next result extends to the context of semigroup actions the notion of measure theoretical entropy for invariant measures with respect to a single dynamical system introduced by A. Katok in [15].

**Theorem C.** *Assume that any element of $G_1^*$ is a continuous map acting freely on a compact metric space $X$. Then:*

(1) $h_{top}(\mathbb{S}, \eta_{\underline{a}}) = \sup_{\{\nu \in \mathcal{M}(X)\,:\, \Pi(\sigma,\nu)\neq\emptyset\}} \sup_{\mu \in \Pi(\sigma,\nu)} \Big\{ h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu)) \Big\}$.

(2) *If, in addition, the semigroup action $\mathbb{S}$ is expansive and $\mathcal{F}_G$ has finite entropy,*

$$h_{top}(\mathbb{S}, \eta_{\underline{a}}) = \sup_{\{\nu \in \mathcal{M}(X)\,:\, \Pi(\sigma,\nu)\neq\emptyset\}} \sup_{\mu \in \Pi(\sigma,\nu)} \Big\{ \inf_{\psi \in C^0(\Sigma_p^+ \times X)} \Big[ P^{(a)}_{top}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi \, d\mu \Big] \Big\}.$$

(3) $h_{top}(\mathbb{S}, \eta_{\underline{a}}) = \sup_{\nu \in \mathcal{M}(X)} \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \int_{\Sigma_p^+} s_\nu(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, \delta) \, d\eta_{\underline{a}}$.

## 4. A characterization of the topological pressure of semigroup actions

In this section we prove that the equality (2.7) holds for every continuous finitely generated semigroup action on a compact metric space, where $P_{\text{top}}(\mathcal{F}_G, \psi, \underline{a})$ denotes the usual topological pressure defined by Bowen with respect to the metric $d_{\Sigma_p^+ \times X}((\omega, x), (\omega', x')) := \max\{d_\Sigma(\omega, \omega'), D(x, x')\}$ on $\Sigma_p^+ \times X$ and the continuous potential $\psi \colon \Sigma_p^+ \times X \to \mathbb{R}$.

**Proposition 4.1.** *Assume that any element of $G_1^*$ is a continuous map on a compact metric space $X$. Then $P_{top}(\mathbb{S}, \varphi, \eta_{\underline{a}}) = P^{(a)}_{top}(\mathcal{F}_G, \psi_\varphi, \underline{a})$ for every $\varphi \in C^0(X)$.*

*Proof.* Firstly, we recall that [3, equation (2.28)] ensures that $P^{(a)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a}) = P_{\text{top}}(\mathcal{F}_G, \phi_{\varphi,\underline{a}})$, where $\phi_{\varphi,\underline{a}} \colon \Sigma_p^+ \times X \to \mathbb{R}$ is given by

$$\phi_{\varphi,\underline{a}}(\omega, x) := \log\left(\underline{a}_{\omega_1}\, e^{\psi_\varphi(\omega,x)}\right)$$

and $P_{\text{top}}(\mathcal{F}_G, \phi_{\varphi,\underline{a}})$ stands for the topological pressure of $\mathcal{F}_G$ with respect to the potential $\phi_{\varphi,\underline{a}}$. We claim that $P_{\text{top}}(\mathcal{F}_G, \phi_{\varphi,\underline{a}}) = P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}})$.

We start observing that

$$\begin{aligned} P_{\text{top}}(\mathcal{F}_G, \phi_{\varphi,\underline{a}}) &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_F \sum_{(\omega,x)\in F} e^{\sum_{j=0}^{n-1} \phi_{\varphi,\underline{a}}(\mathcal{F}_G^j(\omega,x))} \\ &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_F \sum_{(\omega,x)\in F} \prod_{j=0}^{n-1} \underline{a}_{\omega_{j+1}}\, e^{\varphi(g_{\omega_j}\cdots g_{\omega_1}(x))} \\ &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_F \sum_{(\omega,x)\in F} \eta_{\underline{a}}([\omega]_n)\, e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j}\cdots g_{\omega_1}(x))} \qquad (4.1) \end{aligned}$$

where the supremum is taken over all maximal $(n,\varepsilon)$−separated sets $F$ on $\Sigma_p^+ \times X$. Consequently, if $E_1 \subset \Sigma_p^+$ is a maximal $(n,\varepsilon)$−separated set and one considers, for every $\omega \in E_1$, a maximal $(\underline{g}_\omega, n, \varepsilon)$−separated set $E_2(\omega) \subset X$ where the supremum

$$\sup_B \sum_{x\in B} e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j}\cdots g_{\omega_1}(x))}$$

is attained, then the set $F = \bigcup_{\omega\in E_1} \{\omega\} \times E_2(\omega)$ is $(n,\varepsilon)$−separated by $\mathcal{F}_G$ (recall the definition of the metric $d_{\Sigma_p^+\times X}$). Noticing that any $\omega \in E_1$ determines a unique cylinder of length $n + [-\log\varepsilon]$ and that, as $E_1$ is an $(n,\varepsilon)$−maximal separated set, the union of these cylinders covers $\Sigma_p^+$, we conclude that

$$\begin{aligned} \sum_{(\omega,x)\in F} \eta_{\underline{a}}([\omega]_n)\, e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j}\cdots g_{\omega_1}(x))} &= \sum_{\omega\in E_1} \sum_{x\in E_2(\omega)} \eta_{\underline{a}}([\omega]_n)\, e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j}\cdots g_{\omega_1}(x))} \\ &= \sum_{\omega\in E_1} \eta_{\underline{a}}([\omega]_n) \sup_B \sum_{x\in B} e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j}\cdots g_{\omega_1}(x))} \\ &= p^{[-\log\varepsilon]} \sum_{[\omega]_n} \eta_{\underline{a}}([\omega]_n) \sup_B \sum_{x\in B} e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j}\cdots g_{\omega_1}(x))} \\ &= p^{[-\log\varepsilon]} \int_{\Sigma_p^+} \sup_B \sum_{x\in B} e^{\sum_{j=0}^{n-1} \varphi(g_{\omega_j}\cdots g_{\omega_1}(x))}\, d\eta_{\underline{a}}(\omega). \end{aligned}$$

Consequently, taking into account (4.1), we obtain $P_{\text{top}}(\mathcal{F}_G, \phi_{\varphi,\underline{a}}) \geq P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}})$.

Reciprocally, let $F \subset \Sigma_p^+ \times X$ a maximal $(n,\varepsilon)$−separated set where the supremum in (4.1) is attained, and set $E := \pi_1(F)$. It is immediate that $E \subset \Sigma_p^+$ is an $(n, 2\varepsilon)$−generating set. Moreover, for every $(\omega, x) \in F$, the set $\pi_1(B((\omega,x), n, \varepsilon))$ is contained in an $(n + [-\log\varepsilon])$−cylinder in $\Sigma_p^+$.

Therefore,

$$\begin{aligned}\sum_{(\omega,x)\in F}\eta_{\underline{a}}([\omega]_n)\,e^{\sum_{j=0}^{n-1}\varphi(g_{\omega_j}\cdots g_{\omega_1}(x))} &= \sum_{[\omega']_n}\ \sum_{\{(\omega,x)\in F:\,\omega\in[\omega']_n\}}\eta_{\underline{a}}([\omega]_n)\,e^{\sum_{j=0}^{n-1}\varphi(g_{\omega_j}\cdots g_{\omega_1}(x))}\\ &= \sum_{[\omega']_n}\eta_{\underline{a}}([\omega']_n)\sum_{\{(\omega,x)\in F:\,\omega\in[\omega']_n\}} e^{\sum_{j=0}^{n-1}\varphi(g_{\omega'_j}\cdots g_{\omega'_1}(x))}\\ &\le \sum_{[\omega']_n}\eta_{\underline{a}}([\omega']_n)\,p^{[-\log\varepsilon]}\sup_B\sum_{x\in B}e^{\sum_{j=0}^{n-1}\varphi(g_{\omega'_j}\cdots g_{\omega'_1}(x))}\\ &= p^{[-\log\varepsilon]}\int_{\Sigma_p^+}\sup_B\sum_{x\in B}e^{\sum_{j=0}^{n-1}\varphi(g_{\omega_j}\cdots g_{\omega_1}(x))}\,d\eta_{\underline{a}}(\omega),\end{aligned}$$

which proves the claim and finishes the proof of the proposition. □

## 5. A more general Bufetov formula

Assume that any element of $G_1^*$ is a continuous map acting freely on a compact metric space $X$, and fix a random walk $\eta_{\underline{a}}$. Given a continuous map $\varphi\colon X\to\mathbb{R}$, the equalities

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) = \sup_{\{\mu:\ (\mathcal{F}_G)_*(\mu)=\mu\}}\left\{h_\mu(\mathcal{F}_G)+\int_{\Sigma_p^+\times X}\log\left(\underline{a}_{\omega_1}\,e^{\psi_\varphi(\omega,x)}\right)d\mu(\omega,x)\right\} = P_{\text{top}}(\mathcal{F}_G,\phi_{\varphi,\underline{a}}) \tag{5.1}$$

due to (2.5) and Proposition 4.1, may be seen as a more general version of Bufetov formula [8]. Indeed, when $\underline{a}=\underline{p}$ and $\varphi\equiv 0$, it becomes

$$h_{\text{top}}(\mathbb{S}) = h_{\text{top}}(\mathcal{F}_G)-\log p. \tag{5.2}$$

The formula (5.1) also generalizes the statement of Theorem 1.1 of [20]: when $\underline{a}=\underline{p}$, it simplifies to

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{p}}) = P_{\text{top}}(\mathcal{F}_G,\psi_\varphi)-\log p.$$

Suppose, in addition, that any element of $G_1^*$ is a Ruelle-expanding map. Then the skew product $\mathcal{F}_G$ is also Ruelle-expanding (cf. [9]), and so it is expansive (cf. Section 2.5) and has an equilibrium state $\mu_{\varphi,\underline{a}}$ with respect to the potential

$$\phi_{\varphi,\underline{a}}:(\omega,x)\quad\mapsto\quad\log\left(\underline{a}_{\omega_1}\,e^{\psi_\varphi(\omega,x)}\right).$$

Thus,

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) = h_{\mu_{\varphi,\underline{a}}}(\mathcal{F}_G)+\int_{\Sigma_p^+\times X}\log\left(\underline{a}_{\omega_1}\,e^{\psi_\varphi(\omega,x)}\right)d\mu_{\varphi,\underline{a}}(\omega,x). \tag{5.3}$$

In particular, if $\phi\equiv 0$, we get

$$h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) = h_{\mu_{\underline{a}}}(\mathcal{F}_G)+\int_{\Sigma_p^+\times X}\log\left(\underline{a}_{\omega_1}\right)d\mu_{\underline{a}}(\omega,x) \tag{5.4}$$

where $\mu_{\underline{a}}$ is the (unique) equilibrium state of $\mathcal{F}_G$ with respect to the Hölder potential

$$(\omega,x)\quad\mapsto\quad\log\left(\underline{a}_{\omega_1}\right).$$

In the case $\underline{a}=\underline{p}$, the formula (5.3) may be rewritten as

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{p}}) = h_{\mu_{\varphi,\underline{p}}}(\mathcal{F}_G)-\log p+\int_{\Sigma_p^+\times X}\psi_\varphi\,d\mu_{\varphi,\underline{p}}$$

which, taking into account that

$$\int_{\Sigma_p^+\times X}\psi_\varphi\,d\mu_{\varphi,\underline{p}} = \int_X\varphi\,d(\pi_2)_*(\mu_{\varphi,\underline{p}}),$$

becomes

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{p}}) = h_{\mu_{\varphi,\underline{p}}}(\mathcal{F}_G) - \log p + \int_X \varphi \, d(\pi_2)_*(\mu_{\varphi,\underline{p}}). \tag{5.5}$$

Finally, if $\phi \equiv 0$ in (5.5), then it simplifies to

$$h_{\text{top}}(\mathbb{S},\eta_{\underline{p}}) = h_{\mu_{\underline{p}}}(\mathcal{F}_G) - \log p \tag{5.6}$$

and we restore the original Bufetov formula (5.2) since $\mu_{\underline{p}}$, the equilibrium state of $\mathcal{F}_G$ with respect to the potential $(\omega,x) \mapsto -\log p$, is in fact the measure $\mu^{(\underline{p})}$ of maximal entropy of $\mathcal{F}_G$.

## 6. A characterization of stationary measures

This section is devoted to prove the necessary and sufficient condition stated in Theorem A for a Borel probability measure on $X$ to be $\mathbb{P}-$stationary with respect to a fixed random walk $\mathbb{P}$.

6.1. **$\mathcal{F}_G-$invariant measures.** In what follows assume that $\mathbb{S}$ is a finitely generated semigroup action by continuous maps on a compact metric space $X$ and that $h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) < \infty$. We will adapt the proof of Theorem 9.11 of [27], valid for invariant probability measures by a single dynamics, to the context of semigroup actions and the quenched and annealed topological pressures.

**Lemma 6.1.** *Assume that $h_{top}(\mathbb{S},\eta_{\underline{a}}) < \infty$ and consider a Borel measure $\mu$ on $\Sigma_p^+ \times X$.*

(1) *If $\mu$ is an $\mathcal{F}_G-$invariant probability measure, then*

$$P^{(q)}_{top}(\mathcal{F}_G,\psi,\underline{a}) \geq \int_{\Sigma_p^+\times X} \psi \, d\mu \qquad \forall \psi \in C^0(\Sigma_p^+ \times X).$$

(2) *If*

$$P^{(a)}_{top}(\mathcal{F}_G,\psi,\underline{a}) \geq \int_{\Sigma_p^+\times X} \psi \, d\mu \qquad \forall \psi \in C^0(\Sigma_p^+ \times X)$$

*then $\mu$ is an $\mathcal{F}_G-$invariant probability measure.*

*Proof.* Consider an $\mathcal{F}_G-$invariant probability measure $\mu$. As

$$h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) = h_{\mathcal{F}_G}(\mu \mid (\pi_1)_*(\mu)) \geq 0$$

(cf. [18] or [3]), it follows from (2.6) that

$$P^{(q)}_{\text{top}}(\mathcal{F}_G,\psi,\underline{a}) \geq \int_{\Sigma_p^+\times X} \psi \, d\mu \qquad \forall \psi \in C^0(\Sigma_p^+ \times X).$$

Conversely, take a Borel measure $\mu$ on $\Sigma_p^+ \times X$ and assume that $P^{(a)}_{\text{top}}(\mathcal{F}_G,\psi,\underline{a}) \geq \int_{\Sigma_p^+\times X} \psi \, d\mu$ for every $\psi \in C^0(\Sigma_p^+ \times X)$. Firstly, recall that $P^{(a)}_{\text{top}}(\mathcal{F}_G,0,\underline{a}) = h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) < \infty$ and let us show that $\mu$ is a probability. Given $\psi \geq 0$ in $C^0(\Sigma_p^+ \times X)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\Sigma_p^+\times X} n\,\psi \, d\mu &= -\int -n\,\psi\,d\mu \geq -P^{(a)}_{\text{top}}(\mathcal{F}_G,-n\psi,\underline{a}) \\ &\geq -[P^{(a)}_{\text{top}}(\mathcal{F}_G,0,\underline{a}) + \max_X\,(-n\,\psi)] \\ &= n \min_{\Sigma_p^+\times X} \psi - P^{(a)}_{\text{top}}(\mathcal{F}_G,0,\underline{a}) \end{aligned}$$

which is positive for large $n$. Consequently, $\mu$ is a positive functional on $C^0(\Sigma_p^+ \times X)$. On the other hand, for every $n \in \mathbb{N}$,

$$n\,\mu(\Sigma_p^+ \times X) = \int_{\Sigma_p^+\times X} n \, d\mu \leq P^{(a)}_{\text{top}}(\mathcal{F}_G,n,\underline{a}) = P^{(a)}_{\text{top}}(\mathcal{F}_G,0,\underline{a}) + n$$

and

$$n\,\mu(\Sigma_p^+ \times X) = -\int_{\Sigma_p^+ \times X} -n\,d\mu \geq -P^{(a)}_{\text{top}}(\mathcal{F}_G, -n, \underline{a}) = n - P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}).$$

This proves that $\mu(\Sigma_p^+ \times X) = 1$. The $\mathcal{F}_G-$invariance condition follows from (2.5) if we take into account that, for every $\psi$ in $C^0(\Sigma_p^+ \times X)$ and $n \geq 0$,

$$-P^{(a)}_{\text{top}}(\mathcal{F}_G, n(\psi \circ \mathcal{F}_G - \psi), \underline{a}) \leq \int_{\Sigma_p^+ \times X} n(\psi - \psi \circ \mathcal{F}_G)\,d\mu \leq P^{(a)}_{\text{top}}(\mathcal{F}_G, n(\psi - \psi \circ \mathcal{F}_G), \underline{a})$$

and the fact that $P^{(a)}_{\text{top}}(\mathcal{F}_G, n(\psi - \psi \circ \mathcal{F}_G), \underline{a}) = P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \underline{a}) < \infty$ is independent of $n$. □

6.2. **Stationary measures.** Given a random walk $\mathbb{P}$, a Borel probability measure $\nu$ in $X$ is said to be $\mathbb{P}-$*stationary* if, for every measurable set $A$, we have

$$\nu(A) = \int_{\Sigma_p^+} \nu(g_\omega^{-1}(A))\,d\mathbb{P}(\omega).$$

This is equivalent to say that, for every continuous map $\varphi : X \to \mathbb{R}$,

$$\int_X \varphi(x)\,d\nu(x) = \int_{\Sigma_p^+} \left( \int_X \varphi(g_\omega(x))\,d\nu(x) \right) d\,\mathbb{P}(\omega).$$

As $X$ is a compact metric space, the set of $\mathbb{P}-$stationary probability measures is nonempty for every $\mathbb{P}$ (see [17, Lemma I.2.2]). Indeed, a $\mathbb{P}-$stationary probability measure may be obtained as an accumulation point in the space $\mathcal{M}(X)$ of Borel probabilities in $X$, endowed with the weak* topology, of the sequence of probabilities defined by

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} (\mathcal{P}^*)^j\,\delta_x$$

where $x$ is any point in $X$, $\mathcal{P} : C^0(X) \to C^0(X)$ is the operator

$$\mathcal{P}(\varphi)(x) = \int_{\Sigma_p^+} \varphi(g_\omega(x))\,d\,\mathbb{P}(\omega)$$

and, for every probability measure $\gamma \in \mathcal{M}(X)$ and every measurable set $A$ with corresponding indicator map $\chi_A$, we have

$$\mathcal{P}^*\gamma(A) = \int_X \mathcal{P}(\chi_A)\,d\gamma(x) = \int_X \int_{\Sigma_p^+} \chi_A(g_\omega(x))\,d\,\mathbb{P}(\omega)\,d\gamma(x).$$

The set of $\mathbb{P}-$stationary probability measures is a compact convex set with respect to the weak* topology. Its extreme points are called ergodic. When convenient, we will use the following criterium (cf. [17, Lemma I.2.3] and [19, Proposition 1.4.2]):

(1) $\nu$ is $\mathbb{P}-$stationary is and only if the product probability measure $\mu = \mathbb{P} \times \nu$ is $\mathcal{F}_G-$invariant.
(2) $\nu$ is $\mathbb{P}-$stationary and ergodic is and only if the product probability measure $\mu = \mathbb{P} \times \nu$ is $\mathcal{F}_G-$invariant and ergodic.

6.3. **Proof of Theorem A.** Assume that $\nu$ is $\eta_{\underline{a}}-$stationary, or, equivalently, that $\mu = \eta_{\underline{a}} \times \nu$ is $\mathcal{F}_G-$invariant. From Lemma 6.1, we know that, for every $\psi \in C^0(\Sigma_p^+ \times X)$, we have $P^{(q)}_{\text{top}}(\mathcal{F}_G, \psi, \underline{a}) \geq \int_{\Sigma_p^+ \times X} \psi\,d\mu$. Therefore, given $\varphi \in C^0(X)$, if we take the map

$$\begin{array}{rcl} \psi_\varphi : \Sigma_p^+ \times X & \to & \mathbb{R} \\ (\omega, x) & \mapsto & \varphi(x). \end{array}$$

which belongs to $C^0(\Sigma_p^+ \times X)$ and, as $\mu = \eta_{\underline{a}} \times \nu$, satisfies

$$\int_{\Sigma_p^+ \times X} \psi_\varphi \, d\mu = \int_X \varphi \, d\nu$$

then we conclude that

$$P^{(q)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a}) \geq \int_{\Sigma_p^+ \times X} \psi_\varphi \, d\mu = \int_X \varphi \, d\nu.$$

In particular, recalling that $P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) = P^{(a)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a})$ (cf. Proposition 4.1) and $P^{(a)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a}) \geq P^{(q)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a})$, we deduce that, if $\nu$ is $\eta_{\underline{a}}-$stationary, then

$$P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) \geq \int_X \varphi \, d\nu \qquad \forall \varphi \in C^0(X).$$

Conversely, suppose that, for every $\varphi \in C^0(X)$, we have $P^{(q)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a}) \geq \int_X \varphi \, d\nu$. Consider the probability $\mu = \eta_{\underline{a}} \times \nu$ and associate to a map $\psi \in C^0(\Sigma_p^+ \times X)$ the function

$$\begin{aligned} \varphi_\psi : X &\to \mathbb{R} \\ x &\mapsto \int_{\Sigma_p^+} \psi(\omega, x) \, d\eta_{\underline{a}}(\omega). \end{aligned}$$

Then $\varphi_\psi \in C^0(X)$ and, as $\mu = \eta_{\underline{a}} \times \nu$,

$$\int_{\Sigma_p^+ \times X} \psi \, d\mu = \int_{\Sigma_p^+ \times X} \psi(\omega, x) \, d\eta_{\underline{a}}(\omega) \, d\nu(x) = \int_X \left( \int_{\Sigma_p^+} \psi(\omega, x) \, d\eta_{\underline{a}}(\omega) \right) d\nu(x) = \int_X \varphi_\psi \, d\nu.$$

Therefore, using that $\int \psi \, d\tilde{\mu} = \int \varphi_\psi \, d\tilde{\mu}$ for every $\mathcal{F}_G-$invariant probability measure $\tilde{\mu}$, we show that

$$P^{(q)}_{\text{top}}(\mathcal{F}_G, \psi, \underline{a}) = P^{(q)}_{\text{top}}(\mathcal{F}_G, \psi_{\varphi_\psi}, \underline{a}) \geq \int_X \varphi_\psi \, d\nu = \int_{\Sigma_p^+ \times X} \psi \, d\mu$$

which confirms, by Lemma 6.1, that $\mu$ is $\mathcal{F}_G-$invariant. So $\nu$ is $\eta_{\underline{a}}-$stationary.

## 7. Measure theoretical entropy of a semigroup action

In this section we will introduce the concept of metric entropy of a semigroup action and discuss some equivalent variants of this notion.

7.1. **Relative metric entropy of a semigroup action.** In this section we will recall the notion of relative metric entropy proposed by Abramov and Rokhlin in [1], which was later adopted in [18] and [17] and whose connection with the topological entropy of the semigroup action was explored in [10].

Let $\mathbb{P}$ be a $\sigma-$invariant probability measure and $\nu$ a probability measure invariant by any generator in $G_1^*$. Given a measurable finite partition $\beta$ of $X$, $n \in \mathbb{N}$ and $\omega = \omega_1\omega_2 \cdots \in \Sigma_p^+$, define

$$\begin{aligned} \beta_1^n(\omega) &= g_{\omega_1}^{-1}\beta \bigvee g_{\omega_1}^{-1}g_{\omega_2}^{-1}\beta \bigvee \cdots \bigvee g_{\omega_1}^{-1}g_{\omega_2}^{-1}\cdots g_{\omega_{n-1}}^{-1}\beta \\ \beta_0^n(\omega) &= \beta \bigvee \beta_1^n(\omega) \quad \text{and} \quad \beta_1^\infty(\omega) = \bigvee_{n=1}^{\infty} \beta_1^n(\omega). \end{aligned} \tag{7.1}$$

Then the conditional entropy of $\beta$ relative to $\beta_1^\infty(\omega)$, denoted by $H_\nu(\beta|\beta_1^\infty(\omega))$, is a measurable function of $\omega$ and $\mathbb{P}-$integrable. Set $h_\nu(\mathbb{S}, \mathbb{P}, \beta) = \int_{\Sigma_p^+} H_\nu(\beta|\beta_1^\infty(\omega)) \, d\mathbb{P}(\omega)$. Then

$$h_\nu(\mathbb{S}, \mathbb{P}, \beta) = \lim_{n \to +\infty} \frac{1}{n} \int_{\Sigma_p^+} H_\nu(\beta_0^n(\omega)) \, d\mathbb{P}(\omega). \tag{7.2}$$

where $H_\nu(\beta_0^n(\omega))$ is the entropy of the partition $\beta_0^n(\omega)$. According to [1], we may define the metric entropy of $\nu$ as follows.

*Definition* 7.1. The *relative metric entropy of the semigroup action* $\mathbb{S}$ *with respect to* $\mathbb{P}$ *and* $\nu$ is given by

$$h_\nu^{(1)}(\mathbb{S},\mathbb{P}) = \sup_\beta h_\nu(\mathbb{S},\mathbb{P},\beta).$$

For instance, if $\mathbb{P}$ is a Dirac measure $\delta_{\underline{j}}$ supported on a fixed point $\underline{j} = jj\cdots$, where $j \in \{1,\cdots,p\}$, then $h_\nu^{(1)}(\mathbb{S},\delta_{\underline{j}}) = h_\nu(g_j)$. If, instead, $\mathbb{P}$ is the symmetric random walk, that is, the Bernoulli $(\frac{1}{p},\cdots,\frac{1}{p})-$product probability measure $\eta_{\underline{p}}$, then

$$h_\nu^{(1)}(\mathbb{S},\eta_{\underline{p}}) = \sup_\beta \lim_{n\to+\infty} \frac{1}{n}\left(\frac{1}{p^n}\sum_{|\omega|=n} H_\nu(\beta_0^n(\omega))\right).$$

Denote by $\mathcal{M}_G$ the set of Borel probability measures on $X$ invariant by $g_i$, for all $i \in \{1,\cdots,p\}$. In [10], it was shown that

$$\sup_{\nu\in\mathcal{M}_G} h_\nu^{(1)}(\mathbb{S},\mathbb{P}) \le h_{\text{top}}(\mathbb{S}) + (\log p - h_{\mathbb{P}}(\sigma)).$$

Therefore, when $\mathbb{P} = \eta_{\underline{p}}$, we obtain $h_{\eta_{\underline{p}}}(\sigma) = \log p$ and

$$\sup_{\nu\in\mathcal{M}_G} h_\nu^{(1)}(\mathbb{S},\mathbb{P}_{\underline{p}}) \le h_{\text{top}}(\mathbb{S}).$$

This inequality may, however, be strict, as exemplified in [10, Example 5.3]. In the next section we will introduce a more appropriate notion of metric entropy of a semigroup action.

7.2. **Main definition.** The characterization of stationary measures obtained in Subsection 6.2 motivates the following definition of measure theoretical entropy of a semigroup action.

*Definition* 7.2. Given a Borel probability measure $\nu$ in $X$ and a random walk $\eta_{\underline{a}}$ associated to some nontrivial probability vector $\underline{a}$, the *metric entropy of the semigroup action* $\mathbb{S}$ *with respect to* $\nu$ *and* $\eta_{\underline{a}}$ is

$$h_\nu(\mathbb{S},\eta_{\underline{a}}) = \inf_{\varphi\in C^0(X)} \Big\{P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) - \int_X \varphi\,d\nu\Big\}. \tag{7.3}$$

*Remark* 7.3. Observe that, if $\underline{a} = \underline{p}$, then, for all $\varphi \in C^0(X)$,

$$\begin{aligned} P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) &= P_{\text{top}}^{(a)}(\mathcal{F}_G,\psi_\varphi,\eta_{\underline{a}}) \\ &= \sup_{\{\mu:\,(\mathcal{F}_G)_*(\mu)=\mu\}} \left\{h_\mu(\mathcal{F}_G) + \int_{\Sigma_p^+\times X} \log\big[\underline{a}_{\omega_1}\, e^{\psi_\varphi(\omega,x)}\big]\,d\mu(\omega,x)\right\} \\ &= -\log p + \sup_{\{\mu:\,(\mathcal{F}_G)_*(\mu)=\mu\}} \left\{h_\mu(\mathcal{F}_G) + \int_X \varphi(x)\,d(\pi_2)_*(\mu)(x)\right\} \\ &\ge -\log p + \int_X \varphi\,d\nu \end{aligned}$$

for every probability measure $\nu \in \mathcal{M}(X)$ which is a marginal on $X$ of some $\mathcal{F}_G-$invariant probability measure $\mu$. Therefore, under these assumptions,

$$h_\nu(\mathbb{S},\eta_{\underline{p}}) \ge -\log p.$$

*Remark* 7.4. From Subsection 6.3, we conclude that

$$\nu \text{ is } \eta_{\underline{a}}-\text{stationary} \quad\Rightarrow\quad h_\nu(\mathbb{S},\eta_{\underline{a}}) \ge 0.$$

Thus, as the set of $\eta_{\underline{a}}-$stationary probability measures is nonempty (see Subsection 6.2), we obtain

$$\sup_{\nu\in\mathcal{M}(X)} h_\nu(\mathbb{S},\eta_{\underline{a}}) \ge 0.$$

7.3. **Intervention of the annealed pressure.** For a Borel probability measure $\nu$ on $X$, let $\Pi(\nu,\sigma)$ the space of the Borel $\mathcal{F}_G-$invariant probability measures $\mu$ on $\Sigma_p^+ \times X$ such that the marginal $(\pi_1)_*(\mu)$ of $\mu$ on $\Sigma_p^+$ is a $\sigma-$invariant probability measure and the marginal $(\pi_2)_*(\mu)$ of $\mu$ on $X$ is the fixed probability $\nu$.

*Definition* 7.5. Given a random walk $\eta_{\underline{a}}$ associated to some nontrivial probability vector $\underline{a}$ and a Borel probability measure $\nu$ on $X$,

$$h_\nu^{(2)}(\mathbb{S},\eta_{\underline{a}}) = \begin{cases} \sup_{\mu\,\in\,\Pi(\sigma,\nu)} \left\{h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu))\right\} & \text{if } \Pi(\sigma,\nu)\neq\emptyset; \\ 0 & \text{otherwise.}\end{cases}$$

Notice that, if $\nu$ is $\eta_{\underline{a}}-$stationary, then $\mu = \eta_{\underline{a}} \times \nu$ is $\mathcal{F}_G-$invariant, so $\mu$ belongs to $\Pi(\sigma,\nu)$. Therefore, in this case we obtain

$$h_\nu^{(2)}(\mathbb{S},\eta_{\underline{a}}) \geq h_{\eta_{\underline{a}}\times\nu}(\mathcal{F}_G) \; - h_{\eta_{\underline{a}}}(\sigma).$$

If, additionally, $\nu$ is invariant by all the generators of $G$, then

$$h_{\eta_{\underline{a}}\times\nu}(\mathcal{F}_G) - h_{\eta_{\underline{a}}}(\sigma) = h_\nu^{(1)}(\mathbb{S},\eta_{\underline{a}})$$

(cf. [1]), and so

$$h_\nu^{(2)}(\mathbb{S},\eta_{\underline{a}}) \geq h_\nu^{(1)}(\mathbb{S},\eta_{\underline{a}}).$$

7.3.1. **Proof of Theorem C(1).** By Proposition 4.1, we have $h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) = P_{\text{top}}^{(a)}(\mathcal{F}_G,0,\eta_{\underline{a}})$, hence it is immediate that

$$\begin{aligned} h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) &= P_{\text{top}}^{(a)}(\mathcal{F}_G,0,\eta_{\underline{a}}) \\ &= \sup_{\{\mu\,:\;(\mathcal{F}_G)_*(\mu)=\mu\}} \left\{h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu))\right\} \\ &= \sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}} \; \sup_{\mu\,\in\,\Pi(\sigma,\nu)} \left\{h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu))\right\} \\ &= \sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}} h_\nu^{(2)}(\mathbb{S},\eta_{\underline{a}}). \end{aligned}$$

*Remark* 7.6. Assume, in addition, that any element of $G_1^*$ is Ruelle-expanding and take the (unique) annealed equilibrium state $\mu^{(\underline{a})}$, whose existence has been referred in Section 5 (and whose properties were studied in [3] for actions of $C^2-$expanding endomorphisms and in [12] for actions of Ruelle-expanding maps). It satisfies the equalities

$$h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) = P_{\text{top}}^{(a)}(\mathcal{F}_G,0,\eta_{\underline{a}}) = h_{\mu^{(\underline{a})}}(\mathcal{F}_G) - h_{(\pi_1)_*(\mu^{(\underline{a})})}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu^{(\underline{a})})).$$

Define $\nu^{(\underline{a})} = (\pi_2)_*(\mu^{(\underline{a})})$. It may happen that $\nu^{(\underline{a})}$ is not $\eta_{\underline{a}}-$stationary, but obviously its set $\Pi(\sigma,\nu^{(\underline{a})})$ is nonempty. Moreover,

$$\begin{aligned} \sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}} h_\nu^{(2)}(\mathbb{S},\eta_{\underline{a}}) &\geq h_{\nu^{(\underline{a})}}^{(2)}(\mathbb{S},\eta_{\underline{a}}) \\ &\geq h_{\mu^{(\underline{a})}}(\mathcal{F}_G) - h_{(\pi_1)_*(\mu^{(\underline{a})})}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu^{(\underline{a})})) \\ &= P_{\text{top}}^{(a)}(\mathcal{F}_G,0,\eta_{\underline{a}}) \\ &= h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}). \end{aligned}$$

Therefore,

$$\sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}} h_\nu^{(2)}(\mathbb{S},\eta_{\underline{a}}) = h_{\nu^{(\underline{a})}}^{(2)}(\mathbb{S},\eta_{\underline{a}}) = h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}).$$

7.3.2. **Example**. Suppose that any element of $G_1^*$ is a $C^2-$expanding endomorphism acting freely on a compact connected Riemannian manifold $X$, and fix a random walk $\eta_{\underline{a}}$. Recall from [9] that, if $\underline{a} = \underline{p}$, then the measure of maximal entropy of $\mathcal{F}_G$ is the (unique) equilibrium state $\mu^{(\underline{p})}$ of the annealed topological pressure with respect to the map $\psi \equiv 0$ and the random walk $\eta_{\underline{p}}$. Apart from the equalities $h_{\text{top}}(\mathcal{F}_G) = h_{\mu^{(\underline{p})}}(\mathcal{F}_G) = \log\left(\sum_{i=1}^p \deg g_i\right)$, where $\deg g_i$ stands for the degree of the map $g_i$ (well defined since $g_i$ belongs to $End^2(X)$), the measure $\mu^{(\underline{p})}$ also satisfies

$$(\pi_1)_*(\mu^{(\underline{p})}) = \eta_{\underline{m}}$$

where

$$\underline{m} = \left(\frac{\deg\, g_1}{\sum_{i=1}^p \deg g_i}, \frac{\deg\, g_2}{\sum_{i=1}^p \deg g_i}, \cdots, \frac{\deg\, g_p}{\sum_{i=1}^p \deg g_i}\right).$$

Besides,

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \eta_{\underline{p}}) = h_{\text{top}}(\mathbb{S}, \eta_{\underline{p}}) = h_{\text{top}}(\mathcal{F}_G) - \log p = h_{\mu^{(\underline{p})}}(\mathcal{F}_G) - h_{(\pi_1)_*(\mu^{(\underline{p})})}(\sigma) + h^{\eta_{\underline{p}}}((\pi_1)_*(\mu^{(\underline{p})})).$$

Moreover, the marginal $\nu^{(\underline{p})} = (\pi_2)_*(\mu^{(\underline{p})})$ on $X$ is precisely the measure constructed in [7] as a weak$^*$ limit of the sequence of measures obtained by averaging the pre-images of any point $x \in X$ according to the random walk $\eta_{\underline{p}}$, namely

$$\frac{1}{\lambda^n} \int \sum_{g_{\omega_n} \cdots g_{\omega_1}(y) = x} \delta_y \; d\eta_{\underline{p}}(\omega_1, \ldots, \omega_n) \tag{7.4}$$

where $\lambda = \deg(g_1) + \ldots + \deg(g_p)$. See [26] for more details. Equally relevant is the fact that $\nu^{(\underline{p})}$ describes the distribution of the periodic orbits of the semigroup action, as shown in [9, Section 9]. These properties generalize to the context of free semigroup actions of $C^2-$expanding maps the analogous information concerning the measure of maximal entropy of a single Ruelle-expanding map (cf. [25]).

7.4. **The role of the entropy per site.** Given a Borel probability measure $\nu \in \mathcal{M}(X)$, the space of Borel probability measures $\mu$ on $\Sigma_p^+ \times X$ such that the marginal $(\pi_1)_*(\mu)$ of $\mu$ on $\Sigma_p^+$ is a $\sigma-$invariant probability measure and the marginal $(\pi_2)_*(\mu)$ of $\mu$ on $X$ is the fixed probability $\nu$ is nonempty since it contains the product measure $\eta_{\underline{a}} \times \nu$. However, this measure may not be $\mathcal{F}_G-$invariant.

*Definition* 7.7. Given a random walk $\eta_{\underline{a}}$ and a probability measure $\nu \in \mathcal{M}(X)$, define

$$h_\nu^{(3)}(\mathbb{S}, \eta_{\underline{a}}) = \begin{cases} \sup_{\mu \in \Pi(\sigma,\nu)} \left\{\inf_{\psi \in C^0(\Sigma_p^+ \times X)} \left[P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi \, d\mu\right]\right\} & \text{if } \Pi(\sigma,\nu) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Lemma 6.1(2) that, if $\mu$ has a $\sigma-$invariant marginal $(\pi_1)_*(\mu)$ and $(\pi_2)_*(\mu) = \nu$ but is not $\mathcal{F}_G-$invariant, then there exists $\psi \in C^0(\Sigma_p^+ \times X)$ such that $P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) < \int_{\Sigma_p^+ \times X} \psi \, d\mu$. On the other hand, Lemma 6.1(1) implies that if such a $\mu$ is $\mathcal{F}_G-$invariant, then $P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi \, d\mu \geq 0$ for every $\psi \in C^0(\Sigma_p^+ \times X)$. Consequently, if the set $\Pi(\sigma,\nu)$ is nonempty (which happens if $\nu$ is $\eta_{\underline{a}}-$stationary, but this is not a necessary condition), then the supremum in the previous definition is well defined and $h_\nu^{(3)}(\mathbb{S}, \eta_{\underline{a}}) \geq 0$.

For an $\mathcal{F}_G-$invariant probability measure $\mu \in \Pi(\sigma,\nu)$, define

$$H(\mu) := h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu)).$$

**Lemma 7.1.** *If $\mathbb{S}$ is an expansive semigroup action, then $H$ is upper semi-continuous.*

*Proof.* According to (2.5), one may write

$$H(\mu) = h_\mu(\mathcal{F}_G) + \int_{\Sigma_p^+} \log\left(\underline{a}_{\omega_1}\right) d(\pi_1)_*(\mu)(\omega).$$

Notice now that the first term varies upper semi-continuously with $\mu$, due to the fact that $\mathcal{F}_G$ is an expansive map ([27, Theorem 9.13]), and the second term is a continuous function in $\mu$. ☐

The next result generalizes Theorem 9.12 of [27]. Denote by $\mathcal{M}_{\mathcal{F}_G}$ the set of Borel $\mathcal{F}_G-$invariant probability measures on $\Sigma_p^+ \times X$.

**Lemma 7.2.** *Assume that $\mathcal{F}_G$ has finite topological entropy and let $\mu$ be in $\mathcal{M}_{\mathcal{F}_G}$. Then the following assertions are equivalent:*

(1) *$H$ is upper semi-continuous at $\mu$.*
(2) $H(\mu) = \inf_{\psi \in C^0(\Sigma_p^+ \times X)} \left[P_{top}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi\, d\mu\right]$.

*Proof.* We will pursue an argument similar to the one in [27, pages 222–223]; it is included for the reader's convenience.

Let $\mu$ be in $\mathcal{M}_{\mathcal{F}_G}$ and assume that it satisfies the equality

$$H(\mu) = \inf_{\psi \in C^0(\Sigma_p^+ \times X)} \left[P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi\, d\mu\right].$$

Given $\varepsilon > 0$, choose $\psi \in C^0(\Sigma_p^+ \times X)$ such that

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi\, d\mu < H(\mu) + \frac{\varepsilon}{2}$$

and consider the neighborhood

$$V_\mu\left(\psi; \frac{\varepsilon}{2}\right) = \left\{\xi \in \mathcal{M}_{\mathcal{F}_G} \colon \left|\int_{\Sigma_p^+ \times X} \psi\, d\xi - \int_{\Sigma_p^+ \times X} \psi\, d\mu\right| < \frac{\varepsilon}{2}\right\}.$$

Take $\xi \in V_\mu(\psi; \frac{\varepsilon}{2})$; then, by (2.3) and the fact that $H(\xi) \geq 0$ for every $\xi \in \mathcal{M}_{\mathcal{F}_G}$,

$$H(\xi) \leq P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi\, d\xi < P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi\, d\mu + \frac{\varepsilon}{2} < H(\mu) + \varepsilon.$$

Therefore $H$ is upper semi-continuous at $\mu$.

From the definition (2.3), it is immediate that

$$H(\mu) \leq \inf_{\psi \in C^0(\Sigma_p^+ \times X)} \left\{P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi\, d\mu\right\}.$$

We observe that this inequality implies, with no assumption neither on the upper semi-continuity of $H$ nor on the finiteness of the topological entropy of $\mathcal{F}_G$, that for every probability measure $\nu \in \mathcal{M}(X)$, we have

$$h_\nu^{(2)}(\mathbb{S}, \eta_{\underline{a}}) \leq h_\nu^{(3)}(\mathbb{S}, \eta_{\underline{a}}). \tag{7.5}$$

Assume now that $H$ is upper semi-continuous at $\mu$ and let us prove the opposite inequality. As $\mathcal{F}_G$ has finite topological entropy, we may take $c > H(\mu)$ and consider the set

$$\mathcal{C} = \{(\xi, t) \in \mathcal{M}_{\mathcal{F}_G} \times \mathbb{R} \colon 0 \leq t \leq H(\xi)\}.$$

From Theorem 8.1 of [27], the linear properties of the map $(\pi_1)_*$ and the behavior with $\mu$ of the entropy per site $h^{\underline{a}}((\pi_1)_*(\mu))$ (cf. (2.5)), we deduce that $\mathcal{C}$ is a convex set. Moreover, using the weak* topology in the dual space $(C^0(\Sigma_p^+ \times X))^*$, we may consider $\mathcal{C}$ as a subset of the product $(C^0(\Sigma_p^+ \times X))^* \times \mathbb{R}$ with the induced topology. Then, by the upper semi-continuity of $H$ at $\mu$, the point $(\mu, c)$ does not belong the closure of $\mathcal{C}$, say $\overline{\mathcal{C}}$. As the two convex closed sets $\overline{\mathcal{C}}$ and $\{(\mu, c)\}$ are disjoint, by the

Hahn-Banach separation theorem there is a continuous linear functional $F : (C^0(\Sigma_p^+ \times X))^* \times \mathbb{R} \to \mathbb{R}$ such that

$$F(\xi, t) < F(\mu, t) \qquad \forall (\xi, t) \in \overline{\mathcal{C}} \tag{7.6}$$

As we are considering the weak* topology in the dual space $(C^0(\Sigma_p^+ \times X))^*$, there exist some $\psi_0 \in C^0(\Sigma_p^+ \times X)$ and some $\theta \in \mathbb{R}$ such that the functional $F$ is given by

$$F(\xi, t) = \int_{\Sigma_p^+ \times X} \psi_0 \, d\xi + t\,\theta$$

Therefore, (7.6) implies that

$$\int_{\Sigma_p^+ \times X} \psi_0 \, d\xi + t\,\theta < \int_{\Sigma_p^+ \times X} \psi_0 \, d\mu + c\,\theta \qquad \forall (\xi, t) \in \overline{\mathcal{C}}.$$

In particular,

$$\int_{\Sigma_p^+ \times X} \psi_0 \, d\xi + H(\xi)\,\theta < \int_{\Sigma_p^+ \times X} \psi_0 \, d\mu + c\,\theta \qquad \forall (\xi, t) \in \overline{\mathcal{C}}. \tag{7.7}$$

Taking $\xi = \mu$, we get $H(\mu)\,\theta < c\,\theta$, so we must have $\theta > 0$. Hence, we may rewrite (7.7) as

$$\int_{\Sigma_p^+ \times X} \frac{\psi_0}{\theta} \, d\xi + H(\xi) < \int_{\Sigma_p^+ \times X} \frac{\psi_0}{\theta} \, d\mu + c \qquad \forall \xi \in \mathcal{M}_{\mathcal{F}_G}.$$

Consequently, as

$$P^{(a)}_{\mathrm{top}}(\mathcal{F}_G, \frac{\psi_0}{\theta}, \underline{a}) = \sup_{\xi \in \mathcal{M}_{\mathcal{F}_G}} \left\{ H(\xi) + \int_{\Sigma_p^+ \times X} \frac{\psi_0}{\theta} \, d\xi \right\}$$

we conclude that

$$P^{(a)}_{\mathrm{top}}(\mathcal{F}_G, \frac{\psi_0}{\theta}, \underline{a}) \le c + \int_{\Sigma_p^+ \times X} \frac{\psi_0}{\theta} \, d\mu.$$

Rearranging, we obtain

$$\begin{aligned} c \;&\ge\; P^{(a)}_{\mathrm{top}}(\mathcal{F}_G, \frac{\psi_0}{\theta}, \underline{a}) - \int_{\Sigma_p^+ \times X} \frac{\psi_0}{\theta} \, d\mu \\ &\ge\; \inf_{\psi \in C^0(\Sigma_p^+ \times X)} \left\{ P^{(a)}_{\mathrm{top}}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi \, d\mu \right\}. \end{aligned}$$

As $c$ is arbitrary in $]H(\mu), +\infty[$, we finally deduce that

$$H(\mu) \ge \inf_{\psi \in C^0(\Sigma_p^+ \times X)} \left\{ P^{(a)}_{\mathrm{top}}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi \, d\mu \right\}.$$

□

7.4.1. **Proof of Theorem C(2)**. Assume that $\mathbb{S}$ is an expansive semigroup action and that $\mathcal{F}_G$ has finite topological entropy. Take a random walk $\eta_{\underline{a}}$ and a probability measure $\nu$ on $X$ such that $\Pi(\sigma, \nu) \neq \emptyset$. Applying Lemmas 7.1 and 7.2 we deduce that

$$h^{(3)}_\nu(\mathbb{S}, \eta_{\underline{a}}) = h^{(2)}_\nu(\mathbb{S}, \eta_{\underline{a}}). \tag{7.8}$$

Hence, it is straightforward from Theorem C(1) that

$$h_{\mathrm{top}}(\mathbb{S}, \eta_{\underline{a}}) = \sup_{\{\nu \in \mathcal{M}(X)\,:\; \Pi(\sigma,\nu) \neq \emptyset\}} h^{(3)}_\nu(\mathbb{S}, \eta_{\underline{a}}).$$

## 8. Variational principle for the topological entropy

In this section we will start proving Theorem B. Assume that any element of $G_1^*$ is a continuous map acting freely on a compact metric space $X$, that the semigroup action $\mathbb{S}$ is expansive and that $\mathcal{F}_G$ has finite entropy (which happens, for instance, if any element of $G_1^*$ is Ruelle-expanding). Associate to each $\varphi \in C^0(X)$ the map $\psi_\varphi \in C^0(\Sigma_p^+ \times X)$ given by $\psi_\varphi(\omega, x) = \varphi(x)$. Then

$$P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi_\varphi, \underline{a}) = P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) \qquad \text{and} \qquad \int_{\Sigma_p^+ \times X} \psi_\varphi \, d\mu = \int_X \varphi \, d(\pi_2)_*(\mu).$$

Therefore, given $\nu \in \mathcal{M}(X)$ such that $\Pi(\sigma, \nu) \neq \emptyset$ and $\mu \in \Pi(\sigma, \nu)$, we get

$$\inf_{\psi \in C^0(\Sigma_p^+ \times X)} \Big[ P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi \, d\mu \Big] \leq \inf_{\varphi \in C^0(X)} \Big[ P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) - \int_X \varphi \, d\nu \Big].$$

So, from the proof of Theorem C(2) and the existence of $\eta_{\underline{a}}-$stationary measures, we conclude that

$$\begin{aligned} h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) &= \sup_{\nu \in \mathcal{M}(X)} \ \sup_{\mu \in \Pi(\sigma,\nu)} \ \inf_{\psi \in C^0(\Sigma_p^+ \times X)} \Big[ P_{\text{top}}^{(a)}(\mathcal{F}_G, \psi, \underline{a}) - \int_{\Sigma_p^+ \times X} \psi \, d\mu \Big] \\ &\leq \sup_{\{\nu \in \mathcal{M}(X) \,:\, \Pi(\sigma,\nu) \neq \emptyset\}} \ \inf_{\varphi \in C^0(X)} \Big[ P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) - \int_X \varphi \, d\nu \Big] \\ &= \sup_{\{\nu \in \mathcal{M}(X) \,:\, \Pi(\sigma,\nu) \neq \emptyset\}} h_\nu(\mathbb{S}, \eta_{\underline{a}}). \end{aligned}$$

We now prove the opposite inequality. Set

$$\alpha_1 = \sup_{\{\nu \in \mathcal{M}(X) \,:\, \Pi(\sigma,\nu) \neq \emptyset\}} \ \sup_{\mu \in \Pi(\sigma,\nu)} \Big[ h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu)) \Big] \tag{8.1}$$

$$\alpha_2 = \sup_{\{\nu \in \mathcal{M}(X) \,:\, \Pi(\sigma,\nu) \neq \emptyset\}} \ \inf_{\varphi \in C^0(X)} \Big[ P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) - \int_X \varphi \, d\nu \Big]. \tag{8.2}$$

Recall from Theorem C(1) that $\alpha_1 = h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}})$. Assume that $\alpha_1 < \alpha_2$ and consider $b$ in $]\alpha_1, \alpha_2[$. As $b < \alpha_2$, there is $\nu_b \in \mathcal{M}(X)$ such that $\Pi(\sigma, \nu_b) \neq \emptyset$ and

$$b < \inf_{\varphi \in C^0(X)} \Big[ P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) - \int_X \varphi \, d\nu_b \Big].$$

Hence,

$$b < P_{\text{top}}(\mathbb{S}, \varphi, \eta_{\underline{a}}) - \int_X \varphi \, d\nu_b \qquad \forall \, \varphi \in C^0(X). \tag{8.3}$$

In particular, when $\varphi \equiv 0$, we get

$$b < P_{\text{top}}(\mathbb{S}, 0, \eta_{\underline{a}}). \tag{8.4}$$

As $\alpha_1 < b$, for every $\nu \in \mathcal{M}(X)$ such that $\Pi(\sigma, \nu) \neq \emptyset$ one has

$$\sup_{\mu \in \Pi(\sigma,\nu)} \Big[ h_\mu(\mathcal{F}_G) - h_{(\pi_1)_*(\mu)}(\sigma) + h^{\underline{a}}((\pi_1)_*(\mu)) \Big] < b < P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \eta_{\underline{a}}),$$

which contradicts the definition of annealed topological pressure. This contradiction shows that $\alpha_1 = \alpha_2$, which implies that

$$h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) = \sup_{\{\nu \in \mathcal{M}(X) \,:\, \Pi(\sigma,\nu) \neq \emptyset\}} h_\nu(\mathbb{S}, \eta_{\underline{a}}). \tag{8.5}$$

In view of [9], we have the following consequence:

**Corollary 8.1.** *Assume that all elements of $G_1^*$ are Ruelle-expanding. The probability measure $\nu^{(\underline{a})} = (\pi_2)_*(\mu^{(\underline{a})})$, where $\mu^{(\underline{a})}$ is the annealed equilibrium state for $\mathcal{F}_G$ with respect to $\psi \equiv 0$ and $\eta_{\underline{a}}$, is the unique measure of maximal entropy of the free semigroup action when the fixed random walk is $\eta_{\underline{a}}$.*

*Proof.* By the proof of Theorem C(1), $\nu^{(\underline{a})}$ is a measure of maximal entropy for $\mathbb{S}$. On the other hand, taking into account (8.1), for a measure $\nu$ to maximize the operator

$$\sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}}\;\sup_{\mu\,\in\,\Pi(\sigma,\nu)}\;\Big[h_\mu(\mathcal{F}_G)-h_{(\pi_1)_*(\mu)}(\sigma)+h^{\underline{a}}((\pi_1)_*(\mu))\Big]$$

it must be the marginal on $X$ of the unique annealed equilibrium state $\mu^{(\underline{a})}$ of $\mathcal{F}_G$ (which is Ruelle-expanding and topologically mixing as well; cf. [9, Lemma 7.1]) with respect to $\psi\equiv 0$ and $\underline{a}$. So, $\nu=\nu^{(\underline{a})}=(\pi_2)_*(\mu^{(\underline{a})})$. □

## 9. A more general variational principle

Assume that any element of $G_1^*$ is a continuous map acting on a compact metric space $X$, and fix a random walk $\eta_{\underline{a}}$. Recall from Definition 7.2 that, for every Borel probability $\nu$ in $X$ we have

$$h_\nu(\mathbb{S},\eta_{\underline{a}})=\inf_{\varphi\,\in\,C^0(X)}\Big\{P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}})-\int_X\varphi\;d\nu\Big\}.$$

Therefore, for every continuous potential $\varphi\colon X\to\mathbb{R}$,

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}})\geq h_\nu(\mathbb{S},\eta_{\underline{a}})+\int_X\varphi\;d\nu$$

and so

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}})\geq\sup_{\nu\,\in\,\mathcal{M}(X)}\Big\{h_\nu(\mathbb{S},\eta_{\underline{a}})+\int_X\varphi\;d\nu\Big\}.$$

Conversely, if each element of $G_1^*$ is a continuous map, then

$$\begin{aligned}
&P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}})=P^{(a)}_{\text{top}}(\mathcal{F}_G,\psi_\varphi,\eta_{\underline{a}})\\
=&\sup_{\{\mu\,:\;(\mathcal{F}_G)_*(\mu)=\mu\}}\Big\{h_\mu(\mathcal{F}_G)-h_{(\pi_1)_*(\mu)}(\sigma)+h^{\underline{a}}((\pi_1)_*(\mu))+\int_{\Sigma_p^+\times X}\psi_\varphi\,d\mu\Big\}\\
=&\sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}}\;\sup_{\mu\,\in\,\Pi(\sigma,\nu)}\Big\{h_\mu(\mathcal{F}_G)-h_{(\pi_1)_*(\mu)}(\sigma)+h^{\underline{a}}((\pi_1)_*(\mu))+\int_X\varphi\,d\nu\Big\}\\
=&\sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}}\Big\{\Big[\sup_{\mu\,\in\,\Pi(\sigma,\nu)}\Big\{h_\mu(\mathcal{F}_G)-h_{(\pi_1)_*(\mu)}(\sigma)+h^{\underline{a}}((\pi_1)_*(\mu))\Big\}\Big]+\int_X\varphi\,d\nu\Big\}\\
=&\sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}}\Big\{h^{(2)}_\nu(\mathbb{S},\eta_{\underline{a}})+\int_X\varphi\,d\nu\Big\}\\
\leq&\sup_{\{\nu\,\in\,\mathcal{M}(X)\,:\;\Pi(\sigma,\nu)\neq\emptyset\}}\Big\{h^{(3)}_\nu(\mathbb{S},\eta_{\underline{a}})+\int_X\varphi\,d\nu\Big\}
\end{aligned}$$

where the last inequality is a consequence of (7.5).

**Claim 1.** *For every $\nu\in\mathcal{M}(X)$ such that $\Pi(\sigma,\nu)\neq\emptyset$, we have $h^{(3)}_\nu(\mathbb{S},\eta_{\underline{a}})\leq h_\nu(\mathbb{S},\eta_{\underline{a}})$.*

*Proof.* Given $\mu\in\Pi(\sigma,\nu)$, by Proposition 4.1 we deduce that

$$\begin{aligned}
\inf_{\psi\,\in\,C^0(\Sigma_p^+\times X)}\Big\{P^{(a)}_{\text{top}}(\mathcal{F}_G,\psi,\eta_{\underline{a}})-\int_X\psi\;d\mu\Big\}\;&\leq\;\inf_{\varphi\,\in\,C^0(X)}\Big\{P^{(a)}_{\text{top}}(\mathcal{F}_G,\psi_\varphi,\eta_{\underline{a}})-\int_X\psi_\varphi\;d\mu\Big\}\\
&=\;\inf_{\varphi\,\in\,C^0(X)}\Big\{P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}})-\int_X\varphi\;d\nu\Big\}.
\end{aligned}$$

Therefore,

$$\sup_{\mu\,\in\,\Pi(\sigma,\nu)}\;\inf_{\psi\,\in\,C^0(\Sigma_p^+\times X)}\Big\{P^{(a)}_{\text{top}}(\mathcal{F}_G,\psi,\eta_{\underline{a}})-\int_X\psi\;d\mu\Big\}\leq\inf_{\varphi\,\in\,C^0(X)}\Big\{P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}})-\int_X\varphi\;d\nu\Big\}.$$

□

From Claim 1 we conclude that

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) = \sup_{\{\nu\in\mathcal{M}(X):\,\Pi(\sigma,\nu)\neq\emptyset\}} \Big\{h^{(3)}_\nu(\mathbb{S},\eta_{\underline{a}}) + \int_X \varphi d\nu\Big\} \leq \sup_{\{\nu\in\mathcal{M}(X):\,\Pi(\sigma,\nu)\neq\emptyset\}} \Big\{h_\nu(\mathbb{S},\eta_{\underline{a}}) + \int_X \varphi d\nu\Big\}.$$

This ends the proof of Theorem B.

*Remark* 9.1. The previous result ensures that

$$\begin{aligned} P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) &= \sup_{\{\nu\in\mathcal{M}(X):\,\Pi(\sigma,\nu)\neq\emptyset\}} \Big\{h^{(2)}_\nu(\mathbb{S},\eta_{\underline{a}}) + \int_X \varphi d\nu\Big\} \\ &= \sup_{\{\nu\in\mathcal{M}(X):\,\Pi(\sigma,\nu)\neq\emptyset\}} \Big\{h^{(3)}_\nu(\mathbb{S},\eta_{\underline{a}}) + \int_X \varphi d\nu\Big\} \\ &= \sup_{\{\nu\in\mathcal{M}(X):\,\Pi(\sigma,\nu)\neq\emptyset\}} \Big\{h_\nu(\mathbb{S},\eta_{\underline{a}}) + \int_X \varphi d\nu\Big\}. \end{aligned}$$

Therefore, assuming, in addition, that the action $\mathbb{S}$ is expansive, an argument similar to the one presented in Remark 7.6 shows that the supremum is attained at the marginal $\nu^{(\underline{a})}_\phi = (\pi_2)_*(\mu^{(\underline{a})}_{\psi_\phi})$ on $X$ of the annealed equilibrium state $\mu^{(\underline{a})}_{\psi_\phi}$ of $\mathcal{F}_G$ with respect to the potential $\psi_\phi$.

## 10. Invariant measures for the semigroup action

In [17], stationary measures stand for the invariant measures by the action. This is reasonable, in the setting of group actions, every marginal on $X$ of a Borel probability measure invariant by the skew product is $\eta-$stationary for some shift-invariant probability measure $\eta$ (see [2, Theorems 1.7.2 and 2.1.8] or [22]). However, the previous results attest that the relevant measures for semigroup actions are indeed the marginals on $X$ of invariant measures by the corresponding skew product, although they may be non-stationary (cf. [7]).

The variational principle (cf. Theorem B) gives an intrinsic necessary condition for this property to be valid in the context of continuous, free, finitely generated semigroup actions of continuous maps. More precisely, a Borel probability measure $\nu$ in $X$ is the marginal of a Borel probability measure invariant by the skew product $\mathcal{F}_G$ only if

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) \geq \int_X \varphi\, d\nu \qquad \forall\, \varphi\in C^0(X,\mathbb{R}).$$

In fact, consider $\varphi\in C^0(X)$, the corresponding map $\psi_\varphi \in C^0(\Sigma_p^+\times X)$ given by $\psi_\varphi(\omega,x)=\varphi(x)$, and let $\nu$ be a Borel probability measure in $X$ such that $\Pi(\sigma,\nu)\neq\emptyset$. Take a Borel probability measure $\mu$ invariant by the skew product $\mathcal{F}_G$ and such that $(\pi_2)_*(\mu)=\nu$. Then, by Lemma 6.1(1),

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) = P^{(a)}_{\text{top}}(\mathcal{F}_G,\psi_\varphi,\underline{a}) \geq \int_{\Sigma_p^+\times X} \psi_\varphi\, d\mu = \int_X \varphi\, d(\pi_2)_*(\mu).$$

This assertion suggests to consider a Borel probability measure $\nu$ on $X$ to be *invariant by the free semigroup action* $\mathbb{S}$ if and only if

$$P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) \geq \int_X \varphi\, d\nu \qquad \forall\, \varphi\in C^0(X). \tag{10.1}$$

As $P_{\text{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) = P^{(a)}_{\text{top}}(\mathcal{F}_G,\varphi,\underline{a})$, we abbreviate the content of Theorem A and the previous definition saying that, within free semigroup actions of continuous maps:

(1) A Borel probability measure $\nu$ on $X$ is $\eta_{\underline{a}}-$stationary if and only if

$$P^{(q)}_{\text{top}}(\mathcal{F}_G,\varphi,\underline{a}) \geq \int_X \varphi\, d\nu \quad \forall\, \varphi\in C^0(X).$$

(2) A Borel probability measure $\nu$ on $X$ is $\mathbb{S}-$invariant if and only if

$$P^{(a)}_{\text{top}}(\mathcal{F}_G, \psi_\varphi, \underline{a}) \geq \int_X \varphi \, d\nu \qquad \forall\, \varphi \in C^0(X).$$

10.1. **Ergodic optimization.** Let $\mathcal{M}_{\mathbb{S}}(X)$ denote the space of Borel probability measures invariant by $\mathbb{S}$. A question that arises naturally is related with ergodic optimization (we refer the reader to [4] for a survey on this subject and to a vast list of references therein). This concerns the description of $\mathbb{S}-$*maximizing measures* for a given potential $\varphi \in C^0(X)$, that is, the set of measures $\nu_\varphi \in \mathcal{M}_{\mathbb{S}}(X)$ such that

$$\int_X \varphi \, d\nu_\varphi = \sup_{\nu \in \mathcal{M}_{\mathbb{S}}(X)} \Big\{ \int_X \varphi \, d\nu \Big\}. \tag{10.2}$$

We note that maximizing measures always exist by the continuity of the map $\nu \mapsto \int \varphi \, d\nu$ and the compactness of the space $\mathcal{M}_{\mathbb{S}}(X)$. Moreover, it is natural to maximize the right-hand side of (10.2) instead of $\sup_{\nu \in \mathcal{M}(X)}\{\int \varphi \, d\nu\}$ or $\sup_{\{\nu \,:\, \nu(X)=1\}}\{\int \varphi \, d\nu\}$, as the first could generate measures that are not natural (e.g. with negative entropy) and the second is attained at all measures supported in $\varphi^{-1}(\{\max \varphi\})$. Finally, we observe that the set of maximizing measures can be large, as happens whenever $\varphi$ is constant.

Given a potential $\varphi \in C^0(X)$, if any element of $G_1^*$ is Ruelle-expanding then there exists an equilibrium state $\nu_\beta$ with respect to the potential $\beta\varphi$ (cf. Corollary 8.1 and Remark 9.1), where $\beta > 0$ represents the inverse of the temperature in the thermodynamic formalism. Using this information, we will construct special $\mathbb{S}-$maximizing measures for $\varphi$.

**Proposition 10.1.** *Assume that any element of $G_1^*$ is a Ruelle-expanding map acting on a compact metric space $X$, and fix a random walk $\eta_{\underline{a}}$. If $h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) < \infty$, then, given $\varphi \in C^0(X)$, any accumulation point $\nu_0$ in the weak$^*$ topology of $(\nu_\beta)_{\beta>0}$ as $\beta \to +\infty$ is an $\mathbb{S}-$maximizing measure for $\varphi$ and satisfies*

$$h_{\nu_0}(\mathbb{S}, \eta_{\underline{a}}) \geq h_\nu(\mathbb{S}, \eta_{\underline{a}}) \qquad \forall\, \mathbb{S} - \textit{maximizing measure } \nu.$$

*Proof.* It is clear from the definition (10.1) that

$$\frac{1}{\beta} P_{\text{top}}(\mathbb{S}, \beta\varphi, \eta_{\underline{a}}) \geq \sup_{\nu \in \mathcal{M}_{\mathbb{S}}(X)} \int_X \varphi \, d\nu.$$

Conversely, by the variational principle (cf. Theorem B) and the existence of equilibrium states (cf. Remark 9.1), which are in $\mathcal{M}_{\mathbb{S}}(X)$, we also know that, for every $\beta > 0$,

$$\begin{aligned} P_{\text{top}}(\mathbb{S}, \beta\varphi, \eta_{\underline{a}}) &= \sup_{\{\nu \in \mathcal{M}(X)\,:\, \Pi(\nu,\sigma)\neq\emptyset\}} \Big\{ h_\nu(\mathbb{S}, \eta_{\underline{a}}) + \int_X \beta\varphi \, d\nu \Big\} \\ &= \sup_{\nu \in \mathcal{M}_{\mathbb{S}}(X)} \Big\{ h_\nu(\mathbb{S}, \eta_{\underline{a}}) + \int_X \beta\varphi \, d\nu \Big\} \\ &\leq h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) + \beta \sup_{\nu \in \mathcal{M}_{\mathbb{S}}(X)} \int_X \varphi \, d\nu. \end{aligned}$$

Consequently,

$$\limsup_{\beta \to +\infty} \frac{1}{\beta} P_{\text{top}}(\mathbb{S}, \beta\varphi, \eta_{\underline{a}}) \leq \sup_{\nu \in \mathcal{M}_{\mathbb{S}}(X)} \int_X \varphi \, d\nu$$

since, as $h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) < \infty$, we have

$$\lim_{\beta \to +\infty} \frac{1}{\beta} h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) = 0.$$

Therefore, if $\nu_\beta$ is the equilibrium state for $\mathbb{S}$ with respect to $\beta\varphi$ and $\nu_0$ is an accumulation point of $(\nu_\beta)_\beta$, then $\nu_0$ is an $\mathbb{S}-$maximizing measure: indeed, as

$$\frac{1}{\beta}\,P_{\mathrm{top}}(\mathbb{S},\beta\varphi,\eta_{\underline{a}}) = \frac{1}{\beta}\,h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}}) + \int_X \varphi\, d\nu_\beta \tag{10.3}$$

and

$$0 \le \frac{1}{\beta}\,h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}}) \le \frac{1}{\beta}\,h_{\mathrm{top}}(\mathbb{S},\eta_{\underline{a}})$$

taking $\beta \to +\infty$ in (10.3) we obtain

$$\lim_{\beta\to+\infty}\frac{1}{\beta}\,P_{\mathrm{top}}(\mathbb{S},\beta\varphi,\eta_{\underline{a}}) = \sup_{\nu\in\mathcal{M}_{\mathbb{S}}(X)}\int_X \varphi\,d\nu \qquad \text{and} \qquad \sup_{\nu\in\mathcal{M}_{\mathbb{S}}(X)}\int_X \varphi\,d\nu = \int_X \varphi\,d\nu_0.$$

Additionally, as each $\nu_\beta$ is, by definition, $\mathbb{S}-$invariant, we deduce that

$$\begin{aligned}
h_{\nu_0}(\mathbb{S},\eta_{\underline{a}}) + \int_X \beta\varphi\,d\nu_0 &\le \sup_{\nu\in\mathcal{M}_{\mathbb{S}}(X)}\Big\{h_\nu(\mathbb{S},\eta_{\underline{a}}) + \int_X \beta\varphi\,d\nu\Big\}\\
&= P_{\mathrm{top}}(\mathbb{S},\beta\varphi,\eta_{\underline{a}})\\
&= h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}}) + \int_X \beta\varphi\,d\nu_\beta\\
&\le h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}}) + \beta \sup_{\nu\in\mathcal{M}_{\mathbb{S}}(X)}\int_X \varphi\,d\nu\\
&= h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}}) + \int_X \beta\varphi\,d\nu_0
\end{aligned}$$

and so

$$h_{\nu_0}(\mathbb{S},\eta_{\underline{a}}) \le h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}}) \qquad \forall \beta > 0.$$

Therefore, if the metric entropy is upper semi-continuous, then

$$h_{\nu_0}(\mathbb{S},\eta_{\underline{a}}) = \inf_{\beta>0} h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}}).$$

**Lemma 10.1.** *The metric entropy of $\mathbb{S}$ with respect to $\eta_{\underline{a}}$ is upper semi-continuous.*

*Proof.* Fix $\nu$ be in $\mathcal{M}(X)$ and recall that, by definition,

$$h_\nu(\mathbb{S},\eta_{\underline{a}}) = \inf_{\varphi\in C^0(X)}\Big\{P_{\mathrm{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) - \int_X \varphi\,d\nu\Big\}.$$

Given $\varepsilon > 0$, choose $\varphi \in C^0(X)$ such that

$$P_{\mathrm{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) - \int_X \varphi\,d\nu < h_\nu(\mathbb{S},\eta_{\underline{a}}) + \frac{\varepsilon}{2}$$

and consider the neighborhood

$$V_\nu\left(\varphi;\frac{\varepsilon}{2}\right) = \left\{\xi\in\mathcal{M}(X)\colon \left|\int_X \varphi\,d\xi - \int_X \varphi\,d\nu\right| < \frac{\varepsilon}{2}\right\}.$$

Take $\xi \in V_\nu(\varphi;\frac{\varepsilon}{2})$; then

$$h_\xi(\mathbb{S},\eta_{\underline{a}}) \le P_{\mathrm{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) - \int_X \varphi\,d\xi < P_{\mathrm{top}}(\mathbb{S},\varphi,\eta_{\underline{a}}) - \int_X \varphi\,d\nu + \frac{\varepsilon}{2} < h_\nu(\mathbb{S},\eta_{\underline{a}}) + \varepsilon.$$

Therefore $H$ is upper semi-continuous at $\nu$. □

The previous computations also show that, if $\nu \in \mathcal{M}(X)$ is another $\mathbb{S}-$maximizing measure, then $h_\nu(\mathbb{S},\eta_{\underline{a}}) \le \inf_{\beta>0} h_{\nu_\beta}(\mathbb{S},\eta_{\underline{a}})$. Thus, $h_\nu(\mathbb{S},\eta_{\underline{a}}) \le h_{\nu_0}(\mathbb{S},\eta_{\underline{a}})$.

□

## 11. Proof of Theorem C(3)

In this section we extend Theorem 1.1 of [15] to the context of semigroup actions generated by continuous maps acting freely on a compact metric space, thus presenting another intrinsic way to estimate the metric entropy of a semigroup action.

*Definition* 11.1. Given a random walk $\eta_{\underline{a}}$ associated to some nontrivial probability vector $\underline{a}$ and a Borel probability measure $\nu$ on $X$, the *entropy of the semigroup action* $\mathbb{S}$ *with respect to* $\nu$ *and* $\eta_{\underline{a}}$ is defined by

$$h_\nu^{(4)}(\mathbb{S},\eta_{\underline{a}}) = \lim_{\delta\to 0}\,\lim_{\varepsilon\to 0}\,\limsup_{n\to\infty}\,\frac{1}{n}\log\int_{\Sigma_p^+} s_\nu(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,\delta)\;d\eta_{\underline{a}}(\omega) \tag{11.1}$$

where $\omega = \omega_1\,\omega_2\cdots\omega_n\cdots$,

$$s_\nu(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,\delta) = \inf_{\{E\subseteq X\,:\,\nu(E)>1-\delta\}} s(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,E)$$

and $s(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,E)$ denotes the maximal cardinality of the $(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon)-$separated subsets of $E$.

Observe that the previous limit is well defined due to the monotonicity of the function

$$(\varepsilon,\delta)\mapsto\frac{1}{n}\log\int_{\Sigma_p^+} s_\nu(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,\delta)\;d\eta_{\underline{a}}(\omega)$$

on the unknowns $\varepsilon$ and $\delta$. Moreover, if the set of generators is $G_1=\{Id,f\}$, we recover the notion proposed by Katok for a single dynamics $f$.

*Remark* 11.2. We observe that Definition 11.1 could be made in terms of spanning sets. More precisely, given $\varepsilon>0$, a positive integer $n$ and $\underline{g}=g_{\omega_n}\dots g_{\omega_1}$, we say that a subset $A$ of $E\subset X$ is a $(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,E)-$spanning set if for any $x\in E$ there exists $y\in A$ so that $D_{\underline{g}}(x,y)<\varepsilon$. By the compactness of $X$, given $\varepsilon$, $n$ and $\underline{g}$ as before, there exists a finite $(\underline{g},n,\varepsilon,E)-$spanning set.

We denote by $b(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,E)$ the minimum cardinality of a $(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,E)-$spanning. For $\delta>0$ we set

$$b_\nu(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,\delta) = \inf_{\{E\subseteq X\,:\,\nu(E)>1-\delta\}} b(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,E).$$

It is not difficult to see that

$$h_\nu^{(4)}(\mathbb{S},\eta_{\underline{a}}) = \lim_{\delta\to 0}\,\lim_{\varepsilon\to 0}\,\limsup_{n\to\infty}\,\frac{1}{n}\log\int_{\Sigma_p^+} b_\nu(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,\delta)\;d\eta_{\underline{a}}(\omega).$$

Concerning the proof of Theorem C(3), firstly notice that, for any $\nu\in\mathcal{M}(X)$ and $\delta>0$, one has $\nu(X)>1-\delta$ and so, for every $\varepsilon>0$, $n\in\mathbb{N}$ and $g_{\omega_n}\dots g_{\omega_1}$,

$$s(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon)\geq s_\nu(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,\delta).$$

Therefore, for every $\nu\in\mathcal{M}(X)$,

$$\begin{aligned} h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) &= \lim_{\varepsilon\to 0}\,\limsup_{n\to\infty}\,\frac{1}{n}\log\int_{\Sigma_p^+} s(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon)\;d\eta_{\underline{a}}(\omega)\\ &\geq \lim_{\delta\to 0}\,\lim_{\varepsilon\to 0}\,\limsup_{n\to\infty}\,\frac{1}{n}\log\int_{\Sigma_p^+} s_\nu(g_{\omega_n}\dots g_{\omega_1},n,\varepsilon,\delta)\;d\eta_{\underline{a}}(\omega)\\ &= h_\nu^{(4)}(\mathbb{S},\eta_{\underline{a}}). \end{aligned}$$

Thus,

$$h_{\text{top}}(\mathbb{S},\eta_{\underline{a}})\geq \sup_{\nu\in\mathcal{M}(X)} h_\nu^{(4)}(\mathbb{S},\eta_{\underline{a}}).$$

Regarding the other inequality, fix arbitrary $\delta > 0$ and $\zeta > 0$. As $h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) = P_{\text{top}}(\mathcal{F}_G, \log \underline{a})$, there exists an $\mathcal{F}_G-$invariant and ergodic probability $\mu$ on $\Sigma_p^+ \times X$ such that

$$h_\mu(\mathcal{F}_G) + \int \log \underline{a}\; d\mu \geq h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) - \zeta. \tag{11.2}$$

Set $\nu = (\pi_X)_*\mu \in \mathcal{M}_X$. By [15] (Theorem 1.1 of this reference is stated for homeomorphisms but the proof is valid for continuous maps)

$$h_\mu(\mathcal{F}_G) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \inf_{E \in \mathcal{E}} s_\mu(\mathcal{F}_G, n, \varepsilon, E) \tag{11.3}$$

where $\mathcal{E} = \{E \subset \Sigma_p^+ \times X \colon \mu(E) > 1 - \delta\}$. Since $(\pi_1)_*\mu$ is $\sigma-$invariant and ergodic, by Birkhoff's ergodic theorem we know that for $(\pi_1)_*\mu-$almost every $\omega$

$$\int \log \underline{a}\; d\mu = \int \log \underline{a}(\omega)\; d(\pi_1)_*\mu(\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \underline{a}(\omega_j). \tag{11.4}$$

Using the fact that

$$\nu(A) > 1 - \delta \quad \Rightarrow \quad \mu(\Sigma_p^+ \times A) > 1 - \delta,$$

by Lemma 2 of [8] one can obtain that, for any $\delta, \varepsilon > 0$ and positive integer $n$,

$$b_\mu(n, \varepsilon, \delta, \mathcal{F}_G) \leq K(\varepsilon, \delta) \sum_{|\underline{g}|=n} b_\nu(\underline{g}, n, \varepsilon, \delta). \tag{11.5}$$

The relations (11.2), (11.3), (11.4) and (11.5) now yield

$$\begin{aligned} h_\mu(\mathcal{F}_G) + \int \log \underline{a}\; d\mu &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log b_\mu(\mathcal{F}_G, n, \varepsilon, E) + \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \underline{a}(\omega_j) \\ &\leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \Big[ \sum_{|\underline{g}|=n} b_\nu(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon)\, \eta_{\underline{a}}([\omega]_n) \Big] \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_{\Sigma_p^+} b_\nu(g_{\omega_n} \dots g_{\omega_1}, n, \varepsilon, \delta)\, d\eta_{\underline{a}}(\omega) \\ &= h_\nu^{(4)}(\mathbb{S}, \eta_{\underline{a}}). \end{aligned}$$

Since $\zeta > 0$ may be chosen arbitrarily small, we conclude that

$$h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) \leq \sup_{\nu \in \mathcal{M}(X)} h_\nu^{(4)}(\mathbb{S}, \eta_{\underline{a}}).$$

This ends the proof of Theorem C(3).

*Remark* 11.3. If each generator of $G_1^*$ is Ruelle-expanding, then the measure theoretical entropy of a semigroup action can also be computed using periodic orbits of the sequential dynamics observed in the semigroup. Indeed,

$$h_{\text{top}}(\mathbb{S}, \eta_{\underline{a}}) = P_{\text{top}}^{(a)}(\mathcal{F}_G, 0, \underline{a}) = P_{\text{top}}(\mathcal{F}, \phi_{0, \underline{a}})$$

where $\phi_{\varphi,\,\underline{a}}(\omega,x) := \log\left(\underline{a}_{\omega_1}\, e^{\psi_\varphi(\omega,x)}\right)$. As $\mathcal{F}_G$ is also Ruelle expanding, and topologically mixing in each piece of the Ruelle-decomposition (in particular admits a finite Markov partition), then

$$\begin{aligned} h_{\text{top}}(\mathbb{S},\eta_{\underline{a}}) = P_{\text{top}}(\mathcal{F},\phi_{0,\,\underline{a}}) &= \lim_{n\to\infty}\frac{1}{n}\log\sum_{\mathcal{F}_G^n(\omega,x)=(\omega,x)} e^{\sum_{j=0}^{n-1}\,\phi_{0,\,\underline{a}}(\mathcal{F}_G^j(\omega,x))} \\ &= \lim_{n\to\infty}\frac{1}{n}\log\sum_{\sigma^n(\omega)=\omega}\;\sum_{g_{\omega_n}\cdots g_{\omega_1}(x)=x}\;\prod_{j=0}^{n-1} a_{\omega_j} \\ &= \lim_{n\to\infty}\frac{1}{n}\log\sum_{\sigma^n(\omega)=\omega}\;\sum_{g_{\omega_n}\cdots g_{\omega_1}(x)=x}\eta_{\underline{a}}([\omega]_n) \\ &= \lim_{n\to\infty}\frac{1}{n}\log\int_{\Sigma_p^+}\sharp\,\text{Fix}(g_{\omega_n}\dots g_{\omega_1})\,d\eta_{\underline{a}}. \end{aligned}$$

**Acknowledgements.** FR has been financially supported by BREUDS. PV was partially supported by CNPq-Brazil. MC has been financially supported by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020. The authors are grateful to A. O. Lopes for useful comments and for calling their attention to the reference [23].

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