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Fleet and Revenue Management in Car Rental: Quantitative Approaches for Optimization Under Uncertainty

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As the man said, for every complex problem there's a simple solution, and it's wrong.
Umberto Eco, in Foucault's Pendulum

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Abstract

Car rental is a relevant and growing business. Tourism, for example, is a sector whose boom has positively impacted this industry in the past years. Heavily relying on operational efficiency, namely to maximize fleet occupation, car rental companies are still in need of better decision-support tools in order to survive and be profitable in this competitive market, pushed by an uncertain and price-sensitive demand. However, this business has unique and interesting characteristics that hinder the application of methodologies traditionally used in other sectors. These include the high mobility of the fleet, its "re-usability", and the relative flexibility of its size and mix.

Car rental has been the focus of some significantly relevant fleet management studies and one of the main applications of revenue management techniques. Nevertheless, academic work is only starting to focus on and exploit the interconnections between these two fields, commonly tackled separately.

The research work here presented aims to extend current knowledge by closing this gap, and found its main practical motivation and inspiration in the company *Guerin Car Rental Solutions*. The main objectives of this thesis are i) to understand how the car rental business can benefit from its unique characteristics and interconnections to improve its fleet and revenue management, and ii) to propose innovative quantitative approaches to tackle this problem, especially under uncertainty. Throughout the work, there is also a clear focus in developing approaches that are realistic and thus applicable in the real-world.

The contributions of this thesis are aligned with these research objectives. First, we propose a conceptual framework for the car rental fleet and revenue management problem, which allows to structure the field of research and identify relevant gaps and directions. Based on this and on the practical work at *Guerin* to develop a pricing decision-support tool, quantitative approaches to tackle the integrated problem of capacity and pricing are developed. Two methods – a dynamic programming approach and a matheuristic – are proposed for the deterministic version of the problem, as well as a mathematical programming model. This model is extended when considering uncertainty and other increasingly realistic assumptions and constraints. To solve it, we propose an innovative matheuristic that simultaneously generates scenarios and solutions that suit different risk-profiles of decision-makers.

Overall, the contributions of this work are twofold. First, this thesis tackles a new and relevant problem, extending the knowledge in this field. Emerging transportation systems that have similar characteristics, such as car sharing, can benefit from this. Secondly, the innovative methodological contributions – whose techniques range from mathematical and constraint programming to heuristics, dynamic programming and matheuristics – have the potential to be extended and applied to different problems.

Resumo

O aluguer de automóveis é um negócio relevante e em crescimento. O setor do turismo, por exemplo, viu um crescimento que teve um impacto muito positivo neste negócio nos últimos anos. Estando as empresas de aluguer de automóveis fortemente dependentes de eficiência operacional, nomeadamente da ocupação da frota, há ainda uma carência de ferramentas de apoio à decisão que permitam atingir melhores resultados num mercado altamente competitivo, impulsionado por uma procura incerta e sensível ao preço. Este negócio é, no entanto, caracterizado por fatores distintivos e interessantes que impedem a aplicação direta de metodologias tradicionalmente utilizadas noutros setores. Estes fatores incluem uma alta mobilidade da frota, a possibilidade de ser re-utilizada, e a relativa flexibilidade do tamanho e *mix* da frota.

A gestão da frota nas empresas de aluguer de automóveis tem vindo a ser alvo de estudos relevantes. Este negócio tem sido também uma das principais áreas de aplicação de técnicas de gestão de receita. No entanto, a academia apenas agora começa a explorar as interconexões entre estes dois campos, usualmente tratados em separado.

O trabalho de investigação aqui apresentado é impulsionado pela necessidade de contribuir para o conhecimento na área de forma a fechar este *gap*, encontrando motivação prática na empresa *Guerin Car Rental Solutions*. Os principais objetivos desta tese são i) compreender a forma como o aluguer de automóveis pode beneficiar das características e interconexões distintivas do setor para melhorar a gestão de frota e de receita, e ii) propor abordagens quantitativas inovadoras para o problema, especialmente considerando incerteza. Ao longo do trabalho desenvolvido, há ainda um foco claro em escolher abordagens realistas e por isso aplicáveis num contexto real.

As contribuições desta tese estão alinhadas com estes objetivos de investigação. Primeiramente, foi proposto um *framework* conceptual para o problema de gestão de frota e de receita para empresas de aluguer de automóveis, o que permitiu estruturar este campo e identificar *gaps* na literatura e direções de investigação relevantes. Baseado nisto e no trabalho prático na *Guerin* para desenvolver uma ferramenta de apoio às decisões de atribuição de preços, foram desenvolvidas abordagens quantitativas para o problema integrado de capacidade e atribuição de preços. Dois métodos – uma abordagem de programação dinâmica e uma matheurística – são propostos para a versão determinística do problema, assim como um modelo de programação matemática. Este modelo é estendido para considerar incerteza e outros pressupostos e restrições mais realistas. Para o resolver, é proposta uma matheurística inovadora que gera simultaneamente cenários e soluções que se adequam a diferentes perfis de risco do decisor.

No geral, as contribuições deste trabalho seguem duas direções. Por um lado, esta tese propõe um problema novo e relevante, contribuindo para o desenvolvimento do conhecimento nesta área. Sistemas de transportes emergentes que partilham algumas características, como os sistemas de mobilidade automóvel partilhada, podem beneficiar deste desenvolvimento. Por outro lado, as contribuições metodológicas inovadoras – cujas técnicas vão de programação matemática e programação com restrições a heurísticas, programação dinâmica e matheurísticas – têm o potencial de ser continuadas por outros e aplicadas a problemas diferentes.

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Introduction and overview

Car rental is a relevant, growing business worldwide. In Portugal, in 2017, this sector was responsible for more than 25% of car sales due to a boom in tourist demand, thus driving the growth in the automobile industry (Ferreira, 2017). In the USA, 2017 brought new records for the car rental sector, reaching $28.63 billion in total revenue. Although this represents the lowest year-on-year growth in the past years, it was achieved with a smaller fleet. This resulted on the first year-on-year revenue-per-unit growth in four years and the highest value since 1992, when data started being collected (Auto Rental News, 2017). This may be seen as a result of better operational efficiency that maximizes fleet utilization and also of better revenue management practices.

Moreover, this business has unique and interesting characteristics that differentiate it from other transportation sectors. The products sold and priced – the rentals – are complex even at their most basic definition, since they may involve different pick-up and drop-off locations and a large range of possible lengths-of-rental. These products share the same capacity or resources – the fleet – which become available again after the rental ends and may be re-utilized. These issues increase significantly the complexity of the problem and hinder the application of fleet management methodologies developed for other contexts. Also, this is a business characterized by its flexibility. The ability to move the fleet to meet its uncertain demand and relative adaptability of its size and mix, make this one of the most interesting businesses to apply revenue management techniques. Moreover, these characteristics highlight the interconnections between these two areas of decision-support: fleet and revenue management.

Car rental companies depend heavily on operational efficiency. Also, they face significant challenges at the competition level, which tend to intensify in a near future. With a clear need for differentiation, several companies are deciding to achieve that based on price (although not exclusively), creating low-cost brands to compete in these markets.

This thesis tackles fleet and revenue management in the car rental business. More specifically, it aims to close a gap in the literature regarding this relevant and interesting application. The first part of the thesis is focused on motivation, from practical and theoretical points of view. For this, we describe an application work developed alongside a car rental company and present a thorough literature review, which allowed for a conceptual framework for the sector to be proposed and research directions identified. One of these directions is related with the integration of fleet and revenue management issues. Therefore, the second part of the thesis is composed of incremental quantitative approaches to the integration of capacity and pricing problems, first on a deterministic viewpoint and later considering uncertainty.

Despite the main objective of contributing and extending the knowledge on the car

rental sector, this research also presents relevant methodological contributions which can be applied in other transportation sectors and even in different problems.

The remainder of this chapter is organized as follows. First, in Section 1.1 the car rental business will be briefly characterized, mainly based on the car rental company that provided the practical motivation for this research: *Guerin Car Rental Solutions*. In Section 1.2, due to the significant impact of uncertainty, to the range of possible methodological approaches to tackle it, and to the general lack of consensus regarding important concepts in this area, a brief discussion on uncertainty and robustness is presented. Then, the research objectives and methodological approach will be discussed in Section 1.3. Finally, Section 1.4 gives an overview of the thesis, describing the main ideas and contributions of each chapter.

1.1.   The car rental business

The car rental business is structured around its capacity – the fleet of vehicles available to meet demand. This capacity is shared by a pool of locations, often within a non-negligible distance. The decisions regarding this fleet span across different levels of strategy, yet are interconnected.

The decisions tackled in this thesis concern capacity and pricing. Capacity decisions include decisions on how many vehicles (fleet size) of each type (fleet mix) to acquire, how to acquire these vehicles (purchase or leasing), where to make these vehicles available at the start and throughout the season (fleet deployment), and how to achieve this deployment (with actual rentals or by "empty transferring" vehicles by driver or truck). Other main decision is related with which rentals to fulfill, depending on the capacity and on demand. Here, upgrades must be considered as a tool to manage demand – a common practice in the industry where a more-valued vehicle is offered for the same price of the requested less-valued vehicle. Finally, pricing decisions are made for each type of rental, often depending on the antecedence of the rental request. These decisions and their interconnections will be further detailed in Chapter 2.

One of the main challenges for these companies is to tackle the uncertainty in demand. Besides being highly price-sensitive, demand in this context is uncertain and not thoroughly studied in the literature. Therefore, there is no clear indication regarding the best methodology to tackle this specific issue. Section 1.2 will further develop this topic.

1.2.   Considerations on uncertainty and robustness

The notions of uncertainty and robustness are closely linked, being the latter a capacity to withstand the former and other fallibility issues. In operational research, and decision-support in general, uncertainty and robustness are critical issues. Robustness is a recent trend of concern both in academic research, especially for real-life contexts, and in practical applications. Nevertheless, there is not much consensus on the meaning, measures and forms of response to this concern.

For this research, considering uncertainty is paramount. However, the adequate approach to follow is not straightforwardly derived from the problem characteristics, scope

or context. In order to reflect upon this issue and its implications on the methodology followed in this research project, three important bodies of work with a strong focus on uncertainty and, especially, *robustness* will be visited. Our aim is to understand the main characteristics of a robust solution in each perspective so that we tackle this topic within a clear scope and with a strong fit to the problem. The first perspective that will start the discussion will be anchored on an invited review by Bernard Roy published in 2010 in the European Journal of Operational Research (EJOR) (Roy, 2010). This is a thorough review on the topic of robustness, most relevant to gain awareness of the variety of frailties that can affect robustness and how they have been dealt with in the literature. Then, the Robust Optimization perspective on the topic will be presented, based on two basilar works from the field (Ben-Tal et al., 2009; Bertsimas and Sim, 2004). In this perspective, robustness has a specific meaning, related with safeguarding the solution against the "worst-case scenario", that directly influences the concept behind this state-of-the-art methodology; it is therefore also interesting to compare the conceptual frameworks and baseline-definitions that support both approaches. The third perspective will also be anchored on an invited review published in EJOR, authored by Mingers and Rosenhead (2004), and two other relevant previous works by Jonathan Rosenhead regarding the Robustness Analysis methodology (Rosenhead et al., 1973; Rosenhead, 1980). This approach can bring different insights to the discussion as it focuses on keeping options "open" and the consequent flexibility of a solution to respond to future challenges.

In Roy (2010), the author reviews the topic of robustness, explaining its main purpose and origins, and attempts to categorize the research done in the field according to it. He also proposes three new measures of robustness. Nevertheless, the most interesting part of this work is the analysis and discussion on this issue and on the perspectives of different research streams. The debate around the multiple meanings of robustness illustrates the numerous facets of this concept, often context-specific. It is therefore critical to fully understand what these facets may be in order to be able to identify the ones critical for each specific research application.

In this paper, Roy uses the following as the meaning of *robust*: "an adjective referring to a capacity for withstanding "vague approximations" and/or "zones of ignorance" in order to prevent undesirable impacts, notably the degradation of the properties to be maintained." (p. 629). This is felt to be a reasonable and broad definition of the term, independent from (and hence not limiting) the form of response. The author highlights that the concern for robustness transcends the analysis a posteriori of the impacts; it represents a need for resistance that must be taken a priori, when the formulation of the problem begins.

The notion of frailty points is used by the author to name the vague approximations and zones of ignorance that appear in the (imperfect) *formal representation* of the real-life context. The inventory of these frailty points should thus be the first step in accounting for robustness. The author proposes four perspectives to scrutinize the formal representation for frailty points: i) how imperfect knowledge is tackled (it can be: ignored, modelled using e.g. probability distributions, or incorporated in the procedure when it has been conceived to take into account imprecise data), ii) whether questionable meaning is attributed to data,

iii) how complex aspects of reality are modelled, namely how model parameters are chosen, and iv) how technical parameters are introduced (imposed by the processing procedure).

It is also stated that *uncertainty* does not include all possible frailty points (e.g. approximations due to simplifications). The author claims that the scenario-set approach is overall used when one limits the notion of robustness to only considering uncertainty and suggests replacing scenarios for the concept of version, which is defined by a combination of the options arising from the frailty points.

The author structures the research in three "territories" for robustness concern and categorizes the literature in the field according to them. The main split between territories is the motivation and origin for the formal representation: whether it stemmed from a standard OR model – standard territory (in this territory usually a scenario approach is used since the real-life context is often unexplicit and robustness is often linked to a single optimization criterion), whether the real-life context is the starting problem – concrete territory (herein the formulation itself is based on robustness concerns, involving e.g. the a priori identification of frailty points and the elaboration of the version set, in a relatively complex manner), or whether it is mixed between the first two. The forms of response to the robustness concerns are different (as are the *raisons d'être*) for the three territories.

Three measures of robustness are proposed, based on the ones existing in the literature: absolute robustness (value of the solution in the worst case scenario), absolute deviation (absolute regret in the worst case scenario) and relative deviation (relative regret in the worst case scenario). The added value lies in the introduction of two boundaries: a threshold that the decision-maker asks (not) to exceed in the greatest possible number of scenarios, and a guaranteed value under which the decision-maker refuses to go, regardless of the scenario. Therefore, in this perspective, robust solutions are all that, while ensuring that a minimum value achieved in all scenarios, maximize the number (or proportion, or weighted proportion) of scenarios in which the absolute robustness (or absolute deviation or relative deviation, depending on the measure) exceeds a certain value.

Robust Optimization is a methodology for handling uncertain data (Ben-Tal et al., 2009). Due to the relevance that this methodology has been gaining in the past years in the mathematical optimization field, we will focus on the adopted meaning of robustness and the analysis of the impact on the solution value of this approach.

In a mathematical optimization problem, even small uncertainty may in fact have a strong impact, for example, by rendering the nominal optimal solution infeasible. The authors postulate that a methodology is needed for detecting such cases and generating for them robust solutions. In this perspective, robust solutions are those that are "immunized against the effect of data uncertainty" (Preface, p. xi). Therefore, *robust feasible* solutions are sought (solutions that remain feasible whatever the uncertain data reveals itself to be, within an uncertainty set $\mathcal{U}$) and a "worst-case philosophy" dominates. Therefore, the quality of a robust feasible solution is measured by the guaranteed value and the *robust optimal* solution is the best possible within the worst-case. This is generally a conservative approach. Bertsimas and Sim (2004) discuss the trade-off made by accepting sub-optimal solutions in order to ensure that the solution remains feasible when data changes. Within

the same perspective and methodology, they propose an approach where this trade-off is more controllable than in other earlier approaches. It has the advantage of being a linear formulation, that withstands parameter uncertainty with a solution that is either feasible deterministically or with very high probability. As a note it is worth mentioning that this work by Bertsimas and Sim is categorized in Roy (2010) as part of the "standard territory" of robustness research.

Mingers and Rosenhead (2004) is a work somewhat different from the previous ones and was chosen for this discussion because it opens a new "avenue" to follow the different threads of the meaning of robustness. In this work, the authors review the use of problem structuring methods (PSMs), which approach problems that are, by nature, ill-structured and more strategic or high-level (in the sense that they provide the inputs for well-structured problems). These problems are characterized by multiple actors and perspectives, critical intangibles and incommensurable interests, and *key uncertainties*. Robustness analysis is a method briefly presented in this review as an "approach that focuses on maintaining useful flexibility under uncertainty" (p. 532). In order to further understand this base-concept of robustness, two previous works by Jonathan Rosenhead were studied in more detail.

Rosenhead is interested in strategic planning methodologies under uncertainty[1] and it is within this context that a different light is shed on the term robustness. In Rosenhead et al. (1973), the authors express concern for the non-implementation of the traditionally *optimal* recommendations made by operational research professionals to companies, which is pointed as the main motivation for a *robust* approach to strategic decisions. In this perspective, robustness is intrinsically linked with decision flexibility; the most robust initial decision of a plan is the one that leaves more options open for the future. In fact, strategic planning involves a sequence of decisions, where later decisions may be revised after earlier ones have been taken and uncertainty is partly realized. The authors state that "a plan whose initial decisions limit the future as little as possible has an evolutionary advantage in an uncertain world". In this work, a measure of robustness of a certain initial decision is presented as a ratio: the fraction of the set of possible solutions (initially considered with an "acceptable" performance) that remains attainable after this decision is implemented. Also, the concept of *stability* of a decision in introduced – an initial solution is stable if it has a "good performance" on the long-run, even if afterwards no other decision is taken.

The methodology of *Robustness Analysis* is proposed as a general structuring procedure for planning under uncertainty in Rosenhead (1980), based on the previously introduced concepts. In this work, some interesting reflections regarding the author's concept of robustness are worth mentioning in this brief report on the topic. Firstly, in this perspective, robustness is a *relative* measure of flexibility. The value of the ratio mentioned above– e.g. consider a decision with a robustness score of 0.7 – is of little consequence and insight when analyzed alone; however, it is meaningful to state that this is a more robust decision than one with a score of e.g. 0.3. Consequently, robustness is a criterion that should be

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[1]In Rosenhead et al. (1973), *uncertainty* is defined as the impossibility to attribute probabilities to the outcomes of the decisions (confronting with *risk* situations, where the connection decision-outcome is probabilistic).

incorporated on the decision-making process and not only utilized as a a posteriori evaluation of "optimal" decisions if no feedback loops are considered. Secondly, this concept of robustness is more related with strategic flexibility than tactical flexibility. While the latter is related to a system's ability to respond to different operating modes in its *current configuration*, the former enables a system to assume *different configurations* as a whole. Finally, to further contrast with the Robust Optimization perspective, in this context the author assumes that a high level of detail and accuracy is not needed as errors would need to be significant and persistent to drastically change the robustness scores.

Discussion

On the one hand, the work from Roy (2010) is very interesting in the sense that it organizes the discussion around robustness, despite its small bias towards more real-life-context-based approaches to robustness. We believe that the research developed in this thesis is situated in the standard or mixed territory. Despite its focus on the real-life application, its main robustness concern actually relates with uncertainty (especially in demand) and not with other frailty points. In fact, the only other type of frailty point identified comprises the technical parameters introduced when using metaheuristics as part of the solution method. Moreover, there is a significant concern, common to all mathematical representations of reality, that the formal representation does not accurately represent the system of values in place to evaluate the solutions or that this system changes over time. Nevertheless, we believe that this frailty point can be generally tackled by ensuring that a set of solutions with similar performance (instead of a single solution) are provided to the decision-maker, since, as Roy (2010) states, decision-aiding should not dictate a solution rather than provide insights that support the decision through well-argued solutions and conclusions.

On the other hand, in the context of Robust Optimization, robust solutions are the ones that remain feasible even in the worst-case. When comparing with the previous definition, it can be seen that this is a somewhat more conservative approach. Nevertheless, the "budget of uncertainty" is controlled in this approach by the uncertainty set defined. In fact, in this methodology the probability distributions are not required to "define" the uncertainty; an uncertainty set is defined by its bounds. As aforementioned, in this research project only the robustness that withstands uncertainty is a matter of concern (acknowledging yet not considering other frailty points). Within this scope and in a generic way, robust solutions are those that perform well (at least are feasible with very high probability) no matter the realization of the uncertainty. The analysis of the quality of a robust solution is itself challenging as a trade-off is needed between the ability to resist to uncertainty and the value attained by the solution.

The discussion so far has conceptually mapped the concept of robustness on one axis: changes on performance (and/or feasibility) when uncertainty is disclosed (with an emphasis or not in the worst case scenario). The concept of robustness introduced by Rosenhead brings a second axis to the discussion: flexibility on the decisions downstream in the planning process. The *Robust Analysis* methodology may not be the most suitable framework to tackle the problem at hand. In fact, the focus of this methodology is on *strategic* flexibility, i.e. how to make robust here-and-now decisions considering a long-term, strategic

planning environment. Despite some decisions, such as capacity decisions like fleet size and mix, being more strategic in nature (lasting for a selling season), the car rental fleet management problem is characterized by an intrinsic flexibility on other fleet management decisions that can be made within shorter decision time horizons to react to unexpected events, such as leasing vehicles or empty transferring cars between locations. Nevertheless, Robust Analysis brings important insights for the concept of robustness that should be noted and applied in this case as well: the easiness or flexibility of a certain solution to modified when uncertainty is disclosed should also contribute for its robustness.

Finally, it should also be noticed that all bodies of research argue that the robustness concern should be present from the beginning of the formulation or during the solution building procedure, versus other a posteriori evaluations of robustness.

To tackle uncertainty in this research project, considering the overall approach and the scope delimited before, two main avenues of research may be followed: stochastic and robust optimization. The methodology of robust optimization was briefly introduced before. In stochastic optimization, the uncertain data are assumed to be random, following a probability distribution or with a known number of outcomes of randomness (scenario approach). In these optimization models, the goal is to maximize or minimize the *expected* result. Both methodologies have advantages and shortfalls when considering the implementation in this context. Stochastic optimization is simpler to formulate (in the sense that it does not require to handle complex uncertainty sets) and is less conservative. However, according to Ben-Tal et al. (2009), it requires uncertain data to be of a stochastic nature and that probability distributions or known outcomes are associated to it. Moreover, one must be ready to accept probabilistic guarantees such as the ones given by chance constraints. According to the authors, the conservatism of robust optimization can be seen as an advantage in some applications (e.g. when designing a construction), although in car rental fleet management this is probably not such a relevant argument. Also, in stochastic optimization, unlike robust optimization, to increase the relevance of a certain scenario (i.e. its probability) it is necessary to reduce the probability of other scenarios, which may be a disadvantage.

Therefore, it is also important to understand how can robustness be ensured when using stochastic optimization. In fact, using stochastic optimization, one is already taking into account the major frailty point of data uncertainty in the formulation and solution procedure. If uncertainty is accepted as stochastic and a probability distribution or scenarios are associated with the uncertain data, which seems reasonable to assume in this case, the discussion is whether probabilistic guarantees are sufficient for the robustness definition.

As main conclusion for this discussion, we decided to tackle uncertainty in the car rental fleet and revenue management problem mainly in the context of (and with a general focus more related with) stochastic optimization. Nevertheless, robustness concerns are considered, in terms of: i) protecting solutions against worst cases, ii) including this robustness concern in the solution procedure (versus an a posteriori approach), iii) offering a set of good solutions instead of a single solution, and iv) seizing the flexibility of some fleet management decisions to respond to uncertainty.

1.3. Research objectives and methodological approach

The main objective of this research is to contribute to the field of fleet and revenue management in car rental by proposing innovative optimization frameworks that tackle existing gaps in the literature, namely regarding the integration of capacity and pricing decisions. Throughout this thesis, we will build on the existing literature and further develop mathematical models, extending their realism and increasing their industry applicability, and implement innovative solution methods and algorithms to ensure resolution in reasonable time frames and validation regarding the quality of the solutions in the presence of uncertainty.

The first part of the work is based on understanding the need for research regarding fleet and revenue management in car rental, both on a theoretical and a practical viewpoint. For this, we review the literature on the topic and propose a conceptual framework for the problem, highlighting relevant research directions. Additionally, we present the work developed alongside the car rental company *Guerin* to develop a pricing decision-support tool as an indicator of the practical motivation of this work.

After clearly stating the relevant research directions in this area, we aim to follow an incremental approach to the development of solution methods. Therefore, we tackle the deterministic integrated problem first and afterwords expand to consider uncertainty. Generally, mathematical modeling will be used to represent the problems, both in deterministic versions and when considering uncertainty. Exact solutions of these models will be tested using the solvers available. Our main goal is to seize one of the key advantages of modeling a problem, which is the ability to fully understand and accurately depict its decisions, objectives and constraints. Nevertheless, due to the expected complexity of the formulations, to the large size of real world problems and to the need to solve the problems under reasonable time-frames, approximate quantitative solution approaches will also be developed, namely matheuristics that hybridize metaheuristics and mathematical programming.

1.4. Thesis synopsis

Figure 1.1 presents an overview of the remaining chapters of the thesis, which consist of a collection of papers. This figure and the chapters are organized according to the methodological approach previously introduced. The first block of chapters focuses on the motivation and the second block focus on quantitative approaches to tackle the problem. This section provides an overview of the main objectives of these articles and the contributions associated with each of them.

The first two chapters synthesize the motivation for this research. Chapter 2 presents a literature review for the car rental fleet and revenue management problem. A conceptual framework for this problem is proposed, helping to identify existing gaps, trends and four future relevant research directions. These are related with increasing realism to make research applicable in reality and with the integration of different problems, including capacity and pricing, due to close interactions in their decisions, overlapping time horizons and the inherent flexibility of decisions in this business. Then, in Chapter 3, the work de-

Fleet and revenue management integration in car rental

Motivation   **Quantitative approaches for capacity-pricing integration**

Theoretical motivation   Deterministic   With uncertainty

Chapter 2: Literature
review and conceptual
framework   **Chapter 4:** Dynamic
                              programming approach
Practical motivation   **Chapter 6:** Co-
                                                            evolutionary matheuris-
                                                            tic based on a two-space
Chapter 3: Pricing sys-   BRKGA
tem at Guerin   **Chapter 5:** Matheuristic
                              based on a BRKGA

Figure 1.1: Overview of thesis structure

veloped alongside the car rental company *Guerin* is presented. The main objective is to develop a decision-support tool for pricing in internet sales channels. This work allows for the interactions between pricing and fleet management decisions to become clearly mapped. Moreover, the practical need for decisions that take in consideration both fleet-related issues and pricing is established.

The three following chapters comprise the block related with quantitative approaches for the integration of capacity and pricing. Chapters 4 and 5 concern deterministic approaches and Chapter 6 refers to an approach that considers uncertainty. In Chapter 4 a dynamic programming approach, often used to tackle similar problems, is developed and tested. However, the *rental* context, where capacity is re-usable, poses significant limitations on the applicability of the method. This work has nonetheless brought significant insights regarding both the problem structure and the methods applied. Based on these, a matheuristic approach is proposed, presented in Chapter 5. It hybridizes a metaheuristic – Biased Random Key Genetic Algorithm (BRKGA) – with mathematical programming and is able to obtain good results in reasonable time for realistic instances. This work is extended to consider uncertainty in Chapter 6. Here, a stochastic matheuristic is proposed, based on a co-evolutionary BRKGA that simultaneously generates solutions and scenarios and provides the decision-maker with the ultimate tool to manage capacity considering pricing effects in car rental.

Finally, Chapter 7 summarizes the most substantial results obtained and gives directions for future research.

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Theoretical motivation: Literature review and conceptual framework

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This chapter presents a paper that structures the car rental fleet and revenue management framework and establishes the scope of the problems that raise interest in this research. It is of the utmost importance for this thesis since it allowed to define the lines of action throughout the research project, soundly based on the existing literature review. It thus presents the main motivation for research within an academic viewpoint: the existing gaps and relevant research directions.

Fleet and revenue management in car rental companies: A literature review and an integrated conceptual framework

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Abstract   This paper aims to present, define and structure the car rental fleet management problem, which includes operational fleet management issues and problems traditionally studied under the revenue management framework. The car rental business has challenging and distinctive characteristics, which are mainly related with fleet and decision-making flexibility, and that render this problem relevant for academic research and practical applications. Three main contributions are presented: an in-depth literature review and discussion on car rental fleet and revenue management issues, a novel integrating conceptual framework for this problem, and the identification of research directions for the future development of the field.

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2.1. Introduction

This paper aims to present, define and structure the fleet management research focused on the problems faced by car rental companies. The focus on the car rental context arises from the interesting and challenging idiosyncrasies of its fleet and decision-making processes, which have some structural differences when compared to other transportation sectors more traditionally studied in the literature. Moreover, car rental is a growing business, comprising $ 27.11bn in revenue in 2015 in the U.S. – which represented a 4% improvement over the previous year – while the average car rental fleet grew 5% (Auto Rental News, 2015). This growth trajectory has been steady since 2010 and is forecasted to continue. From 2016 to 2021, the global car rental industry is expected to grow 5.6% (CAGR), due to increasing tourism activities, the globalization of operations, and the global rise of income levels (ReportsnReports, 2015).

The car rental fleet management problem embeds decisions that are traditionally framed within different strategic levels and studied by different research areas. The main decisions are related with clustering locations that will share the same fleet, deciding on the fleet size and composition, distributing fleet amongst rental stations, deciding on prices, selecting which reservations to accept, and assigning these reservations to specific vehicles. In a real-world setting, these decisions are not only linked by close interactions but also by overlapping decision-making time horizons. In fact, one of the main characteristics of the car rental fleet that motivates this study is its inherent flexibility. On the one hand, the fleet is significantly easy to move and re-locate, enabling e.g. the use of strategic fleet balancing decisions often referred to as "empty transfers". On the other hand, there is also a flexibility on the decision-making process that often renders the traditional hierarchical overview of fleet decisions too rigid. For example, the acquisition and removal of cars to and from the fleet is significantly flexible, as these contracts are often incentivized with small lead times, and frequently throughout the year.

Due to the relatively small number of papers that deal with this problem so far, this review aims to be exhaustive within its scope, which comprises quantitative methods that were developed to support decisions related with car rental fleet management. It is structured in three main parts. Firstly, the seminal works that launched the interest in the field are reviewed, which are generally accounts of the early implementation of fleet and revenue management systems in car rental companies. The second part is devoted to the main works, which have structured the field and set the ground for future works. These will be the focus of the third part, which is divided in smaller sections related with the type of decisions: the clustering of rental locations in groups that share the same fleet (pools) and the fleet management within each pool, which comprises operational decisions, revenue management decisions and the integration of both.

Arising from the literature review, a conceptual framework is proposed to structure the car rental fleet management problem. The literature in the area is scarce and somewhat concentrated in only a few of the problems; however, the interest in this field has been growing in the past years and expanding to different sub-problems within this scope. The framework herein proposed aims to contextualize the relations between the different sub-problems, and is motivated by the need to support the development of methodologies that

are applicable in real-world settings.

One of the main contributions of this work is also the proposal of four research directions. These are based on the framework and literature review and are related with the increase of the realism and applicability of the existing methods and the exploration of different levels of integration of the sub-problems.

The remainder of this paper is thus structured as follows. In Section 2.2, the problem is introduced and described in detail; moreover, some "lessons learned" from research in other transportation sectors are discussed. Then, in Section 2.3, the literature review on the car rental field is presented and discussed. The framework for the problem is then presented in Section 2.4 and, based on it, the research gaps are identified. Four main research directions for the future are also proposed in this section. Finally, some conclusions are drawn and the main contributions and limitations of this work are discussed in Section 2.5.

2.2. Problem and contextualization

In this section, the problem of fleet management in the car rental industry, which incorporates several interconnected sub-problems, is presented. The goal of this section is to informally describe and contextualize the main business decisions, with no specific intention to structure and thus limit the problem definition.

Fleet management is indeed a mature topic of research in other transportation fields. From some of these fields, such as the airline industry and maritime or rail-freight transportation, parallels can be drawn with the car rental business and thus useful lessons can be learned. Nevertheless, there are structural differences that support the need for a more specific treatment of the car rental business and these will also be presented.

2.2.1 Fleet management in the car rental business

The goal of this section is to broadly present the car rental fleet management problem and its main decisions. In fact, the car rental business profitability is heavily dependent on its fleet and all decisions that concern it. These fleet decisions span across all strategic levels of the company, from the network design decisions to specific-vehicle maintenance requirements. The following description focuses on the main decisions dealt with within this scope regarding the network design, the definition and utilization of the fleet, and the management of booking requests and consequent schedules for each vehicle.

Network In bigger car rental companies, the rental stations are usually aggregated in pools – groups of stations that share the same fleet. These pools are independent from other administrative divisions (e.g. regional divisions) although they can overlap; therefore, there is a certain flexibility to change and adjust them. In fact, this is not a "one-time decision" by nature; the pool design may be frequently reshuffled as a means to e.g. meet seasonal changes in demand patterns across locations. There is a specific set of cars assigned to each pool, to be shared by the rental stations that form it. Within the pool, the specific location of the car at a certain time depends on its status: if it is fulfilling a reservation, idle at a

certain station, or under maintenance at a certain workshop. Ultimately, some cars may even be outside their pool, if a customer picks-up a car in a rental station that belongs to the pool but returns it to a station outside the pool.

Fleet definition   A car rental fleet is composed of a number of cars of different types (rental groups). These groups may be substitutable, which will be discussed later when the reservation proceedings are described, thus connecting the decisions of "how many cars of a certain group to have?" for the different rental groups. Overall and generically, the size of the fleet is mostly determined by the company's strategic positioning and available investment, which is a one-time decision and out of the scope of the fleet management problem studied. Nevertheless, the operational adjustments made to increase and decrease the fleet, usually within a pool, are critical for a proper fleet management and will be herein analysed.

A fleet management problem deeply linked, or even included, in deciding the size adjustments, is related with the vehicles to acquire and remove from the fleet. Actually, buying and selling vehicles can play a very relevant part on the company's profitability. This part of the process is extremely dependent on the type of car rental company. Some car rental companies are part of vertically integrated business groups, and thus have a close access to a manufacturer and/or to a wholesale reseller. For these companies, acquiring vehicles can be compared to a leasing contract, where a specific service deadline is defined for each car; they can also have access to discounted prices or other amenities. As for removing the cars from the fleet, they are more protected against fluctuations in the used car market, for example, as the responsibility to dispose of the stock falls on the reseller company of the group, or at least is shared with it. For the remainder of the companies, however, *how* the vehicles are acquired and removed from the fleet is as important as *when*. These decisions can also be significantly flexible, yet this depends on the mode of acquisition/removal. In fact, although some acquisition contracts must be made with some antecedence, the assignment of new cars to the respective pools can be made with a short notice, if needed. The decisions on the removal of cars from the fleet are also extremely important, especially since they can and should be made vehicle-by-vehicle. If the company wants to sell back the used car, specific information, such as the odometer values, is critical to decide on "sell dates" (Lacetera et al., 2011).

Fleet utilization   Another critical decision is how to divide the existing fleet among the rental stations, within a pool. This is a critical aspect since the majority of the operational costs in car rental are related with idle fleet. That is to say, the ideal operational goal of car rental companies would be to have 100% of the fleet occupied 100% of the time. These decisions are extremely flexible, as the fleet levels at each station are constantly being changed due to incoming returns and pick-ups. Moreover, due to imbalances on demand and the possibility to *rent here, return there*, there may exist the need to empty reposition the vehicles between stations, either with a driver repositioning a specific car or by transferring a batch of vehicles by truck. These transfers, which are critical for balancing the fleet levels across the pool, or among pools, are extremely costly. These costs may be

reduced with proper planning methods.

Booking requests and vehicle scheduling   Fleet management in car rental companies also includes the task of assigning specific vehicles to booking requests. These requests can be made with some antecedence (reservations), enabling a pre-plan of this assignment, or by walk-in customers, which require a vehicle on the fly from a specific rental station. In some companies, this assignment is decided by the rental station staff. However, for other companies, this is planned in a somewhat centralized level (e.g. pool level), especially for rental groups which have a smaller number of cars available (e.g. luxury cars). Furthermore, it is also important to schedule the planned maintenance for the fleet vehicles. This can be done simultaneously with the scheduling of the reservations.

The main characteristics of a booking request are: the desired renting group, the pickup (or check-out) date and station, and the return (or check-in) date and station. If there are no cars from the desired group available, it is common practice to offer an *upgrade*, i.e. a vehicle from a "better" rental group for the price of the originally requested group. Also, as a last resource to avoid a lost sale, some companies offer a possibility to *downgrade*, i.e. get a vehicle from a "worse" rental group for a discounted price. It is because of these strategies that it is critical for car rental companies to manage their fleet integrating all rental groups.

Due to the close links between demand and fleet occupation, and the importance that occupation has on the operational efficiency and cost structure of the company, the decision to accept or reject booking requests is also important when managing the fleet. Although some companies may fulfil all booking requests in order of arrival as long as there is capacity available, other companies manage demand by saving capacity for more profitable reservations that may arrive later, either by using complex segmentation, capacity allocation or pricing methods or simply by heuristically prioritizing reservations.

Uncertainty   The fleet management problem in car rental is severely affected by uncertainty. Demand uncertainty is the issue most recognized and addressed by car rental companies, by investing in accurate forecast methods, for example. Nevertheless, other factors bring uncertainty to the problem, with significant impact on fleet management. For example, when acquiring new vehicles and removing old vehicles from the fleet, the costs and profits associated with these decisions have a degree of uncertainty that can have a significant impact on the final decisions. Also, the availability of the fleet is often influenced by uncertainties such as unplanned vehicle maintenance and repairs or delayed car returns.

2.2.2   Lessons learned from other transportation sectors

A myriad of sectors and industries have been using quantitative approaches to optimize or improve their fleet management processes. From maritime transportation to humanitarian aid, the need to efficiently manage a fleet of vehicles is extended across strategic levels and business functions. Academic research is more prominent in certain fields, namely in those where transportation is the core business; nevertheless, interesting and innovative applications have been rising. The process of retrieving the lessons learned from other sectors

is often hindered by the lack of standardized problem names. For example, 'assignment' is a concept with several interpretations in terms of scope, inputs and outputs. 'Assignment' can either refer to allocating parts of the fleet to a specific location, to assigning a specific vehicle to a specific order/demand, or to assigning a type of vehicle to a type of order/demand. Despite these difficulties, in this section there is an attempt to understand what type of research has been developed in the fleet management context in different sectors, what parallels can be found with the car rental business, and what differences and challenges hinder its straightforward application on this field.

Airline industry   The airline industry is a traditional "comparison sector" in the car rental literature. In terms of operational fleet decisions, there is indeed a strong body of research in this field regarding fleet, tail and crew assignment, amongst other issues. For a thorough review on assignment in the airline industry see Sherali et al. (2006). Moreover, as a pioneer in the field of revenue management, the airline industry is also often referred when dealing with capacity control decisions in car rental.

The focus on fleet management problems in the airline sector has been mainly motivated by practical issues the airlines have been faced with. For example, Salazar-González (2014) describe the analytical approaches used in a real application: a research project developed alongside a regional carrier, which worked on issues such as fleet assignment, as well as aicraft routing and other problems. These approaches were validated for real-world instances and are currently being employed by the company.

Fleet assignment in airline relates to assigning aircraft types with different capacities and different characteristics to the previously scheduled flights, based on availabilities, operational costs and potential revenues. This is different from the assignment of a specific physical aircraft to a flight, which is called tail assignment. Flight scheduling, on the other hand, is related with the flight network specification, including departure and arrival locations and times, working as an input to the assignment problem. Other critical fleet management issue in this field is the rotation problem, in which an individual aircraft can be assigned specific routes among those prescribed for its own type, while satisfying maintenance constraints. (Sherali et al., 2006; Clausen et al., 2010)

The tail assignment problem has strong similarities with the vehicle-reservation assignment problem in car rental, as the "services" for each specific physical vehicles are being scheduled. Also, the allocation of vehicle types to flight legs can be comparable with tactical decisions in car rental. In fact, time-space networks and the representation of connections and flight legs with arcs is often used in this field; this is also frequently considered in some car rental fleet decisions in Section 2.3. However, the main difference that motivates the specific study of the car rental problem is the mobility of the fleet, namely the possibility to empty reposition the vehicles. This flexibility comes from the costs that, although important, are still low enough to make these decisions profitable; in the airline industry, however, the costs to do the same are prohibitive. Moreover, in the car rental industry, the process of buying and selling vehicles comprises important decisions that, due to the business characteristics, turn out to be essentially operational, namely as far as timings and acquisition modes are concerned. In the airline industry, these variables are

not usually considered alongside other fleet management issues and the decisions are taken in a more hierarchical and sequential way, once again due to the heavier costs associated with fleet decisions in this sector.

Maritime transportation   Research on maritime transportation can bring interesting insights to car rental fleet management, especially regarding the decisions on size/mix. In this industry, the decisions on how many ships of each type are needed to meet demand are made periodically and often multi-stage approaches are used, thus representing the fleet renewal problem. Pantuso et al. (2014) present a thorough survey on these problems. The authors also state that models on fleet size often include the decision on assigning specific ships to pre-determined routes as well. In this industry, demand uncertainty is high, such as in car rental. However, the supply is much slower to adapt to peaks in demand as the lead time to acquire new ships is significantly higher than the one to acquire new cars. Nevertheless, one of the lessons learned from this industry should relate with the detail given to the mode of acquisition and disposal of the vehicles from the fleet. As examples, issues often considered are the possibility of chartering in and chartering out, or laying up, i.e. keeping the ship idle at a specific location with reduced crew and costs. Also in this field, it is critical to accurately represent fleet heterogeneity.

In summary, research developed on maritime transportation fleet management may bring significant insights for the car rental business, especially when the heterogeneous fleet renewal process is considered. Nevertheless, once again, the flexibility in buying and moving car rental fleet brings important advantages to the process and significant improvements may be gained by considering it explicitly (versus directly applying the research developed for maritime transportation to car rental).

Rail-freight   In this context, the empty vehicle redistribution is core. Deciding the railcar distribution, i.e. where to send empty railcars to meet the next order, can be seen as an assignment to specific customers. Yet this is usually made at bulk, and not specifically for each vehicle. This problem has been studied since the 1990s (Spieckermann and Voß, 1995; Sherali and Suharko, 1998); nevertheless, it is still relevant and motivating research up to this date (Gorman et al., 2011). From these works, many parallels can be driven with the car rental business, as the mobility and flexibility of the vehicles is similar. One important idiosyncrasy of the car rental business, however, is that "orders" must be met without delay, at the risk of being lost, and cannot be, for example, backordered. In rail-freight, there is also plenty of research on fleet size, a problem commonly integrated with empty reposition decisions, such as in Sayarshad et al. (2010); nevertheless, this sector usually deals with a more homogeneous fleet than in car rental and substitutability between fleet types is not such a critical issue.

Trucking   In the truckload carrying industry, several fleet management problems arise that have some parallels with the car rental industry. Powell (1991) reviewed optimization models and algorithms for problems such as the assignment of drivers to pending loads or the distribution of vehicles among locations and dynamically moving them to meet new

demand. Several constraints make this problem significantly different from the car rental one, such as maximum tour length restrictions, time windows on pick-ups and deliveries, and the possibility to seize backhaul opportunities. Nevertheless, research in this area has been growing and, for example, fleet size and balance has been on the focus of recent works (Żak et al., 2011; Carbajal et al., 2012).

Shared-use vehicle services   Shared-use vehicle services, such as car or bike sharing, are not a *traditional* transportation sector per se, since the academic and practical interest in this topic has emerged in the past decades. Nevertheless, since the mid-1980s, there is a growing interest in the academic community in this sector, related with different operational problems (Shaheen, 2013) that have also several similarities with the car rental fleet management problem. In fact, these similarities may allow for lessons from vehicle sharing systems to be learned by the car rental industry and vice versa.

Barth and Shaheen (2002) broadly define shared-use vehicle systems as fleets of vehicles used by different users throughout the day. The authors propose a framework for classifying these systems, which can follow different operational modes, such as car sharing, where a network of strategic parking locations is available for the user to pick-up and return the car, and station-car, where vehicles are deployed near metropolitan rail stations to be used by rail commuters. Gavalas et al. (2015) present a comprehensive review of algorithms for the management of shared-use vehicle systems, enabling a comparison of both sectors in terms of issues, goals, scope and approaches. The authors review the methods applied regarding the design of the vehicle sharing system – including network location, fleet size and deployment between stations –, which shows several similarities with the car rental business. The authors also focus on customer incentivisation schemes to help the balancing the fleet and on the operational improvement of the vehicles empty transfers.

In fact, key lessons can be learned from this sector especially regarding the deployment of the fleet between stations in order to meet unbalanced demand. Nevertheless, although the motivation is similar in both sectors, the business and operational approaches differ significantly (e.g. pricing schemes and vehicle reposition strategies). The main differences between the two sectors are related with the heterogeneity of the car rental fleet, and the "scale and scope" of the operations. The latter issue arises from the fact that vehicle-sharing systems have a daily focus, as defined by Barth and Shaheen (2002), and are usually centered in a single metropolitan area, while the car rental scope is usually wider in terms of time horizon and geography. These differences may lead, for example, to a greater need of detail (e.g. pricing in the car rental industry is usually almost individualized per reservation).

Nevertheless, the differences between the two sectors are becoming blurry as shared-use vehicle systems get operationally more complex (e.g. by introducing heterogeneous fleets). Therefore, besides the lessons learned, also some methodologies developed for the car rental industry may be adapted to meet future needs of shared-use vehicle systems.

Other sectors   Other sectors deal with interesting fleet management problems, although research is not as developed as in the above-mentioned industries. For example, in un-

derground mining, there is an interesting fleet management problem which involves dispatching load-haul-dump vehicles that move in forward or reverse mode on bi-directional segments of the network. In these systems, there are constraints on the capacity of the network, and so the continuous monitoring of the traffic on the network and re-optimization is critical. (Gamache et al., 2005; Beaulieu and Gamache, 2006)

In humanitarian aid, fleet management is also critical due to its impact on overhead costs. Martinez et al. (2011) focus on the importance of developing research in this specific field and, using a case-based approach, attempt to understand how this type of fleet is actually managed by international humanitarian organizations, what are the critical factors that affect it, and how it impacts the aid programs. Identifying optimal vehicle procurement policies, for example, is an important problem in this sector (Eftekhar et al., 2014).

Another interesting and different sector is presented by Perrier et al. (2007), who survey winter road maintenance research, including fleet sizing and fleet replacement models and algorithms for plowing and snow disposal. Also in fleet sizing, an interesting case study regarding the fleet of towing tractors in airports is presented by Du et al. (2016). The model developed in this work considers relevant realistic constraints such as vehicle lifetime allowed and different removal options, such as selling.

There are traditional sectors where fleet management optimization models and algorithms have been developed with detail for the past years, such as the airline industry or maritime transportation. From this body of research, the car rental sector can and has been getting insights and its basilar foundations; nevertheless, there are business specificities that motivate specific research in this field.

Therefore, in the car rental industry, although many lessons can be learned from traditional transportation sectors, there is also the need to develop quantitative methods for the fleet management problem that take in consideration the business specificities, thus enabling its actual use in the real-world context.

2.2.3 Overview on the discussed problems

In Section 2.2.1, the fleet management problem was described for car rental companies. Several sub-problems were identified:

- The network division in pools of stations,
- The fleet size and the mix between different types of vehicles,
- How and when to acquire new vehicles for the fleet and how, when and which vehicles to remove,
- The distribution of the pool fleet among rental stations and how to make the vehicles available at the corresponding rental station,
- At what price to sell a specific product (combination of vehicle type, start date and location, end day and location, antecedence of the request),
- Which booking requests to accept and reject,
- What type of vehicle to supply for each accepted booking request,

- What booking requests to schedule to a specific vehicle, considering its specific requirements and attributes (e.g. maintenance requirements, planned removal date).

Some of the problems wherein considered are often studied under the revenue management framework, namely the ones where pricing and capacity allocation decisions are concerned. In the next sections, we will be arguing that fleet and revenue management issues should be integrated, due to two main reasons. On the one hand, there is often an ambiguous boundary between the two fields: for example, some authors defend that some decisions "traditionally" seen as more operational (e.g. deciding the number of vehicles) are actually functions of revenue management, and early records of the implementation of revenue management systems in car rental companies also included this type of operational decisions. On the other hand, even if one is able to define a clear boundary between the two fields, there are strong links between decisions belonging to different domains. In car rental, it is possible to observe that the main drivers of revenue are the demand, which is highly price-sensitive, and the occupation levels, which have a strong impact on the allocation of operational costs. For example, decisions on prices for different rental stations have significant impact on the demand levels in each station, leading to a need to re-balance the fleet levels in the pool in order to meet demand. In a different perspective, a specific pricing strategy can also be used in order to "push" demand for the rental stations where availability is higher.

It is also important to consider that these connections between "revenue management issues" and "operational/fleet management issues" are present in other industries as well. For example, Guerriero et al. (2012) address the problem of accepting/rejecting requests for a fleet of trucks of a logistics operator, which is seen as a capacity control revenue management problem, considering as well the operational/fleet management problem of empty repositioning trucks between locations.

2.3. Literature review on car rental fleet management

Literature on car rental fleet and revenue management is not plentiful, yet it has been blooming in the past decade. This literature review aims to present the different research streams in this field and discuss its main developments and opportunities. It is structured as follows. As an introductory note, the main contributions of the review presented in this paper are presented, comparing with the only review previously published for car rental fleet management. Then, the seminal works in the area, from the 1970s-90s, are presented. These describe the first experiments of car rental companies with revenue/yield management as well as fleet management decisions, following the ones in the airline industry. Afterwards, two basilar works from the 2000s that made significant contributions, either by structuring the field or by adapting the general models to the reality of the business, are discussed. The remainder of the works on the field are then presented, organized by problem/issue. To the best of our knowledge, the 23 papers analysed within the next section comprise the full body of work to date on car rental fleet management. Figure 2.1 represents graphically the structure of the proposed literature review and will be used as a pointer in the remainder of this section.
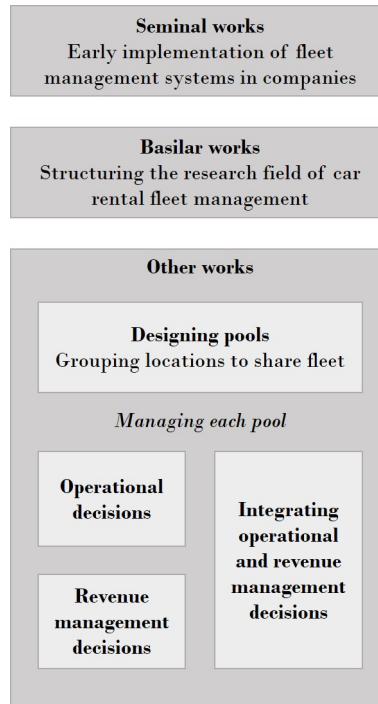
Figure 2.1: Literature review structure. This figure represents the structure of this section; the blocks represent the type of papers reviewed, which are divided in three main categories. The last category is divided in smaller sub-categories, by sub-problem

A discussion is then presented, based on revenue management and operational issues in the car rental context, with a small note on the general size of the problems. The approaches and methods used for the different problems are also reviewed.

2.3.1 Literature review

Yang et al. (2008) propose a review of the literature on the car rental logistic problem. Due to the scarcity of literature on this topic to date, the authors compare some specific problems with the ones faced by the airline industry. In fact, similar problems are relevant in both industries. However, there are significant idiosyncrasies of the car rental business that justify a more detailed analysis of the sector and there is sufficient potential of growth in this area that justifies a more challenging/critical approach to the proposed frameworks. The review by Yang et al. is descriptive and heavily dependent on the work of Pachon et al. (2006). Nevertheless, it suggests some interesting future research directions, such as the focus on vehicle-reservation assignment, which would later be developed (Hertz et al., 2009; Oliveira et al., 2014). It also pinpoints the importance of better demand forecast models, which include more realistic features such as no-shows; it will be argued in this chapter, however, that the need to include uncertainty should be considered in a broader manner than forecast models.

Thus, a more recent, exhaustive, critical and ground-building review was in need, which

could enable a reviewed and comprehensive framework of the car rental fleet management problem.

Seminal works: arising from the airline industry

Research in the field of car rental fleet and revenue management arose from the industry and the first accounts describe the implementation of decision-aid systems in main car rental companies. Interestingly, since early on, the boundary between fleet management and revenue management often appears as somewhat blurry and the same decisions are often considered under different frameworks.

The first academic work in the car rental setting was published in 1977 and presented a decision support system (DSS) developed for pool control in Hertz Rent-a-Car. The implementation and analytical models developed are presented, involving strategic and tactical decisions, such as pool design, fleet size, and fleet deployment. (Edelstein and Melnyk, 1977)

The implementation of a yield management DSS also in Hertz Rent-a-Car is described in Carroll and Grimes (1995). Four main questions are answered by this system: "how many cars should Hertz have?", "where should it deploy its cars?" (fleet management), "what products should it offer?", and "what products should it sell?" (revenue management). Some important lessons can be derived from this early yet realistic and applied work. Regarding fleet size, the authors confirm the importance of the relationships with manufacturers and resellers, when planning acquisitions and removals. In fact, the structure of manufacturers' purchase plans and the means to dispose of used cars through retail car sales or through wholesale markets are pinpointed as complicating factors of the problem. The authors also make an important note regarding the two levels or perspectives of fleet size. It is important to distinguish between the strategic, overall definition of fleet size (long-term), and the adjustments made (either long or short-term), in which the system is focused. The revenue management focus is present in the two last questions, where product segmentation and capacity control mechanisms are implemented.

In the same decade, National Car Rental is also reported to achieve significant gains with the implementation of a revenue management system (Geraghty and Johnson, 1997). This system also controls the fleet planning process, such as empty transfers between stations, accelerating or retarding returns of vehicles, and redirecting new cars for the rental locations. This paper is somewhat more detailed in the methods used. It includes upgrading and overbooking decisions, as well as an heuristic to set prices based on an elasticity model that relates historic rate and demand variability.

Later, also Dollar Thrifty Automotive Group described their efforts on Revenue Management, namely their efforts on measuring the impact of these decisions using the 'Performance Monitor' system. (Blair and Anderson, 2002; Anderson and Blair, 2004)

As a conclusion, it can be observed that the first academic works concerning car rental fleet management were generally focused on describing the practical implementation of decision-aiding quantitative methods and tools in car rental companies, as part of complex decision support systems that deal with several issues of fleet and revenue management. Therefore, this is a field that was born from a practical need in the industry, where most

relevant problems and issues were generally tackled within an integrated system.

Basilar works: structuring the field

The basilar modelling framework for the car rental fleet management problem is set in Pachon et al. (2006). The main contribution of this paper is a sequential and hierarchical structure, which divided the planning process in pool segmentation – clustering rental locations in pools that share the same fleet –, strategic fleet planning – deciding fleet size for each pool –, and tactical fleet planning – deciding fleet levels for each location within a pool, and consequent "empty transfers" between locations. Besides the modelling framework, the authors propose solution methods to deal with its computational burden. Although presenting some over-simplifying assumptions, the formulations presented were the cornerstone for future works. In the strategic fleet planning step, when deciding the optimal fleet size, the authors also decide on the number of acquisitions and removals, although only in the form of leasing from and returning to the manufacturer. Also, substitution between vehicle types is not considered and hence the problems are separable by car type.

The authors consider that determining the optimal fleet size and mix for each location on a daily basis is a "primary function of revenue management". This supports the claim for a comprehensive overview of these problems and leads future works towards this goal, by considering the integration of "traditional" revenue management functions of pricing and capacity control in the fleet management framework. The operational problem of assigning accepted booking requests to specific vehicles, which is not considered in this framework, is also a clear example of ambiguity between fleet and revenue management in car rental, since it links operational issues, such as empty repositioning, and revenue management issues, such as capacity control.

Other core work in the car rental fleet management area is developed by Fink and Reiners (2006) that propose a realistic approach to the fleet size and mix problem, considering acquisitions and removals. This paper presents a model for this problem that includes several real-world issues that make this a realistically implementable model, such as considering multi-periods, a country-wide network, groups with partial substitutability, among other characteristics. Other contributions include a detailed description of the problem faced by car rental companies, with key details such as the typical life cycle of a car. Also, the authors propose a system architecture for a DSS that includes the optimization model. A relevant simplifying assumption in this work is that its scope excludes the relationship with car manufacturers and resellers. Therefore, the acquisitions and removals are, as before, seen as leasing contracts with virtual depots for car pickup and return, not accurately representing the actual buy-and-sell process.

Other relevant works: developing the field
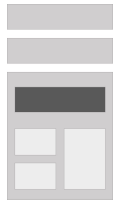
Designing fleet-sharing pools   Following the framework proposed by Pachon et al. (2006) and starting with the arguably most long-term and strategic decision – pool segmentation –, it is possible to conclude that it has not received much attention from the research community. Yang et al. (2009) study the problem of grouping locations in pools with the objective of minimizing the number of pools with a similar approach to the one proposed by Pachon et al. (2006), yet they also consider the decision on the pool logistic center, i.e. the rental station that will be coordinating the shared fleet. The authors propose a model and an approximation algorithm. One key issue pointed out by the authors is that defining the pool should encompass some flexibility, as this design is not directly correlated with administrative delimitations. This supports the claim that capacity decisions can easily be reviewed periodically, even on the design of the pools of locations that share resources.

Managing the fleet in each pool
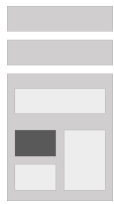
Focus on fleet/operational decisions   Within each pool, the decisions most often considered are the fleet size – how many vehicles to have in a specific pool – and deployment – how to distribute the fleet among locations, and how to empty reposition the fleet to achieve that. In fact, there are two perspectives in the literature regarding fleet size, defending that it should be set either i) considering each pool independently or ii) all pools simultaneously, at the time of pool design. Perspective ii) is presented in Pachon et al. (2006) while perspective i) is adopted by the remaining papers that deal with fleet size, where all rental locations are considered as part of one inseparable pool. The problems of fleet size and fleet deployment, typically with different decision time horizons, are often solved in an integrated manner. You and Hsieh (2014) model these two problems with a mixed-integer non-linear formulation and proposed a hybrid genetic-based algorithm to solve it. The main limitation of this work is the oversimplifying assumptions made. For example, the authors consider that all rentals take only one day, and thus, at the end of the day, the cars are all returned to a certain station.

Also Li and Tao (2010) deals with both problems, presenting a two-stage dynamic programming model where the fleet size is the first-stage decision and the vehicle transfer policy is the second-stage decision, as well as an heuristic approximation that shows good performance in determining fleet size. This work assumes that there are no lost sales, as it is possible to subcontract capacity. Other assumptions, however, can be challenged for their realism, namely that there are only two rental stations and that all rentals last only one day.

Song and Earl (2008) propose an event-driven model, not specific for car rental, that integrates also fleet size and transfer. The authors show that the policy for empty repositioning is of threshold control type; the explicit form of the cost function under threshold control is derived and used to calculate optimal fleet size and threshold values. Uncertainty in empty vehicle repositioning time is modelled using an exponential distribution yet it is

shown that the method carries over to a range of distributions. No lost sales are assumed and an extension for hub-and-spoke systems is presented. Nevertheless, as the focus is not solely in car rental, some assumptions may not be completely adjustable, namely considering the system as only "two-depot" and considering that the arrival of loaded vehicles (which is, in the car rental context, the check-in of a reservation) is determined only by travel time.

Pachon et al. (2003) study the fleet deployment problem for car rental companies, considering fleet size as a given parameter. A stochastic model representing the problem is proposed, and then decomposed into two-sub-problems: deployment – decide fleet levels in each station –, and transportation – decide how to reposition cars among stations. The deployment sub-problem is formulated and solved as a static inventory control problem and the transportation sub-problem as a linear optimization program. A heuristic is developed to reduce the gap of the decomposition approximation. Some extensions are also considered: the cost of unsatisfied demand and excess fleet, service level constraints, and price elasticity of demand, where the authors present the sufficient conditions of optimality and then retrieve from the literature a price-elasticity demand function that fulfils them. Despite considering only one type of car and one-day rentals, the models proposed are still today significantly relevant.

The operational decision of assigning vehicles to reservations is usually studied as an isolated problem. Ernst et al. (2011) present a mathematical formulation for the assignment problem and its Lagragean dual problem. To solve this formulation, the authors use the Wedelin method by incrementally updating the Lagrangean multipliers. They also propose an heuristic based on the upper and lower bounds found, that shows a good performance on building the schedules and also on providing good lower bounds. This model considers multiple types of vehicles with substitution, planned maintenance requirements and planned vehicle disposals for specific vehicles. The schedules are meant to be rebuilt daily, although protecting already accepted reservations.

In Oliveira et al. (2014), a network-flow model formulation of this problem is presented, considering interdependencies between rental groups, vehicle maintenance and disposal, and also different reservation priorities. The authors propose a relax-and-fix heuristic procedure, which includes a constraint based on local branching that enables and controls modifications between iterations.

Hertz et al. (2009) solve the assignment problem in car rentals assuming that each day it is possible to buy and/or to subcontract more cars to satisfy the requests and also considering that some maintenance hours had to be scheduled for each vehicle within certain constraints. The authors propose an heuristic solution that combines two tabu search procedures with graph optimization techniques. The main difference is related with the capacity constraints; there is herein a tacit understanding that the fleet size is a tactical or short-term decision: if requests exceed the stock, it is possible not only to upgrade but also to subcontract or buy new cars. The constraints on maintenance are detailed and consist on the maximum time of use without maintenance and a capacity constraint on maintenance work, which is characterized by duration and number of workers. The main difference between this work and Oliveira et al. (2014) is that Hertz et al. (2009) do not consider that it is possible to reposition vehicles for demand to be fulfilled. Therefore, the formulation is

focused on the dimension of "time" rather than "space": the reservations are not characterized by their starting and ending location, and the vehicle availability is analysed in terms of time (*when* it will be available), not considering the location *where* it will be available. This work arises from the ROADEF'99 international challenge, where these details were set. The authors also describe four other heuristic approaches presented in the challenge to the same problem, and compare the results obtained.

Focus on revenue management decisions   Research in these car rental issues has recently been blooming, namely under the revenue management framework, especially regarding capacity controls. That is to say, the problem of whether to accept or reject the booking requests that arrive. Conejero et al. (2014) actually tackle this problem without explicitly considering it a part of the "revenue management functions". The authors model this problem as a time-expanded network and propose an iterative algorithm to solve it. A first algorithm checks for admissibility (i.e. whether a reservation can be accepted) by finding a maximum flow on an auxiliary network, based on the Ford-Fulkerson approach; the authors then propose an iterative method based on a simplification of the auxiliary network. This paper is focused on the impact of one-way reservations in the fleet (im)balance. As the main application of this work is for the rental of *electric* cars, this is especially critical, due to the constraint on space for charging on drop-off. The main limitation of this work is the non-existence of empty repositioning flows. The aim of the work was indeed to balance the fleet without recurring to the repositioning; nevertheless, it would be interesting to analyse the profitability of its implementation.

Guerriero and Olivito (2014) study the issue of accepting or rejecting reservations using revenue management techniques. The authors propose a dynamic programming formulation and use linear approximations – i.e., static models solved "dynamically" by updating demand and capacity information – to derive acceptance policies based on booking limits and bid prices. The authors consider the existence of walk-in booking requests and the possibility of upgrading. The performance of both policies is compared under different circumstances.

The main focus of Steinhardt and Gönsch (2012) is the integration of the accept/reject decisions with planned upgrades. The authors propose a dynamic programming formulation, and two decomposition approaches (in days and in resources) and heuristics to solve the problem. This work also has a significant contribution to the utmost relevant discussion on the concepts, importance and implementation of upgrading mechanisms in car rental (see Section 2.3.2 below).

Regarding pricing decisions, Oliveira et al. (2015) describe the implementation of a DSS to update prices for a car rental company in the websites of e-brokers that compare prices in the market. The decision on price updates are controlled by an adaptive heuristic procedure, which is based on actual and desired occupation levels.

Integrating both perspectives In Haensel et al. (2012), the capacity control problem is integrated with fleet management decisions, more specifically fleet deployment and fleet repositioning. In this paper, a two-stage stochastic programming model for booking limits and transfer decisions for one type of car is proposed. The first-stage decisions are related with the capacity control (booking limits) and vehicle transfers and the second-stage decisions, after the uncertain demand is disclosed, represent the number of capacity actually "sold". A small case study is used to compare the deterministic and stochastic versions of the model. One simplification that can arguably cause significant changes in the structure of the problems is the fact that only round-trips, which start and end in the same rental station, are allowed.

To the best of our knowledge, only Madden and Russell (2012) deal with pricing decisions integrated with fleet management issues in the car rental context. In this work, the authors tackle the issue of pricing together with fleet deployment. In fact, it is interesting to investigate the similarities and differences between the two approaches – quantity-based and price-based revenue management – in the context of car rental (see Section 2.3.2). Madden and Russell (2012) propose an integer model based on a time-space network of rental locations, each with supply and demand for various car types based on the pricing level, that optimizes the choice of price levels together with relocation decisions. The dimensionality of the problem derives from the discrete approach to the choice of price levels and thus a linear programming formulation solved on a rolling horizon basis is proposed as an approximation. This unique formulation is based on the idea that pricing should help re-balance the fleet, through its impact on demand. Nevertheless, with this approach there is still the need to accurately describe the relationship between price levels and demand, a vulnerability which often makes the implementation impractical.

A different approach to revenue management, specifically for the car rental business is proposed by Anderson et al. (2004). The goal is to define acceptable prices and number of cars available for rent at a given price . The authors show that car rental is similar to "swing contracts" in electricity or gas markets, as the company is holder of swing-like options on car rentals. Prices are random variables, function of the remaining time to the start of rental and available inventory, modelled by a stochastic differential equation. In this work, only one type of vehicle is considered, and no upgrades are allowed. Some notes regarding the behaviour of prices are interesting to analyse, such as: although prices fluctuate, they seem to be bounded above, due to the "competitive, winner take all, nature of car rental market" and the price elasticity of consumers, and below, due to the marginal costs. As in other approaches, still, the slope of the demand curve is needed as a parameter.

2.3.2 Discussion

Revenue management issues in car rental fleet management

Based on the literature on car rental *fleet* and *revenue* management, it is possible to conclude that there is some degree of ambiguity between these two functions, derived from the many conceptual links that exist between the two types of decisions in a real-world setting. It is therefore important to clarify what types of decisions revenue management tradition-

ally studies and critically assess how it has been applied in this sector, both in practice and in academia.

Van Ryzin and Talluri (2005) categorize revenue management as *quantity-based* if its primary tactical tool for managing demand is based on capacity-allocation decisions or *price-based* if it is based on prices. The choice between these two tools is dependent on the business context and on the flexibility the company has to change each of the variables, among other factors. Some industries traditionally use more quantity-based revenue management, such as airlines, while others use more price-based revenue management, such as retail.

Most previous research works on car rental focuses on capacity controls (quantity-based revenue management) (Conejero et al., 2014; Guerriero and Olivito, 2014; Steinhardt and Gönsch, 2012; Haensel et al., 2012). With the following section, we will support the claim that tackling pricing decisions is also important and adequate for car rental companies and has the potential to bring some added value to the discussion. First, the main logical reasoning to favour quantity-based revenue management will be de-constructed, building on general basilar works on both streams of research. Then, some relevant works that have been attempting to integrate or provide a common framework for the dichotomy quantity-price will be presented.

Overview on capacity allocation and pricing decisions   Netessine and Shumsky (2002) introduce the field of yield management, focusing on capacity allocation decisions. The authors present the main motivation for firms to practice yield management and present the traditional tools for capacity control (booking limits, protection levels and overbooking) as well as other extensions. Herein, the authors discuss at a high level the main idiosyncrasy of car rental: the variation and mobility of capacity.

Also Van Ryzin and Talluri (2005) discuss the application of capacity control tools and their applicability to different sectors. Although not considering the specific case of car rentals, the discussion around airline companies and the reasons why they use capacity-controls may be of some interest. For airline companies, it is argued that "traditional" airlines (versus "low-cost" airlines) commit to prices on an aggregate origin-destination level and not on a departure-by-departure basis, which hinders the utilization of price-based revenue management tools. Moreover, the allocation of the resources to the different fares is extremely flexible, though subject to the capacity of the flight.

In fact, it is in the differences between these two business models that one may find the support for a different reasoning in car rental. As mentioned before, car rental companies are not subject to the same capacity constraints as airline companies, as it is easier to acquire, move and remove capacity. Moreover, car rental companies usually price their products not on an aggregate level, differentiating, for example, weekdays and weekends. Furthermore, even in the airline business, there have been changes in the past decade with the emergence of low-cost carriers and the proliferation of their pricing approach to the rest of the sector, causing a change of paradigm in practice. For example, nowadays it is easy to verify that even "traditional" airlines price on a departure-by-departure basis. Related with this, McAfee and te Velde (2006) present an interesting study that confronts the theories

in the literature on airline dynamic pricing strategies with data depicting the companies' actual pricing behaviour.

Another reason that could support a hypothetical claim that quantity-based revenue management is the only adequate approach in car rental is that this is a highly competitive market and therefore companies are price-takers. Nevertheless, Talluri and Van Ryzin (2006) dedicate a chapter of their book to the relationship between economics and revenue management and claim that, although revenue management can be seen, at first, as a kind of anomaly from the classical economic models (for example, the wide dispersion of prices in the airline market may not be expected under intense competition), in the real-world contexts there are many economic forces at play that should be considered. For example the authors demonstrate that, even in a perfect competition setting, if there is a pre-commitment to capacity and demand is uncertain, price dispersion, either among companies or within the same company, is the unique competitive equilibrium; this is derived from the structure of the competitive market and not from the revenue maximization goal per se. For perfect competitive markets, this is also true when peak-loads exist, even if uncertain, or advance purchase discounts are applied. Other perfect competition, monopoly and oligopoly situations, possibly more adequate for the car rental business, are also analysed by the authors.

In fact, nowadays, pricing is getting more and more dynamic, since it is possible to gather data in real time and since the internet allows the price-updating process to be significantly easier and faster (Bitran and Caldentey, 2003). In this work, Bitran and Caldentey review the main pricing models in revenue management and their importance *within* the capacity and inventory decisions and claim that prices are very efficient variables that managers can use for controlling demand. Also, Şen (2013) shows that the use of dynamic pricing strategies may have a significant impact on the revenue of companies, even if simple dynamic heuristics are used to change prices based on the remaining product inventory. The author aims to emphasize the impact and benefits of this practice, which had been, on this perspective, not as present as needed in the revenue management literature, mainly due to the inherent computational difficulty of the method.

One may thus conclude that the logic and arguments that excluded pricing decisions from the demand-control toolbox of revenue management in this context have been vanishing with recent developments in the technology and business models used by car rental companies.

Integration of capacity allocation and pricing decisions   To the best of our knowledge, the first seminal work that attempted to integrate pricing decisions with allocation decisions in a similar context was authored by Weatherford (1997). Here, different types of joint pricing and allocation problems for a perishable-asset problem are studied, considering either the presence and absence of demand diversion and nesting. With this approach, the prices, which were given as inputs for traditional models, are considered as decision variables alongside the capacity allocation.

More recently, Feng and Xiao (2006) study the integration of pricing and capacity allocation for perishable products with significant contributions, namely the notion of maximum concave envelope for an arbitrary set of prices. In this problem, at any time one or

more customer classes are served and other classes may be declined. After choosing a class to serve, the pricing decision occurs, selecting from within a specific price set. Demand for each price is modelled as a continuous Poisson process and its intensity is dependent on time. Aiming to fill the capacity at the highest possible prices, the suppliers must decide simultaneously which classes to serve and at what prices. Despite the significant contributions to solving this problem, the authors recognize that the assumption on the ability to set prices might be too restricted. Also, a possible challenge of this integration is highlighted, as this perspective may invalidate the most favoured nested policy in capacity allocation, which defends that if a certain class is served then all higher classes must be served.

Following the efforts to integrate the two perspectives, Maglaras and Meissner (2006) propose a common formulation for a dynamic pricing strategy and a dynamic capacity allocation rule that controls when to accept or reject new requests for a multi-product situation. Another significant contribution from this work is a useful simplification for the multi-product dynamic setting: an equivalent formulation in terms of resource consumption rather than demand rates that significantly reduces the dimensionality of the problem.

Finally, some authors defend that, when given the choice between price-based or quantity-based revenue management – which has been most used in car rental –, it is possible to argue that pricing is the most advantageous approach, as it achieves the same function as quantity-based tools – rationing supply and limiting sales – but doing so in a more profitable way (Gallego and Van Ryzin, 1997). Nevertheless, the authors favour integrated approaches, defending that "there is a growing consensus among researchers and practitioners alike that the pricing decisions that induce demand cannot be separated from traditional, capacity-oriented yield management decisions; these two decision are inextricably linked".

It is thus possible to conclude that there are relevant arguments that support the utilization of price as a tool to control demand in car rental, especially if integrated with quantity-based approaches such as capacity allocation.

Operational issues in car rental fleet management

There are some key operational issues in car rental which are interesting to analyse due to their relevance in different works. Upgrading is a very important tool used in the car rental business and is often overlooked or oversimplified in academic works. The empty repositioning of vehicles is also extremely relevant for most of the fleet management decisions and is considered with different levels of detail in the literature. There is also a significant variance in the costs included in the objective functions, so the cost components will also be analysed in this section. A brief overview of the profit/value of rentals for more operational models (rather than revenue-oriented ones) is also presented. Then, a discussion is proposed on how the uncertainty that affects different processes is tackled. Finally, there is a small note for the disparity on the time-horizon assumed for different fleet and revenue management decisions in car rental. This discussion will be based on the basilar and later works presented before.

Upgrades Upgrading strategies are very common in the car rental business. They are built on the concept of *substitution* among different car types. When a car type requested

by a customer is not available and a car of a "more desired type" is offered at the price of the original car, it is called an upgrade (Steinhardt and Gönsch, 2012). In this work, the authors discuss at length the upgrading strategies and their impact in car rental and state that the two main considerations are fairness and scope. The issue of fairness implies that upgrade priority is given to customers who purchase higher quality products. The scope is related with the extent of the substitution relationships between groups/products. The authors distinguish between and consider both full cascading – a group can be upgraded to any higher group (approach followed by (Pachon et al., 2006; Hertz et al., 2009)) – and limited cascading – upgrades are only allowed to the next higher group (considered by Conejero et al. (2014)).

On the one hand, the concept of fairness is often overlooked or dealt with only implicitly. On the other hand, the scope of the upgrade is often discussed and other intermediate extents are considered. For example, Guerriero and Olivito (2014) and Oliveira et al. (2014) consider that the allowed upgrades are mapped into a matrix, and in Fink and Reiners (2006) upgrades are allowed up to two higher groups. Some authors do not explicit the upgrading strategy followed yet mention that substitution is allowed (Madden and Russell, 2012; Ernst et al., 2011) There is also a lack of a common notation for the upgrading strategies: for example, Steinhardt and Gönsch (2012)'s "full cascading" is also labelled as "nested demand" in Pachon et al. (2006).

Steinhardt and Gönsch (2012) also discuss two different upgrade mechanisms: an ad hoc mechanism, where the firm must immediately decide to upgrade when an upgradeable product is sold, and a mechanism that postpones the decision until the customer picks up the car. The generality of works does not refer the choice between these two mechanisms, as it seems to be most dependent on the problem.

In fact, upgrades are critical not only for the business but also for the model formulation. If there is no substitution between car types, the model can be separated by type and the complexity is significantly decreased, which is a reason why some works consider only one car type and, consequently, no upgrades (You and Hsieh, 2014; Haensel et al., 2012; Li and Tao, 2010; Song and Earl, 2008). Nevertheless, due to their frequency in real-world settings, realistic models do consider, at whatever extent, upgrading strategies. The choice of this extent contains a trade-off in itself: although higher upgrade flexibility leads to a higher fleet utilization, in the long-term the customers might "learn" the strategy and start to require lower-valued groups leading to revenue degradation (Fink and Reiners, 2006).

Finally, other options considered to fulfil demand for unavailable car groups are downgrades, i.e. as a last resource, offering a car from a lower group at a lower price, in Oliveira et al. (2014), and sub-contracting capacity (Hertz et al., 2009).

Empty transfers   Vehicle empty repositioning is a critical part of most fleet management problems in car rental. In the literature, however, this process is modelled following different representations, regarding both transportation time and mode. As for the duration of the transportation, some authors aim to approximate the actual transportation time, requiring that a matrix can be defined and given with the time the empty transfers take between all possible locations (Fink and Reiners, 2006; Guerriero and Olivito, 2014; Oliveira et al., 2014). This approach is more realistic although it can demand higher pre-processing

difficulties when defining the time matrix. Other works assume that all vehicles can be transported overnight and simplify the modelling process (Pachon et al., 2006, 2003; You and Hsieh, 2014; Li and Tao, 2010). The main downside of this approach is not the capping assumption on the transfer time, which can be guaranteed by the pool design, but resides on the limitation imposed on the transferring schedules. For the vehicle-reservation problem, for example, it might have a significant impact if transfers between close stations are allowed during the day.

The empty transfers can also be materialized in different modes. Fink and Reiners (2006) distinguish transfers by truck, by driving the car itself, and using combined options such as driving the car up to a point, from where it is sent by truck to the final destination. Also Song and Earl (2008) consider different modes that are characterised by different speeds; yet this is applied on a context related with containers rather than cars. No other work that is focused specifically in the car rental business considers different types of transfer when tackling the problem.

Costs Most fleet management problems in the car rental context are formulated as cost minimization problems, and even in the ones formulated as revenue maximization problems the costs play a significant role, especially if they are realistically defined. Depending on the specific problem and the degree of detail of the models in the literature, different types of costs are considered. The list below presents the most critical ones found throughout the car rental fleet and revenue management literature; thus, each work generally considers a combination of these costs:

- *Acquisition costs*: usually considered per vehicle (Pachon et al., 2006; Hertz et al., 2009). Since the buy-and-sell relationships are out of the scope of most works, adding and removing cars from the fleet is often modelled as a leasing-type activity, not including acquisition costs, or including them in a "per vehicle basis", overlooking the economies of scale, contracts and other realistic characteristics;

- *Holding costs*:
    - Leasing/sub-contracting costs, per unit of time (Pachon et al., 2006; Hertz et al., 2009; Song and Earl, 2008);
    - Operating/stocking costs, per day (You and Hsieh, 2014; Hertz et al., 2009);
    - Maintenance costs, per maintenance session and depending on type of car (Hertz et al., 2009), or per car and per day, depending on the current location (Song and Earl, 2008);
    - Penalty per day of delay in returning the car, if it is leased (Fink and Reiners, 2006; Pachon et al., 2006).

- *Empty transfer costs* (following from the discussion above):
    - Transfer cost per car, depending or not on the origin-destination pair (Pachon et al., 2006; Guerriero and Olivito, 2014);
    - Transfer cost per unit of distance travelled in km (You and Hsieh, 2014);

- Fixed transfer cost (not dependent on the number of cars), depending on the origin-destination pair (You and Hsieh, 2014).

- *Lost sales cost* (You and Hsieh, 2014).

Profit / Value of rentals Most works that are operational-oriented (i.e., not developed under the revenue management framework) do not explain how the profit gained from each rental is pre-processed. Considering the business process, one assumes that the profit of a rental is a given parameter dependent on the origin, destination, starting date, length of rent, and car type requested (Oliveira et al., 2014). In some works, nevertheless, some simplifications are assumed, in accordance with the problem and other important assumptions. For example, You and Hsieh (2014) consider a given constant daily fee for all reservations, which is increased if the car is not returned to the same place where it was picked-up (note that in this work only one car type and one-day reservations are considered).

Uncertainty Some papers address deterministic versions of the problems in car rental fleet management (Conejero et al., 2014; Madden and Russell, 2012; Ernst et al., 2011; Hertz et al., 2009; Yang et al., 2009). All of those that consider uncertainty in the process, focus on demand for a specific product, which is thus the most relevant uncertain factor in these problems. Even in deterministic versions, for those papers that have a practical application, the given demand is said to be estimated based on historical data and forecasting techniques. Most works that consider demand to be uncertain state that it follows a certain distribution, such as Poisson (Haensel et al., 2012; Song and Earl, 2008) possibly altered by seasonality effects (You and Hsieh, 2014), Normal with different scenarios for its mean and variance (Guerriero and Olivito, 2014), discrete uniform (Li and Tao, 2010), or others (Steinhardt and Gönsch, 2012).

Moreover, there are other parts of the process subject to uncertainty. For example, Fink and Reiners (2006) claim that there is a significant level of uncertainty in the turnaround process (between rentals) that can be caused by delayed check-ins, need for repair, no-shows, among other factors. Nevertheless, they do not include this uncertainty in the model. Song and Earl (2008) consider uncertainty in the empty transfer times as well, modelling them with a probability distribution.

Time span of decisions It is important to understand what time horizon is usually used for each type of problem. In fact, it will be shown that there are discrepancies between works that address the same problem and that there are overlaps between problems that are usually considered in separate strategic levels, on the modelling framework proposed by Pachon et al. (2006).

The pool segmentation main problem is to decide how to group rental locations into fleet-sharing pools. The reported time horizons for this decision were of 3 to 6 months (Yang et al., 2009). In fact, this decision is not necessarily rigid and can be updated more than once in a year to deal with changes in demand, among other factors.

Fleet size and mix is a decision that is taken monthly or each trimester (Pachon et al., 2006). However, decisions on acquisitions and removals, which logically impact fleet size,

are reported to be taken in significantly different horizons, such as weekly (Fink and Reiners, 2006).

The fleet deployment within each pool is tackled in shorter time-spans, yet shows some discrepancies and sometimes overlaps with other decisions. It can be addressed daily (Pachon et al., 2006, 2003; You and Hsieh, 2014), weekly or every other week (Haensel et al., 2012), and considering a one-month horizon (Madden and Russell, 2012). If one considers the former, seminal works, these decisions are made in a five-day horizon (Geraghty and Johnson, 1997) and the decisions that influence size in a two-months (Geraghty and Johnson, 1997) to two-years (Carroll and Grimes, 1995) horizon.

The decisions on capacity allocation, namely whether or not to accept/reject requests, are taken considering one or two week horizons (Guerriero and Olivito, 2014; Steinhardt and Gönsch, 2012; Haensel et al., 2012). Pricing decisions are said to be made considering a one month horizon (Madden and Russell, 2012).

Even though a significant amount of works do not clearly define the time horizon considered, it is possible to conclude that most decisions not only share important links but can also be made in overlapping time horizons.

Other issues   Other issues of the real-world setting of car rental fleet management problems are related with the behaviour of the consumer. No-shows – reservations made beforehand that are not fulfilled because the customer does not pick-up the vehicle – and cancellations – similar to no-shows, yet the customers notifies the company with some advance – are not usually considered. Walk-in customers – customers that arrive to a rental station and request a vehicle, without a previous reservation – are considered more often. Overbooking is a "typical" revenue management technique, yet, if applied, has significant operational implications and is usually not considered in vehicle-reservation assignment problems, in which it is critical. For example, Oliveira et al. (2014) consider that all reservations that were confirmed must be met and assume that there is always enough capacity to do so. Nevertheless, the importance of these issues is highly dependent on the problem considered.

As for each company's strategy to deal with lack of capacity, some works consider that there are no lost sales, i.e. all demand must be met, even if some capacity has to be subcontracted at a significantly higher cost (Li and Tao, 2010; Hertz et al., 2009; Song and Earl, 2008).

As for more operational issues, usually maintenance constraints are only considered when tackling the assignment of reservations to specific vehicles (Oliveira et al., 2014; Hertz et al., 2009).

Size of the problems

Most of the works discussed in the last sections applied numerical examples to validate the results. Some of these were inspired or derived from real-world settings and problems faced by specific car rental companies. It is interesting to understand the size of the problems that was considered adequate to depict the reality of car rental companies, considering the different problems. Table 2.1 presents some of the main factors that influence the size of the problems for some of the works discussed before. The factors are characterized

by the maximum values found in the instances of each paper. Not all factors that influence size are present but only the parameters derived from the real-world structure of the problem/business, which can help describe the different works in terms of their practical application.

The works that dealt with capacity allocation, i.e. accepting/rejecting booking requests, are not included in this table since the different approaches do not favour an unbiased comparison. Moreover, it is important to understand that the differences observed are often due to the level of complexity. For example, papers that dealt with fleet size/mix and deployment in an integrated manner show some discrepancies because of their significantly different goals: deriving general threshold policies versus developing models to solve real instances. Nevertheless, it was felt that this type of analysis could bring some insights, not as a comparison tool but as an overview tool for the assessment of the field.

Table 2.1: Factors that influence the size of instances tested

Problem	Paper	Factors that influence size
Pool segmentation	Pachon et al. (2006)	27 rental stations
	Yang et al. (2009)	27 rental stations[1]
Fleet size/mix[2]	Pachon et al. (2006)	12 pools, 2 car groups
Fleet size/mix and fleet deployment	Fink and Reiners (2006)	"A few hundred stations" (p. 285), 18.000 vehicles, 15 car groups, 20.000 rental requests
	Song and Earl (2008)	2 rental stations, 1 car group
	Li and Tao (2010)	2 rental stations, 1 car group
	You and Hsieh (2014)	38 rental stations, 1 car group
Fleet deployment	Pachon et al. (2006)	6 rental stations, 1 car group
	Pachon et al. (2003)	6 rental stations, 1 car group
Fleet deployment and pricing	Madden and Russell (2012)	13 rental stations, 5 lengths of rent, 5 car groups, 8 price levels, 3 market segments
Fleet assignment	Oliveira et al. (2014)	2600 rental requests, 39 vehicles, 5 car groups, 40 rental stations
	Ernst et al. (2011)	7700 rental requests, 2100 vehicles, 140 car groups, 23 rental stations
Fleet assignment and fleet size/mix	Hertz et al. (2009)	210 rental requests, 12 car groups

[1]Of these 27 stations, 11 are potential pool logistic centers.
[2]With integration of all pools in one problem

In fact, although some works already consider real-sized problems and instances, most of the assumptions made in this field still limit the applicability of the research in real-world settings. For example, it is interesting to notice that some works consider only one pool yet deal with more rental stations than the problems that aim to divide these into pools. Moreover, some assumptions and simplifications are arguably more realistic than others; fleet heterogeneity is said to be a critical characteristic of the car rental business and its inclusion on most problems significantly alters its structure, namely because of the possible upgrading strategies. In fact, the rental network design, especially the number of stations, and the characterisation of the fleet, especially the number of car groups, are generally felt to be the most important characteristics to increase realism and applicability. Nevertheless, for most problems, these are issues that have a significant impact on the size of problems and hence are often simplified.

2.3.3 Methods

Several quantitative methods have been applied in this field. Table 2.2 presents the methods and approaches followed by the works discussed in Section 2.3.1. The seminal works were excluded from this analysis since they mostly focus on decision support systems developed for specific companies, especially their structure and architecture, and the methods used are often not discussed in detail.

It is important to consider that, as it was explained in Sections 2.3.1 and 2.3.2, different assumptions and levels of "realism" were considered among the different papers. This has a significant influence in the choice of methodology, not only due to the complexity and dimensionality of the problems but also due to structural issues, such as nested upgrading strategies. Regarding the problems, it was decided to include only the general designation. However, some of the works tackling the same problem differed significantly on the assumptions made and issues considered. Regarding pool segmentation, the works of Yang et al. (2009) and Pachon et al. (2006) differ on the decisions, as the former work decides not only how the rental locations should be grouped in pools but also which location in each pool should be the pool logistics coordination center. As for fleet size, some works consider more detailed supply conditions on the acquisitions/removals issues, e.g. differentiating between leasing and buying (Hertz et al., 2009). It is important to mention that in this work the fleet size aspect is not the "core" decision: the increase of the size arises from situations of unavailability, due the requisite to fulfil all demand. However, other works consider simplified versions of the process (Pachon et al., 2006; Fink and Reiners, 2006) and others do not even consider this issue (You and Hsieh, 2014). Also regarding fleet assignment, as it was previously mentioned, Hertz et al. (2009) do not consider the possibility to reposition empty vehicles.

Table 2.2: Methods and approaches

Paper	Problems	Approach
Pachon et al. (2006)	Pool segmentation Fleet size/mix Fleet deployment	Separated optimization models (deterministic for pool segmentation and fleet size/mix; stochastic for fleet deployment) – all the following works integrate the problems Column generation algorithm for pool segmentation Decomposition approach for fleet deployment
Yang et al. (2009)	Pool segmentation	Optimization model Heuristic solution method
Fink and Reiners (2006)	Fleet size/mix Fleet deployment	Optimization model (solved using minimum cost network flow model optimization)
Song and Earl (2008)	Fleet size/mix Fleet deployment	Dynamic programming model
Li and Tao (2010)	Fleet size/mix Fleet deployment	Dynamic programming model (two-stage) Heuristic solution method
You and Hsieh (2014)	Fleet size/mix Fleet deployment	Optimization model (MIP stochastic model) Hybrid genetic-based algorithm
Pachon et al. (2003)	Fleet deployment	Optimization model (stochastic) Heuristic solution method (based on a decomposition approach)
Madden and Russell (2012)	Fleet deployment Pricing	Optimization model (MIP deterministic model) Solution method based on linear programming approximations solved on a rolling horizon
Hertz et al. (2009)	Fleet size/mix Fleet assignment	Optimization model Meta-heuristic solution method (combines tabu-search and graph optimization)
Oliveira et al. (2014)	Fleet assignment	Optimization model (network flow model) Math-heuristic solution method (relax-and-fix approach based on a rolling horizon)
Ernst et al. (2011)	Fleet assignment	Optimization model (mathematical formulation + Lagrangean dual problem) Heuristic solution method
Anderson et al. (2004)	Capacity allocation	Real options analytical model Numerical solution method
Steinhardt and Gönsch (2012)	Capacity allocation	Dynamic programming model Decomposition approach

Continued from previous page

Paper	Problems	Approach
Haensel et al. (2012)	Fleet deployment Capacity allocation	Optimization model (two-stage stochastic programming model)
Conejero et al. (2014)	Capacity allocation	Iterative algorithm based on flows on networks
Guerriero and Olivito (2014)	Capacity allocation	Dynamic programming model Solution method based on linear programming approximations and revenue management policies
Oliveira et al. (2015)	Pricing	Adaptive heuristic procedure

It is possible to conclude that a large range of quantitative techniques has been used to address the different problems regarding car rental fleet management. It is also important to mention that in some works, such as Fink and Reiners (2006), simulation models were used as a tool to evaluate the robustness and quality of the solutions attained.

Although there is a tendency to use certain tools for specific problems (e.g. dynamic programming for capacity allocation), there is not a clear "methodological dominance". This seems reasonable in a field that is still growing and whose motivation arises from diverse areas, such as pricing or logistics, that traditionally recur to different methods.

It is also possible to observe that most works present a two-fold approach, where a model is presented and, due to its complexity or the dimensionality of the instances, a solution method to solve the model is proposed.

2.4. A framework for the car rental fleet management problem

2.4.1 Proposed framework

Building on the works of Pachon et al. (2006), a framework is herein proposed, with a different aim than the former and thus with a different perspective on the problem. Rather than providing a modelling framework, the goal was to structure the fleet management problem in the full car rental context, including revenue management and operational issues, in a more holistic yet detailed "map of action". That is to say, to consider, structure and frame the interactions between the usually isolated sub-problems studied by the companies, structuring a framework that is business-oriented rather than methodology-oriented.

The main goal of this framework is to open possibilities of research rather than to categorize the existing literature. Therefore, there is an attempt to follow the flow of the business decisions and to avoid restricting the decisions, e.g. based on geographical level or decision time horizon. The only "geographical" decision considered is the pool segmentation, since it is a building-block that can be easily disregarded if one chooses to consider the rental stations as a whole group and not divided in pools. Figure 2.2 represents the proposed framework.
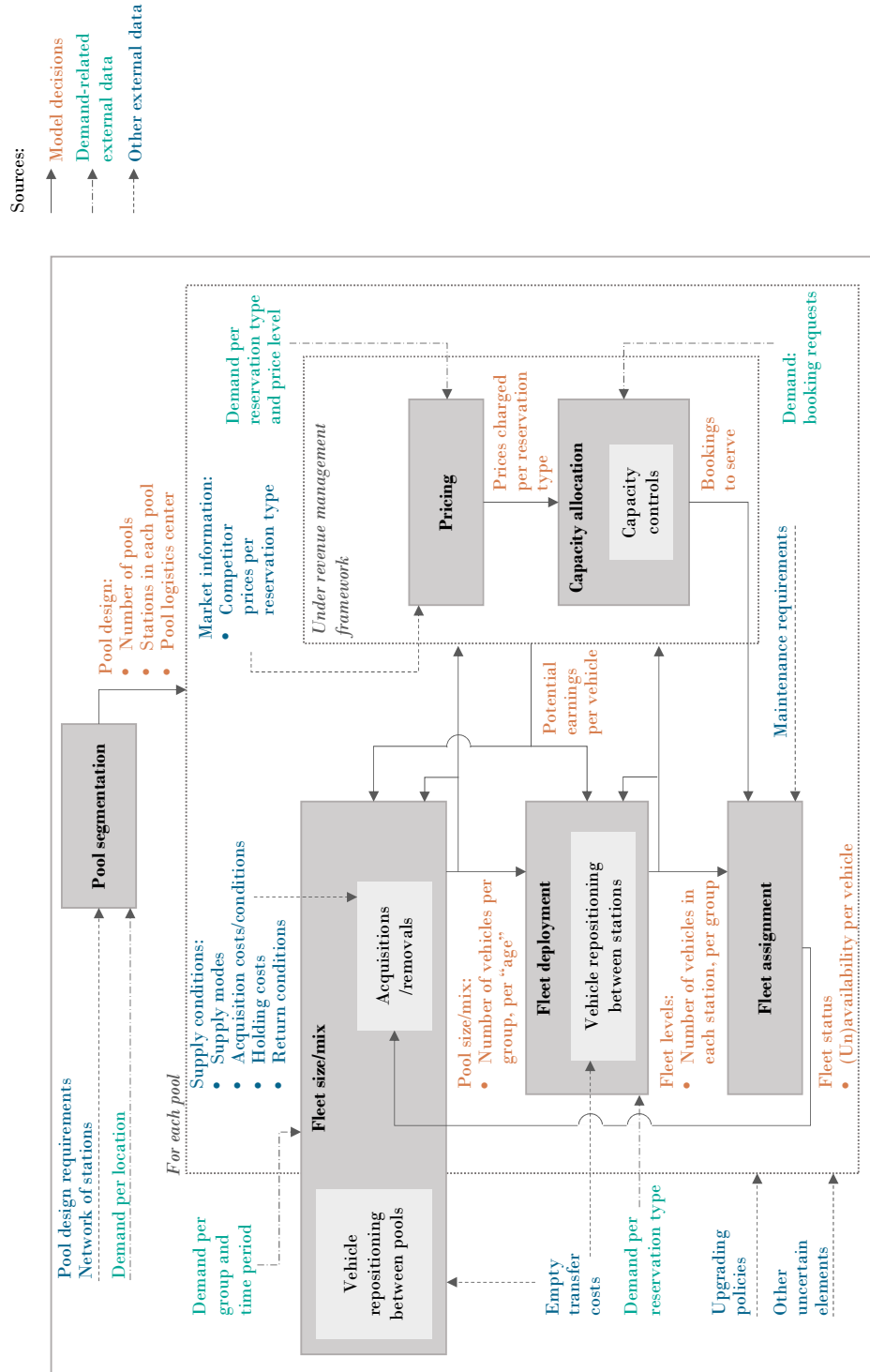
Figure 2.2: Framework for the car rental fleet management problem. The blocks represent different sub-problems and the links between them the existing conceptual connections. Other relevant inputs are also represented

The building blocks are connected by their main inputs/outputs, often in overlapping decision time horizons. Also, the key characteristic – flexibility – is present throughout the business process. In fact, most decisions are not "rigid" and can be frequently updated, from the most strategic ones, such as pool segmentation, to the most tactical ones, such as fleet assignment. That is represented in the framework with feedback loops in the processes where this is seen as critical. Moreover, the demand is a general input to the process, even though in different levels, and it should include not only unconstrained and constrained demand modelling/forecasting but also realistic issues such walk-in customers, no-shows and cancellations. The modelling of this core input is itself a research stream most relevant in this field.

As for the sources of the inputs/outputs of the building blocks, they can either be model decisions (inputs that are provided by other building blocks) or exogenous to these building blocks. Furthermore, the latter are divided in demand-related and non-demand-related inputs, due to the importance of demand modelling in this context.

As can be observed in Figure 2.2, the *Pool segmentation* building block receives as input exogenous information related with the network of rental stations of the car rental company, such as location and demand. The main decisions are related with the pool design. These will provide relevant information for all the subsequent building blocks, as they will focus on a single pool.

Fleet size/mix aims to decide how many vehicles of each group should comprise the fleet. As this involves a careful planning of the *Acquisitions and removals* – highly dependent on the time that the vehicles have been in the fleet – , it is significant to also decide on the best "vehicle age" mix. Besides "age"-related issues, other supply conditions must be considered within this block, such as supply modes (e.g. leasing), conditions (e.g. contract return date and/or "vehicle age") and costs (fixed or variable). To determine an adequate fleet size, it is also important to consider how the demand per vehicle group is distributed throughout the time horizon. The potential earnings are also critical inputs to decide on fleet size/mix. This information is highly dependent on the price setting and capacity allocation policies of the company. Moreover, if more than one pool is considered, it is important to consider that some vehicles may be currently in other pools (e.g. if the customer returned it to a different station) and it is necessary to plan the *Vehicle repositioning between pools*.

The main decisions of this block are the main inputs for *Fleet deployment*, which aims to determine how many vehicles of each group should be in each station. For this, the potential earnings are also essential inputs, as well as the empty transfer costs, since *Vehicle repositioning between stations* is a critical part of this problem. Here, the demand should be considered per reservation type. A "reservation type" encompasses reservations that require the same vehicle group and start and end in the same periods of time and in the same rental stations. This is important in order to understand not only the time distribution of demand but also its geographical distribution.

For *Fleet assignment*, it is important to know how the fleet is deployed between stations but also which reservations should be fulfilled by these vehicles, which is determined by the *Capacity allocation*. Maintenance constraints and requirements (especially dates of unavailability of specific vehicles) are also relevant exogenous information to this block. The main decision translates into a schedule of bookings to fulfil by each vehicle, which

also provides useful information regarding unavailability when planning the fleet removals.

The problems that are usually tackled under the revenue management framework receive (and provide) significant inputs to the already mentioned building blocks, namely the number of available vehicles. *Pricing* is an issue that also requires external inputs such as market information regarding competitor prices, with a significant level of detail: usually, per reservation type. The high level of detail when encompassing demand inputs in this problem is also critical, as demand is not only dependent on the reservation type but is also highly sensitive to the price level. The *Capacity allocation* perspective on the revenue management car rental problem aims to select the bookings to serve within a list of booking requests (specific reservations made by customers for a certain price and with a certain antecedence), considering the limited availability of the fleet and defining the adequate *Capacity controls*, such as bid prices.

For all the building blocks of the framework that deal with problems within a specific pool, the upgrading policies are a relevant input, as it was previously discussed, since they connect the problems for different vehicle groups. Also, relevant uncertainties that affect the process, besides demand, should be considered (e.g. late reservation returns or unplanned maintenance).

From a complete and detailed view of this framework, together with an analysis of the existing literature, it is possible to identify potential research gaps and interests.

2.4.2 Research gaps and directions

Research gaps

Despite the fact that this is a growing field, where a significant body of research is still under development, it is interesting to understand what the most and the least studied problems are. Figure 2.3 presents a "heat map" of the field, based on the framework proposed on Section 2.4.1. The intensity of the grey-scale increases proportionally with the amount of research, measured in number of papers[3]. Also, the connections between the different problems are analysed: the intensity of grey-scale is related to the number of papers that aim to integrate the connected problems. The papers considered are the ones presented in Table 2.2; therefore, it is important to notice that even the highest intensity refers only to eight papers.

The issue of fleet deployment is the most studied field, often in connection with the sizing issue. Nevertheless, the latter problem is not usually studied considering important characteristics of car rental fleet management: the inter-pool vehicle repositioning – due to the general assumption of "one pool perspective" when integrating with fleet deployment – and, especially, acquisitions and removals. As it was previously discussed, this is a significant aspect of the problem which is often oversimplified or even overlooked.

The pool segmentation problem has received little attention, as well as the fleet assignment; nevertheless, for the latter, the trend seems to be reverting in the past years. As for

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[3]Regarding the building-block "Acquisitions/Removals", included in "Fleet size/mix", only the papers where this issue was directly addressed (e.g. explicitly considering acquisition mode or removal location or date) were considered.
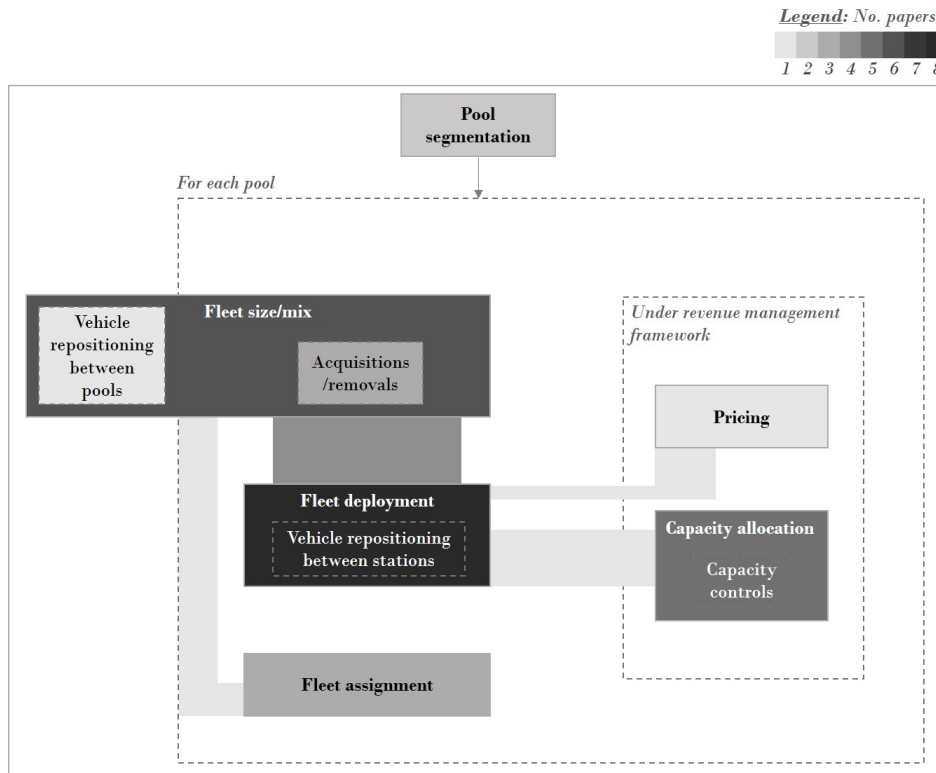
Figure 2.3: Heat map of the car rental fleet management literature. Based on the proposed framework, this heat map aims to show where most research has been focused and what areas are still developing. The grey intensity scale illustrates the number of papers focused on each problem and the integration of problems: from light grey (1 paper) to dark grey (8 papers)

the former, this may be because the resulting pools are often confused with administrative divisions, which are much more strategic and difficult to update and adapt to demand and other external factors. Within the fleet deployment building-block, it is worth mentioning that the two main decisions – deciding fleet levels in each station and how to reposition vehicles among the stations – often lead to a decomposition in two sub-problems to facilitate the resolution (Pachon et al., 2006, 2003; You and Hsieh, 2014).

It is also interesting to notice that the relationship between operational problems and the ones within the revenue management framework, which are significantly dependent on operational decisions such as fleet size and deployment, namely capacity allocation, is still not very much developed. Also, as discussed before, pricing is a core issue for this sector and it is only addressed by one paper. Capacity allocation decisions, on the other hand, are the most studied problem within the revenue management framework.

Research directions

Firstly, it is important to establish that, as this is a problem that is still growing in the academic field, there are yet no saturated problems. Nevertheless, based on the review of the

literature in the field, building on the proposed framework and keeping a close connection with the real-world applications in the car rental business, the following research directions are identified. Figure 2.4 "maps" these research directions to the framework presented in Figure 2.2, graphically referencing the sub-problems.

RD1) **Increased realism to specific problems**: due to the early stages of development of this field, there is the need to develop and extend the existing models, in order to encompass realistic assumptions, namely demand uncertainty, supply and resell relations, fleet heterogeneity and maintenance constraints;

RD2) **Operational integration**: the conceptual integration of the logistic/operational issues of fleet management (not studied under the revenue management framework) is mainly supported by the close interactions of the decisions, the discussion on their overlapping time horizons, and the inherent flexibility of all the decisions considered, even if in different degrees. The conceptual integration of the problems does not necessarily mean that a "monolithic" solution approach should be used. In fact, the challenge of dimensionality will be present and innovative solutions to deal with it will be needed;

RD3) **Overall integration**: extending the integration proposed above, the encompassing integration of all building-blocks from the framework is supported by the reasons mentioned before and by the close links between the additional decision levels, discussed in Section 2.3.2. Despite the expected advantages of this full overview of the problem, the inherent challenge of dimensionality would be increased;

RD4) **Horizontal middle-level integration**: since the middle-level problems (fleet size/mix, deployment, assignment, pricing and capacity allocation) are the ones where the connections seem to be most critical and where the time horizons overlap the most, it would be interesting, especially in the perspective of a real-world application, to aim for an integration of these problems.

2.5. Conclusions

The main goal of this paper was to present, define and structure the car rental fleet management problem. In this process, three main contributions may be highlighted due to their relevance not only to the aforementioned goal but especially to support future developments in this field of research. Firstly, the topic was reviewed, encompassing a thorough discussion on the operational and revenue management issues of the problem. Secondly, based on the literature review, a framework was proposed to structure the car rental fleet management problem. Finally, based on this framework, four main research directions for the future are discussed. Overall, we believe the framework proposed, and its resulting contributions, will assist the field in its development where the focus on the business characteristics allied with a strong methodological background will allow the application in real-world settings of the research developed. Moreover, fleet management in some

RD1) Increased realism   **RD2)** Operational integration

RD3) Overall integration   **RD4)** Horizontal integration
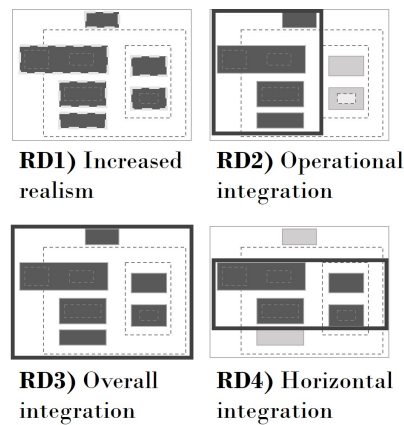
Figure 2.4: Research directions referenced to the proposed framework

innovative transportation models, namely shared mobility systems, can also benefit from the contributions developed in this field, due to similarities found in the important differentiating characteristics: fleet and decision flexibility. Some interesting future work may therefore also lie in the expansion of this framework and resulting work to these systems.

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Practical motivation: Pricing decision support system

In this chapter, we present the paper that resulted from a short-term project developed alongside the car rental company Guerin. *This project focused on tactical pricing issues and allowed to better understand the business, its characteristics, current practices and main needs in terms of decision-support tools. Here, the main relationships between pricing and capacity are highlighted (although not tackled), thus providing the practical motivation for the following chapters.*

Pricing for Internet Sales Channels in Car Rentals

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Abstract   Internet sales channels, especially e-brokers that compare prices in the market, have a major impact on car rentals. As costs are heavily correlated with unoccupied fleet, occupation considerations should be integrated with swift responses to the market prices. This work was developed alongside Guerin, a Portuguese car rental, to build a tool that quickly updates prices on e-brokers websites to increase total value. This paper describes the specificities of the problem and their implication on the solution, and presents an adaptative heuristic to update prices and the system's architecture.

3.1.   Introduction

Similarly to other tourism-related sectors, the car rental business has been deeply impacted in the past years by internet sales, namely by the development of a new sales channel: broker websites that compare the offers of different competitors. The impact of this channel

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is especially relevant for car rentals due to the lack of differentiation of their product, if compared, for example, with the hotel business. As the vehicles and pick-up stations are the same and the clients are able to compare all the offers in the market with full transparency, the price takes an even more determinant role in their decision.

The ultimate goal of every company is to maximize its revenue. For car rental companies, the main slice of the costs is related with unoccupied fleet. Therefore, the "revenue challenge" deals not only with uncertain demand (highly dependent on the companies' positioning versus the other prices on the market); it also deals with the need to maximize the occupation of the fleet for each day, ensuring the cars were booked at the highest possible price. Moreover, for the e-brokers channel, it is of the utmost importance to be watchful and agile in order to respond to changes on the market prices. This requires processing a massive amount of data in an extremely short time-frame.

This paper presents the work developed in *Guerin Car Rental Solutions*, a Portuguese car rental company, to build a tool that allows for a swift, systematic, regular and profitable update of all the company's pricing positions in the market. This tool is based in an heuristic procedure that is adaptative in the way it continuously corrects the prices responding to the changes in the market conditions (demand and competitors' prices), in order to attract the right (number of) customers at each point in time and thus increasing the value collected for the fleet each day.

3.2. The problem

3.2.1 Brief description

The main decision of this problem is setting the price to charge for a specific search that a customer makes online a certain number of days in advance (e.g. 30 days beforehand). In fact, this problem is highly dependent on the *antecedence* of the search versus the start of the reservation. The e-brokers channel is typically used by the *leisure* segment, allowing prices to vary over time (this is not true for the corporate segment, for example, where other reservation formats, such as pre-established contracts, may limit the price variation). Therefore, there is a need to balance the goal of occupying the fleet and the goal to do so at the highest possible price.

A search is characterized by the e-broker website where it is made, the starting date (vehicle pick-up) of the reservation, the rental length (in days), the pick-up station, and the type of vehicle required. The *number of days in advance* is calculated based on the starting date of the reservation (see Figure 3.1).

The goal is to develop a tool that is able to calculate, in short intervals (e.g. every two hours) the prices to set for each search the customers can make.

3.2.2 Important characteristics

One of the characteristics of this problem that influences the most the pricing decision-process is the *transparency* between competitors. When searching online on the e-brokers website, the customers search for a specific vehicle, location and dates and the results are

Figure 3.1: Example of a search

retrieved for all competitors with an offer in the market, usually organized by price. This leads to a *best deal* effect, in which the competitor offering the lowest price gets the most attention, even if the price difference to the others is marginal. This can only be surpassed by some competitors, whose considerable market share and customer awareness can trigger customer preference even if their price is not the lowest in the market.

However, one can note that although the customers have full transparency between competitors, their searches are usually focused on specific dates and locations, *hindering the transparency between prices of the same company*. For example, if a customer wishes to rent a car for a leisure trip, he/she is not likely to change the trip dates (or the region of the car pick-up) in order to get a cheaper deal (this is not necessarily true for other businesses such as airline). Therefore, there is a *flexibility* for the car rental to set the prices of different searches independently of each other.

Nevertheless, it is important to bear in mind that the prices of the different searches are not completely independent: 1) the main slice of the costs of car rental companies derives from unoccupied fleet; 2) fleet occupation and price influence each other; low prices increase the pace at which new reservations are made, increasing the pace of fleet occupation (and high prices have the opposite effect) – it is thus possible to use the prices to accelerate/decelerate occupation rate; 3) a fleet can be occupied by a myriad of different searches (and thus prices) – in fact, the fleet of a certain vehicle type in a certain region is, in a certain day, occupied by reservations made in different e-brokers that started in different days and stations and will have different durations. The main challenge faced was thus related to the amount of data to process in order to calculate the price for every *search*, in a short period of time (two hours), including processing the current prices in the market for each competitor for each search, and the occupation of the fleet(s) affected. For the company considered in this paper, the amount of different searches for which updated prices must be calculated at each iteration is in the range of the tens of thousands.

3.2.3 Literature review

This problem is herein regarded as a revenue management problem. In Talluri and Van Ryzin (2006), revenue management is related with three different types of "demand-management decisions": structural decisions (related with the configuration of the sale – e.g. auction), price decisions (related with the price to set for different products, customers, product life-cycle, ...) and quantity decisions (e.g., related with the allocation of resources to segments). The authors state the importance of recognizing the business context so as to understand the relevance of price and quantity flexibility, which will have a deep impact on revenue management strategies. For example, in the hotel business, it may difficult to increase capacity of rooms whilst prices are significantly easier to change. In fact, in the problem tackled in this paper, the main focus is on *price decisions*, as the flexibility to change prices is significantly superior to the one to change capacity. In Netessine and Shumsky (2002), business characteristics that justify why companies adopt revenue management programs are reviewed. It is possible to verify that the problem herein described presents these characteristics. Firstly, as in most service-oriented problems, excess resources are impossible to store (in this case, if a car is not used some day, the capacity is lost). Also, pricing decisions are made with uncertain demand, and different customers with different willingness to pay have different demand curves while sharing the same resources. Moreover, the company is profit-oriented and able to freely implement the decisions.

The fact that this problem is related with a web-based sales channel has a deep impact on the problem definition. Already in 1998, it is argued that internet-based marketplaces decrease the customer cost to obtain information and compare offers, which promotes price competition; moreover, they increase the ability of the seller to charge different prices to different customers (or to charge different prices over time), which reduces customer surplus and increases company's profit (Bakos, 1998).

Revenue management is historically linked with the airline business. In Belobaba (1989), the implementation of a revenue management system at Western Airlines is described, with a seat inventory control based on the Expected Marginal Seat Revenue (EMSR) decision model, which sets and revises booking limits to the number of seats available at different prices. This model takes into account the uncertainty of demand and is based on the value of the expected revenue per seat per class or segment, thus defining their protection levels (number of seats hold for sale for certain segments) and, consequently, booking limits. Over the time, revenue management has been applied to different sectors and situations. In Bertsimas and Popescu (2000), the maximization of revenue in a network environment is tackled by defining dynamic policies for the allocation of shared resources to different types of uncertain demand. The authors propose a solution based on approximate dynamic programming, which may have applications on airlines, hotels, car rentals, amongst other businesses.

Revenue management on car rental business was early on tackled by Carroll and Grimes (1995), who described the implementation process of the Yield Management System (YMS) in Hertz, designed to help decisions related with pricing, fleet planning and fleet deployment between stations. Hertz's YMS segmented the market with different-valued offers and helped decide when to make these offers. It also protected fleet for higher-value reser-

vations when supply was short. Also in Geraghty and Johnson (1997) the revenue management program of a different company is described – National Car Rental. This system was designed to *manage capacity*, tackling fleet planning issues, planned upgrades (with inventory protected using the above-mentioned EMSR model), overbooking, and creating a *Reservations Inventory Control procedure* for selection of the most profitable reservations amongst similar ones with different lengths of rent. It also tackled *pricing issues*: for different segments, the model recommended prices to maximize occupation based on a target utilization. This model for pricing, which inspired the heuristic proposed, is composed of two parts. Firstly, an elasticity model is created relating historic rates with variations in demand. Then, the comparison between occupation (or demand) *forecast* and *target* leads to the required change in booking pace. From the elasticity model it is thus possible to retrieve the rate that will induce the required change in the pace of demand.

More recently, a revenue management problem in a car rental network is tackled in Haensel and Mederer (2011). The authors propose a stochastic programming approach to optimize fleet deployment between locations and capacity controls to protect fleet for higher revenue reservations, under a demand of probabilistic nature; in this context, the prices are not considered as decisions. Other works apply revenue management techniques to the car rental context, such as Guerriero and Olivito (2014). Here, the authors aim to allocate vehicles of different categories to different customer segments, following a framework on which the company must decide if it is more profitable to accept a rental request or not and thus define acceptance policies. A parallel can also be found with pricing problems. For example, in Maglaras and Meissner (2006) dynamic pricing strategies are studied, which might be applied to the car rental context. In this work, a common formulation for allocation of capacity and dynamic pricing is presented.

3.3. The proposed solution

The proposed solution to set the prices for the e-brokers sales channel is a heuristic designed to be swift, adaptative to the market and fleet conditions, and fast to implement. Research on revenue management and pricing in car rental is growing significantly on the stream of optimal policies and controls for allocating the right customers to the right-priced offers. However, to the best of the authors' knowledge, there is still a gap on the development of methods that address the link between the need to protect fleet and the need to take in consideration competitors' prices and the company's competitive position. This was a specific requirement of the work herein developed.

This heuristic is based on the concept of *goal occupation*, which is described in this chapter. A full decision tool / system was also designed and implemented and is also described in this chapter, representing the functioning conditions surrounding the heuristic procedure. This tool allows the car rental company to parametrize the heuristic procedure based on its business sensitivity, as well as to monitor key fleets and seasons of the year, providing useful indicators related to regional fleet balancing, real margins applied by e-brokers and market price fluctuations.
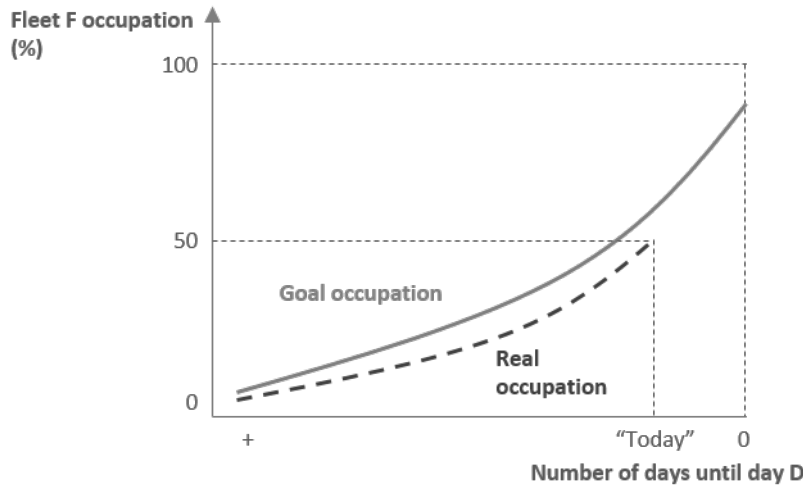
Figure 3.2: Representation of the evolution of goal and real occupation of a fleet for a specific day

3.3.1 Heuristic procedure based on goal occupation

The main objective of this problem is to set the prices to charge for each search, at each time distance to the reservation, so that the revenue of the car rental company is maximized. For that, one needs only to ensure that the capacity of each fleet is only booked to the maximum of its capacity by the different types of searches that influence it. However, the relation between the price to charge, the minimum price in the market for the same search, and the amount of reservations that one is able to get from it is intrinsically hard to define realistically. Therefore, the concept of *goal occupation curve* is introduced: the percentage of vehicles from a fleet that the company aims to have occupied with reservations for a certain date, at a certain time distance. For example, the company wants to have enough reservations to occupy 60% of fleet F on day D, 30 days before day D (see Figure 3.2).

Following this goal occupation curve allows the company to increase the value attained for the fleet sold. That is to say that the company is able to define how much vehicles to hold for the "late" clients that are willing to pay more for them. This concept is parallel to the *target utilization* (Geraghty and Johnson, 1997). The heuristic proposed is thus based on the concept introduced in Geraghty and Johnson (1997) that states that changes in the rates influence the difference between target and forecast occupation, which should be minimized. In our problem, as this procedure is designed to be run progressively and frequently (every two hours) and to be adaptive, actual occupation is used (not forecast). Therefore, in this heuristic, the prices charged are a mean to minimize the distance between the observable fleet occupation and the *goal occupation*, for a certain time distance to the reservation. Based on the discussion regarding the problem characteristics, it is assumed that companies are only able to get clients if their price is the lowest in the market. Therefore, if the real occupation is lower than the goal, the price should be set to be the lowest in the market (although never allowing for the price to fall below a certain minimum price).

Conversely, when real occupation is higher than the goal, the company should increase the price in order to reduce occupation rate, yet striving to achieve the desired pricing position versus competitors. If the occupation becomes extremely high, however, the increase should become independent of the competitors' prices, so as to hinder more (undesired) occupation.

Real occupation is calculated based on the concept of *most constrained day*. In fact, a certain *search* whose price is being settled will be translated in a reservation that may last for more than one day. In order to be conservative, all calculations consider the most occupied day in the reservation. Also, to calculate the new price is critical to "see" the prices the customer "sees" for each player. Therefore, when deciding the price to settle, the *margin* the e-broker is applying on the company's price should also be added.

This procedure settles the price for each search independently of the other searches, although several searches influence the same fleet occupation, as seen in the discussion above. The only searches that are linked are the ones that share all characteristics except for the group, and whose hierarchy of the groups was previously set (e.g. the price of a compact car should always be lower than the price of a luxury model). The searches are thus mostly considered independent in order to agilize the procedure since its main consequence is a higher degree of conservatism. For example, if a certain fleet in under-occupied, this heuristic will decrease the price for all searches that influence this fleet (as if they were the only ones affecting it). The effect may be higher than expected due to this "over-kill". However, as this procedure is adaptative, if this happens, in the next run the fleet will be slightly over-occupied and the prices will be adapted to this new context. In fact, this *easiness to adapt* is one of the most important characteristics of this procedure.

3.3.2   Full system overview

Figure 3.3 aims to describe the working flows of the full system designed. The system is divided in two areas: a User Area and a Work Area. The first is where the heuristic is parametrized, as well as the frequency to run each search. It also allows the user to choose which key groups to monitor. The second is where the connection with the main input and output systems is established and the procedure itself is based.

There are four main types of inputs to this system. The first two types of inputs are provided by external systems while the latter are user-defined:

- The *prices* currently available for the customers on each e-broker website, for the company and all its competitors;
- The *occupation* of the fleets for each day in this horizon;
- The main parameters of the system;
- The key groups to monitor.

Every pre-defined interval of time (hourly or every two hours), the system follows three steps. Previously, the user has defined the schedule of each search, based on its characteristics. For this, the user may define several command lines, such as "recalculate prices for all searches/reservations starting next month" or "for all that take place on 2015 High Season". From this, the first step of the system is to *list all the searches for whom*
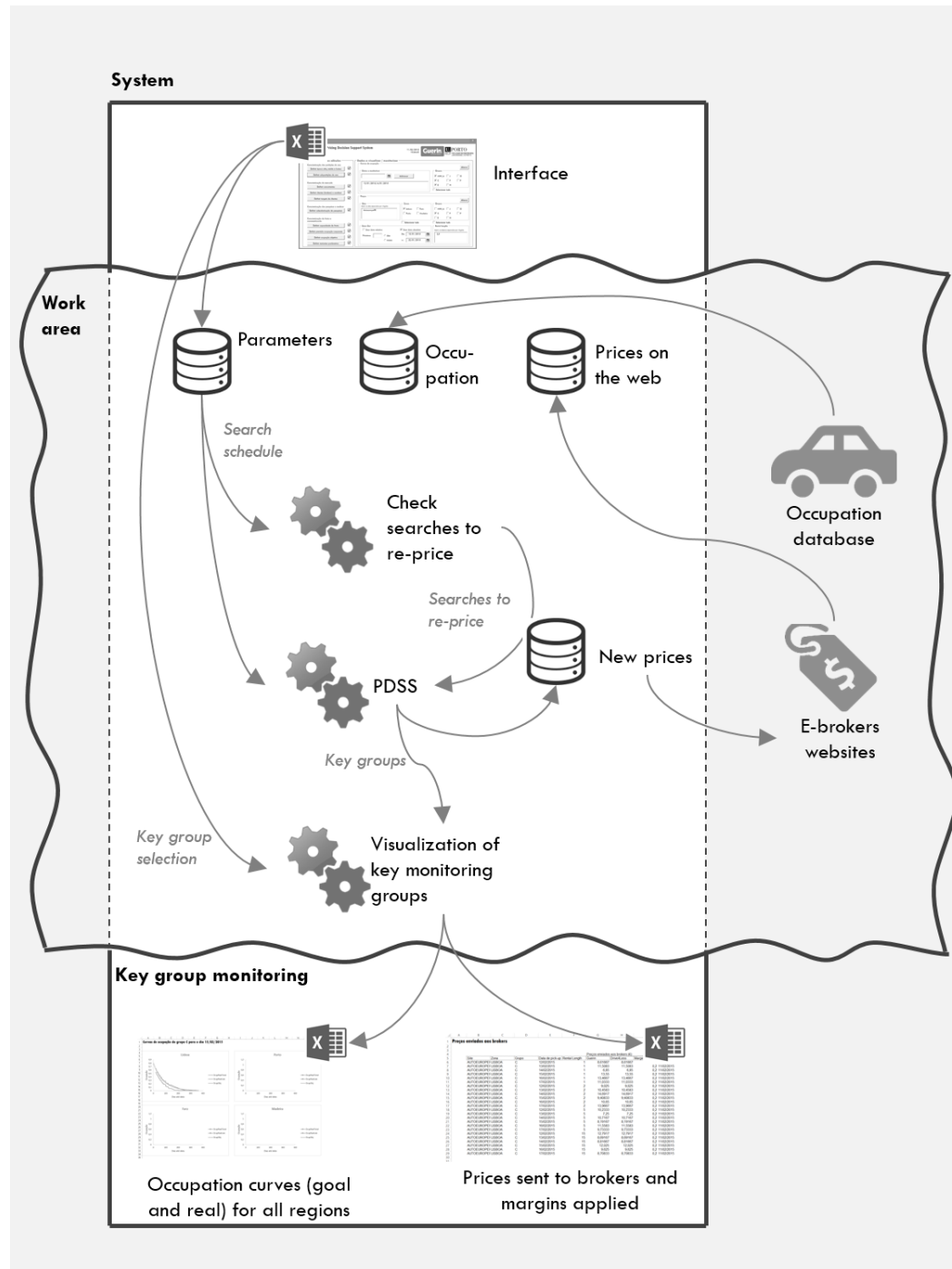
Figure 3.3: System overview

to recalculate prices at this moment. Secondly, the heuristic procedure recalculates prices for all the listed searches and sends them to the e-brokers. Finally, for the key monitoring groups, the main indicators are made available to the user.

It is important to note that companies do not have much control over the margins applied by e-brokers and thus a potential margin is computed and used in step 2: this margin is equal to the last company's price retrieved from the e-brokers' website divided by the recalculated price sent to the e-broker on the last iteration.

For some pre-selected key groups, the evolution of the occupation and price/margins is also monitored. For example, the user may follow the evolution of a certain fleet's occupation for a specific week and compare different regions through interactive graphic displays.

3.4. Conclusions and future work

This system, currently being implemented on *Guerin*, will enable the company to adapt its prices to the occupation of the fleet and the fluctuations of the market, seizing significantly more value from the customer willingness to pay. Moreover, it will be possible to monitor key periods of the year or key vehicle groups, as well as the margins added to the prices by the e-brokers.

The future work lays ahead in two different dimensions. On the business side of the problem, there is a need to measure the actual impact of this system and, based on that, refine key parameters of the problem such as the *goal objective curve*. This is possible since this system is designed to save all the data for future research.

On the academic side of the problem, since this is, at the best of the authors knowledge, yet to be fully explored by the academic community, future work is needed to build models that are able to bring even more value to the company by maximizing the revenue and tackling all the prices that relate to the same fleet occupation in an integrated way. Moreover, as these models will lead to (realistic) large instances, solution methods to solve them will be required. In a second stage, it will also be interesting to include *fleet sizing and deployment* issues in the problem, integrating decisions of vehicle transfer between regions when occupations are unbalanced.

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Deterministic capacity-pricing integration: Dynamic programming approach

In the first paper, in Chapter 2, four relevant research directions were identified. One of these is related with the integration of "horizontal decisions": the ones commonly tackled in fleet management (such as fleet size/mix or fleet deployment, henceforth named capacity decisions) and the ones tackled under the revenue management framework, capacity allocation or pricing. Resulting from the practical know-how acquired in Chapter 3, pricing was preferred over capacity allocation due to its direct translation to the current practices in the business. Therefore, this and the following chapters deal with the integration of capacity decisions and pricing.

The first approach to solve the integrated capacity-pricing problem is presented in this paper. This is a deterministic approach, based on dynamic programming, which has been successfully used in similar problems. As a proof of concept, some simplifications are applied to reduce the size of the problem, namely considering a homogeneous fleet. This approach did not prove to be able to tackle realistic instances for this problem. Nonetheless, its development brought valuable insights regarding e.g. problem structure. It is worth to notice the discussion on the impact of the "rental context", where inventory is re-usable, versus a more traditional "sales context", where inventory is depleted.

A dynamic programming approach for integrating dynamic pricing and capacity decisions in a rental context

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Abstract   Car rental companies have the ability and potential to integrate their dynamic pricing decisions with their capacity decisions. Pricing has a significant impact on demand, while capacity, which translates fleet size, acquisition planning and fleet deployment throughout the network, can be used to meet this price-sensitive demand. Dynamic programming has been often used to tackle dynamic pricing problems and also to deal with similar integrated problems, yet with some significant differences as far as the inventory depletion and replenishment are considered. The goal of this work is to understand what makes the car rental problem different and hinders the application of more common methods. To do so, a discrete dynamic programming framework is proposed, with two different approaches to calculate the optimal-value function: one based on a Mixed Integer Non Linear Program (MINLP) and one based on a Constraint Programming (CP) model. These two approaches are analyzed and relevant insights are derived regarding the (in)ability of discrete dynamic programming to effectively tackle this problem within a rental context when realistically sized instances are considered.

4.1. Introduction

This work deals with the integration of dynamic pricing decisions with resource capacity, deployment and consumption decisions within the car rental context. The goal is to decide, for the time horizon of a specific selling season:

- How many cars to have in the fleet,
- When to acquire them,
- How to deploy them among rental stations throughout the time horizon,
- How to assign them to rentals (that start and end throughout the time horizon and rental stations),
- How to price these rentals.

Car rental companies face a significantly price-sensitive demand. Since online sale channels have allowed companies to change their prices virtually instantaneously and with no cost, dynamic pricing is becoming a critical demand-management tool, not only in this sector but also in airlines, hotels and other businesses that rely on revenue management techniques (including pricing) to seize price-sensitivity and other demand segmentation characteristics.

In car rental, unlike the above-mentioned (more traditionally studied) sectors, the fleet is highly flexible and mobile, since the vehicles (resources) are easy(ier) to transfer, deploy and acquire. However, there is a myriad of products – the rentals – that share the same fleet capacity. The rentals are broadly characterized by their start and end date and location. Other elements (such as vehicle group required, for example) may characterize a type of rental. Nonetheless, for the sake of simplicity and clarity, throughout this work the fleet

is assumed to be homogeneous and the pricing decisions, made for each rental type on its broader definition, can be considered as "reference prices" to which others are indexed (e.g. variations according to antecedence of purchase or extra conditions required). For a more detailed review on car rental fleet and revenue management works, the reader may refer to Oliveira et al. (2016).

Recently, some interesting works have been dealing with the integration of dynamic pricing with capacity and inventory decisions (Adida and Perakis, 2010; Simchi-Levi et al., 2014). This integration has been becoming relevant for companies that can change and update their pricing policies and inventory and capacity decisions in an increasingly easier way, due to the improvement of the above-mentioned technological systems. The methodological approach applied often involves dynamic programming due to its affinity with the problem. Also, for the stochastic problem, other non-parametric approaches such as robust optimization have been developed. For a thorough and relevant review regarding dynamic pricing, especially when learning processes regarding demand are considered, the reader should refer to den Boer (2015).

The work herein presented aims to tackle a similar problem, which differs on the type of capacity/inventory decisions made. In previously studied cases, the capacity/inventory was decided and considered to be available at the start of the horizon (or at multiple points throughout the horizon, through multiple capacity decisions) and then depleted by the demand until the end of the horizon. In the car rental (actually any rental) context, the capacity is not only affected by these decisions but also by "returning" (re-usable) resources. That is to say, the resource is not depleted by demand but only temporarily occupied and it will become available capacity again, possibly at a different location. This difference has a significant impact on the structure of the problem and motivated the research presented in this paper.

The goal of this work is to study the possibility to develop a solution method based on one of the most applied methodologies in the similar problem presented above – dynamic programming – and understand its advantages, limitations and drawbacks in this context. Besides its common application within the dynamic pricing and revenue management setting, dynamic programming has also been used on works that deal with car rental operational and fleet management problems, such as fleet size and deployment (Li and Tao, 2010).

In this work, a general discrete dynamic programming framework is developed as well as two approaches to calculate the value of the decisions at each stage and state, which are presented in Section 4.2. Then, in Section 4.3, some illustrative numeric examples are used to analyze the limitations and advantages of the method. Finally, in Section 4.4, some conclusions are drawn.

4.2. Discrete dynamic programming formulation

One important characteristic of this problem is that most decisions are intrinsically integer, such as the number of vehicles to acquire and deploy or the number of fulfilled rentals. Due to the detail considered for rental types, which aggregate rentals that start and end at specific

locations and times, the order of magnitude of these decisions is relatively small and an approximate result obtained by relaxing the integrality constraints might be significantly impaired. Therefore, a discrete formulation was developed.

Dynamic programming provides a general framework to solve different problems, where a multistage structure is latent and can be used to decompose a complex problem into simpler sub-problems. Within this context, a *stage* represents a point where decisions are made. The goal is to formulate the problem so that at any stage the only information needed to make decisions is summarized on one or more *state* variables. The state at a specific stage is fully defined (on the deterministic case herein considered) by the state and decisions of the previous state, translated on a *state transition function*. At each stage, an *optimal-value function* can be defined, dependent on the current state and on the decisions made. Dynamic programming is then based on the recursive computation of the optimal-value function (Bradley, 1977).

In this section, the stages, state variables and spaces, and transition functions will be defined. Also, two approaches to calculate the optimal-value function will be presented.

The notation for indexes and sets used throughout this paper is as follows:

n	Index for stage;
e	Index for state;
r	Index for type of rental;
s, c	Indexes for rental station;
p	Index for price level;
$\mathcal{E}^n$	Set of states possible to achieve in stage n;
$\mathcal{S}$	Set of rental stations;
$\mathcal{R}$	Set of types of rental;
$\mathcal{R}_n^{\text{start}}$	Set of types of rentals that start at stage n;
$\mathcal{R}_{n,s}^{\text{start}}$	Set of types rentals that start at station s at stage n;
$\mathcal{P}$	Set of possible price levels.

Also, the following parameters will be considered:

T	Number of time periods on the time horizon;
HC_n	Holding cost for the fleet of vehicles existent at stage n (cost per vehicle);
TT_{sc}	Empty transfer time between station $s \in \mathcal{S}$ and station $c \in \mathcal{S}$;
TC_{scn}	Cost of initiating an empty transfer from station $s \in \mathcal{S}$ to station $c \in \mathcal{S}$ at stage n (cost per vehicle);
$DEM_r(q_r)$	Demand for type of rental $r \in \mathcal{R}$, dependent on the price q_r that is charged for this type of rental;
DEM_{rp}	Demand for type of rental $r \in \mathcal{R}$ if price level $p \in \mathcal{P}$ is charged for this type of rental (alternative notation);

PRI_p Monetary value associated with price level $p \in \mathcal{P}$.

4.2.1 Stages

In the car rental context, the start and end dates that characterize the rental types can be aggregated in e.g. weeks. The same unit can be used for the capacity decisions due to the flexibility of some vehicle acquisition modes, such as leasing. These time units mark the decision points throughout the time horizon and are the most notorious element that contributes to the multistage structure of the problem.

The computation will follow the *backward induction* method, as it will start at the last time period and end at the first. That is to say, the calculation will start at $n = 0$, where n defines the number of stages missing to end the time period, and end at $n = T$.

The decisions made at each stage n are represented by the following variables:

u_r^n Number of rentals of type $r \in \mathcal{R}_n^{\text{start}}$ fulfilled at stage n;

q_r^n Price charged for rentals of type $r \in \mathcal{R}_n^{\text{start}}$;

w_s^n Number of vehicles acquired to be available in rental station $s \in \mathcal{S}$ at stage n;

y_{sc}^n Number of vehicles to deploy from station $s \in \mathcal{S}$ to station $c \in \mathcal{S}$ by an empty transfer that starts at stage n.

4.2.2 State variables, transition function and state spaces

At any stage, the state variables should provide all the information required to make the previously mentioned decisions. Dynamic formulations for inventory problems and integrated pricing and capacity problems use the stock or inventory available at each stage as the state variables.

In this case, in order to decide on number of rentals fulfilled (u-type decision), two types of information are required: the amount of demand for this type of rental, which is solely dependent on the pricing decision made at the same stage, and the stock of vehicles of each rental type available at the starting station, which depends on decisions from previous periods and should thus be summarized on the state variables.

At each station $s \in \mathcal{S}$ and stage n, this stock depends on the previous stock, the vehicles that leave the station (either occupied by rentals or transfers) and the vehicles that arrive (either at the end of rentals or transfers or by their acquisition) and can be computed by the following equation:

$$\text{stock}_s^n = \text{stock}_s^{n+1} - \text{rentals that leave}_s^n - \text{transfers that leave}_s^n$$
$$+ \text{rentals that arrive}_s^n + \text{transfers that arrive}_s^n + \text{vehicles aquired}_s^n \tag{4.1}$$

As previously discussed, since the state variables fully define the state, the transition function should only depend on the previous state. However, since the rentals and empty transfers may last for more than one time period, Equation 4.1 requires information from stages other than the immediately subsequent.
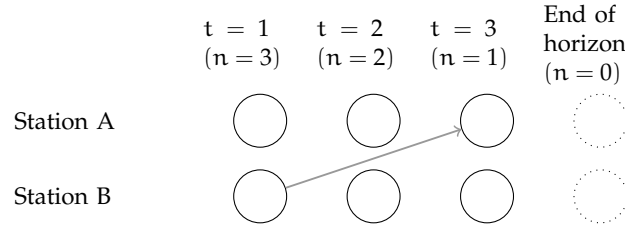
Figure 4.1: Space-time network of an explanatory example, with 2 rental stations and 3 time periods. The arrow represents a rental type that starts in station B in $t = 1$ and ends in station A in $t = 3$.
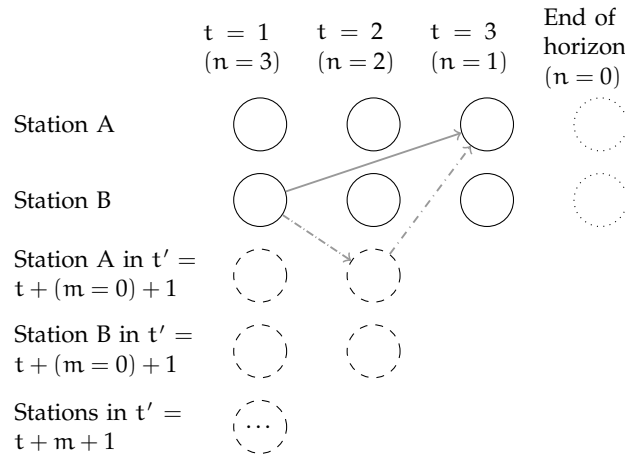
Figure 4.2: Space-time network of Figure 4.1, extended to include the representation of the additional state variables. The solid arrow is now decomposed in two dashed arrows using the additional state variable so that the state transition depends solely on the previous period.

Therefore, an artifact was developed and a second type of state variable introduced to represent the capacity occupied by current rentals or transfers that will be available in a later period. Figures 4.1 and 4.2 present an example to better explain these variables. As exemplified with the solid arrow, if there is a rental type that starts in $t = 1$ (station B) and ends in $t = 3$ (station A), the arrival of these rentals will affect the stock of vehicles available at station A in $t = 3$ (Figure 4.1). However, this decision occurs on a stage other than the immediately consequent. With an additional stock variable, it is possible to memorize, for any stage, how many vehicles are currently occupied and will be available in a certain station in a certain number of time periods. In the example presented, as shown by the dashed arrows in Figure 4.2, in $t = 2$, rentals of this type will increase the stock of vehicles currently occupied that will be available in station A in the next period. Then, in $t = 3$, the stock in station A will be increased by these units.

For each stage, the state variables can thus be defined as:

x_s^n Number of vehicles available in station $s \in \mathcal{S}$ (stock), at stage n;

o_{sm}^n Number of vehicles that are currently occupied and will be available in station $s \in \mathcal{S}$, at stage $n + m + 1$.

Thus, at each stage n, the state transition function t^n takes the following form:

$$
\begin{aligned}
\text{state}^{n-1} = \quad & t^n(u_r^n, q_r^n, w_s^n, y_{sc}^n, \text{state}^n) \\[4pt]
= \quad & \begin{cases}
x_s^{n-1} = x_s^n - \displaystyle\sum_{r \in \mathcal{R}_{s,n}^{\text{start}}} u_r^n - \sum_{c \in \mathcal{S}} y_{sc}^n + o_{s0}^n + w_s^n, \quad \forall s \in \mathcal{S} \\[14pt]
o_{sm}^{n-1} = o_{s,m+1}^n + \displaystyle\sum_{r \in \mathcal{R}_n^{\text{start}} \cap \mathcal{R}_{n+m+1,s}^{\text{end}}} u_r^n + \sum_{c:c \in \mathcal{S}, TT_{cs}=m+2} y_{cs}^n, \\[14pt]
\hspace{4cm} \forall s \in \mathcal{S}, m = \{0, ..., n-2\}
\end{cases}
\end{aligned}
\tag{4.2}
$$

State space: As for the state space, it was assumed that at the beginning of the time horizon ($n = T$), the state variables will be null (meaning that there are no vehicles occupied or in stock). For the remaining stages, an upper bound $XMAX$ must be defined for the x-type stock variables and another $OMAX$ for the o-type occupation variables. Each state is a combination of the possible values of these state variables. Therefore, the following equation defines the number of possible states NS, per stage $n < T$:

$$
NS = \max[1, (OMAX + 1)^{|\mathcal{S}|(n-1)}] \times (XMAX + 1)^{|\mathcal{S}|}
\tag{4.3}
$$

Therefore, there are $NS \times |\mathcal{S}|$ x-type state variables per stage n and $NS \times |\mathcal{S}|(n-1)$ o-type state variables per stage $n > 1$.

4.2.3 Optimal-value calculation

At each stage n and state s_n, the maximum possible return over the n missing stages is given by the optimal-value function v_n. As previously discussed, this function v_n is defined recursively, based on the return f of this stage, which depends on the current decisions and state, and on the optimal-value function of the previous stage. Since the overall goal is to optimize the profit, the recursive optimization problem is broadly given by:

$$
v^n(\text{state}^n) = \max \quad \left\{ f^n(u_r^n, q_r^n, w_s^n, y_{sc}^n, \text{state}^n) + v^{n-1}(t^n(u_r^n, q_r^n, w_s^n, y_{sc}^n, \text{state}^n)) \right\}
$$
$$
\text{s.t. Constraints on decisions} \tag{4.4}
$$

The return function f, in this case the profit obtained, is given by the difference between the revenue obtained from the price charged for each of the fulfilled rentals and the costs of holding the fleet of vehicles existent at this stage and the cost of initiating empty transfers:

$$
f^n = \sum_{r \in \mathcal{R}_n^{\text{start}}} u_r^n \times q_r^n - z^n \times HC_n - \sum_{s \in \mathcal{S}} \sum_{c \in \mathcal{S}} y_{sc}^n \times TC_{scn}
\tag{4.5}
$$

The auxiliary decision variable z summarizes the total fleet and is defined, at each stage, by the sum of the vehicles acquired (decision variable), the vehicles in stock (state variable) and the vehicles occupied on rentals or transfers (state variable):

$$z^n = \sum_{s \in \mathcal{S}} \left(w_s^n + x_s^n + \sum_{m=0}^{n-1} o_{sm}^n \right) \tag{4.6}$$

The constraints on decisions are as follows:

- The price-dependent demand ($DEM_r(q_r)$) is an upper bound on the number of rentals that can be fulfilled:

$$u_r^n \leq DEM_r(q_r), \quad \forall r \in \mathcal{R}_n^{\text{start}} \tag{4.7}$$

- The overall capacity in a station limits the rentals fulfilled and the empty transfers leaving the station:

$$\sum_{r \in \mathcal{R}_{n,s}^{start}} u_r^n + \sum_{c \in \mathcal{S}} y_{sc}^n \leq x_s^n + w_s^n, \quad \forall s \in \mathcal{S} \tag{4.8}$$

- Also, an auxiliary constraint ensures that no empty transfers start and end in the same station:

$$y_{ss} = 0, \quad \forall s \in \mathcal{S} \tag{4.9}$$

- All decisions should lead to integer values.

- Additionally, the resulting state must be possible to achieve (i.e. be defined).

Two relevant characteristics of this optimization problem are the integrality of the decision variables and the non-linearity of the objective function. Therefore, two adequate optimization models and consequent resolution strategies were applied: a Mixed Integer Non Linear Program (MINLP) and a Constraint Programming (CP) model.

For each stage and state, the MINLP model proposed is a straightforward adaptation of the optimization problem summarized in (4.4) and (4.5). The main difference is related with the price decision variable q that is transformed into a binary variable q_{rp} that indicates whether or not a specific *price level p* from a previously defined set $\mathcal{P}$, which is associated with a monetary value PRI_p, is chosen for rental type r. This causes minor adaptations in the objective function and on the demand constraint (4.7). Also, binary variables st_e are added, to indicate whether or not the state achieved on the consequent stage from the decisions made will be state $e \in \mathcal{E}^{n-1}$ or not. This requires additional constraints to associate the binary variables with the consequent states and to ensure that at least one possible state is achieved. The model is thus as follows[1]:

---

[1]The symbol $[\text{state}]_{t^n}$ indicates that the state expression was calculated based on the transition function and thus involves decision variables, while the symbol $[\text{state}]_e$ refers to an input/parameter: the state variables associated with state e

$$\max f^n + v^{n-1} = \Big[\sum_{r \in \mathcal{R}_n^{\text{start}}} u_r^n \times \sum_{p \in \mathcal{P}} \big(q_{rp}^n \times PRI_p \big) - z^n \times HC_n - \sum_{s \in \mathcal{S}} \sum_{c \in \mathcal{S}} y_{sc}^n \times TC_{scn} \Big]$$

$$+ \Big[\sum_{e \in \mathcal{E}^{n-1}} st_e \times v^{n-1}(st_e) \Big]$$

s.t. (4.6), (4.8),(4.9)

$$u_r^n \le DEM_{rp} + M(1 - q_{rp}), \quad \forall r \in \mathcal{R}_n^{\text{start}}, p \in \mathcal{P}$$

$$[x_s^{n-1}]_{t^n} \le [x_s^{n-1}]_e + M(1 - st_e), \quad \forall e \in \mathcal{E}^{n-1}, s \in \mathcal{S}$$

$$[x_s^{n-1}]_{t^n} \ge [x_s^{n-1}]_e - M(1 - st_e), \quad \forall e \in \mathcal{E}^{n-1}, s \in \mathcal{S}$$

$$[o_{sm}^{n-1}]_{t^n} \le [o_{sm}^{n-1}]_e + M(1 - st_e), \quad \forall e \in \mathcal{E}^{n-1}, s \in \mathcal{S}, m \in \{0, ..., n-2\}$$

$$[o_{sm}^{n-1}]_{t^n} \ge [o_{sm}^{n-1}]_e - M(1 - st_e), \quad \forall e \in \mathcal{E}^{n-1}, s \in \mathcal{S}, m \in \{0, ..., n-2\}$$

$$\sum_{e \in \mathcal{E}^{n-1}} st_e \ge 1$$

$$u_r^n \in \mathcal{Z}_0^+, \forall r \in \mathcal{R}_n^{\text{start}}; \quad z^n \in \mathcal{Z}_0^+; \quad y_{sc}^n \in \mathcal{Z}_0^+, \forall s, c \in \mathcal{S}$$

$$q_{rp}^n \in \{0, 1\}, \forall r \in \mathcal{R}_n^{\text{start}}, p \in \mathcal{P}; \quad st_e \in \{0, 1\}, \forall e \in \mathcal{E}^{n-1} \tag{4.10}$$

The second approach is based on Constraint Programming (CP), which aims to solve combinatorial problems by combining search and constraint solving, following the basis of logic programming (Rossi et al., 2006). This modeling and solving approach is suitable for integer, finite domain decision variables that are related by a set of constraints. Due to the characteristics of its associated search procedures, non-linearity presents no issue for CP models. Also, logic constraints such as "if-then-else" and implication statements can be implemented, which simplifies the model when compared with (4.10). For the sake of comparison between approaches, the price decision variable also refers to *price levels*, yet in this case it indicates directly the level, instead of having a binary variable per level. A similar reasoning is applied to the decision variable indicating the consequent state. The variable domains were considered as follows:

$$
\begin{aligned}
u_r &= \{0, ..., DUB_{s:\text{start}_r}\}, && \forall r \in \mathcal{R}_n^{\text{start}} \\
q_r &\in \mathcal{P}, && \forall r \in \mathcal{R}_n^{\text{start}} \\
w_s &= \{0, ..., DUB_s\}, && \forall s \in \mathcal{S} \\
y_{sc} &= \{0, ..., x_s^n\}, && \forall s, c \in \mathcal{S} \\
z &= \{0, ..., \sum_{s \in \mathcal{S}} DUB_s\} \\
st &\in \mathcal{E}^{n-1}, && \forall e \in \mathcal{E}^{n-1}
\end{aligned}
$$

The demand upperbound per station DUB_s was calculated by:

$$DUB_s = \sum_{r \in \mathcal{R}_{n,s}^{\text{start}}} \Big(\max_{p \in \mathcal{P}} DEM_{rp} \Big) \tag{4.11}$$

The CP model is then similar to the previous one:

$$\max f^n + v^{n-1} = \Big[\sum_{r \in \mathcal{R}_n^{\text{start}}} u_r^n \times q_r^n \times PRI_p - z^n \times HC_n - \sum_{s \in \mathcal{S}} \sum_{c \in \mathcal{S}} y_{sc}^n \times TC_{scn} \Big] + \Big[v^{n-1}(st) \Big]$$

s.t. (4.6),(4.8),(4.9)

$$u_r^n \leq DEM_{rq_r}, \quad \forall r \in \mathcal{R}_n^{\text{start}}$$

$$[x_s^{n-1}]_{t^n} = [x_s^{n-1}]_e \Rightarrow st = e, \quad \forall e \in \mathcal{E}^{n-1}, s \in \mathcal{S}$$

$$[o_{sm}^{n-1}]_{t^n} = [o_{sm}^{n-1}]_e \Rightarrow st = e, \quad \forall e \in \mathcal{E}^{n-1}, s \in \mathcal{S}, m \in \{0,...,n-2\} \qquad (4.12)$$

Base case: To start the recursive calculation, it is important to define the base case for $n = 0$. Since in this problem it represents the end of the horizon, when no more rentals or vehicles are considered, it was assumed that $v^0 = 0$.

4.3.   Illustrative numeric examples

Scope

The goal of this section is to provide some numerical examples that illustrate the drawbacks and limitations of this method in order to support the discussion on its adequacy. The main characteristics of the problem that influence will be identified to understand the potential and limits of its application.

From the discussion on the number of states and state variables, it was possible to verify that four main characteristics of the problem could significantly influence the effectiveness of the method proposed: the upper bound on the number of vehicles in stock in each station $XMAX$, the upper bound on the number of vehicles currently occupied to be available in a specific future period of time and station $OMAX$, the number of stations S and the number of time periods (i.e. stages).

From Equation (4.3) it is possible to observe that the number of states in a stage easily explodes. Therefore, as an example, considering two rental stations and three periods of time: for $XMAX = 10$ and $OMAX = 5$[2], the maximum number of states in a stage goes above 4.300, which leads to more than 17.000 state variables. If these numbers are doubled ($XMAX = 20, OMAX = 10$), the number of states becomes bigger than 53.000, with over 213.000 state variables.

The main issue is that this makes the effectiveness of the model highly dependent on two characteristics that are not intrinsic to the problem (although the maximum stock could have a clear parallel with the number of parking spaces available), and indirectly on the scale of the problem.

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[2]It is reasonable to assume that $OMAX \leq XMAX$.

Data

Instances: These numeric experiments are based on three cases that were adapted from instances provided for the Capacity-Pricing Problem in car rental[3], which present a "photograph" of the rental system at a certain time, showing the demand for each type of rental, as well as the remaining parameters. The instances chosen were the ones where the number of rental types was i) the smallest, ii) the biggest and iii) the median value. It is important to analyze how the approach performs varying this indicator since the number of rentals is one of the most relevant drivers of complexity to solve each sub-problem and, at the same time, it has virtually no impact on the number of states and stages, i.e. on the number of sub-problems to solve.

Experiment environment: The algorithms and MINLP and CP models were developed in C++/IBM ILOG Concert Technology and were run on a workstation computer with 48 Gigabyte of RAM memory, with 2 CPUs (Intel(R) Xeon(R) X5690 @ 3.46 GHz), with a 64-bit Operating System. The MINLP Solver used was CPLEX 12.6.1 and the CP solver used was CPLEX CP Optimizer 12.6.1.

Due to the number of stages and states, a time limit was set for calculating the optimal-value function. This makes it possible that the value obtained is not the optimum, yet it was considered as a mandatory control of the overall run time, since the MINLP or CP solver could experience difficulties in finding the optimum value or proving its optimality. The time limit chosen was 3 seconds. Preliminary experiments indicated that within this limit both solvers would often reach and prove the optimal result and that increasing it to 5 or 10 seconds would not significantly impact the results obtained. Nevertheless, it was considered that the possibility of no solution being found for a few specific stages and states was significant and impactful and therefore a "back-up" mechanism was developed so that in this case the time limit was slightly increased. Moreover, in the last stage (corresponding to the first time period), since only one state is possible, the time limit for its calculation was increased to 60 seconds in order to improve the probability of achieving and proving the optimum.

Results and discussion

Figure 4.3 presents the best value obtained for each instance, with different combinations of the parameters *XMAX* and *OMAX*. These numeric examples illustrate the potential and limitations of the approach proposed since they represent small configurations of the problem and already embody significant insights.

Firstly, if one compares the overall values obtained by both approaches, the one that uses the Constraint Programming model to calculate the optimal-value function (henceforward referred to as "CP approach" for the sake of brevity) obtains better results than the one that uses the Mixed Integer Non Linear Program ("MINLP approach"), especially for

---

[3]*Capacity-Pricing Model: car rental instances*, 2017, available at doi: http://dx.doi.org/10.17632/g49smv7nh8.1
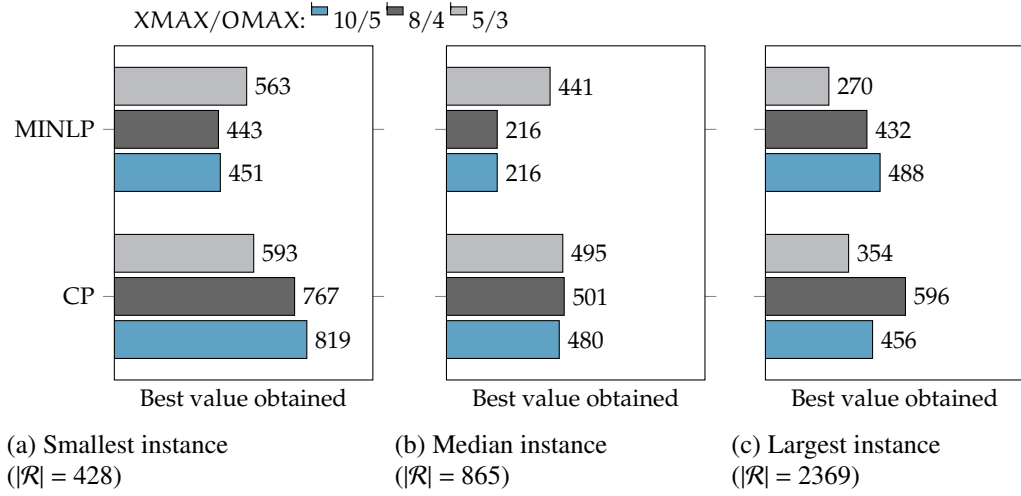
Figure 4.3: Best profit values obtained by each approach, for each instance, with different combinations of *XMAX/OMAX* parameters.

smaller instances.[4]

An interesting analysis can be made regarding the effect of the parameters *XMAX* and *OMAX* (directly connected with the number of states). It could be expected that an increase of these parameters would lead to higher values being obtained, since they represent a constraint on the "original problem" and their increase can be compared to a gradual relaxation of this constraint. Nevertheless, for the MINLP approach, this only happens for the biggest instance. In the remaining instances, increasing these parameters leads to a lower value. This might be explained by the effect of the time limitation imposed to each sub-problem. Due to this limit, the solver may not reach the optimum solution. Increasing the parameters makes the problem more complex to solve and thus makes the optimum more difficult to achieve. In fact, when the parameters are increased, the number of states increases and the sub-problems (which have decision variables and constraints dependent on the number of states) get more complex to solve.

As for the CP approach, a similar tendency is not as visible and it becomes difficult to draw conclusions regarding the relative strength of each of the two contradictory effects of increasing the parameters *XMAX* and *OMAX*: 1) the "original problem" is more relaxed and thus better solutions could be achieved, and 2) the problems become more complex and, due to the time limit, the optimum value is more difficult to achieve.

From this analysis rises an interest in observing the actual run times to understand if the hypothesis related with the optimum being reached more easily is supported. Two bounds on expected time to solve can be drawn. The first is based on the number of optimization problems to solve and the time limit to solve them, not considering the previously mentioned extra-time rules. The second is the actual upper bound on time that considers all

---

[4]Throughout this discussion, the notion of instance *size* will be associated with the intrinsic parameter being analyzed: the number of rental types. It it thus especially related with the complexity of the optimization problems solved for each stage and state (not the number of stages and states *per se*)

(a) *XMAX* = 10, *OMAX* = 5   (b) *XMAX* = 8, *OMAX* = 4   (c) *XMAX* = 5, *OMAX* = 3
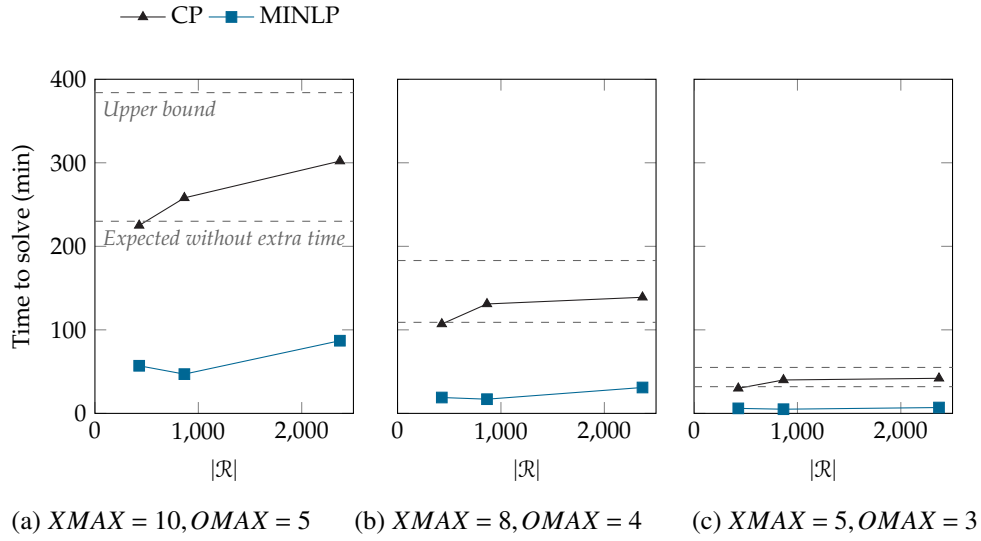
Figure 4.4: Time to solve the numeric examples using the two approaches to calculate the optimal-value function, plotting the instances by the number of rental types $|\mathcal{R}|$, for three possible combinations of the parameters *XMAX* and *OMAX*

the extra-time rules. In order to reach the latter, the extra time would have to be used to its maximum for all optimization problems. Figure 4.4 presents the time to solve these numeric examples and reach the values presented in Figure 4.3.

As expected, with an increase in instance size, there is a trend to increase the time to solve. Also, as it can be easily observed, the MINLP approach is consistently faster than the CP approach. In fact, the former is consistently below the expected bound (not considering extra time) while for the latter this only happens for the smallest instance. This means that the MINLP approach was often able to prove optimality in less than the time limit imposed, while the CP approach often used the extra time allowed. This does not fit in a straightforward way with the results previously discussed when comparing the best values obtained by each approach. In fact, the integrated analysis of Figures 4.3 and 4.4 supports the claim that the CP approach quickly finds good solutions yet takes longer to reach the optimum (or prove optimality) and that the ability and speed to prove optimality varies significantly more in the MINLP approach: from extremely fast to prove optimality to returning a feasible solution with a high gap. This seems reasonable considering the characteristics of each solution method and the fact that the complexity of the optimization problems varies significantly among stages (within the same instance).

Table 4.6 summarizes and compares the key differences and results from the two approaches. Overall, it is possible to conclude that the time limit imposed has a significant impact. Nevertheless, although it can lead to poorer overall results, if a time limit is not imposed the time to solve could make both approaches nonviable.

Table 4.6: Comparison of key results and differences

	CP approach	**MINLP approach**
Overall best profit values	Generally higher	Generally lower
Effect of increasing size-influencing parameters	No significant effect on profit	Lower profit values
Time to solve	Significantly slower	Significantly faster
Conclusions	Quickly finds good solutions yet has difficulty proving optimality, increasing significantly the time to solve.	Achieving/proving optimality ranges from extremely fast to impossible within time limit, making it difficult to obtain better solution values.

4.4.   Conclusions

In this work, a dynamic programming approach was developed to deal with the integrated dynamic pricing and capacity problem in the car rental business. This methodology has been successfully applied to similar problems and from the multistage structure of the problem (and the consequent "stock available" type of control) can be seen as an adequate method. Nevertheless, the fact that the capacity is "re-usable" in the rental context raises significant applicability issues that were analyzed.

The first drawback of applying dynamic programming to this context is that the number of states and state variables easily explodes. Already with these small numeric examples (in terms of number of time periods and rental stations, and considering deterministic demand) this method shows computational limitations. This is mainly due to the fact that the problem is related with a *rental* context – and this is why car rental is not like any other pricing and capacity/inventory model: the number of states explodes because stock can "return" after being depleted and that makes it necessary to keep track of occupied vehicles, which relates with decisions from time periods other than the immediately previous one.

An additional limitation is that the number of states is based on parameters that are not derived from the original problem, although they may have a close parallel to actual operational constraints, such as the stock of vehicles in a station being limited by the available parking spots. Although it was possible to observe that increasing the maximum number of vehicles in stock and occupied (and thus increasing the number of states) may hinder getting a better solution due to time limitations, not increasing these parameters for a real-world application of the methodology is not a viable option. In fact, the values herein proposed fail to fully reflect the reality of the problem. Ideally, these parameters should have no impact on the optimum value. Nevertheless, from a quick analysis of the order of magnitude of the demand values, it is easily established that in these numeric examples they have had impact.

These conclusions do not support the claim that dynamic programming is an adequate method to tackle this problem. Nevertheless, this discussion was able to bring some insights related with the problem structure as well as the potential and limitations of CP and MINLP when embedded in a discrete dynamic programming approach.

As future work, other methodologies will be applied to this rental context, especially considering the case of uncertain demand and realistically sized problems.

Acknowledgments

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Deterministic capacity-pricing integration: Matheuristic approach

This paper presents a deterministic approach to the capacity-pricing integration based on a matheuristic. The problem is fully defined by a mathematical programming model and the solution method proposed is based on its decomposition, motivated by the insights gained throughout the research regarding the structure of the problem. This approach performs significantly well in tackling realistic instances for the deterministic version of the problem.

Integrating pricing and capacity decisions in car rental: a matheuristic approach

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Abstract   Pricing and capacity decisions in car rental companies are characterized by high flexibility and interdependence. When planning a selling season, tackling these two types of decisions in an integrated way has a significant impact. This paper tackles the integration of capacity and pricing problems for car rental companies. These problems include decisions on fleet size and mix, acquisitions and removals, fleet deployment and repositioning, as well as pricing strategies for the different rental requests. A novel mathematical model is proposed, which considers the specific dynamics of rentals on the relationship between inventory and pricing as well as realistic requirements from the flexible car rental business, such as upgrades. Moreover, a solution procedure that is able to solve real-sized instances within a reasonable time frame is developed. The solution procedure is a matheuristic based on the decomposition of the model, guided by a biased random-key genetic algorithm (BRKGA) boosted by heuristically generated initial solutions. The positive impact on profit, of integrating capacity and pricing decisions versus a hierarchical/sequential approach, is validated.

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5.1.   Introduction

Car rental companies face several decisions related with their capacity, including decisions on fleet size/mix, acquisitions and deployment, which are significantly connected with the pricing of the rentals that are fulfilled. This paper proposes a new mathematical model for the integration of these problems, as well as a solution procedure that is able to solve realistically sized instances within a reasonable time frame.

Motivation

The car rental business is a relevant sector within the current mobility systems, which has been significantly growing in the past years. In the U.S., the revenue gains have grown 4% in 2015, with an average fleet growth of 5% (Auto Rental News, 2015). Moreover, the use of rental cars is also expected to grow in the future beyond the traditional corporate and leisure utilization, towards becoming an occasional alternative to owning a private car (Gao et al., 2016).

This is a business that faces interesting and challenging operations management issues, in which quantitative methods that support decision-making are becoming critical. In 2013, the CEO of Hertz, one of the main global players in the market, highlighted how technology is becoming the key competitive advantage of car rental companies and how it has been taking a central space even in the governing structure of the organizations (McKinsey & Company, 2013). These challenges are important for practitioners yet the literature has only recently gained momentum in structuring and studying the interesting fleet operations and revenue management problems faced by car rentals. The main differentiation of this business, when compared with more traditionally studied transportation sectors, is its flexibility. The inherent flexibility of the fleet (mobility of the vehicles), the flexibility associated with acquiring vehicles for the fleet (and removing them) and the flexibility of the decision-making processes, associated with a highly competitive, price-sensitive and efficiency-dependent market, make this a relevant and interesting sector to study.

Brief Problem Description and Previous Works

This work deals with the integration of two of the main decisions that car rentals face: determining the *capacity* of their fleet – which includes decisions on acquisition modes and timings, as well as fleet deployment between locations, in order to meet demand – and determining the *price* of the variety of rentals that are requested. Currently, these problems are mainly tackled separately, often within a sequential or hierarchical framework. Pachon et al. (2006) propose the primary modeling framework for fleet planning in the car rental industry: a sequential and hierarchical structure of mathematical models and solution methods to solve in different steps the problems of *pool segmentation* (where the rental stations are clustered in fleet-sharing pools), *strategic fleet planning* (where the size of the fleet of each pool is decided), and *tactical fleet planning* (where the fleet levels in each station of the pool are decided – deployment – as well as the required vehicle transfers).

Recently, some works have been developed that attempt to integrate the most linked decisions, especially fleet size and fleet deployment (You and Hsieh, 2014; Li and Tao,

2010), although often simplifying the problem and hindering the practical application of the developed research. However, in Fink and Reiners (2006), fleet sizing is studied in detail, using a realistic modeling approach that included acquisitions and removals of vehicles as well as other issues, such as partial substitution between vehicle groups, in order to turn it applicable to real-world situations.

In car rental, fleet management often intertwines operational issues, such as the ones discussed so far, with other problems usually tackled under the revenue management framework. In fact, due to the inherent flexibility of the fleet, this sector is often studied in revenue management, especially as far as capacity allocation is concerned. For example, Guerriero and Olivito (2014) derive different acceptance policies for car rental booking requests while Steinhardt and Gönsch (2012) integrate these approaches with operations issues related with planned upgrades. As for pricing, it is considered as an emerging tool used by practitioners to manage demand, since it is increasingly easy and cheap for companies to dynamically and swiftly change the prices through online booking channels (McAfee and te Velde, 2006; Oliveira et al., 2015). Heterogeneity of customer preferences influence most rental businesses, inclusively on the antecedence of the requests and especially in what regards pricing and revenue management. In many businesses, rental customers are divided into two main groups: customers that require the service with some antecedence and "walk-in" customers, with different willingness to pay and different service expectations. In the car rental problem tackled in this paper, the antecedence of the rental requests is considered to have several discretized levels and may significantly influence the demand, alongside price.

Oliveira et al. (2017b) present a thorough literature review on fleet management and revenue management issues on car rental and propose a conceptual framework for the different levels of decision. One group of decisions deals with *pool segmentation*, as proposed by Pachon et al. (2006). Then, for each pool, five interconnected decision blocks are defined. Three of these blocks are related with operations fleet management problems such as *fleet size/mix*, which broadly decides how many vehicles of each type will compose the fleet (including decisions on acquisitions and removals), *fleet deployment*, which deals with the distribution of the vehicles among locations and how they are repositioned between them, and *fleet assignment*, which assigns specific vehicles to the existing rental requests. As for the problems usually tackled by revenue management, two main blocks are defined that represent the two perspectives of the field: *pricing*, where the price of each rental is defined, and *capacity allocation*, which decides which fixed-price rental requests should be fulfilled with the existing capacity. This paper can be positioned within this conceptual framework, since it integrates three main decision blocks for a single pool of locations: *fleet size/mix*, *fleet deployment* and *pricing*.

As previously discussed, the integration between fleet size/mix and fleet deployment has been often considered in the car rental fleet management literature. The integration of these problems (herein commonly referred to as *capacity problems*) with pricing decisions is mentioned as a research direction with considerable potential (Oliveira et al., 2017b). Some interesting works aim to fulfill this gap, such as Haensel et al. (2012) where fleet deployment is integrated with capacity allocation decisions by simultaneously deciding on booking limits and vehicle transfers for a homogeneous fleet. To the best of our knowledge,

only one paper aims to integrate pricing with fleet management decisions (in the case, fleet deployment as well) for the car rental business. Madden and Russell (2012) propose an interesting formulation where the price is decided based on the discrete choice of price levels and where the direct impact of price on demand is used to balance fleet levels.

The potential of this integration, which is starting to be explored in the literature, derives from the close connections between the two problems and the overlapping decision-making time horizons. In fact, pricing decisions influence and are influenced by the availability of the fleet, which is dependent on fleet occupation and on fleet size and location.

Also in other sectors, the relationship between pricing and capacity, the ability of price to manage demand, and the potential of their integration is being explored. Zhang and Zhang (2010) investigate the role of congestion tolls in an airport as a demand management tool as well as a financing source, focusing on the impact of carriers with a significant market position transposing these costs for higher price tickets. Also, in Wang et al. (2004), the problem of locating a park-and-ride facility is integrated with the pricing decision. The relationship between pricing and production decisions is thoroughly explored in Bajwa et al. (2016). In Ghoniem and Maddah (2015), pricing is integrated with assortment and inventory decisions for substitutable products in a retail environment. Some interesting insights are identified regarding the lack of structure of the solutions obtained, which reflects the potential of optimization approaches, and regarding the importance of this integration to significantly improve profitability. The impact of price-driven product substitution for a company selling to different customer segments, within a context of integrated pricing and production decisions, is further studied in Kim and Bell (2011), with a significant effort on demand and substitution modelling.

The solution method proposed in this work is a matheuristic, since it hybridizes a metaheuristic with mathematical models. This approach decomposes the original mathematical model in terms of its decisions. The metaheuristic guides the search over the decisions on pricing strategy, while the remaining decisions are solved using mathematical models generated by fixing the pricing strategies on the original monolithic model. In fact, approaches that combine decomposition strategies with metaheuristics are currently being used to solve difficult combinatorial problems. The decomposition takes advantage of special structures of the problem enabling these approaches to outperform "less hybridized" methodologies. Raidl (2015) proposes an interesting discussion on this topic, showing promising possibilities for these approaches. The combination of genetic algorithms with decomposition strategies to solve complex problems has been used with success, for example, by Paes et al. (2016) to tackle the unequal area facility location problem.

The metaheuristic used to guide the decomposition is a biased random-key genetic algorithm (BRKGA) (Gonçalves and Resende, 2011). BRKGA is a variation of the random-key genetic algorithm (RKGA) where there is a bias on the choice of one of the parents towards one with a better fitness (instead of an entirely random selection). BRKGA has been used with success in several complex problems. Moreover, this type of methodology has the ability to encompass problem-specific knowledge and to use it to boost its performance. This is demonstrated, for example, in the work of Ramos et al. (2016), where BRKGA is used to tackle the container loading problem and includes procedures that take into account static stability constraints derived from mechanical equilibrium conditions.

An important part of the solution method is the generation of initial solutions for the first population of BRKGA, which is conventionally entirely random. Other works have used this type of boost for BRKGA. When tackling the two-stage stochastic Steiner tree problem, Hokama et al. (2014) use a constructive heuristic to generate the entire initial population of the algorithm. It is nonetheless more common to generate only a part of the initial population, thus ensuring that there is still randomness associated with it. For example, when tackling the three-dimensional bin packing problem with heterogeneous bins, Li et al. (2014) generate four solutions using a constructive heuristic. These solutions are added to the initial population, whose remainder individuals are randomly generated. Furthermore, even one heuristically generated solution added to the initial population can have significant impact. In Stefanello et al. (2013), a genetic algorithm is proposed to solve the problem of pricing network of roads, i.e. defining tolls to be applied in some arcs of the network. One solution is generated by relaxing integrality constraints and is added to the initial population, thus boosting the overall performance.

Contributions

The car rental business is characterized and seizes its natural advantage of being able to decide on capacity levels with significant flexibility. Nevertheless, other characteristic that fully differentiates this sector from other sectors mentioned above (such as retail) is the *rental*-type of transaction considered. In this context, capacity is not only affected by initial or frequent capacity/inventory decisions but also by "returning" vehicles, which are temporarily used but become available again in the future, possibly at a different location. This impacts significantly the structure of the problem.

Traditionally, car rental companies tend to separate these problems deciding on fleet size first and then managing the demand through pricing decisions that accelerate or decelerate occupation and deploying the fleet to meet demand. This work points out to the fact that integrating these decisions will allow for significant improvements due to the flexibility gained by also using fleet size as a tool to manage demand. The main disadvantage of the integration – the computational burden – is tackled by the use of an innovative solution procedure.

This work has thus three main contributions:

- A new mathematical model for the integration of capacity (including decisions on acquisitions, fleet size and mix, and deployment) and pricing decisions for car rentals.

- An innovative and high-performing solution method for the problem that is able to obtain good solutions for real-sized instances within realistic time frames. This method is based on a decomposition of the mathematical model. A genetic algorithm is used to guide the search over part of the decision variables. The value (fitness) of these *partial solutions* is evaluated by fixing them in the original model and solving it for the remaining decisions. Moreover, a structured and robust way of heuristically generating initial solutions for the genetic algorithm is proposed, showing a significant power to boost the search.

- A quantitative proof that the integration of these problems brings measurable improvements for companies, when compared with a sequential approach.

The solution method proposed was conceived according to a modular and quick-to-answer design, so that it can be easily implemented in a decision support system to help car rental companies make more profitable decisions. Nevertheless, the work developed in this paper already brings relevant managerial insights, especially regarding the potential of integrating pricing and capacity decisions for a selling season. Based on this research, a company is able to ascertain whether an integrated approach brings advantages over a sequential approach, mainly based on market size and number and type of products to price.

Moreover, a parallel can be built between the car rental business and *car sharing* systems, namely in what concerns the mobility/flexibility of the fleet and decision-making processes and the role of pricing in managing demand. Therefore, this model and solution procedure can be extended to be applied in this increasingly relevant urban mobility topic.

Paper Structure

This paper is structured as follows. Firstly, the capacity-pricing problem for car rentals and the novel mathematical model will be presented (Section 5.2) and the proposed solution method will be explained (Section 5.3). Then, in Section 5.4, the computational tests and their results are discussed and finally, in Section 5.5, some conclusions and future research directions are drawn.

5.2.   Problem definition

The work here presented was inspired by the case of a Portuguese car rental company. In this section, the problem will be introduced by providing an overall scenario of this company's business and an overview of the scope of the problem at hand. Then, the Capacity-Pricing Model will be fully defined by its mathematical formulation as an Integer Non-Linear Programming Model (INLP).

5.2.1   Problem Statement: The Case of a Portuguese Car Rental Company

This work aims to support the decisions of a car rental company that is planning a selling season (1-3 months) and must decide on the acquisition and fleet capacity plan, which are interconnected with the overall pricing strategy. Indeed, in this kind of business, when demand exists, companies can generally increase their profit either by maintaining the prices and increasing the fleet, or raising the prices and keeping the fleet size as it is.

The car rental company that inspired this work is based in Portugal, where it has approximately 40 rental stations, divided into four regions. All regions share the same fleet. The company uses these regions as "units of location" when tackling the tactical/strategic problems that will be detailed in this section. This is due to the fact that moving vehicles between stations within the same region is negligible in terms of both cost and time, unlike inter-region movements.

The fleet of the company is composed of approximately 10, 000 vehicles and is divided in up to 5 vehicle groups, depending on the selling season. Besides being differentiated by group, the fleet is also divided in owned and leased fleet. The purchase of owned vehicles is planned with a certain advance. Usually, these vehicles are available in the beginning of the selling season and are sold within one year. Leased vehicles, however, are used to face peaks in demand and can be available for shorter and more flexible periods of time, with a higher cost for the company.

This fleet is used to serve the different types of rentals requested, which are characterized by start and end locations and dates, as well as required vehicle group. Depending on the selling season, this company can deal with 450-2, 500 different rental types, priced individually and differentiated according with the antecedence of the request. For each rental type, the number of requests – the demand –, which may later occupy the acquired vehicles, is highly dependent on pricing.

A common practice to help meet demand in different locations throughout the season is to perform "empty transfers". An "empty transfer" occurs when a vehicle is moved from one location to another not as part of a rental but to meet demand, with a non-negligible cost and travel time. The company performs these transfers by either truck or using a driver.

Other practice used by this and other car rental companies to meet demand is to offer upgrades when the requested group is not available. This means that the companies offer a more-valued vehicle than what was requested for the same price. This allows them to maximize the utilization of the fleet and to meet demand. However, regular upgrades are commonly avoided as they incentivize the strategic consumer behavior of renting vehicles that do not meet expectations in hope of being offered an upgrade. A proper fleet planning can help provide the required vehicles where they are needed so that upgrades are only used as a last resource.

Due to maintenance costs, there is a high emphasis on the company's ability to maximize the occupation of vehicles. Currently, the company follows a hierarchical approach: first, it decides the capacity and afterwards makes the pricing decisions with the goal to maximize the fleet occupation. The main objective of this work is increase the company's profitability by integrating the capacity with the pricing decisions, since the latter have a strong impact on demand and, consequently, can help make better capacity decisions.

5.2.2 Mathematical Model

In order to fully describe the problem presented above, a mathematical model was developed. The main decision variables are related with the acquisition of vehicles and with the prices of different rental types. The number of rental requests is highly dependent on the pricing decisions. These requests may later become *fulfilled rentals* and occupy the acquired vehicles. In order to fully understand and map this interaction between capacity and pricing, other decisions are considered, such as the stock of vehicles in each location and time period and the number of vehicles "empty transferred" between locations.

As mentioned above, the fleet of vehicles, besides being differentiated by group, is also divided in owned and leased fleet. It is assumed that the total number of vehicles purchased for the owned fleet are available at the beginning of the time horizon. The leased vehicles

may become available for shorter periods throughout the time horizon.

The objective is to maximize the company's profit in the time horizon. The profit is the difference between the revenues obtained with the rentals fulfilled and the costs of leasing/acquiring fleet, performing the empty transfers and maintaining the owned fleet, as well as a penalization factor for upgrades.

This model has four main groups of constraints, which will be further explained in Section 5.2.2.4:

- Stock calculating constraints, where the stock of vehicles of each group in each time period and station is computed;
- Capacity/Demand constraints, where these are established as a limitation on the number of rentals fulfilled and empty transfers realized;
- Business-related constraints, where the limitations regarding possible upgrades and available purchase budget are established;
- Other auxiliary constraints.

The goal of introducing pricing decisions, and corresponding changeable demand levels, in the capacity planning is not to produce operational decisions that are updated on an online manner and swiftly react to changes in the system. This objective, although very important for car rental companies, is considered to be out of scope for this study. Within the tactical/strategic scope herein considered, pricing decisions and demand information are used in an offline manner to provide better quality to a model that aims to produce season-lasting decisions, such as fleet size and mix. In a real-world application, applying this model to support such decisions would not be conflicting with using a more operational model where requests appear in an online fashion and decisions like performing empty transfers or offering upgrades are revised and dynamically optimized.

5.2.2.1 Indices and Parameters

$t, t' = \{0, ..., T\}$	Index for the set $\mathcal{T}$ of time periods, where $t = 0$ represents the initial conditions of the time horizon (season) and "overlaps" with $t = T$ for the previous season
$g, g1, g2 = \{1, ..., G\}$	Index for the set $\mathcal{G}$ of vehicle groups
$s, s1, s2, c = \{1, ..., S\}$	Indices for the set $\mathcal{S}$ of rental locations
$r = \{1, ..., R\}$	Index for the set $\mathcal{R}$ of rental types (characterized by check-out and check-in location and time period, and vehicle group requested)
$sout_r$	Check-out location of rental type r
sin_r	Check-in location of rental type r
$dout_r$	Check-out time period of rental type r
din_r	Check-in time period of rental type r
gr_r	Vehicle group requested by rental type r

$a = \{0,...,A\}$	Index for the set $\mathcal{A}$ of antecedences allowed (number of time periods between the rental request and the start of the rental), where $a = 0$ represents a "walk-in" customer
$p = \{1,...,P\}$	Index for the set $\mathcal{P}$ of price levels allowed
PRI_{pg}	Pecuniary value associated with price level p for vehicle group g (for example, for group $g = 2$, price level $p = 1$ has a pecuniary value of $PRI_{1,2} = 20 \, €$)
DEM_{rap}	Demand for rental type r, at price level p, with antecedence a
COS_g	Buy cost of a vehicle of group g. The value considered is the net cost: purchase gross cost minus salvage value derived from its sale after one year (see Section 5.2.1)
LEA_g	Leasing cost (per time unit) of a vehicle of group g
OWN_g	Ownership cost (per time unit) of a vehicle of group g
LP_g	Leasing period for a vehicle of group g
PYU	Penalty charged for each upgrade
UPG_{g1g2}	Whether a vehicle of group $g1$ can be upgraded to a vehicle of group $g2$ (= 1) or not (= 0)
TT_{s1s2}	Transfer time from location $s1$ to location $s2$
TC_{gs1s2}	Transfer cost of a vehicle of group g from location $s1$ to location $s2$
BUD	Total budget for the purchase of vehicles
M	Big-M large enough coefficient

Other sets:

$\mathcal{R}^{g^-}$	Rental types that do not require group g
$\mathcal{R}^{in}_{st}$	Rental types whose check-in is at location s at time $\in [t-1,t[$
$\mathcal{R}^{out}_{st}$	Rental types whose check-out is at location s at time $\in [t-1,t[$
$\mathcal{R}^{use}_t$	Rental types that require a vehicle to be in use at t (i.e., $dout < t \wedge din \geq t$)

Inputs from previous seasons (previous decision periods):

INX^O_{gs}	Initial number of owned (O) vehicles of group g located at s, at the beginning of the season ($t = 0$)
$ONY^{L/O}_{gts}$	Number of owned (O) or leased (L) vehicles of group g on on-going empty transportation (previously decided), being transferred to location s, arriving at time t
$ONU^{L/O}_{gts}$	Number of owned (O) or leased (L) vehicles of group g on on-going rentals (previously decided), being returned to location s at time t

5.2.2.2 Decision Variables

w_{gs}^O Number of vehicles of group g acquired for the owned fleet available at $t = 0$ in location s

w_{gts}^L Number of vehicles of group g acquired by leasing to be available at time t in location s

q_{rap} $= 1$ if price level p is charged for rental type r with antecedence a; $= 0$ otherwise

$x_{gts}^{L/O}$ Number (stock) of leased (L) or owned (O) vehicles of group g located at s at time t

$y_{s1s2gt}^{L/O}$ Number of leased (L) or owned (O) vehicles of group g empty transferred at time t from location $s1$ to location $s2$

$u_{rag}^{L/O}$ Number of fulfilled rentals requested as rental type r with antecedence a that are served by a leased (L) or owned (O) vehicle of group g

$f_{gt}^{L/O}$ Auxiliary variable: total leased (L) or owned (O) fleet of group g at time t

5.2.2.3 Objective Function

Equation 5.1 represents the objective function of the model, which aims to maximize the profit of the company, comprising the activities of renting, purchasing and leasing vehicles and managing the fleet. The first element of the objective function represents the revenue earned from the fulfilled rentals, which is given by the price charged (dependent on the group requested) times the number of rentals served using leased and owned fleet. This term of the objective function renders the model non-linear, since two decision variables are multiplied.

The second term represents the cost of purchasing the owned fleet – a one-time cost. The following terms are related with the costs of leasing the vehicles (recurrent throughout the leasing period) and the ownership costs (also recurrent). The latter are significantly smaller than the former and aim to represent the regular costs of maintaining the owned fleet. Then, the empty transfer costs are represented, which depend on the group of each vehicle transferred and the origin-destination pair. Finally, an artificial marginal cost to offer upgrades is included in order to ensure that this practice only exists if there are no available cars from the required group.

$$
\begin{aligned}
\max \quad & \text{Profit from fulfilled rentals} - \text{Buying cost} - \text{Leasing cost} - \text{Ownership cost} \\
& - \text{Empty transfer cost} - \text{Penalty for upgrading} = \\
\max \quad & \sum_{r=1}^{R} \sum_{a=1}^{A} \left(\left(\sum_{g=1}^{G} u_{rag}^L + u_{rag}^O \right) \sum_{p=1}^{P} q_{rap} PRI_{p,gr_r} \right) - \sum_{g=1}^{G} \left(\sum_{s=1}^{S} w_{gs}^O \right) COS_g \\
& - \sum_{g=1}^{G} \left(\sum_{t=1}^{T} f_{gt}^L \right) LEA_g - \sum_{g=1}^{G} \left(\sum_{t=1}^{T} f_{gt}^O \right) OWN_g \\
& - \sum_{s1=1}^{S} \sum_{s2=1}^{S} \sum_{g=1}^{G} \left(\sum_{t=1}^{T} \left(y_{s1s2gt}^L + y_{s1s2gt}^O \right) \right) TC_{gs1s2} - \sum_{g=1}^{G} \sum_{r \in \mathcal{R}^{g^-}} \sum_{a=1}^{A} \left(u_{rag}^L + u_{rag}^O \right) PYU \quad (5.1)
\end{aligned}
$$

5.2.2.4   Constraints

Stock calculating constraints: Equations 5.2, 5.3, 5.4, 5.5 and 5.6 represent the calcu-
lation of the "stock" of available vehicles of a certain group, in a specific location, at a
specific time. These constraints also link the problem for the different time periods and
locations.

Equations 5.2 aim to characterize the stock of owned vehicles of each group, in each
location, for each time period except the initial one. The stock is equal to the one of
the previous period, increased by expected arrivals from rentals and empty transfers that
started on previous seasons (parameters) and by the arrival of vehicles that were being
employed in rentals that started this season and were meanwhile returned to this specific
location, decreased by the vehicles that were meanwhile occupied by rentals that started
in this location, increased also by the vehicles that were being empty-transferred from
other locations and have meanwhile arrived, and finally decreased by the vehicles that
were transferred to other locations.

$$
\textbf{s.t. } x^O_{gts} = x^O_{g,t-1,s} + ONY^O_{gts} + ONU^O_{gts}
$$

$$
+ \sum_{r \in \mathcal{R}^{in}_{s,t}} \sum_{a=1}^{A} u^O_{r,a,g} - \sum_{r \in \mathcal{R}^{out}_{s,t}} \sum_{a=1}^{A} u^O_{r,a,g}
$$

$$
+ \sum_{c=1}^{S} y^O_{c,s,g,t-TT_{cs}-1} - \sum_{c=1}^{S} y^O_{s,c,g,t-1} \qquad \forall g, t > 0, s \tag{5.2}
$$

Equations 5.3 and 5.4 represent a similar situation yet applied to the leased fleet. One
of the main differences of this type of fleet is that acquisitions may occur throughout the
season. Therefore, a similar structure can be seen when confronting with Equations 5.2,
but with the addition of some terms related with the acquisition of leased vehicles. In
Equations 5.3 and Equations 5.4, the stock is increased with the corresponding leasing
acquisitions. Then, since leased vehicles must be removed from the fleet after the leasing
period (*LP*) is over, Equations 5.4, valid for all time periods greater than the leasing period,
also decrease the stock by the number of returned leased vehicles.

$$
x^L_{gts} = x^L_{g,t-1,s} + ONY^L_{gts} + ONU^L_{gts}
$$

$$
+ \sum_{r \in \mathcal{R}^{in}_{s,t}} \sum_{a=1}^{A} u^L_{r,a,g} - \sum_{r \in \mathcal{R}^{out}_{s,t}} \sum_{a=1}^{A} u^L_{r,a,g}
$$

$$
+ \sum_{c=1}^{S} y^L_{c,s,g,t-TT_{cs}-1} - \sum_{c=1}^{S} y^L_{s,c,g,t-1}
$$

$$
+ w^L_{g,t-1,s} \qquad \forall g, 0 < t \le LP_g, s \tag{5.3}
$$

$$x_{gts}^{L} = x_{g,t-1,s}^{L} + ONY_{gts}^{L} + ONU_{gts}^{L}$$

$$+ \sum_{r \in \mathcal{R}_{s,t}^{in}} \sum_{a=1}^{A} u_{r,a,g}^{L} - \sum_{r \in \mathcal{R}_{s,t}^{out}} \sum_{a=1}^{A} u_{r,a,g}^{L}$$

$$+ \sum_{c=1}^{S} y_{c,s,g,t-TT_{cs}-1}^{L} - \sum_{c=1}^{S} y_{s,c,g,t-1}^{L}$$

$$+ w_{g,t-1,s}^{L} - w_{g,t-LP_g-1,s}^{L} \qquad \forall g, t > LP_g, s \tag{5.4}$$

Equations 5.5 and 5.6 calculate this stock for the beginning of the season ($t = 0$). As for the owned fleet (Equations 5.5), the initial stock will be equal to the stock existent in the previous season (parameter) and the number of purchased vehicles. The leased fleet (Equations 5.6) is considered to be initially null.

$$x_{g0s}^{O} = INX_{gs}^{O} + w_{gs}^{O} \qquad \forall g, s \tag{5.5}$$

$$x_{g0s}^{L} = 0 \qquad \forall g, s \tag{5.6}$$

Capacity / Demand constraints: At a given location and time period, the number of rentals fulfilled and the empty transfers that start at that location and time are limited by the stock of available cars (Equations 5.7). Equations 5.8 ensure that this number is also limited by the demand for the rental type, at the chosen price level.

$$\sum_{r \in \mathcal{R}_{s,t}^{out}} \sum_{a=1}^{A} u_{rag}^{L/O} + \sum_{c=1}^{S} y_{scgt}^{L/O} \leq x_{gts}^{L/O} \qquad \forall g, t, s \tag{5.7}$$

$$\sum_{g=1}^{G} \left(u_{rag}^{L} + u_{rag}^{O} \right) \leq DEM_{rap} + M(1 - q_{rap}) \qquad \forall r, a, p \tag{5.8}$$

Business-related constraints: The upgrading policies (i.e., which groups can be upgraded to which groups) are translated into Equations 5.9.

$$\sum_{a=1}^{A} \left(u_{rag}^{L} + u_{rag}^{O} \right) \leq UPG_{gr_r,g} \times M \qquad \forall r, g \tag{5.9}$$

Also, the number of purchased vehicles in each time period is limited by the total available budget (Equations 5.10).

$$\sum_{s=1}^{S} \sum_{g=1}^{G} w_{gs}^{O} COS_g \leq BUD \tag{5.10}$$

Other constraints: Equations 5.11 ensure that only one price level is chosen per rental type and antecedence.

$$\sum_{p=1}^{P} q_{rap} = 1 \qquad \forall r, a \tag{5.11}$$

In order to facilitate the construction of the objective function, an auxiliary decision variable was created that represents the totality of the leased (L) and owned (O) fleet of a certain group in each time period. Equations 5.12 define it as the sum of the stock of vehicles at the rental locations, the vehicles that are currently being used in rentals and the cars currently being transferred between locations.

$$f_{gt}^{L/O} = \sum_{s=1}^{S} x_{gts}^{L/O} + \sum_{r \in \mathcal{R}_t^{use}} \sum_{a=1}^{A} u_{r,a,g}^{L/O}$$
$$+ \sum_{s1=1}^{S} \sum_{s2=1}^{S} \sum_{t'=\max\{0, t-TT_{s1s2}\}}^{t-1} y_{s1,s2,g,t'}^{L/O} \qquad \forall g, t \tag{5.12}$$

Finally, Equations 5.13 represent the domain of the decision variables. Except for the binary variable that selects the price level to be charged, all variables are integer and non-negative.

$$
\begin{aligned}
q_{rap} &\in \{0, 1\} & &\forall r, a, p \\
w_{gts}^{L} &\in \mathbb{Z}_0^+ & &\forall g, t, s \\
w_{gs}^{O} &\in \mathbb{Z}_0^+ & &\forall g, s \\
y_{s1s2gt}^{L/O} &\in \mathbb{Z}_0^+ & &\forall s1, s2, g, t \\
x_{gts}^{L/O} &\in \mathbb{Z}_0^+ & &\forall g, t, s \\
u_{rag}^{L/O} &\in \mathbb{Z}_0^+ & &\forall r, a, g \\
f_{gt}^{L/O} &\in \mathbb{Z}_0^+ & &\forall g, t
\end{aligned}
\tag{5.13}
$$

A brief discussion on some insights regarding model structure are presented on Appendix 5.A. This discussion is based on an analogy between the formulation proposed and the transportation problem model and helps further understand the inherent structure of this problem.

5.3. Proposed Solution Method

Since the Capacity-Pricing Model is significantly complex and hard to solve for real-sized instances, inclusively due to the non-linearity of the objective function, a solution method was proposed to obtain good quality solutions within a reasonable time-frame.

The overall idea of the method is based on the decomposition of the original model in

pricing decisions and the *remaining decisions*, exploring the structure of the mathematical model. A metaheuristic – in this case, a genetic algorithm – is used to search for good pricing strategies. Here, for simplicity, the term *pricing strategy* will be used to represent a set of feasible values for the pricing decision variables q_{rap}, $\forall r, a, p$ (see Section 5.2.2.2). To assess how good a pricing strategy is, the values corresponding to the pricing decisions are fixed and the mathematical model is solved for the remaining variables. The resulting objective value quantifies the profit that can be obtained with the pricing strategy.

Figure 5.1 shows the overview of the proposed solution method. Biased Random-Key Genetic Algorithm (BRKGA) is the metaheuristic used to search good pricing decisions and is represented by the central diamond shape. In this section, the BRKGA framework will be detailed, including the structure of the chromosomes and population. Fitness calculation is the process within the genetic algorithm that assesses how good each pricing strategy is and, as mentioned above, comprehends solving the mathematical model with the pricing decisions fixed. In fact if the prices are fixed inputs and not decision variables, the problem becomes an integer linear problem and hence easier to solve. Such problems are special cases of Mixed Integer Programs (MIP) and due to ubiquity of this acronym, it will be used to represent this problem. In order to speed up the process, the linear program (LP) that results from relaxing the integrality constraints of this MIP is considered as a substitute approximation for the fitness evaluation.

From this search procedure, the best pricing strategy is retrieved and the final value to the remaining variables is calculated by fixing the pricing strategy and solving to optimality the resulting MIP model (bottom rectangle in Figure 5.1).

BRKGA's "generation zero" is conventionally entirely random. In this solution method, specific knowledge about this problem, such as the natural decomposition scheme that arises from forbidding upgrades, was used to improve the performance of the BRKGA's search. Initial heuristic pricing strategies were generated and fed into the "generation zero". These solutions are achieved by decomposition, relaxation and construction, and are represented by the rectangles on top, which are shown as inputs to the BRKGA procedure. This will also be detailed in this section, with a full discussion on the modeling choices made, which are also represented in the top right corner of Figure 5.1.

5.3.1   BRKGA Framework

A Biased Random-Key Genetic Algorithm (BRKGA) was used to guide the search over different pricing strategies. In genetic algorithms, a solution is considered as an *individual* belonging to a *population* and encoded in a *chromosome*. The objective function value of the solution is translated into the chromosome's fitness. A population, composed of a set of individuals, is evolved over some generations. Each generation involves the creation of a new population through the combination of pairs of individuals of the previous generation (the parents), as well as random mutation. The fitness value is herein critical for the selection of elements to combine and produce the following generation. Genetic algorithms with random-keys use random real numbers between 0 and 1 as genes. A deterministic procedure, the *decoder*, translates each chromosome into a solution of the original problem and evaluates it in terms of its fitness. (Gonçalves and Resende, 2011).
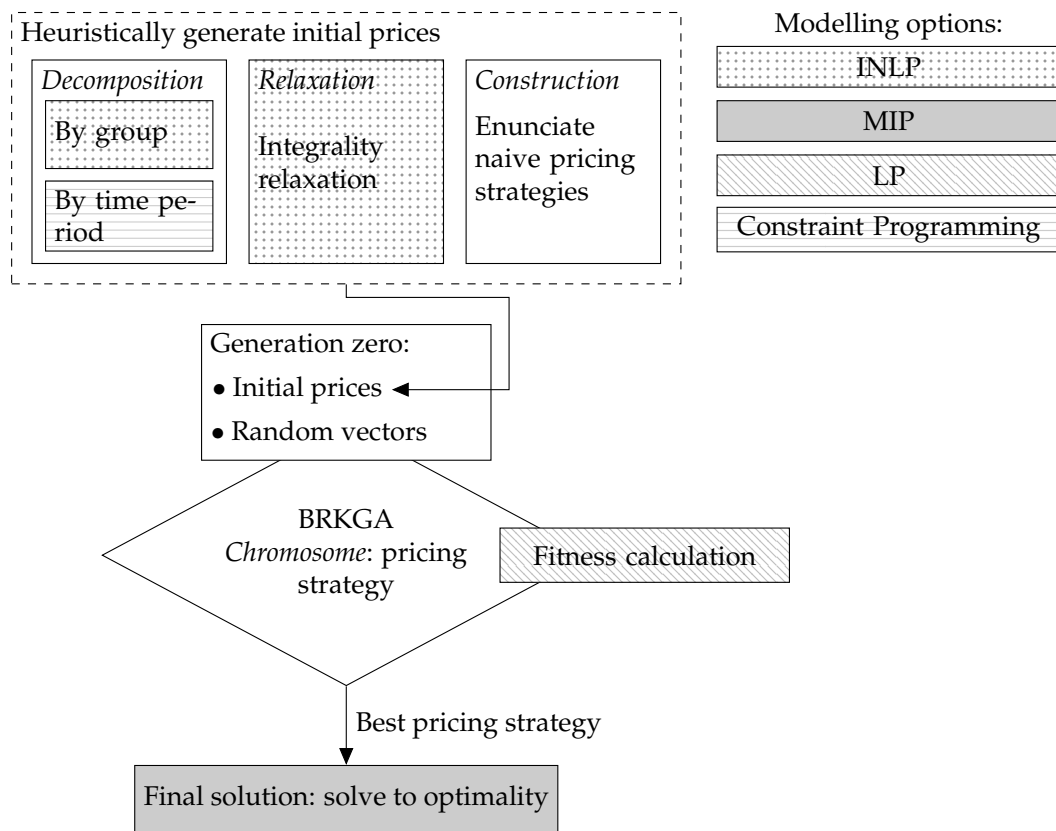
Figure 5.1: Overview of the proposed solution method.

	r_1			r_2			r_3			r_4		
	a_1	a_2	a_3	a_1	a_2	a_3	a_1	a_2	a_3	a_1	a_2	a_3
chromosome:	0.86	0.22	0.74	0.15	0.78	0.19	0.52	0.66	0.98	0.41	0.12	0.05
price level 1 (> 0.50)	X		X		X		X	X	X			
price level 2 (< 0.50)		X		X		X				X	X	X

Figure 5.2: Chromosome structure and translation into a pricing strategy. Example of choice among 2 possible price levels for 4 rental types (r) and 3 antecedences (a).

In this case, the solutions that compose a population and that were translated into chromosomes are the *pricing strategies*. The value of each pricing strategy would be the result of solving the MIP model with the price as a fixed input. However, in order to accelerate the procedure, an approximation was used to evaluate the fitness of each chromosome: the linear program (LP) resulting from relaxing all integrality constraints. To obtain the final solution, the MIP model is run with the integrality constraints considering as price input the best pricing strategy found by the BRKGA.

5.3.1.1   General Idea and Motivation

The general idea of the proposed solution method is to use BRKGA to generate and evolve pricing strategies. Each pricing strategy is evaluated in terms of the optimum outcome for all the decisions, by solving the Capacity-Pricing Model to optimality with prices as inputs. This allows to decompose the main problem in easier sub-problems. At the same time, the fact that this decomposition and the search within the consequent sub-problems are guided by a metaheuristic gives the solution method, at least theoretically, a certain validity and consistency. Moreover, by using a population-based method, it is expected that local optima will be avoided.

5.3.1.2   Chromosome Structure

A chromosome represents a pricing strategy, i.e., the price levels chosen for each rental type, requested with a certain antecedence. A chromosome is a vector of genes, which can take on a value – an allele – between 0 a 1. In this structure, each gene in the chromosome relates to the combination of a rental type with an antecedence level, therefore each chromosome has $|\mathcal{R}| \times |\mathcal{A}|$ genes, where $|\mathcal{R}|$ is the number of rental types and $|\mathcal{A}|$ the number of antecedence levels. The allele of the gene, i.e. the random number (n) associated with it, is then compared with the threshold that comes from dividing the range $[0, 1]$ in $|\mathcal{P}|$ equal partitions, where $|\mathcal{P}|$ is the number of possible price levels allowed:

$$\text{price level} = \left\lfloor \frac{n}{1/|\mathcal{P}|} \right\rfloor + 1 \tag{5.14}$$

Figure 5.2 illustrates this translation process for a simple example.

5.3.1.3   Fitness Evaluation

In order to understand the value of each pricing strategy, the fitness of the chromosome is evaluated. As mentioned above, the objective is to understand what the optimum result of using each pricing strategy is, considering the impact it has in all other decisions. To achieve this, one should solve the MIP that results from fixing on the Capacity-Pricing Model the pricing strategy given by the chromosome. Preliminary tests showed that, although the MIP model is fairly quick to solve, the solution times (around a few minutes) were not adequate when considering a population of considerable size that should evolve for some generations within a reasonable time frame. Therefore, to significantly speed up the process, the linear relaxation of the MIP (LP) was used as an approximation.

For this approximation to be valid, it is important to guarantee that not only the LP obtains an objective value similar to the MIP but also that the fitness ranking by which the chromosomes are sorted in a population is similar. In fact, in BRKGA, the evolution of a population consists, on a simplified view, in three main steps: (1) the best elements of the population (the elite) are directly copied to the next generation, (2) new chromosomes are generated from the cross-over of two elements of the current generation (elite or not), and (3) new chromosomes are randomly generated and inserted (mutant chromosomes). The fitness is used to sort the elements of a population so that the top (elite) and bottom elements are identified and steps (1) and (2) take place.

Therefore, to validate this approximation, 100 chromosomes were randomly generated and evaluated using the MIP model and its LP relaxation, based on Instance 1 (see Section 5.4.1). As expected, the LP was always solved in a few seconds, while the MIP was given a time limit of 2 minutes and could prove optimality in approximately one third of the cases. Figure 5.3 shows the boxplot for each situation. In fact, the values of the objective function were very similar, even in the cases where the MIP could not prove optimality. As expected, the LP approximation obtained results more similar to the MIP when the latter was able to prove optimality. However, in both cases, the differences are very small (always less than 0.14%).

As for the order by which these 100 chromosomes are sorted, there are some differences when using the objective function value of the LP or of the MIP. Nevertheless, these differences do not appear to be significant. Figure 5.4 shows this by plotting the chromosomes in the order sorted by *LP approximation* against the *MIP objective function value*. With this, it is possible to conclude that, although the ranking order is not exactly the same using the two approaches (if it were the graph would show a monotonically decreasing plot), where there are differences in ranking position there are no major differences in the objective function value. For example, the main difference is between positions 33 and 35 and here the difference of the MIP objective function value between the three chromosomes in these positions is only of 0.03%. Therefore, solving the LP relaxation provides a valid approximation for the MIP objective value in the context of this solution method.
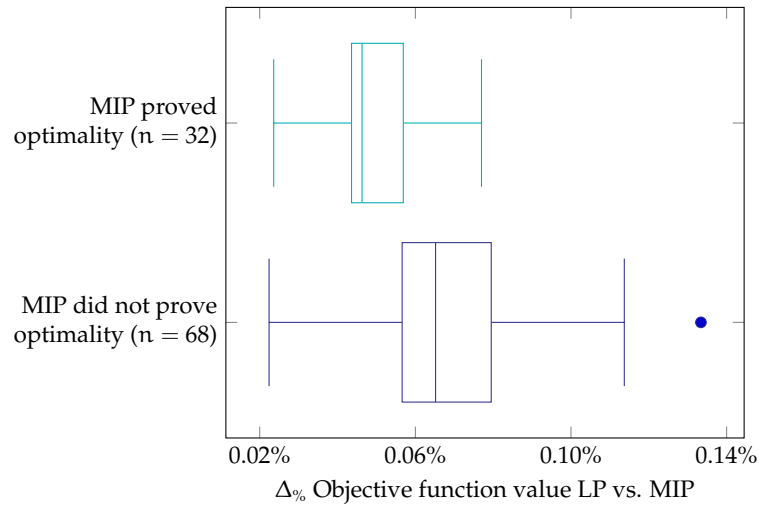
Figure 5.3: Box plot for the percent variation between objective value of LP relaxation vs. MIP for 100 random chromosomes.
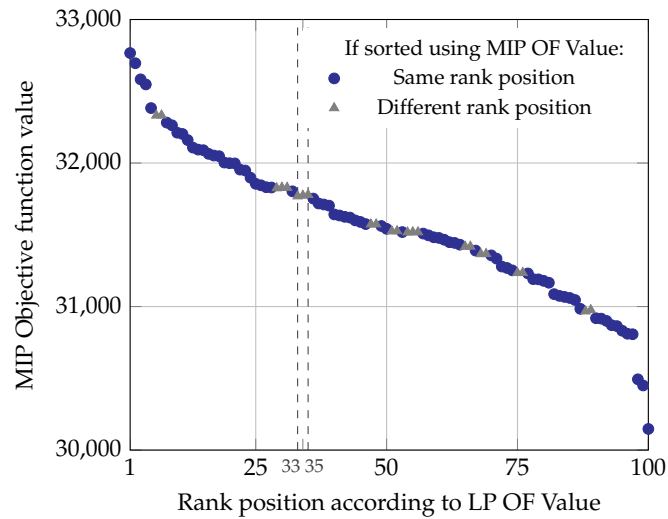
Figure 5.4: MIP objective function (OF) value of 100 random chromosomes sorted using the LP approximation.

5.3.2 Generation Zero: Heuristically Generated Initial Pricing Strategies

In order to boost the performance of the BRKGA, some specific pricing strategies were added to the (conventionally entirely random) initial generation, or "generation zero". The goal was to use specific knowledge of the problem to provide solutions that could have a good performance and could otherwise be missed. This specific knowledge is especially related with practical-driven simplifications or relaxations of the original problem. For example, if upgrades are not allowed, the problem becomes separable by vehicle group and hence easier to solve and the resulting pricing strategy may show significant potential to improve and evolve in this framework.

In this work, the addition of initial pricing strategies was structured according to their sources. The initial prices were thus obtained by three types of procedures:

- *Decomposition* of the main problem in separable sub-problems;
- *Relaxation* of integrality constraints;
- *Construction* of naive strategies.

Decomposition: One of the "natural decompositions" of the Capacity-Pricing Model was previously mentioned and consists in solving the problem for each vehicle group individually. Although the resulting sub-problems are still INLPs, they are smaller and easier to solve, and provide significant information in a practical context. Another decomposition approach often used in multi-period problems is to separate the problem by time period. In this case, it corresponds to solving the problem with a "myopic" perspective, considering one week at a time (if the week is used as time unit) and using the decisions of the previous week as inputs of the following one. Two approaches were used, with different "myopia degrees" that were materialized in how the leasing costs were accounted for. In the most myopic approach, only the leasing costs for that specific week were considered whether in the other approach if a leasing was decided in that week the leasing costs for the entire leasing period were imputed to the decision week. Other "myopic" aspect of both these approaches is that purchases for the owned fleet are only considered on the first week. In conclusion, three initial pricing strategies are generated by decomposition: one by group decomposition and two by time period decomposition.

Relaxation: The initial pricing strategies generated by this method do not necessarily arise from specific knowledge about the problem, but from the behavior of the Capacity-Pricing Model. Some preliminary experiments were conducted in order to understand if relaxing the integrality constraints of specific (integer) decisions would have a significant impact on both solving speed and solution quality. From these experiments, four different relaxation approaches were selected. The first consisted in relaxing the integrality of all decision variables, except the binary price selecting variables. The remaining three consisted on relaxing the integrality of all decision variables, except the binary price selecting variables and one of the three main decisions: acquisitions (w), stock (x) and rentals fulfilled (u). Each of these four approaches are still based on non-linear models, yet are easier to solve than the original one.

Table 5.5: Modelling options comparison.

| | Decomposition | | |
	By group	By time period	Relaxation
(M)INLP multi-period	✓*	–	✓*
(M)INLP single-period	–	✓*	✓*
CP multi-period	✓*	–	✗
CP single-period	–	✓	✗

∗ subject to size limitations

Construction: These initial pricing strategies are generated not based on the Capacity-Pricing Model but on construction heuristics and aim to represent the naive or obvious solutions that could otherwise be missed. It is not expected that these strategies allow for a significant improvement boost, yet, since the processing time of enunciating these strategies is negligible, it is worth considering them, as they are often the strategies "at hand" to be used by companies. There are two ways of constructing naive strategies: one is to price every rental type requested at every antecedence level with the same price ($|\mathcal{P}|$ pricing strategies are thus generated, where $|\mathcal{P}|$ is the number of possible price levels), and the other is to apply always increasing or decreasing price levels to a rental type, depending on the antecedence (which leads to two other naive pricing strategies).

Modeling Options

The *decomposition* and *relaxation* methods to obtain initial pricing solutions are based on non-linear models. Although these models are simpler and easier than the original Capacity-Pricing Model, preliminary experiments revealed difficulties in tackling some of the bigger real-sized instances (see Section 5.4.1). To face this, it was necessary to define and compare basilar modeling options.

Table 5.5 compares four types of models that can be used to generate initial prices: multi- and single-period (M)INLP and Constraint Programming (CP) models. Single-period models are introduced to tackle the *decomposition by time period*. The *construction* of naive strategies is not represented in the table since it consists in enunciating pricing strategies and not in solving mathematical models.

The most immediate option would be to use the Capacity-Pricing Model (INLP multi-period) and a single-period corresponding version. The multi-period model suits the *decomposition by group*, with the addition of a group of constraints to ensure that no upgrades are allowed, and the four types of integrality *relaxation* considered. In both cases, preliminary tests showed that there is a practical limitation on the size of the instance tackled, especially due to the compile-time of the non-linear segment of the objective function. However, this time can be considered a "fixed cost", since the objective function is common to these five initial prices to be generated (one by *group decomposition* and four by

integrality *relaxation*) and thus only has to be compiled once.

The single-period model could be used to generate the two initial pricing strategies based on *time-period decomposition*, with similar instance size limitations. Nevertheless, since the objective function is different, a new "fixed cost" should be considered. By definition, it could also be used while *relaxing* the integrality of different decision variables.

Summarizing, each of the first two lines of Table 5.5 encompass a fixed resolution time to compile the non-linear model (which leads to limitations in the instance size) and a variable time to solve per initial price.

As the instance size limitations can hinder the generation of *decomposition* and *relaxation* initial pricing strategies for bigger instances and are mainly caused by the non-linearity of the objective function, a different modeling (and, consequently, solving) approach was considered: the adaptation of the multi-period and the single-period (M)INLP models to a multi-period and a single-period Constraint Programming (CP) models. Constraint Programming was considered due to its ability to deal with non-linearity issues, which were consuming the most time in the previously considered models.

First and foremost, Constraint Programming is a "paradigm for solving combinatorial search problems" (Rossi et al., 2006) and a modeling approach suitable for integer decisions. The basic idea of CP is that variables have finite integer domains, related by a set of constraints that must be satisfied and that define the finite solution space. Therefore, it cannot be used to tackle the generation by *relaxation*. Preliminary tests showed that there were practical limitations on the instance size for the multi-period CP model for *group decomposition*. Moreover, these limitations were more significant than the ones found for the INLP model (i.e., some instances that could be tackled by the INLP could not be tackled by the CP multi-period model). However, for the single-period CP model, no significant size limitations were found.

Concluding, for the *decomposition by group* and *relaxation* the original Capacity-Pricing Model with additional constraints was selected as preferred modeling approach, while the CP single-period model was used to heuristically generate initial prices based on *time decomposition*. The CP single-period model is presented as an Appendix (Section 5.B) and was developed with two alternative objective functions, depending on the degree of myopia, as previously discussed.

In order to ensure a reasonable run time, a practical limit was set to establish for which instances *relaxation* and *group decomposition* initial prices could be generated. Nonetheless, few instances are expected to surpass this limit. The limit was calculated based on computational tests that showed that there is an exponential relationship between the size of the instances (measured by the number of rental types $|\mathcal{R}|$ times the number of vehicle groups $|\mathcal{G}|$) and the time to generate *relaxation* and *group decomposition* pricing strategies (see Figure 5.5). In order to keep run time for these procedures under the limit of 100 minutes, for instances with $|\mathcal{R}| \times |\mathcal{G}| > 8,250$ it will not be advantageous to generate prices by *relaxation* and *group decomposition*. As observed in Figure 5.5, only the two biggest instances, which are considerably bigger than the remaining ones, would be included in this set. The impact of the generation of these initial pricing strategies on the overall run time will be further discussed on Section 5.4.
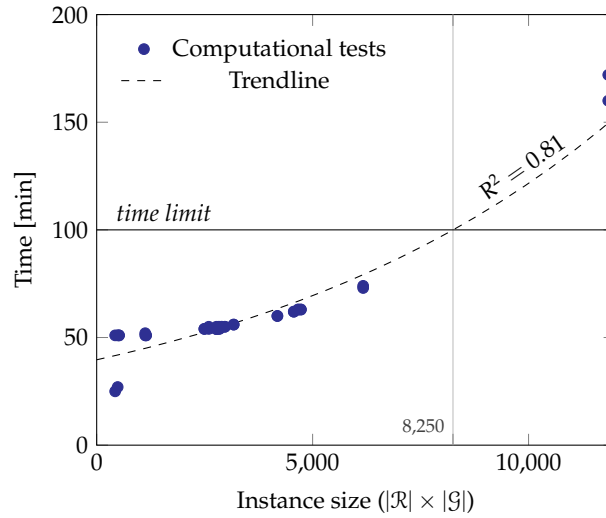
Figure 5.5: Time to generate group decomposition and relaxation initial prices, using (M)INLP models.

5.4. Computational Tests, Results and Discussion

This section aims to present and discuss the computational tests performed and results obtained, as a means to validate the different components of the solution method, as well as its overall relevance. This section describes the real-sized instances generated, based on the ones available in the literature, to perform these tests. Then, a baseline of comparison is established, in order to understand the impact of the capacity and pricing integration versus the (typical) sequential/hierarchical approach. Finally, the most relevant results will be presented and discussed, including a comparison with an exact approach to the original INLP using a non-linear solver.

5.4.1   Instances

In Oliveira et al. (2014), twenty instances for the vehicle-reservation assignment problem in car rentals are presented. These instances are based on real data retrieved from a Portuguese car rental company and contain real information regarding detailed reservation requests and vehicles. The data regarding reservation requests was used to generate fourty realistic instances. This section explains how these instances were generated, with special focus on types of rentals and demand data.

Instances for the vehicle-reservation assignment problem:

The vehicle-reservation assignment problem presented in Oliveira et al. (2014) consists in assigning specific vehicles to fulfill reservation requests in order to maximize the profit of the car rental company. The instances made available (Oliveira et al., 2016) provided, among other parameters and information, full lists of reservation requests that the company had received up to specific dates. These requests were characterized by start and end date
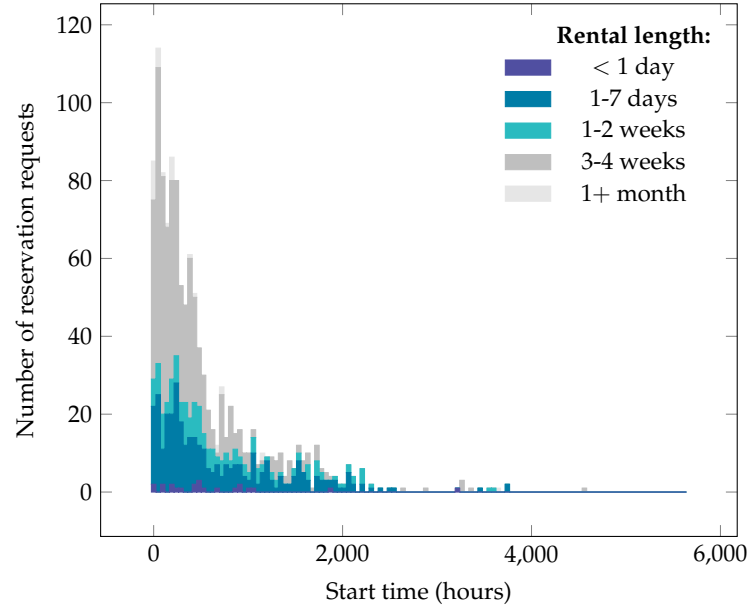
Figure 5.6: Example of original instance for the vehicle-reservation assignment problem – distribution throughout the time horizon according to start date, by rental length.

(and hour), start and end rental station, vehicle group requested, the profit expected from fulfilling the request, a priority status (related with customer confirmation) and an indication whether the customer would, if needed, accept a downgrade, which are used by the company as a last resource (upgrades were assumed to be always accepted).

As expected, in these instances the density of requests is higher for closer dates. Figure 5.6 exemplifies for a specific instance how the reservations are distributed in time according to their start date and rental length.

Adaptation towards realistic rental types:

In order to build significant and realistic rental types for the capacity-pricing integrating model, the reservation requests listed on the above-mentioned original instances were aggregated by rental types. All listed reservation requests that shared the following characteristics were aggregated by rental types: group of the vehicle required, starting week, ending week, starting zone, and ending zone.

As for the start and end time, rentals were aggregated in a weekly basis due to the strategic level of the decisions considered in this model, as discussed in Section 5.2. Moreover, only reservations that start within the time horizon of twelve weeks were considered.

As for the start and end location, the original list of reservations detailed the specific rental station. Once more, due to the type and impact of decisions considered in these models, the start rental stations mentioned in the original instances were aggregated in four zones. As for the end zone, since in the original instances it was almost always coincident with the start zone, it was also randomly determined.

From this aggregation, other parameters were also defined: the number of rental zones

and the number of required groups are dependent on the rental types for each instance.

Demand inputs:

The aggregation of the reservations described previously in this section provided a measure of the actual demand for the different types of rentals, based on the number of listed reservations that fell into each aggregated bin (rental type).

Nevertheless, in this model, the demand input DEM_{rap} for each rental type r depends on the price level $p = \{1, ..., P\}$ and on the antecedence $a = \{0, ..., A\}$ with which the rental request was made. Therefore, there was a need to generate different levels of demand for each rental type, related with the variation of these two indices. The demand given by the aggregation of reservations on the original instances (OD) provides a realistic reference for each rental type, and sets the reference demand for the first price level ($p = 1$). The reference demand (RD) for the following prices levels ($p > 1$) is strictly decreasing and is obtained by the following equation, where $|\mathcal{P}|$ stands for the number of price levels and α is a parameter that controls the gap between the levels.

$$RD_p = \begin{cases} OD, & p = 1 \\ RD_{p-1} - \frac{OD}{\alpha|\mathcal{P}|}, & p > 1 \end{cases} \tag{5.15}$$

Note that $\alpha \geq 1$ ensures that the reference demands are strictly decreasing and never null. However, if needed, it is also possible to model the demand-price relationship of a luxury product, where the demand increases as the price increases, by setting $\alpha < 0$. In this specific case, based on preliminary results and type of business, the value $\alpha = 2$ was chosen.

After setting the reference demand for each price level, one needs to generate the demand per antecedence as is detailed in the following equations, where β represents a randomly generated number such that $\beta \in]0, 1[$:

$$DEM_{rap} = \begin{cases} RD_p, & a = 0 \\ RD_{p+1} + \lfloor \beta(DEM_{r,p,a-1} - RD_{p+1} + 1) \rfloor, & a > 0 \wedge p < P \\ \lfloor \beta \times DEM_{r,p,a-1} \rfloor, & a > 0 \wedge p = P \end{cases} \tag{5.16}$$

The reference demand for each price level is associated with the first antecedence level. For the following antecedence levels, the demand value will be built from the reference demand of the next price level, to which will be added a fraction (β) of the gap between this and the demand value of the previous antecedence. On the last price level, a similar reasoning is applied, where the reference demand upon which the value is built is zero. This calculation ensures that all values that the demand takes across price levels are greater than the reference demand of the next price level. Figure 5.7 represents a possible demand profile for a rental type which had an original demand $OD = 329$.

The generated demand profiles are based on realistic data from a car rental company that operates in Portugal, which is a relatively small market. In order to validate the results
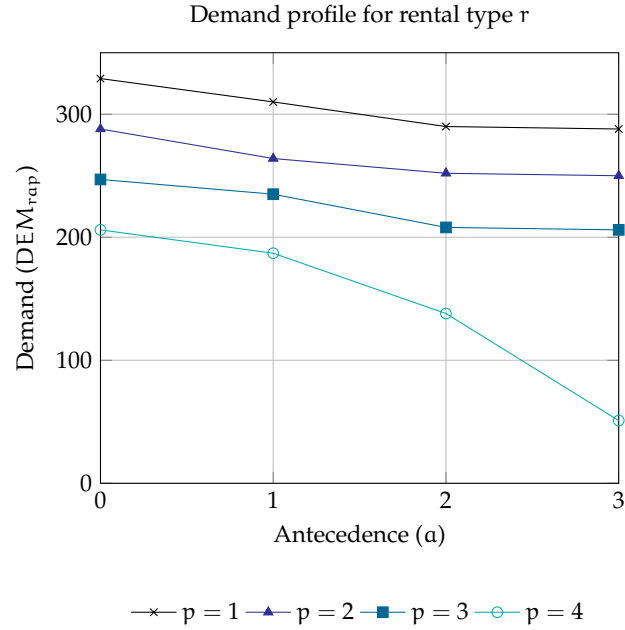
Figure 5.7: Example of a generated demand profile for a specific rental type.

for bigger markets, a scale factor was also considered when generating the demand profiles. More specifically, two different instances were generated from each of the original listings of requests: the first is directly derived from the original instances ("scale factor" of 1) and represents a small/medium market such as Portugal, while the second has a "scale factor" of 100, which is multiplied to the former demand profile, thus representing the challenges faced by a company operating on a significantly bigger market.

Remaining inputs:

Some parameters were unknown in the original instances or not fully adaptable to this model and were thus randomly generated, based on previously defined minimum and maximum values and respecting the relationship and hierarchy between vehicle groups, when applicable – for example, for the monetary value associated with each price level and group. The cost parameters were also generated in a similar fashion, yet maintaining a reasonable comparison between them when needed, e.g. the daily leasing costs are always significantly higher than the daily ownership/maintenance costs.

The upgrades were allowed in a fully nested way, i.e. the vehicle groups follow an hierarchy and rentals that require a least valued group can be upgraded to all groups that are more valued.

The budget was also randomly set yet for all instances it was proportional to the number of rental types and the scale factor of demand.

Table 5.11, on Appendix 5.D, details the main characteristics of each of the forty instances generated following the methodology described in this section. The instances are available at Oliveira et al. (2017a).
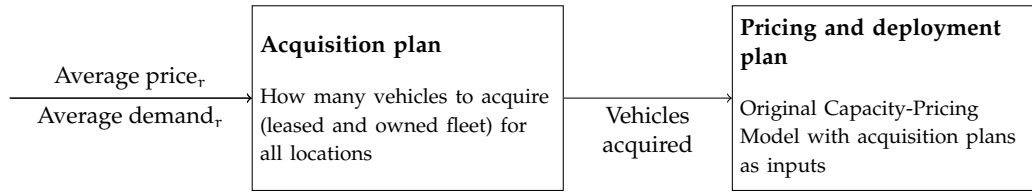
Figure 5.8: Structure of the sequential baseline resolution strategy.

5.4.2   Baseline: Hierarchical Resolution Strategy

In order to justify the advantages of integrating capacity and pricing problems, a baseline resolution strategy was developed for comparison, based on a more traditional sequential or hierarchical decision-making process. The goal of this comparison is to determine the potential of integrating these problems. Figure 5.8 depicts the overall hierarchical approach. In this framework, the first decisions made are the ones related with the acquisitions (capacity), based on average prices and demand for each rental type. These decisions are made for the fleet as a whole, without considering the deployment between locations. The aggregated number of fulfilled rentals is also decided in order to account for upgrading decisions. On a second phase, the deployment, empty transfers and consequent stock decisions are made, as well as the pricing decisions, also implying the decision on the number of rentals fulfilled. The second phase thus consists on solving the original Capacity-Pricing Model where the acquisitions (for the overall pool of locations) are fixed inputs. The mathematical programming models used for the first and second phase are adaptations from the model presented in Section 5.2.2 and are detailed in Appendix 5.C.

5.4.3   Structure of the Tests

Proposed solution method: To assess the performance of the solution method proposed in Section 5.3, each of the forty instances (see Section 5.4.1) was run twice. Firstly, the BRKGA was run with a fully random generation zero. Secondly, the heuristically generated initial prices were added to this generation when running the BRKGA. This makes it possible to measure the impact of using these initial prices.

To implement the BRKGA, the "brkgaAPI" (Toso and Resende, 2015) was adapted. Apart from the size of the chromosomes that is strictly dependent on the number of rental types and vehicle groups of the instance (see Section 5.3.1), the remaining main parameters were kept constant for all instances. The default values suggested in Toso and Resende (2015) were adequate for the problem herein considered and thus used (Table 5.6). No parallel decoding was applied and one independent population was considered. This was due to the fact that the decoding procedures are based in mathematical models and are therefore significantly more complex and time-consuming than the ones usually used with this metaheuristic. Finally, the stopping criterion chosen for the BRKGA procedure was the solving time (1 hour).

As for the heuristic generation of initial prices, a time limit was set for the group and time period decomposition and relaxation. The construction-generated prices are virtually

Table 5.6: Main BRKGA parameters that are similar for all instances

Parameter		Value
p	size of population	50
pe	fraction of population to be the elite-set	0.20
pm	fraction of population to be replaced by mutants	0.10
ρ	probability that offspring inherits an allele from elite parent	0.70

immediate to generate. The group decomposition and the relaxation involve solving INLP models, as discussed in Section 5.3.2. For each of these models, a time limit of 10 minutes was set. As for the time period decomposition, two series of single-period Constraint Programming models were solved (for the two types of objective functions), with a time limit of also 10 minutes.

As for the final MIP run used to solve to optimality the best pricing strategy, a time limit of 10 minutes was applied, although it was never reached.

Comparison baseline: Besides the proposed solution method, a baseline was developed based on the sequential resolution of the same problem (see Section 5.4.2). Each instance was solved as well using this method. In order to be conservative when assessing the performance of the proposed method, the time limit for the baseline was set to be slightly bigger than the actual maximum total time to solve when using the integrated method – 2 hours and 40 minutes (9600 seconds).

Exact approach to the Capacity-Pricing Model: In order to validate the need for a non-exact solution method, the mathematical model presented in Section 5.2.2 was solved using a non-linear solver for each instance, with the same time limit set for the proposed method.

Technical details: The algorithms, Mathematical Programming models and Constraint Programming models were developed in C++/IBM ILOG Concert Technology and were run on a HP Z820 Workstation computer with 128 Gigabyte of RAM memory, and with 2 CPUs (Xeon E5-2687W 0 @ 3.10 GHz). The MIP and MINLP Solver used was CPLEX 12.6.3 and the CP solver used was CPLEX CP Optimizer 12.6.3.

5.4.4 Results and Discussion

In the remainder of this section, the results will be presented and discussed. Four main issues will be discussed in more detail: the advantages and disadvantages of feeding heuristically generated initial prices to BRKGA's "generation zero", which initial solutions perform the best, the overall performance of the integrating approach versus the sequential baseline, and the advantages and disadvantages of using a non-exact approach to this problem.
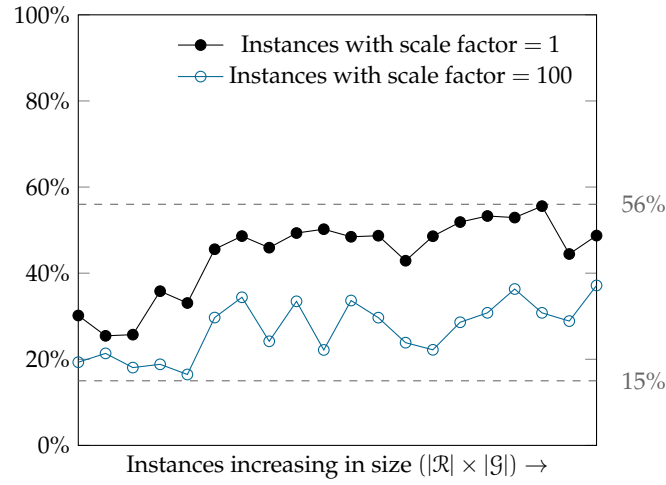
Figure 5.9: Improvement on final solution MIP objective function value: heuristically generated initial prices added to generation zero versus fully random generation zero.

Impact of Using Initial Prices

One of the novelties of the proposed approach was the use of heuristically generated initial pricing strategies to compose part of the initial generation of the BRKGA framework. It is thus important to understand how this part of the methodology impacts the overall performance in terms of solution quality and solving time.

First and foremost, using heuristically generated initial prices is significantly beneficial for the solution quality. Figure 5.9 presents the improvement on the final solution obtained by the full methodology (including heuristically generated initial prices) versus a similar BRKGA procedure but with a *fully random* generation zero. The detailed values that support this figure can be found in Table 5.12, on Appendix 5.D. As observed, adding heuristically generated initial prices to the initial generation improves significantly the final solution, with results at least 15% better and, for some instances, more than 50%.

This level of improvement is due to the fact that the heuristically generated initial prices represent significantly good solutions. This statement is supported by two facts: **(1)** the fitness of the initial pricing strategies is consistently the best fitness of the initial generation, and **(2)** BRKGA has less room to evolve when these good solutions are directly added to the initial generation.

In fact, the best of these initial pricing strategies (evaluated in terms of their fitness – see Section 5.3.1.3) is for all instances the best of the initial generation **(1)**. Table 5.13 and Table 5.14, in Appendix 5.D, present the detailed results for the fully random BRKGA run and the run with additional heuristic initial prices. These tables include the best fitness obtained for the initial and last generations, the number of generations that the method was able to evolve within the time limit, and the final MIP run objective function value and gap. Moreover, for the run with additional heuristic initial prices (Table 5.14), the best fitness obtained by these added initial prices is presented and it is possible to observe that it always matches the best fitness of the initial generation.
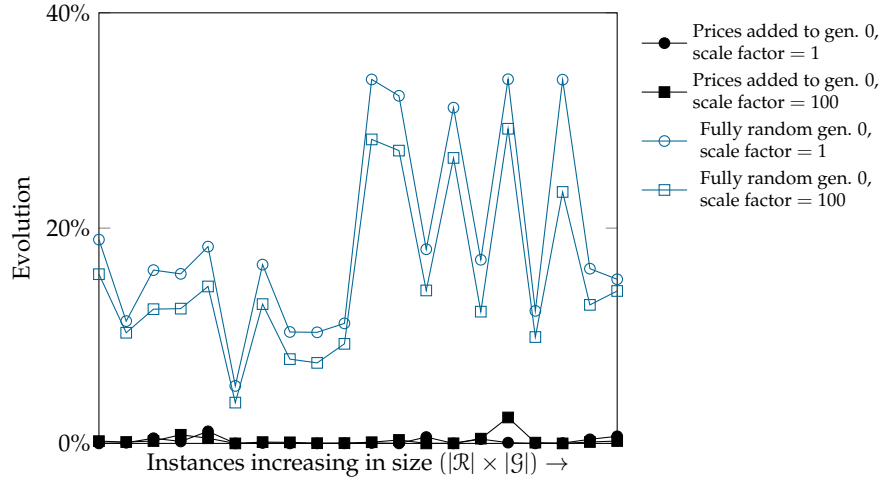
Figure 5.10: Evolution of BRKGA: last generation best fitness versus initial generation best fitness.

Moreover, there are significant differences when comparing the evolution that BRKGA is able to achieve (i.e., how much the best fitness of each generation increases from the initial to the last). Figure 5.10 depicts this evolution (in percentage of increase from initial to last generation), discriminating the scale factor of the instances (i.e., the size of the market considered) and whether the "fully random generation zero" version of BRKGA or the "initial prices added to generation zero" version of BRKGA was used. Although the size of the market did not influence significantly this evolution, the type of methodology used did. When these (good) initial pricing strategies were added to the initial generation, the evolution was significantly smaller (**2**).

Other interesting conclusion of Figure 5.9 is that there is a difference on the improvement achieved by heuristically generating initial prices when comparing the size of the markets considered (scale factor). The improvement is bigger for instances that represent markets of the size as the one in study (scale factor = 1), for the same number of rental types and vehicle groups. This means that the procedures to generate initial pricing strategies are especially efficient for smaller markets. This is an expected result since generating initial prices often involves solving complex models with a limit on solution time: the smaller the instance, the better solutions are obtained.

Furthermore, there is a difference when comparing sizes of instances (indicated by the number of rental types and vehicle groups): the improvement achieved by adding initial prices to the initial generation tends to be more significant for bigger instances. This might be explained by the fully-random BRKGA loosing impetus when the solution space increases significantly.

Nevertheless, the boost on solution quality obtained by inserting heuristically generated initial prices on generation zero of the BRKGA procedure comes with a price to pay: the additional time to solve. Figure 5.11 shows the average time to solve each component for all instances. In Appendix 5.D, the discriminated values for each instance can be found on Table 5.15. These results reflect the time limits discussed on Section 5.4.3. Group
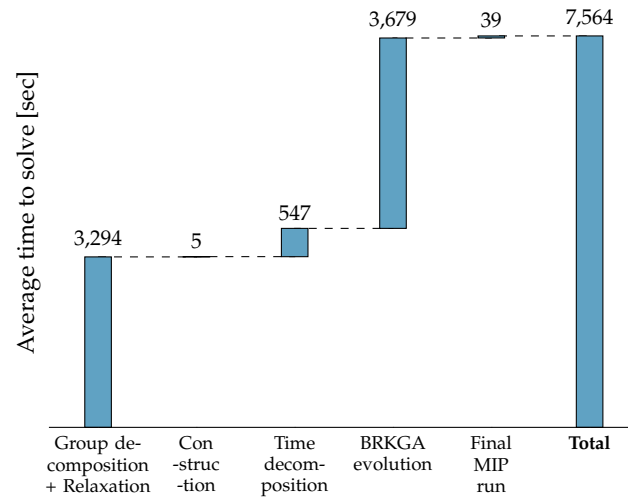
Figure 5.11: Average time to solve each each component of the proposed method.

decomposition and relaxation, which imply solving (M)INLP models, take more time than the other procedures to generate initial prices. However, as it will be discussed in the following paragraphs, they have a good performance. Also as expected, the MIP run to generate the final complete solution is fairly quick.

Overall, the generation of initial prices increases significantly the solving time. Nevertheless, this increase seems to be more than justified by the boost on solution quality obtained. Moreover, although the BRKGA is not able in average to evolve much of these already very good initial solutions, it is important to ensure that local optima are being avoided and that the variability of the solutions are improved as much as possible within a realistic time-frame. In fact, it is important to bear in mind that the ultimate goal of such a methodology is to be applied in a decision support system to help companies make better plans. Considering the time horizon of this problem, which aims to plan for a full selling season, 2 or 3 hours, even if multiplied by a finite number of different runs to test different scenarios or strategic options (see Section 5.5 for a more detailed discussion on this topic), seem to be a small price to pay.

Initial Heuristic Prices – Comparing Sources

Since the different sources of the initial prices added to the initial generation of BRKGA take a significantly different amount of time to solve, it is important to analyze their performance in terms of solution quality. The performance is evaluated by the following measures: (1) the number of times the initial price with the best fitness for some instance was generated by this specific source, and (2) how close were in average the fitness values of the initial prices generated by this source to the best fitness of the instance (translating each fitness value into a percentage of the instance's best fitness), as well as the stability (or variability) of this closeness (measured by its standard deviation). Figure 5.12 presents these measures for the four sources of heuristically generated initial prices. Also, Table 5.16 in
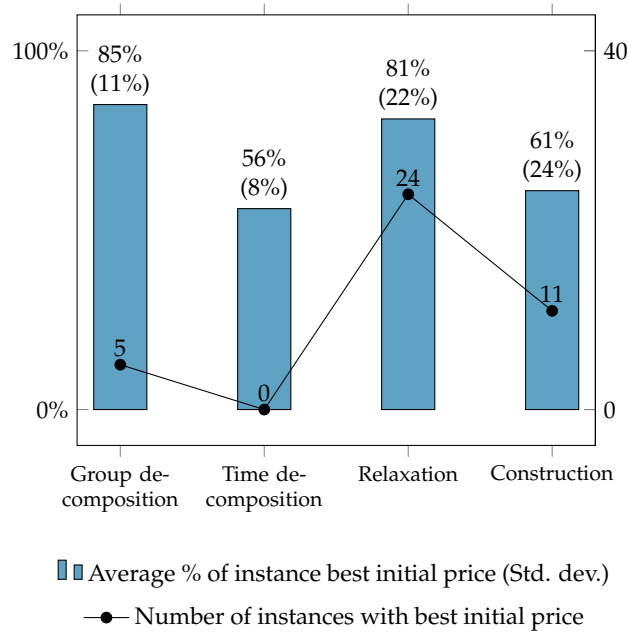
Figure 5.12: Performance of the sources of initial prices: frequency of generation and closeness to the best initial price generated for each instance.

Appendix 5.D details for each instance the best fitness obtained by each source.

The standard deviation of the measure "percentage of best price" is included to clarify the variability of the quality of the initial prices generated by each source. Nevertheless, this variability could be ascertained by the other two measures. In fact, if a certain source provides a significant number of times the *best price* of the instance but has a relatively low average *percentage of best price* – as happens for *construction* – it is due to a high variability of the quality of the generated prices.

Except for *time decomposition*, all sources provide more than once the initial price with the best fitness of the instance. *Relaxation* achieves this for more than half of the instances. *Construction* of naive/obvious solutions seems to be quite powerful too. Nevertheless, as discussed above, it is not very stable. As expected, for the same instance if some obvious pricing strategy is very good some other is bound to be quite bad. *Group decomposition* is not often the best initial pricing strategy source yet it is very stable and, in average, significantly close to the best initial price. As for *time decomposition*, it is never the source of the best price and the average percentage of the best price is also penalized for that. Nevertheless, it shows the less variability of all sources and also consumes less time than *group decomposition* and *relaxation*.

In conclusion, all sources show some advantages, with results that seem to be highly dependent on the instance, and it thus seems reasonable to keep all of them in the procedure. Once again keeping the "big picture" of the ultimate application in mind, the modular structure of this part of the methodology also renders it easy to be translated into a decision support system, where these modules or sources can be turned on and off depending on the time/performance trade-off of the decision-maker.
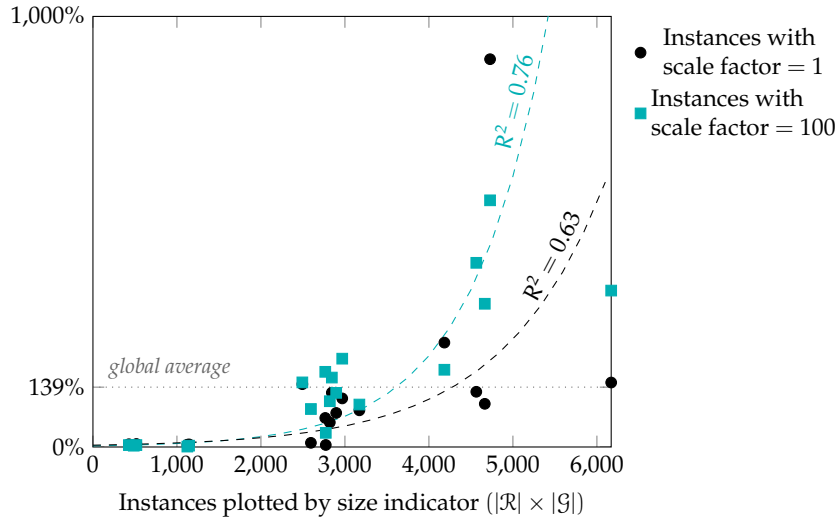
Figure 5.13: Improvement of proposed integrating method versus baseline sequential approach, in terms of best objective value.

Performance versus Baseline

Comparing the proposed methodology with the sequential baseline presented in Section 5.4.2 establishes a quantitative proof of the value of integrating capacity and pricing decisions in car rental. This is, in fact, one of the main contributions of this work, as the integration of these decisions, although conceptually supported by other works (see Section 5.1), is now quantitatively justified versus a more traditional sequential or hierarchical approach.

The sequential baseline defined in Section 5.4.2 gives an upper bound on the value that can be actually obtained by companies, since it has (slightly) more processing time and is already using the novel model proposed, which is a detailed and enhanced representation of the problem. That is to say that it was designed to make the comparison between approaches fair and the relative performance of the integration measured in a conservative way.

Figure 5.13 shows for each instance the improvement in terms of the final solution objective value obtained by the integrating versus the baseline (sequential) approach. It is possible to observe that the improvement is extremely significant, growing exponentially with the size of the instances. For every instance, the proposed approach is better than the sequential approach. Moreover, it was able to solve the two biggest instances, which could not be solved by the sequential method and are thus not represented. Considering the instances that could be solved, the average of improvement in objective value is 139%, yet it can go up to approximately 900%. Table 5.17, in Appendix 5.D, details the improvement achieved for each instance.

Exact vs. Non-Exact Approach to the Capacity-Pricing Model

The mathematical model presented in Section 5.2.2 was implemented in a non-linear solver in order to evaluate the extent to what a straightforward exact approach could perform well

and thus assess the need for non-exact solution methods for this problem.

Table 5.7 shows the overall results and Table 5.18, on Appendix 5.D, details these results. For 13 out of 40 instances, using this exact approach would lead to slightly better or similar results than using the proposed method. It is interesting to notice that the solver was always stopped by the time limit, even when running these instances.

In 27 out of 40 instances the proposed heuristic method outperforms the non-linear solver. In fact, for 7 of these instances, the exact solver is not even able to find a feasible solution different from the "trivial solution" (where all decision variables are set to 0 and the best objective value is also 0). For the 20 instances where the solver is able to find other feasible solutions, the proposed heuristic method achieves in average 204% of improvement on the objective value. It is interesting to notice that the size of the instances (measured by the number of rental types $|\mathcal{R}|$ times the number of vehicle groups $|\mathcal{G}|$) is likely not the only factor influencing the ability of the exact solver to find good solutions.

Overall, these results support the importance of developing heuristic solution methods as the one presented in this work to tackle the Capacity-Pricing Model.

Table 5.7: Overall comparison of the best values obtained by the proposed solution method (here denoted *BRKGA*) and by the non-linear solver (here denoted *INLP*).

	# instances	Average size indicator ($\|\mathcal{R}\| \times \|\mathcal{G}\|$)	Average improvement on best objective value: BRKGA vs. INLP
BRKGA **worst** performance than INLP	**4**	2,339	-2%
BRKGA **similar** performance to INLP	**9**	695	0%
BRKGA **better** performance than INLP	**27**		
− *INLP: 1+ feasible solutions*	20	3,693	204%
− *INLP: trivial solution*	5	3,441	-
− *INLP: no feasible solutions*	2	11,845	-

5.5. Conclusions

This paper tackled the integration of capacity and pricing problems in car rentals, which are significantly relevant – both academically and for practitioners. A new integrating mathematical model was proposed, as well as a solution procedure based on its decomposition, guided by a biased random-key genetic algorithm. The value of integrating these problems was established and empirically measured by successfully comparing the results

of the proposed solution method with the ones obtained by a hierarchical and sequential approach.

The solution methodology herein proposed may be used by companies to support their decisions, since it was built on a realistic model and is relatively fast to produce good solutions. In fact, as previously discussed, its average solving times and modular structure allow for it to be used as part of a decision support system where the final user could run the procedure several times for different scenarios, such as different levels of investment on the fleet or different demand forecasts. At the same time, it would be possible to select different sources of heuristically generated initial prices to better control time to solve and variability of solutions.

As for future work, the modeling and integration of the demand-price relationship in the model could be developed. In order to obtain more realistic and robust solutions, the stochasticity of demand could be considered. Moreover, further economic studies could help develop a more precise demand input for the model, thus leading to more accurate results.

Finally, as it was previously mentioned, this sector shares important characteristics with emerging mobility systems, such as carsharing, namely fleet mobility and flexibility, heavy dependency on efficiency and high occupation rates, and the ability to use prices to manage demand. This work can thus be extended, e.g. by allowing the free floating of the fleet (dropping-off vehicles in any location, not only previously established locations), to help carsharing companies better manage their fleet and pricing schemes. In Wagner et al. (2016), the challenges posed by the spatial flexibility of free float are identified and the authors propose a model to explain the variation of activity through the proximity of certain points of interest. Building on this type of demand-modelling techniques, the work developed in this paper can be further extended to the rapidly expanding market of carsharing.

Acknowledgements

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Appendix 5.A   Insights on Model and Problem Structure

The mathematical model developed on Section 5.2.2 brings some structural insights to the problem at hand. An analogy can be made between specific sections of this formulation and transportation problems modeled with linear programming. One of the main differences resides in the fact that most of the conventional parameters of transportation problems (capacities, needs, unit costs, . . .) are, in this formulation, decision variables. Nevertheless, interesting insights can help understand the problem structure and model behavior.

Figure 5.14 depicts this analogy for a section of the problem consisting of a specific *location* and *time*. For clarity, the problem is here simplified: it considers only owned fleet (O), two vehicle groups $g = \{1, 2\}$, two types of rentals $r = \{1, 2\}$ and two different antecedence levels $a = \{1, 2\}$, and it is focused on a specific location $s = 1$ and time period $t = 1$. In this small example, both rental types $r = \{1, 2\}$ refer to rentals that start in the same location and period of time, yet $r = 1$ requests a vehicle of group $g = 1$ and $r = 2$ of group $g = 2$. Also, the upgrading policy of the company states that a rental type requesting group $g = 1$ can be upgraded to a vehicle of group $g = 2$, but not the other way around.

The origins of the transportation problem (nodes on the left) are the different vehicle groups plus a virtual node that represents all rental requests that will not be fulfilled, i.e. not assigned to a group. The capacity in the origins represents the number of vehicles of the specific group available at the specific location and time and is a function of the acquisition decision for this group. For the virtual "no-group" origin node, the capacity is unlimited.

The destinations (nodes on the right) are the requests for the rental types, with a certain antecedence. The need of each destination is the demand of each rental type, for each antecedence, which is a function of the corresponding pricing decision (here the chosen price level is represented as $pd_{ra} = \sum_{p \in \mathcal{P}} p \times q_{rap}$).

The unitary link "parameters" are here also highly dependent on the pricing decision. They represent the profit obtained from fulling a rental, which is a function of the price charged. If an upgrade is offered, there is a penalization to account for. All links with origin in "no-group-node" have a null profit.

In this problem, for a specific time and location, the flow between origins – available fleet – and destinations – rental requests – represents the actual number of rentals fulfilled. The limitations on the number of rentals by the stock of available vehicles and by the demand for each type of request reflect the main constraints of the problem, presented before. This type of analogy allows to better understand the structure of the problem and the relationship between the different decisions.

Appendix 5.B   Constraint Programming Single-Period Model

Considering the indices and parameters and based on the Capacity-Pricing Model presented in Section 5.2.2, the following single-period Constraint Programming model was developed.

Note that each single-period model tackles a different set of rental types $\mathcal{R}_t : dout = t$, consisting of the ones whose starting date (*dout*) falls within the considered time period t.
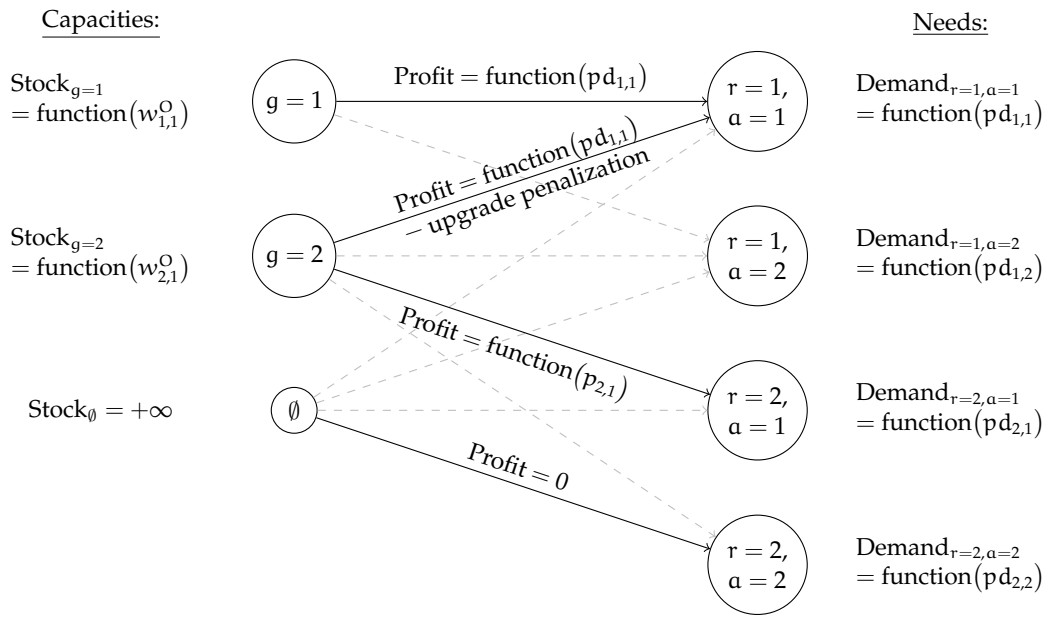
Figure 5.14: Representation as a "transportation" problem, with capacities on the origin nodes (stock of vehicle groups) and needs on the destination nodes (demand for related rental types): zoom in a specific location ($s = 1$) and time ($t = 1$). The notation follows the mathematical model notation presented in Section 5.2.2.1 and, for simplicity, includes the notation pd_{ra} to indicate the price level decided for rental type r and antecedence a $\left(pd_{ra} = \sum_{p \in \mathcal{P}} p \times q_{rap} \right)$.

Decision Variables

The following table presents the decision variables, as well as their domains.

$q_{ra} = \{1,...,P\}$ — price level charged for rental type r with antecedence a

$w_{gs}^{L} = \{0,...,ubw^{L}\}$ — Number of vehicles of group g acquired by leasing available at location s. The domain upper bound is based on the maximum demand for all rental types, considering the sum of all antecedence levels:

$$ubw^{L} = \sum_{r \in \mathcal{R}_t} \sum_{a} \max_{p} \{DEM_{rap}\}$$

$w_{gs}^{O} = \{0,...,ubw^{O}\}$ — Number of vehicles of group g acquired for the owned fleet available in location s (only for $t = 0$). The domain upper bound is based on the available budget:

$$ubw^{O} = \left\lceil \frac{B}{C_g} \right\rceil$$

$x_{gs}^{L/O} = \{0,...,2ubw^{L/O}\}$ — Number of leased (L) or owned (O) vehicles of group g located at s. The domain upper bound is based on purchases, yet not limited to them (interconnected time periods).

$y_{s1s2g}^{L/O} = \{0,...,2ubw^{L/O}\}$ — Number of leased (L) or owned (O) vehicles of group g transferred from location $s1$ to location $s2$ (starting on this time period). The domain is limited by the stock available.

$u_{rag}^{L/O} = \{0,...,ubu\}$ — Number of fulfilled rentals requested as rental type r with antecedence a that are served by a leased (L) or owned (O) vehicle of group g. The domain upper bound is based on the maximum demand for each specific rental type and antecedence:

$$ubu = \max_{p} \{DEM_{rap}\}$$

$f_{g}^{L/O} = \{0,...,prevW^{L/O} + ubu\}$ — Auxiliary variable: total leased (L) or owned (O) fleet of group g. The domain upper bound is given by the purchases of previous time periods plus the upper bound of the current rentals:

$$prevW^{L/O} = \sum_{t' < t} \sum_{g} \left[w_{gs}^{L/O} \right]_{t'}$$

Also, since the single-period model was developed to be solved sequentially for all time periods, all inputs (parameters) that are given by the result of the decision variables from the previous time period will be noted with the prefix P. Note that the variable that represents the decision on the number of rentals to fulfill that start in the specific t, $u_{rag}^{L/O}$ has $|\mathcal{R}_t|$ elements on the first index while the input $Pu_{rag}^{L/O}$ that stores these decisions for past time periods, has a corresponding number of $|\mathcal{R}|$ elements.

CP Model

Absolutely myopic objective function:

max   Profit from fulfilled rentals − Buying cost − Leasing cost (for the time period)
      − Ownership cost − Empty transfer cost − Penalty for upgrading =

$$
\max \quad \sum_{r=1}^{|R_t|} \sum_{a=1}^{A} \left(\left(\sum_{g=1}^{G} u_{rag}^{L} + u_{rag}^{O} \right) PRI_{q_{ra}g_{r}} \right) - \sum_{g=1}^{G} \left(\sum_{s=1}^{S} w_{gs}^{O} \right) COS_g
$$

$$
- \sum_{g=1}^{G} f_g^L LEA_g - \sum_{g=1}^{G} f_g^O OWN_g - \sum_{s1=1}^{S} \sum_{s2=1}^{S} \sum_{g=1}^{G} \left(y_{s1s2gt}^L + y_{s1s2gt}^O \right) TC_{gs1s2}
$$

$$
- \sum_{g=1}^{G} \sum_{r \in \mathcal{R}^{g^-}} \sum_{a=1}^{A} \left(u_{rag}^L + u_{rag}^O \right) PYU \quad (5.17)
$$

Alternative: less myopic objective function (different leasing cost term):

max   Profit from fulfilled rentals − Buying cost − Leasing cost (for the entire leasing period)
      − Ownership cost − Empty transfer cost − Penalty for upgrading =

$$
\max \quad \sum_{r=1}^{|R_t|} \sum_{a=1}^{A} \left(\left(\sum_{g=1}^{G} u_{rag}^{L} + u_{rag}^{O} \right) PRI_{q_{ra}g_{r}} \right) - \sum_{g=1}^{G} \left(\sum_{s=1}^{S} w_{gs}^{O} \right) COS_g
$$

$$
- \sum_{g=1}^{G} f_g^L LEA_g LP_g - \sum_{g=1}^{G} f_g^O OWN_g - \sum_{s1=1}^{S} \sum_{s2=1}^{S} \sum_{g=1}^{G} \left(y_{s1s2gt}^L + y_{s1s2gt}^O \right) TC_{gs1s2}
$$

$$
- \sum_{g=1}^{G} \sum_{r \in \mathcal{R}^{g^-}} \sum_{a=1}^{A} \left(u_{rag}^L + u_{rag}^O \right) PYU \quad (5.18)
$$

Stock calculating constraints:

Owned fleet:

$$
\text{s.t. } x_{gs}^{O} = \begin{cases} INX_{gts}^{O} + w_{gs}^{O}, & t = 0 \\[2ex] \left[Px_{gs}^{O} \right]_{t-1} + ONY_{gts}^{O} + ONU_{gts}^{O} \\ + \sum_{r \in \mathcal{R}_{s,t}^{in}} \sum_{a=1}^{A} Pu_{r,a,g}^{O} - \sum_{r \in \mathcal{R}_{s,t}^{out}} \sum_{a=1}^{A} Pu_{r,a,g}^{O} & t > 0 \\ + \sum_{c=1}^{S} y_{c,s,g,t-TT_{cs}-1}^{O} - \sum_{c=1}^{S} y_{s,c,g,t-1}^{O}, \end{cases} \quad \forall g, s \quad (5.19)
$$

Leased fleet:

$$
x_{gs}^L = \begin{cases}
0, & t = 0 \\
\\
\begin{aligned}
& \left[Px_{gs}^L\right]_{t-1} + ONY_{gts}^L + ONU_{gts}^L \\
& + \sum_{r \in \mathcal{R}_{s,t}^{in}} \sum_{a=1}^A Pu_{r,a,g}^L - \sum_{r \in \mathcal{R}_{s,t}^{out}} \sum_{a=1}^A Pu_{r,a,g}^L \\
& + \sum_{c=1}^S y_{c,s,g,t-TT_{cs}-1}^L - \sum_{c=1}^S y_{s,c,g,t-1}^L \\
& \hspace{4cm} + wL_{gs},
\end{aligned} & 0 < t \le LP_g \\
\\
\begin{aligned}
& \left[Px_{gs}^L\right]_{t-1} + ONY_{gts}^L + ONU_{gts}^L \\
& + \sum_{r \in \mathcal{R}_{s,t}^{in}} \sum_{a=1}^A Pu_{r,a,g}^L - \sum_{r \in \mathcal{R}_{s,t}^{out}} \sum_{a=1}^A Pu_{r,a,g}^L \\
& + \sum_{c=1}^S y_{c,s,g,t-TT_{cs}-1}^L - \sum_{c=1}^S y_{s,c,g,t-1}^L \\
& \hspace{2cm} + w_{gs}^L - \left[Pw_g^L s\right]_{t-LP_g-1},
\end{aligned} & t > LP_g
\end{cases} \qquad \forall g, s \qquad (5.20)
$$

Capacity on origins / needs on destinations constraints:

$$
\sum_{g=1}^G \left(u_{rag}^L + u_{rag}^O\right) \le DEM_{raq_{ra}} \qquad \forall r \in \mathcal{R}_t, a \qquad (5.21)
$$

$$
\sum_{r \in \mathcal{R}_{s,t}^{out}} \sum_{a=1}^A u_{rag}^{L/O} + \sum_{c=1}^S y_{scg}^{L/O} \le x_{gs}^{L/O} \qquad \forall g, s \qquad (5.22)
$$

Business-related constraints:

$$
UPG_{gr_r,g} = 0 \Rightarrow \sum_{a=1}^A \left(u_{rag}^L + u_{rag}^O\right) = 0 \qquad \forall r \in \mathcal{R}_t, g \qquad (5.23)
$$

$$
\text{(only for } t = 0\text{)} \qquad \sum_{s=1}^S \sum_{g=1}^G w_{gs}^O COS_g \le BUD \qquad (5.24)
$$

Other constraints:

$$
f_g^{L/O} = \sum_{s=1}^S x_{gs}^{L/O} + \sum_{r \in \mathcal{R}_t^{use}} \sum_{a=1}^A Pu_{r,a,g}^{L/O}
$$
$$
+ \sum_{s1=1}^S \sum_{s2=1}^S \sum_{t'=t-TT_{s1s2}}^{t-1} \left[Py_{s1,s2,g}^{L/O}\right]_{t'} \qquad \forall g \qquad (5.25)
$$

Appendix 5.C   Sequential Resolution Strategy

The sequential resolution strategy presented in Section 5.4.2 was developed as a baseline to assess the performance of the integration strategy and consists on solving two models sequentially: an acquisition plan model and a pricing and deployment plan model. The mathematical formulation of these models will be presented in this appendix.

5.C.1   Acquisition Plan Model

Considering the indices and parameters and based on the Capacity-Pricing Model presented in Section 5.2.2, the following MIP adaptation was developed for the acquisition plan. Moreover, this model also requires as inputs the average price level ($avgq_g$) and average demand ($avgDEM_g$) per rental type, which were linearly derived from each instance (see Section 5.4.1).

Decision Variables:

w^L_{gt} Number of vehicles of group g acquired by leasing at time $t = \{0,...,T-1\}$

w^O_g Number of vehicles of group g acquired for the owned fleet available at time $t = 0$

$x^{L/O}_{gt}$ Number of leased (L) or owned (O) vehicles of group g at time t

$u^{L/O}_{rg}$ Number of fulfilled rentals requested as rental type r that are served by a leased (L) or owned (O) vehicle of group g

$f^{L/O}_{gt}$ Auxiliary variable: total leased (L) or owned (O) fleet of group g at time t

Mathematical Integer Program (MIP)

$$\max \quad \sum_{r=1}^{R}\Big(\sum_{g=1}^{G} u^L_{rg} + u^O_{rg}\Big)PRI_{avgq_g,gr_r} - \sum_{g=1}^{G}\Big(\sum_{s=1}^{S} w^O_{gs}\Big)COS_g$$

$$- \sum_{g=1}^{G}\Big(\sum_{t=1}^{T} f^L_{gt}\Big)LEA_g - \sum_{g=1}^{G}\Big(\sum_{t=1}^{T} f^O_{gt}\Big)OWN_g - \sum_{g=1}^{G}\sum_{r\in\mathcal{R}^{g^-}}\Big(u^L_{rg} + u^O_{rg}\Big)PYU \quad (5.26)$$

Stock calculating constraints:

$$\textbf{s.t. } x^O_{gt} = x^O_{g,t-1} + \sum_{s=1}^{S}\Big(ONY^O_{gts} + ONU^O_{gts}\Big)$$

$$+ \sum_{s=1}^{S}\sum_{r\in\mathcal{R}^{in}_{s,t}} u^O_{rg} - \sum_{s=1}^{S}\sum_{r\in\mathcal{R}^{out}_{s,t}} u^O_{rg} \qquad \forall g, t>0 \qquad\qquad (5.27)$$

$$x^L_{gt} = x^L_{g,t-1} + \sum_{s=1}^{S}\Big(ONY^L_{gts} + ONU^L_{gts}\Big)$$

$$+ \sum_{s=1}^{S} \sum_{r \in \mathcal{R}_{s,t}^{in}} u_{rg}^{L} - \sum_{s=1}^{S} \sum_{r \in \mathcal{R}_{s,t}^{out}} u_{rg}^{L}$$

$$+ w_{g,t-1}^{L} \qquad \forall g, 0 < t \leq LP_g \qquad (5.28)$$

$$x_{gt}^{L} = x_{g,t-1}^{L} + \sum_{s=1}^{S} \left(ONY_{gts}^{L} + ONU_{gts}^{L} \right)$$

$$+ \sum_{s=1}^{S} \sum_{r \in \mathcal{R}_{s,t}^{in}} u_{rg}^{L} - \sum_{s=1}^{S} \sum_{r \in \mathcal{R}_{s,t}^{out}} u_{rg}^{L}$$

$$+ w_{g,t-1}^{L} - w_{g,t-LP_g-1}^{L} \qquad \forall g, t > LP_g \qquad (5.29)$$

$$x_{g0}^{O} = \sum_{s=1}^{S} INX_{gs}^{O} + w_{g}^{O}, \qquad \forall g \qquad (5.30)$$

$$x_{g0}^{L} = 0 \qquad \forall g \qquad (5.31)$$

Capacity on origins / needs on destinations constraints:

$$\sum_{g=1}^{G} \left(u_{rg}^{L} + u_{rg}^{O} \right) \leq avgDEM_r \qquad \forall r \qquad (5.32)$$

$$u_{rg}^{L/O} \leq x_{g,dout_r,sout_r}^{L/O} \qquad \forall r, g \qquad (5.33)$$

Business-related constraints:

$$u_{rg}^{L} + u_{rg}^{O} \leq UPG_{gr_r,g} \times M \qquad \forall r, g \qquad (5.34)$$

$$\sum_{g=1}^{G} w_{g}^{O} COS_g \leq BUD \qquad (5.35)$$

Other constraints:

$$f_{gt}^{L/O} = x_{gt}^{L/O} + \sum_{r \in \mathcal{R}_t^{use}} u_{rg}^{L/O} \qquad \forall g, t \qquad (5.36)$$

$$w_{gt}^{L} \in \mathbb{Z}_0^{+} \qquad \forall g, t \in \{0, ..., T-1\}$$

$$w_{g}^{O} \in \mathbb{Z}_0^{+} \qquad \forall g$$

$$x_{gt}^{L/O} \in \mathbb{Z}_0^{+} \qquad \forall g, t$$

$$u_{rg}^{L/O} \in \mathbb{Z}_0^{+} \qquad \forall r, g$$

$$f_{gt}^{L/O} \in \mathbb{Z}_0^{+} \qquad \forall g, t \qquad (5.37)$$

5.C.2   Pricing and Deployment Plan Model

Considering the indices and parameters, decision variables and the Capacity-Pricing Model presented in Section 5.2.2, the following adaptation was developed. The adaptation consists on an extension of the model (with the same decision variables), with the addition of two constraint groups. The main difference resides on the overall acquisition plan (aggregated for all locations), which is an input that comes from the MIP model presented above. This input will be mentioned as Pw. The additional constraints are:

$$\sum_{s=1}^{S} w_{gts}^{L} = Pw_{gt}^{L} \qquad \forall g,t \tag{5.38}$$

$$\sum_{s=1}^{S} w_{gs}^{O} = Pw_{g}^{O} \qquad \forall g \tag{5.39}$$

Appendix 5.D   Complete Tables of Results

Table 5.11: Main characteristics of the generated instances

Instance	Base instance (Oliveira et al., 2014)	Scale factor	# rental types ($|\mathcal{R}|$)	# vehicle groups ($|\mathcal{G}|$)	Size indicator ($|\mathcal{R}| \times |\mathcal{G}|$)
1	8	1	428	1	428
2	8	100	428	1	428
3	18	1	486	1	486
4	18	100	486	1	486
5	3	1	517	1	517
6	3	100	517	1	517
7	5	1	562	2	1,124
8	5	100	562	2	1,124
9	12	1	572	2	1,144
10	12	100	572	2	1,144
11	20	1	831	3	2,493
12	20	100	831	3	2,493
13	11	1	865	3	2,595
14	11	100	865	3	2,595
15	19	1	922	3	2,766
16	19	100	922	3	2,766
17	13	1	924	3	2,772
18	13	100	924	3	2,772
19	1	1	564	5	2,820
20	1	100	564	5	2,820
21	4	1	948	3	2,844
22	4	100	948	3	2,844
23	7	1	724	4	2,896
24	7	100	724	4	2,896
25	6	1	742	4	2,968

Continued from previous page

Instance	Base instance (Oliveira et al., 2014)	Scale factor	# rental types ($\|\mathcal{R}\|$)	# vehicle groups ($\|\mathcal{G}\|$)	Size indicator ($\|\mathcal{R}\| \times \|\mathcal{G}\|$)
26	6	100	742	4	2,968
27	9	1	793	4	3,172
28	9	100	793	4	3,172
29	14	1	1,046	4	4,184
30	14	100	1,046	4	4,184
31	16	1	1,141	4	4,564
32	16	100	1,141	4	4,564
33	17	1	933	5	4,665
34	17	100	933	5	4,665
35	15	1	1,182	4	4,728
36	15	100	1,182	4	4,728
37	2	1	1,234	5	6,170
38	2	100	1,234	5	6,170
39	10	1	2,369	5	11,845
40	10	100	2,369	5	11,845

Table 5.12: Improvement on final solution MIP objective function value: heuristically generated initial prices added to generation zero versus fully random generation zero

Instance of scale factor = 1	Instance of scale factor = 100	Size indicator ($\|\mathcal{R}\| \times \|\mathcal{G}\|$)	Improvement for scale factor = 1	Improvement for scale factor = 100
1	2	428	30.2%	19.3%
3	4	486	25.5%	21.4%
5	6	517	25.7%	18.1%
7	8	1,124	35.8%	18.8%
9	10	1,144	33.1%	16.5%
11	12	2,493	45.6%	29.7%
13	14	2,595	48.6%	34.4%
15	16	2,766	45.9%	24.2%
17	18	2,772	49.3%	33.5%
19	20	2,820	50.2%	22.2%
21	22	2,844	48.4%	33.7%
23	24	2,896	48.7%	29.7%
25	26	2,968	42.9%	23.9%
27	28	3,172	48.6%	22.2%
29	30	4,184	51.9%	28.6%
31	32	4,564	53.3%	30.8%
33	34	4,665	52.9%	36.3%
35	36	4,728	55.6%	30.8%
37	38	6,170	44.5%	28.9%
39	40	11,845	48.7%	37.2%
		average	44.3%	27.0%

Table 5.13: Proposed solution method – results for fully random BRKGA (without heuristically generated initial prices)

Instance	Best fitness generation zero	Best fitness last generation	# generations	Final MIP OF value	Final MIP gap
1	42,804	56,153	182	56,147	0%
2	3,879,500	4,908,510	245	4,908,490	0%
3	51,435	68,815	301	68,814	0%
4	4,862,010	5,997,660	184	5,997,660	0%
5	65,714	87,933	293	87,933	0%
6	6,142,810	7,877,400	289	7,877,400	0%
7	34,446	45,570	221	45,559	0%
8	3,084,800	3,924,010	310	3,923,860	0%
9	36,584	48,961	236	48,951	0%
10	3,177,430	4,106,230	280	4,106,070	0%
11	56,201	64,770	63	64,750	0%
12	4,945,060	5,645,490	59	5,645,440	0%
13	59,889	70,108	49	70,096	0%
14	5,306,280	5,955,530	54	5,955,480	0%
15	61,698	71,712	53	71,699	0%
16	5,470,600	6,174,840	53	6,174,680	0%
17	65,041	75,846	53	75,821	0%
18	5,755,090	6,499,770	49	6,499,680	0%
19	32,557	38,720	45	38,692	0%
20	2,852,850	3,301,400	41	3,301,250	0%
21	63,118	73,281	49	73,271	0%
22	5,592,730	6,290,130	50	6,290,060	0%
23	38,718	44,815	41	44,798	0%
24	3,334,590	3,752,240	43	3,752,090	0%
25	46,298	54,649	47	54,625	0%
26	4,097,910	4,680,220	43	4,680,120	0%
27	41,062	48,569	61	48,555	0%
28	3,593,020	4,116,970	77	4,116,700	0%
29	64,546	72,491	34	72,462	0%
30	5,629,730	6,185,670	36	6,185,630	0%
31	77,776	85,802	28	85,780	0%
32	6,795,650	7,303,810	28	7,303,520	0%
33	51,107	56,805	31	56,783	0%
34	4,417,890	4,826,680	33	4,826,640	0%
35	78,408	86,524	30	86,503	0%
36	6,777,220	7,307,780	31	7,307,720	0%
37	75,955	84,580	32	84,558	0%
38	6,614,440	7,294,210	39	7,294,080	0%
39	161,692	170,322	16	170,274	0%
40	14,656,500	15,212,500	15	15,212,500	0%

Table 5.14: Proposed solution method – results for BRKGA with heuristically generated initial prices

Instance	Best fitness initial solutions	Best fitness generation zero	Best fitness last generation	# generations	Final MIP OF value	Final MIP gap
1	73,087	73,087	73,087	137	73,087	0%
2	5,856,040	5,856,040	4,908,510	150	5,856,790	0%
3	86,301	86,301	68,815	139	86,329	0%
4	7,279,250	7,279,250	5,997,660	110	7,279,330	0%
5	110,504	110,504	87,933	129	110,548	0%
6	9,291,210	9,291,210	7,877,400	118	9,301,070	0%
7	61,889	61,889	45,570	72	61,878	0%
8	4,647,930	4,647,930	3,924,010	113	4,662,610	0%
9	65,101	65,101	48,961	66	65,139	0%
10	4,669,800	4,669,800	4,106,230	94	4,782,210	0%
11	93,649	93,649	64,770	23	94,248	0%
12	7,305,510	7,305,510	5,645,490	31	7,321,720	0%
13	103,769	103,769	70,108	18	104,172	0%
14	7,969,430	7,969,430	5,955,530	25	8,003,950	0%
15	104,247	104,247	71,712	19	104,624	0%
16	7,658,140	7,658,140	6,174,840	26	7,667,490	0%
17	113,191	113,191	75,846	18	113,213	0%
18	8,664,240	8,664,240	6,499,770	26	8,674,780	0%
19	58,140	58,140	38,720	19	58,121	0%
20	4,024,910	4,024,910	3,301,400	18	4,033,130	0%
21	108,267	108,267	73,281	19	108,768	0%
22	8,388,140	8,388,140	6,290,130	32	8,406,700	0%
23	66,509	66,509	44,815	18	66,613	0%
24	4,826,660	4,826,660	3,752,240	18	4,865,230	0%
25	77,619	77,619	54,649	20	78,055	0%
26	5,797,090	5,797,090	4,680,220	19	5,796,600	0%
27	71,358	71,358	48,569	28	72,141	0%
28	5,006,880	5,006,880	4,116,970	34	5,030,720	0%
29	110,077	110,077	72,491	13	110,046	0%
30	7,949,970	7,949,970	6,185,670	21	7,954,810	0%
31	131,509	131,509	85,802	16	131,486	0%
32	9,550,530	9,550,530	7,303,810	21	9,552,340	0%
33	86,853	86,853	56,805	15	86,827	0%
34	6,577,990	6,577,990	4,826,680	20	6,579,660	0%
35	134,592	134,592	86,524	17	134,573	0%
36	9,548,500	9,548,500	7,307,780	22	9,557,750	0%
37	122,101	122,101	84,580	20	122,145	0%
38	9,387,120	9,387,120	7,294,210	26	9,399,680	0%
39	253,300	253,300	170,322	9	253,255	0%
40	20,864,700	20,864,700	15,212,500	12	20,864,600	0%

Table 5.15: Time to solve each component of the proposed method

| | Time [sec] | | | | | |
Instance	Group decomposition + Relaxation	Construction	Time period decomposition	BRKGA	Final MIP run	Total
1	1,512	5	605	3,612	1	5,735
2	3,037	4	511	3,624	0	7,177
3	1,643	4	468	3,601	1	5,717
4	3,047	4	422	3,626	0	7,100
5	3,040	5	510	3,612	1	7,167
6	3,034	5	469	3,607	0	7,114
7	3,057	5	516	3,626	9	7,213
8	3,109	5	473	3,624	2	7,212
9	3,057	5	518	3,652	3	7,234
10	3,069	0	565	3,641	3	7,279
11	3,217	5	573	3,750	69	7,615
12	3,215	5	579	3,609	5	7,412
13	3,242	6	481	3,723	43	7,496
14	3,291	5	625	3,699	3	7,623
15	3,270	5	578	3,643	38	7,534
16	3,265	5	535	3,756	4	7,566
17	3,265	5	480	3,646	21	7,417
18	3,271	5	531	3,651	4	7,463
19	3,279	5	525	3,725	41	7,575
20	3,269	0	632	3,635	7	7,544
21	3,283	5	530	3,726	46	7,589
22	3,267	5	582	3,648	4	7,506
23	3,289	5	578	3,792	109	7,772
24	3,317	5	684	3,744	5	7,754
25	3,296	5	531	3,648	51	7,530
26	3,300	5	585	3,705	5	7,600
27	3,342	0	534	3,679	46	7,601
28	3,338	5	489	3,655	14	7,501
29	3,613	5	584	3,615	92	7,910
30	3,584	5	596	3,643	6	7,834
31	3,729	5	593	3,811	85	8,223
32	3,703	5	596	3,738	6	8,048
33	3,775	4	546	3,681	185	8,191
34	3,759	5	549	3,738	4	8,054
35	3,784	5	491	3,671	117	8,069
36	3,773	5	597	3,611	25	8,011
37	4,364	5	599	3,751	195	8,913
38	4,455	5	510	3,729	29	8,728
39	10,319*	6	530	3,764	224	14,843
40	9,592*	6	585	3,756	58	13,997

* Run despite size limitation discussed on Section 5.3.2. Not included in average.

Table 5.16: Comparison of the best fitness obtained by each approach to generate initial prices (best value for each instance highlighted).

Instance	Group decomposition	Time decomposition	Relaxation	Construction
1	73,040	43,363	**73,087**	66,141
2	5,762,000	3,057,180	**5,856,040**	5,380,860
3	86,238	38,051	**86,301**	78,033
4	6,945,850	3,260,770	**7,279,250**	6,649,860
5	**110,504**	46,330	110,341	100,056
6	8,911,330	4,788,130	**9,291,210**	8,546,980
7	**61,889**	35,081	61,107	55,109
8	4,070,060	2,531,770	**4,647,930**	4,305,980
9	**65,101**	37,676	64,189	58,167
10	**4,669,800**	2,836,990	4,632,280	4,493,570
11	83,492	55,080	**93,649**	88,474
12	5,115,850	4,694,010	**7,305,510**	6,871,040
13	83,727	57,776	**103,769**	97,267
14	5,382,840	5,137,320	**7,969,430**	7,538,400
15	83,720	62,568	**104,247**	97,992
16	5,700,040	4,808,470	5,717,840	**7,658,140**
17	92,592	52,242	**113,191**	104,677
18	5,909,810	4,851,550	**8,664,240**	8,235,850
19	**58,140**	33,132	54,507	52,179
20	3,676,720	2,717,660	3,308,830	**4,024,910**
21	92,313	62,265	**108,267**	101,476
22	5,782,670	5,087,590	**8,388,140**	7,997,890
23	63,790	41,376	**66,509**	61,862
24	3,903,230	3,395,280	**4,826,660**	4,654,030
25	73,544	45,364	**77,619**	74,272
26	4,767,000	3,895,570	4,724,890	**5,797,090**
27	64,423	42,227	**71,358**	66,208
28	3,891,500	3,136,800	3,661,660	**5,006,880**
29	86,325	63,046	**110,077**	102,936
30	5,766,210	5,217,100	5,781,400	**7,949,970**
31	102,272	77,070	**131,509**	123,660
32	7,005,540	6,272,680	7,005,540	**9,550,530**
33	77,371	54,172	**86,853**	81,755
34	4,521,370	4,108,220	**6,577,990**	6,232,290
35	105,184	73,569	**134,592**	125,208
36	6,945,900	6,363,110	6,945,900	**9,548,500**
37	104,925	73,008	117,303	**122,101**
38	6,748,610	5,572,490	6,748,610	**9,387,120**
39	224,797	130,562	159,451	**253,300**
40	17,314,300	11,503,500	15,803,700	**20,864,700**

Table 5.17: Improvement of proposed integrating method versus baseline sequential approach

Instance of scale factor = 1	Instance of scale factor = 100	Size indicator ($|\mathcal{R}| \times |\mathcal{G}|$)	Improvement for scale factor = 1	Improvement for scale factor = 100
1	2	428	6.4%	4.4%
3	4	486	4.2%	3.0%
5	6	517	6.9%	4.4%
7	8	1,124	5.3%	1.0%
9	10	1,144	6.0%	3.1%
11	12	2,493	145.9%	150.0%
13	14	2,595	9.6%	88.2%
15	16	2,766	67.5%	174.6%
17	18	2,772	4.7%	32.9%
19	20	2,820	57.1%	106.3%
21	22	2,844	126.1%	161.2%
23	24	2,896	79.1%	125.8%
25	26	2,968	112.8%	205.1%
27	28	3,172	85.2%	98.6%
29	30	4,184	242.4%	179.4%
31	32	4,564	128.2%	427.8%
33	34	4,665	100.3%	332.7%
35	36	4,728	900.6%	572.9%
37	38	6,170	149.7%	363.3%
39	40	11,845	-	-
		average	117.8%	158.7%

Table 5.18: Comparison of the best values obtained by the non-linear solver (INLP) and BRKGA with heuristically generated initial prices.

Instance	Size indicator ($|\mathcal{R}| \times |\mathcal{G}|$)	BRKGA best value	INLP best value	Improvement BRKGA vs. INLP
1	428	73,082	73,059	0%
2	428	5,854,440	5,852,920	0%
3	486	86,349	86,306	0%
4	486	7,276,740	7,275,560	0%
5	517	110,562	110,482	0%
6	517	9,299,180	9,309,570	0%
7	1124	61,709	61,790	0%
8	1124	4,661,830	4,652,980	0%
9	1144	64,730	65,042	0%
10	1144	4,814,190	4,855,770	-1%

Continued from previous page

Instance	Size indicator ($|\mathcal{R}| \times |\mathcal{G}|$)	BRKGA best value	INLP best value	Improvement BRKGA vs. INLP
11	2493	96,716	83,401	16%
12	2493	7,346,430	0	
13	2595	105,274	106,594	-1%
14	2595	7,954,920	5,742,290	39%
15	2766	105,954	104,982	1%
16	2766	8,108,450	5,192,100	56%
17	2772	113,830	115,564	-2%
18	2772	8,663,270	7,672,260	13%
19	2820	56,125	44,669	26%
20	2820	4,042,860	2,921,580	38%
21	2844	108,628	110,285	-2%
22	2844	8,378,750	6,100,250	37%
23	2896	66,729	39,086	71%
24	2896	4,675,110	0	
25	2968	78,988	44,493	78%
26	2968	5,827,720	0	
27	3172	71,909	66,283	8%
28	3172	5,400,970	3,901,760	38%
29	4184	102,913	75,652	36%
30	4184	7,949,700	0	
31	4564	130,884	71,048	84%
32	4564	9,555,990	667,525	1332%
33	4665	82,036	58,443	40%
34	4665	6,234,340	0	
35	4728	134,346	64,661	108%
36	4728	9,549,240	1,688,550	466%
37	6170	122,200	28,783	325%
38	6170	9,400,800	689,945	1263%
39	11845	253,245		
40	11845	20,864,600		

Capacity-pricing integration under uncertainty: Matheuristic approach

The goal of this paper is to tackle the capacity-pricing integration problem under uncertainty. Building on the previous paper (Chapter 5), a stochastic mathematical programming model is proposed, as well as a matheuristic approach. This paper represents the maturest work in this thesis, resulting from a research evolution. Therefore, some options regarding problem modeling are adjusted and the scope is more sharply defined, when comparing with the previous papers. The main adjustment in scope is related with the strategic level of price and capacity decisions, translated on the aggregation level of prices and rental types. More specifically, different antecedence levels for requests are not considered in this paper. The stochastic view of the problem makes it easier to understand this final "narrowing/focusing process" on scope, resulting in clear contributions, both problem-related and methodological.

A co-evolutionary matheuristic for the car rental capacity-pricing stochastic problem

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Abstract   When planning a selling season, a car rental company must decide on the number and type of vehicles in the fleet to meet demand. The demand for the rental products is uncertain and highly price-sensitive and thus capacity and pricing decisions are interconnected. Moreover, since the products are rentals, capacity "returns". This creates a link between capacity with fleet deployment and other tools that allow the company to meet demand, such as upgrades, transferring vehicles between locations or temporarily leasing additional vehicles.

We propose a methodology that aims to support decision-makers with different risk profiles plan a season, providing good solutions and outlining their ability to deal with uncertainty, when little information about it is available. This matheuristic is based on a co-evolutionary genetic algorithm, where parallel populations of solutions and scenarios

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co-evolve. The fitness of a solution depends on the risk profile of the decision-maker and its performance against the scenarios, which is obtained by solving a mathematical programming model. The fitness of a scenario is based on its contribution in making the scenario population representative and diverse. This is measured by the impact the scenarios have on the solutions.

Computational experiments show the potential of this methodology in terms of the quality of the solutions obtained and the diversity and representativeness of the set of scenarios generated. Its main advantages are that no information regarding probability distributions is required, it supports different decision-making risk profiles and it provides a set of good solutions for an innovative complex application.

Keywords Revenue management · Pricing · Car rental fleet management · Genetic algorithms · Stochastic programming

6.1. Introduction

When planning a selling season, a car rental company must decide on the fleet size and mix, i.e. the capacity it will have to meet demand throughout the season and rental locations. The demand is uncertain and highly price-sensitive. Therefore, the prices charged by a company are connected with and should influence the capacity decisions. Capacity decisions are also connected with other instruments that allow the company to "meet" its demand, which range from offering upgrades to transferring vehicles between locations or temporarily leasing additional vehicles.

The goal of this work is to provide decision-makers with profitable solutions to capacity and pricing decisions, assessing and increasing their ability to deal with the different realizations of uncertainty, represented by scenarios, when little information regarding those is available. The methodology developed is based on a co-evolutionary genetic algorithm, where parallel populations of solutions and scenarios co-evolve, depending on each other for the fitness evaluation of their individuals. On the one hand, this method aims at obtaining a representative and diverse population of scenarios, measured according to the impact they have on the population of solutions. On the other hand, the solutions evolve according to different decision-making risk profiles that assess its performance against the population of scenarios.

6.1.1 Previous works

This work deals with the integration of capacity and pricing decisions under uncertainty within the context of the car rental business. In this section, the relevance of the application and methodological scope of the work will be discussed. Firstly, the recently growing body of research on car rental fleet management and pricing will be briefly reviewed. This is an innovative and different application due to the fact that the capacity is *rented* rather than sold. However, previous works that tackled the integration of pricing and capacity, although not directly applicable, can bring relevant insights to this problem. A stochastic

approach to the problem is considered, where the uncertainty is represented by scenarios. Stochastic problems with similar characteristics are briefly reviewed in terms of methodological approaches. Moreover, fundamental previous works that laid the foundation for the methodological idea developed in this paper will be presented.

6.1.1.1 Car rental fleet management and pricing

The car rental fleet management problem is initially structured in Pachon et al. (2003, 2006). Fink and Reiners (2006) extends the operational issues within fleet management and deployment, considering essential and realistic practical needs. In Oliveira et al. (2017c), the link with revenue management issues is introduced and the body of research developed in this field is reviewed and structured. Existing gaps and relevant future research directions are discussed, including the integration of pricing and/or capacity allocation (revenue management issues) with operational decisions related with fleet size/mix and deployment. The need to consider uncertainty in demand in order to approximate the model to reality is also highlighted.

In a previous paper – Oliveira et al. (2017d) – we tackled the first research direction. A mathematical model for the deterministic integration of dynamic pricing and capacity decisions was proposed. Due to the complexity of the problem, a matheuristic was proposed. This matheuristic is based on a decomposition of the problem, where the price decisions are directly encoded in the chromosomes and the remaining decisions and the fitness of the full solution are obtained by solving a mathematical programming model. Moreover, some performance-boosting initial population generation procedures were proposed.

In this work, we propose to tackle the even more complex problem that arises when uncertainty is incorporated. Moreover, additional realistic requirements (such as price hierarchy) are included and demand is modeled considering its relationship with competitor prices.

6.1.1.2 Integration of capacity, inventory and pricing decisions

Pricing decisions have often been tackled independently of capacity and inventory decisions. A recent and growing body of research on the integration of these topics has been arising.

Den Boer (2015) presents an interesting and thorough literature review on the topic of dynamic pricing, especially focused on learning processes. Following the structure proposed by the author, the car rental pricing problem herein considered can be seen as a dynamic pricing problem with inventory effects, more specifically "jointly determining selling prices and inventory–procurement". In Gallego and van Ryzin (1994), the dynamic pricing problem for inventories with price-sensitive and stochastic demand is tackled, including an extension where initial stock is considered as a decision variable. The *rental* facet of the problem at hand hinders the direct application of the insights drawn. Focusing on perishable assets, a dynamic pricing problem under competition is studied in Gallego and Hu (2014). Here, the dynamics of a oligopoly are considered, dealing with substitutability among assets. These characteristics are more similar to the car rental market,

where vehicles that are available at a certain day (or the corresponding available days-of-use) "expire" since they cannot be used in a future time period. Relevant results are obtained regarding dynamic pricing strategies. As this, other important works have dealt with similar environments with insightful outcomes. Adida and Perakis (2010) present an interesting work, where different joint dynamic pricing and inventory control models that deal with demand uncertainty (which depends linearly on price) are considered, within a make-to-stock manufacturing context. This work compares stochastic and robust optimization approaches, introduces different formulations and compares their computational performance.

Nevertheless, the car rental business is characterized by the return of its "sold inventory" in a pre-determined future time period and location. This causes significant changes to the problem structure and renders the problem even more complex to solve. In Oliveira et al. (2017a), a dynamic programming approach is developed for a deterministic and simpler version of this problem and this question is further discussed.

Additionally, the relationship between demand and price in this context is difficult to determine due to the effect of competition and to the myriad of products offered (rental types) that share the same resources (vehicle fleet). Therefore, new approaches are needed to tackle this problem.

6.1.1.3 Representing uncertainty by scenarios

Scenarios can be important tools for companies dealing with relevant uncertainties. Moreover, the process of scenario generation is critical for the practical relevance of the results obtained.

Scenario generation consists on defining discrete outcomes (realizations) for all random variables and time periods (Høyland and Wallace, 2001), especially useful for stochastic problems. Mitra and Di Domenica (2010) review the scenario generation methods applied in the literature for stochastic programming models, including sampling-based generation (e.g. Monte Carlo, boostrap or conditional sampling methods), statistical methods (e.g. property matching or regressions) and simulation-based generation (e.g. Vector Auto Regressive methods), as well as other less used methods (e.g. hybrid methods). The authors discuss relevant, desirable characteristics that all scenario generation methods should incorporate: including a variety of factors and existing correlations, considering the purpose of the model (to understand e.g. if it is more relevant to capture variance or higher moments), being consistent with any theory and with empirical data observations. Kaut and Wallace (2003) evaluate different scenario generation methods and propose two properties (and corresponding methodologies to test them) that a method should satisfy to be applicable and relevant to a given problem. Most of these techniques involve a considerable amount of knowledge about the uncertainty and random variables, e.g. their probability distribution.

6.1.1.4   Methodological approaches

The car rental capacity-pricing problem can be represented by a mathematical model with a non-linear objective function, which quantifies the profit obtained by the company. This is due to the fact that both the number of rentals that the company is able to fulfill and the price it charges for them are decision variables. This renders the problem complex, especially in the stochastic case.

Solving non-linear integer stochastic mathematical programming models is becoming a promising approach to obtain good and accurate solutions for complex real-world situations, such as hazard management of post-fire debris flows or transportation network protection against extreme events such as earth-quakes (Krasko and Rebennack, 2017; Lu et al., 2018). Often, solution approaches are required to deal with the inherent complexity, such as decomposition or (meta)heuristics, even when non-linearity is not an issue to consider (Özcan, 2010; Yan et al., 2008; Puga and Tancrez, 2017).

Genetic algorithms have been proposed to tackle complex stochastic problems (Gu et al., 2010; Wang et al., 2011). In these works, random variables are often associated with probability distributions, thus scenarios are generated by random sampling or simulation. Furthermore, the hybridization of genetic algorithms and linear programming has been successfully used to develop alternative stochastic methodologies (Reis et al., 2005).

In this field, scenario generation is heavily dependent on the knowledge of probability distributions for the random variables and consists on selecting a small set of scenarios that represent it well, which is highly complicated in the multivariate case (Löhndorf, 2016). The author presents an empirical analysis of popular scenario generation methods for stochastic optimization. State-of-the-art methods are compared in terms of solution quality, using a problem where analytical solutions are available. Their adequacy is dependent on the problem characteristics and probability distributions. Guastaroba et al. (2009) focus on optimal portfolio selection problem and compare scenario generation techniques for this problem. One of the conclusions is that the adequacy of the method depends on the risk profile of the decision-maker.

6.1.1.5   Core methodological previous works

For this problem, using scenarios to represent uncertainty has a practical interest in terms of the application of the method, since scenarios can be useful to help decision-makers understand and act upon the outputs. Nevertheless, the only information regarding the uncertain parameters available for this problem is the bounds on the values they can take. Therefore, a methodology that tackles this lack of information is needed.

"Robust Discrete Optimization" is a mathematical programming framework for making robust (i.e., worst-case based) decisions for integer problems (Kouvelis and Yu, 1997) and, unlike the more known Robust Optimization approach, makes use of scenarios without associating them with probabilities. The main disadvantage of the Robust Discrete Optimization approach proposed by Kouvelis and Yu (1997) is that models often become intractable, especially when the number of scenarios is large. Following the definition of Robust Discrete Optimization problem, Herrmann (1999) proposes a metaheuristic based

on genetic algorithms for providing worst-case scenario solutions, especially adequate for problems where the set of scenarios is too large for each element to be evaluated individually, or even known. In this work, the author proposes the co-evolution of solutions and scenarios in two parallel spaces, as follows.

Considering that SO is the set of all solutions and SC the set of all possible scenarios, the value obtained by a solution $i \in SO$ when scenario $j \in SC$ occurs is given by $F(i, j)$. The goal is to find the solution that performs best for the worst-case, which is translated (in a minimization problem) to:

$$\min_{i \in SO} \max_{j \in SC} F(i, j) \tag{6.1}$$

The author thus proposes a two-space genetic algorithm where scenarios (SC) and solutions (SO) co-evolve in different populations (P_{SO} and P_{SC}) composed of individuals whose fitness depends not only on its characteristics but also on the characteristics of the other population. This genetic algorithm favors solutions with better worst-case performances and scenarios with worse "best solutions". The fitness of a solution i_0 is evaluated as $\max_{j \in P_{SC}} F(i_0, j)$ (worst scenario for this solution), while the fitness of a scenario j_0 is evaluated as $\min_{i \in P_{SO}} F(i, j_0)$ (best solution for this scenario). The groundbreaking idea in this work is that using efficient genetic algorithms to evolve populations of scenarios requires only an initial sample that will evolve and is thus expected to adequately represent the full set, which would otherwise take significantly more effort to explore. Simultaneously, the solutions evolve to perform better, considering the worst-case scenario.

This work is continued by Jensen (2001) that proposes a ranking-based evaluation for scenario fitness that performs better and fixes symmetry and bias issues of the original approach. In Cramer et al. (2009), an entirely random scenario population is used, eliminating the "co-evolutionary" characteristic of the method. Despite reducing complexity, the performance is not better than the approach proposed by Jensen (2001) and requires defining an adequate or sufficient number of random scenarios.

We aim to extend the idea of a two-space genetic algorithm to evolve solutions and scenarios to other decision-making risk profiles. Considering the *expected value* as the goal to evaluate solutions (stochastic approach) rather than the *worst-case value* significantly impacts the evolution of the scenario population. This focuses the evolutionary drive in obtaining a representative population, rather than converging to the worst-case scenario. To achieve this, recent developments on the field of instance generation were considered. In Gao et al. (2016), an evolutionary algorithm is proposed for generating instances that are diverse with respect to different features of the problem. It aims to "diversify" points in N-dimensions by ranking candidates based on distance to nearest neighbors in each axis. Using this technique with elitism leads to new children being added to the population only if they extend the extreme values or lie in a large gap between existing points. Also in Deb et al. (2002), the concept of crowding distance is used to estimate the density of solutions surrounding a particular point in a population. It compares to the largest cuboid enclosing the point without enclosing any other points, with a similar reference to nearest neighbors in each axis.

6.1.2 Contributions

The main contributions of this paper are related with the mathematical model and the solution methodology proposed.

- We propose a new two-stage stochastic model (extending the deterministic model proposed in Oliveira et al. (2017d)):

  - Its main innovative feature is that the *stochastic* capacity-pricing problem for car rentals is modeled. Few papers focus on the integration of pricing with capacity decisions, using tactical information and uncertainty to deal with strategic decisions, especially in the complex rental context, where inventory is not depleted but only temporarily unavailable.
  - The issue of vehicle group price hierarchy is included, on a more realistic approach to the problem.
  - Demand uncertainty and price-sensitivity are modeled in an innovative and efficient way, with a significant fit with the problem at hand and its strategic scope. Nevertheless, the overall model can still be adapted to consider other alternative demand models in the future.

- We propose an innovative solution method to tackle the problem, based on the decomposition of the stochastic model in first-stage and second-stage decisions:

  - Solutions to the first-stage decisions and scenarios are generated in parallel with mutual impact on fitness evaluation, requiring little information on random variables to do so.
  - The fitness depends on the profit obtained by each pair (solution, scenario), which is calculated using a mathematical programming model.
  - The methodology is easily adaptable to different decision-making risk profiles.
  - Specific problem know-how can be used in the initial populations to boost the evolutionary procedure (e.g. providing extreme scenarios).
  - It can be implemented and run in reasonable time in a decision-support system.

Overall, this methodology has a relevant fit with the problem at hand, making it useful in real-world applications. Moreover, it is a methodology that can be easily extended to other problems where information regarding uncertainty is scarce.

6.1.3 Paper structure

This paper is structured as follow. Firstly, the problem will be stated and the mathematical model presented (Section 6.2). Then, Section 6.3 presents the co-evolutionary matheuristic developed and in Section 6.4 the results of the computational tests are discussed. Finally, conclusions are drawn and future work and promising research directions are discussed (Section 6.5).

6.2.   Problem Definition

Car rental companies preparing a season must decide on the size and mix of their fleet, i.e. the capacity they will have to face demand in that season. In order for this capacity to be used efficiently, some operational issues that will take place during the season must be considered, as well as the uncertain demand.

A car rental company has several rental stations that share the same fleet. Within the scope of this problem, these stations are often aggregated in regions or locations such that transferring a vehicle between stations within the same region is negligible in terms of time or cost, unlike transfers from one region to another. Also the unit of time considered can be seen as an aggregated measure within this scope, e.g. one week.

The fleet is composed by distinct vehicles, aggregated in vehicle groups that differ in several aspects, namely customer valuation. Nevertheless, the "products" that car rental companies "trade" are *rental types*. Each rental type is characterized not only by the vehicle group requested by the customer but also by start and end time periods and start and end locations (which may be different). Different rental types (products) "compete" for the same fleet (capacity). Moreover, if not conflicting in time, two rental types can use the same vehicle. The demand for each rental type is independent, uncertain and highly price-sensitive. Since it is increasingly easier for costumers to compare the prices of all companies offering a certain rental type, for those companies where brand loyalty is not a dominant effect, demand is usually only attracted by having the lowest price in the market. Due to consumer value perception, the company must also consider constraints on the hierarchy of prices for rental types that are similar in all characteristics except for the vehicle group requested. That is to say, price hierarchy for rentals that start and end at the same time and place must respect the hierarchy of vehicle value, i.e., all other parameters being equal, depending on the vehicle groups considered, a rental price for a more-valued vehicle cannot be less than the price of less-valued one.

Before the season starts, the company must decide on how many vehicles of each group to purchase to meet the (uncertain) demand and where to make them available at the start of the season. Since demand is heavily influenced by the pricing strategy of the company, it must also decide previously the price it will charge for each rental type. After the season starts and demand is revealed, the company has other tools to meet demand that must be considered since they impact the capacity decisions. On the one hand, since two rentals can use the same vehicle as long as they do not overlap in time, it is critical to decide on fleet deployment throughout the season and network of locations. This deployment is achieved either by actual rentals that start and end in different locations (whose number, limited by demand and capacity, is decided by the company) or by empty transferring vehicles by truck or driver. Pricing is a relevant tool to influence demand and, consequently, fleet deployment and utilization. Also, the company has the possibility to upgrade rentals: offering a more-valued vehicle than requested for the same price. Upgrades are a common practice in this business. Nevertheless, they are used sporadically as a "last resource" to avoid the situation where customers request a less-valued vehicle because they are expecting an upgrade. Finally, to meet temporary peeks in demand, the company may lease more vehicles for a significantly higher cost.

This problem is here modeled as a two-stage stochastic model, where the uncertain parameters are related with demand and competitors' prices. The decisions made before the season starts define the first-stage and, after uncertainty is revealed, the recourse actions or second-stage decisions include the deployment decisions, upgrading and leasing. The goal of this work is to provide decision-makers with profitable solutions to the first stage decisions, describing their ability to deal with the different realizations of uncertainty, represented by scenarios. Due to this more strategic setting, aggregated levels of demand and prices are considered, and more operational "online" pricing decisions (such as updating prices throughout the season) are excluded from the scope.

6.2.1 Problem modeling

In this section, the uncertain integrated pricing and capacity problem in car rental is fully defined using a mathematical programming model. This model is extended from the deterministic model presented in (Oliveira et al., 2017d). However, this model differs not only because it considers some parameters to be uncertain but also because it models more accurately the relationships between demand, price decided and minimum price in the market. Moreover, it considers that the price charged for a rental requiring a vehicle of a certain group may be limited by the price charged for a rental that only differs on the vehicle group requested (price hierarchy). The notation used is presented in Table 6.1.

Table 6.1: Notation

Indices, parameters and other notation	
$\theta = \{1, ..., \Theta\}$	Index for the set of scenarios
$t, t' = \{0, ..., T\}$	Indices for the set $\mathcal{T}$ of time periods[1]
$g, g1, g2 = \{1, ..., G\}$	Indices for the set $\mathcal{G}$ of vehicle groups
$s, s1, s2, c = \{1, ..., S\}$	Indices for the set $\mathcal{S}$ of rental stations
$r, r' = \{1, ..., R\}$	Indices for the set $\mathcal{R}$ of rental types (characterized by check-out station and time period, check-in station and time period, and group requested)
so_r	Check-out station of rental type r
do_r	Check-out time period of rental type r
si_r	Check-in station of rental type r
di_r	Check-in time period of rental type r
gr_r	Vehicle group requested by rental type r
$p_r = \{1, ..., P_r\}$	Index for the set $\mathcal{P}_r$ of price levels allowed for rental type r
LBP_r	Lower bound on prices for rentals of type r
UPB_r	Upper bound on prices for rentals of type r

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[1] $t = 0$ represents the initial conditions of the time period and "overlaps" with $t = T$ for the previous period

	Continued from previous page
PRI_{rp}	Pecuniary value charged for rental type r related with price level p. The value for $p = 1$ corresponds to the lower bound on price (LBP_r) and for $p = P_r$ corresponds to the upper bound on price (UBP_r). The intermediate levels are a discretization of this range.
$COM_{r\theta}$	Minimum price charged by the competitors for rental type r in scenario θ
$DEM_{r\theta}^A$	Demand for rental type r in scenario θ, when it is **a**bove the minimum price in the market for a similar product $COM_{r\theta}$
$DEM_{r\theta}^B$	Demand for rental type r in scenario θ, when it is **b**elow the minimum price in the market for a similar product $COM_{r\theta}$, with $DEM_{r\theta}^A \leq DEM_{r\theta}^B$
MGP	Marginal price difference
PLM_{g1g2}	Whether the price charged for a vehicle of group $g1$ should be lesser than or equal to the price charged for a vehicle of group $g2$, considering the same check-out and check-in locations and time periods, (= 1) or not (= 0)
COS_g	Buying cost of a vehicle of group g. The value considered is the net cost: purchase gross cost minus salvage value derived from its sale after one year
OWN_g	Ownership cost per time unit of a vehicle of group g
LEA_g	Leasing cost (per time unit) of a vehicle of group g
LP_g	Leasing period for a a vehicle of group g
PYL_g	Penalty charged for each day that a leasing return of group g is late
PYU	Penalty charged for each upgrade
UPG_{g1g2}	Whether a vehicle of group $g1$ can be upgraded to a vehicle of group $g2$ (= 1) or not (= 0)
TT_{s1s2}	Transfer time from station $s1$ to station $s2$
TC_{gs1s2}	Transfer cost of a vehicle of group g from station $s1$ to station $s2$
BUD	Total budget for the purchase of vehicles
M	Big-M large enough coefficient
E_θ	Mathematical expectation with respect to scenario θ

Inputs from previous periods

Assumption: For all periods, $\mathcal{T}, \mathcal{G}, \mathcal{S}, \mathcal{R}$ are the same, as well as rental types $r \in \mathcal{R}$.

INX_{gs}^O	Initial number of owned (O) vehicles of group g located at station s, at the beginning of the time period ($t = 0$)

Continued from previous page	
$ONY^{L/O}_{gts}$	Number of owned (O) or leased (L) vehicles of group g on on-going transportation (previously decided), being transferred to station s, arriving at time t
$ONU^{L/O}_{gts}$	Number of owned (O) or leased (L) vehicles of group g on on-going rentals (previously decided), being returned to station s at time t

Other sets

$\mathcal{R}^{g^-}$	Rental types that do not require group g
$\mathcal{R}^{in}_{st}$	Rental types whose check-in is at station s at time t
$\mathcal{R}^{out}_{st}$	Rental types whose check-out is at station s at time t
$\mathcal{R}^{use}_{t}$	Rental types that require a vehicle to be in use at time t

6.2.1.1 Demand modeling

The relationship between demand, the price decision and the competitors' prices for each rental type is based on the following almost-"winner-takes-all" assumptions, suitable for car rental companies that are not "market leaders" or owners of widely recognized brands:

1) If a company has the lowest price in the market, it will have a certain level of demand that corresponds to most of the pool of the customers (there will be always customers that are willing to pay a premium for the brand, and will not choose the lowest price in the market). If the price is set below the threshold point where the company starts to be the lowest in the market, the demand level will not increase from this level.

2) If a company does not provide the lowest price in the market, it will only attract a marginal share of the market. If the price is set above the threshold point where the company ceases to be the lowest in the market, the demand level will not decrease from this level.

3) The minimum competitor price in the market is an uncertain parameter, within a limited range.

4) The highest demand level (associated with the lowest price in the market) is an uncertain parameter, within a limited range.

5) The lowest demand level (associated with a price that is not the lowest in the market) is an uncertain parameter, within a limited range associated with the highest demand level.

These assumptions, graphically represented in Figure 6.1, aim to capture and adapt to the scope of this work the price-sensitive and uncertain nature of car rental demand. Ainscough et al. (2009) present an interesting, although limited, survey on car rental consumers
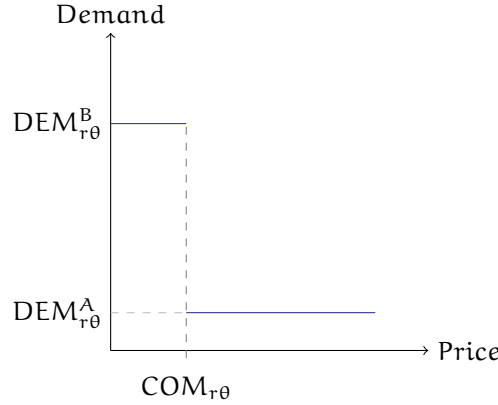
Figure 6.1: Relationship between the price decision for a specific rental type r and the minimum competitor price and demand values, which are parameters dependent on scenario θ

where the effects of rental agency brand and price are studied. It concluded that the rental agency brand has a positive impact on willingness to rent, which makes these assumptions more suitable for companies that do not own a widely recognized brand. Nevertheless, this approach can scale this effect of "brand loyalty", even within these companies, by adequate parametrization. Furthermore, in the study, the conclusions support the hypothesis that higher prices lead to lower willingness to rent, with no support that they lead to a higher perception of service quality. These conclusions sustain the relatively simple assumptions made within the strategic scope of this model.

One could argue that these assumptions do not lead to a price decision but only to a "sell/ no-sell" decision, i.e. being above or below the threshold price. This would be completely valid if different products, or rental types, did not share the same resources and if price hierarchy and substitution issues between groups were not considered.

6.2.1.2 Mathematical model

Decision variables:

w_{gs}^O Number of vehicles of group g acquired for the owned fleet available at time $t = 0$ in station s

q_{rp} $= 1$ if the price charged for rental type r is associated with price level p; $= 0$ otherwise

$w_{gts\theta}^L$ Number of vehicles of group g acquired by leasing at time t to be available at station s in scenario θ

$y_{s1s2gt\theta}^{L/O}$ Number of leased (L) or owned (O) vehicles of group g transferred from station $s1$ to station $s2$ in scenario θ; the transfer begins at t

$u_{rg\theta}^{L/O}$ Number of rentals of type r that are served by a leased (L) or owned (O) vehicle of group g in scenario θ

$x_{gts\theta}^{L/O}$ Number of leased (L) or owned (O) vehicles of group g located at station s at the start of time period t in scenario θ

$f^L_{gt\theta}$ Auxiliary variable: total leased fleet of group g at time t in scenario θ

$z_{r\theta}$ Auxiliary variable: = 1 if the price charged for rental type r is above the minimum value in the market in scenario θ; = 0 otherwise

Optimization model:

$$\max \quad -\sum_{g=1}^{G}\left(\sum_{s=1}^{S}w^O_{gs}\right)\left(COS_g + T \times OWN_g\right)$$

$$+ E_\theta\left[\sum_{r=1}^{R}\left[\left(\sum_{g=1}^{G}u^L_{rg\theta} + u^O_{rg\theta}\right)\sum_{p=1}^{P}q_{rp}PRI_{rp}\right] - \sum_{g=1}^{G}\left(\sum_{t=1}^{T}f^L_{gt\theta}\right)LEA_g\right.$$

$$\left. -\sum_{s1=1}^{S}\sum_{s2=1}^{S}\sum_{g=1}^{G}\left(\sum_{t=1}^{T}\left(y^L_{s1s2gt\theta} + y^O_{s1s2gt\theta}\right)\right)TC_{gs1s2} - \sum_{g=1}^{G}\sum_{r\in\mathcal{R}^{g^-}}\left(u^L_{rg\theta} + u^O_{rg\theta}\right)PYU\right] \quad (6.2)$$

$$\textbf{s.t.} \quad \sum_{s=1}^{S}\sum_{g=1}^{G}w^O_{gs}COS_g \leq BUD \tag{6.3}$$

$$\sum_{p=1}^{P}q_{rp} = 1 \qquad \forall r \tag{6.4}$$

$$\sum_{p=1}^{P}q_{rp}PRI_{rp} \leq \sum_{p=1}^{P}q_{r'p}PRI_{r'p} \qquad \forall r,r' : \Big\{ so_r = so_{r'} \wedge si_r = si_{r'}$$

$$\wedge\, do_r = do_{r'} \wedge di_r = di_{r'}$$

$$\wedge\, PLM_{gr_r,gr_{r'}} = 1 \Big\} \tag{6.5}$$

$$\sum_{g=1}^{G}\left(u^L_{rg\theta} + u^O_{rg\theta}\right) \leq DEM^A_{r\theta}$$

$$+\left(DEM^B_{r\theta} - DEM^A_{r\theta}\right)(1 - z_{r\theta}) \qquad \forall r,\theta \tag{6.6}$$

$$COM_{r\theta} \geq \sum_{p=1}^{P}q_{rp}PRI_{rp} - Mz_{r\theta}, \qquad \forall r,\theta \tag{6.7}$$

$$\sum_{r\in\mathcal{R}^{out}_{st}}u^{L/O}_{rg\theta} + \sum_{c=1}^{S}y^{L/O}_{scgt\theta} \leq x^{L/O}_{gts\theta} \qquad \forall g,t,s,\theta \tag{6.8}$$

$$u^L_{rg\theta} + u^O_{rg\theta} \leq UPG_{gr_r,g} \times M \qquad \forall r,g,\theta \tag{6.9}$$

$$x^O_{g0s\theta} = INX^O_{gs} + w^O_{gs} \qquad \forall g,s,\theta \tag{6.10}$$

$$x^L_{g0s\theta} = 0 \qquad \forall g,s,\theta \tag{6.11}$$

$$x^O_{gts\theta} = x^O_{g,t-1,s,\theta} + ONY^O_{gts} + ONU^O_{gts}$$
$$+ \sum_{r \in \mathcal{R}^{in}_{s,t-1}} u^O_{rg\theta} - \sum_{r \in \mathcal{R}^{out}_{s,t-1}} u^O_{rg\theta}$$
$$+ \sum_{c=1}^{S} y^O_{c,s,g,t-TT_{cs}-1,\theta} - \sum_{c=1}^{S} y^O_{s,c,g,t-1,\theta} \qquad \forall g, t > 0, s, \theta \qquad (6.12)$$

$$x^L_{gts\theta} = x^L_{g,t-1,s,\theta} + ONY^L_{gts} + ONU^L_{gts}$$
$$+ \sum_{r \in \mathcal{R}^{in}_{s,t-1}} u^L_{rg\theta} - \sum_{r \in \mathcal{R}^{out}_{s,t-1}} u^L_{rg\theta}$$
$$+ \sum_{c=1}^{S} y^L_{c,s,g,t-TT_{cs}-1,\theta} - \sum_{c=1}^{S} y^L_{s,c,g,t-1,\theta}$$
$$+ w^L_{gts\theta} \qquad \forall g, 0 < t < LP_g, s, \theta \qquad (6.13)$$

$$x^L_{gts\theta} = x^L_{g,t-1,s,\theta} + ONY^L_{gts} + ONU^L_{gts}$$
$$+ \sum_{r \in \mathcal{R}^{in}_{s,t-1}} u^L_{rg\theta} - \sum_{r \in \mathcal{R}^{out}_{s,t-1}} u^L_{rg\theta}$$
$$+ \sum_{c=1}^{S} y^L_{c,s,g,t-TT_{cs}-1,\theta} - \sum_{c=1}^{S} y^L_{s,c,g,t-1,\theta}$$
$$+ w^L_{gts\theta} - w^L_{g,t-LP_g,s,\theta} \qquad \forall g, t \geq LP_g, s, \theta \qquad (6.14)$$

$$f^L_{gt\theta} = \sum_{s=1}^{S} x^L_{gts\theta} + \sum_{r \in \mathcal{R}^{use}_t} u^L_{rg\theta}$$
$$+ \sum_{s1=1}^{S} \sum_{s2=1}^{S} \sum_{t'=\max\{0,t-TT_{s1s2}\}}^{t-1} y^L_{s1,s2,g,t',\theta} \qquad \forall g, t, \theta \qquad (6.15)$$

$$w^O_{gs} \in \mathbb{Z}^+_0 \qquad \forall g, s$$
$$q_{rp} \in \{0,1\} \qquad \forall r$$
$$w^L_{gts\theta} \in \mathbb{Z}^+_0 \qquad \forall g, t, s, \theta$$
$$y^{L/O}_{s1s2gt\theta} \in \mathbb{Z}^+_0 \qquad \forall s1, s2, g, t, \theta$$
$$x^{L/O}_{gts\theta} \in \mathbb{Z}^+_0 \qquad \forall g, t, s, \theta$$
$$u^{L/O}_{rg\theta} \in \mathbb{Z}^+_0 \qquad \forall r, g, \theta$$
$$f^L_{gt\theta} \in \mathbb{Z}^+_0 \qquad \forall g, t, \theta$$
$$z_{r\theta} \in \{0,1\} \qquad \forall r, \theta \qquad (6.16)$$

The objective function (Eq. 6.2) represents the profit obtained by the fulfilled rentals. It considers: the one-time cost of purchasing vehicles and the cost per time period of maintaining this owned fleet, the revenue earned – the price of the rentals multiplied by the number of rentals fulfilled –, and other costs such leasing vehicles, performing empty transfers between stations and an artificial cost to penalize upgrades.

Constraint 6.3 establishes the purchasing budget. Constraints 6.4 limit the selection of price levels to a single level per rental type (lower and upper bounds on price are guaranteed by the definition of the PRI_{pr} parameters, as explained in the notation section).

A novel issue introduced in this model is the hierarchy among vehicle groups concerning price. Besides being an essential requirement from the business perspective, it introduces some changes to the structure of the problem, relevant for the methodology. More specifically, it is required that the prices for rental types that are similar in everything except vehicle group required follow some hierarchical rules. The goal is to avoid that a luxury vehicle is sold for a smaller price than a compact vehicle, for the same dates and locations. The unitary matrix $PLM_{rr'}$ describes the relationship between groups, indicating whether the price of a group is limited by the price of other. Constraints 6.5 translate this requirement.

The following constraints have been added or significantly altered compared to the model in Oliveira et al. (2017d), based on the assumptions presented and Figure 6.1. Constraints 6.6 limit the number of rentals fulfilled to the existing demand and Constraints 6.7 relate the price charged for the rental type, the minimum price that the competitors are charging and the demand levels, using binary variables $z_{r\theta}$. It is assumed that if the company prices a rental marginally lower than the minimum price in the market for a given scenario ($z_{r\theta} = 0$), it will attract a major slice of the market. If not ($z_{r\theta} = 1$), it will secure only a residual slice of the market.

Constraints 6.8 limit the vehicles that exit a certain station in a time period by the stock available. Constraints 6.9 reflect the upgrading policies, i.e. which vehicle groups can be upgraded to which vehicle groups.

Constraints 6.10 to 6.14 define the evolution of the stock variables, as in the previous work. At the beginning of the time period, the stock of owned fleet is given by the initial purchases (6.10) and there is no stock of leasing fleet (6.11). For later time periods, in each station and for each rental group, the stock of owned fleet in each scenario is given by the previous stock, increased by the arrival of transfers and rentals from the previous season (parameters) and from previous time periods of the current season and decreased by others that start on this time period (6.12). For the leasing fleet, the stock is also changed by leasing acquisitions (6.13) and, when the leasing period expires, removals from the fleet (6.14). It is assumed that the removal takes place in the same station as the acquisition.

Finally, the auxiliary variables that summarize total leased fleet per group and time period are calculated (Constraints 6.15) and the domain of all decisions variables is established (Constraints 6.16).

6.3. Solution method

As presented in Section 6.1, the methodology proposed in this work is based on the idea of co-evolution between a population of solutions and a population of scenarios to achieve solutions that have a good performance across a diverse set of scenarios.

The evolutionary components of the algorithm are based on a biased random-key genetic algorithm (BRKGA) framework (Gonçalves and Resende, 2011), since it is a widely used and well-performing genetic algorithm, which is structured so that the evolutionary procedures are independent of the problem and is available on an API (Toso and Resende, 2015). Appendix 6.A, in the Supplementary Materials, details the adaptations made to the "problem-independent" part of the original BRKGA framework in order to establish two parallel spaces of evolution.

In this section, the basic co-evolutionary procedures will be discussed and the problem-dependent parts of the genetic algorithm – decoding chromosomes and calculating fitness – will be detailed for both types of populations.

6.3.1 Co-evolution of solutions and scenarios

The main goal of this solution method is to obtain good solutions for the stochastic problem defined in Section 6.2, for which the only information regarding uncertain parameters are the lower and upper bounds that their values can take. In order to obtain scenarios that have a diverse impact on the solutions, a set of scenarios will be generated by an evolutionary procedure that is parallel to the one of the solutions.

The specific goal of the evolution of the solution population is to obtain values for the *first-stage decisions* of the stochastic model in Section 6.2.1 that lead to good performance in terms of total profit, when compared with the scenario population. The definition of "good performance" depends on the risk profile of the decision-maker and different alternatives will be discussed later in this section. The goal of the evolution of the scenario population is to diversify the impact of its elements on the profit of solutions.

Therefore, the link between these two types of populations is in the calculation of total profit, which involves calculating (or approximating) the *second-stage value function*. For this, the profit resulting from each <first-stage solution, scenario> pair is computed. This process will be further discussed in the remainder of Section 6.3.

6.3.2 Solution population

Decoder

Solution chromosomes encode solutions to the first-stage decision variables, which must be decided before uncertainty is revealed. These comprehend the purchase of vehicles for the owned fleet and the pricing decisions. When deciding the structure of these chromosomes, it was decided to favor feasibility, i.e. to ensure that the structure always leads to feasible solutions. The first segment of the chromosome refers to the pricing decisions (q_{rp} variables) and it is organized by time period. For each time period t, there is a gene corresponding to each rental type that starts in t. These rental types are ordered so that

rental types that require vehicle groups that limit (in terms of price) other rental types are decoded first.

Considering the bounds and other limits on prices, some relevant pre-processing steps are applied to the definition of the possible price levels per rental. Let r and r' be two rental types similar in all requirements except for vehicle group and whose vehicle required by r' limits the price of the vehicle group required by r (i.e., $r_{sout} = r'_{sout} \wedge r_{sin} = r'_{sin} \wedge r_{dout} = r'_{dout} \wedge r_{din} = r'_{din} \wedge PLM_{r_g, r'_g} = 1$). Let the parameter IPL be the price level interval between levels (e.g. 1 monetary unit). Due to the order of the chromosome, the price for rental type r' is calculated before r.

The maximum value to charge will be given by the smaller value among the upper bound for the specific rental or the price charged for the rental r' that limits it. The minimum value is determined by the lower bound of the rental type. After defining the range of possible prices, the discretization depends on parameter IPL. Considering that this range may not be divisible by IPL, it was decided that the first $|\mathcal{P}_r| - 1$ levels will be spaced by this value while the latter level $|\mathcal{P}_r|$ will correspond to the higher limit, no matter what the distance is to the previous level.

Therefore, the number of price levels for rental type r is given by:

$$|\mathcal{P}_r| = \left\lceil \frac{\min\left(\sum_{p \in \mathcal{P}_{r'}} q_{r'p} PRI_{r'p}, UBP_r \right) - LBP_r}{IPL} \right\rceil + 1 \tag{6.17}$$

The values associated with each price level $p \in \mathcal{P}_r$ are:

$$PRI_{rp} = \begin{cases} LBP_r + (p-1)IPL, & p < |\mathcal{P}_r| \\ \min\left(\sum_{p \in \mathcal{P}_{r'}} q_{r'p} PRI_{r'p}, UBP_r \right), & p = |\mathcal{P}_r| \end{cases}$$

This process is exemplified in Figure 6.2.

The decoding procedure consists of dividing $[0, 1[$ in $|\mathcal{P}_r|$ equal intervals. Based on the value of the gene g_i and in which interval it falls, the price is selected. This fulfills Constraints 6.4.

$$p = \left\lfloor \frac{g_i}{1/|\mathcal{P}_r|} \right\rfloor \Rightarrow q_{rp} = 1 \tag{6.18}$$

The following segment of the chromosome, $|\mathcal{S}| \times |\mathcal{G}| + 1$ genes, corresponds to purchase decisions (w_{gs}^O variables). Each gene g_i corresponds to a combination of station and vehicle group, as exemplified in Figure 6.3. The fraction of its value over the sum of the values of the chromosomes in the purchases segment $\mathcal{W}$ corresponds to the fraction of the budget that will be assigned to purchase vehicles of this group to be available at this station. The extra gene corresponds to non-assigned budget. The values are thus given by:

$$w_{gs}^O = \left\lfloor \left(\frac{g_i}{\sum_{j \in \mathcal{W}} g_j} B \right) / COS_g \right\rfloor \tag{6.19}$$
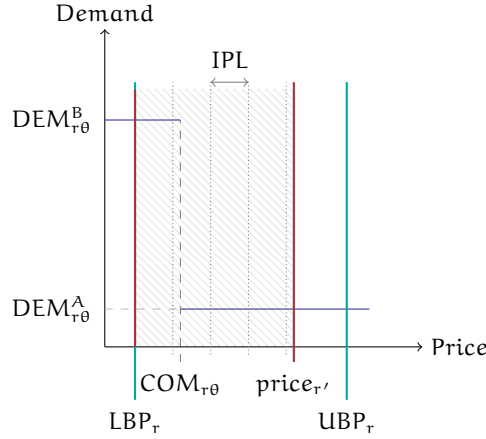
Figure 6.2: Definition of price range (hatched area) and corresponding price levels (dotted lines in gray and red lines – the bounds), for a rental type r, whose price is limited by rental type r'. The price decided for r' is here represented as $price_{r'}$ for simplicity
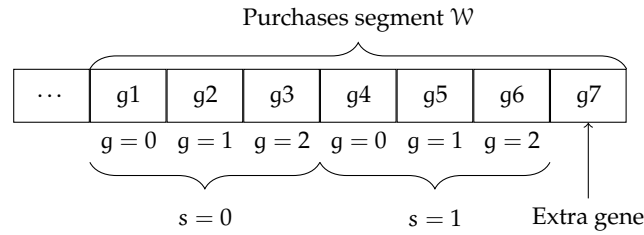

Figure 6.3: Structure of a segment of the solution chromosome, corresponding to owned fleet purchase decisions; example for 2 stations and 3 vehicle groups

Including the budget as a limit on the purchases incorporates Constraints 6.3 on the decoding procedure, thus contributing to the above-mentioned feasibility goal. Nevertheless, it requires that these segments of the chromosome are read twice (once for the calculation of the denominator and once for each numerator), which can lead to a poorer efficiency of the algorithm.

Fitness evaluation

The fitness of an individual determines its ability to survive in a population. For solutions, the goal is to favor those that perform well when faced with the scenario population.

Performance of a solution vs. a specific scenario: As previously introduced, the performance of a specific set of first-stage decisions (a solution) when a specific scenario is revealed is measured by the profit resulting from solving to optimality (or approximately) the second-stage problem. That is to say, by deciding the best number of rentals to be fulfilled and the best plan for fleet deployment, vehicle leasing and empty transfers – the recourse decisions, which are made after uncertainty is revealed in form of a scenario – and

establishing the resulting profit. By fixing the first-stage decisions encoded in the solution chromosome in the mathematical model presented in Section 6.2.1, it becomes a linear model (for sake of brevity, henceforth designated as second-stage MIP) and hence easier to solve. Nevertheless, to speed up the process, an approximation was considered that results from relaxing the integrality constraints on all decision variables of the second-stage MIP, resulting on an LP formulation. The validity of this approximation will be further discussed on Section 6.4.1.

Performance of a solution across scenarios: Since we lack information regarding the probability distribution of the scenarios, the performance of a solution across all scenarios in the scenario population is computed as the non-weighted average of its performance for each scenario. Nevertheless, decision-makers with different risk profiles value different metrics of performance. Therefore, in order to enrich the information that can be given to the decision-maker, three different decision criteria for solution fitness were established. Consider SO and SC to be the set of solutions and scenarios, respectively, within the corresponding populations. Consider $F(i, j)$ to be the profit obtained by solution $i \in SO$ when scenario $j \in SC$ is revealed:

- **Laplace criterion**: This is the baseline criterion of expected value in a stochastic approach. As previously explained, due to lack of probability information, the non-weighted average of the total profit obtained for all scenarios is considered as the fitness of a solution i:

$$\text{fitness}_i = \frac{\sum\limits_{j \in SC} F(i, j)}{|SC|} \tag{6.20}$$

- **Pessimist criterion**: Some robust approaches to decision-making favor solutions that perform well when the worst-case scenario is revealed. Since this is a maximization problem, the fitness of a solution according to this criterion is the worst (minimum) profit it obtains across scenarios:

$$\text{fitness}_i = \min_{j \in SC} F(i, j) \tag{6.21}$$

- **Optimist criterion**: An optimist approach is also considered, where the fitness of a solution is the best (maximum) profit it obtains across scenarios:

$$\text{fitness}_i = \max_{j \in SC} F(i, j) \tag{6.22}$$

6.3.3 Scenario population

Decoder

Scenario chromosomes encompass information on the uncertain parameters discussed in Section 6.2: for each rental type, the level of demand if the price is above the minimum in the market ($DEM_{r\theta}^A$), the level if the price is below ($DEM_{r\theta}^B$) and the minimum price

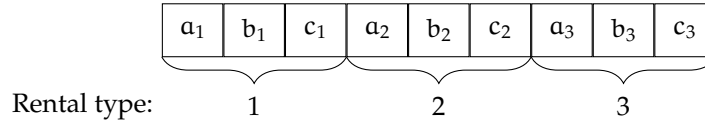| a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3

Rental type: 1   2   3

Figure 6.4: Structure of a scenario chromosome, for 3 rental types

of the competitors in the market ($COM_{r\theta}$). As Figure 6.4 exemplifies, in each scenario, each rental type r is associated with three genes: a_r, b_r and c_r. Therefore, a scenario chromosome has $3|\mathcal{R}|$ genes. A typical instance of this problem considers 400-2000 rental types, therefore the chromosome size can be a limitation. To decode this chromosome, the following additional inputs are required:

DF_r Demand forecast for the market of rental type r. It is an upper bound on $DEM_{r\theta}^B$, the demand achieved if the price is the lowest in the market;

$\Delta^{\max}$ Maximum difference between the forecasted demand (DF_r) and $DEM_{r\theta}^B$;

Δ^{A-B} Maximum difference between $DEM_{r\theta}^A$ and $DEM_{r\theta}^B$;

LBC_r Lower bound on possible competitor prices;

UBC_r Upper bound on possible competitor prices;

λ Parameter to scale the exponential relation between competitor price and the gene value.

The genes of type b_r are related with uncertain parameter $DEM_{r\theta}^B$ and define the fraction of $\Delta^{\max}$ that is considered in this scenario. Genes of type a_r define a similar fraction for Δ^{A-B}. As for the minimum competitor price $COM_{r\theta}$, it is non-linearly related with genes of type c_r, as represented in Figure 6.5. The parameter c_r represents the distance between the minimum competitor price, in the scenario, to its lower bound (considering its full range). As there are several competitors and the minimum prices tend to be consistently closer to their lower bound than to their upper bound, this connection is modeled by an exponential function. In this figure, the impact of using an exponential relationship in the decoding is exemplified, as well as of the choice of parameter λ. For a gene with value $c_r = 0.5$, if a "direct translation" were to be used the percentual distance of $COM_{r\theta}$ to its lower bound would be 50% (straight dashed line in Figure 6.5). Using an exponential relationship ($\lambda = 5$), this value reduces to 8.2%, and increasing λ will lead to even smaller values. As shown in Figure 6.5, with this exponential mapping the probability that the price is closer to the lower bound than to the upper bound is increased.

Summarizing, for each rental type r, based on the genes of the chromosome as presented in Figure 6.4, the values of the uncertain parameters are thus obtained:

$$DEM_{r\theta}^B = DF_r(1 - a_r\Delta^{\max}) \tag{6.23}$$

$$DEM_{r\theta}^A = DEM_r^B(1 - b_r\Delta^{A-B}) \tag{6.24}$$

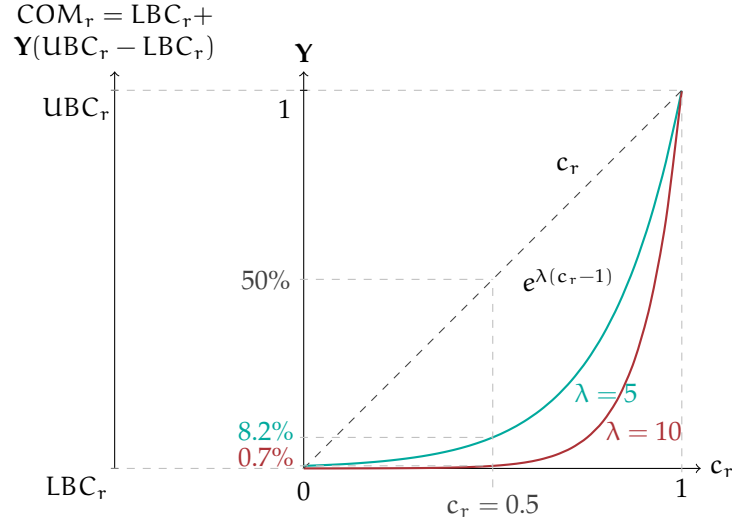$$COM_{r\theta} = LBc_r + (UBc_r - LBc_r)e^{\lambda(c_r-1)} \tag{6.25}$$

Figure 6.5: Exponential relationship between genes of type c_r and the uncertain parameter $COM_{r\theta}$ (and its lower and upper bounds LBC_r and UBC_r), exemplified for $\lambda = 5$ and $\lambda = 10$, compared with a linear relationship

Theoretically extreme cases: In this specific problem, it is possible to theoretically define the extreme cases. In theory, the best scenario is the one where demand is always the highest possible and competitor prices are the lowest (and vice-versa for the worst scenario). These two extreme scenarios, TBS (Theoretically Best Scenario) and TWS (Theoretically Worst Scenario) are included in the first generation of the scenario population, which is otherwise randomly generated. When using this method with other real-world problems, if some scenarios but not the totality of them are known, they can also be added to the initial scenario generation. Also, if the extreme cases are not previously known, the evolution of the scenario population is expected to converge to include those values.

Fitness

Having established the goal of obtaining a diverse and representative population of scenarios, the fitness evaluation must ensure that individuals that contribute the most for this goal survive in the population. *Diversity* is considered in terms of impact that the scenarios have on the profit of the solutions. A diverse scenario population consists of scenarios that result in different profits for the same set of solutions.

The fitness of an individual scenario translates its contribution to the population diversity and is based on the distance to other scenarios, in terms of difference in total profit obtained by the solutions. The methodology to compute distance is based on research in feature-based diversity optimization for instance generation discussed in Section 6.1.

Each scenario j is mapped on a bi-dimensional space, according to two correlated features: the best value obtained by a solution when it is unveiled $\max_{i \in SO} F(i, j)$ and the worst value obtained $\min_{i \in SO} F(i, j)$. Figure 6.6 exemplifies this procedure. For each feature, or axis, the scenarios that represent extremes are given a very high fitness value, in order to
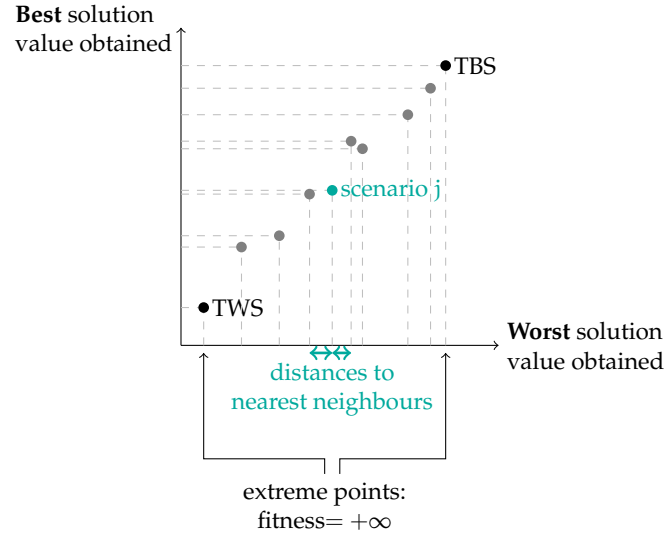
Figure 6.6: Example of fitness calculation and distance quantification for the scenario population

favor scenarios that broaden the "space" occupied by population. For the remaining scenarios, the nearest neighbors are identified and the product between the distances to each of the neighbors is computed. The fitness of a scenario is thus the maximum value between the product of distances in both axes. With this, scenarios that "fill in gaps" within the space the population occupies are favored. Algorithm 1 details the steps of this calculation.

In this case, evaluating the fitness of a scenario j' implies knowing not only the value of $F(i, j')$ for all $i \in SO$ but also the F value of the other scenarios $j \in SC$ for all $i \in SO$. This highlights the relevance of firstly computing the matrix $F(i, j)$ for each combination of solution and scenario and afterwards calculate the fitness values (see Figure 6.13b), so as to apply Equation 6.2 only once per pair (i, j).

6.4.   Computational experiments, results and discussion

The goal of the computational experiments discussed in this section is to validate the value of the methodology proposed in terms of: i) the quality of the solutions proposed, ii) the diversity and representativeness of the generated set of scenarios, which support the "robustness" of the solutions, and iii) the applicability and utility of the method when integrated in a decision-support system.

Instances: To test the methodology proposed, a set of instances were adapted from the set of realistic instances for the (deterministic) car rental capacity-pricing problem, available in Oliveira et al. (2017b). The adaptation procedure is detailed in Appendix 6.B, in the Supplementary Materials.

In the intensive computational tests, the instances are run with ten different seeds for each of the three different solution fitness criteria. The first eight instances from the original

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Algorithm 1 Pseudocode for scenario fitness calculation

Require: Matrix $|SO| \times |SC|$ of profit values $F(i, j)$

    tuple $B(id, best) \leftarrow \emptyset$

    tuple $W(id, worst) \leftarrow \emptyset$

    for all $j \in SC$ **do**

        $B_j \leftarrow (j, \max_{i \in SO} F(i, j))$

        $W_j \leftarrow (j, \min_{i \in SO} F(i, j))$

    end for

    sort B in ascending order

    sort W in ascending order

    $Bdist \leftarrow \emptyset$

    $Wdist \leftarrow \emptyset$

    for all $j \in \{2, ..., |SC| - 1\}$ **do**

        $Bdist_{B_j.id} \leftarrow (B_{j+1}.best - B_j.best) \times (B_j.best - B_{j-1}.best)$

        $Wdist_{W_j.id} \leftarrow (W_{j+1}.worst - B_j.worst) \times (B_j.worst - B_{j-1}.worst)$

    end for

    fitness $\leftarrow \emptyset$

    for all $j \in SC$ **do**

        if $j = B_1.id$ OR $j = B_{|SC|}.id$ OR $j = W_1.id$ OR $j = W_{|SC|}.id$ **then** fitness$_j \leftarrow +\infty$

        else fitness$_j \leftarrow \max(Bdist_j, Wdist_j)$

        end if

    end for

    return fitness

---

set were used for the comprehensive tests (30 runs per instances). Six other instances (selected to represent different sizes) were run once (for one seed and one fitness direction criterion) in order to draw some conclusions regarding computational time. This will be further discussed later in this section. Table 6.3 presents the main characteristics of the instances used. The size of the instance is approximated by the number of rental types and vehicle groups it considers. For the same size indicator, two different market size factors are considered: small and large. The difference between these two instances is that the reference values for the demand, as well as the budget, are in a *large market* 100 times larger than in a *small market*. As will be discussed throughout this section, this has a significant impact on the complexity of the instances.

Parameters: The stopping criterion for the genetic algorithm is the maximum number of generations and was set to 3000. The number of chromosomes in the solution and scenario populations were set to 20, after preliminary tests showed that these values allowed for the algorithm to perform well. The genetic algorithm was based on the brkgaAPI released (see Appendix 6.A for the detailed alterations) and the remaining parameters were set to match the proposed default parameters (Gonçalves and Resende, 2011; Toso and Resende, 2015).

Technical details: The tests were run on a server Intel(R) Xeon(R) X5690 with 3.46GHz (2 processors), and 48GB RAM. The MIP and LP solver used was CPLEX 12.6.3 and the algorithm was coded in C++.

Table 6.3: Main characteristics of the instances

Instance	Size indicator $(\|\mathcal{R}\| \times \|\mathcal{G}\|)$	Market size factor	# runs
1	428	Small	30
2	428	Large	30
3	486	Small	30
4	486	Large	30
5	517	Small	30
6	517	Large	30
7	1124	Small	30
8	1124	Large	30
17	2772	Small	1
18	2772	Large	1
29	4184	Small	1
30	4184	Large	1
37	6170	Small	1
38	6170	Large	1

6.4.1   Preliminary tests: validating the LP approximation on fitness calculations

In order to speed up the solution method proposed, an approximation was considered for calculating the best profit that a certain solution can achieve if a certain scenario is realized. This approximation consists on relaxing the integrality constraints on all decision variables of the second-stage MIP that provides the optimum profit value required, resulting on a linear program (LP). The goal of these preliminary tests is to validate this approximation.

The most relevant impact of using an approximation is not on the profit values obtained per se but on the differences caused on the rank of the solutions and scenarios according to their fitness. Therefore, to validate this approximation, the fitness of the last generation of solutions and scenarios was calculated based on the LP and on the second-stage MIP, for each run of the first eight instances. The consequent order of the solutions and scenarios was compared for both cases. The order according to which the individuals were ranked was the same using the two approaches, for both solution and scenario populations. Moreover, the differences in fitness value were negligible, with no significant differences between solution fitness criteria. Table 6.10 in Appendix 6.C, in the Supplementary Materials, presents the detailed results. This validates the approximation considered in this solution method.

6.4.2   Solution evolution

The goal of the tests presented in this subsection is to assess the ability of the method to generate good solutions. For this, we study the solution fitness improvement over generations and overall run times. To further understand the quality of these solutions, approximations of the Expected Value of Perfect Information (EVPI) and of the Value of the Stochastic Solution (VSS) are analyzed.

Table 6.4 presents the main results associated with the evolution of the solution populations, as well as the overall computing times. The fitness of the solutions is assessed according to the different criteria presented in Section 6.3.2, as well as the final best value. As mentioned before, the fitness value is computed by solving the LP approximation of the second-stage MIP, while the final best values presented result from solving the MIP models of the last generation of the genetic algorithm to optimality.

Table 6.4: Overall results of best solution fitness and final value and computational times

Instance	Solution fitness criteria	Improvement on solution fitness (last vs. initial generation)	Final best value (according to fitness criterion)	Time (sec)
1	Laplace	76%	37,728.8	1,288
	Pessimist	1026%	5,777.5	1,407
	Optimist	54%	80,555.2	1,310
2	Laplace	24%	4,037,570.0	2,450
	Pessimist	83%	1,001,230.0	2,413
	Optimist	54%	8,067,420.0	2,451
3	Laplace	59%	45,993.1	1,458
	Pessimist	441%	8,039.4	1,593
	Optimist	51%	98,043.0	1,493
4	Laplace	25%	4,933,750.0	2,950
	Pessimist	76%	1,309,700.0	2,928
	Optimist	55%	9,797,250.0	2,914
5	Laplace	54%	55,429.3	1,641
	Pessimist	239%	10,395.6	1,787
	Optimist	51%	119,184.0	1,673
6	Laplace	25%	6,071,930.0	3,155
	Pessimist	70%	1,681,320.0	3,110
	Optimist	52%	11,925,700.0	3,150
7	Laplace	187%	32,385.5	8,915
	Pessimist	991%	3,965.8	9,353
	Optimist	79%	67,439.3	9,018
8	Laplace	44%	3,476,000.0	16,191
	Pessimist	116%	778,926.0	16,195
	Optimist	75%	6,784,230.0	16,238

Laplace average: 62%
Pessimist average: 380%
Optimist average: 59%

The percent improvement on solution fitness throughout the genetic algorithm is also presented. It is possible to observe that this metric is influenced by the fitness criterion selected. For runs where the Pessimist criterion guided the solution evolution, this improvement is significantly larger in average (380% vs. 62% and 59%). This effect is consistently present in all instances. This difference can be explained by the characteristics of the problem, namely the impact on profit of fleet utilization levels, linked with the random
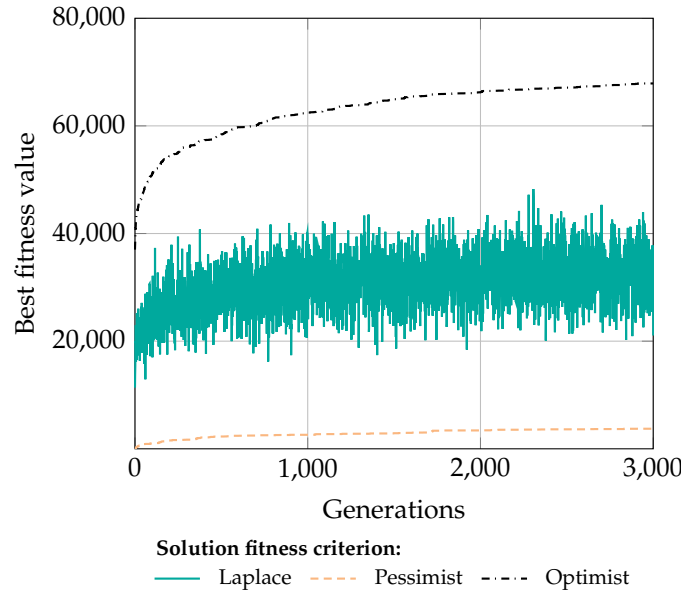
Figure 6.7: Evolution of solution fitness, for the different solution fitness criteria (example from instance 7)

construction of the initial generation of solutions. If demand and market prices are high (in the best scenarios), random solutions, at worst, lead to a fleet smaller than what it should be. Nevertheless, there is a high level of utilization of the small fleet capacity and all vehicles tend to generate profit. However, if demand and market prices are low (in the worst scenarios), similar random solutions will overestimate fleet capacity, which will lead to a lower utilization of the fleet and consequently higher costs for lower revenues. Therefore, the room for improvement is larger when considering the performance on worse scenarios.

Figure 6.7 shows an example of solution fitness evolution throughout the genetic algorithm. Three runs with different solution fitness criteria are compared. As expected, the scale of the fitness values is significantly different for each criterion. Nevertheless, it is possible to observe the evolution and convergence of the fitness values. For Laplace criterion, despite significant oscillations, there is a convergence around an average value. These oscillations are related with changes in the scenario population, which is also evolving. For Pessimist and Optimist criteria, the evolution profile is not oscillating since the extreme scenarios (TWS and TBS) are known for this problem, and were included in the initial generation of the scenario population (see Section 6.3.3).

Finally, Table 6.4 also quantifies the average computational times. It is possible to see they are influenced by instance size and market size factor. Despite this increase in run time, it is important to notice that, due to the strategic scope of the decisions here considered, these values are reasonable for the application of this methodology as a decision-support tool. This issue will be further discussed on Section 6.4.4.
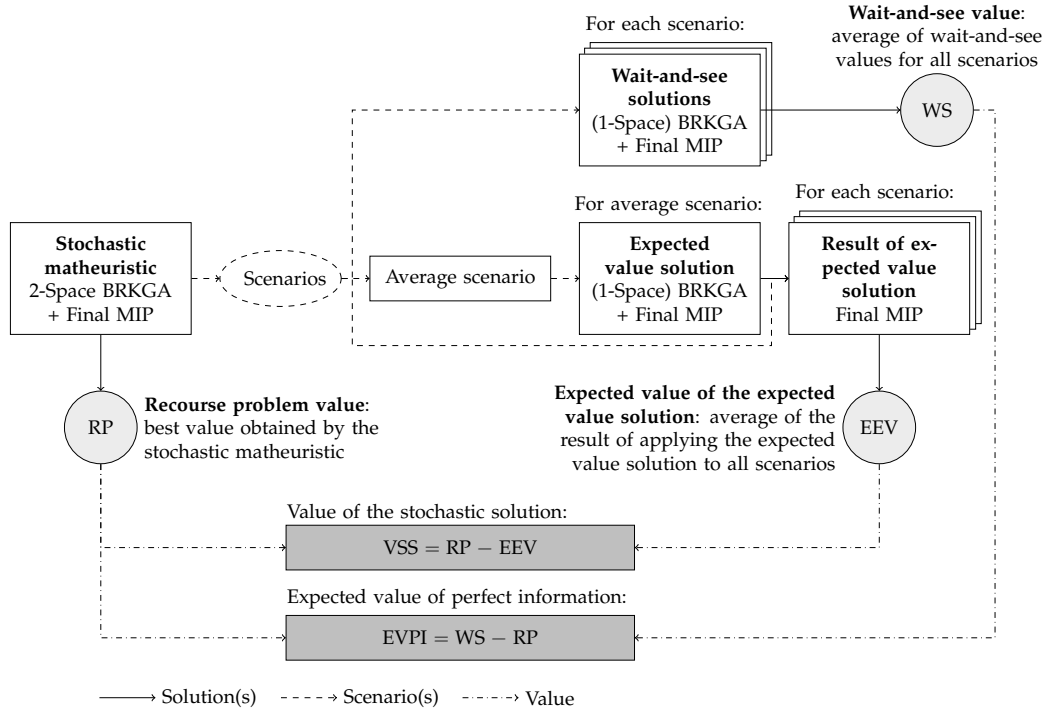
Figure 6.8: Framework for calculating VSS and EVPI, considering the Laplace criterion for solution fitness (for a maximization problem)

6.4.2.1 Expected value of perfect information (EVPI) and Value of the stochastic solution (VSS)

The solution method is a matheuristic procedure and thus the results it obtains – in terms of solutions and scenarios – are not proven to be optimal. Nevertheless, interesting insights can be drawn by developing an approximated measure of the Expected Value of Perfect Information (EVPI) and the Value of the Stochastic Solution (VSS), assuming the the set of scenarios generated is representative. In this context, the terms EVPI and VSS, as well as other terminology from Stochastic Programming such as recourse problem value or wait-and-see solutions, will be loosely used for simplicity, representing an approximation derived from a non-exact methodology. Figure 6.8 presents the framework for obtaining these values. The goal is to obtain an approximation of how important it is to have good forecasts in this problem (EVPI) and of how much can a company profit from applying this stochastic method instead of a deterministic one (VSS).

EVPI is the difference between the best value obtained by the stochastic method or recourse problem (RP) and the wait-and-see value (WS). We use the Laplace criterion for solution fitness when computing the RP throughout this section, in order to be comparable with the other values that are also results of the (non-weighted) average of scenarios. The WS is the average for all scenarios of the profit values obtained if the decisions were made knowing the scenario in advance (perfect information). The stochastic method is a *matheuristic* that consists on evolving a Two-Space BRKGA where solutions and scenar-

ios co-evolve and, for the last generation populations, solving MIP models to calculate the exact impact of each scenario on each solution. In order to obtain a comparable value, a similar matheuristic was run to generate wait-and-see solutions. The difference is that this One-Space BRKGA evolves only the solution population, considering a fixed scenario (deterministic).

VSS compares the best value obtained by the stochastic method (RP) with the expected result across scenarios of applying a deterministic solution that results from considering only an average scenario (EEV). A similar One-Space BRKGA was also used to generate the expected value solution. The impact of this solution across scenarios was computed by solving MIP models.

For this computation, the first six instances, which are similar in size, were used. Table 6.5 presents the results obtained.

Table 6.5: Average measures of Expected Value of Perfect Information (EVPI) and Value of the Stochastic Solution (VSS), using indicators of Recourse Problem value (RP), Wait-and-See value (WS) and Expectation of using Expected Value solution (EEV)

In-stance	RP	WS	EEV	EVPI = WS − RP	EVPI$_\%$= EVPI / RP	VSS = RP − EEV	VSS$_\%$ = VSS / EEV
1	37,729	40,991	30,506	3,263	9%	7,223	24%
2	4,037,570	4,244,760	3,730,540	207,190	5%	307,030	8%
3	45,993	49,663	37,902	3,670	8%	8,091	21%
4	4,933,750	5,183,560	4,530,280	249,810	5%	403,470	9%
5	55,429	59,697	45,522	4,268	8%	9,907	22%
6	6,071,930	6,335,700	5,506,880	263,770	4%	565,050	10%

Regarding the EVPI, it is possible to conclude that, assuming the set of scenarios is representative and considering that genetic algorithms as described above are used to make decisions, improvements between 4% and 9% can be expected if the uncertainty is removed. This value is helpful for companies to understand how much to invest in better forecasting methods, for example.

As for the VSS, it measures in a more direct way the impact of using this stochastic method instead of a similar deterministic one. These values are significantly impacted by the market size factor of instances. For smaller market sizes (instances 1, 3 and 5), considering uncertainty results on 21%-24% improvements on expected profit. Large market instances (2, 4 and 6) show less expressive results, with nonetheless significant improvements of 8%-10%.

6.4.3   Scenario evolution

The goal of the tests presented in this subsection is to assess the quality of the final scenario population obtained by the proposed method. For this, two main characteristics are studied: diversity – how different the scenarios in the population are, and representativeness – how well the set of scenarios represents the possible ranges of impact on the solutions.

As mentioned when discussing the results of solution fitness evolution on Section 6.4.2, the oscillation on solution fitness when using the Laplace criterion (see Figure 6.7) is caused by the non-monotonous evolution of the scenario population. In fact, due to the strategy for increasing diversity on the scenario population, scenarios are ranked in terms of how different is the impact they have on the solutions. This means that a single individual entering or leaving the population can cause the fitness of another individual to drastically change. However, in this work, quantifying the value or fitness of each scenario is a means to evolve towards a population with certain characteristics. The focus is not the individual fitness of scenarios but on the structure of the population as a whole. Two main characteristics of the scenario population are considered:

- Diversity, since the goal is to obtain scenarios that are different in terms of the impact they have on the solutions, and

- Representativeness, since the scenario population does not comprise all possible scenarios and we need to ensure that the performance of the solutions against these is indicative of their performance against all possible scenarios.

6.4.3.1 Diversity

Diversity of a scenario population is connected with the scenario fitness calculation presented in Section 6.3.3. Figure 6.9 is based on the bi-dimensional system previously introduced (exemplified in Figure 6.6), where each scenario in a population is mapped according to the worst and the best values it "causes" on the solution population. Figure 6.9 compares the initial and last generation of the scenario population. The initial population (green triangles) has two clear extreme points (TWS and TBS – see Section 6.3.3) and the remaining scenarios, which were randomly generated, are close to each other. The last generation (gray circles) is spread between these two extremes, showing the effect of evolution. The hatched area represents an *a fortiori* impossible region (the worst value cannot be better than the best value obtained). The fact the population is not spread across the remaining space is due to the fact that the two features that are used to map scenarios are correlated, i.e. the higher the worst value is, the higher the best value tends to be. Especially considering that the solution population converges to similar and well-performing solutions, in the last generation it is not expected that a single scenario is simultaneously adverse for one solution and favorable for other. Based on this mapping, a possible measure of population diversity is the average euclidean distance between the coordinates of the scenarios projected in this 2-axes system. There is an average fourfold increase in this metric when comparing initial and final populations for all runs. Table 6.11 in Appendix 6.D, in the Supplementary Materials, details these results.

6.4.3.2 Representativeness

Since there is no better information available regarding uncertainty than the bounds used in the scenario decoder, it is difficult to evaluate the representativeness of scenario populations in terms of the *accuracy* of the generation method. Nevertheless, it is possible to evaluate it in terms of its *precision*. In fact, representative scenario populations should have a similar
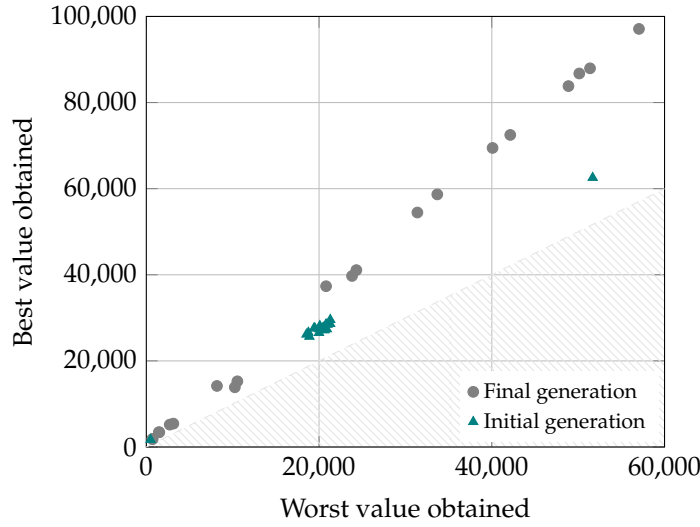
Figure 6.9: Representation of the scenarios in the initial and final generation of the scenario population (example from instance 3, run with Laplace solution fitness criterion). The two axes refer to the best and the worst values achieved by the solutions in the solution population for each scenario. Considering the notation introduced in Section 6.3.2, for a scenario j: $\max_{i \in SO} F(i, j)$ and $\min_{i \in SO} F(i, j)$, correspondingly. The hatched area represents an *a fortiori* impossible region

impact on the solutions. To test this, the best solutions found for each instance were evaluated against the last generation of 20 scenarios from the same run and against the scenarios generated in different runs (for the same instance and solution fitness criterion). This last set of scenarios is composed of 200 elements, corresponding to the last generations of 20 scenarios for each of the 10 runs. The average values of the profit obtained by solving the second-stage MIP problems across the two different sets of scenarios were compared. Table 6.6 presents the results per instance and solution fitness direction. In average, the impact is similar, ranging from 0% to 6%. Nevertheless, there is a slight tendency of the co-generated scenarios to underestimate profit obtained. Although the average values in Table 6.6 are all positive, for some runs/seeds the difference is negative. Therefore, the fact that a single run of the method results in scenarios that slightly underestimate profit for the best solution is an average trend rather than a general output.

Besides analyzing the precision of the scenario population in terms of the average impact on profit, it is important to evaluate this precision on its *impact across scenarios*. For this, the solution fitness criteria play an important role, since the criterion used significantly influences the structure of the final solutions generated. For the Pessimist and Optimist criteria, the performance across all scenarios is not considered at any point of the evolutionary procedure and it becomes an essential point of study in this section. When considering the problem at hand, a relevant first-stage decision that is critical for this performance is the fleet size. If the decision is made solely considering the worst-case scenario (Pessimist criterion), where demand is lower, the company will tend to acquire a smaller fleet so that it does not incur in costs with non-utilized fleet. Therefore, if the demand increases con-

Table 6.6: Difference in average profit obtained by the best solution when facing all scenarios generated in different runs (of the same solution fitness direction) vs. facing co-generated scenarios

| | Percent increase in average profit | | |
| | Solution fitness criteria | | |
Instance	Laplace	Pessimist	Optimist
1	1.6%	0.7%	1.3%
2	0.6%	0.2%	0.5%
3	2.6%	1.0%	6.0%
4	1.9%	0.5%	1.0%
5	6.0%	3.5%	2.9%
6	1.8%	1.7%	2.0%
7	4.1%	4.1%	1.9%
8	2.2%	0.7%	3.3%

siderably (best-case scenarios), the company may not able to seize all the possible revenue due to the lack of vehicles. In the opposite case, Optimist decisions lead the higher fleet size decisions, causing unnecessary costs in scenarios where demand is lower.

Figure 6.10 presents how the best solutions generated in instance 7 perform across all scenarios generated for the same solution fitness criterion. The conclusions drawn are similar for all instances.

The solutions perform similarly across all scenarios when the solution fitness criterion is the same. This supports the representativeness of the scenario populations generated by this method in terms of the impact they have on the "robustness" of solutions. That is to say, a solution co-generated with a small set of scenarios has the same behavior than a solution co-generated with another set of scenarios. This effect is more visible on solutions generated by the Laplace and Optimist criteria (for simplicity, henceforward named as "Laplace solutions" and "Optimist solutions"). For those generated by the Pessimist criterion ("Pessimist solutions"), as the scenarios get closer to the best case, differences between solutions become more evident. Since the solution evaluation depends only on the performance on the worst case, it is expected that the evolution of solutions leads to different performances on the best cases.

There is also an observable difference between the performance profile of Pessimist solutions when compared to those of Optimist or Laplace solutions, which is related with the goal of each evolution strategy. Pessimist solutions perform better than Laplace and Optimist solutions on the worst cases yet not on the best cases or in average. This observation is coherent with the previous discussion on the solution fitness criteria impact on fleet size decisions. In fact, throughout all best solutions retrieved from all instances, there is a relevant difference between the non-used budget on Pessimist solutions (21%) and Laplace and Optimist solutions (both 1%). For Laplace solutions, a use of the budget similar to Optimist solutions is expected, since the best scenarios only compensate the bad in average if the company has enough cars to sell when demand increases.
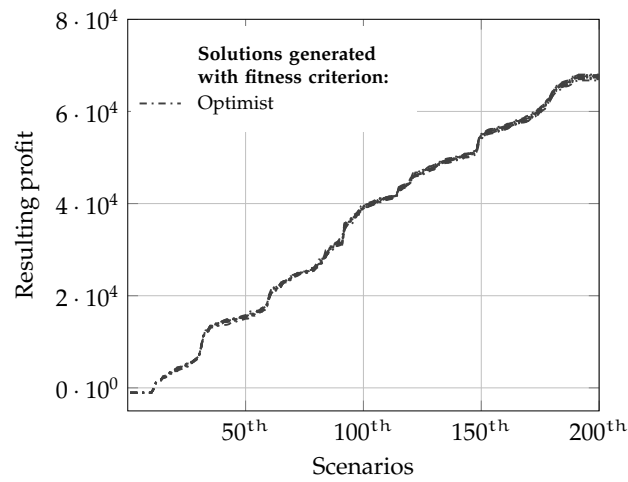
Moreover, the similarity between Laplace and Optimist solutions is visible not only
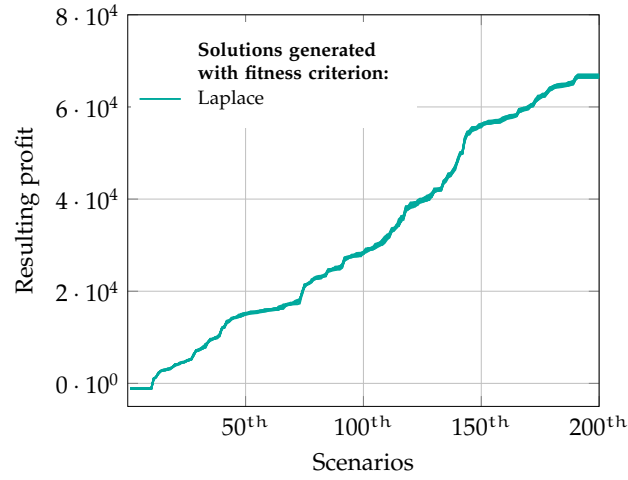
(a) Laplace criterion
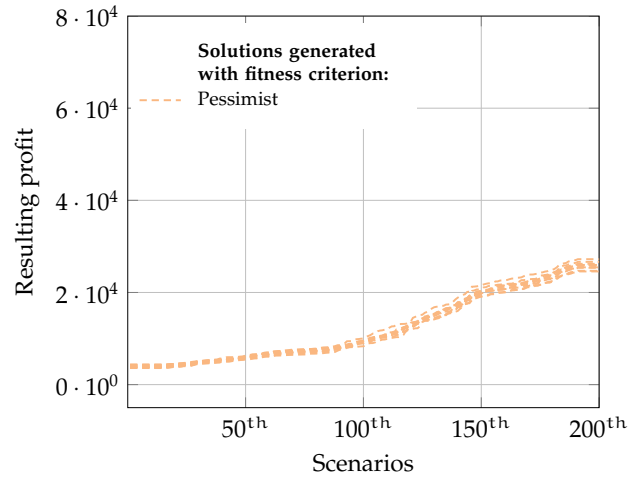
(b) Pessimist criterion

(c) Optimist criterion

Figure 6.10: Resulting profit of applying the best solutions generated in each run against the full set of scenarios that were generated considering the same solution fitness criterion (instance 7 – small market)

on performance but in part of the solution structure as well. Figure 6.11 shows how the solutions in Figure 6.10 translate into fleet purchase decisions. Besides performing similarly across scenarios, Laplace solutions have a similar structure among themselves in terms of fleet size and mix decisions, as well as Optimist solutions. Also, there is some similarity between the structure of solutions generated by these two criteria. Pessimist solutions, however, have a distinct structure from Laplace and Optimist solutions (in terms of vehicles purchased for Location 1, for example). Moreover, they show larger differences among themselves, including total fleet size values. These differences are, as discussed, a clear indicator of performance dissimilarity in scenarios with higher demand.

6.4.4 Decision support

Ultimately, the goal of this methodology is to support decision-makers, by providing a set of good solutions that are appropriate for different risk profiles and by helping the visualization of the different impact uncertainty can have on these solutions. In this section, the outputs of the method that are relevant for decision-makers are discussed, as well as possible computational limitations.

The number of solutions that the decision-maker can obtain with this method depends on the time or computational resources available. Even if these are scarce, with only one run per solution fitness criterion, three different and good solutions can be compared. As an example, Figure 6.12 shows the best results achieved in these conditions for instance 8. The vertical axis represents the resulting profit from each of the best solutions if each scenario occurs. Following the direction of improvement of scenarios, vertical lines show points where the lead among the three best solutions changes. For the worst scenarios, the Pessimist solution performs slightly better than the others. Then, there are a few scenarios where the resulting profit is similar for the three solutions. These scenarios are followed by a significant portion of scenarios where the Laplace solution outperforms the others. Finally, for the best scenarios, the Optimist solution has the best results, closely followed by the Laplace solution, while the Pessimist solution falls behind.

In order to make a decision, it is important to understand in what these solutions differ in terms of structure. Table 6.7 presents the main characteristics of the three solutions exemplified in Figure 6.12, in terms of the capacity decisions. It is possible to see that the Pessimist solution has a higher percentage of budget that is not used for purchases, leading to a smaller fleet size. This partly explains why this solution is not able to perform as well in scenarios with high demands. Other structural insights are related with the vehicle groups. It is possible to see that all solutions favor Group 1 (the less-valued vehicle group), yet this effect is magnified for Optimist and Laplace solutions. As for the rental locations where the purchased vehicles are made available, in this case, there are only slight differences between solutions. Nevertheless, this metric can be relevant for the decision-maker in other situations.

The tools developed in this methodology can be applied in interesting features of a decision-support system. For example, it would be possible for a decision-maker to change the solutions found and test these new solutions against the pool of scenarios that were generated in the process. Also, the decision-maker can feed some scenarios to the initial

(a) Laplace solutions
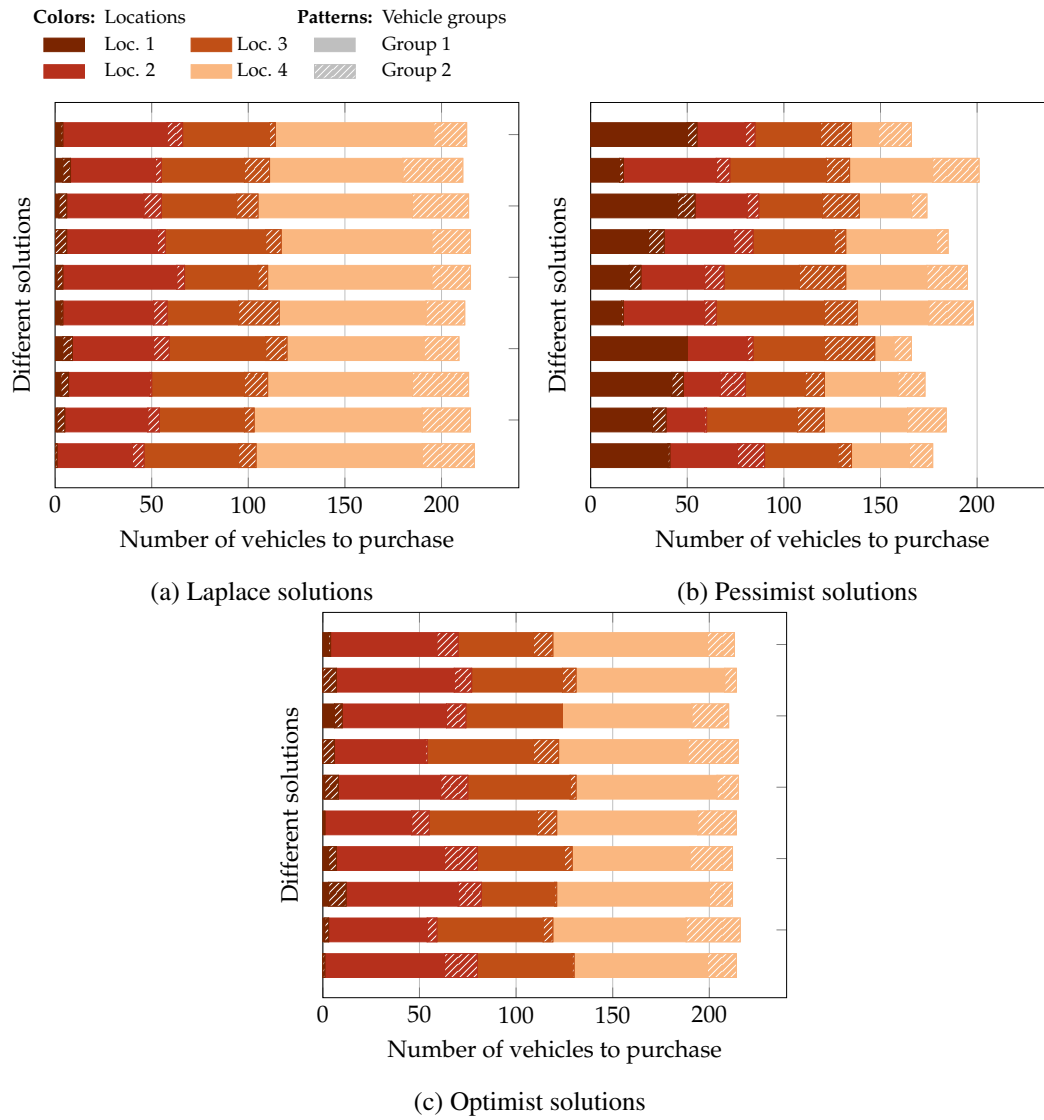
(b) Pessimist solutions

(c) Optimist solutions

Figure 6.11: Fleet size/mix decisions for the solutions presented in Figure 6.10: number of vehicles to purchase per vehicle group and available location
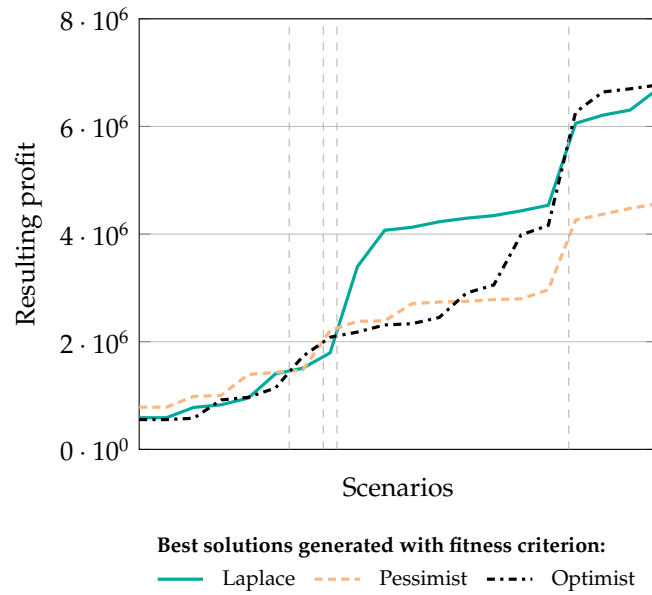
Figure 6.12: Final best solutions generated (example from instance 8)

Table 6.7: Characteristics of fleet capacity spending for the solutions presented in Figure 6.12

		Solutions in Figure 6.12		
		Laplace	**Pessimist**	**Optimist**
Non-allocated budget		1%	30%	1%
Weight of vehicle group in purchases	Group 1	74%	68%	79%
	Group 2	26%	32%	21%
Weight of locations as destinations of purchased vehicles	Location 1	30%	28%	25%
	Location 2	32%	28%	29%
	Location 3	19%	17%	20%
	Location 4	20%	27%	26%

population. Nevertheless, some limitations must apply, namely the need to maintain a minimum number of randomly generated individuals on the population.

A possible drawback of the applicability of this methodology is the significant increase of computational effort required if the size of the instances increase. As mentioned in the beginning of Section 6.4, the intensive computational tests performed required instances to be run with ten different seeds for each of the three different solution fitness criteria. All instances available are realistic yet different in size. In order to assess the computational time required for larger instances, six instances with larger and different sizes were run once (for one seed and one fitness direction criterion only). These include the largest instances available.

Table 6.8 summarizes these results. As expected, the run time increases not only with the size of the instance but also for large markets, with higher demand. Nevertheless, considering the strategic scope of the problem at hand, this methodology could still be employed for these extreme cases, probably with a limitation on the number of runs. However, it could be argued that for more complex seasons and markets, such decision-support tools are even more needed. Even if a decision support tool takes 12 hours or a day, it is providing needed support for a season-lasting decision such as fleet size and mix and it may still be useful.

Table 6.8: Average run times

Instance	Size indicator ($|\mathcal{R}| \times |\mathcal{G}|$)	Market size factor	Average time (sec)
1	428	Small	1,335
2	428	Large	2,438
3	486	Small	1,515
4	486	Large	2,931
5	517	Small	1,700
6	517	Large	3,138
7	1,124	Small	9,096
8	1,124	Large	16,208
17	2,772	Small	26,982*
18	2,772	Large	69,480*
29	4,184	Small	32,607*
30	4,184	Large	43,021*
37	6,170	Small	46,072*
38	6,170	Large	43,591*

* Based on a single run.

6.5.   Conclusions

This study presents not only a new approach to deal with an innovative application, but also methodological contributions that can be applied beyond this scope. This methodology can be adapted to provide good solutions to complex two-stage stochastic problems

where the information on uncertainty is scarce. It does not require the decision-maker to define the scenarios or probabilities associated with them, but only to establish upper and lower bounds for the uncertain parameters. The scenarios are generated and evolve alongside the solutions, and are fine-tuned to be representative and diverse in relation to these solutions. This is of particular interest in practical applications where the number of uncertain parameters is large and the explicit definition of uncertainty scenarios is difficult to obtain. Moreover, this method provides the decision-maker with a set of possible solutions, clearly associated with the impact that the different scenarios have on them.

In the future, regarding this innovative application, the model can be extended towards a more tactical (possibly weekly) scope, in order to develop a decision-making support tool for the decisions not considered here. In the car rental problem, this could refer to decisions regarding multi-stage-oriented pricing and deployment actions.

Moreover, this methodology can be further developed. If adapted to thoroughly-tested problems with known analytical solutions, intensive computational tests can help improve the efficiency of the genetic algorithm in terms of quality of the solutions achieved and run times, as well as to validate the conclusions drawn in this work.

Acknowledgments

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Appendix 6.A   BRKGA API adaptation towards co-evolution

The BRKGA framework was adapted to the "two-space genetic algorithm" idea proposed by Herrmann (1999), where solutions and scenarios form two different populations that co-evolve. This framework, presented in Figure 6.13a, is often used as a "black-box", where only the problem-dependent parts (i.e. the translation of chromosomes into solutions and the calculation of their fitness) need to be custom built.

Nevertheless, its adaptation towards a two-space algorithm involved some changes to accommodate the connections between the two types of populations, including two types of mutually dependent fitness evaluations, two types of decoders, and other specific changes to the evolution code (e.g. ensuring that the fitness of a solution is re-calculated in each generation, even if it is copied from a past generation), as presented in Figure 6.13b. In the Appendix, a full register of the changes made to the BRKGA framework can be found.

## Appendix 6.B   Adaptation of literature instances

The alterations required are related with different modeling of prices, demand and their relationship and refer to the parameters presented with the mathematical model (Section 6.2.1) and with the scenario decoding procedure (Section 6.3.3). For each rental type, the demand forecast for total market value (DF_r) and the upper bound on price charged by the company (UBP_r) were retrieved from the highest corresponding values in each literature instance, while the lower bound on price (LBP_r) was set in proportion with the rental length. The upper and lower bound of competitor prices (LBC_r and UBC_r) were considered the same as the bounds for the prices of the company. Table 6.9 summarizes the values set for the additional parameters. All remaining parameters were obtained directly in the literature instances.

Table 6.9: Values set to the additional parameters required, when adapting the literature instances

	Parameter	Value set
Mathematical model	MGP	2
	LBP_r	$5(di_r - do_r + 1)$
Decoding procedures	IPL	1
	Δ^{max}	0.2
	Δ^{A-B}	0.9
	λ	5
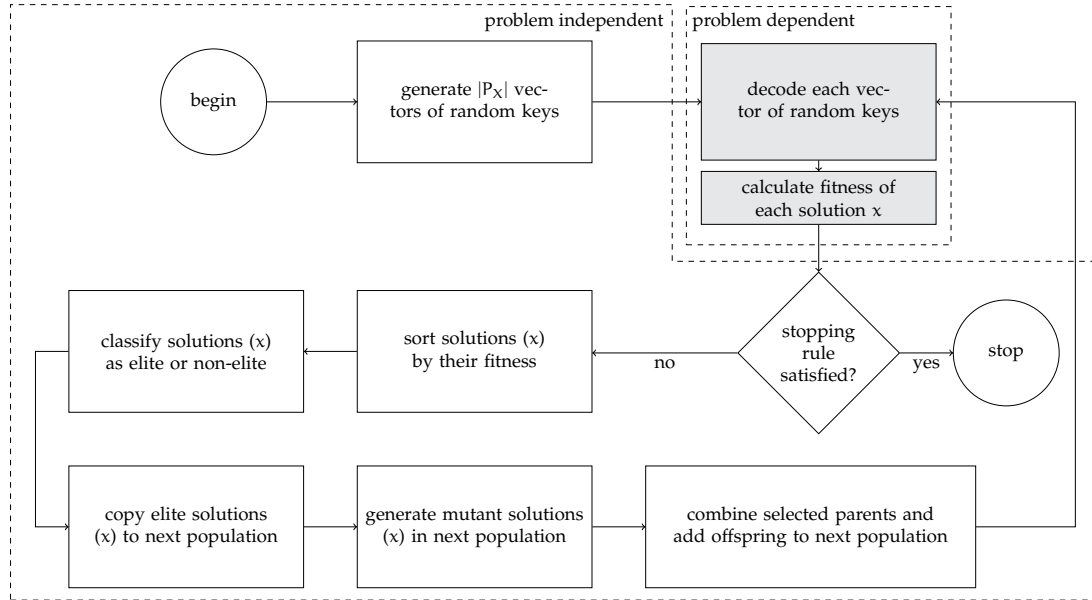
(a) BRKGA flowchart, adapted from Gonçalves and Resende (2011)
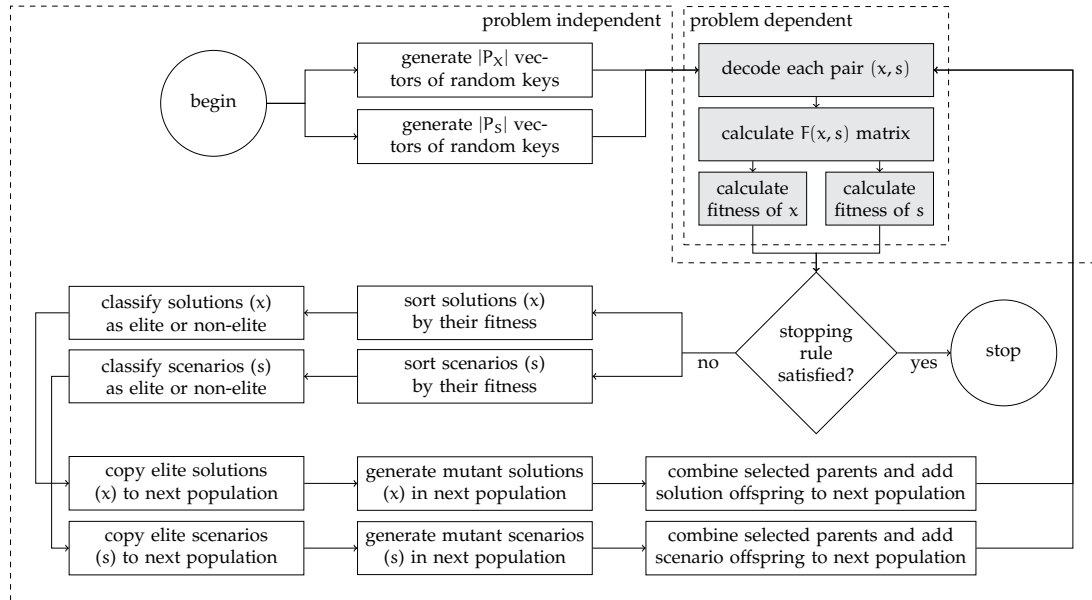
(b) Changes to the original "one-space" BRKGA flowchart (confront with Figure 6.13a) towards a "two-space" algorithm

Appendix 6.C   Ranking differences between MIP models and their LP relaxation

Table 6.10: Comparative results of classifying and ranking solutions based on their fitness, calculated using MIP models or their LP relaxation

| Instance | Solution fitness criteria | Count of individuals ranked differently using MIP and LP models | | Average delta (%) in fitness value | |
		Solutions	Scenarios	Solutions	Scenarios
1	Laplace	0	0	1.85e-05%	1.42e-03%
	Pessimist	0	0	2.28e-05%	8.17e-04%
	Optimist	0	0	1.06e-05%	4.13e-03%
2	Laplace	0	0	1.48e-05%	2.23e-03%
	Pessimist	0	0	1.35e-05%	1.51e-03%
	Optimist	0	0	0%	1.13e-03%
3	Laplace	0	0	8.48e-06%	2.44e-03%
	Pessimist	0	0	1.98e-06%	1.01e-03%
	Optimist	0	0	9.92e-06%	1.32e-03%
4	Laplace	0	0	3.07e-06%	1.26e-02%
	Pessimist	0	0	9.35e-06%	2.21e-03%
	Optimist	0	0	0%	0%
5	Laplace	0	0	6.59e-06%	7.16e-04%
	Pessimist	0	0	8.30e-05%	8.06e-04%
	Optimist	0	0	0%	9.04e-04%
6	Laplace	0	0	2.69e-06%	5.62e-04%
	Pessimist	0	0	9.19e-06%	2.21e-03%
	Optimist	0	0	0%	5.07e-04%
7	Laplace	0	0	5.76e-05%	1.44e-02%
	Pessimist	0	0	2.16e-04%	8.74e-03%
	Optimist	0	0	4.79e-05%	4.18e-03%
8	Laplace	0	0	6.07e-05%	4.05e-03%
	Pessimist	0	0	7.18e-05%	5.05e-03%
	Optimist	0	0	0%	9.82e-05%

Appendix 6.D   Diversity increase measure

Table 6.11: Measures of diversity increase in scenario populations. The average euclidean distance is computed considering each scenario j represented by a system of coordinates $(\min_{i \in SO} F(i, j), \max_{i \in SO} F(i, j))$, where $F(i, j)$ is the optimal profit from implementing solution i and scenario j occurring

Instance	Solution fitness criteria	Average euclidean distance between scenarios		Increase
		Initial generation	Final generation	
1	Laplace	7,877	33,677	4.28
	Pessimist	7,877	21,369	2.71
	Optimist	7,877	31,153	3.96
2	Laplace	736,418	3,194,527	4.34
	Pessimist	736,418	2,436,218	3.31
	Optimist	736,418	3,414,816	4.64
3	Laplace	9,590	39,416	4.11
	Pessimist	9,590	27,601	2.88
	Optimist	9,590	41,223	4.30
4	Laplace	865,175	3,770,988	4.36
	Pessimist	865,175	2,900,457	3.35
	Optimist	865,175	4,072,062	4.71
5	Laplace	11,609	48,034	4.14
	Pessimist	11,609	34,059	2.93
	Optimist	11,609	48,691	4.19
6	Laplace	1,062,182	4,592,790	4.32
	Pessimist	1,062,182	3,798,226	3.58
	Optimist	1,062,182	4,903,228	4.62
7	Laplace	5,739	27,156	4.73
	Pessimist	5,739	15,042	2.62
	Optimist	5,739	27,377	4.77
8	Laplace	539,122	2,712,308	5.03
	Pessimist	539,122	1,915,444	3.55
	Optimist	539,122	2,700,682	5.01
			Average	**4.02**

Conclusions and future work

This thesis tackles the integration of fleet and revenue management in the car rental business. The integration of these blocks of decision-support, traditionally tackled in a separate or sequential manner, is based on a theoretical and academic ground and arises from practical needs of car rental companies that operate in this market. The theoretical motivation is discussed in Chapter 2, which presents a literature review and proposes a new conceptual framework for these problems. From these, relevant research gaps and research directions are identified, including the integration of capacity and pricing and the consideration of uncertainty for higher realism and applicability. Chapter 3 describes the work developed alongside *Guerin Car Rental Solutions*, where an algorithm and a decision-support system are designed to update prices considering competitor prices and fleet occupation. This work provided the practical motivation for the integration of capacity and pricing. The first approach to tackle the deterministic version of this problem is described in Chapter 4 based on dynamic programming. Despite limitations on the size of problems that could be tackled, this approach led to relevant insights regarding the problem structure. In Chapter 5, a new mathematical programming model is proposed, as well as a matheuristic based on the decomposition of the problem. This matheuristic hybridizes a genetic algorithm with mathematical programming. In Chapter 6, uncertainty is considered in the capacity-pricing problem. The mathematical programming model is extended to include uncertain demand and to consider other realistic issues. Moreover, an innovative matheuristic is proposed, also based in genetic algorithms and mathematical programming, which simultaneously generates solutions and scenarios.

The main contributions of each chapter are mainly related with the expansion of knowledge and advancement of the car rental application field and of the state-of-the-art of relevant methodologies. This chapter summarizes the contributions of this thesis present in each chapter and discusses relevant future work.

Firstly, Chapter 2 aims to structure the field of fleet and revenue management in car rental, and presents a published paper:

* Oliveira, B.B., Carravilla, M.A., Oliveira, J.F., 2017. **Fleet and revenue management in car rental companies: A literature review and an integrated conceptual framework.** *Omega* 71, 11–26.

The first highlight of this paper is an in-depth literature review and discussion on car rental fleet management issues. Moreover, a novel conceptual framework for the car rental fleet management problem is proposed, which integrates operational and revenue-oriented decisions and interactions. From the analysis of the framework and relevant literature, four

future research directions are identified. They are related with the integration between decisions, namely "horizontal-level" ones such as capacity and pricing, and the applicability of models, namely the consideration of relevant realistic elements such as uncertainty. The framework proposed has the potential to support the development of this field of research, and can be expanded to other innovative transportation models, especially shared mobility systems. These systems have similar characteristics to car rental, in terms of flexibility and decision-making and can benefit from this and other works.

Chapter 3 described the practical motivation for this research, arising from the development of a pricing system for *Guerin Car Rental Solutions*, presented in the following paper:

* Oliveira, B.B., Carravilla, M.A., Oliveira, J.F., Raicar, P., Acácio, D., Ferreira, J., Araújo, P., 2015. **Pricing for internet sales channels in car rentals**, in: Póvoa A., de Miranda J. (eds), *Operations Research and Big Data*. Studies in Big Data, vol 15, pp. 139–147. Springer, Cham

The algorithm developed and the decision-support system designed enable the company to swiftly adapt and improve its pricing position in relation to its fleet occupation and the prices of competitors, increasing its revenue. The specific future work is related with improving information that feeds key parameters of the algorithm and with the improvement of the quality of the solutions obtained by using more advanced methodologies. Besides these possible improvements to the developed system, this work raised other interesting avenues of research, namely the potential to integrate capacity-inducing decisions and pricing.

After establishing the theoretical and practical motivation for this research, the following chapters describe methodologies to deal with the integration of capacity and pricing decisions in car rental. Chapter 4 describes the first attempt to tackle the deterministic version of the problem based on dynamic programming, resulting on the following paper:

* Oliveira, B.B., Carravilla, M.A., Oliveira, J.F., 2017. **A dynamic programming approach for integrating dynamic pricing and capacity decisions in a rental context**, in: Vaz, A.I.F., Almeida, J., Oliveira, J.F., Pinto, A.A. (Eds.), *Operational Research. APDIO 2017*. Springer Proceedings in Mathematics & Statistics, vol 223, pp. 297–311. Springer, Cham

This paper presented a new dynamic programming model to tackle the capacity-pricing integration in car rental. Some limitations regarding the size of the problem hinder the application of this type of methodology to tackle this problem. Nevertheless, relevant insights regarding the problem structure were drawn from this work. The most important is related with the fact the rental context that makes capacity re-usable (and not depleted by demand throughout time) is the main characteristic that makes this problem unique and that raises the need for innovative solution methods. Moreover, this work allowed for a relevant methodological discussion on the potential and limitations of Constraint Programming and Mixed Integer Non Linear Programming as part of a discrete dynamic programming approach. Future work is focused on developing methodologies that are able to deal with the complexity of the problem and its realistic size.

Chapter 5 meets the "future work" mentioned above, by presenting a well-performing innovative approach for the same problem, that is able to deal with its complexities and realistic size. The work is presented in the following paper:

* Oliveira, B.B., Carravilla, M.A., Oliveira, J.F., 2017. **Integrating pricing and capacity decisions in car rental: a matheuristic approach**, submitted and under second round of review at *Omega*.

Different methodological and problem-related highlights are present in this paper. Firstly, a new mathematical model that integrates car rental capacity and pricing decisions is proposed, allowing for the problem to be fully defined. In addition, a matheuristic approach that obtains good solutions for real-world sized instances is developed. It is based on a model decomposition, guided by a genetic algorithm. To improve the procedure performance, a structured design of search-boosting initial solutions for the genetic algorithm is proposed. Finally, to validate the practical motivation for this problem, the value of applying this integrated approach when compared with a hierarchical/sequential one was quantified. Future work is twofold: first, the further development of demand modeling, by including competition aspects and uncertainty, would improve the applicability and realism of the model; secondly, once again the parallel with other innovative sectors, such as shared mobility systems, arise as a potential extension of this work.

The last paper is presented in Chapter 6 and tackles the integrated capacity-pricing problem under uncertainty:

* Oliveira, B.B., Carravilla, M.A., Oliveira, J.F., Costa, A.M., 2018. **A co-evolutionary matheuristic for the car rental capacity-pricing stochastic problem**, submitted and under review at *European Journal of Operational Research*.

The main contributions of this paper follow two main directions. On the one hand, a new two-stage mathematical programming model is proposed, extending the one presented in the previous chapter. On the other hand, an innovative methodology to tackle the problem is proposed, based on the decomposition of the model in decisions made before and after uncertainty is revealed. As far as the model is concerned, its main innovative features are the consideration of uncertainty, the modeling of demand price-sensitivity, and the increased realism of considering vehicle group price hierarchy constraints. As for the methodology, its main highlight is that it generates solutions to the first-stage decisions and scenarios in parallel, requiring little information on random variables to do so. This methodology is easily adaptable to different decision-making risk profiles and can be implemented and run in reasonable time in a decision-support system. Moreover, it can be extended to other two-stage stochastic problems where information on uncertainty is scarce. Future work is also identified concerning the application, mainly related with the extension of the model towards a more tactical scope, including decisions regarding multi-stage-oriented pricing and deployment actions.

Overall, this body of research can lead to two relevant directions of future work: a problem-oriented extension and a methodology-oriented extension. In the former, the work

developed for car rental can be extended to innovative transportation systems, such as car sharing. This emerging transportation trend is becoming key to improve city logistics and important characteristics are similar in these two fields: fleet mobility and flexibility, importance of keeping high occupation rates, and the ability to use prices to manage demand. The extension should mainly focus on fleet free-floating, which is the absence of mandatory pick-up and drop-off points. This motivated the submission of a research proposal to FCT (Foundation of Science and Technology, Portugal) in 2017, in cooperation with the University of Coimbra. As for the latter, the innovative methodology developed to generate scenarios in parallel with solutions should be further developed and validated, applying to different stochastic problems where bounds on solution quality and scenarios are known. From here, the methodology could be enhanced in terms of solution and scenario quality and run time, making it applicable to several real-world problems where uncertainty information is scarce. This extension motivated the submission of a research proposal to ARC (Australian Research Council) in 2018, in cooperation with the University of Melbourne.