Combinatorics and degenerations in algebraic geometry: Hurwitz numbers, Mustafin varieties and tropical geometry

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PhD thesis

submitted by

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ZUSAMMENFASSUNG

Degenerationstechniken sind mächtige Werkzeuge in der Untersuchung verschiedener Objekte der algebraischen Geometrie. Sie ermöglichen es geometrische Objekte mit Hilfe kombinatorischer Methoden zu untersuchen. Diesem großen Nutzen geht jedoch häufig die große Herausforderung voraus den Zusammenhang zur Kombinatorik zu konkretisieren. Das junge Gebiet der tropischen Geometrie bietet einen konzeptionellen Rahmen, um diesen Zusammenhang herzustellen. In dieser Arbeit untersuchen wir verschiedene Probleme der algebraischen Geometrie mit Hilfe degenerativer und kombinatorischer Methoden. Diese Probleme lassen sich wiederum in drei Themengebiete unterteilen: Enumeration von Überlagerungen (Hurwitz Zahlen), Degenerationen von projektiven Räumen (Mustafin varietäten), Modulräume tropischer Kurven und treue Tropikalisierungen.

Im Themengebiet der Enumeration von Überlagerungen gliedern wir diese Arbeit in drei Teile, welche sich mit zwei Problemstellungen beschäftigen.

In den ersten zwei Teilen untersuchen wir drei Varianten der *doppelten Hurwitz Zahlen* auf ihr polynomielles Verhalten: Einfache, monotone und strikt monotone doppelte Hurwitz Zahlen. Diese Zahlen enumerieren verzweigte Überlagerungen der komplexen projektiven Gerade für gegebenes Verzweigungsdatum und

folgen einem stückweise polynomiellen Verhalten im Konfigurationsraum des Verzweigungsdatums. Für diese Arbeit führen wir mit Hilfe von kombinatorischen Interpolationen einen gemeinsamen Rahmen für diese Varianten ein, so dass eine simultane Studie dieser Zahlen möglich ist.

Im ersten dieser Teile führen diese Untersuchung mit Hilfe von tropischer Geometrie und Methoden der Ehrhart Theorie durch. Insbesondere stellen wir Algorithmen zur Verfügung welche diese polynomielle Struktur berechnen. Desweiteren decken wir eine rekursive Struktur im sogenannten *Geschlecht* 0 Fall auf, die die Veränderung des polynomiellen Verhaltens bei Bewegung im Konfigurationsraum beschreibt. Dieser Teil der Arbeit basiert auf dem Preprint [Hah17a] und verallgemeinert ihn.

Im zweiten dieser Teile untersuchen wir das polynomielle Verhalten der oben aufgeführten Hurwitz Zahlen aus darstellungstheoretischer Perspektive. Bereits Adolf Hurwitz stellte einen Zusammenhang zwischen Hurwitz Zahlen und der symmetrischen Gruppe her. Moderne Entwicklungen erlauben eine Erweiterung dieses Zusammenhangs und eine Studie von Hurwitz Zahlen in der Sprache des Fockraums. Dieser erlaubt es die Kombinatorik von Degenerationen in der Sprache von Operatoren zu verfolgen. Wir benutzen den sogenannten *fermionischen Fockraum*, um explizite Formeln für die Polynome der oben aufgeführten Hurwitzzahlen herzuleiten. Dies führt uns zu einer rekursiven Stuktur des polynomiellen Verhaltens bei Bewegung im Konfigurationsraum auf der Ebene der erzeugenden Funktionen dieser Varianten von Hurwitz Zahlen. Dieser Teil basiert auf dem Preprint [HKL17], welcher eine gemeinsame Arbeit des Autors mit Reinier Kramer und Danilo Lewanski darstellt.

Im dritten Teil beschäftigen wir uns mit sogenannt *Pillowcase-Überlagerungen.* Dies sind Überlagerungen einer Orbifaltigkeit mit vier Orbifaltigkeitspunkten, deren grundlegender topologischer Raum der Torus ist. Überlagerungen des Torus und der Pillowcase-Orbifaltigkeit hängen mit der Theorie der quasimodularen Formen zusammen [Dij95; KZ95; EO06]. Eine Formulierung dieses Zusammenhangs für den Torus in der Sprache der tropischen Geometrie wurde in [BBBM14] und [GM16] erarbeitet. In dieser Arbeit legen wir den Grundstein für eine solche Formulierung für die Pillowcase-Orbifaltigkeit, indem wir einen tropischen Korrespondenzsatz beweisen, der Pillowcase-Überlagerungen mit Hilfe von tropischer Geometrie enumeriert. Dieser Teil basiert auf fortlaufender Arbeit des Autors.

Im Themengebiet der Degenerationen von projektiven Räumen beschäftigen wir uns mit sogenannten *Mustafin Varietäten.* Dies sind Degenerationen von projektiven Räumen, welche von einer Wahl von Punkten im Bruhat-Tits Gebäude der projektiven linearen Gruppen induziert werden, was äquivalent zu einer Wahl von endlich vielen invertierbaren Matrizen der gleichen Dimension ist. In einer wegweisenden Arbeit wurde in [CHSW11] ein Zusammenhang von Mustafin Varietäten zu der Theorie der gemischten Unterteilungen von Polytopen hergestellt, was wiederum zu einer Verbindung zur tropischen Geometrie führte. In dieser Arbeit vertiefen wir diese Verbindung und nutzen tropische konvexe Hüllen und tropische Schnitttheorie, um Mustafin Varietäten zu untersuchen. Unsere Arbeit führt uns insbesondere zu einer vollständigen Klassifikation der sogenannten *Komponenten der speziellen Faser.* Eine Anwendung unserer Methode ist die Aufdeckung einer Verbindung der Theorie der Mustafin Varieäten mit sogenannten *vor-verknüpften Grassmannschen*, die wichtige Objekte in der Theorie der "Limit linear series" darstellen [Oss06]. Diese Arbeit basiert auf dem Preprint [HL17], welcher eine gemeinsame Arbeit des Autors mit Binglin Li darstellt.

Im Themengebiet bezüglich der Modulräume tropischer Kurven und treuen Tropikalisierungen beschäftigen wir uns mit dem tropischen Analogon des folgenden algebro-geometrischen Zusammenhangs: Jede nicht-hyperelliptische Geschlecht 3 Kurve kann als ebene Quartik in die projektive Ebene eingebettet werden. Dies ist für die naiven Definitionen der tropischen Analoga nicht wahr, was in [BJMS15] mit Hilfe aufwändiger computergestützter Methoden bewiesen wurde. In dieser Arbeit schlagen wir eine Lösung für dieses Problem vor, welche die Definition einer ebenen tropische Kurve erweitert und damit das tropische Analgon für nicht-realisierbar-hyperellitpische Kurven des oben genannten algebro-geometrischen Zusammenhangs herstellt. Wir erreichen dies mit Hilfe eines konstruktiven Beweises. Desweiteren untersuchen wir Einbettungen von abstrakten realisierbar-hyperelliptischen tropischen Geschlecht 3 Kurven und beweisen, dass diese Kurven nicht als tropische ebene Kurven eingebettet werden können. Unsere Methoden umfassen die Theorie der tropischen Modifikationen und Verfeinerungen, das Konzept treuer Tropikalisierungen und die Theorie von Divisoren auf abstrakten tropischen Kurven. Diese Arbeit basiert auf dem Preprint [HMRT18], welcher eine gemeinsame Arbeit des Autors mit Hannah Markwig, Yue Ren und Ilya Tyomkin darstellt.

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CONTENTS

CHAPTER 1

INTRODUCTION

In its classical setting, the field of *algebraic geometry* is concerned with the study of the geometry of zero loci of multivariate polynomial equations. From a modern perspective, algebraic geometry investigates geometric structures related to these zero loci by means of modern abstract algebra. Among many powerful techniques used for this aim is the notion of *degenerations*. Such a degeneration describes a transformation of a certain object of interest into a new object, such that this process preserves some of the relevant initial structure. Degenerations can enable the study of geometric objects from a combinatorial view point which then gives new insights into the underlying geometric structure. This approach features the challenge to master the underlying combinatorics of degenerations.

The use of degeneration techniques has been especially succesful in the field of *enumerative geometry*, in which we study enumerative questions in algebraic geometry. One of the earliest such question is the famous Problem of Apollonius, which asks for the number of circles, which are tangent to three given circles. While the solution to this problem can be obtained by fairly elementary considerations, there are other problems which are equally easy to pose but much harder to answer. As a consequence, the modern field of enumerative geometry is equipped with far more sophisticaded methods (such as degeneration technniques). As mentioned above, one of the main challenges concerning the use of degeneration techniques lies in the combinatorics. *Tropical geometry* offers a new and attractive framework to organise the combinatorics of degenerations in algebraic geometry. A common way to describe tropical geometry is as a combinatorial shadow of algebraic geometry. From a geometric view point "tropical geometry describes worst possible degenerations of the complex structure on an $n-$fold X" [Mik06a]. These degenerations then yield polyhedral structures, which inherit many structural properties. We call this process *tropicalisation*. This enables the practise of the following philosophy, which underlies tropical geometry: In order to answer a question in geometry, we translate it (if possible) to a question concerning the associated polyhedral structures. We then study these structures from the perspective of combinatorics and discrete geometry and solve the problem in this polyhedral setting. This way, we recover a solution for the initial question in the geometric setting. Among the origins of tropical geometry is the foundational work in [Mik05], in which Mikhalkin proved his famous correspondence theorem, which expresses the number of plane curves of fixed degree and genus passing through a fixed number of points in terms of tropical curves, which are certain piecewise linear graph embeddings. Another foundational contribution was made in [SS04] by Speyer and Sturmfels, which introduces tropical linear spaces and relates them to the theory of matroids. In the last decade tropical geometry has had several successes in various fields, such as proofs of Kontsevich's formula [GM07b] and the Caporaso-Harris formula [GM07a] in enumerative geometry, a proof of the maximal rank conjecture for quadrics [JP16] in algebraic geometry and a proof of the log-concavity conjecture in matroid theory [HK12].

Moreover, many connections to other fields such as non-archimedean geometry [Pay09; BPR16; BPR13; GRW15; GRW16] or complexity theory [ABGJ17] were established.

In this thesis, we take a combinatorial approach to several problems in algebraic geometry with a view towards enumerative geometry and moduli spaces.

Our key tool will be degeneration techniques, especially in the context of tropical geometry. We now make this more precise.

1.1 A thematic overview

The topics in this thesis can be clustered into three themes, forming one chapter each:

1. Enumeration of covers: Hurwitz numbers (chapter 3);
2. Degenerations of projective spaces: Mustafin varieties (chapter 4);
3. Moduli spaces of tropical curves and faithful tropicalisations (chapter 5).

We give brief overviews on each of those themes and refer to the respective chapter introduction for more in-depth discussions.

1.1.1 Enumeration of covers: Hurwitz numbers

Hurwitz numbers are topological invariants counting branched coverings of Riemann surfaces with fixed ramification data. These enumerative objects were introduced by Adolf Hurwitz in the late 19th century [Hur91] and are related to various different branches of mathematics, such as algebraic geometry, combinatorics, representation theory of the symmetric group, free probability theory, quantum field theory and many more. Degeneration techniques are an important tool in the study of Hurwitz numbers and tropical geometry proves to be a framework to master the underlying combinatorics [CJM10; CJM11; BBM11]. They admit definitions in several settings equivalent to the definition in terms of branched coverings, e.g. one can define Hurwitz numbers in terms of factorisations in the symmetric group with fixed cycle types.

Single Hurwitz numbers. The invariants Hurwitz originally studied are now called *single Hurwitz numbers*. These numbers count genus g, degree d coverings of the complex projective line $\mathbb{P}^1$ with fixed ramification profile ν over

∞ and simple ramification over b other points, where b is determined by the Riemann-Hurwitz formula. After almost a century of seldom attention by the mathematial community, surprising and fruitful connections to Gromov-Witten theory found in the 1990s sparked new interest in the study of Hurwitz numbers, where some cases have proved to be of particular interest. Especially the celebrated ELSV formula [ELSV99; ELSV01], which expresses single Hurwitz numbers in terms of intersection numbers on the Deligne-Mumford compactification of the moduli space of genus g curves with $n = \ell(\nu)$ marked points $\overline{\mathcal{M}}_{g,n}$ launched a number of interactions between Hurwitz numbers and Gromov-Witten theory. An immediate consequence of this formula is the fact that single Hurwitz numbers are polynomial in the entries of ν up to a combinatorial factor. On the other hand, the ELSV formula has been helpful in understanding the intersection theory of $\overline{\mathcal{M}}_{g,n}$, as Kazarian derived in [Kaz09] many important results like a proof of Witten's conjecture, Virasoro constraints and a proof of Faber's λ_g conjecture directly from the ELSV formula.

Double Hurwitz numbers. A natural generalisation of single Hurwitz numbers was introduced by Okounkov in [Oko00] as what is nowadays called *simple double Hurwitz numbers*. These numbers count genus g, degree d coverings of the complex projective line $\mathbb{P}^1$ with fixed ramification profile μ over 0, ν over ∞ and simple ramification over b other points, where b is determined by the Riemann-Hurwitz formula. We denote simple double Hurwitz numbers by $h_g(\mu, \nu)$. One of the main open problems regarding double Hurwitz numbers is to derive an ELSV-type formula, expressing simple double Hurwitz numbers in terms of the intersection theory of a suitable moduli space. In their foundational paper [GJV05], Goulden, Jackson and Vakil studied this problem with the philosophy that possible polynomial behaviour of simple double Hurwitz numbers should be viewed as evidence for an ELSV-type formula and that understanding such behaviour should give indications on the conrete shape of such a formula. In fact, they proved that simple double Hurwitz numbers behave piecewise polynomially in the entries of μ and ν, giving the configuration space of these partitions a chamber structure carrying the polynomial behaviour. This lead Goulden, Jackson and Vakil to give a concrete conjecture on an ELSV-type formula for simple double Hurwitz numbers when $\mu = (d)$, which they proved in genus 0 and 1.

In [SSV08], the study of the nature of this polynomiality was deepened in genus 0. In their work Shadrin, Shapiro and Vainshtein proved that when moving from a chamber to one of its neighbours, the difference of the corresponding polynomials may be expressed in terms of simple double Hurwitz numbers with smaller input data. This is called a *recursive wall-crossing structure*. Later, this result was generalised to arbitrary genus in [CJM10; CJM11] using tropical geometry and in [Joh15] using a representation theoretic appraoch involving the so-called *fermionic Fock space* and *semi-infinite wedge formalism*.

As mentioned above, Hurwitz numbers admit a description in terms of factorisations in the symmetric group. In this setting, single and simple double Hurwitz numbers essentially count factorisations of permutations into a certain number of transpositions. There are several modifications on the conditions imposed on these transposition, leading e.g. to so-called *monotone*, *strictly monotone* or *pruned* Hurwitz numbers. These numbers provide rich structures and interesting applications [DK15; DN17; ALS16; GGN14; GGN16; KLS16; Hah17b].

Pillowcase covers. The generating function of the enumerative problem of counting torus coverings played an important role in Dijkgraaf's work [Dij95] on mirror symmetry for elliptic curves. It is an interesting fact (used in Dijkgraaf's work) that these functions are quasimodular forms, which was proved rigorously in [KZ95]. A connection to tropical geometry was uncovered in [BBBM14] in the context of tropical mirror symmetry for elliptic curves. This led to a refined quasimodularity statement, which was claimed in [BBBM14] and later proved in a more general sense in [GM16] using so-called *global graphs*. The enumeration of torus coverings is closely connected to Masur-Veech volumes of moduli spaces of flat surfaces.

A similar problem is the enumeration of pillowcase coverings, which is related to the moduli space of quadratic differentials. The pillowcase orbifold is obtained by equipping the torus with a complex involution. It was shown that the corresponding generating series is a quasimodular form in [EO06].

Many of the numerous variants and generalisations of Hurwitz numbers discussed above feature similar interesting properties. This thesis answers questions about the structure of generalisations of Hurwitz numbers and their tropical counterparts. For more details, see section 1.2.1.

1.1.2 Degenerations of projective spaces: Mustafin varieties

While a lot of progress has been made in the study of degenerations of curves, the nature of degenerations in higher dimensions have remained more mysterious. One approach of studying the higher dimensional case is considering degenerations of projective spaces, which are spaces surrounding higher dimensional varieties: If we can degenerate projective spaces, this induces degenerations for all contained embedded subvarieties. Moreover, an understanding of the properties of the degenerated surrounding spaces may give insights into the properties of the degenerated subvarieties. A concrete embodiment of this idea was introduced by Mustafin in his work on non-achimedean uniformisation [Mus78], where he generalised Mumford's construction in his seminal paper on the uniformisation of curves [Mum72] to higher dimensions. After appearing as Deligne schemes in works of Faltings [Fal01] and Keel-Tevelev [KT06], the degenerations constructed by Mustafin are nowadays known as *Mustafin varieties*. These varieties have a rich combinatorial structure made visible in pioneering work of Cartwright, Häbich, Sturmfels and Werner [CHSW11]. In particular, they are related to the beautiful theory of mixed subdivisions which yields a strong connection between Mustafin varieties and the combinatorics of tropical convex hull computations — another manifestation of tropical geometry as a tool to master the combinatorics of degenerations. Here, we deal with the combinatorics of special fibers of Mustafin varieties, see section 1.2.2.

1.1.3 Moduli spaces of tropical curves and faithful tropicalisations

When diving into the foundations of tropical geometry, we immediatly face the fact that the tropicalisation process looses information about the source object. Moreover, the combinatorial structure of the tropicalisation of a curve highly depends on its embedding. The idea of finding tropicalisations which lose as little information as possible has led to the notion of a *faithful tropicalisation*, which grew out of the connection between tropical geometry and Berkovich space theory (see e.g. [Pay09], [BPR16],[GRW15]). In particular, one can associate a *(Berkovich) skeleton* to any compact curve C. This skeleton carries the structure of a metric graph. In contrast to the tropicalisation, this skeleton does not depend on the embedding of C. Moreover, the relationship between Berkovich spaces and tropical geometry yields a natural map from the skeleton of C to the tropicalisation of C for any embedding. Those tropicalisations for which this map is an isometry of metric graphs are called *faithful*, i.e. they loose as little information of possible.

The problem of losing information in tropicalisations in the context of tropical moduli was illustrated in [BJMS15]: Starting with a non-hyperelliptic genus 3 curve and considering its canonical embedding, we obtain a plane quartic. In other words: There is a surjective map from the moduli space of plane quartics to the non-hyperelliptic locus of the moduli space of genus 3 curves. In [BJMS15] it was proved that this is not true for the naive definition of the tropical moduli space of plane quartics. More precisely, there are abstract tropical genus 3 curves that cannot be realised as a plane tropical quartic. In fact, the following result was proved in [BJMS15] for a reasonable measurement.

Theorem 1.1.1 ([BJMS15])**.** *About* 29.5% *of all abstract tropical genus* 3 *curves can be realised as a plane tropical quartic.*

Here, we suggest an natural extension of the definition of tropical plane curve for which the situation with quartics and genus 3 curves is in accordance with algebraic geometry, see section 1.2.3.

1.2 Outline of results

Here, we outline the results of this thesis. We note that more detailed discussions of our results in each theme can be found in the introductions of their respective chapters.

1.2.1 Enumeration of covers: Hurwitz numbers

In this chapter, which represents the largest part of this thesis, we conduct a study of Hurwitz numbers. We concentrate on two topics:

1. Piecewise polynomiality properties of Hurwitz-type counts

2. Tropical pillowcase covers

Piecewise polynomiality properties of Hurwitz-type counts

As mentioned before the study of piecewise polynomiality properties of simple double Hurwitz numbers has been an extensive subject of research. Due to the possibility of implied connections to Gromov-Witten theory it is natural to ask whether other variants of double Hurwitz numbers behave piecewise polynomially and admit a recursive wall-crossing structure, as well. We approach this problem from two perspectives: We take a *tropical approach* and a *Fock space approach*. Both can be view as capturing information about degenerations as explained in the following.

A tropical approach The relation between Hurwitz numbers and tropical geometry was first discovered in [CJM10], which led to a proof of the above mentioned wall-crossing formulae for simple double Hurwitz numbers in [CJM11] heavily involving tropical geometry. One way to establish this relationship is to view the branched coverings involved in the Hurwitz numbers computation as maps between curves and those maps can be tropicalised to maps between tropical curves. This connection was generalised to monotone double Hurwitz numbers in

[DK15]. Motivated by the success in the simple double Hurwitz numbers case, we use this connection to monotone Hurwitz numbers, extend it to strictly monotone Hurwitz numbers as well and develop algorithms computing the polynomials for these variants in terms of tropical curves and Ehrhart theory in algorithm 3.2.22 and algorithm 3.2.31. We note that the output of these algorithms are polynomials for strictly monotone double Hurwitz numbers as well. This is explained as we derive piecewise polynomial behaviour for strictly monotone double Hurwitz numbers in our work concerning the Fock space approach.

Furthermore, we derive a recursive wall-crossing structure for a refinement of monotone and strict monotone double Hurwitz numbers in genus 0 in theorem 3.2.39.

Our study of Hurwitz numbers from a tropical perspective is done in a unified approach by proving all results for interpolations between simple, monotone and strictly monotone double Hurwitz numbers (in the flavour of [GGN16]) which specialise to the extremal cases.

This work can be found in section 3.2, which is partially based on the preprint [Hah17a]. While the work in [Hah17a] is only concerned with interpolations of simple and monotone Hurwitz number, we extend it here to interpolations involving strictly monotone Hurwitz numbers as well.

A Fock space approach In order to derive a recursive wall-crossing structure for all genera, we take a representation theoretic approach in terms of the so-called *Fock space*. Using the Fock space approach, we can degenerate and disassemble the counting problem in such a way that each Fock space operator is in charge of one of the branch points. It was used in [Joh15] in order to derive wall-crossing formulae for simple double Hurwitz numbers (which differ from the ones in [CJM11]). Furthermore, a way of computing monotone and strictly monotone Hurwitz numbers in terms of the *fermionic Fock space* was shown in [KLS16]. We use these results and derive formulae for montone and strictly monotone double Hurwitz numbers, which imply piecewise polynomial behavior — this implication was unknown for strictly monotone double Hurwitz numbers prior to this work.

Moreover, we use these formulae to derive recursive wall-crossing formulae for a refinement of the generating functions of monotone and strictly monotone Hurwitz numbers in all genera.

As in the tropical approach, these results are proved in a unified manner by deriving them for an interpolation between simple, monotone and strictly monotone double Hurwitz numbers in such a way that the results specialise to the extremal cases.

This study was conducted by the author in a joint work with Reinier Kramer and Danilo Lewanski. It can be found in section 3.3, which is based on the preprint [HKL17].

Tropical pillowcase covers

We use the connection between Hurwitz numbers and tropical geometry discovered and explored in [CJM10; BBM11; MR15] in order to formulate and proof a correspondence theorem for pillowcase covers to *tropical pillowcase covers*. In particular, we introduce the notion of tropical pillowcase covers and give a weighted bijection to the algebro-geometric ones. In forthcoming work, we plan to address relations to quasimodularity questions and to Masur-Veech weights (see e.g. [GM16]).

This work can be found in section 3.4 and is based on ongoing work by the author.

1.2.2 Degenerations of projective spaces: Mustafin varieties

We study the combinatorics of the special fiber of Mustafin varieties. In particular, we give a new combinatorial interpretation of the special fiber of Mustafin varieties in terms of images of rational maps studied in [Li17], which appear as vision maps in multiview geometry [AST13], in theorem 4.1.1. This interpretation is goverened by tropical convex hull computations. Applications of tropical intersection theory lead us to the following theorem

Theorem 1.2.1. *The images of rational maps studied in [Li17] completely classify the irreducible components of the special fibers of Mustafin varieties.*

Moreover, we give an algorithm computing the number of irreducible components for any point configurations in terms of linear algebra. We further use Falting's interpretation of Mustafin varieties as a moduli space [Fal01] in order to reveal a connection to the theory of *limit linear series*. This theory, which was developed by Eisenbud and Harris in [EH86], is concerned with the behaviour of linear series under degenerations and has had many success (e.g. a proof that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$). In his work in [Oss06], Osserman developed a theory which yields a compactified version of the Eisenbud-Harris limit linear series theory. Here the main object of study are so-called *(pre)linked Grassmannians*. *Linked Grassmannians* were introduced in [Oss06] in the context of limit linear series. They were later generalised by Osserman to prelinked Grassmannians in [Oss14]. It was proved in [HO08], that linked Grassmannians are flat and reduced for weak constraints on the base scheme. For prelinked Grassmannians no criterion for flatness or reducedness is known. (Pre)linked Grassmannians are schemes representing a certain moduli functor parametrising tuples of rank r subbundles and are degenerations of Grassmannians. The focal point of study in understanding (pre)linked Grassmannians are their so-called *simple points*. As mentioned above, we reveal a connection between Mustafin varieties and (pre)linked Grassmannians. In a first step, we prove that Mustafin varieties given by a so-called *convex point configurations* are (pre)linked Grassmannians for rank 1 subbundles (see theorem 4.3.4). This leads us to a family of flat and reduced (pre)linked Grassmannians of rank 1. Using the connection to tropical geometry, we prove that the set of simple points is a dense open subset in these cases (see proposition 4.3.6).

This study was conducted by the author in a joint work with Binglin Li. It can be found in chapter 4, which is based on the preprint [HL17].

1.2.3 Moduli spaces of tropical curves and faithful tropicalisations

As mentioned before the relationship between the moduli space of genus 3 curves and the moduli space of plane quartics is not reflected by their naive tropical analogues. In this thesis, we aim to repair this defect by defining the notion of tropical plane curves more carefully.

Naively, tropical plane curves are defined to tropical curves in the standard tropical plane $\mathbb{R}^2$. In order to achieve our goal, we broaden this definition by allowing tropical curves in other planes. Before we expand on this, we discuss the implications of restricting to the standard tropical plane in the case of plane quartics: As mentioned before in algebro-geometric setting every non-hyperelliptic genus 3 curves is realised by a plane quartic. This is achieved by considering the so-called *canonical embedding* of any such non-hyperelliptic genus 3 curve. Consindering these canonically embedded curves, it turns out that their tropicalisations do not behave well in many case, i.e. they are *not* faithful. Moreover, as any embedding of a non-hyperelliptic genus 3 curve into the plane is a canonical embedding, we need to consider tropicalisations of quartics in planes in higher dimensions.

More precisely, for any abstract tropical genus 3 curves Γ, we want to find a classical genus 3 curve C with Berkovich skeleton Γ, such that there is an embedding of C into a plane, which yields a faithful tropicalisation. In [BPR16], it is proved that any curve admits an embedding yielding a faithful tropicalisation. However, this is not constructive and gives no indication of the degree of the embedding.

In this thesis, we approach this task of finding embeddings into planes yielding faithful tropicalisations by considering linear re-embeddings of canonically embedded curves. One way of producing such linear re-embeddings is the theory of *tropical modifications* and *tropical refinements*: In his 2006 ICM lecture [Mik06b], Grigory Mikhalkin proposed so-called modifications as a way of locally repairing embeddings of tropical plane curves. By choosing a lift of such a modification, we obtain linear re-embeddings of tropical plane curves and their surrounding planes (also called tropical refinement [IMS09; MMS12]). These linear re-embeddings can unfold the structure lost in the initial tropicalisation. This was made precise in [CM16b], where an algorithm was developed to construct faithful tropicalisations of elliptic curves in dimension 4.

In particular, we introduce the notion of *realisably hyperelliptic curves* and construct plane quartics in higher dimensions which tropicalise faithfully, such that any not realisably hyperelliptic tropical genus 3 curve is realised as a tropicalisation of a plane quartic (see theorem 5.1.1). Moreover, we use the theory of divisors on tropical curves in order to prove that realisably hyperelliptic abstract tropical genus 3 curves are not realised as tropicalisations of plane quartics.

This study was conducted by the author in a joint work with Hannah Markwig, Yue Ren and Ilya Tymokin. It can be found in chapter 5, which is based on the preprint [HMRT18].

CHAPTER 2

PRELIMINARIES

In this chapter, we review the basic notions of algebraic geometry, enumerative geometry, moduli space theory, combinatorics and degeneration techniques needed for this thesis. We begin in section 2.1 by recalling the basics of tropical geometry needed for each of the three themes we discuss. This is continued in section 2.2, where we discuss the foundations of Hurwitz theory. We introduce the basics of the theory surrounding Mustafin varieties in section 2.3. Finally, we review the specifics of the relationship between tropical plane quartics and abstract tropical genus 3 curves in section 2.4.

2.1 Tropical geometry

In this section, we recall the basics of tropical geometry required for this thesis. For a more detailed exposition on tropical geometry see e.g. [MS15].

In a sense, tropical geometry can be viewed as geometry of the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where $a \oplus b := \min(a, b)$ and $a \odot b := a + b$ or equivalently

$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where $a \oplus b := \max(a, b)$ and $a \odot b = a + b$. The former is the so-called min-*convention*, the latter the so-called max-*convention*. Unless specified otherwise, we will use the max-convention in this thesis. In principal, these notions are equivalent, however it is sometimes useful to use one or the other (or both simultaneously). We define *tropical polynomials* in n variables $x_1, \ldots, x_n$ to be finite expressions

$$F := \bigoplus_{\underline{v}} a_{\underline{v}} \odot \underline{x}^{\odot \underline{v}},$$

where we take the tropical sum over finitely many vectors $\underline{v} \in \mathbb{Z}^n$ and

$$\underline{x}^{\odot \underline{v}} = (v_1 \cdot x_1) \odot \cdots \odot (v_n \cdot x_n).$$

For a tropical polynomial F, the corresponding *tropical hypersurface* is given by

$$\operatorname{trop}(F) := \{\underline{y} \in (\mathbb{R} \cup \{-\infty\}) \mid \text{the maximum in } F(y) \text{ is attained at least twice}\}.$$

The theory of tropical hypersurfaces is closely connected to the beautiful theory of subdivisions of Newton polytopes. For more details see e.g. [RST05; Gat06; BIMS15]. Moreover, tropical hypersurfaces are related to algebro-geometric hypersurfaces by considering algebraic geometry over non-archimedean fields: Let K be a non-archimedean field with valuation $\operatorname{val} : K \to \mathbb{Q}$. For a Laurent polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, such that

$$f = \sum_{\underline{v}} a_{\underline{v}} \underline{x}^{\underline{v}},$$

where $\underline{x}^{\underline{v}} = x_1^{v_1} \cdots x_n^{v_n}$, we associate a tropical polynomial

$$F = \bigoplus_{\underline{v}} (-\operatorname{val}(a_{\underline{v}})) \odot \underline{x}^{\odot \underline{v}}.$$

In order to state the relationship between the algebro-geometric hypersurface $V(f)$ given as the zero locus of f in the torus $K^\times$ and the tropical hypersurfaces associated to F, we define the *tropicalisation map*

$$\begin{aligned}\mathrm{Trop}\colon (K^{\times})^n &\to \mathbb{R}^n\\ \underline{x} &\mapsto (-\mathrm{val}(x_1), \dots, -\mathrm{val}(x_n)).\end{aligned}$$

The connection is now given by the tropical fundamental theorem, which in the following form is also called Kapranov's theorem.

Theorem 2.1.1. *Let f be a Laurent polynomial over an algebraically closed non-trivially valuated field K and F the associated tropical polynomial, then*

$$\mathrm{trop}(F) = \overline{\mathrm{Trop}(V(f))},$$

where $\overline{\mathrm{Trop}(V(f))}$ denotes the Euclidean closure of $\mathrm{Trop}(V(f))$ in $\mathbb{R}^n$.

For homogenous Laurent polynnomials f, we will often consider the tropicalisation map to the $(n-1)$–dimensional *tropical torus* $\mathbb{TP}^n = \mathbb{R}^n/_{\mathbb{R}\mathbf{1}}$, where $\mathbf{1} = (1, \dots, 1)$ is the all-one vector.

Remark 2.1.2. As mentioned above, the theory of tropical hypersurfaces is related to the theory of Newton subdivisions. More precisely, the combinatorics of tropical hypersurfaces can be obtained by considering the dual complex of the induced subdivision of the Newton polytope of the polynomial. We call a tropical hypersurface which is dual to a subdivision of the d-simplex a *tropical degree d hypersurface*. In particular, a tropical curve which is dual to a subdivision of a d–triangle is called a *tropical degree d curve*.

Similarly, we can *tropicalise* any subvariety of the torus $K^{\times}$. This is the *extrinsic definition of tropical varieties*, i.e. a tropical variety is the tropicalisation of an algebro-geometric subvariety of the torus. In turns out that these tropicalisations always yield weighted polyhedral structures with special properties. This yields an intrinsic definition of tropical varieties in terms of polyhedral complexes satisfying the so-called *balancing condition*.

Definition 2.1.3. An *embedded tropical curve* C is a one dimensional polyhedral complex in $\mathbb{TP}^n$ with rational slopes, such that

1. for each e we associate a multiplicity $m_e \in \mathbb{Z}_{>0}$,
2. for each x, we have $\sum m_e v_e = 0$, where we take the sum over all edges e containing x and $v_e \in \mathbb{TP}^n$ is the primitive integral vector pointing into the direction for e. This condition is called the *balancing condition.*

Remark 2.1.4. It is important to note that for any curve $C \subset (K^\times)^n$ its tropicalisation is an embedded tropical curve. An example of a tropicalised plane quartic can be found in figure 5.5.

Moreover, the Berkovich theory and the theory of semistable models yields a way of tropicalising abstract curves: Let X be a smooth curve over K and $\mathfrak{X}$ a semistable regular model over the valuation ring R. We assume $\mathfrak{X}$ has bad reduction, then we consider the dual graph of the special fiber. For every component we introduce a vertex and assign the genus of the component to that vertex. Two vertices are adjacent if the respective components intersect. Moreover, we can give this graph a metric structure by assigning a length to each edge. This is achieved by considering the completion of the local ring at the intersection point, which is isomorphic to $R[[x,y]]/\langle xy - t^l \rangle$, where t is a uniformiser. Then l is the length of the edge. The graph we obtain this way is an abstract tropical curve, which we call the *tropicalisation of the abstract curve* X.

This construction is related to Berkovich theory: Starting with a curve X as before, we can consider its analytification X^{an}, which one can think of as the limit of all tropicalisations of X with respect to embeddings into quasiprojective toric varieties [Pay09]. The Berkovich space X^{an} is an infinite and infinitely branched graph with a metric structure, which does not depend on a chosen embedding. By choosing a semistable vertex set V (see [BPR13]), we obtain an induced finite metric graph with vertex set V which we denote by $\Sigma(X,V)$. Furthermore, for any V there exists a deformation retract

$$r_{\Sigma(X,V)} : X^{an} \twoheadrightarrow \Sigma(X,V).$$

If V is minimal, we call $\Sigma(X, V)$ a *minimal skeleton* of X. By [BPR13], these minimal skeletons exist. Further, the stable reduction theorem ensures uniqueness for non-rational smooth curves [BPR13]. In these cases we write $\Sigma(X)$. It turns out that the minimal skeleton of X coincides with the tropicalisation of X as an abstract curve in the sense above.

We consider an embedding i of X into a quasiprojective toric variety and denote the corresponding tropicalisation by $\mathrm{Trop}(X, i)$ (see e.g. [Pay09, section 2] for tropicalisations of toric embeddings). For any such i, there is a natural map

$$\mathrm{trop} : X^{an} \to \mathrm{Trop}(X, i).$$

In particular, we obtain a map

$$\mathrm{trop} : \Sigma(X) \to \mathrm{Trop}(X, i). \tag{2.1}$$

We call the tropicalisation of a curve X with respect to its embedding i *faithful* if equation (2.1) is an isometry on its image.

The above discussion can be viewed as the extrinsic definition of abstract tropical curves. In the following we give the respective intrinsic definition.

Definition 2.1.5. An *abstract tropical curve* is a connected metric graph Γ with unbounded edges called ends, together with a function associating a genus $g(V)$ to each vertex V. The ends are considered to have infinite length. The genus of an abstract tropical curve Γ equals $g(\Gamma) := b_1(\Gamma) + \sum_V g(V)$, where $b_1(\Gamma)$ is the first Betti number. An *isomorphism* of a tropical curve is an automorphism of the underlying graph that respects the lengths and the genus at vertices. The *combinatorial type* of a tropical curve is obtained by disregarding the metric structure.

We will use several aspects of tropical geometry in each of the three themes which underly this thesis. We subdivide this section according to their affilliation to each of the themes and begin in section 2.1.1 by discussing the theory of tropical covers, which we will use in chapter 3 and chapter 5. In section 2.1.2 we discuss the notions of tropical convexity and their relationship to tropical linear spaces, which will be in important in our study of Mustafin varieties in chapter 4. We further introduce the notion of divisors on tropical curves in section 2.1.3, which will be important in our study of tropical moduli spaces in chapter 5.

2.1.1 Tropical covers

In this section, we introduce the basics of tropical covers. Similarly to the theory of branched coverings between Riemann surfaces, we consider morphisms between abstract tropical curves. Moreover, we introduce tropical counterparts to the notions of ramification and branching and begin by defining morphisms between abstract tropical curves.

Definition 2.1.6. A continuous map $\pi : C_1 \to C_2$ between abstract tropical curves C_i is a *tropical cover* if

1. for each edge e of C_1, the image of the map π restricted to e is either an edge or a vertex of C_2,

2. for each edge e of C_1, the map π restricted to e is a dilitation by some non-negative integer $\omega_{e,\pi}$,

3. for any inner vertex v of C_1, let $e_1, \dots, e_k$ be the edges adjacent to $\pi(v)$, denote by $e_i^1, \dots, e_i^{l_i}$ the edges adjacent to v, such that $\pi(e_i^k) \subset e_i$, then we have the following *balancing condition*

$$\sum_{l=1}^{l_i} \omega_{e_i^l,\pi} = \sum_{l=1}^{l_j} \omega_{e_j^l,\pi}.$$

   We call the number on each side of the equations the local degree $d_{v,\pi}$ of π at v. (Literature in non-archimedean geometry refers to maps satisfying a balancing condition as *harmonic morphisms* [Cap14; ABBR15].)

Moreover, we call π a *tropical morphism* if it is a tropical cover satisfying the following condition.

4. for any inner vertex v of C_1, if l (resp. k) denotes the number of edges e of C_2 (resp. C_1 with $\omega_{e,\pi} > 0$) adjacent to $\pi(v)$ (resp. v), then we have the Riemann-Hurwitz condition

$$k - d_{v,\pi}(2g(\pi(v)) + l - 2) + 2g(v) - 2 \geq 0$$

   We denote the number on the left by $r_{v,\pi}$.

The map π is called *minimal* if it does not contract any leaves. We call two tropical morphisms $\pi : C_1 \to C_2$ and $\pi' : C_1' \to C_2'$ isomorphic if there exist an isomorphisms $f_1 : C_1 \to C_1'$ and $f_2 : C_2 \to C_2'$, such that the following diagram commutes

$$\begin{tikzcd} C_1 \arrow[r, "f_1"] \arrow[d, "\pi"] & C_1' \arrow[d, "\pi'"] \\ C_2 \arrow[r, "f_2"] & C_2' \end{tikzcd}$$

The *combinatorial type* of a tropical cover $\pi : C_1 \to C_2$ is the induced map between the combinatorial types of C_1 and C_2.

Remark 2.1.7. The Riemann-Hurwitz condition in the notion of a tropical morphism is motivated by the Riemann-Hurwitz condition for branched morphisms between Riemann surfaces. We make this more precise in remark 2.2.18.

We are now ready to introduce a notion of ramification for morphisms between tropical curves.

Definition 2.1.8. Let $\pi : C_1 \to C_2$ be a tropical morphism. A subset E of C_1, such that $\pi(E)$ is a point, is called a *ramification component* of π if E is a conencted component of $\pi^{-1}(\pi(E))$ and contains

1. an inner edge of C_1 or
2. an inner vertex v of C_1 with $r_{v,\pi} > 0$ or
3. a leaf v of C_1 adjacent to an edge e of weight $\omega_{e,\pi} > 1$.

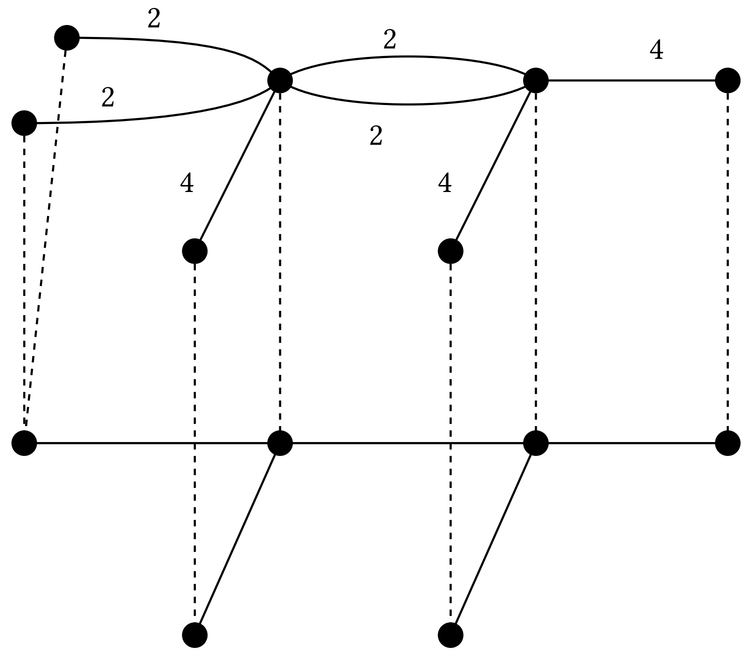


Figure 2.1: Example of a tropical morphism in the sense of definition 2.1.6. The numbers on the edges correspond to the weight of the map. The lengths of the edges of C_1 and C_2 are determined by each other via the weight condition.

If $p \in C_2$, such that $\pi^{-1}(p)$ does not contain a ramification component of π, we call p *unramified*. Moreover, let v be a leaf with $\pi^{-1}(v) = \{v_1, \ldots, v_k\}$ and v_i is a leaf adjacent to an edge e of weight $\omega_i = \omega_{e,\pi}$, then we call $(\omega_1, \ldots, \omega_k)$ the ramification profile of v.

2.1.2 Tropical convexity and tropical linear spaces

One of our main combinatorial tools in this thesis is the notion of tropical convexity. We restrict ourselves to basic notions and results and refer to [MS15] chapter 5.2 and [Jos14] chapter 5 for a more detailed introduction. Our proof of theorem 4.1.1 involves the identification of certain lattice points in so-called *tropical convex hulls*. We achieve this by means of tropical intersection theory of tropical linear spaces (i.e. tropical varieties of degree 1). Tropical intersection theory is a well-developed theory, for more details see e.g. [AR10] or [MS15].

We remark that in this subsection and in chapter 4, we will use the min-convention in the definition of tropical convexity. However, this induces computations of tropical linear spaces in the sense of the max-convention. We make this more precise in the following.

Tropical Convexity

In a sense tropical convexity is the notion of convexity over the tropical semiring $(\overline{\mathbb{R}}, \oplus, \odot)$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, $a \oplus b = \min(a, b)$ and $a \odot b = a + b$. We make this more precise in the following definition:

Definition 2.1.9. Let S be a subset of $\mathbb{R}^n$. We call S *tropically convex*, if for any choice $x, y \in S$ and $a, b \in \mathbb{R}$ we get $a \odot x \oplus b \odot y \in S$.
The *tropical convex hull* of a given subset V of $\mathbb{R}^n$ is given as the intersection of all tropically convex sets in $\mathbb{R}^n$ containing S. We denote the tropical convex hull of V by $\mathrm{tconv}(V)$.

This definition implies that every tropical convex set S is closed under tropical scalar multiplication. Thus, if $x \in S$ then so is $x + \lambda\mathbf{1}$, where $\lambda \in \mathbb{R}$ and $\mathbf{1} = (1, \ldots, 1)$. Therefore, we usually identify S with its image in the $(n-1)$-dimensional tropical torus $\mathbb{R}^n/\mathbb{R}\mathbf{1}$.

We are interested in tropical convex hulls of a finite number of points. We begin by treating the case of two points.

Proposition 2.1.10 ([DS04])**.** *The tropical convex hull of two points $x, y \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$ is a concatenation of at most $n-1$ ordinary line-segments. The direction of each line segment is a zero-one-vector.*

The proof of this proposition is constructive and describes the points in the tropical convex hull explicitly. We will use this fact in section 4.2.2 to prove our statements in the case of a 2–point configuration. To give this explicit description for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, we note that after relabelling and adding multiples of $\mathbf{1}$, we may assume $0 = y_1 - x_1 \le y_2 - x_2 \le \cdots \le y_n - x_n$. Then the tropical convex hull consists of the concatenation of the lines connecting the following points:

$$
\begin{aligned}
x =&(y_1 - x_1) \odot x \oplus y = (y_1, y_1 - x_1 + x_2, y_1 - x_1 + x_3, \dots, y_1 - x_1 + x_n)\\
&(y_2 - x_2) \odot x \oplus y = (y_1, y_2, y_2 - x_2 + x_3, \dots, y_2 - x_2 + x_n)\\
&\dots\dots\\
&(y_{n-1} - x_{n-1}) \odot x \oplus y = (y_1, y_2, \dots, y_{n-1}, y_{n-1} - x_{n-1} + x_n)\\
&(y_n - x_n) \odot x \oplus y = (y_1, \dots, y_n)
\end{aligned}
$$

Some of these points might coincide, however they are always consecutive points on the line segment. Next, we introduce a useful description of tropical convex hulls in terms of bounded cells of a tropical hyperplane arrangement: Fix a subset $\Gamma = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n/\mathbb{R}\mathbf{1}$, with $v_i = (v_{i1}, \dots, v_{in})$. Consider the *standard tropical hyperplane* at v_i in the max-plus algebra:

$$
\begin{aligned}
H_{v_i} = \{w \in \mathbb{R}^n/\mathbb{R}\mathbf{1} : \text{the } \textit{maximum} \text{ of } w_1 - v_{i1}, \dots, w_n - v_{in}\\
\text{is attained at least twice}\}.
\end{aligned}
$$

Taking the common refinement, we obtain a polyhedral complex structure on $\mathbb{R}^n/\mathbb{R}\mathbf{1}$, i.e. a subdivision into convex polyhedra. The union of the bounded cells of this complex coincides with the tropical convex hull $\mathrm{tconv}(\Gamma)$ (see e.g. chapter 5.2 in [MS15]).

Remark 2.1.11. The standard tropical hyperplane at a point v in the max-plus algebra is the tropicalisation of the standard hyperplane at a point $\tilde{v} \in (\mathbb{K}^\times)^n$, which tropicalises to v. Thus, this description implies that the computations involved in computing the tropical convex hull in the min-convention translates to combinatorics of tropicalised linear spaces in the max-convention.

In the following example we compute the tropical convex hull of three points

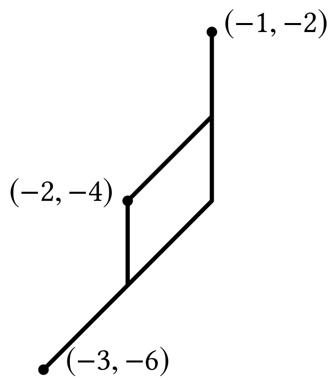


Figure 2.2: The tropical convex hull of $v_1 = (0, -1, -2)$, $v_2 = (0, -2, -4)$ and $v_3 = (0, -3, -6)$ in the tropical torus.

in the tropical torus.

Example 2.1.12. We pick 3 points

$$v_1 = (0, -1, -2),$$
$$v_2 = (0, -2, -4),$$
$$v_3 = (0, -3, -6).$$

Viewed as points in the tropical torus, we can identify these points with $\widetilde{v}_1 = (-1, -2)$, $\widetilde{v}_1 = (-2, -4)$ and $\widetilde{v}_3 = (-3, -6)$. The tropical convex hull is illustrated in figure 2.2.

Remark 2.1.13. The tropical convex hull of finitely many points is also called a *tropical polytope*. Tropical polytopes can be thought of as tropicalisations of polytopes over the field of real Puiseux series $\mathbb{R}\{\{t\}\}$ (see proposition 2.1 in [DY07]). One can generalise this notion to arbitrary tropical polyhedra and polyhedra over $\mathbb{R}\{\{t\}\}$, which in turn has applications in linear programming and complexity theory (see e.g. [ABGJ15]).

Tropical linear spaces

There are two notions of tropical linear spaces: An intrinsic purely combinatorial one given in terms of matroids and an extrinsic algebro-geometric one given as the tropicalisation of linear spaces. We will only talk about the extrinsic notion and refer to [MS15] for a more detailed discussion.

Example 2.1.14. Let L be the linear space in K^2 given by $x + y = 1 = 0$. The tropicalisation is the standard tropical line illustrated in figure 2.3.

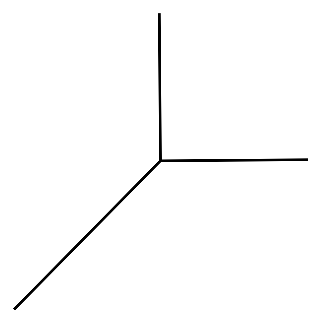


Figure 2.3: The standard tropical line.

In this thesis, we want to study intersections of tropicalised linear spaces. There are two notions of tropical intersection products: So-called *stable intersection* and *intersection products* as defined in [AR10]. However, it was proved in [Rau16] and [Kat12] that those two notions are equivalent. We start by defining stable intersection.

Definition 2.1.15. Let Σ_1, Σ_2 be tropicalised linear spaces in ${}^{\mathbb{R}^n}\!/\!_{\mathbb{R}\mathbf{1}}$. The stable intersection $\Sigma_1 \cap_{st} \Sigma_2$ is the polyhedral complex

$$\Sigma_1 \cap_{st} \Sigma_2 = \bigcup_{\substack{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2 \\ \dim(\sigma_1 + \sigma_2) = n}} \sigma_1 \cap \sigma_2.$$

All multiplicities are 1.

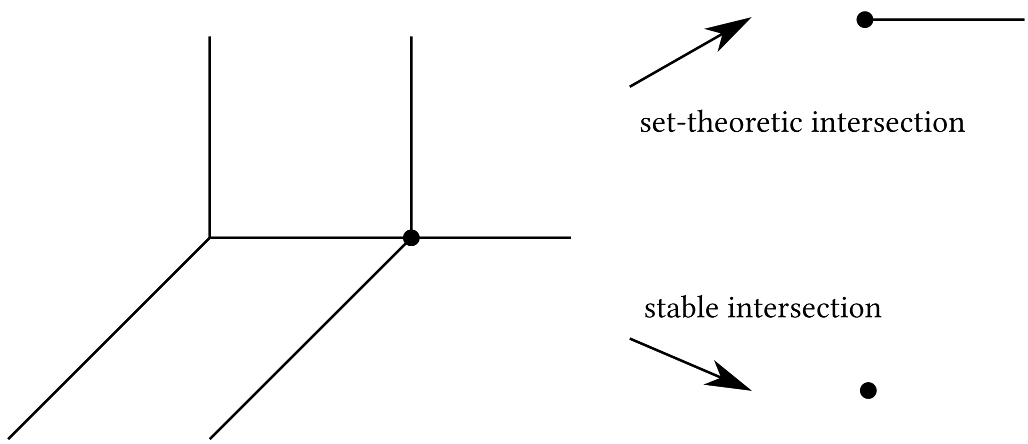


Figure 2.4: The difference between set-theoretic and stable intersection.

Example 2.1.16. We illustrate the difference between set-theoretic intersection and stable intersection in the example of two lines not in tropical general position. The two lines in figure 2.4 intersect set-theoretically in the half-bounded line segment as illustrated in the upper right. However, the stable intersection only yields a single point as illustrated in the lower right.

In algebraic geometry, two general linear spaces of respective codimension m_1 and m_2 intersect in codimension $m_1 + m_2$. A similar fact holds for tropical linear spaces.

Definition 2.1.17. A quadratic matrix $M \in \mathbb{R}^{r\times r}$ is *tropically singular*, if the minimum in

$$\det(M) = \bigoplus_{\sigma\in\mathcal{S}_r} m_{1\sigma(1)} \odot \cdots \odot m_{r\sigma(r)}$$

is attained at least twice. A point configuration $m_1, \ldots, m_n$ in $\mathbb{R}^d$ is in *tropical general position*, if every maximal minor of the matrix $((m_{ij})_{ij})$ is non-singular.

Theorem 2.1.18. *[MS15] Let X_1, X_2 be linear spaces contained in the $n-$dimensional torus T and let Σ_1, Σ_2 be weighted balanced rational polyhedral complexes whose supports are* $\mathrm{Trop}(X_1)$ *and* $\mathrm{Trop}(X_2)$ *respectively. There exists a Zariski dense subset*

$U \subset T$, *consisting of elements* $\mathbf{t} = (t_1, \ldots, t_n)$ *with* $\mathrm{val}(\mathbf{t}) = \mathbf{0}$, *such that*

$$\mathrm{Trop}(\mathrm{X}_1 \cap \mathbf{t}\mathrm{X}_2) = \Sigma_1 \cap_{st} \Sigma_2.$$

Let H be the standard tropical hyperplane at a point v in ${}^{\mathbb{R}^n}\!/_{\mathbb{R}\mathbf{1}}$, then we denote the $k-$fold stable self-intersection (i.e. $\underbrace{H \cap_{st} \cdots \cap_{st} H}_{\text{k times}}$) by H^k.

Moreover, for a polyhedral complex P of dimension d, we call its subcomplex P' consisting of all polyhedra in P of dimension smaller or equal to k, where $k < d$, *the* $k-$*skeleton of* P. The following fact was proved in [AR10].

Lemma 2.1.19. *Let* H *be the standard tropical hyperplane in* ${}^{\mathbb{R}^n}\!/_{\mathbb{R}\mathbf{1}}$ *rooted at a point* v. *Its* $k-$*fold stable self-intersection* H^k *is given by its* $(d-k)$*-skeleton with all weights equal to* 1.

Remark 2.1.20. Let $v_1, \ldots, v_n \in {}^{\mathbb{Z}^d}\!/_{\mathbb{Z}\mathbf{1}}$ in tropical general position, let $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$ and let H_i be the standard tropical hyperplane at v_i. Then the stable intersection coincides with set-theoretic intersection in the sense that

$$H_1^{m_1} \cap_{st} \cdots \cap_{st} H_n^{m_n} = H_1^{m_1} \cap \cdots \cap H_n^{m_n}.$$

Finally, since the standard tropical hyperplane is a tropicalised linear space, we obtain the following proposition as a consequence of the previous discussion.

Proposition 2.1.21. *Let* $v_1, \ldots, v_n \in {}^{\mathbb{Z}^d}\!/_{\mathbb{Z}\mathbf{1}}$ *and let* $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$, *such that* $\sum_{i=1}^{n} m_i = d - 1$. *Further, let* H_i *be standard tropical hyperplane at* v_i, *then*

$$H_1^{m_1} \cap_{st} \cdots \cap_{st} H_n^{m_n} = \{pt\}$$

is exactly one point with multiplicity one. If $v_1, \ldots, v_n$ *are in tropical general position, the set-theoric intersection*

$$H_1^{m_1} \cap \cdots \cap H_n^{m_n} = \{pt\}$$

coincides with the intersection product.

2.1.3 Divisors on abstract tropical curves

In this section, we introduce the fundamentals of the theory of divisors on abstract tropical curves. For a more in depth exposition, see e.g. [Bak08; HMY12]. We begin by defining the notion of a divisor on a tropical curve.

Definition 2.1.22. Let Γ be an abstract tropical curve. A *divisor* D is a finite $\mathbb{Z}$–linear combination $D = \sum_{x \in \Gamma} D(x) \cdot x$ of points of Γ.

Inspired by the algebro-geometric setting, we introduce tropical analoga to the classical notions.

Definition 2.1.23. Let D be a divisor on Γ. The *degree* of D is the sum of the coefficients $\sum_{x \in \Gamma} D(x)$. We call D *effective* if $D(x) \geq 0$ for all $x \in \Gamma$ and we write $D \geq 0$. The *support* of the divisor D is defined by $\operatorname{supp}(D) := \{x \in \Gamma \mid D(x) \neq 0\}$.

We further want to introduce the notion of *tropical linear systems*. We begin by defining tropical rational functions and their associated divisors.

Definition 2.1.24. Let Γ be a tropical curve. A *tropical rational function* is a function $f : \Gamma \to \mathbb{R}$, which is piecewise linear on each edge with finitely many pieces and integral slopes. For any point $x \in \Gamma$, we define the *order* $\operatorname{ord}_x(f)$ *of* f *at* x as the sum of all outgoing slopes at x. The principal divisor associated to f is given by

$$(f) := \sum_{x \in \Gamma} \operatorname{ord}_x(f) \cdot x.$$

Now that we have defined the notion of tropical rational functions, we are ready to define linear systems of tropical curves.

Definition 2.1.25. We call two divisors D and D' on an abstract tropical curve Γ *linearly equivalent* and write $D \sim D'$ if $D - D' = (f)$ for a tropical rational function f. For a divisor D, we denote the set of tropical rational functions f, such that the divisor $D + (f)$ is effective by $R(D)$ and define its *linear system* by $|D| := \{D + (f) \mid f \in R(D)\}$.

This leads us to a definition of very ampleness.

Definition 2.1.26. An effective divisor $D \geq 0$ on an abstract tropical curve Γ is called *very ample* if $R(D)$ seperates points, i.e. if for any $x, x' \in \Gamma$ there exists $f \in R(D)$, such that $f(x) \neq f(x')$.

Example 2.1.27. One of the most important examples of divisors is the *canonical divisor* K_Γ of a curve Γ, which is defined by

$$K_\Gamma := \sum_{x \in \Gamma} (\mathrm{val}(x) - 2)x.$$

2.2 Hurwitz numbers and Hurwitz-type counts

We begin by introducing the basic notions of Hurwitz numbers.

Definition 2.2.1. Let d be a positive integers, μ, ν two ordered partitions of d and let g be a non-negative integer. Moreover, let $q_1, \ldots, q_m$ be points in $\mathbb{P}^1$, where $b = 2g - 2 + \ell(\mu) + \ell(\nu)$. We define a Hurwitz cover of type (g, μ, ν) to be a holomorphic map $\pi : C \longrightarrow \mathbb{P}^1$, such that:

(1) C is a connected genus g curve,

(2) π is a degree d map, with ramification profile μ over 0, ν over ∞ and $(2, 1, \ldots, 1)$ over q_i for all $i = 1, \ldots, b$,

(3) π is unramified everywhere else,

(4) the pre-images of 0 and ∞ are labeled, such that the point labeled i in $\pi^{-1}(0)$ (respectively $\pi^{-1}(\infty)$) has ramification index μ_i (respectively ν_i).

We define an isomorphism between two covers $\pi : C_1 \longrightarrow \mathbb{P}^1$ and $\pi' : C_2 \longrightarrow \mathbb{P}^1$ to be a homeomorphism $\phi : C_1 \longrightarrow C_2$ respecting the labels, such that the following diagram commutes:

$$\begin{tikzcd} C_1 \arrow[r, "\phi"] \arrow[d, "\pi"] & C_2 \arrow[d, "\pi'"] \\ \mathbb{P}^1 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1 \end{tikzcd}$$

Then we define the simple double Hurwitz numbers as follows:

$$h_{b;\mu,\nu} = \sum \frac{1}{|\mathrm{Aut}(\pi)|},$$

where the sum goes over all isomorphism classes of Hurwitz covers of type (g, μ, ν). This number does not depend on the position of the q_i. The degree is implicit in the notation $h_{b;\mu,\nu}$, as $d = \sum \mu_i = \sum \nu_j$. The genus g of simple branch points is determined by the Riemann-Hurwitz formula, so $b = 2g - 2 + \ell(\mu) + \ell(\nu)$ as above.

When we drop the condition on C to be a *connected* curve and the labels in condition (4), we obtain the notion of *disconnected simple double Hurwitz numbers* $h^{\circ}_{b;\mu,\nu}$.

Remark 2.2.2. The notions of simple double Hurwitz numbers and disconnected simple double Hurwitz numbers determine each other by the inclusion-exclusion principle.

For $\sigma \in S_d$, we denote its cycle type by $C(\sigma) \vdash d$. We define the following factorisation counting problem in the symmetric group.

Definition 2.2.3. Let d, g, μ, ν be as in definition 2.2.1. We call $(\sigma_1, \tau_1, \dots, \tau_b, \sigma_2)$ a factorisation of type (g, μ, ν), if:

(1) $\sigma_1,\ \sigma_2,\ \tau_i \in \mathcal{S}_d$,

(2) $\sigma_2 \cdot \tau_b \cdot \dots \cdot \tau_1 \cdot \sigma_1 = \mathrm{id}$,

(3) $b = 2g - 2 + \ell(\mu) + \ell(\nu)$

(4) $C(\sigma_1) = \mu,\ C(\sigma_2) = \nu$ and $C(\tau_i) = (2, 1, \dots, 1)$,

(5) the group generated by $(\sigma_1, \tau_1, \dots, \tau_b, \sigma_2)$ acts transitively on $\{1, \dots, d\}$,

(6) the disjoint cycles of σ_1 and σ_2 are labeled, such that the cycle i has length μ_i.

We denote the set of all factorisations of type (g, μ, ν) by $\mathcal{F}(g, \mu, \nu)$.

A well-known fact is the following theorem, which is essentially due to Hurwitz.

Theorem 2.2.4. *Let g, μ, ν as in the previous definition, then*

$$h_{b;\mu,\nu} = \frac{1}{d!} \left|\mathcal{F}(g, \mu, \nu)\right| .$$

Remark 2.2.5. We can modify theorem 2.2.4 for disconnected simple double Hurwitz numbers by dropping the transitivity condition in definition 2.2.3.

As proved in [GGN14] monotone double Hurwitz numbers appear as the coefficients of the HCIZ-integral. They can be defined as counts of factorisations as in definition 2.2.3 by imposing an additional condition on the transpositions:

Definition 2.2.6. Let k be a non-negative integer. We call $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ a *monotone factorisation* of type (g, μ, ν), if it is a factorisation of type (g, μ, ν), such that:

(7a) If $\tau_i = (r_i\ s_i)$ with $r_i < s_i$, we have $s_i \leq s_{i+1}$ for all $i = 1, \ldots, b-1$.

Let $\vec{\mathcal{F}}(g, \mu, \nu)$ be the set of all monotone factorisations of type (g, μ, ν). Then we define the *monotone double Hurwitz number* to be:

$$h^{\leq}_{b;\mu,\nu} = \frac{1}{d!} \left|\vec{\mathcal{F}}(g, \mu, \nu)\right| .$$

Remark 2.2.7. Instead of requiring $s_i \leq s_{i+1}$ is condition (7a), we can also require $s_i < s_{i+1}$, which yields the notion of *strictly monotone double Hurwitz numbers.*

In [GGN16], a combinatorial interpolation between simple and monotone double Hurwitz numbers was introduced. The idea is to impose the monotonicity condition (7) only on the first k transpositions.

Definition 2.2.8. Let d, g, μ, ν be as in definition 2.2.1 and let k be a non-negative integer. We define a mixed factorisation of type (g, μ, ν, k) to be a factorisation $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ of type (g, μ, ν) satisfying the following additional condition:

(7b) If $\tau_i = (r_i\ s_i)$ with $r_i < s_i$, we have $s_i \leq s_{i+1}$ for all $i = 1, \ldots, k-1$.

Let $\mathcal{F}(g, \mu, \nu, k)$ be the set of all mixed factorisations of type (g, μ, ν, k). Then we define the *mixed Hurwitz number* to be:

$$h^{(2),\leq}_{k,b-k;\mu,\nu} = \frac{1}{d!} \left| \mathcal{F}(g, \mu, \nu, k) \right| .$$

Fixing the length μ and ν, we can view mixed Hurwitz numbers as a function

$$\begin{aligned} h^{(2),\leq}_{k,b-k} : \mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)} &\to \mathbb{Q} \\ (\mu, \nu) &\mapsto h^{(2),\leq}_{k,b-k;\mu,\nu}, \end{aligned}$$

where $\ell(\mu)$ (resp. $\ell(\nu)$) is the length of μ (resp. ν). For each $I \subset \{1, \ldots, \ell(\mu)\}$, $J \subset \{1, \ldots, \ell(\nu)\}$ we obtain linear equations $\sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$, where the μ_i (resp. ν_j) are the coordinates in $\mathbb{N}^{\ell(\mu)}$ (resp. $\mathbb{N}^{\ell(\nu)}$). The equations induce a hyperplane arrangement $\mathcal{W}$ in $\mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)}$. By considering the complement on $\mathcal{W}$ this hyperplane arrangement divides $\mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)}$ into chambers C.

Theorem 2.2.9 ([GJV05], [GGN16])**.** *The function $h^{(2),\leq}_{k,b-k}$ described above is piecewise polynomial, i.e. for each chamber C there exists a polynomial $P^{(2),\leq}_{k,b-k}(C) \in \mathbb{Q}[\underline{M}, \underline{N}]$, where $\underline{M} = M_1, \ldots, M_{\ell(\mu}$ and $\underline{N} = N_1, \ldots, N_{\ell(\nu)}$, such that $h^{(2),\leq}_{k,b-k;\mu,\nu} = P^{(2),\leq}_{k,b-k}(C)(\mu, \nu)$.*

2.2.1 Tropical Hurwitz numbers

In this section, we recall the notions of tropical Hurwitz numbers and their correspondence to algebro-geometric Hurwitz numbers introduced in [CJM10] and [BBM11]. We further summmarise the tropical interpretation of monotone Hurwitz numbers in [DK15].

Hurwitz numbers in terms monodromy graphs

As a first step we associate maps between graphs to certain factorisations in the symmetric group in the following construction.

Construction 2.2.10. Let $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ be a factorisation of type (g, μ, ν) with σ_1 as in equation (3.3). We associate a graph with labeled vertices and edges and with a map to the interval $[0, \ldots, b+1]$ as follows:

Constructing the graph.

1. We start with $\ell(\mu)$ vertices over 0, labeled by $\sigma_1^1, \ldots, \sigma_1^{\ell(\mu)}$. We will call these vertices *in-ends*. Moreover, we attach an edge e_v to each vertex v over 0 which maps to $(0,1)$. We label these edge attached to the vertex labeled σ_1^j by the same label.

2. We define $\Sigma_{i+1} = \tau_i \cdots \tau_1 \sigma_1$ for $i = 1, \ldots, m$, $\Sigma_0 = \sigma_1$ and $\Sigma_{b+1} = (\sigma_2)^{-1}$. Comparing Σ_i and Σ_{i+1}, the transposition τ_i either joins two cycles of Σ_i or cuts one cycle in two.

Assuming the preimage of $[0, i)$ has been constructed, repeat the steps (3)–(4) until $i = b$:

(3a) **[Join]** If τ_i joins the cycles Σ_{i-1}^s and $\Sigma_{i-1}^{s'}$ to a new cycle Σ', we create a vertex over i labeled τ_i. This vertex is joined with the edges corresponding to Σ_{i-1}^s and $\Sigma_{i-1}^{s'}$. These edges map to some interval (a_s, i) and $(a_{s'}, i)$ respectively, where a_s (resp. $a_{s'}$) is the image of the other vertex adjacent to the edge corresponding to Σ_{i-1}^s (resp. $\Sigma_{i-1}^{s'}$). We call those edges the *incoming edges at* τ_i. Moreover, we attach an edge to τ_i mapping to $(i, i+1)$, which we label by Σ'. We call this edge the *outgoing edge at* τ_i.

(3b) **[Cut]** If τ_i cuts Σ_{i-1}^s into Σ' and Σ'', we create a vertex over i labeled τ_i. We attach one edge connecting τ_i to the edge corresponding to Σ_{i-1}^s, which maps to (a_s, i) as above and attach two edges mapping to $(i, i+1)$ labeled Σ' and Σ'' respectively. As above, we call the edge mapping to (a_s, i) the *ingoing edge at* τ_i and the edges mapping to $(i, i+1)$ *outgoing edges at* τ_i.

(4) We extend those edges which so far are only adjacent to one vertex, such

that the edge e maps to $(a_e, i+1)$, where a_e is the image of the vertex adjacent to e.

(5) When $i = b$ is reached, the leaves of the graph which are not adjacent to in-ends correspond to the cycles of Σ_{b+1}. We create vertices over $b+1$ which we label $(\sigma_2^{-1})^1, \ldots, (\sigma_2^{-1})^{\ell(\nu)}$ and connect the corresponding edges to those vertices.

Relabelling the graph.

(6) We drop the labels τ_i at the vertices of $1, \ldots, b$.

(7) We label the in-ends (resp. out-ends) by $1, \ldots, \ell(\mu)$ (resp. $(1, \ldots, \ell(\nu))$ according to the labels of σ_1 and σ_2.

(8) If a vertex or an edge is labeled by a cycle σ, we replace the label by the length of the cycle.

We obtain a graph Γ with a map to $[0, b+1]$. We call Γ together with the map the *monodromy graph of type* (g, μ, ν) *associated to* $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$.

An example for a more general version of this construction can be found in example 3.2.5.

Remark 2.2.11. We can view the graph Γ as an abstract tropical curve, where we can choose arbitrary lengths. Furthermore, the line associated to the interval $[0, b+1]$ is an abstract tropical curve as well (it is called the *tropical projective line*: the point 0 in the interval corresponds to 0 in the projective line, the point $[b+1]$ in the interval corresponds to ∞ in the projective line). The map $\Gamma \to [0, b+1]$ can be viewed as a tropical cover of degree d, where the edges adjacent to vertices over 0 (resp. ∞) yield the profile μ (resp. ν). The ramification points in Γ are the 3−valent vertices and the branch points in $[0, b+1]$ are their images.

With the next definition we obtain a classification of the graphs we obtain from construction 2.2.10.

Definition 2.2.12. A monodromy graph Γ of type (g, μ, ν) is a graph with a map to $[0, b+1]$ (where $b = 2g - 2 + \ell(\mu) + \ell(\nu)$) with the following properties:

Graph/Map conditions.

1. The graph Γ is a connected.
2. The first Betti number of Γ is g.
3. The map sends vertices to integers, we call the image i of a vertex its position. Moreover, the map sends edges to open intervals. For a vertex of position i, we call edges mapped to (a, i) for $a < i$ *incoming edges at* i and edges mapped to (i, a) for $a > i$ *outgoing edges at* i.
4. The graph has $\ell(\mu) + \ell(\nu)$ leaves. There are $\ell(\mu)$ leaves mapped to 0 labeled by $1, \ldots, \ell(\mu)$ and $\ell(\nu)$ leaves over $b+1$ labeled by $1, \ldots, \ell(\nu)$.
5. Over each integer $i \in [0, b]$, there is exactly one vertex which locally looks like one of the graphs in figure 2.5. We call these vertices *inner vertices.*

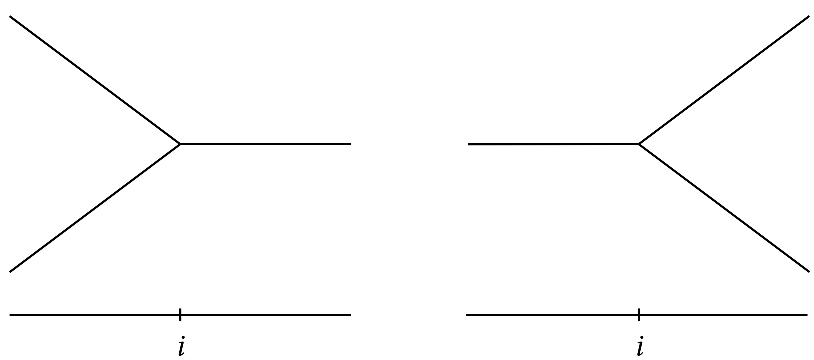


Figure 2.5: Local structure of the map for a monodromy graph.

Weight conditions.

4. We assign a positive integer weight $\omega(e)$ to each edge e. The in-end labeled i has weight μ_i. The out-end labeled j has weight ν_j.

5. At each inner vertex, the sum of the weights of incoming edges equals the sum of the weights of outgoing edges.

This is the *balancing condition for monodromy graphs*, which comes from the obeservation that by definition a monodromy graph is a combinatorial type of a tropical morphism in the sense of definition 2.1.6. An isomorphism of monodromy graphs $\Gamma \longrightarrow [0, b+1]$ and $\Gamma' \longrightarrow [0, b+1]$ of type (g, μ, ν) is a graph isomorphism $f : \Gamma \rightarrow \Gamma'$, such that

$$\begin{tikzcd} \Gamma \arrow[rr, "f"] \arrow[dr] & & \Gamma' \arrow[dl] \\ & {[0, b+1]} & \end{tikzcd}$$

commutes.

We now define tropical simple double Hurwitz numbers in terms of monodromy graphs.

Definition 2.2.13. Let d be a positive integers, μ, ν two ordered partitions of d and let g be a non-negative integer. Furthermore, let $b = 2g - 2 + \ell(\mu) + \ell(\nu)$, then we define the *tropical simple double Hurwitz numbers*

$$\mathbb{T}h^{(2)}_{b;\mu,\nu} = \sum \frac{1}{|\mathrm{Aut}\,(\Gamma \rightarrow [0, b+1])|} \prod \omega(e),$$

where we sum over all monodromy graphs $\Gamma \longrightarrow [0, b+1]$ of type (g, μ, ν) and take the product of all inner edges e of Γ.

Their relation to the algebro-geometric counterparts is given by the following theorem. It is proved by analysing how many factorisations of type (g, μ, ν) yield the same monodromy graph of type (g, μ, ν).

Theorem 2.2.14 ([CJM10]). *Let d be a positive integers, μ, ν two ordered partitions of d and let g be a non-negative integer. Furthermore, let $b = 2g - 2 + \ell(\mu) + \ell(\nu)$, then*

$$h^{(2)}_{b;\mu,\nu} = \mathbb{T}h^{(2)}_{b;\mu,\nu}$$

A similar approach was taken in [DK15] in order to give a tropical interpretation for monotone double Hurwitz numbers, when $\nu = (a, \ldots, a)$ for some positive integer a. We generalise this approach to monotone double Hurwitz numbers (and more general cases) in section 3.2.1.

Hurwitz numbers in terms of degenerated Riemann surfaces

A more general approach to the liason between Hurwitz numbers and tropical geometry was discovered in [BBM11]. In the following we explain a more intrinsic way of relating tropical geometry to Hurwitz numbers.

We begin by defining Hurwitz numbers in a more general sense.

Definition 2.2.15. Fix two Riemann surfaces S_1, S_2 and let g be a non-negative and d a positive integer. Moreover, let $\mathcal{M} = (\mu_1, \ldots, \mu_r)$ be a collection of partitions μ_i of d. Finally, let $\mathcal{Q} = \{q_1, \ldots, q_r\}$ be a set of points in S_2. We define a Hurwitz cover of type $(g, \mathcal{M}, \mathcal{Q}, S_2)$ to be a holomorphic map $\pi : S_1 \longrightarrow S_2$, such that

1. S_1 is of genus g,
2. π is a degree d map, with ramification profile μ_i over q_i,
3. π is unramified everywhere else.

We define an isomorphism between two covers $\pi : S_1 \longrightarrow S_2$ and $\pi' : S_1' \longrightarrow S_2'$ to be a homeomorphism $\phi : S_1 \longrightarrow S_1'$ respecting the labels, such that the following diagram commutes:

$$\begin{tikzcd} S_1 \arrow[r, "\phi"] \arrow[d, "\pi"] & S_1' \arrow[d, "\pi'"] \\ S_2 \arrow[r, "\mathrm{id}"] & S_2' \end{tikzcd}$$

Then we define the Hurwitz numbers associated to the above data as follows:

$$h_g(\mathcal{M}, S_2) = \sum \frac{1}{|\mathrm{Aut}(\pi)|},$$

where the sum goes over all isomorphism classes of Hurwitz covers of type $(g, \mathcal{M}, \mathcal{Q}, S_2)$ for a fixed set of points $\mathcal{Q}$.

Remark 2.2.16. As implicit in the notation the Hurwitz numbers, $h_g(\mathcal{M}, S_2)$ do not depend on the choice of $\mathcal{Q}$.

Next, we define tropical Hurwitz numbers in this more general setting and begin by introducing the following notion for a tropical morphism $\pi : C_1 \to C_2$, where $b_1(C_2) = g(C_2)$: Let v be an inner vertex of C_1, such that $\pi(v)$ is adjacent to the edges $e_1, \ldots, e_{k_v}$ of C_2. We choose a configuration $\mathcal{P}_v = \{p'_1, \ldots, p'_{k_v}\}$ of k_v points on $\mathbb{P}^1$, we define $\mu'_v(p'_i)$ as the partition of $d_{v,\pi}$ defined by π at v over e_i and denote the collection of these partitions by μ'_v.

Definition 2.2.17. Let $\pi : C_1 \to C_2$ be a minimal tropical morphism, where $b_1(C_2) = g(C_2)$, then we define its multiplicity

$$m(\pi) = \frac{1}{|\mathrm{Aut}(\pi|} \prod \omega_{e,\pi} \prod \left(\prod_{i=1}^{k_v} \left|\mathrm{Aut}(\mu'(p'_i))\right| \right) h_{g(v)}(\mu'_v, \mathbb{P}^1),$$

where the first product is taken over all inner edges of C_1 and the second product over all inner vertices of C_1. For a non-negative integer g, an abstract tropical curve C_2, where $b_1(C_2) = g(C_2)$ and a tuple of partitions $\mathcal{M} = (\mu_v)_v$ indexed by the leaves v of C_2, we define the *tropical Hurwitz number* associated to the data C_2 and $\mathcal{M}$ by

$$\mathbb{T}h_g^{C_2}(\mathcal{M}) = \sum m(\pi),$$

where we take the sum over all minimal tropical morphisms $\pi : C_1 \to C_2$, such that

1. C_1 is an abstract tropical curve of genus g,
2. π ramifies with profile μ_v over the leaf v,
3. π is unramified outside the leaves of C_2.

In order to understand the relation between tropical and algebro-geometric Hurwitz numbers in this general sense, we associate to the data $C_2, \mathcal{M}$ in definition 2.2.17 the data $S_2, \mathcal{M}, \mathcal{Q}$ in definition 2.2.15:

1. In order to construct the surface S_2, associate a sphere S_v with $l_v = \mathrm{val}(v)$ boundary circles B_{v,e_i}, which we label by the adjacents edges $e_1, \dots, e_{l_v}$. For any edge e connecting the vertices v and v', we glue the spheres S_v and $S_{v'}$ along the boundary circles $B_{v,e}$ and $B_{v',e}$ along a homeomorphism $g_e : B_{v,e} \to B_{v,e'}$. We obtain S_2.

2. For any leaf v, we mark a point q_v on S_v not contained in a boundary circle. We set $Q = \{(q_v)_v\}$.

3. For any leaf v, associate its ramification profile μ_v to the point q_v and we keep $\mathcal{M}$.

Remark 2.2.18. One of the more subtle observations in this construction is that every tropical morphism which yields non-zero weight satisfies the Riemann-Hurwitz condition in definition 2.1.6. Thus, the Riemann-Hurwitz condition is a necessary condition for tropical covers to be realisable.

We are now ready to state a (slightly less general version) of the correspondence theorem proved in [BBM11]. This theorem is proved by associating tropical covers to algebro-geometric covers in the same manner we associated the above data and by analysing how many algebro-geometric covers yield the same tropical covers. This will be revisited in section 3.4.

Theorem 2.2.19 ([BBM11])**.** *Let g be an non-negative integer, C an abstract tropical curve, where $b_1(C) = g(C)$ and $\mathcal{M}$ a tuple of partitions of the same positive integer d indexed by the leaves of C, then*

$$\mathbb{T}h_g^C(\mathcal{M}) = h_g(\mathcal{M}, S).$$

2.2.2 Semi-infinite wedge formalism

In this section, we introduce the notion of the semi-infinite wedge formalism. For a more extensive discussion, we refer to section 2 in [Joh15], which we will follow roughly. We need the following conventions. We write $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$. For partitions μ, ν, we set $m := \ell(\mu)$, and $n := \ell(\nu)$. We also define the functions $\varsigma(z) := e^{z/2} - e^{-z/2}$ and $\mathcal{S}(z) := \frac{\varsigma(z)}{z}$. We begin by defining the semi-infinite wedge space $\mathcal{V}$.

Definition 2.2.20. Let V be a vector space with basis labeled by the half-integers. We use an underscore to correspond the corresponding basis vector, such that $\underline{1/2}$ is the basis vector labeled by $1/2$, thus

$$V = \bigoplus_{i \in \mathbb{Z}} \underline{i + 1/2}.$$

Then the *semi-infinite wedge space* $\mathcal{V}$ is defined by

$$\mathcal{V} = \bigwedge^{\frac{\infty}{2}} V = \bigoplus_{(i_k)_k} \underline{i_1} \wedge \underline{i_2} \wedge \cdots ,$$

where the sum is taken over all decreasing sequences of half integers $(i_k)_k \in \mathbb{Z}'$, such that for some integer c

$$i_k + k - 1/2 = c \tag{2.2}$$

for sufficiently large k. The constant c is known as the *charge*. We give this space $\mathcal{V}$ an inner product $(\cdot, \cdot)$ by declaring the basis indexed by the sequences $(i_k)_k$ to be orthonormal.

Remark 2.2.21. We note that as suggested by the used expressions (e.g. *charge*), the terminology surrounding the semi-infinite wedge formalism is motivated by physics, in particular the notion of Dirac's electron sea. The vector space V mimicks all possible energy levels of a single electron. In physics, Pauli's exclusion law roughly states that two electrons *cannot* occupy the same energy level (in a certain system), which should give a motivation as to why to consider collections of electrons in terms of the wedge formalism. For more details, we refer to the sections 2.3.2 and 2.3.3 [Joh15].

Example 2.2.22. A special role is played by the vacuum vector $|0\rangle \in \mathcal{V}$, which corresponds to the basis vector given by the sequence $-1/2, -3/2, -5/2, \ldots$.

We are mostly concerned with the charge 0 subspace of $\mathcal{V}$, which we denote by $\mathcal{V}_0$. The basis vectors of charge 0 are conveniently indexed by partitions, as follows: From a sequence $(i_k)_k$, we define a new sequence $\lambda_k = i_k + k - 1/2$. Then $(\lambda_k)_k$ is non-increasing and since $(i_k)_k$ is in the charge 0 subspace $\lambda_k = 0$ for sufficiently large k. Thus, the basis vector corresponding to $(i_k)_k$ can be index by a partition λ:

$$v_\lambda = \lambda_1 - 1/2 \wedge \lambda_2 - 3/2 \wedge \lambda_3 - 5/2 \wedge \cdots \wedge \lambda_i - i + 1/2 \wedge \cdots$$

Operators on $\mathcal{V}$

The next step is to introduce operators acting on $\mathcal{V}$. They are analogues from the finite dimensionsal situation: If W is a finite dimensional representation of a lie algebra $\mathfrak{g}$, then so is the space $\bigwedge^k W$, where $\mathfrak{g}$ acts of $\bigwedge^k W$ by the Leibniz rule

$$g \cdot (w_1 \wedge \cdots \wedge w_k) = \sum_{i=1}^{k} w_1 \wedge \cdots \wedge g \cdot w_i \wedge \cdots \wedge w_k.$$

In particular, for a basis $e_1, \ldots, e_n$ of W, we can view $n \times n$ matrices as the Lie algebra $\mathfrak{gl}(W)$, which acts on $\bigwedge^k W$ naturally.

We now extend this situation to the infinite dimensional setting by giving a meaning to the action of certain $\infty \times \infty$ matrices on $\mathcal{V}$.

Definition 2.2.23. The algebra $\mathfrak{gl}_\infty$ consists of $\infty \times \infty$ matrices with only finitely many non-zero entries: For $i, j \in \mathbb{Z}'$, let E_{ij} be the standard matrix with only one non-zero entry

$$E_{ij}\underline{k} = \delta_{jk}\underline{i},$$

then we obtain

$$\mathfrak{gl}_\infty = \bigoplus_{i,j\in\mathbb{Z}'} E_{ij}.$$

Since the matrices in $\mathfrak{gl}_\infty$ have only finitely many non-zero entries, matrix multiplication is well defined. The usual commutator makes $\mathfrak{gl}_\infty$ into a Lie algebra and the Leibniz rule makes $\mathcal{V}$ into a representation.

The operators we will use are not contained in $\mathfrak{gl}_\infty$, thus we need a more general notion of operators acting on $\mathcal{V}$.

Definition 2.2.24. The algebra $\mathcal{A}_\infty$ consists of those $\infty \times \infty$ matrices with only finitely many non-zero diagonals

$$\mathcal{A}_\infty = \left\{ \sum_{i,j\in\mathbb{Z}'} a_{ij}E_{ij} : \ a_{ij} = 0 \text{ for } |j-i| \gg 0 \right\}$$

In order to define the action of the operators in $\mathcal{V}$, we make the following observation: We an recover a decreasing sequence $(i_k)_k$ of half integers from its set of values $S = \{i_1, i_2, \dots\}$. Now let $\mathbb{Z}'^-$ and $\mathbb{Z}'^+$ be the negative and positive elements of $\mathbb{Z}'$. Then we see that the sets

$$S \cap \mathbb{Z}'^+, \ S^c \cap \mathbb{Z}'^- \tag{2.3}$$

are finite, where S^c is the complement of S in $\mathbb{Z}'$. Moreover, we can reconstruct a decreasing sequence i_k satisfing equation (2.2) from any S satisfying equation (2.3). Using this correspondence we denote the vector $\underline{i_1} \wedge \underline{i_2} \wedge \cdots$ by v_S for the corresponding set S.

We now define the action of the operators in $\mathcal{A}_\infty$. This is done by re-defining the action of the matrices E_{ij} for $i = j$. We note that for operators with only zeros along the main diagonal we obtain a well-defined action by the linear continuation of the action of the E_{ij}. For E_{kk} we give the following definition.

Definition 2.2.25. Let k be positive integer and S a set satisfying equation (2.3), then

$$E_{kk} \cdot v_S = \begin{cases} v_S, & \text{if } k \geq 0, k \in S \\ -v_s, & \text{if } k \leq 0, k \notin S \\ 0 & \text{else} \end{cases}$$

Remark 2.2.26. Before applying these notions to the theory of Hurwitz numbers, we make the following remarks:

1. The above definition of the action of E_{kk} substracts the naive action of E_{kk} on the vacuum from the naive action of E_{kk} on v_S. In fact, let E'_{kk} denote the naive action of E_{kk} in the sense of definition 2.2.23, then

$$E_{kk} \cdot v_s = E'_{kk} \cdot v_S - E'_{kk} \cdot |0\rangle ,$$

2. With the above definition of E_{kk} every operator in $\mathcal{A}_\infty$ has a well-defined operation on $\mathcal{V}$.

3. The above action is not a representation of $\mathcal{A}_\infty$, but it is a projective representation.

Applications to Hurwitz theory

In order to apply the theory of the semi-infinite wedge space $\mathcal{V}$ to the theory of Hurwitz numbers, we need to explore its relation to the representation theory of $\mathcal{S}_n$. We begin by definining a couple of operators.

Definition 2.2.27. Let $k \in \mathbb{Z}$ and $k \neq 0$, then

$$\alpha_n = \sum_{k \in \mathbb{Z}'} E_{k,k-n}.$$

Furthermore, let μ be a partition, then

$$a_{-\mu} = \prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i} v_\emptyset.$$

We define the *energy operator* E and the *charge operator* C by

$$E = \sum_{k \in \mathbb{Z}'} k E_{kk}, \; C = \sum_{k \in \mathbb{Z}'} E_{kk}.$$

For a vector v_S, let E_S (resp. C_S) be its eigenvector with respect to E (resp. C), then we call E_S the *energy* and C_S the *charge* of v_S (see also definition 2.2.20). Lastly, we define

$$\mathcal{F}_r = \sum_{k \in \mathbb{Z}'} \frac{k^r}{r!} E_{kk}.$$

For an operator M, we denote its *vacuum expectaction* by

$$\langle M \rangle = (\langle 0| \,, M \,|0\rangle) \tag{2.4}$$

By the above discussion, we obtain the following theorem

Theorem 2.2.28. *Let μ, ν be two partitions of the same positive integer, let $g \geq 0$ and $b = 2g - 2 + \ell(\mu) + \ell(\nu)$, then*

$$h^{\circ}_{b;\mu,\nu} = \frac{1}{\prod \mu_i} \frac{1}{\prod \nu_j} \left\langle \prod \alpha_{\mu_i} \mathcal{F}_2^b \prod \alpha_{-\nu_j} \right\rangle .$$

Moreover, let $h_{\mu,\nu}(z) = \sum_{b=1}^{\infty} h^{\circ}_{b;\mu,\nu} \frac{z^b}{b!}$, then

$$h_{\mu,\nu}(z) = \frac{1}{\prod \mu_i} \frac{1}{\prod \nu_j} \left\langle \prod \alpha_{\mu_i} e^{z\mathcal{F}_2} \prod \alpha_{-\nu_j} \right\rangle . \tag{2.5}$$

We now introduce new operators, which allow us to rewrite equation (2.5). For n any integer, and z a formal variable, define the operators

$$\mathcal{E}_n(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{n}{2})} E_{k-n,k} + \frac{\delta_{n,0}}{\varsigma(z)}, \tag{2.6}$$

and we observe for $n \neq 0$

$$\alpha_n = \mathcal{E}_n(0) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} E_{k-n,k}.$$

Their commutation formulae are known to be

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma\left(\det\begin{bmatrix} a & z \\ b & w \end{bmatrix}\right)\mathcal{E}_{a+b}(z+w), \qquad\qquad [\alpha_k, \alpha_l] = k\delta_{k+l,0}.$$

We will also use the $\mathcal{E}$-operator without the correction in energy zero, i.e.

$$\tilde{\mathcal{E}}_0(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}E_{k,k} = \sum_{r=0}^{\infty}\mathcal{F}_r z^r$$

In order to express the Hurwitz numbers in terms of the semi-infinite wedge formalism, we introduce the following notation.

Notation 2.2.29. Let F be a formal series in $z_1, \ldots, z_n$, then we denote by

$$[z_1^{a_1}\cdots z_n^{a_n}]F$$

the coefficient of $z_1^{a_1}\cdots z_n^{a_n}$ in F.

As seen in [OP06], the simple double Hurwitz number can be computed as

$$h^{\circ}_{b;\mu,\nu} = \frac{b!\cdot[u^b]}{\prod\mu_i\prod\nu_j}\left\langle\prod_{i=1}^{m}\mathcal{E}_{\mu_i}(0)\prod_{j=1}^{n}\mathcal{E}_{-\nu_j}(u\nu_j)\right\rangle.$$

The monotone double Hurwitz numbers have the following expression, derived in [ALS16]:

$$h^{\leq}_{b;\mu,\nu} = \frac{[u^b]}{\prod\mu_i\prod\nu_j}\left\langle\prod_{i=1}^{m}\alpha_{\mu_i}\mathcal{D}^{(h)}(u)\prod_{j=1}^{n}\alpha_{-\nu_j}\right\rangle,$$

where the operator $\mathcal{D}^{(h)}(u)$ has the vectors v_λ as eigenvectors with the generating series for the complete homogeneous polynomials h evaluated at the content $\mathbf{cr}^\lambda$ of the Young tableau λ as eigenvalues:

$$\mathcal{D}^{(h)}(u).v_\lambda = \sum_{v=0} h_v(\mathbf{cr}^\lambda)u^v v_\lambda\,.$$

Explicitly, the operator $\mathcal{D}^{(h)}$ can be expressed in terms of the operators $\mathcal{E}$ as

$$\mathcal{D}^{(h)}(u) = \exp\left(\left[\frac{\tilde{\mathcal{E}}_0\left(u^2\frac{d}{du}\right)}{\varsigma\left(u^2\frac{d}{du}\right)} - E\right].\log(u)\right).$$

Let $\mathcal{O}_x^{(h)}(u)$ indicate the conjugation $\mathcal{D}^{(h)}(u)\alpha_x\mathcal{D}^{(h)}(u)^{-1}$. Since

$$\mathcal{D}^{(h)}(u)^{-1}.|0\rangle = |0\rangle,$$

one can insert the operator $\mathcal{D}^{(h)}(u)^{-1}$ on the right and insert

$$1 = \mathcal{D}^{(h)}(u)\mathcal{D}^{(h)}(u)^{-1}$$

between every consecutive pair of operators $\alpha_{-\nu_i}$, obtaining

$$h^{\le}_{b;\mu,\nu} = \frac{[u^b]}{\prod\mu_i\prod\nu_j}\left\langle\prod_{i=1}^m\mathcal{E}_{\mu_i}(0)\prod_{j=1}^n\mathcal{O}^{(h)}_{-\nu_j}(u)\right\rangle.$$

The operators $\mathcal{O}^{(h)}$ have been computed in [KLS16] to be equal to

$$\mathcal{O}^{(h)}_{-\nu}(u) = \sum_{v=0}^{\infty}\frac{(v+\nu-1)!}{(\nu-1)!}[z^v]\mathcal{S}(uz)^{\nu-1}\mathcal{E}_{-\nu}(uz),$$

so the result is the following lemma, which is the first key observation for the discussion in section 3.3.

Lemma 2.2.30. *Let g be a non-negative number, μ and ν partitions of the same positive integer. The monotone double Hurwitz number corresponding to this data can be computed as*

$$h^{\le}_{b;\mu,\nu} = \frac{[u^b]}{\prod\mu_i}\sum_{\substack{v\vdash b\\ \ell(v)=n}}\prod_{j=1}^n\frac{(v_j+\nu_j-1)!}{\nu_j!}[z_1^{v_1}\dots z_n^{v_n}]$$
$$\prod_{j=1}^n\mathcal{S}(uz_j)^{\nu_j-1}\left\langle\prod_{i=1}^m\mathcal{E}_{\mu_i}(0)\prod_{j=1}^n\mathcal{E}_{-\nu_j}(uz_j)\right\rangle.$$

Similarly, the strictly monotone double Hurwitz numbers can be expressed in the same way, substituting for $\mathcal{D}^{(h)}$ the operator $\mathcal{D}^{(\sigma)}(u) := \mathcal{D}^{(h)}(-u)^{-1}$. This reads

$$h^{<}_{b;\mu,\nu} = \frac{[u^b]}{\prod \mu_i \prod \nu_j} \left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{O}^{(\sigma)}_{-\nu_j}(u) \right\rangle ,$$

where

$$\mathcal{O}^{(\sigma)}_{-\nu}(u) = \sum_{v=0}^{\nu} \frac{\nu!}{(\nu - v)!} [z^v] \mathcal{S}(uz)^{-\nu-1} \mathcal{E}_{-\nu}(uz) ,$$

and we obtain the following lemma in a fashion analogous to lemma 2.2.30.

Lemma 2.2.31. *Let g be a non-negative number, μ and ν partitions of the same positive integer. The strictly monotone double Hurwitz number corresponding to this data can be computed as*

$$h^{<}_{b;\mu,\nu} = \frac{[u^b]}{\prod \mu_i} \sum_{\substack{v \vdash b \\ 0 \le v_j \le \nu_j}} \prod_{i=1}^{n} \frac{(\nu_j - 1)!}{(\nu_j - v_j)!} [z_1^{v_1} \dots z_n^{v_n}] \prod_{j=1}^{n} \mathcal{S}(uz_j)^{-\nu_j - 1} \left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(uz_j) \right\rangle .$$

Remark 2.2.32. We see that the monotone and strictly monotone double Hurwitz numbers are computed as linear combinations of vacuum exepectations similar to the ones appearing in the equation for simple double Hurwitz numbers.

2.2.3 Pillowcase covers and quasimodular forms

In this section we introduce the notion of quasimodular forms and their connections to pillowcase covers as discovered by [EO06]. We begin by defining quasimodular forms.

Definition 2.2.33. Let $\mathbb{H}$ denote the upper half-plane of the complex numbers. A function $f : \mathbb{H} \to \mathbb{C}$ is called an *almost holomorphic modular form of weight*

$k \in \mathbb{Z}$ if for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and $\tau = u + iv \in \mathbb{H}$ we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c + d\tau)^k f(\tau)$$

and f has the form

$$f(\tau) = \sum_{j=0}^{r} \frac{f_j(\tau)}{v^j},$$

where f_j is holomorphic in $\mathbb{H} \cup \{\infty\}$. The constant term f_0 is called a *quasimodular form of weight k* and r is called the depth of f.

The following well-known structural result is a key motivation in the study of quasimodular forms, where we use the notion of the *Eisenstein series*

$$E_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m + n\tau)^{2k}}.$$

We will also make use of the coordinate change $q = e^{2\pi i\tau}$ and write $E_{2k}(q)$

Theorem 2.2.34. *The ring of quasimodular forms is a polynomial ring over $\mathbb{C}$ generated by the Eisenstein series E_2, E_4 and E_6.*

We now introduce an enumerative problem concerning pillowcase covers, which is connected to the moduli space of quadratic differentials as seen in [EO06]. In order to do this, let L be a lattice in $\mathbb{C}$, let $T = {}^{\mathbb{C}}\!/\!_{L}$ be a complex torus and we fix the complex involution $\pm : \mathbb{C} \to \mathbb{C},\ z \mapsto -z$. Then the so-called *pillowcase orbifold* is a sphere with four quotient $({}^{\mathbb{Z}}\!/\!_{2})-$orbifold points given by the following quotient

$$\mathfrak{P} = {}^{T}\!/\!_{\pm}.$$

Let μ be a partition and let ν be a partition of an odd number into odd parts.

We want to enumerate degree $2d$ covers

$$\pi : \mathfrak{C} \to \mathfrak{P},$$

where $\mathfrak{C}$ is an orbifold curve. This translates (see the discussion at the beginning of section 3.4) to a degree $2d$ cover $\tilde{\pi} : C \to T$ of the torus T, such that the following diagram with the natural horizontal maps commutes

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\tilde{\pi}"] & \mathfrak{C} \arrow[d, "\pi"] \\ T \arrow[r] & \mathfrak{P}. \end{tikzcd}$$

Moreover, we impose the following ramification data. Viewed as a map to the sphere, π has profile $(\nu, 2^{d-|\nu|/2})$ over $0 \in \mathfrak{P}$ and profile (2^d) over the other three orbifold points. Moreover, we require π to have profile $(\mu_i, 1^{2d-\mu_i})$ over some $\ell(\mu)$ given points of $\mathfrak{P}$ and unramified elsewhere. The Riemann-Hurwitz formula yields

$$\chi(\mathfrak{C}) = \ell(\mu) + \ell(\nu) - |\mu| - |\frac{\nu}{2}|,$$

which determines the genus of $\mathfrak{C}$. Two covers $\pi_i : \mathfrak{C}_i \to \mathfrak{P}_i$ are identified if there is an isomorphism $f : \mathfrak{C}_1 \to \mathfrak{C}_2$, such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{C}_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & \mathfrak{C}_2 \arrow[dl, "\pi_2"] \\ & \mathfrak{P} & \end{tikzcd}$$

We denote the automorphism group of π by $\mathrm{Aut}(\pi)$ and consider the following generating series

$$Z(\mu, \nu; q) = \sum_{\pi} \frac{q^{\deg(\pi)}}{|\mathrm{Aut}(\pi)|},$$

where we sum over all covers $\pi : \mathfrak{C} \to \mathfrak{P}$ with ramification data specified by μ and ν as above. The degree of each such cover is even.

For $\mu = \nu = \emptyset$, any cover has the form

$$\pi : \mathbb{T} \xrightarrow{\pi'} \mathbb{T} \to \mathbb{T}/_{\pm}$$

with π' unramified and we see that $|\mathrm{Aut}(\pi)| = 2|\mathrm{Aut}(\pi')|$. Moreover, by [EO06], we obtain

$$Z(\emptyset, \emptyset; q) = \prod_n (1 - q^{2n})^{-\frac{1}{2}}$$

by enumerating the π'. We define the new generating series

$$Z'(\mu, \nu, q) = \frac{Z(\mu, \nu; q)}{Z(\emptyset, \emptyset; q)}. \tag{2.7}$$

This enumerates covers without unramified components. By usual inclusion-exclusion, one can extract a generating series for connected covers from equation (2.7). We denote this generating series by $Z^\circ(\mu, \nu, q)$. The generating series $Z'(\mu, \nu; q)$ has an astonishing structure as proved by Eskin-Okounkov:

Theorem 2.2.35 ([EO06]). *The generating series $Z'(\mu, \nu; q)$ is a polynomial in $E_2(q^2)$, $E_4(q^4)$ and $E_6(q^4)$ and of weight $\ell(\mu) + |\mu| + |\nu|/2$.*

2.3 Mustafin varieties

In this section we introduce the foundations of the theory of Mustafin varieties.

2.3.1 Bruhat-Tits Buildings and Tropical Convexity

We begin by recalling some of the relations between Bruhat-Tits buildings and tropical convexity. For a short summary of Bruhat-Tits buildings, we refer to Section 2 of [CHSW11], for more details on the relation between buildings and tropical convexity see e.g. [JSY07; Wer11]. We denote the Bruhat-Tits building associated to $\mathrm{PGL}(V)$ by $\mathfrak{B}_d$ and by $\mathfrak{B}_d^0$ the set of lattice classes in $\mathfrak{B}_d$. We call two lattice classes $[L], [M]$ in $\mathfrak{B}_d^0$ adjacent if there exist representatives $L' \in [L]$ and $M' \in [M]$, such that $\pi M' \subset L' \subset M'$.

We pick a basis $e_1, \dots, e_d$ of V. The associated apartment A is the geometric realisation of a simplicial complex on a vertex set isomorphic to the integral part $\mathbb{Z}^d/\mathbb{Z}\mathbf{1}$ of the tropical torus. The vertex set consists of all homothety classes of lattices in V having diagonal form in $e_1, \dots, e_d$. More precisely, the following map is a bijection:

$$\begin{aligned} f : A \cap \mathfrak{B}_d^0 &\longrightarrow \mathbb{Z}^d/\mathbb{Z}\mathbf{1} \\ \{\pi^{m_1}Re_1 + \cdots + \pi^{m_d}Re_d\} &\longmapsto (-m_1, \dots, -m_d) + \mathbb{Z}\mathbf{1}. \end{aligned} \tag{2.8}$$

We write $\operatorname{tconv}(\Gamma)$ for the convex hull of the image of the point configuration under this map.

Definition 2.3.1. We call a point configuration $\Gamma \subset \mathfrak{B}_d^0$ convex if for $[L], [L'] \in \Gamma$, any vertex of the form $[\pi^a L \cap \pi^b L']$ is also in Γ.
The convex hull $\operatorname{conv}(\Gamma)$ of a point configuration Γ is the intersection of all convex sets containing Γ.

The following lemma essentially goes back to [KT06], is a special case of lemma 21 in [JSY07] and can be found in this version as lemma 4.1 in [CHSW11].

Lemma 2.3.2. *Let Γ be a point configuration in one apartment. The map in equation* (2.8) *induces a bijection between the lattices in* $\operatorname{conv}(\Gamma)$ *in the Bruhat-Tits building $\mathfrak{B}_d$ and the lattice points in* $\operatorname{tconv}(\Gamma)$.

Remark 2.3.3. We call a point configuration $\Gamma \subset \mathfrak{B}_d^0$ that is contained in the same apartment A in tropical general position, if the point configuration

$$\{f(a) : a \in A\} \subset \mathbb{Z}^d/\mathbb{Z}\mathbf{1} \subset \mathbb{R}^d/\mathbb{R}\mathbf{1}$$

is in tropical general position.

2.3.2 Images of rational maps

Let W be a d–dimensional vector space over k and let $(W_i)_{i=1}^n$ be a tuple of subvectorspaces of W, such that

$$\bigcap_{i=1}^{n} W_i = \langle 0 \rangle.$$

In [Li17], images of rational maps of the form

$$\mathbb{P}(W) \dashrightarrow \mathbb{P}({}^{W}\!/\!_{W_1}) \times \cdots \times \mathbb{P}({}^{W}\!/\!_{W_n})$$

were studied. In particular, the Hilbert function and the multidegrees were computed. For every non-empty $I \subset [n] := \{1, \ldots, n\}$ we define

$$d_I = \dim \bigcap_{i \in I} W_i.$$

For every $h \in \mathbb{Z}_{\geq 0}$ we define $M(h)$ to be the set

$$\begin{aligned} M(h) = &\{(m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n : \sum_{i=1}^{n} m_i = h \text{ and} \\ &d - \sum_{i \in I} m_i > d_I \text{ for every non-empty } I \subset [n]\}. \end{aligned}$$

Remark 2.3.4. There are several equivalent notions of the multidegree of a variety X in a product of projective spaces $\left(\mathbb{P}_k^{d-1}\right)^n$. One possibility is to consider the multigraded Hilbert polynomial h_X: Let x^u be a monomial of maximal degree in h_X and c_u be the coefficient. The multidegree function takes value $\frac{c_u}{u!}$ at u, where $u! = u_1! \cdots u_n!$.

Another way to describe the multidegree is to consider the intersection of X with a system of u_i general linear equations in the i–th factor of $\prod_{i=1}^n \mathbb{P}\left({}^{k^d}\!/\!_{W_i}\right)$, where we choose the u_i such that the intersection is finite. Then the value of the multidegree function at $u = (u_1, \ldots, u_n)$ is the cardinality of this intersection product. For a more thorough introduction in terms of Chow classes, see e.g.

[CLZ16]. We denote the set of multidegrees by

$$\mathrm{multDeg}(X) = \{u \in \mathbb{Z}^n_{\geq 0} : \text{the multidegree function is non-zero at } u\}.$$

Theorem 2.3.5. *[Li17] Assume k is algebraically closed. Set $p = \max\{h : M(h) \neq 0\}$. The dimension of $X(W, W_1, \ldots, W_n)$ is p. Its multidegree function takes value one at the integer vectors in $M(p)$ and 0 otherwise. The Hilbert function of $X(W, W_1, \ldots, W_n)$ is*

$$\sum_{S \subset M(p)} (-1)^{|S|-1} \prod_{j=1}^{n} \binom{u_i + \ell_{S,i}}{\ell_{S,i}},$$

where the u_i are the variables and $\ell_{S,i}$ is the smallest $i-th$ component of all elements of S. Moreover, $X(W, W_1, \ldots, W_n)$ is Cohen-Macaulay.

We end this section with an example.

Example 2.3.6. Let $W = \mathbb{C}^3$ and e_1, e_2, e_3 the standard basis. Moreover, let $W_1 = (0), W_2 = (e_2, e_3), W_3 = (e_2, e_3)$ then

$$X = X(W, W_1, W_2, W_3) = \mathbb{P}^2 \times pt \times pt.$$

In the notation of theorem 2.3.5, we see $p = 2 = \dim(X)$ with multidegree $\mathrm{multDeg}(X) = M(p) = M(2) = \{(2, 0, 0)\}$. The multidegree function takes value 1 at $(2, 0, 0)$ and 0 else.

2.3.3 Mustafin Varieties

In this section, we review the theory developed in [CHSW11], where many interesting structural results about $\mathcal{M}(\Gamma)$ and its special fiber were proved. We state results needed for our approach and refer to [CHSW11] for a more detailed discussion. We begin by fixing an algebraic set-up and defining Mustafin varieties: Let R be a discrete valuation ring, K the quotient field and k the residue field. We fix a uniformiser π. As an example take $\mathbb{K} = \mathbb{C}((\pi))$ as the ring of formal Laurent series over $\mathbb{C}$ with discrete valuation $v(\sum_{n\geq l} a_n \pi^n) = l$ for $l \in \mathbb{Z}$ and $a_n \in \mathbb{C}$ with

$a_l \neq 0$. Then $R = \{\sum_{n \geq l} a_n \pi^n : k \in \mathbb{Z}_{\geq 0}\}$ and $k = \mathbb{C}$. Moreover, let V be a vector space of dimension d over K. We define $\mathbb{P}(V) = \operatorname{ProjSym}(V^*)$ as parameterising lines through V. We call free R–modules $L \subset V$ of rank d *lattices* and define $\mathbb{P}(L) = \operatorname{ProjSym}(L^*)$, where $L^* = \operatorname{Hom}_R(L, R)$. Note, that we will only consider lattices up to homothety, i.e. $L \backsim L'$ if $L = c \cdot L'$ for some $c \in K^\times$.

Definition 2.3.7. Let $\Gamma = \{L_1, \dots, L_n\}$ be a set of rank d lattices in V. Then $\mathbb{P}(L_1), \dots, \mathbb{P}(L_n)$ are projective spaces over R whose generic fibers are canonically isomorphic to $\mathbb{P}(V) \simeq \mathbb{P}^{d-1}_{\mathbb{K}}$. The open immersions

$$\mathbb{P}(V) \hookrightarrow \mathbb{P}(L_i)$$

give rise to a map

$$\mathbb{P}(V) \longrightarrow \mathbb{P}(L_1) \times_R \cdots \times_R \mathbb{P}(L_n).$$

We denote the closure of the image endowed with the reduced scheme structure by $\mathcal{M}(\Gamma)$. We call $\mathcal{M}(\Gamma)$ the *associated Mustafin variety.* Its special fiber $\mathcal{M}(\Gamma)_k$ is a scheme over k.

While the generic fiber of such a scheme is isomorphic to $\mathbb{P}^{d-1}$, the special fiber has many interesting properties.

Proposition 2.3.8 ([CHSW11]).

1. *For a finite set of lattices Γ, the Mustafin variety $\mathcal{M}(\Gamma)$ is an integral, normal, Cohen-Macaulay scheme which is flat and projective over R. Its generic fiber is isomorphic to $\mathbb{P}^{d-1}_{\mathbb{K}}$ and its special fiber is reduced, Cohen-Macaulay and connected. All irreducible components are rational varieties and their number is at most $\binom{n+d-2}{d-1}$, where $n = |\Gamma|$.*

2. *If Γ is a convex set in $\mathfrak{B}^0_d$, then the Mustafin variety is regular and its special fiber consists of n smooth irreducible components that intersect transversely. In this case the reduction complex of $\mathcal{M}(\Gamma)$ is induced by the simplicial subcomplex of $\mathfrak{B}_d$ induced by Γ.*

3. *If Γ is a point configuration in one apartment, the components of the special*

fiber correspond to maximal cells of a subdivision of the the simplex $|\Gamma| \cdot \Delta_{d-1}$ *induced by* Γ.

4. *An irreducible component mapping birationally to the special fiber of* $\mathbb{P}(L_i)$ *is called a primary component. The other components are called secondary components. For each* $i = 1, \ldots, n$ *there exists such a primary component. A projective variety* X *arises as a primary component for some point configuration* Γ *if and only if* X *is the blow-up of* $\mathbb{P}_k^{d-1}$ *at a collection of* $n-1$ *linear subspaces.*

5. *Let* C *be a secondary component of* $\mathcal{M}(\Gamma)_k$. *There exists a vertex* v *in* $\mathfrak{B}_d^0$, *such that*

$$\mathcal{M}(\Gamma \cup \{v\})_k \longrightarrow \mathcal{M}(\Gamma)_k$$

restricts to a birational morphism $\tilde{C} \to C$, *where* $\tilde{C}$ *is the primary component of* $\mathcal{M}(\Gamma \cup \{v\})_k$ *corresponding to* v.

We now choose coordinates in the same way as in [CHSW11]. Consider the diagonal map

$$\Delta : \mathbb{P}(V) \longrightarrow \mathbb{P}(V)^n = \mathbb{P}(V) \times_K \cdots \times_K \mathbb{P}(V).$$

The image of Δ is the subvariety of of $\mathbb{P}(V)^n$ cut out by the ideal generated by the 2×2 minors of a matrix $X = (x_{ij})_{\substack{i=1,\ldots,d \\ j=1,\ldots,n}}$ of unknowns, where the jth column corresponds to coordinates in the jth factor.
Start with an element $g \in \mathrm{GL}(V)$, it is represented by an invertible $n \times n$ matrix over K. It induces a dual map $g^t : V^* \to V^*$ and thus a morphism $g : \mathbb{P}(V) \to \mathbb{P}(V)$. For n elements $g_1, \ldots, g_n \in \mathrm{GL}(V)$, the image of

$$\mathbb{P}(V) \xrightarrow{\Delta} \mathbb{P}(V)^n \xrightarrow{g_1^{-1} \times \cdots \times g_n^{-1}} \mathbb{P}(V)^n$$

is the subvariety of $\mathbb{P}(V)^n$ cut out by the multihomogeneous prime ideal

$$I_2((g_1, \ldots, g_n)(X)) \subset K[X],$$

where $(g_1, \ldots, g_n)(X)$ is the matrix whose jth column is given by

$$g_j \begin{pmatrix} x_{1j} \\ \vdots \\ x_{dj} \end{pmatrix}.$$

Consider a reference lattice $L = Re_1 + \cdots + Re_d$. For any configuration $\Gamma = \{L_1, \ldots, L_n\}$ of lattices in $\mathfrak{B}_d^0$, we choose g_i, such that $g_i L = L_i$ for all i. The following diagram commutes:

$$\begin{tikzcd}
\mathbb{P}(V) \arrow[r, "(g_1^{-1}\times\cdots\times g_n^{-1})\circ\Delta"] \arrow[d] & \mathbb{P}(V)^n \arrow[d] \\
\prod_R \mathbb{P}(L_i) \arrow[r, "(g_1^{-1}\times\cdots\times g_n^{-1})"] & \mathbb{P}(L)^n
\end{tikzcd}$$

It follows immediately that the Mustafin Variety $\mathcal{M}(\Gamma)$ is isomorphic to the subscheme of $\mathbb{P}(L)^n \cong (\mathbb{P}_R^{d-1})^n$ cut out by the multihomogeneous ideal

$$I_2\left((g_1, \ldots, g_n)(X)\right) \cap R[X]$$

in $R[X]$.

Definition 2.3.9. A configuration $\Gamma = \{L_1, \ldots, L_n\}$ in $\mathfrak{B}_d$ is of *monomial type* if there exist bases for the R-modules $L_1, \ldots, L_n$, such that the multi-homogeneous ideal in $k[X]$ defining $\mathcal{M}(\Gamma)_k$ is generated by monomials in the dual bases.

As proved in [CHSW11] point configurations Γ in one apartment and tropical general position yield interesting properties of the special fiber of the corresponding Mustafin varieties.

Proposition 2.3.10 ([CHSW11])**.** *Let Γ be a point configurations. The following are equivalent:*

1. *The configuration Γ is in general position.*

2. *The special fiber $\mathcal{M}(\Gamma)_k$ is of monomial type.*

3. *$\mathcal{M}(\Gamma)_k$ is defined by a monomial ideal in $k[X]$ for the coordinates chosen before.*

4. *The number of secondary components of $\mathcal{M}(\Gamma)_k$ equals $\binom{n+d-2}{d-1} - n$.*

Thus, in the case of tropical general position, the number of components of $\mathcal{M}(\Gamma)_k$ is $\binom{n+d-2}{d-1}$.

2.3.4 (Pre)linked Grassmannians

In this section, we discuss the basic theory surrounding (pre)linked Grassmannians, developed in the Appendix of [Oss06] and the Appendix of [Oss14] in the context of the theory of limit linear series. We begin with the following situation: Let S be any base scheme and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ vector bundles on S, each of rank d. We have morphisms $f_i : \mathcal{E}_i \to \mathcal{E}_{i+1}$, $g_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$ and a positive integer $r < d$. For an $S-$scheme T, we denote the pull-backs of the vector bundles by $\mathcal{E}_{i,T}$ and the maps induced by the pull-backs by $f_{i,T} : \mathcal{E}_{i,T} \to \mathcal{E}_{i+1,T}$. The functor studied in [Oss06] is described as follows:

Definition 2.3.11. We define $\mathcal{LG}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$ to be the functor associating to each S-scheme T the set of sub-bundles $V_1, \ldots, V_n$ of $\mathcal{E}_{1,T}, \ldots, \mathcal{E}_{n,T}$ of rank r that satisfy $f_{i,T}(V_i) \subset V_{i+1}$ and $g_{i,T}(V_{i+1}) \subset V_i$ for all i.

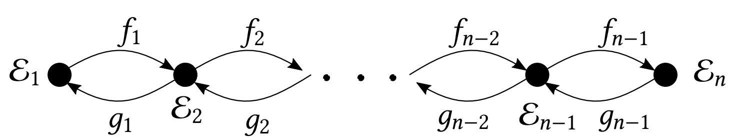


Figure 2.6: The underlying graph of the linked Grassmannian together with the input data.

This functor is representable by a projective scheme:

Lemma 2.3.12. *The functor $\mathcal{LG}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$ is representable by a projective scheme*

$$\mathrm{LG} = \mathrm{LG}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$$

over S, which is naturally a closed subscheme of a product G of Grassmannians schemes over S; G is smooth and projective over S of relative dimension $nd(d-r)$. More precisely: Let G_i be the Grassmannian of rank r sub-bundles of $\mathcal{E}_i$, then LG *is a closed subscheme of $G = G_1 \times \cdots \times G_n$.*

Remark 2.3.13. If there is no confusion about the data, we denote $\mathrm{LG}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$ simply by LG.

In order to study the dimension of LG, additional hypothesis were made in [Oss06].

Definition 2.3.14. We say that $\mathrm{LG}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$ is a **linked Grassmannian of length** n if S is integral, Cohen-Macaulay and the following conditions on f_i, g_i are satisfied:

(I) There exists an $s \in \Gamma(S, \mathcal{O}_S)$, such that for all i

$$f_i \circ g_i = s \cdot \mathrm{id}_{\mathcal{E}_{i+1}} \text{ and } g_i \circ f_i = s \cdot \mathrm{id}_{\mathcal{E}_i}.$$

(II) On the fibers of the $\mathcal{E}_i$ at any point in the zero-locus of s, we have

$$\mathrm{Ker} f_i = \mathrm{Im} g_i \text{ and } \mathrm{Ker} g_i = \mathrm{Im} f_i.$$

Equivalently, for any i and given integers r_1 and r_2 such that $r_1 + r_2 < d$, the closed subscheme of S obtained as the locus where f_i has rank less than or equal to r_1 and g_i has rank less than or equal to r_2 is empty.

(III) At any point of S, we have

$$\mathrm{Im} f_i \cap \mathrm{Ker} f_{i+1} = \{0\} \text{ and } \mathrm{Im} g_{i+1} \cap \mathrm{Ker} g_i = \{0\}.$$

Equivalently, for any integer r' and any i, we have locally closed subschemes of S corresponding to the locus where f_i has rank exactly r' and $f_{i+1}f_i$ has rank less than or equal to $r'-1$ (similarly for g_i). Then we require all those subschemes to be empty.

We will call a tuple $(\mathcal{E}_i, \{f_i, g_i\}_i)$ satisfying these conditions an *$s-$linked chain.*

An important notion for our discussion are *the exact points of a linked Grassmannian.*

Definition 2.3.15. A point of a linked Grassmannian is exact if the corresponding collection of subvectorspaces V_i satisfies the conditions that $\ker g_i|_{V_{i+1}} \subset f_i(V_i)$ and $\ker f_i|_{V_i} \subset g_i(V_{i+1})$ for all i.

The following proposition was proved in lemma A.11 and lemma A.12 in [Oss06].

Proposition 2.3.16. *For linked Grassmannians, we have the following description of exact points:*

(i) The exact points form an open subscheme of LG *and are naturally described as the complement of the closed subscheme on which* $\mathrm{rk} f_i|_{V_i} + \mathrm{rk} g_i|_{V_{i+1}} < r$ *for some* i.

(ii) In the case $s = 0$, *we can describe exact points as those with* $\mathrm{rk} f_i|_{V_i} + \mathrm{rk} g_i|_{V_{i+1}} = r$ *for all* i *(which is even true for arbitrary scheme-valued points).*

Exact points have the following properties:

(iii) The exact points are dense in LG *and indeed dense in every fiber of* $\mathrm{LG} \to S$.

(iv) Every exact point $x \in \mathrm{LG}$ *is a smooth point of* LG *over* S.

The non-exact points of the fibers may also be described nicely:

(v) The non-exact points of a fiber are precisely the intersections of the components of that fiber.

Remark 2.3.17. In our case, linked Grassmannians will always be cosindered over $S = \mathrm{Spec}(R)$, where R is a DVR. Thus, we will only talk about exact points and non-exact points of the special fiber, since the generic fiber is a Grassmannian as seen in the proof of lemma 2.3.12 in [Oss06].

The following theorem was proved in [HO08].

Theorem 2.3.18 ([HO08]). *Let S be integral and Cohen-Macaulay and let* LG *be any linked Grassmannian over S. Then* LG *is flat over S, reduced and Cohen-Macaulay, with reduced fibers.*

In the Appendix of [Oss14], the notion of linked Grassmannians of length n was generalised to arbitrary underlying graphs. We make this more precise in the following definition and give a quick summary of the results we will need in section 4.3.

Definition 2.3.19. Let G be a finite directed graph, connected by (directed) paths, d an integer and S a scheme. Suppose we are given the following data: $\mathcal{E}_\bullet$, consisting of vector bundles $\mathcal{E}_v$ of rank d over S for each $v \in V(G)$ and morphisms $f_e : \mathcal{E}_v \to \mathcal{E}_{v'}$ for each edge $e \in E(G)$, where e points from v to v'. For a directed path P in G, we denote by f_P the composition of the morphisms f_e along the edges e in the path.
Given an integer $r < d$ suppose further that we also have the following condition satisfied: For any two paths P, P' in G with the same head and tail, there are sections $s, s' \in \Gamma(S, \mathcal{O}_S)$ such that

$$s \cdot f_P = s' \cdot f_{P'}.$$

Moreover, if P (resp. P') is minimal (i.e. a shortest path with respect to the canonical graph metric), then s' (resp. s) is invertible.
We define the **prelinked Grassmannian** $\mathrm{LG}(r, \mathcal{E}_\bullet)$ to be the scheme representing the functor associating to an S−scheme T the set of all collections $(\mathcal{F}_v)_{v \in V(G)}$ of rank-r subbundles of the $\mathcal{E}_{v,T}$ satisfying the property that for all edges $e \in E(G)$,

we have $f_e(\mathcal{F}_v) \subset \mathcal{F}_{v'}$, where e points from v to v'. It was proved in [Oss14] that this functor is representable. Moreover, $\mathrm{LG}(r, \mathcal{E}_\bullet)$ is a projective scheme over S and compatible with base change.

Remark 2.3.20.

1. Note that we allow the path consisting of one vertex and consider $f_P = \mathrm{id}$ in this case. The condition implies that for any cycle P, we obtain $f_P = s \cdot \mathrm{id}$ for some scalar s, which corresponds to definition 2.3.14 (I).

2. When G is the graph consisting of vertices $v_1, \ldots, v_n$ and directed edges $e_{i,i+1}$ pointing from v_i to v_{i+1} and $e_{i+1,i}$ pointing from v_{i+1} to v_i we are in the situation of definition 2.3.11.

The notion of exact points generalises easily to prelinked Grassmannians, but they are harder to study. To overcome this obstacle, the notion of simple points was introduced in [Oss14]. When the graph G together with the data $\mathcal{E}_\bullet, f_\bullet$ is an $s-$linked chain, the notion of simple points coincides with the notion of exact points (see [Oss14]).

Definition 2.3.21. Let k be a field over S and $(F_v)_v$ a $k-valued$ point of a prelinked Grassmannian $\mathrm{LG}(r, \mathcal{E}_\bullet)$. We say that $(F_v)_v$ is simple if there exist $v_1, \ldots, v_r \in V(G)$ and $s_i \in F_{v_i}$ for $i = 1, \ldots, r$, such that for every $v \in V(G)$, there exist paths P^{v_i} with each P^{v_i} going from v_i to v and such that $f_{P^{v_1}}(s_1), \ldots, f_{P^{v_r}}(s_r)$ form a basis for F_v.

We end this section by giving the two main statements for simple points:

Proposition 2.3.22 ([Oss14]). *The following statements hold for simple points of a prelinked Grassmannian:*

1. *The simple points form an open subset of* $LG(r, \mathcal{E}_\bullet)$.

2. *On the locus of simple points* $LG(r, \mathcal{E}_\bullet)$ *is smooth over* S *of relative dimension* $r(d - r)$.

2.4 Tropical plane quartics and tropical genus 3 curves

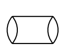
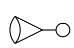
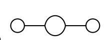
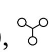
In this section, we discuss the relationship of tropicalised plane quartics and tropical genus 3 curves. For the sake of explicitness in our study in chapter 5, we pick as the ground field the field K of generalised Puiseux series in t (with real exponents), with valuation val sending a series to its least exponent [Mar10]. We use the max-convention.

2.4.1 The moduli spaces of tropical curves of genus 3

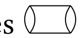
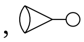
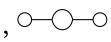
Given a combinatorial type of abstract tropical curves, the set of all tropical curves with this type can be parameterised by the quotient of an open orthant in a real vector space under the action of automorphisms of the underlying graph that preserve the genus at vertices. Cones corresponding to different combinatorial types can be glued together by keeping track of possible degenerations under the poset of combinatorial types. In this way, the tropical moduli space M_g^{trop} of curves of genus g inherits the structure of an abstract cone complex. For more details on tropical moduli spaces of curves, see e.g. [ACP15; CCUW17; Cha12; GKM09; Mik07].

Definition 2.4.1. We say that an abstract tropical curve is *maximal* it is appears in the interior of a top-dimensional cone of M_3^{trop}.

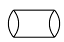
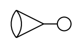
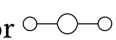
The combinatorial types of maximal tropical curves of genus 3 are:

 (000),  (020),  (111),  (212),  (303).

For the full poset of combinatorial types and cones, see [Cha12, figure 1].

A tropical curve is called hyperelliptic, if it has a divisor of degree 2 and rank 1. Equivalently, a tropical curve is hyperelliptic, if it admits a degree 2 cover of a tree (see [Cha13, theorem 3.13]). The cover must be balanced resp. harmonic, but does not have to satisfy the Riemann-Hurwitz condition (see definition 2.1.6). Hyperelliptic maximal tropical curves of genus 3 have types  ,  ,  or  , where in the first three pictures edges which form a 2-edge cut need to have the same lengths. A picture with the poset of all types of hyperelliptic curves of genus 3 is shown in [Cha13, figure 2].

Definition 2.4.2. We say that a tropical curve is *realisably hyperelliptic*, if it is hyperelliptic and the corresponding degree 2 cover is realisable, i.e. the local Hurwitz number at each vertex is non-zero.

Equivalently in our case of maximal genus 3 curves, we cannot have a 3-valent vertex mapping to a 3-valent vertex with each adjacent edge having weight 2 — such a local picture is excluded by the Riemann-Hurwitz condition for realisable tropical covers [ABBR15; Cap14]. Realisably hyperelliptic maximal tropical curves of genus 3 have types  ,  or  where edges which form a 2-edge cut need to have the same lengths (see figure 2.7).

2.4.2 Modifications, re-embeddings and coordinate changes

For our purposes, it is sufficient to consider modifications along tropical lines without a vertex. Here, we assume without restriction that we modify at a tropical line defined by the tropical polynomial $L = \max\{0, Y\}$. The graph of L considered as a function on $\mathbb{R}^2$ consists of two linear pieces. At the break line, we attach a two-dimensional cell spanned in addition by the vector $(0, 0, -1)$ (see e.g. [AR10, Construction 3.3]). We assign multiplicity 1 to each cell and obtain a balanced fan in $\mathbb{R}^3$. It is called the *modification of* $\mathbb{R}^2$ *along* L. If $\Gamma \subset \mathbb{R}^2$ is a plane tropical curve, we bend it analogously and attach downward ends to get the modification of Γ along L, which now is a tropical curve in the modification of $\mathbb{R}^2$ along L.

Let $\ell = m + y \in K[x, y]$ be a lift of L, i.e. $-\operatorname{val}(m) = 0$. We fix an irreducible polynomial $q \in K[x, y]$ defining a curve in the torus $(K^*)^2$. The tropicalisation of the variety defined by $I_{q,\ell} = \langle q, z - \ell\rangle \subset K[x, y, z]$ is a tropical curve in the

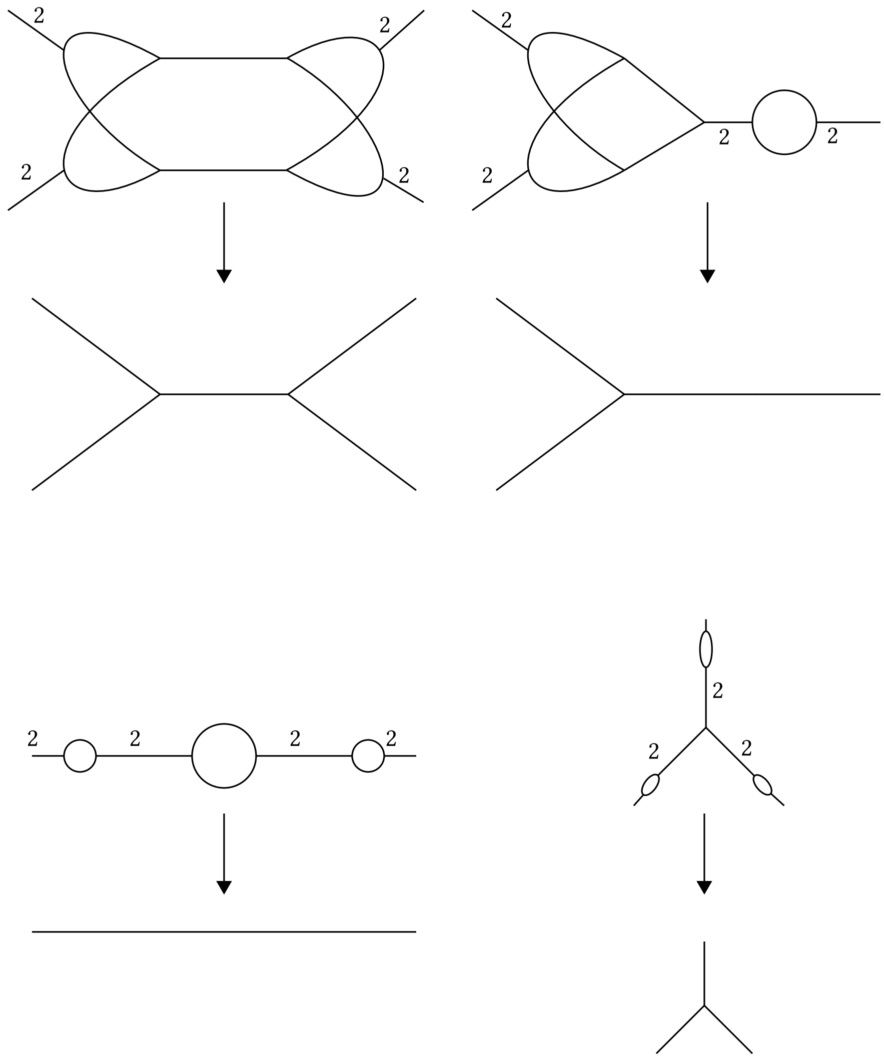


Figure 2.7: The hyperelliptic covers for the maximal genus 3 curves. We mark the edges of weight 2, while all unmarked edges are of weight 1. The two covers at the top of the picture and the cover on the bottom left are tropical morphisms and realisable as all local Hurwitz numbers are 1. The cover on the bottom right is not realisable as the Riemann-Hurwitz condition is not fulfilled.

modification of $\mathbb{R}^2$ along L. We call it the *linear re-embedding* of the tropical curve $\mathrm{Trop}(V(q))$ *with respect to* ℓ.

For almost all lifts ℓ, the linear re-embedding equals the modification of $\mathrm{Trop}(V(q))$ along L, i.e. we only bend $\mathrm{Trop}(V(q))$ so that it fits on the graph of L and attach downward ends. However, for some choices of lifts ℓ, the part of $\mathrm{Trop}(V(I_{q,\ell}))$ in the cell of the modification attached to the graph of L contains more attractive features. We are most interested in these special linear re-embeddings. We describe $\mathrm{Trop}(V(I_{q,\ell}))$ by means of two projections (see figure 2.8):

1. the projection π_{XY} to the coordinates (X, Y) produces the original tropical curve $\mathrm{Trop}(V(q))$.

2. the projection π_{XZ} gives a new tropical plane curve $\mathrm{Trop}(V(\tilde{q}))$ inside the projections of the cells $\{Y \geq 0, Z = Y\}$ and $\{Y = 0, Z \leq 0\}$, where $\tilde{q} = q(x, z - m)$. The polynomial $\tilde{q}$ generates the elimination ideal $I_{q,\ell} \cap K[x, z]$.

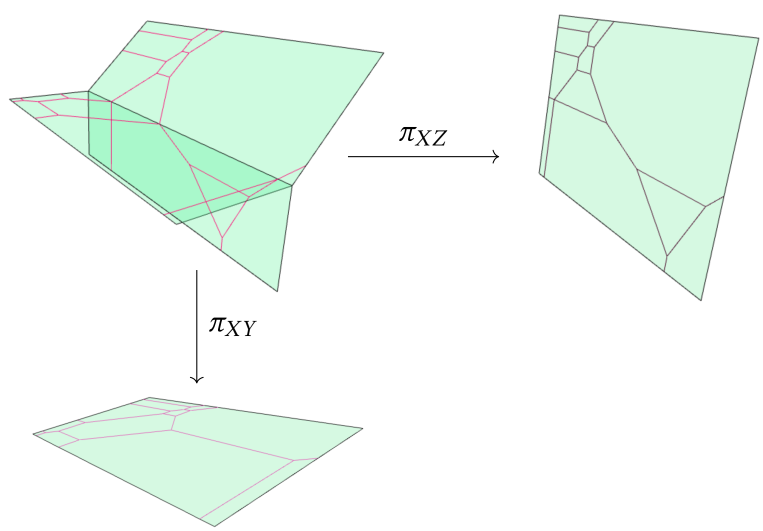
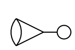


Figure 2.8: A tropical curve of type  and its two projections

By [CM16b, lemma 2.2] the projections above define $\mathrm{Trop}(V(I_{q,\ell}))$. The content of the lemma is to recover the parts of $\mathrm{Trop}(V(I_{q,\ell}))$ which are not contained in the interior of top-dimensional cells — for the images of the lower codimension cell the preimage under the projection is not unique. The projections are given by linear coordinate changes of the original curve. We can study the Newton subdivision of the projected curve in terms of these coordinate changes: A term $a \cdot x^i y^j$ of q is replaced by $a \cdot x^i (z - m)^j$, and so it contributes to all terms of the form $x^i z^k$ for $0 \leq k \leq j$. This is called the "feeding process" and is visualised in Figure 2 of [LM17]. From the feeding, we can deduce expected valuations of the coefficients. The subdivision corresponding to the expected valuations is dual to the projection of the modified curve. We care for cases in which there is cancellation and the expected valuation is not taken, these are the special re-embeddings that will make hidden geometric properties of $\mathrm{Trop}(V(q))$ visible.

2.4.3 Faithful tropicalisation and the forgetful map

As explained in the discussion prior to definition 2.1.5, the relation of Berkovich analytic spaces [Ber90] and tropical geometry [Pay09] hands us a way to overcome the challenge that extrinsic tropicalisation depends on the chosen coordinates. As such a faithful tropicalisation is the best candidate to reflect relevant geometric properties of the algebraic curves [BPR16]. This is due to the Berkovich skeleton, which can be obtained from a given (extended) skeleton by contracting it to its minimal expression [BPR13]. Notice that a smooth tropicalised curve in $\mathbb{R}^2$ (i.e. one dual to a unimodular triangulation) is faithful. Following [BPR16, Theorem 5.24] we can in general obtain faithfulness by ensuring that the tropical multiplicities of all vertices and edges on the skeleton equal one. For vertices, this means that the corresponding initial degeneration has to be irreducible, for edges, the weight has to be one. Since in all our constructions, only the first summands of the Puiseux series coefficients of a defining polynomial for a plane quartic are important, we can easily achieve this.

If we send an algebraic curve X of genus g to its minimal skeleton, we obtain an element in M_g^{trop}. If we have a faithful tropicalisation for X, we can construct this element in M_g^{trop} from $\mathrm{Trop}(X)$: we equip the graph $\Gamma = \mathrm{Trop}(X) \subset \mathbb{R}^n$ with a genus function on its vertices, and a length function on its edges. For the genus function, we use an extended Berkovich skeleton $\Sigma(X)$ coming from a semistable model of X with a horizontal divisor that is compatible with Γ [GRW16; GRW15]. To each vertex V in Γ we assign the sum of the genera of all semistable vertices of $\Sigma(X)$ mapping to V under $\mathrm{trop}\colon \Sigma(X) \to \mathrm{Trop}(X)$. The semistable vertices correspond exactly to the components of the central fiber X_0 [BPR13], so we equip them with the genus of the assocciated component. For a tropicalisation of a smooth curve in $\mathbb{R}^2$, where Γ is dual to a Newton subdivision, we assign to each vertex V of Γ the number of interior lattice points of its dual polygon. For the length function, note that every edge e of a tropicalised curve comes with a natural *direction vector* $v(e)$ (defined up to sign). In the case of tropicalised curves in $\mathbb{R}^2$, it is orthogonal to the dual edge in the Newton subdivision, and of the same length. The balancing condition ensures that the sum of the direction vectors of the edges adjacent to a vertex V (with appropriate signs) is zero. The weight of an edge equals the greatest common divisor of the coordinates of its direction vector. We define the *length* of an edge e to be the Euclidean length in $\mathbb{R}^n$ divided by the Euclidean norm of $v(e)$.

The *tropical forgetful map* $\mathrm{ft}^{\mathrm{trop}}$ shrinks all ends and leaf edges of a faithfully tropicalised curve, and equips the remaining graph with the genus function on its vertices and length function on its edges as above. If we start with a faithfully tropicalised quartic curve, we obtain an element in M_3^{trop} as image under the forgetful map.

CHAPTER 3

HURWITZ NUMBERS: PIECEWISE POLYNOMIALITY PROPERTIES AND PILLOWCASE COVERS

3.1 Introduction

In this chapter, we study various variants of Hurwitz numbers from different perspectives. We focus on some of the Hurwitz numbers, which have proved to be of particular interest. Simple double Hurwitz numbers $h_{b,\mu,\nu}$ count coverings of genus g and degree d of the Riemann sphere with two fixed ramification profiles μ and ν over the points 0 and ∞, and over the other b fixed points on $\mathbb{P}^1$, the ramification profile must be simple. Hence μ, ν are partitions of d and, by the Riemann-Hurwitz formula, $b = 2g - 2 + \ell(\mu) + \ell(\nu)$. There exist several modifications of the condition of simplicity on the intermediate ramifications, whose corresponding numbers also provide rich structures.

Two of those will be important in this thesis. Labelling the sheets of the covering from 1 to d, every intermediate simple ramification corresponds to a transposition $(a_i, b_i)_{i=1,\dots,b}$ that can be written such that $a_i < b_i$. The monotone double Hurwitz numbers are defined as the same count as the simple double Hurwitz numbers with the extra requirement that the coverings should satisfy the condition $b_i \leq b_{i+1}$. The strictly monotone Hurwitz case requires that $b_i < b_{i+1}$. For each of these three definitions, specialising ν to the trivial partition (1^d) one obtains the single version of the corresponding double Hurwitz number.

In the following, we give an in-depth exposition on recent developments in Hurwitz theory mentioned in chapter 1.

Simple single Hurwitz numbers

It was observed in [GJ97] that the single Hurwitz numbers in genus zero exhibit polynomiality in the entries of the partition μ, up to a combinatorial prefactor. The generalisation in any genus could later be derived from the celebrated Ekedahl–Lando–Shapiro–Vainshtein (ELSV) formula [ELSV01].

Simple double Hurwitz numbers

In [Oko00], Okounkov proved that the simple double Hurwitz numbers can be expressed in terms of the semi-infinite wedge formalism by exhibiting explicit operators. This rephrasing implies a relation to integrable systems of Kadomtsev-Petviashvili (KP) type—more precisely, the partition function of the double Hurwitz numbers is a tau-function of the KP integrable hierarchy.

A combinatorial approach to the simple double Hurwitz numbers appears in the foundational paper [GJV05] of Goulden, Jackson, and Vakil, in which it is proved that simple double Hurwitz numbers are piecewise polynomial in the entries of μ and ν. Roughly speaking, relative conditions on μ and ν determine hyperplanes (walls) in the configuration space of these partitions. The complement of the walls is divided in several distinct connected components, which are called chambers. The piecewise polynomiality property means that, inside each chamber, there exist a polynomial depending on the chamber whose evaluations at the entries of μ and ν coincide with the Hurwitz numbers under examination. Moreover, in the same paper they proposed a conjecture of *strong piecewise polynomiality*, proposing a lower bound on the degree of the polynomial. This lower bound is considered to be a strong indication of the connection with the intersection theory of moduli spaces, as it shows up as a consequence of the ELSV formula in the case of simple single Hurwitz numbers.

Wall-crossing formulae, semi-infinite wedge and tropical geometry

The chamber structure and wall-crossing formulae in genus zero for double Hurwitz numbers were studied with algebro-geometric methods by Shadrin, Shapiro, and Vainshtein [SSV08]. In particular, it was shown that the difference in polynomials between adjacent chambers may be expressed in terms of Hurwitz numbers with smaller input data, which yields a recursive structure. A tropical approach to double Hurwitz numbers has been developed by Cavalieri, Johnson, and Markwig [CJM10]. More precisely, the notion of *tropical double Hurwitz numbers* was introduced in terms of tropical covers of the tropical projective line and shown to coincide with the definition of algebro-geometric double Hurwitz numbers in a correspondence theorem. (In [BBM11], this was generalised to a tropical notion of Hurwitz numbers counting covers between arbitrary Riemann surfaces). This approach led the same authors to determine the chamber structure and to derive the above mentioned wall-crossing formulae in genus 0 in the same paper by means of tropical geometry. The chamber structure and recursive wall-crossing formulae in arbitrary genus were derived in [CJM11]. However, the strong piecewise polynomiality conjecture remained an open problem. Finally, this was solved by Johnson in [Joh15]. He used the operator language of [Oko00] to derive an explicit algorithm to compute the chamber polynomials, which led to recursive wall-crossing formulae. It was proved in [ACEH17] that double Hurwitz numbers satisfy the CEO topological recursion, which implies the existence of an ELSV-type formula. However, the concrete shape of this formula remains an open question.

Monotone Hurwitz numbers

The monotone Hurwitz numbers were introduced in [GGN14] as a combinatorial interpretation of the asymptotic expansion of the Harish-Chandra–Itzykson–Zuber (HCIZ) random matrix model.

Grothendieck dessins d'enfant or strictly monotone Hurwitz numbers

Dessins d'enfant were introduced by Grothendieck in [Gro97]. Their enumeration counts Hurwitz coverings of genus g and degree d over the Riemann sphere, with two ramifications μ and ν over 0 and ∞, and a single further ramification profile over 1, whose length is determined by the Riemann–Hurwitz formula.

The connection between strictly monotone Hurwitz numbers and dessins d'enfant counting is explained in the following.

Mixed cases

It is natural to interpolate several Hurwitz-type enumerative problems, by allowing different conditions on different blocks of intermediate ramifications. In fact hypergeometric tau functions for the 2D Toda integrable hierarchy were proved to have several explicit combinatorial interpretations [HO15]—one of them is in terms of mixed strictly monotone/weakly monotone double Hurwitz numbers, another one involves a mixed case of combinatorial problems, in which the part relative to the strictly monotone ramifications can be interpreted in terms of Grothendieck dessins d'enfant. This implies indirectly that the enumeration of Grothendieck dessins d'enfant and strictly monotone Hurwitz numbers coincide. A direct proof of this fact through the Jucys correspondence [Juc74] is derived in [ALS16].

A combinatorial study of the mixed monotone–simple double case can be found in [GGN16], in which piecewise polynomiality is proved.

CEO topological recursion and ELSV-type formulae for double Hurwitz numbers

A more recent development in Hurwitz theory is its relationship to *topological recursion in the sense of Chekhov–Eynard–Orantin (CEO)* [EO07]. In principal topological recursion is a technique appearing in string theory. Starting with a spectral curve, the CEO recursion associates a sequence of invariants. For many examples these invariants can be interpreted as counts of enumerative problem. It is a general fact that numbers satisfying the CEO recursion admit an expression in terms of the intersection theory of the moduli space of curves, although this expression may be hard to derive [DOSS14; Eyn14]. After it was proved that simple single Hurwitz numbers satisfy the CEO recursion [EMS11], the connection between the simple, monotone and Grothendieck dessins d'enfants Hurwitz numbers and CEO recursion has been an active field of research [DK15; DDM17; ALS16; DM14; DOPS17; KZ15; ACEH17].

3.1.1 Pillowcase covers and quasimodular forms

Instead of counting covers of the Riemann sphere one can consider Hurwitz numbers which deal with maps to more complicated surfaces. In this thesis, we study the important case of enumerating pillowcase coverings, which – as mentioned before – is related to the moduli space of quadratic differentials. The pillowcase orbifold is obtained by equipping the torus with a complex involution. It was shown that the corresponding generating series is a quasimodular form in [EO06], which yields a possibility to understand the asymptotics of the pillowcase count.

3.1.2 Outline of results

Piecewise polynomiality properties of Hurwitz-type counts

In section 3.2 and section 3.3, we give unified approaches to study the piecewise polynomiality of simple, monotone and strictly monotone double Hurwitz numbers. Inspired by the work in [GGN16], we introduce two interpolations between those enumerative invariants: *Triply interpolated Hurwitz numbers* and *triply mixed Hurwitz numbers.*

A tropical approach We approach the piecewise polynomiality of the above mentioned Hurwitz-type counts from the perspective of tropical geometry in section 3.2. We begin by defining triply interpolated Hurwitz numbers $H^{(2),\leq,<}_{p,q,r;\mu,\nu}$ in terms of the symmetric group. We use this definition to introduce a tropical interpretation of triply interpolated Hurwitz numbers in the flavour of [CJM10] and [DK15]. Using this description, we develop an algorithm to compute the polynomials in each chamber (see algorithm 3.2.22 for the case of genus 0 and algorithm 3.2.31 for the case of arbitrary genus g), which involves Erhart theory. More precisely, these algorithms involve the integration over lattice points in a polytope. Both algorithms specialise to the extremal cases of simple, monotone and strictly monotone double Hurwitz numbers.

Finally, we introduce a Hurwitz-type counting problem generalising triply interpolated Hurwitz numbers in genus 0, which is accessible by our algorithms as well. We give recursive wall-crossing formulae for this generalised counting problem, which in particular implies wall-crossing formulae for triply interpolated Hurwitz numbers.

A semi-infinite wedge approach In section 3.3, we introduce the notion of triply mixed Hurwitz numbers $h^{(2),\leq,<}_{p,q,r;\mu,\nu}$ and derive explicit formulae for their generating functions. In particular, this proves piecewise polynomiality of strictly monotone double Hurwitz numbers and provides a new explicit proof of the piecewise polynomiality of the mixed Hurwitz numbers case, and the obtained expressions allow us to derive wall-crossing formulae. These results specialise to the three types of Hurwitz numbers and to the mixed case of any pair.

Our methods rely on the application of an algorithm for scalar product computations in the infinite wedge space [Joh15], that we taylor slightly for our use. The new key ingredients to run the algorithm in this case are the operators for the monotone and the strictly monotone ramifications, derived in [ALS16].

Tropical pillowcase covers

In chapter 5, we study pillowcases from a tropical point of view. In particular, we introduce the notion of tropical pillowcase covers. We weight these covers by a product of local multiplicities which are Hurwitz numbers induced by the combinatorial data. This leads us to a correspondence theorem, which computes algebro-geometric pillowcase covers in terms of tropical geometry. We hope to relate this to further quasimodularity statements in future work.

3.2 A monodromy graph approach to the piecewise polynomiality of triply interpolated Hurwitz numbers

In this section we combine several methods to study the piecewise polynomiality of triply interpolated double Hurwitz numbers. We begin by introducing triply interpolated Hurwitz numbers and giving a tropical interpretation in the flavour of [DK15].

3.2.1 Triply interpolated Hurwitz numbers via monodromy graphs

We define triply interpolated Hurwitz numbers, which are a combinatorial interpolation between simple, monotone and strictly monotone Hurwitz numbers.

Definition 3.2.1. Let g be a non-negative integer, μ, ν partitions of the same positive integer, p, q, r non-negative integers, such that $p+q+r = 2g-2+\ell(\mu)+\ell(\nu)$. A triply interpolated factorisation of type (g, μ, ν, p, q, r) is a factorisation of type (g, μ, ν) (see conditions (1)–(6) definition 2.2.3), such that:

(7c) If $\tau_i = (r_i\ s_i)$ with $r_i < s_i$, we have $s_{i+1} \geq s_i$ for $i = 1, \dots, p-1$ and $s_{i+1} > s_i$ for $i = p, \dots, p+q-1$.

We denote the set of all triply interpolated factorisations by $\mathcal{F}(g, \mu, \nu, p, q, r)$ and define the triply interpolated Hurwitz number by

$$H^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \frac{1}{d!} \left|\mathcal{F}(g, \mu, \nu, p, q, r)\right| .$$

Remark 3.2.2. We note that for $p = q = 0$, we obtain the simple double Hurwitz numbers, for $q = r = 0$ the monotone double Hurwitz numbers and for $p = r = 0$ the strictly monotone double Hurwitz numbers associated to the data.

In [DK15], lemma 7, the following result was proved for monotone factorisations. It may be easily adapted for triply interpolated factorisations. For the convenience of the reader we give the proof. We work in the symmetric group ring. For an introduction to these techniques and their connection to Hurwitz numbers, see e.g. [CM16a] chapter 9.

Lemma 3.2.3. *Fix a permutation σ of cycle type μ and nonnegative integers p, q, r. The number of triply interpolated factorisation $(\sigma_1, \tau_1, \ldots, \tau_r, \sigma_2)$ of type (g, μ, ν, p, q, r), satisfying $\sigma_1 = \sigma$ does not depend on the choice of σ for $p = 0$ or $q = 0$.*

Proof. Fix a permutation σ of cycle type μ. Let $K^\bullet_\tau(\sigma)$ be the number of triply interpolated factorisations $(\sigma, \tau_1, \ldots, \tau_b, \sigma_2)$ of type $\tau = (g, \mu, \nu, p, q, r)$ where we drop the transitivity condition. We can rewrite the equation as follows

$$\sigma_2^{-1}\tau_b \cdots \tau_1 = \sigma^{-1}.$$

We see that $K^\bullet_{\tau,k}(\sigma)$ is the coefficient of σ^{-1} in

$$C_\nu h_p(J_2, \ldots, J_{|\nu|})(C_\kappa)^r \in \mathbb{C}[\mathcal{S}_d] \text{ for } q = 0, \tag{3.1}$$

$$C_\nu \sigma_q(J_2, \ldots, J_{|\nu|})(C_\kappa)^r \in \mathbb{C}[\mathcal{S}_d] \text{ for } p = 0, \tag{3.2}$$

where $\kappa = (2, 1, \ldots, 1) \vdash |\nu|$, C_w denotes the conjugacy class of permutations with cycle type w, h_i is the complete homogeneous symmetric polynomial of degree i, σ_i is the elementary homogeneous symmetric polynomial of degree i and J_i

denote the Jucys-Murphy elements

$$J_i = (1, i) + \cdots + (i-1, i) \in \mathbb{C}[\mathcal{S}_d]$$

for $i = 2, \ldots, |\nu|$. It is well known, that conjugacy classes and the symmetric polynomials in the Jucys-Murphy elements lie in the center of $\mathbb{C}[\mathcal{S}_d]$. Thus the expressions in equation (3.1) and equation (3.2) are a linear combination of conjugacy classes and therefore all permutations in the same conjugacy class appear with the same coefficient. Thus $K^{\bullet}_{\tau}(\sigma)$ only depends on the conjugacy class of σ. Now let $K^{\circ}_{\tau,k}(\sigma)$ be the number of factorisations as above that satisfy the transitivity condition. If σ is a $d-$cycle, where $d = \sum \mu_i$, then $K^{\bullet}_{\tau}(\sigma) = K^{\circ}_{\tau}(\sigma)$ and the result holds. For any permutation σ, set $\sigma = \Sigma_1 \cdots \Sigma_{\ell(\mu)}$ be the decomposition in disjoint cycles. We can decompose every non-transitive factorisation into a union of transitive factorisations. This leads to the following formula

$$K^{\bullet}_{\tau}(\sigma) = K^{\circ}_{\tau}(\sigma) + \sum_{s=2}^{\ell(\mu)} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_s = [\ell(\mu)] \\ \mu^{(1)} \sqcup \cdots \sqcup \mu^{(s)} = \mu = \nu \\ \nu^{(1)} \sqcup \cdots \sqcup \nu^{(s)}}} \sum_{\substack{p_1 + \cdots + p_s = p \\ q_1 + \cdots + q_s = q \\ r_1 + \cdots + r_s = r \\ g_1 + \cdots + g_s = g}} \prod_{l=1}^{s} K^{\circ}_{\tau_l}(\Sigma_{I_l}),$$

where the summation is over partitions of $[\ell(\mu)] = \{1, \ldots, \ell(\mu)\}$ into disjoint non-empty subsets $I_1 \sqcup \cdots \sqcup I_s$, ordered tuples of partitions $\mu^{(1)} \sqcup \cdots \sqcup \mu^{(s)}$ (resp. $\nu^{(1)} \sqcup \cdots \sqcup \nu^{(s)}$) whose union is μ (resp. ν), such that $|\nu^l| = |\mu^l|$ and

$$g_l = \frac{p_l + q_l + r_l + 2 - \ell(\mu) - \ell(\nu)}{2}$$

and where we use the notation $\tau_l = (g_l, \mu^{(l)}, \nu^{(l)}, p_l, q_l, r_l)$. Moreover, for $I \subset [\ell(\mu)]$, we let Σ_I denote the permutation obtained by taking the product of all cycles Σ_i for $i \in I$. We already proved that $K^{\bullet}_{\tau,k}(\sigma)$ only depends on the cycle type of σ for $p = 0$ or $q = 0$, by induction on the length of μ the terms $K^{\circ}_{\tau}(\Sigma_{I_l})$ only depend on the cycle type of Σ_{I_l}, thus $K^{\circ}_{\tau,k}(\sigma)$ only depends on the cycle type of σ and we are finished. $\square$

Thus, in order to compute $H^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ — when $p = 0$ or $q = 0$ — we do not have to count all triply interpolated factorisations of type (g, μ, ν, p, q, r), rather we may compute the number of triply interpolated factorisations $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ of type (g, μ, ν, p, q, r) with fixed σ_1 and multiply this number by $\frac{1}{d!} \cdot |\{\sigma \in S_d : C(\sigma) = \mu\}|$ to obtain $H^{\leq,<,(2)}_{p,q,r;\mu,\nu}$. We can thus simplify this counting problem with a smart choice of σ_1 (see Equation (3.3)), We translate the counting problem to a problem of counting monodromy graphs as in [CJM10], [CJM11] and [DK15]. In the latter, the choice of σ_1 as in Equation (3.3) was already utilised. To give our description of triply interpolated Hurwitz numbers in terms of monodromy graphs, we make the following choice for fixed $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$:

$$\sigma = (1 \cdots \mu_1)(\mu_1 + 1 \cdots \mu_1 + \mu_2) \cdots \left(\sum_{i=1}^{\ell(\mu)-1} \mu_i + 1 \cdots \sum_{i=1}^{\ell(\mu)} \mu_i \right), \tag{3.3}$$

where the cycle $\sigma_1^s = \left(\sum_{i=1}^{s-1} \mu_i + 1 \cdots \sum_{i=1}^{s} \mu_i \right)$ is labeled by s. We define $M^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ to be the number of triply interpolated factorisations of type (g, μ, ν, p, q, r) with σ_1 as in Equation (3.3). The number of permutations of cycle type μ with labeled cycles is

$$\epsilon(\mu) = \frac{d!}{\mu_1 \cdots \mu_{\ell(\mu)}}$$

and we see that

$$H^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \frac{1}{d!} \epsilon(\mu) M^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \frac{1}{\mu_1 \cdots \mu_{\ell(\mu)}} M^{\leq,<,(2)}_{p,q,r;\mu,\nu} \tag{3.4}$$

for $p = 0$ or $q = 0$. In particular equation (3.4) is true for the extremal cases of monotone and strictly monotone Hurwitz numbers.

We will express $M^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ in terms of monodromy graphs and begin by associating a graph to a triply interpolated factorisation.

Construction 3.2.4. Let $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ be a triply interpolated factorisation of type (g, μ, ν, p, q, r) with σ_1 as in Equation (3.3). We equip the graph we obtain from construction 2.2.10 setps (1)–(5) with additional structure:

Colouring the graph.

(6) We color all edges *normal*.

(7) We color all edges adjacent to in-ends *dashed*.

(8) We repeat steps [(9a)] and [(9b)] for all transpositions $\tau_1, \dots, \tau_{p+q}$.

(9a) [**Cut**] If τ_i is a transposition as in (3a), then we assume $\tau_i = (a\ b)$ with $a < b$ and $a \in \Sigma^s_{i-1}$ and $b \in \Sigma^{s'}_{i-1}$. Then we colour the edge labeled $\Sigma^{s'}_{i-1}$ *bold* and the outgoing egde at τ_i *dashed*.

(9b) [**Join**] If τ_i is a transposition as in (3b), we colour the edge labeled Σ^s_{i-1} *bold*. For $\tau_i = (a\ b)$ with $a < b$, we colour the outgoing edge corresponding to the cycle containing b *dashed*.

Distributing counters.
We distribute a counter to all non-normal edges.

(10) We start by distributing 1 to all edges adjacent to in-ends.

(11) For $\tau_i = (r_i\ s_i)$, where $r_i < s_i$, there is a unique way of expressing s_i as follows:

$$s_i = \sum_{j=1}^{l} \mu_j + c,$$

where $c < \mu_{j+1}$. Then we distribute c to the outgoing dashed/bold edge adjacent to the vertex labeled τ_i.

Relabelling the graph.

(12) We drop the labels τ_i at the vertices of $1, \dots, m$.

(13) We label the in-ends (resp. out-ends) by $1, \dots, \ell(\mu)$ (resp. $(1, \dots, \ell(\nu))$ according to the labels of σ_1 and σ_2.

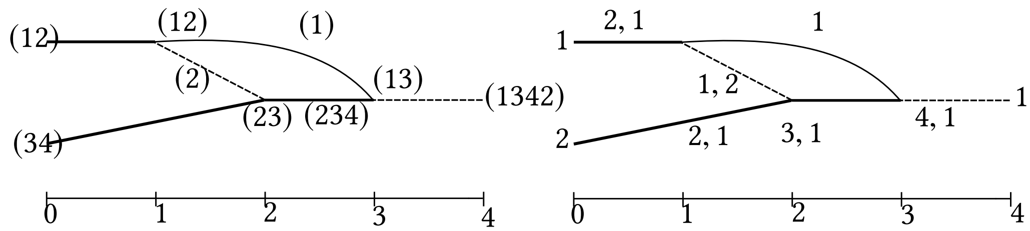


Figure 3.1: In the upper graph, the bold permutations correspond to transposistion τ_i, the other ones correspond to the cycles of the permutations $\tau_i \cdots \tau_1\sigma_1$. In the lower graph, the non-normal edges are bi-labeled, where the first number is the weight and the second is the counter.

(14) If a vertex or an edge is labeled by a cycle σ, we replace the label by the length of the cycle.

We obtain a graph Γ. We call Γ the *triply interpolated monodromy graph of type* (g, μ, ν, p, q, r) *associated to* $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$.

Example 3.2.5. On the left of figure 3.1, we illustrate the cut-and-join process for the following factorisation of type $(1, (2, 2), (4))$:

$$((12)(34), (12), (23), (13), (1243)).$$

In fact, this is a monotone factorisation, which can be view as a triply interpolated factorisation of type $(1, (2, 2), (4), 3, 0, 0)$ and the associated triply interpolated monodromy graph is illustrated on the right.

We now classify the graphs we obtain from construction 3.2.4. Moreover, we will understand how many triply interpolated factorisations yield the same monodromy graph. This result will be our main tool in the discussion of polynomiality.

Definition 3.2.6. A triply interpolated monodromy graph Γ of type (g, μ, ν, p, q, r) is a monodromy graph of type (g, μ, ν) (where $p + q + r = 2g - 2 + \ell(\mu) + \ell(\nu)$) with the following properties:

Colouring conditions.

The following conditions are only applied to edges adjacent to the first $p+q$ inner vertices.

6. We colour the edges of the graph by the three colours: normal, bold and dashed, such that each inner vertex is one of the six types in Figure 3.2.

7. There are no normal in-ends.

8. We call a connected path of bold edges beginning at an in-end a *chain*.

9. Let C and C' be two chains and let f_C (resp. $f_{C'}$) be the position of the first inner vertex of C (resp. C') and let l_C (resp. $l_{C'}$) be the position of the position of the last inner vertex of C (resp. C'). Then we require the intervals $[f_C, l_C]$ and $[f_{C'}, l_{C'}]$ to have empty intersection.

10. The intervals $[f_C, l_C]$ induce a natural ordering on the chains, namely $C < C'$ if $f_C < f_{C'}$. We require this ordering to be compatible with the ordering of the partition μ as follows: Let C_1 and C_2 be two chains of bold edges, i_1 and i_2 the respective in-ends, the we demand $C_1 < C_2$ if and only if $i_1 < i_2$.

The ordering of the chains corresponds to the monotonicity condition as we will see later.

Counter conditions.

11. We distribute a counter to each non-normal edge (thus, those inner edges are bi-labeled by the weight and the counter and the non-normal leaves are tri-labeled where the additional label is a number in $\{1, \ldots, \ell(\mu)\}$ or $\{1, \ldots, \ell(\nu)\}$).

12. The counter for each in-end is set to 1.

13. At each inner vertex v mapping to k, there is a unique incoming bold edge, whose counter we denote by i_k and a unique out-going non-normal edge,

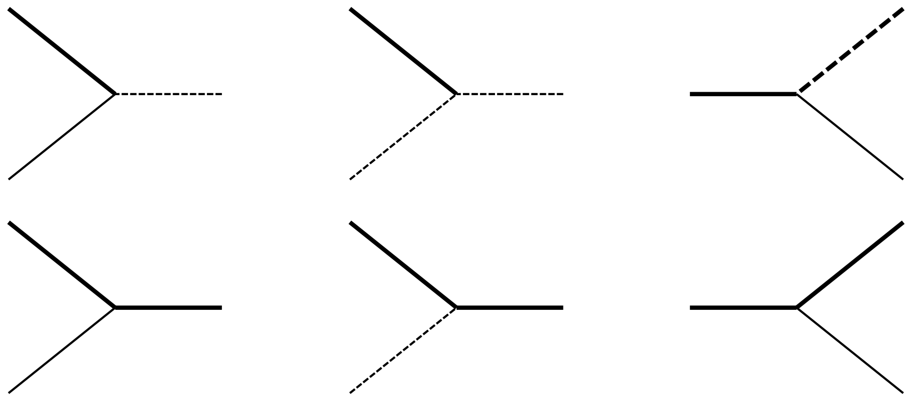


Figure 3.2: Local colouring of the graph.

whose counter we denote by o_k. For $k = 1, \dots, p$ we require $i_k \leq o_k$ and for $k = p+1, \dots, p+q$, we require $i_k < o_k$.

14. Every non-normal edge arises from a unique chain of bold edges: Every bold edge is part of a unique chain and every dashed edge is sourced at a unique chain. Let the non-normal edge e arise from the chain starting at the in-end labeled i. The counter l_e of the non-normal edge e is smaller or equal than μ_i and greater than $\mu_i - \omega(e)$.

The last condition reflects that these cycles corresponding to each such edge e should contain at least $\mu_i - (l_e - 1)$ elements.

Definition 3.2.7. Let Γ be a triply interpolated monodromy graph of type (g, μ, ν, p, q, r). We call the graph we obtain by removing the counters the *reduced monodromy graph of* Γ.
A graph Γ that appears as a triply interpolated monodromy graph of type (g, μ, ν, p, q, r) without counters is called a *reduced monodromy graph of type* (g, μ, ν, p, q, r).

We called the graph we obtained from construction 3.2.4 a triply interpolated monodromy graph. The following lemma justifies the choice of this term.

Lemma 3.2.8. *The graphs obtained from construction 3.2.4 are triply interpolated monodromy graphs in the sense of definition 3.2.6.*

Proof. Let $(\sigma_1, \tau_1, \ldots, \tau_b, \sigma_2)$ be a triply interpolated factorisation of type (g, μ, ν, p, q, r) with σ_1 as in Equation (3.3). The conditions (1)-(7) in Defintion 3.2.6 follow immediately by construction.

The chains of bold edges correspond to the following situation in the symmetric group setting: Suppose $\tau_i = (r_i\ s_i)$ for $r_i < s_i$. Since we chose σ_1 in Equation (3.3), we can group the transpositions τ_i for $i \leq k$. We say τ_i is of type t, if s_i is contained in the cycle of σ_1 labeled t. Now, for $i < j$, let t_i (resp. t_j) be the type of τ_i (resp. τ_j), then $t_i \leq t_j$.

A chain of bold edges starting at the in-end i corresponds to the transpositions of type i. Thus conditions (8)-(10) follow.

The counter conditions (11)-(13) follow by construction. For condition (14) we observe the following: Let e be a non-normal edge which arises from the chain of bold edges C and whose source vertex has position p. Let C start at i and let l_e be the counter of e. Moreover, let the i−th cycle of σ_1 be of the form

$$\left(\sum_{a=1}^{i-1} \mu_a + 1 \cdots \sum_{a=1}^{i} \mu_a \right).$$

Then for $\tau_p = (r_p\ s_p)$ we have $s_p = \sum_{a=1}^{i-1} \mu_a + l_e$. By monotonicity multiplying σ_1 by $\tau_1 \cdots \tau_p$ does not change the images of $\sum_{a=1}^{i-1} \mu_a + l_e, \ldots, \sum_{a=1}^{i} \mu_a - 1$. Thus the cycle of $\tau_p \cdots \tau_1 \sigma_1$ containing $\sum_{a=1}^{i-1} \mu_a + l_e$ has the following structure

$$\left(\cdots \sum_{a=1}^{i-1} \mu_a + l_e \cdots \sum_{a=1}^{i} \mu_a \cdots \right),$$

where the dots left and right indicate other elements. Thus the weight $\omega(e)$ of the weight e fulfils the following inequality

$$\omega(e) \geq \mu_i - l_e + 1$$

or equivalently

$$l_e > \mu_i - \omega(e).$$

Thus condition (14) is fulfilled as well. $\square$

Definition 3.2.9. An automorphism of a triply interpolated monodromy graph Γ is a graph automorphism $f : \Gamma \to \Gamma$, such that:

1. The function f respects labels, weights, colours and counters.

2. The following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & \Gamma \\ \downarrow & & \downarrow \\ [0, m+1] & \xrightarrow{\text{id}} & [0, m+1]. \end{array}$$

We denote the automorphism group of Γ by $\mathrm{Aut}(\Gamma)$.

We are now ready to give a weighted bijection between triply interpolated factorisations and triply interpolated monodromy graphs of type (g, μ, ν, p, q, r).

Lemma 3.2.10. *Let Γ be a triply interpolated monodromy graph of type (g, μ, ν, p, q, r). The number $m(\Gamma)$ of triply interpolated factorizations of type (g, μ, ν, p, q, r) with σ_1 as in Equation (3.3) for which construction 3.2.4 produces Γ is*

$$m(\Gamma) = \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod \omega(e),$$

where we take the product over all dashed and normal edges e, which are not adjacent to out-ends.

We call $m(\Gamma)$ the multiplicity of Γ.

Remark 3.2.11. An immediate consequence of this lemma is that the number $m(\Gamma)$ does not depend on the counters of Γ. We will use this in section 3.2.2.

Proof. Let v be one of the first $p + q$ inner vertices. If v is a cut, the corresponding transposition is uniquely defined by the weights of the outgoing edges and the

counter of the outgoing dashed or bold edge. If two edges are joined at v, the larger entry of the corresponding transposition is uniquely defined by the counter of the outgoing non-normal edge and the source chain of the in-going bold edge. However, we have a number of possibilites for the first element of the transposition, which is exactly the weight of the non-bold ingoing edge.
Now let v be an inner vertex whose position is greater than $p + q$. If v is a cut with ingoing edge e, there are $\omega(e)$ possibilities for τ_v, except when $\omega(e) = 2n$ and both outgoing edges have weight n. Then, there are only n possibilities for τ_v. If both cycles have distinguishable evolution, it matters which cycle has which evolution and obtain a factor of 2. If the cycles have undistinguishable evolution, this corresponds to a contribution of Aut(Γ).
If v is a join with ingoing edges e and e', the number of possibilites for τ_v is $\omega(e) \cdot \omega(e')$. Thus the lemma is proved. □

Example 3.2.12. The multiplicity of the graph in the right of Figure 3.1 is 2.

By our previous discussion we can compute triply interpolated Hurwitz numbers in terms of triply interpolated monodromy graphs.

Proposition 3.2.13. *Let p, q, r be positive integers and μ, ν ordered partitions of the same number d. Then:*

$$H^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \frac{1}{\mu_1 \cdots \mu_{\ell(\mu)}} \sum_{\Gamma} m(\Gamma) \textit{ for } p = 0 \textit{ or } q = 0,$$

where we sum over all triply interpolated monodromy graphs of type (g, μ, ν, p, q, r).

Proof. This is an immediate consequence of lemma 3.2.8, lemma 3.2.10 and Equation (3.4). □

3.2.2 Piecewise polynomiality

We want to use lemma 3.2.10 to study the piecewise polynomiality of triply interpolated Hurwitz numbers in the flavour of the discussion of section 6 of

[CJM10]. We begin by studying the genus 0 case and we will use Erhart theory to generalise these results to higher genera.

The genus 0 case

For the rest of this subsection, we assume $p+q+r=-2+\ell(\mu)+\ell(\nu)$. It is our aim to show that $M^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ is piecewise polynomial and to provide a constructive method to compute the polynomials in each chamber. By Equation (3.4) this also produces a method to compute the polynomials for $H^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ in each chamber when $p=0$ or $q=0$.

Proposition 3.2.14. *The function*

$$\begin{aligned} M^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)} : \mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)} &\to \mathbb{Q} \\ (\mu,\nu) &\mapsto M^{\leq,<,(2)}_{p,q,r;\mu,\nu} \end{aligned}$$

is piecewise polynomial, i.e. for every chamber C induced by the hyperplane arrangement $\mathcal{W}$, there exists a polynomial $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C) \in \mathbb{Q}[\underline{M},\underline{N}]$, such that $M^{\leq,<,(2)}_{p,q,r;\mu,\nu} = m^{\leq,<,(2)}_{p,q,r;\mu,\nu}(C)(\mu,\nu)$ for all $(\mu,\nu) \in C$.

Proof. The proof follows from lemma 3.2.18, corollary 3.2.19 and lemma 3.2.21. □

Remark 3.2.15. Note, that this proposition does not prove that the functions for triply interpolated double Hurwitz numbers, which we denote by $H^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$, are polynomials for $p=0$ or $q=0$, as these function differ from the functions $M^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$ by a factor of $\frac{1}{\prod \mu_i}$. It follows from theorem 2.2.9, that the polynomials $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C)$ contain a factor of $\prod_{i=1}^{\ell(\mu)} \mu_i$ — however this is not true for the contribution of each graph to $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C)$ as can be seen in example 3.2.23. By theorem 3.3.5, the same is true for the interpolation between simple and strictly monotone Hurwitz numbers, i.e. $p=0$. It would be interesting to see how the contributions for each graph add up to a polynomial which is divisible by $\prod \mu_i$.

We start by examining the edge weights in the equation

$$M^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \sum_{\Gamma} m(\Gamma) = \sum_{\Gamma} \prod \omega(e), \tag{3.5}$$

where we sum over all triply interpolated monodromy graphs of type $(0, \mu, \nu, p, q, r)$. We want to group some of the graphs appearing in the sum.

Definition 3.2.16. Let Γ be a reduced monodromy graph of type (g, μ, ν, p, q, r). Then we define the function

$$\begin{aligned} F(\Gamma, \mu, \nu, p, q, r) = \Big| \{ &\text{triply interpolated monodromy graphs} \\ \tilde{\Gamma} \text{ of type } &(g, \mu, \nu, p, q, r) \text{ with reduced monodromy graph } \Gamma \} \Big| . \end{aligned}$$

Then we can rewrite Equation (3.5) as follows

$$M^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \sum_{\Gamma} \prod \omega(e) F(\Gamma, \mu, \nu, p, q, r), \tag{3.6}$$

where we sum over all reduced monodromy graphs of type $(0, \mu, \nu, p, q, r)$.
For the remainder of this subsection, we prove the following three claims constructively:

1. The set of reduced monodromy graphs of type $(0, \mu, \nu, p, q, r)$ only depends on the chamber C (induced by the hyperplane arrangement $\mathcal{W}$) in which μ and ν are contained and not on the specific entries μ_i, ν_j. (lemma 3.2.18)

2. The product $\prod \omega(e)$ appearing in Equation (3.6) is a polynomial. (corollary 3.2.19)

3. The function $F(\Gamma, \nu, \nu)$ is a polynomial in each chamber. (lemma 3.2.21)

Claim (1) was actually observed in [CJM10] for the case of (non-triply interpolated) monodromy graphs. For the convenience of the reader, we repeat the argument. We begin by introducing some notation.

Notation 3.2.17. Let μ be a partition and let $I \subset \{1, \ldots, \ell(\mu)\}$. Then μ_I is the subpartition of μ given by $\mu_I = (\mu_{i_1}, \ldots, \mu_{i_{|I|}})$, where $i_j < i_{j+1}$.

Lemma 3.2.18. *The set of reduced monodromy graphs of type* $(0, \mu, \nu, p, q, r)$ *only depends on the chamber* C.

Proof. Let Γ be a triply interpolated monodromy graph of type $(0, \mu, \nu, p, q, r)$, then we cut Γ along e and obtain two triply interpolated monodromy graphs Γ_1 and Γ_2. Let e point away from Γ_1, then Γ_1 and Γ_2 are of respective type $(0, \mu_{I_1}, \nu_{J_1} \cup \{\omega(e)\}, k_1)$ and $(0, \mu_{I_2} \cup \{\omega(e)\}, \nu_{J_2}, k_2)$ for subsets $I_1, I_2 \subset \{1, \ldots, \ell(\mu)\}$ and $J_1, J_2 \subset \{1, \ldots, \ell(\nu)\}$. and $k_1 + k_2 = k$. Moreover, we have $|\mu_{I_1}| = |\nu_{J_1} \cup \{\omega(e)\}|$ and we obtain

$$\omega(e) = \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j.$$

The only requirement for a reduced monodromy graph to contribute to the sum Equation (3.6) is the positivity of all edge weights. As we saw above, this only depends on the chamber C we pick. ◻

Claim (2) immediately follows:

Corollary 3.2.19. *Every edge weight* $\omega(e)$ *is a linear polynomial in the entries of* μ *and* ν*. Thus* $\prod \omega(e)$ *is a polynomial in the entries of* μ *and* ν *as well.* ◻

Before we can prove claim (3), we need the following definition.

Definition 3.2.20. Let B a path in Γ starting at an in-end, such that

1. There are s edges in B.
2. The first $s - 1$ edges form a chain of bold edges. (see (8) of definition 3.2.6)
3. The last edge is dashed.

We call B a *chain-path of length* s.

The following lemma is our key step towards proposition 3.2.14.

Lemma 3.2.21. *The function $F(\Gamma,\mu,\nu,p,q,r)$ can be expressed as a polynomial in each chamber C.*

Proof. We fix a reduced monodromy graph Γ of type $(0,\mu,\nu,p,q,r)$. Assigning counters to Γ translates to assigning counters to each chain-path in Γ as follows: Fix a chain-path B of length s and distribute the counter l_k to the $k-$th edge e_k in B. Moreover, let B start at the in-end labeled i. Then $(l_1,\ldots,l_s)$ satisfies the counter conditions if and only if

1. $l_1 \leq \cdots \leq l_{s'} < \cdots <\leq l_s$ (see condition (13) in definition 3.2.6) for some $t < s$,
2. $1 \leq l_1 \leq \mu_i$ (see condition (14) in definition 3.2.6),
3. $\max\{1, \mu_i - \omega(e_k)\} \leq l_k \leq \mu_i$. (see condition (14) in definition 3.2.6)

Thus we need to prove, that the cardinality of the set

$$\{(l_1,\ldots,l_s) | l_1 \leq \cdots \leq l_{s'} < \cdots < l_s,\ l_1 = 1,\ \max\{1,\mu_i-\omega(e_k)\} \leq l_k \leq \mu_i\} \quad (3.7)$$

is piecewise polynomial in the entries of μ and ν. We can express this cardinality as the following iterative sum

$$\sum_{\substack{l_2= \\ \max\{1,\mu_i-\omega(e_1)\}}}^{\mu_i} \cdots \sum_{\substack{l_{s'}= \\ \max\{l_{s-1},\mu_i-\omega(e_2)\}}}^{\mu_i} \sum_{\substack{l_{s+1}= \\ \max\{l_{s-1}+1,\mu_i-\omega(e_2)\}}}^{\mu_i} \cdots \quad (3.8)$$
$$\sum_{\substack{l_{s-1}= \\ \max\{l_{s-2}+1,\mu_i-\omega(e_{s-1})\}}}^{\mu_i} \mu_i - \max\{l_{s-1}, \mu_i - \omega(e_s)\}.$$

If we know whether $\max\{\mu_i-\omega(e_1),1\} = \mu_i-\omega(e_1)$ and if we have a total ordering on the $\mu_i - \omega(e_k)$, we can compute this sum using Faulhaber's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j n^{p+1-j},$$

where B_j is the j–th Bernoulli number. Notice, that the right-hand side is a polynomial in n. Thus, the cardinality of the set in Equation (3.7) is a polynomial in μ_i and the edge weights $\omega(e)$ (and since $\omega(e)$ is linear form in the entries of μ and ν, the cardinality is a polynomial in the entries of μ and ν), whenever we know the value of $\max\{\mu_1 - \omega(e_1), 1\}$ and if we have a total ordering on the $\mu_i - \omega(e_k)$, which we can compute iteratively. Now, we show that choosing a chamber C for μ and ν implies those conditions.

Let $I_k \subset \{1, \ldots, \ell(\mu)\}$ and $J_k \subset \{1, \ldots, \ell(\nu)\}$ for $k = 1, \ldots, s$, such that

$$\omega(e_k) = \sum_{j \in I_k} \mu_j - \sum_{j \in J_k} \nu_j$$

for all k. We observe that for the edge e_1, we get

$$\max\{1, \mu_i - \omega(e_1)\} = \mu_i - \omega(e_1)$$

if and only if

$$\sum_{j \in J_1} \nu_j - \sum_{j \in I_1 - \{i\}} \mu_j > 0.$$

This implies that in a fixed chamber C, we know the value of $\max\{1, \mu_i - \omega(e_1)\}$. Moreover, we fix two edges e_j and e_k, such that $j < k$. We see that since e_j and e_k are in the same chain-path and e_k appears later than e_j, we have $I_j \subset I_k$ and $J_j \subset J_k$. Thus $\omega(e_k) > \omega(e_j)$ if and only if

$$\sum_{l \in I_k - I_j} \mu_l - \sum_{l \in J_k - J_j} \nu_l > 0.$$

Thus we can answer whether $\omega(e_k) > \omega(e_j)$ in each chamber.

Let $P_B(C)$ be the polynomial computing the cardinality of the set in Equation (3.7) associated to the chain-path B in the chamber C. Since we can choose counters in each chain-path independentely (they do not intersect, since chains of bold edges do not intersect), in the chamber C, the function $F(\Gamma, \mu, \nu, p, q, r)$ is given by $\prod P_B(C)$, where we take the product over all chain-paths. Since the graph is finite, $F(\Gamma,\mu,\nu,p,q,r)$ is a polynomial in the entries for μ and ν in each chamber C as desired. ◻

We have now derived the following algorithm, which computes the polynomials $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$ for $p+q+r=-2+\ell(\mu)+\ell(\nu)$, which also gives polynomials expressing $H^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$ in each chamber C for $p=0$ or $q=0$.

Algorithm 3.2.22. Computing polynomials in genus 0

procedure $\textsc{Hurwitz}(\ell(\mu),\ell(\nu),p,q,r,C)$ ▹ The polynomial expressing $M^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ and $H^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ in the chamber C for $p=0$ or $q=0$

  $\mathfrak{G}(\ell(\mu),\ell(\nu),C,k) \leftarrow$ Set of monodromy graphs of type $(0,\mu,\nu,p,q,r)$ in C

  for all $\Gamma\in\mathfrak{G}(\ell(\mu),\ell(\nu),C,k)$ **do**

    $E(\Gamma) \leftarrow$ Set of edges of Γ

    for all $e\in E(\Gamma)$ **do**

      $\omega(e) \leftarrow$ linear form in μ and ν expressing the weight of e

    end for

    $W(\Gamma) \leftarrow \prod\omega(e)$, where the product is taken over all non-bold edges which are not adjacent to out-ends

    $CP(\Gamma) \leftarrow$ Set of all chain-paths in Γ

    for all $P\in CP(\Gamma)$ **do**

      $q_P \leftarrow$ Polynomial expressing equation (3.8) in C

    end for

  end for

  $C(\Gamma) \leftarrow \prod_{P\in CP(\Gamma)} q_P$ ▹ Polynomial obtained from the chain paths

  $m(\Gamma) \leftarrow$ The polynomial $W(\Gamma)\cdot C(\Gamma)$ ▹ Polynomial obtained from the edge weights

  $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C) \leftarrow \sum_{\Gamma\in\mathfrak{G}(\ell(\mu),\ell(\nu),C,k)} m(\Gamma)$

  return $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$ ▹ The desired polynomial is $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$

  if $p=0$ or $q=0$ **then**

    return $\frac{1}{\mu_1\cdots\mu_{\ell(\mu)}} m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$ ▹ Polynomial expressing $H^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$

in C

end if

end procedure

$\square$

Example 3.2.23. We use this algorithm to compute the polynomials for $H^{\leq,<,(2)}_{0,2,0,\mu,\nu}$ for $\ell(\mu) = \ell(\nu) = 2$. The possible graphs are illustrated in Figure 3.3. There are four chambers in that case as illustrated in Figure 9 in [CJM10].

We start with the chamber C_1 given by $\mu_1 > \nu_1, \mu_1 > \nu_2, \mu_2 < \nu_1, \mu_2 < \nu_2$. In this chamber the graphs I.a, I.b, IV, V, VI, VII contribute positive multiplicities:

$$
\begin{aligned}
\text{mult(I.a)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} \right), \\
\text{mult(I.b)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} \right), \\
\text{mult(IV)} &= (\mu_1 - \nu_2)\mu_2(\mu_1 - \nu_2), \\
\text{mult(V)} &= (\mu_1 - \nu_1)\mu_2(\mu_1 - \nu_1), \\
\text{mult(VI)} &= (\mu_1 - \nu_2)\mu_2\nu_2, \\
\text{mult(VII)} &= (\mu_1 - \nu_1)\mu_2\nu_1.
\end{aligned}
$$

Adding all these contributions we obtain

$$
m^{\leq,<,(2)}_{0,2,0,\mu,\nu}(C_1) = \mu_1\mu_2(2\mu_1 + \mu_2 - \nu_1 - \nu_2 + 1) = \mu_1\mu_2(\mu_1 + 1).
$$

Next we look at the chamber C_2 given by $\mu_1 < \nu_1, \mu_1 > \nu_2, \mu_2 < \nu_1, \mu_2 > \nu_2$. In this chamber the graphs I.a, I.b, IV, V contribute positive multiplicities:

$$
\begin{aligned}
\text{mult(I.a)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} \right), \\
\text{mult(I.b)} &= \mu_1 \left(\mu_2\nu_2 - \frac{\nu_2^2}{2} + \frac{\nu_2}{2} \right),
\end{aligned}
$$

$$\begin{aligned}
\operatorname{mult}(\mathrm{III}) &= \mu_1\left(\frac{\mu_2^2}{2}+\frac{\mu_2}{2}-\mu_2\nu_2+\frac{\nu_2^2}{2}-\frac{\nu_2}{2}\right),\\
\operatorname{mult}(\mathrm{IV}) &= (\mu_1-\nu_2)\mu_2\nu_2,\\
\operatorname{mult}(\mathrm{VI}) &= (\mu_1-\nu_2)\mu_2(\mu_1-\nu_2).
\end{aligned}$$

Adding all these contributions we obtain

$$m^{\le,<,(2)}_{0,2,0,\mu,\nu}(C_2)=\mu_1\mu_2\left(\nu_1+1\right).$$

Let the chamber C_3 be given by $\mu_1<\nu_1, \mu_1<\nu_2, \mu_2>\nu_1, \mu_2>\nu_2$. In this chamber the graphs I.a, I.b, II, III contribute positive multiplicities:

$$\begin{aligned}
\operatorname{mult}(\mathrm{I.a}) &= \mu_1\left(\mu_2\nu_1-\frac{\nu_1^2}{2}+\frac{\nu_1}{2}\right),\\
\operatorname{mult}(\mathrm{I.b}) &= \mu_1\left(\mu_2\nu_2-\frac{\nu_2^2}{2}+\frac{\nu_2}{2}\right),\\
\operatorname{mult}(\mathrm{II}) &= \mu_1\left(\frac{\mu_2^2}{2}+\frac{\mu_2}{2}-\mu_2\nu_1+\frac{\nu_1^2}{2}-\frac{\nu_1}{2}\right),\\
\operatorname{mult}(\mathrm{III}) &= \mu_1\left(\frac{\mu_2^2}{2}+\frac{\mu_2}{2}-\mu_2\nu_2+\frac{\nu_2^2}{2}-\frac{\nu_2}{2}\right).
\end{aligned}$$

Adding all these contributions we obtain

$$m^{\le,<,(2)}_{0,2,0,\mu,\nu}(C_3)=\mu_1\mu_2(\mu_2+1).$$

Lastly, we consider the chamber C_4 given by $\mu_1>\nu_1, \mu_1<\nu_2, \mu_2>\nu_1, \mu_2<\nu_2$. In this chamber the graphs I.a, I.b, II, V contribute positive multiplicities:

$$\begin{aligned}
\operatorname{mult}(\mathrm{I.a}) &= \mu_1\left(\mu_2\nu_1-\frac{\nu_1^2}{2}+\frac{\nu_1}{2}\right),\\
\operatorname{mult}(\mathrm{I.b}) &= \mu_1\left(\frac{\mu_2^2}{2}+\frac{\mu_2}{2}\right),\\
\operatorname{mult}(\mathrm{II}) &= \mu_1\left(\frac{\mu_2^2}{2}+\frac{\mu_2}{2}-\mu_2\nu_1+\frac{\nu_1^2}{2}-\frac{\nu_1}{2}\right),
\end{aligned}$$

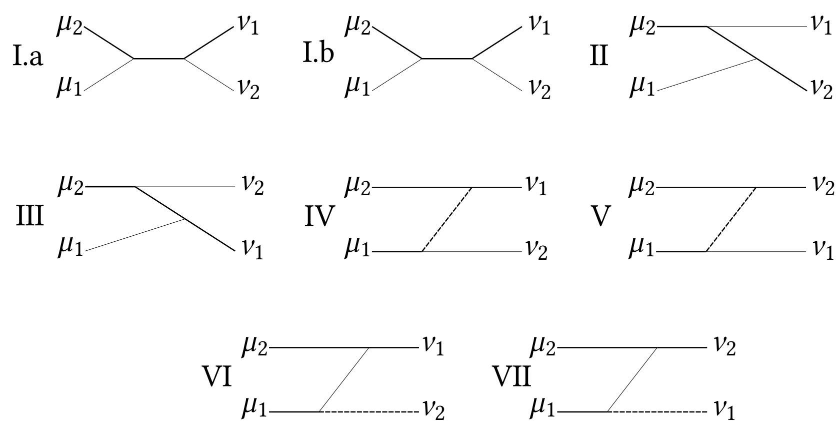


Figure 3.3: The graphs appearing for $(0, \mu, \nu, 2, 0, 0)$ for $\ell(\mu) = \ell(\nu) = 2$.

$$\mathrm{mult}(\mathrm{V}) = (\mu_1 - \nu_1)\mu_2\nu_2,$$
$$\mathrm{mult}(\mathrm{VII}) = (\mu_1 - \nu_1)\mu_2(\mu_1 - \nu_1).$$

Adding all these contributions we obtain

$$m^{\leq,<,(2)}_{0,2,0,\mu,\nu}(C_4) = \mu_1\mu_2\left(\nu_2 + 1\right).$$

Thus we see

$$H^{\leq,<,(2)}_{0,2,0,\mu,\nu}(C_1) = \mu_1 + 1,\ \ H^{\leq,<,(2)}_{0,2,0,\mu,\nu}(C_2) = \nu_1 + 1$$
$$H^{\leq,<,(2)}_{0,2,0,\mu,\nu}(C_3) = \mu_2 + 1,\ \ H^{\leq,<,(2)}_{0,2,0,\mu,\nu}(C_4) = \nu_2 + 1$$

Piecewise polynomiality in arbitrary genus

For higher genera, we once again study

$$M^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \sum m(\Gamma).$$

For the rest of this subsection we let g, p, q, r be non-negative integers, such that $p + q + r = 2g - 2 + \ell(\mu) + \ell(\nu)$ for the considered partitions μ, ν. Let Γ be a triply interpolated monodromy graph of type (g,μ,ν,p,q,r). We introduce a variable x_i for each of the g cycles in Γ. Then by a similar argument as before, each edge weight may be expressed as a linear polynomial in the entries of μ and ν and in the x_i. We still require all edge weights $\omega(e)$ to be greater than zero and thus obtain a hyperplane arrangement H in the entries of μ and ν and x_i. We will refine this hyperplane arrangement along the way and obtain the piecewise polynomiality result that way.

As in the genus 0 case, we rewrite the equation above by passing over to reduced monodromy graphs:

$$M^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \sum_{\Gamma} \prod \omega(e) F(\Gamma,\mu,\nu,p,q,r), \tag{3.9}$$

where we sum over all reduced monodromy graphs of type (g,μ,ν,p,q,r). Since there are additional variables x_i the order on each chain-path is no longer determined just by the entries of μ and ν. Thus, we need to refine the sum further. We begin by restricting to chambers due to a generalisation of lemma 3.2.18 which was proved in theorem 3.9 in [CJM11]:

Theorem 3.2.24 ([CJM11])**.** *The set of reduced monodromy graphs of type (g,μ,ν,p,q,r) is the same for all μ,ν in the same chamber C.*

Thus, we know that $\prod \omega(e)$ is a polynomial in the x_i and the entries of μ and ν in each chamber. Now, we want to express $F(\Gamma,\mu,\nu,p,q,r)$ as a polynomial as well. They key point in the proof of lemma 3.2.21 was the fact that the graph structure imposed an ordering on the edge weights $\omega(e)$ and 1. For higher genera this is no longer true as depending on the values of x_i there may be several orderings on each chain path (see example 3.2.32). To deal with this problem, we introduce the notion of an ordering on a reduced monodromy graph.

Definition 3.2.25. An ordering O on a reduced monodromy graph Γ is a partial ordering on the edge weights and 1, that restricted to each chain path and 1 is a total ordering. We denote by $\mathcal{O}(\Gamma)$ the possible orderings on Γ.

Next, we refine the function $F(\Gamma, \mu, \nu, p, q, r)$.

Definition 3.2.26. Let Γ be a reduced monodromy graph and O an ordering on Γ. Then we define $F(\Gamma, \mu, \nu, p, q, r, \underline{x}, O)$ (where $\underline{x} = x_1, \ldots, x_g$) to be the function counting all possible counter distributions on Γ compatible with O.

We want to argue that $F(\Gamma, \mu, \nu, p, q, r, \underline{x}, O)$ is a polynomial in the x_i and the entries of μ and ν. However, we have to be careful about the values of $\underline{x}$, since not all choices of $\underline{x}$ are compatible with O. Thus, we define $Q(\Gamma, \mu, \nu, O)$ to be the set of all values for x_i fulfilling the ordering O. It is easy to see, that this set is convex and the x_i are bounded since all edge weights have to be positive. We thus obtain the following lemma:

Lemma 3.2.27. *The set $Q(\Gamma, \mu, \nu, O)$ is a polytope with equations given by linear forms in the entries of μ and ν.*

Definition 3.2.28. We denote the hyperplane arrangement in $\mathbb{N}^{\ell(\mu)+\ell(\nu)}$ induced by the combinatorial types of $Q(\Gamma, \mu, \nu, O)$ by $\mathcal{V}(\Gamma, O)$

By same argument as in lemma 3.2.21, we also get the following lemma:

Lemma 3.2.29. *The function $F(\Gamma, \mu, \nu, p, q, r, \underline{x}, O)$ is a polynomial in $\underline{x}$ and the entries of μ and ν for $\underline{x} \in Q(\Gamma, \mu, \nu, O)$.*

We can now rewrite Equation (3.9) as follows:

$$M^{\leq,<,(2)}_{p,q,r;\mu,\nu} = \sum_{\Gamma} \sum_{O \in \mathcal{O}(\Gamma)} \sum_{\underline{x} \in Q(\Gamma,\mu,\nu,O)} \prod \omega(e) F(\Gamma, \mu, \nu, p, q, r, \underline{x}, O).$$

It is well-known that summing a polynomial over a polytope with rational vertices yields a quasi-polynomial (see e.g. [Woo14], [BBDKV14]).
Since $\prod \omega(e) F(\Gamma, \mu, \nu, p, q, r, \underline{x}, O)$ is a polynomial in $\underline{x}$ and the entries of μ and ν and since $Q(\Gamma, \mu, \nu, O)$ is a polytope, $M^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ is a quasi-polynomial in each chamber of the hyperplane arrangement given as the common refinement of $\mathcal{W}$ and the family $(\mathcal{V}(\Gamma, O))_{\Gamma,O}$.

Remark 3.2.30. We note that with our method, we only proved that we obtain piecewise quasi-polynomiality for $H^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$. However, we know by Theorem 2.2.9 that the triply interpolated Hurwitz number is a polynomial for $p = 0$. Moreover, in Algorithm 3.2.31 we pick one chamber C' induced by the refined hyperplane arrangement in C. However, by Theorem 2.2.9 the result does not depend on the choice of the finer chamber in C for $q = 0$. By theorem 3.3.11 the same is true for $p = 0$.

We can now state our algorithm for higher genera, which computes the polynomials for $H^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$ when $p = 0$ or $q = 0$..

Algorithm 3.2.31. Computing polynomials in genus g

procedure HURWITZ($\ell(\mu), \ell(\nu), p, q, r, C$) ▹ The polynomial expressing $M^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ and $H^{\leq,<,(2)}_{p,q,r;\mu,\nu}$ in the chamber C for $p = 0$ or $q = 0$
  $\mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k) \leftarrow$ Set of all monodromy graphs of type (g, μ, ν, p, q, r) in C
  for all $\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k)$ **do**
    $\mathcal{O}(\Gamma) \leftarrow$ Set of all orderings on Γ
    for all $O \in \mathcal{O}(\Gamma)$ **do**
      $Q(\Gamma, \mu, \nu, O) \leftarrow$ Polytope induced by the inequalities for Γ in C and O
      $\mathcal{V}(\Gamma, O) \leftarrow$ Hyperplane arrangement induced by the equations for $Q(\Gamma, \mu, \nu, O)$
    end for
    $C(\mathcal{O}) \leftarrow$ Common refinement of C and the family $\mathcal{V}(\Gamma, O)_{O\in\mathcal{O}(\Gamma)}$
  end for
  $C(\mu, \nu) \leftarrow$ Common refinement of C and the family of chambers we computed above $(C(\mathcal{O}(\Gamma)))_{\Gamma\in\mathfrak{G}(\ell(\mu),\ell(\nu),C,g,k)}$
  Choose some chamber C' in $C(\mathcal{O}(\Gamma))$
  for all $\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k)$ **do**
    $E(\Gamma) \leftarrow$ Set of all edges in Γ
    for all $e \in E(\Gamma)$ **do**

$\omega(e) \leftarrow$ linear form in of μ, ν and $\underline{x}$
end for
$W(\Gamma) \leftarrow \prod \omega(e)$, where we take the product over all non-bold edges $e \in E(\Gamma)$ which are not adjacent to out-ends
$CP(\Gamma) \leftarrow$ State of all chain-paths
for all $O \in \mathcal{O}(\Gamma)$ **do**
for all $P \in CP(\Gamma)$ **do**
$q_P(O) \leftarrow$ Polynomial expressing equation (3.8) with respect to the order O
end for
$c(O) \leftarrow \prod_{P \in CP(\Gamma)} g_P(O)$
end for
$m(\Gamma, C') \leftarrow \sum_{O \in \mathcal{O}(\Gamma)} \sum_{\underline{x} \in Q(\Gamma,\mu,\nu,O)} \prod W(\Gamma)c(O)$
end for
$m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C) = \sum_{\Gamma \in \mathfrak{G}(\ell(\mu),\ell(\nu),C,g,k)} m(\Gamma, C')$ ▷ The desired polynomial $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$
return $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$
if $p = 0$ or $q = 0$ **then**
return $\frac{1}{\mu_1 \cdots \mu_{\ell(\mu)}} m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$ ▷ Polynomial expressing $H^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$ in C
end if
end procedure

Example 3.2.32. In this example, we treat the graph Γ in Figure 3.4. The weight function is

$$\mu_1\mu_2(\mu_2 + \mu_3 - x_2).$$

The only chain path is given by $(\mu_3, \mu_2 + \mu_3, x_1, \mu_1 + x_1, \nu_1)$, thus the counter function is given by:

$$F(\Gamma, \mu, \nu, 3, 0, 0, \underline{x}) = \sum_{l_2=1}^{\mu_3} \sum_{\substack{l_3=\max \\ \{l_2, \mu_3-x_1+1\}}}^{\mu_3} \sum_{\substack{l_4=\max \\ \{l_3, \mu_3-\mu_1-x_1+1\}}}^{\mu_3} \mu_3 - l_4 + 1.$$

There are five different ordering:

$$O_1 : \nu_1 > \mu_2 + \mu_3 > \mu_1 + x_1 > x_1 > \mu_3$$
$$O_2 : \nu_1 > \mu_2 + \mu_3 > \mu_1 + x_1 > \mu_3 > x_1$$
$$O_3 : \nu_1 > \mu_2 + \mu_3 > \mu_3 > \mu_1 + x_1 > x_1$$
$$O_4 : \nu_1 > \mu_1 + x_1 > \mu_2 + \mu_3 > x_1 > \mu_3$$
$$O_5 : \nu_1 > \mu_1 + x_1 > \mu_2 + \mu_3 > \mu_3 > x_1$$

We show, how to compute the contributions for O_1 and O_2. For O_1 we obtain:

$$F(\Gamma, \mu, \nu, 3, 0, 0, \underline{x}, O_1) = \sum_{l_2=1}^{\mu_3} \sum_{l_3=l_2}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1.$$

For the ordering O_2, we get the following formula:

$$F(\Gamma, \mu, \nu, 3, 0, 0, \underline{x}, O_1) = \sum_{l_2=1}^{\mu_3-x_1+1} \sum_{l_3=\mu_3-x_1}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1 + \sum_{l_2=\mu_3-x_1+2}^{\mu_3} \sum_{l_3=l_2}^{\mu_3} \sum_{l_4=l_3} \mu_3 - l_4 + 1.$$

The ordering impose the following inequalities on x_1:

$$O_1 : \mu_2 + \mu_3 - \mu_1 > x_1 > \mu_3$$
$$O_2 : \min\{\mu_2 + \mu_3 - \mu_1, \mu_1\} > x_1 > \max\{0, \mu_3 - \mu_1\}$$

The inequality given by O_2 induces additional hyperplanes not given by equations of type $\sum : i \in I\mu_i - \sum_{j \in J} \nu_j$. The contributions of O_1 and O_2 (which we do not

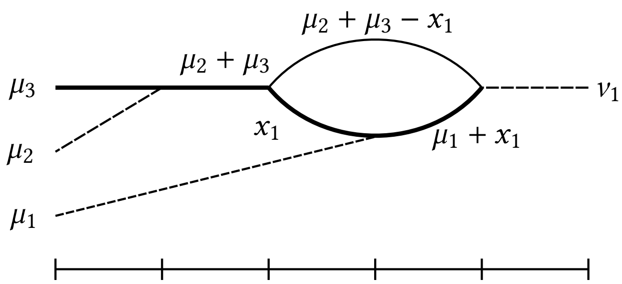


Figure 3.4: Triply interpolated monodromy graph of genus 1.

expand further, since the first sum alone expands to 20 terms) are

$$\sum_{x_1=\mu_3}^{\mu_2+\mu_3-\mu_1}\left(\mu_1\mu_2(\mu_2+\mu_3-x_2)\sum_{l_2=1}^{\mu_3}\sum_{l_3=l_2}^{\mu_3}\sum_{l_4=l_3}^{\mu_3}\mu_3-l_4+1\right)+$$
$$\sum_{x_1=\max\{0,\mu_3-\mu_1\}}^{\min\{\mu_2+\mu_3-\mu_1,\mu_1\}}\left(\mu_1\mu_2(\mu_2+\mu_3-x_2)\sum_{l_2=1}^{\mu_3-x_1+1}\sum_{l_3=\mu_3-x_1}^{\mu_3}\sum_{l_4=l_3}^{\mu_3}\mu_3-l_4+1+\right.$$
$$\left.\sum_{l_2=\mu_3-x_1+2}^{\mu_3}\sum_{l_3=l_2}^{\mu_3}\sum_{l_4=l_3}^{\mu_3}\mu_3-l_4+1\right),$$

which is a polynomial in each chamber of the refined hyperplane arrangement. We note that after computing the polynomial for every graph our method yields the same polynomial in each chamber (see remark 3.2.30), while this may not be true of each graph.

3.2.3 Chamber behaviour in genus 0

In this section, we define a counting problem in the symmetric group generalising triply interpolated Hurwitz numbers in genus 0. We use this to obtain recursive wall-crossing formulae in genus 0. As before, for fixed $\ell(\mu)$ and $\ell(\nu)$, let $m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C)$ be the polynomial expressing $M^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}$ in the chamber C. Moreover, let C_1 and C_2 be adjacent chambers seperated by the wall $\delta=\sum_{i\in I}\mu_i-\sum_{j\in J}\nu_j$ (w.l.o.g $\delta>0$ in C_1). We want to compute the wall-crossing for fixed $\ell(\mu)$ and $\ell(\nu)$

$$WC_{C_1}^{C_2}(0, \ell(\mu), \ell(\nu), p, q, r) = m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C_1) - m^{\leq,<,(2)}_{p,q,r;\ell(\mu),\ell(\nu)}(C_2).$$

We will study the wall-crossing formulae for $p = 0$ or $q = 0$, i.e. for interpolations between monotone and simple double Hurwitz numbers and interpolations between strictly monotone and simple double Hurwitz numbers.

Remark 3.2.33. To study this wall-crossing problem, we define related Hurwitz-type counts generalising triply interpolated Hurwitz numbers. It is an interesting feature of this Hurwitz-type counting problem that the wall-crossing induced by the piecewise polynomial structure can itself be expressed in terms of these Hurwitz numbers with smaller input data. For a precise formulation, see definition 3.2.36 and theorem 3.2.39.

Disclaimer. *As mentioned before we will study the problem for $p = 0$ or $q = 0$. As the discussions are completely parallel, we will only work out the details for $q = 0$ (i.e. the interpolation between monotone and simple double Hurwitz numbers).*

We classify those reduced monodromy graphs Γ having different multiplicity in C_1 than in C_2, since graphs with the same multiplicity cancel in $WC_{C_1}^{C_2}(0, \ell(\mu), \ell(\nu), p, 0, r)$. By our discussion in Section 3.2.2, there are five cases of graphs contributing to $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_1)$, which contribute a different multiplicity than $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_2)$:

(1a) The graphs contributing to $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_1)$ (resp. $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_2)$) having a normal edge of weight δ (resp. $-\delta$) emerging from one of the first p vertices.

(1b) The graphs contributing to $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_1)$ (resp. $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_2)$) having an non-normal edge of weight δ (resp. $-\delta$) emerging from one of the first p vertices.

(1c) The graphs contributing to $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_1)$ (resp. $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_2)$) having an edge of weight δ (resp. $-\delta$) emerging from one of the last r vertices.

(2a) The graphs contributing to $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_1)$ (resp. $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_2)$) with a chain-path (see definition 3.2.20) containing two edges e and e' (e coming before e'), such that $\omega(e) - \omega(e') = \delta$ (resp. $(\omega(e) - \omega(e') = -\delta)$.

(2b) The graphs contributing to $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_1)$ (resp. $m^{\leq,<,(2)}_{p,0,r;\ell(\mu),\ell(\nu)}(C_2)$) with a chain-path containing two edges e and e' (e coming before e'), such that $\omega(e) - \omega(e') = -\delta$ (resp. $\omega(e) - \omega(e') = \delta$).

Conditions (1a), (1b) and (1c) correspond to the fact that every edge weight must be greater than 0. Since $\delta < 0$ in C_2, the graph Γ has multiplicity 0 in that chamber. Conditions (2a) and (2b) correspond to changes in the polynomials computing the counters for each chain path: The polynomial can change if $\mu_i - \omega(e)$ (e contained in a chain path starting at μ_i) or $\omega(e) - \omega(e')$ (e and e' contained in the same chain path) changes sign by crossing δ. Note that $\mu_i - \omega(e) = \omega(e') - \omega(e)$ for e' being the in-end μ_i. Thus, we obtain the two cases (2a) and (2b).
The following idea will be our main tool in this section: We start with a triply interpolated monodromy graph and cut it along some distinguished edge (resp. two distinguished edges). Since the graphs are of genus 0, we obtain two (resp. three) new triply interpolated monodromy graphs. As a first step, we classify the pairs (resp. triples) of graphs we can obtain by this cutting process. Our second step is a regluing process. We will glue our graphs from two (resp. three) smaller graphs. The key observation here is, that in order to obtain a triply interpolated monodromy graph again, this gluing has to respect the following:

1. The ordering of the chains of bold edges.

2. The monotonicity of the counters, if we glue edges to a new chain-path.

In order to formalise this, we introduce a new and more general Hurwitz-type counting problem, where these two conditions are framed in terms of what we call *start and end conditions*. In some sense, these start and end conditions store the information concerning the counter and position of the edge, which we cut in the first place. Analysing this regluing process, we obtain a recursive wall-crossing formula for this more general counting problem. To understand the general idea, we start by decomposing the graphs above into smaller graphs and thus make the mentioned cutting process more precise.

(1a) Let Γ be a triply interpolated monodromy graph of type $(0, \mu, \nu, p, 0, r)$ with a normal edge of weight $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$, emanating from one of the first $p+q$ vertices. We cut the graph along the edge δ and obtain two graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), p_1, 0, r_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, p_2, 0, r_2)$ with $p_1 + p_2 = p$, and $r_1 + r_2 = r$. Starting with two triply interpolated monodromy graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), p_1, 0, r_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, p_2, 0, r_2)$, we want to glue them along the edge corresponding to δ. However, this does not always yield a triply interpolated monodromy graph (e.g: If δ is a normal edge in Γ_1, but a bold edge in Γ_2.) Thus, we need some compatibility condition for these graphs. In fact, in order to obtain a triply interpolated monodromy graph of type $(0, \mu, \nu, p, 0, r)$ with a normal edge of weight δ, the edge δ must be normal in Γ_1 and dashed in Γ_2. Furthermore, if δ emanates from a chain of bold edges starting at μ_l in Γ_1 the edge δ must join with a chain of bold edges starting at μ_j in Γ_2 where $j > l$ (since the dashed and normal edges connect chains of bold edges). This corresponds to the end condition of type $(1, l, i)$ for Γ_1 and the start condition of type $(1, l)$ for Γ_2 in definition 3.2.35, where i is the label we choose for the out-end of Γ_1 corresponding to δ.

(1b) Let Γ be a triply interpolated monodromy graph of type $(0, \mu, \nu, p, 0, r)$ with a non-normal edge of weight $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$. As before, we cut along δ and obtain two graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), p_1, 0, r_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, p_2, 0, r_2)$. Gluing two graphs Γ_1 and Γ_2 of these types along δ, we see that in order to obtain a graph as in (1b), there are two types of conditions: Either δ is dashed in both Γ_1 and Γ_2 and if δ is contained in a chain path starting μ_l in Γ_1, the in-end δ in Γ_2 must join with a chain of bold edges starting at μ_j in Γ_2 with $j > l$. Alternatively, δ is dashed in Γ_1 and bold in Γ_2. Morever if δ has counter c in Γ_1, the first inner edge of the chain of bold edges starting at δ in Γ_2 must have counter $c' > c$. This corresponds to the end condition of type $(2, i, l, c)$ for Γ_1 and start condition of type $(2, l, c)$ in definition 3.2.35, where i is the label we choose for the out-end of Γ_1 corresponding to δ.

(1c) Let Γ be a triply interpolated monodromy graph of type $(0, \mu, \nu, p, 0, r)$ with a normal edge of weight $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$, emanating from one of the last r vertices. As before, we cut along δ and obtain two graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), p_1, 0, r_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, p_2, 0, r_2)$. Here the only condition for the gluing process we require is δ only interacting with one of the last r_j vertices in Γ_j $(j = 1, 2)$. This corresponds to the end and start condition $(3, i)$ in definition 3.2.35, where i is the label we choose for the out-end of Γ_1 corresponding to δ.

(2a) We once again start with a triply interpolated monodromy graph of type $(0, \mu, \nu, p, 0, r)$. We impose the condition that there is a chain path with two edges e_1, e_2 (e_1 appearing before e_2), such that

$$\omega(e_1) - \omega(e_2) = \delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j.$$

For

$$\omega(e_1) = \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j \tag{3.10}$$

and

$$\omega(e_2) = \sum_{i \in I_2} \mu_i - \sum_{j \in J_2} \nu_j, \tag{3.11}$$

and $\omega(e_1) - \omega(e_2) = \delta$ this translates to $I_2 = I_1 \sqcup I^c$ and $J_2 = J_1 \sqcup J^c$. Cutting along the edges e_1 and e_2, we obtain three graphs Γ_1, Γ_2 and Γ_3 of respective types

$$(0, \mu_{I_1}, \left(\nu_{J_1}, \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j\right), p_1, 0, r_1), \tag{3.12}$$

$$(0, \left(\mu_{I^c}, \sum_{i\in I_1} \mu_i - \sum_{j\in J_1} \nu_j\right), \left(\nu_{J^c}, \sum_{i\in I_2} \mu_i - \sum_{j\in J_2} \nu_j\right), p_2, 0, r_2), \tag{3.13}$$

$$(0, \left(\mu_{I_2^c}, \sum_{i\in I_2} \mu_i - \sum_{j\in J_2} \nu_j\right), \nu_{J_2^c}, p_3, 0, r_3), \tag{3.14}$$

where $p_1 + p_2 + p_3 = p$ and $r_1 + r_2 + r_3 = r$. Regluing graphs of these respective types corresponds to the gluing process in (1b). Thus we need an end condition of type $(2, l, c, i)$ for Γ_1, start condition of type $(2, l, i)$ for Γ_2, end condition of type $(2, l, c, i)$ for Γ_2 and start condition of type $(2, l, i)$ for Γ_3. (If $I_1 = p$ and $J_1 = \emptyset$, we only cut at $\delta + \mu_p$ and thus obtain only Γ_2 and Γ_3.)

(2b) Starting with a triply interpolated monodromy graph of type $(0, \mu, \nu, p, 0, r)$, with a chain path containing two edges e_1 and e_2 (with e_1 appearing before e_2, such that

$$\omega(e_1) - \omega(e_2) = -\delta = \sum_{i\in I^c} \mu_i - \sum_{j\in J^c} \nu_j.$$

For $\omega(e_1)$ and $\omega(e_2)$ as in Equation (3.10) and (3.11) respectively, and $\omega(e_1) - \omega(e_2) = \delta$ this translates to $I_2 = I_1 \sqcup I$ and $J_2 = J_1 \sqcup J$. Similar as in (2a), we cut along e_1 and e_2 to obtain three graphs Γ_1, Γ_2 and Γ_3 of respective types

$$(0, \mu_{I_1}, \left(\nu_{J_1}, \sum_{i\in I_1} \mu_i - \sum_{j\in J_1} \nu_j\right), p_1, 0, r_1),$$

$$(0, \left(\mu_I, \sum_{i\in I_1} \mu_i - \sum_{j\in J_1} \nu_j\right), \left(\nu_J, \sum_{i\in I_2} \mu_i - \sum_{j\in J_2} \nu_j\right), p_2, 0, r_2),$$

$$(0, \left(\mu_{I_2^c}, \sum_{i\in I_2} \mu_i - \sum_{j\in J_2} \nu_j\right), \nu_{J_2^c}, p_3, 0, r_3),$$

where $p_1 + p_2 + p_3 = p$ and $r_1 + r_2 + r_3 = r$. Regluing graphs of these respective types, we need to impose the same end and start conditions as in (2a).

Notation 3.2.34. We fix two partitions μ and ν.

1. For a subset $I = \{i_1, \dots, i_n\}$ (where $i_j < i_{j+1}$) of $\{1, \dots, \ell(\mu)\}$ and positive integers δ, j ($j \notin I$), we denote by $(\mu_I, \delta)_j$ the partition $(\mu_{i_1}, \dots, \mu_{i_j}, \delta, \mu_{i_j+1}, \dots, \mu_{i_n})$.

2. Let $(\sigma_1, \tau_1, \dots, \tau_r, \sigma_2)$ be a triply interpolated factorisation with $C(\sigma_1) = \mu$ and $C(\sigma_2) = \nu$. We define $\tau^1(l)$ to be the transposition with the biggest position containing elements of the cycle of σ_2 labeled l. Moreover, let $t^1(l)$ be the position of $\tau^1(l)$.

3. Let $(\sigma_1, \tau_1, \dots, \tau_r, \sigma_2)$ be a triply interpolated factorisation with $C(\sigma_1) = \mu$ and $C(\sigma_2) = \nu$. We define $\tau^2(l)$ to be the transposition with the smallest position containing elements of the cycle of σ_1 labeled l. Morover, let $t^2(l)$ to be the position of $\tau^2(l)$.

Definition 3.2.35. Let μ, ν, p, q, r be data as before. Let $\eta = (\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ be a triply interpolated factorisation of type $(0, \mu, \nu, p, 0, r)$, such that $\tau_i = (r_i\ s_i)$. We begin by defining end-conditions:

1. We say η satisfies end condition $(1, l, i)$ if

   - $t^1(i) \leq k$
   - $\sum_{j=1}^{l-1} \mu_j + 1 \leq s_{t^1(i)} \leq \sum_{k=1}^{l} \mu_i$

   In monodromy graph language, this corresponds to the following picture: The out-end corresponding to ν_i is coloured normal and emanates from the chain of bold edges starting at μ_l (see Figure 3.5).

2. We say η satisfies end condition $(2, l, c, i)$ if

   - $t^1(i) \leq k$

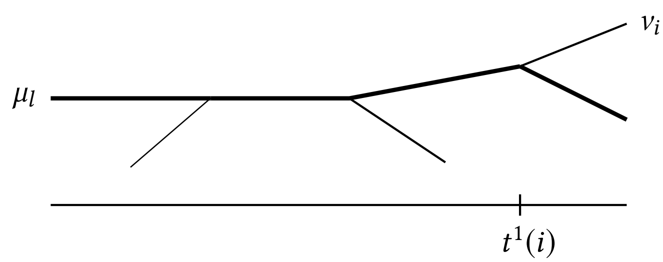


Figure 3.5: Schematic drawing of end condition $(1, l, i)$.

- $s_{t^1(i)} = \sum_{j=1}^{l-1} \mu_j + c$ for $0 \leq c < \mu_l$

In monodromy graph language, this corresponds to the following picture: The out-end corresponding to v_i is coloured dashed, has counter c and emanates from the chain of bold edges starting at μ_l. (See Figure 3.6).

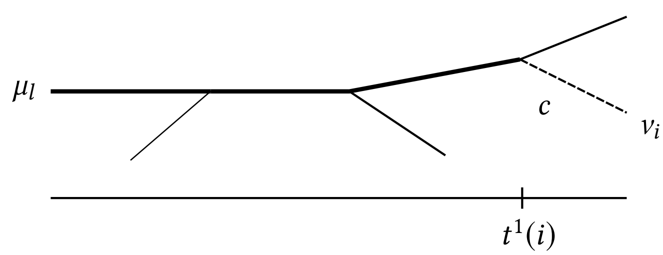


Figure 3.6: Schematic drawing of end condition $(2, l, c, i)$.

3. We say η satisfies end condition $(3, i)$ if
   - $t^1(i) > k$

   In monodromy graph language, this corresponds to the following picture: The out-end corresponding to v_i emanates from a vertex whose position is greater than k.

Now we define start conditions.

1. We say η satisfies start condition $(1, l)$ if

   - $t^2(l) \leq k$
   - $s_{t^2(l)} \geq \sum_{j=1}^{l} \mu_j + 1$

   In monodromy graph language, this corresponds to the following picture: The in-end corresponding to μ_l is coloured dashed and joined with a chain of bold edges emanating at $\mu_{l'}$ for $l' > l$. (See Figure 3.7).

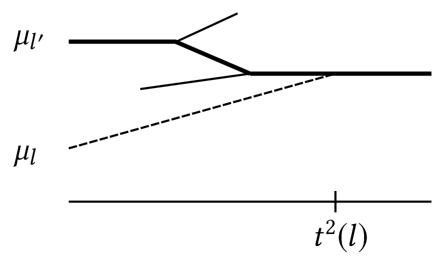


Figure 3.7: Schematic drawing of end condition $(1, l)$ for $l' > l$.

2. We say η satisfies start condition $(2, l, c)$ if

   - $t^2(l) \leq k$
   - $s_{t^2(l)} \geq \sum_{j=1}^{l-1} \mu_j + c$ for $0 \leq c < \mu_l$

   In monodromy graph language, this corresponds to one of the following two pictures: Either μ_l is joined with a chain of bold edges emanating at μ_m for $m > l$ or there is a chain of bold edges starting at μ_l and the first counter is greater or equal to c.

3. We say η satisfies start condition $(3, l)$ if

   - $t^2(l) > k$

   In monodromy graph language, this corresponds to the following picture: The in-end corresponding to μ_l is adjacent to a vertex whose position is greater than k.

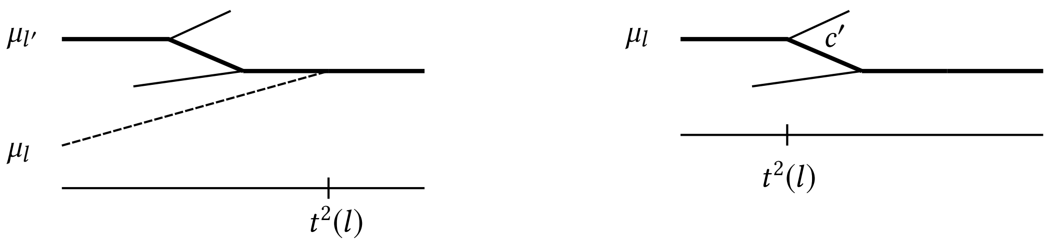


Figure 3.8: Schematic drawing of the two possible graphs for end condition $(2, l, c)$: $l' > l$ in the left graph;$c' \geq c$ in the right graph.

Definition 3.2.36. Let S be a set of start conditions and E a set of end conditions, i.e. for each $i \in [\ell(\mu)]$ (resp. $j \in [\ell(\nu)]$), there exists at most one tuple $(1, l)$, $(2, l, c)$ or $(3, l)$ (resp. $(1, l, i)$, $(2, l, c, i)$ or $(3, i)$) for some $l \in \{1, \ldots, \ell(\mu)\}, c \in \{0, \ldots, \mu_l - 1\}$ (resp. $i \in [\ell(\nu)], l \in [\ell(\mu], c \in \{0, \ldots, \mu_l - 1\}$), which is contained in S, resp. E. Then we define $M^{\leq,<,(2)}_{p,0,r}(\mu, \nu, S, E)$ to be the number of all triply interpolated factorisations $(\sigma_1, \tau_1, \ldots, \tau_m, \sigma_2)$ of type $(0, \mu, \nu, p, 0, r)$ with σ_1 as in Equation (3.3) satisfying the conditions in E and S.

Remark 3.2.37. For $E = S = \emptyset$, we obtain triply interpolated Hurwitz numbers. Moreover, our methods from subsection 3.2.2 can be applied to this generalised version to obtain piecewise polynomiality in the entries of μ and ν and the information in S and E with chambers given by $\mathcal{W}$.
By the same arguments as in Section 3.2.2, $M^{\leq,<,(2)}_{p,0,r}(\mu, \nu, S, E)$ may be expressed as a polynomial in the entries of μ and ν in each chamber induced by the hyperplance arrangement $\mathcal{W}$. We denote the polynomial in the chamber C by $m^{\leq,<,(2)}_{p,0,r}(\mu, \nu, S, E)(C)$.

Before we are ready to state the main theorem of this section, we introduce some notation.

Notation 3.2.38. Let μ be an ordered partition, $i \in \{1, \ldots, \ell(\mu)\}$ and δ an integer, then we define the partition $(\mu, \delta)_i = \left(\mu_1, \ldots, \mu_{i-1}, \delta, \mu_i, \ldots, \mu_{\ell(\mu)}\right)$.
Moreover, let S be a set of start conditions and $I \subset \{1, \ldots, \ell(\mu)\}$, then S_I is the set of all start condition $(1, l), (2, l, c)$ oder $(3, l)$ with $l \in I$.

Figure 3.9: The case (1a) and (2a) simultaneously on the left, the case (1b) and (2a) simultaneously on the right.

Theorem 3.2.39. *Let μ and ν be ordered partitions, p, q, r non-negative integers (yielding genus 0) and C_1, C_2 chambers separated by the wall defined by $\delta = \sum_{i\in I}\mu_i - \sum_{j\in J}\nu_j$ for subsets $I \subset \{1,\ldots,\ell(\mu)\}, J \subset \{1,\ldots,\ell(\nu)\}$. Then*

$$W_{C_1}^{C_2}(\mu,\nu,k) = m^{\leq,<,(2)}_{p,0,r;\mu,\nu}(\mu,\nu,S,E)(C_1) - m^{\leq,<,(2)}_{p,0,r;\mu,\nu}(\mu,\nu,S,E)(C_2)$$

is a sum of products with factors of δ and two or three polynomials $m^{\leq,<,(2)}_{p',0,r'}(\mu',\nu',S',E')(C_1)$ (resp. $m^{\leq,<,(2)}_{p',0,r'}(\mu',\nu',S',E')(C_2)$), where the apostrophes indicate smaller input data. More precisely, the data $p', 0, r', \mu', \nu', S', E'$ has to satisfy $p' < p$, $r' < r$, $\mu' = (\mu_I, \delta)_i$ (for some $i \in I$), $\nu' = (\nu_J, \delta)_j$ (for some $j \in J$), S' is the union of S_I and a start condition corresponding to the entry δ and E' the union of E_J and an end condition corresponding to the entry δ for $m_0^{k'}(\mu',\nu',S',E')(C_1)$ (resp. replacing δ by $-\delta$ for $m_0^{k'}(\mu',\nu',S',E')(C_2)$).

Remark 3.2.40. The same theorem is true for $p = 0$ and arbitrary q where we must make the according adjustments in the definition of start and end conditions (i.e. change inequalities for counters to strict inequalities for counters in the regluing process).

Proof of Theorem 3.2.39. By the discussion at the beginning of this section, we already have an interpretation of the cutting and gluing process in terms of the polynomials $m^{\leq,<,(2)}_{p',0,r'}(\mu',\nu',S',E')(C_1)$ and $m^{\leq,<,(2)}_{p',0,r'}(\mu',\nu',S',E')(C_2)$. The proof is straightforward but involves many cases. We will work out the formula for (2a), since this is the most difficult case. The remaining cases can be worked out analogously.

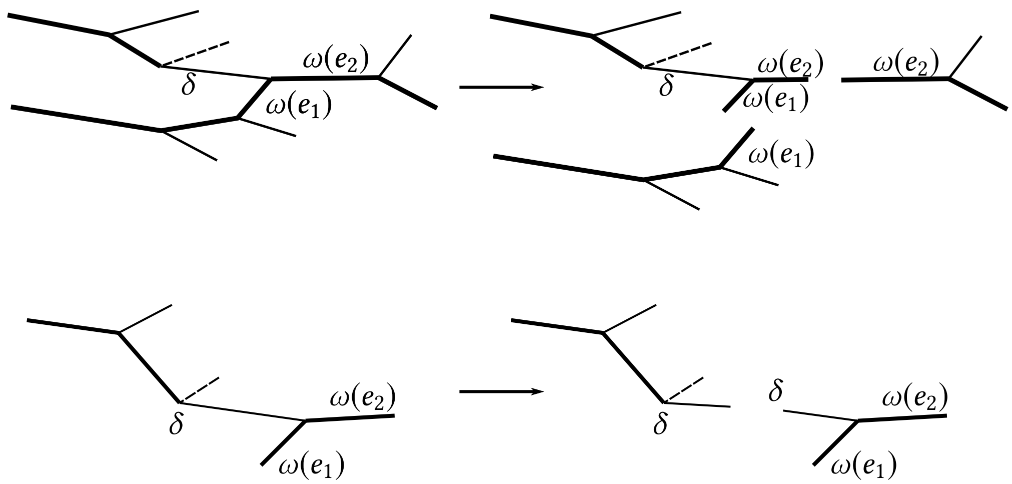


Figure 3.10: Schematic drawing of the cut-and-join process corresponding to cases (1a) and (2a) simultaneously.

As discussed before, we identify those graphs with different multiplicity in C_1 than in C_2. The following is the number of all graphs as in (2a), where e_1 (see Equation (3.10)) is not an edge adjacent to an in-end (and analogously in chamber C_2):

$$\sum_{\substack{I_1\subset I^c, J_1\subset J^c\\ |I_1|>1 \text{ or } J_1\neq\emptyset}} \sum_{p_1+p_2+p_3=k} \binom{m-k}{|I_1|+|J_1|-1-p_1, |I^c|+|J^c|-p_2, |I-I_1|+|J-J_1|-1-p_3}$$
$$\sum_{t\in I_1}\sum_{1\leq l\leq l'\leq \mu_t}\Bigg(m^{\leq,<,(2)}_{p_1,0,r_1}(\mu_{I_1},(\nu_{J_1},\omega(e_1)),E_{J_1}\cup\{(2,t,l,|J_1|+1)\},S_{J_1})(C_1)\cdot$$
$$m^{\leq,<,(2)}_{p_2,0,r_2}((\mu_I,\omega(e_1))_t,(\nu_J,\omega(e_2),E_J\cup\{(2,t,l',|J|+1)\},S_I\cup\{(2,t,l)\})(C_1)\cdot$$
$$m^{\leq,<,(2)}_{p_3,0,r_3}((\mu_{I^c-I_1},\omega(e_2))_t,\nu_{J^c-J_1},E_{I^c-I_1},S_{J^c-J_1}\cup\{(2,t,l')\})(C_1)\Bigg),$$

where $\omega(e_2)=\omega(e_1)+\delta$) and we impose the condition on r_i that we obtain genus 0 in each factor. As mentioned before, by cutting along the two distinguished edges, we obtain three graphs of respective types as in Equation (3.12), (3.13) and (3.14). Each of these types corresponds to one of the three factors. The binomial coefficient counts the number of possible orderings on the vertices not affected by the monotonicity condition. All the other other cases work similarly, however what needs to be checked is that we obtain every graph exactly once. In fact, by the method above, we overcount, since (1a) and (2a) or (1b) and (2a) may appear simultaneously, which corresponds to the local picture illustrated in Figure 3.9: The same graph Γ can be reglued from pieces that we obtain in case (1a) and

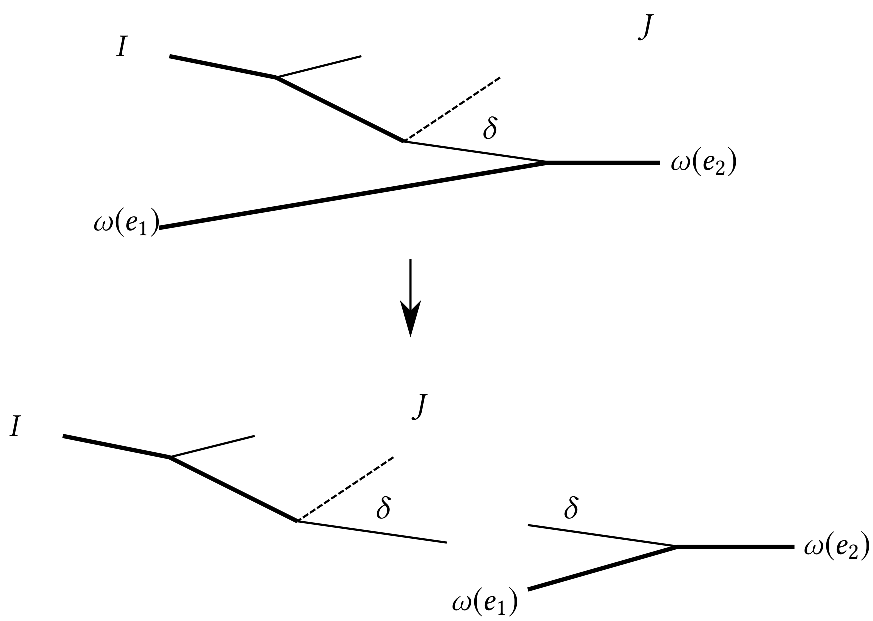


Figure 3.11: Upper graph: Schematic drawing the graph corresponding to the correction term with set of in-ends indexed by $I \cup \{e_1\}$ and set of out-ends indexed by $J \cup \{e_2\}$. Lower graph: Schematic drawing of the graph obtained by cutting along δ.

(2a). By cutting along δ we obtain case (1a), however cutting along e_1 and e_2, we obtain case (2a). This cut-and-join process is illustrated in Figure 3.10. The upper picture is a schematic drawing of the cutting along the edges e_1 and e_2, when case (1a) and (2a) happen at the same time. To compute the correction term, we need to remove all graphs G as shown in the left of the lower picture. This is done by cutting along δ counting all graphs $\tilde{G}$ with out-end δ as in the right hand side of the lower picture and realising that the multiplicity of G is δ times the multiplicity of $\tilde{G}$. In terms of the formula this means that we need to subtract the number of graphs as in upper picture in Figure 3.11

$$\sum_{\substack{h\in I:\\ h<t}} \delta \cdot m^{\leq,<,(2)}_{p_2-1,0,r_2}(\mu_I, (\nu_J, \delta), E_J \cup (1, h, |J|+1), S_I).$$

from the factor

$$m^{\leq,<,(2)}_{p_2,0,r_2}((\mu_I, \omega(e_1))_t, (\nu_J, \omega(e_2), E_J \cup \{(2, t, l', |J|+1)\}, S_I \cup \{(2, t, l)\})(C_1)$$

in each summand. By a similar argument, we see that when the cases (1b) and (2a) appear simultaneously, we need to subtract the following term from

$$m^{\leq,<,(2)}_{p_2,0,r_2}((\mu_I, \omega(e_1))_t, (\nu_J, \omega(e_2), E_J \cup \{(2, t, l', |J|+1)\}, S_I \cup \{(2, t, l)\})(C_1)$$

in each summand:

$$\sum_{\substack{h\in I:\\ h<t}} \sum_{g=1}^{\mu_h} \delta \cdot m^{\leq,<,(2)}_{p_2-1,0,r_2}(\mu_I, (\nu_J, \delta), E_J \cup (2, h, g, |J|+1), S_I).$$

□

3.3 Wall-crossing formulae and piecewise polynomiality for mixed Grothendieck dessins d'enfant, monotone, and simple double Hurwitz numbers

In this section, we take a different approach to the piecewise polynomiality of interpolations between Hurwitz numbers. More precisely, we use the semi-infinite wedge formalism we summarised in section 2.2.2. This will lead to an extensive study of several Hurwitz-type counts on the level of their generating functions. In particular, we will prove piecewise polynomiality for these interpolations between strictly monotone, monotone and simple double Hurwitz numbers, which specialises to those extremal cases. This will lead us to a study of the wall-crossing behaviour of these interpolations, which uncovers an underlying recursive structure of their generating functions.

3.3.1 Triply mixed Hurwitz numbers

We begin by defining a new interpolation between several Hurwitz-type counts, which is more suitable for the study in terms of the semi-infinite wedge formalism. For the rest of this section, we fix $m = \ell(\mu)$ and $n = \ell(\nu)$.

Definition 3.3.1 (Triply mixed Hurwitz numbers)**.** Let g, p, q, r be non-negative integers and let μ and ν be ordered partitions, such that $p + q + r = 2g - 2 + m + n$. We call a tuple $(\sigma_1, \tau_1, \dots, \tau_b, \sigma_2)$, where $b = p + q + r$, a *triply mixed factorisation of type* (g, μ, ν, p, q, r) if it satisfies conditions (1)–(4) in definition 2.2.3 and for $\tau_i = (r_i\ s_i)$, where $r_i < s_i$ we have

(7d) $s_{i+1} \geq s_i$ for $i = q + 1, \dots, q + r - 1$,

(7e) $s_{i+1} > s_i$ for $i = q + r + 1, \dots, b - 1$.

We denote the set of all triply mixed factorisations of type (g, μ, ν, p, q, r) by $\mathcal{F}^{(2),\leq,<}_{p,q,r;\mu,\nu}$ and we define the *triply mixed Hurwitz numbers*

$$h^{(2),\leq,<}_{p,q,r;\mu,\nu} := \frac{1}{d!} \left| \mathcal{F}^{(2),\leq,<}_{p,q,r;\mu,\nu} \right| .$$

Remark 3.3.2. Triply mixed Hurwitz numbers can be thought of as a two-dimensional combinatorial interpolation between different Hurwitz-type counts:

1. For $q = r = 0$, we obtain the simple double Hurwitz numbers.

2. For $p = q = 0$, we obtain the strictly monotone double Hurwitz numbers, denoted by $h^{<}_{g;\mu,\nu}$. In [ALS16], it was proved that this number is equivalent to the Grothendieck dessins d'enfant count as explained in the introduction.

3. For $p = r = 0$, we obtain the monotone double Hurwitz numbers, denoted by $h^{\leq}_{g;\mu,\nu}$.

Triply mixed Hurwitz numbers are a generalisation of the notion of mixed double Hurwitz numbers introduced in [GGN16], which corresponds to the one-dimensional interpolation between simple and monotone double Hurwitz numbers, i.e. $p = 0$.

We note that triply mixed Hurwitz numbers differ from triply interpolated Hurwitz numbers fundamentally: While they differ in the ordering of the simple, monotone and strictly monotone transpositions, there is also a systematic difference. In the case of triply interpolated Hurwitz numbers, we require the monotonicity and strict monotonicity conditions to be compatibale, i.e. $s_{p-1} \leq s_p < s_{p+1}$. In the case of triply mixed Hurwitz numbers, there are no constraints on the relative behaviour of s_{q+r} and s_{q+r+1}.

It is natural to ask whether Hurwitz-type counts behave polynomially in some sense. In particular, we define the subspace

$$\mathcal{H}(m,n) = \{(\underline{M},\underline{N}) : \underline{M} \in \mathbb{N}^m, \underline{N} \in \mathbb{N}^n, \text{ such that} \sum_{i=1}^{m} M_i = \sum_{j=1}^{n} N_j\} \subset \mathbb{N}^m \times \mathbb{N}^n,$$

where $\underline{M} = (M_1, \ldots, M_m)$ and $\underline{N} = (N_1, \ldots, N_n)$ and view triply mixed Hurwitz numbers as a function in the following sense

$$h^{(2),\leq,<}_{p,q,r} \colon \mathcal{H}(m,n) \to \mathbb{Q} \colon (\mu,\nu) \mapsto h^{(2),\leq,<}_{p,q,r;\mu,\nu} .$$

Definition 3.3.3. We define the *hyperplane arrangement* $\mathcal{W}(m,n) \subset \mathcal{H}(m,n)$ induced by the family of linear equations $\sum_{i\in I} M_i = \sum_{j\in J} N_j$ for $I \subset [m]$, $J \subset [n]$, where the variables M_i correspond to $\mathbb{N}^m$ and the variables N_j correspond to $\mathbb{N}^n$. We call the hyperplanes induced by each equation *the walls of the hyperplane arrangement* and the sets of all $(\underline{M},\underline{N})$ at the same side of each wall *the chambers of the hyperplane arrangement.*

Recall that in the interior of the chambers of the hyperplane arrangement, the connected and disconnected Hurwitz numbers agree.

3.3.2 Applying Johnson's algorithm

In this section we use the infinite wedge formalism introduced in 2.2.2 and apply an algorithm described in [Joh15] to evaluate the vacuum expectations expressing monotone and strictly monotone Hurwitz numbers in terms of the $\mathcal{E}-$operators as in equation (2.6). For $I, K \subset [m]$ and $J, L \subset [n]$, where $[n] = \{1, \ldots, n\}$, define

$$\varsigma' \left(\begin{smallmatrix} I & J \\ K & L \end{smallmatrix} \right) = \varsigma \left(\det \begin{bmatrix} |\mu_I| - |\nu_J| & z_J \\ |\mu_K| - |\nu_L| & z_L \end{bmatrix} \right). \tag{3.15}$$

where $\mu_I = \sum_{i \in I} \mu_i$ for a partition μ, and similarly $z_I = \sum_{i \in I} z_i$ for the variables z_i. We define

$$\mathcal{E}'(I, J) = \mathcal{E}_{|\mu_I| - |\nu_J|}(z_J)$$

and observe that their commutator yields

$$[\mathcal{E}'(I, J), \mathcal{E}'(K, L)] = \varsigma' \left(\begin{smallmatrix} I & J \\ K & L \end{smallmatrix} \right) \mathcal{E}'(I \cup K, J \cup L). \tag{3.16}$$

Following [Joh15, Section 3], we choose a chamber $\mathfrak{c}$ of the hyperplane arrangement $\mathcal{W}(m, n)$, and consider the vacuum expectation (equation (2.4))

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(z_j) \right\rangle$$

there. The idea of the algorithm is to commute all positive-energy operators to the right and all negative-energy operators to the left (see definition 2.2.27 for the notion of energy), where they will annihilate the vacuum and the covacuum, respectively. In doing so, we pick up correlators, reducing the total amount of operators in the correlator. This ensures the algorithm terminates.

More explicitly, suppose we have a term of the form

$$\left\langle \prod_{i=1}^{k} \mathcal{E}'(I_i, J_i) \right\rangle, \tag{3.17}$$

where the product is ordered. Take the left-most negative-energy operator, $\mathcal{E}'(I_i, J_i)$. If it is next to the covacuum, the term is zero. Otherwise, commute it to the left. By equation (3.16), this commutation results in two new terms: one where the factors $\mathcal{E}(I_{i-1}, J_{i-1})$ and $\mathcal{E}'(I_i, J_i)$ are switched, and one where they are replaced by $\mathcal{E}'(I_{i-1} \cup I_i, J_{i-1} \cup J_i)$. Both of these terms are again of shape equation (3.17), so the algorithm can continue.

In the end we get the following formula:

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(z_j) \right\rangle = \frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}}^{\text{finite}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}, \tag{3.18}$$

where $CP^{\mathfrak{c}}$ is a finite set of *commutation patterns* that only depends on the chamber $\mathfrak{c}$ of the hyperplane arrangement $\mathcal{W}(m, n)$: the chamber determines the sign of the energy of the $\mathcal{E}'$-operators obtained from the commutators, and hence the operators to be commuted in future steps. The I_ℓ^P, J_ℓ^P, K_ℓ^P, and L_ℓ^P are the four partitions involved in the ℓ-th step of commutation pattern P.

Note that the only difference between the correlators on the left-hand side of equation (3.18) and the ones used in Johnson's paper is in the arguments of the $\mathcal{E}'$-operators with negative energy. This difference only affects slightly the definition of the functions ς' and the prefactor $1/\varsigma(z_{[n]})$.

Combining equation (3.18) with 2.2.30 and 2.2.31 and substituting $uz_j \mapsto z_j$, we have just proved the first main theorem of this paper from which we will derive theorem 3.3.5 and theorem 3.3.18.

Theorem 3.3.4. *Let g be a non-negative integer and let m, n be positive integers such that $(g, n + m) \neq (0, 2)$. Let $\mathfrak{c}$ be a chamber of the hyperplane arrangement*

$\mathcal{W}(m,n)$. *For each $\mu, \nu \in \mathfrak{c}$, we have*

$$h^{\leq}_{g;\mu,\nu} = \frac{1}{\prod \mu_i} \sum_{\substack{v \vdash b \\ \ell(v)=n}} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{\nu_j!} [z_1^{v_1} \dots z_n^{v_n}] \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j-1} \frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}}^{\text{finite}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}$$

and

$$h^{<}_{g;\mu,\nu} = \frac{1}{\prod \mu_i} \sum_{\substack{v \vdash b \\ 0 \leq v_j \leq \nu_j}} \prod_{i=1}^{n} \frac{(\nu_j-1)!}{(\nu_j-v_j)!} [z_1^{v_1} \dots z_n^{v_n}] \prod_{j=1}^{n} \mathcal{S}(z_j)^{-\nu_j-1} \frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}}^{\text{finite}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}.$$

3.3.3 Piecewise polynomiality for monotone and strictly monotone Hurwitz numbers

In this section we begin approaching the problem of piecewise polynomiality of triply mixed Hurwitz numbers. We use a semi-infinite wedge approach to this problem inspired by Johnson's work in [Joh15]. To be more precise, we begin by deriving piecewise polynomiality for monotone Hurwitz numbers (recovering theorem 2.2.9 for $p = 0$) and for strictly monotone Hurwitz numbers directly from the expression in theorem 3.3.4. This shows that triply mixed Hurwitz numbers are piecewise polynomial for the extremal cases of $p = b$, $q = b$, and $r = b$.

Theorem 3.3.5 (Piecewise polynomiality). *Let g be a non-negative integer and let m, n be positive integers such that $(g, n+m) \neq (0,2)$. Let $\mathfrak{c}$ be a chamber of the hyperplane arrangement $\mathcal{W}(m,n)$. Then there exist polynomials $P_g^{\mathfrak{c},\leq}$ and $P_g^{\mathfrak{c},<}$ of degree $4g-3+m+n$ in $m+n$ variables such that*

$$h^{\leq}_{g;\mu,\nu} = P_g^{\mathfrak{c},\leq}(\mu,\nu)\,;$$
$$h^{<}_{g;\mu,\nu} = P_g^{\mathfrak{c},<}(\mu,\nu)$$

for all $(\mu, \nu) \in \mathfrak{c}$.

Remark 3.3.6. The case $(g, n + m) = (0, 2)$ only occurs for $g = 0$ and $\mu = \nu = (d)$ for some positive integer d, which implies that there are no intermediate ramifications ($b = 0$). In this case there is, up to isomorphism, a unique covering $z \mapsto \alpha z^d$, for $\alpha \in \mathbb{C}^\times$, with automorphism group of order d. Hence the Hurwitz number equals $h_{0;(d),(d)} = \frac{1}{d}$ independently of the monotone conditions, reflecting a rational function this time, but indeed again of degree $4g - 3 + m + n = -1$.

Proof. Let us first prove the statement for the monotone case. We fix a chamber $\mathfrak{c}$. By theorem 3.3.4, we can write the monotone Hurwitz numbers as

$$\prod_{i=1}^{m} \mu_i \prod_{j=1}^{n} \nu_j h^{\leq}_{g;\mu,\nu} = \sum_{\substack{v \vdash b \\ \ell(v)=n}} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{(\nu_j-1)!} [z_1^{v_1} \dots z_n^{v_n}]$$
$$\prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j-1} \frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}}^{\text{finite}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix} . \tag{3.19}$$

Let us first prove the following:

Lemma 3.3.7. *For* $(\mu, \nu) \in \mathfrak{c}$*, each summand*

$$[z_1^{v_1} \dots z_n^{v_n}] \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j-1} \frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}}^{\text{finite}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}$$

is a polynomial in the entries of μ *and* ν *of degree bounded by* $2g - 1 + m + n$.

Proof. Let us recall that the expansion of the function $\mathcal{S}(z)$ reads

$$\mathcal{S}(z) = \frac{2\sinh(z/2)}{z} = \sum_{n=0} \frac{z^{2n}}{2^{2n}(2n+1)!} = 1 + \frac{z^2}{24} + \frac{z^4}{1920} + O(z^6).$$

Hence, the coefficient of z_j^{2t} in $\mathcal{S}(z_j)^{\nu_j-1}$ is a polynomial in ν_j of degree t. We show that

$$\frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}}^{\text{finite}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}$$

is a formal power series in $z_1, \ldots, z_n$: Let B_k be the k-th Bernoulli number. The expansion of $1/\varsigma(z)$ reads

$$\frac{1}{\varsigma(z)} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{(1-2^{1-2n})B_{2n}z^{2n-1}}{(2n)!} = \frac{1}{z} - \frac{z}{24} + \frac{7z^3}{5760} + O(z^5).$$

Therefore we need to show that $z_{[n]}$ divides the product of the functions ς' in equation (3.18) for each commutation pattern P. Indeed it suffices to observe that, for every commutation pattern P, in the last step of Johnson's algorithm the correlator is

$$\left\langle \mathcal{E}_a(z_I)\mathcal{E}_{-a}(z_{[n]\setminus I})\right\rangle = \varsigma(az_{[n]}) \left\langle \mathcal{E}_0(z_{[n]})\right\rangle$$

for some I and a depending on P, which is divisible by $z_{[n]}$. Note that the functions ς', by equation (3.15), are odd functions of either $z_i\mu_j$ or $z_i\nu_j$, for some i and j. Therefore the coefficient of $[z_1^{w_1} \ldots z_n^{w_n}]$ is a polynomial in μ_i and ν_j of degree $w_{[n]} + 1$. This concludes the proof of the lemma. □

Now observe that $\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}$ is a polynomial in ν_j of degree v_j and lower degree equal to one if v_j is non-zero, hence each $\prod_{j=1}^n \frac{(v_j-\nu_j-1)!}{(\nu_j-1)!}$ is a polynomial in the entries of ν of degree $2g-2+m+n$, and lower degree equal to the number of v_j that are non-zero. This implies the piecewise polynomiality for $\prod \mu_i \prod \nu_j h^{\leq}_{g;\mu,\nu}$. We are left to prove that each μ_i and each ν_j divides the right-hand side of equation (3.19). For the divisibility by μ_i, observe that, since $\mathcal{E}_{\mu_i}(0)$ has positive energy, in any commutation pattern P it happens that it is commuted with an operator of the form $\mathcal{E}_{\mu_K-\nu_L}(z_L)$ producing a factor $\varsigma(\mu_i z_L)$, which is divisible by μ_i. To prove the divisibility by ν_j, we distinguish two cases:

$v_j \neq 0$ In this case the factor $\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}$ is divisible by ν_j;

$v_j = 0$ Since the operator $\mathcal{E}_{-\nu_j}(z_j)$ has negative energy, in any commutation pattern P it happens that it is commuted with an operator of the form $\mathcal{E}_{\mu_K-\nu_L}(\sum z_L)$ producing a factor

$$\varsigma\left((\mu_K - \nu_L)z_j - \nu_j z_L\right),$$

hence the coefficient of $[z_j^0]$ of the corresponding summand is divisible by v_j.

Note that the division by the factor $\prod \mu_i \prod \nu_j$ decreases the degree of the polynomial by $n+m$. Hence, the total upper bound for the degree of the polynomial $P_g^{\mathfrak{c},\leq}$ is

$$(2g-1+m+n)+(2g-2+m+n)-(m+n)=4g-3+m+n,$$

while the lower bound is given by $(m+n-1)+1-(m+n)=0$. This concludes the proof for the monotone case.

Let us now prove the strictly monotone case. By lemma 2.2.31 we can rewrite the strictly monotone Hurwitz numbers as

$$\prod_{i=1}^{m}\mu_i\prod_{j=1}^{n}\nu_j h^{<}_{g;\mu,\nu}=\sum_{\substack{v\vdash b\\ 0\leq v_j\leq \nu_j}}\prod_{i=1}^{n}\frac{(\nu_j-1)!}{(\nu_j-v_j)!}[z_1^{v_1}\dots z_n^{v_n}]$$
$$\prod_{j=1}^{n}\mathcal{S}(z_j)^{-\nu_j-1}\frac{1}{\varsigma(z_{[n]})}\sum_{P\in CP^{\mathfrak{c}}}^{\text{finite}}\prod_{\ell=1}^{m+n-1}\varsigma'\begin{pmatrix} I_\ell^P & J_\ell^P\\ K_\ell^P & L_\ell^P\end{pmatrix}.$$

Note that the only differences with the monotone case are in the powers of the functions $\mathcal{S}$ and in the prefactor $\frac{\nu!}{(\nu-v)!}$. However, the coefficient of z^{2t} in $\mathcal{S}^{-\nu-1}$ is again a polynomial in ν of degree t, and the prefactor $\frac{\nu!}{(\nu-v)!}$ is again a polynomial in ν of degree v_i. Therefore the entire same argument applies with the same lower and upper bounds on the degrees. This concludes the proof of theorem 3.3.5. □

An example: computing the lowest degree for the monotone case

Let us test our formula computing the lowest degree for the monotone case. Firstly, note that, because the factor $\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}$ is divisible by ν_j for $v_j\neq 0$, the lowest degree occurs for all $v_j=0$ but one. Hence let us consider vectors $v=(0,\dots,b,\dots,0)$, for b in the k-th position for some $k=1,\dots,n$. Then the expression for the monotone case then reads

$$[\deg_{\nu,\mu}=0]\frac{(b+\nu_k-1)!}{\prod\mu_i\prod\nu_j(\nu_k-1)!}[z_1^0\dots z_k^b\dots z_n^0]$$
$$\prod_{i=1}^n \mathcal{S}(z_i)^{\nu_i-1}\left\langle\prod_{i=1}^m \mathcal{E}_{\mu_i}(0)\prod_{j=1}^n \mathcal{E}_{-\nu_j}(z_j)\right\rangle .$$

We can therefore set $z_j=0$ for $j\neq k$. This implies that there is only one possible commutation pattern. Summing over k we obtain that the total lowest degree is

$$[\deg_{\nu,\mu}=0]\sum_{k=1}^n (b+\nu_k-1)\dots(\nu_k+1)[z^{2g-2+m+n}]$$
$$\prod_{\substack{j=1\\ j\neq k}}^n \frac{\varsigma(z\nu_j)}{\nu_j}\prod_{i=1}^m \frac{\varsigma(z\mu_i)}{\mu_i}\frac{\mathcal{S}(z)^{\nu_k-1}}{\varsigma(z)}$$

In order to compute the lowest degree, we have to pick the linear term from each ς-function at the numerator, hence we find that

$$[\deg_{\nu,\mu}=0]h^{\leq}_{g;\mu,\nu}=(b-1)!\sum_{k=1}^n [\deg_\nu=0][z^{2g-2}]\mathcal{S}(z)^{\nu-2} .$$

Recall the generating series of the generalised Bernoulli polynomials $B_k^{(n)}(x)$ [Nør24, p. 145] (cf. also [Rom84, Section 4.2.2]), by

$$\left(\frac{t}{e^t-1}\right)^n e^{xt} =: \sum_{k=0}^\infty B_k^{(n)}(x)\frac{t^k}{k!} ,$$

with specific cases given by $B_k^{(n)}:=B_k^{(n)}(0)$ and the standard Bernoulli numbers $B_k:=B_k^{(1)}$ (with $B_1=-\frac{1}{2}$). These are polynomial in both n and x. In our case, this gives

$$[z^{2g-2}].\mathcal{S}(z)^{\nu-2}=[z^{2g-2}].\left(\frac{e^z-1}{z}\right)^{\nu-2}e^{-\frac{\nu-2}{2}z}=\frac{B_{2g-2}^{(2-\nu)}\left(\frac{2-\nu}{2}\right)}{(2g-2)!}$$

$$= \frac{1}{(2g-2)!} \sum_{k=0}^{2g-2} \binom{2g-2}{k} \Big(\frac{2-\nu}{2}\Big)^{2g-2-k} B_k^{(2-\nu)} \,.$$

Taking the degree zero part in ν corresponds to setting $\nu = 0$, which yields, using [Nør24, Equation 81*],

$$\begin{aligned}
\sum_{k=0}^{2g-2} \frac{1}{k!(2g-2-k)!} B_k^{(2)} &= \sum_{k=0}^{2g-2} \frac{1}{k!(2g-2-k)!} \big((1-k)B_k - kB_{k-1}\big) \\
&= -\Bigg(\sum_{k=0}^{2g-2} \frac{(k-1)B_k}{k!(2g-2-k)!} + \frac{B_{k-1}}{(k-1)!(2g-2-k)!} \Bigg) \\
&= -\sum_{k=0}^{2g-2} \frac{(k-1)B_k}{k!(2g-2-k)!} - \sum_{k=0}^{2g-3} \frac{(2g-2-k)B_k}{k!(2g-2-k)!} \\
&= -\frac{2g-3}{(2g-2)!} \sum_{k=0}^{2g-2} \binom{2g-2}{k} B_k = -\frac{(2g-3)B_{2g-2}}{(2g-2)!}
\end{aligned}$$

Hence, the final expression reads

$$[\deg_{\nu,\mu} = 0] h^{\leq}_{g;\mu,\nu} = -\frac{n\,(2g-3+m+n)!(2g-3)B_{2g-2}}{(2g-2)!} \delta_{g \geq 1},$$

which shows that the lowest degree does not vanish for $g \geq 1$.

3.3.4 Piecewise polynomiality for mixed Hurwitz numbers

After having developed the necessary tools in section 3.3.3, we use the same approach to prove piecewise polynomiality of triply mixed Hurwitz numbers in this section. We start with a number of useful lemmata.

Lemma 3.3.8. *The conjugation with the exponential of $\mathcal{F}_2$ acts on the operator $\mathcal{E}_{-\nu}(z)$ by shifting the variable z by the opposite of the energy. Explicitly:*

$$e^{u\mathcal{F}_2} \mathcal{E}_{-\nu}(A) e^{-u\mathcal{F}_2} = \mathcal{E}_{-\nu}(A + u\nu) \,.$$

Proof. It is a straightforward computation. We denote the adjoint action by $\mathcal{F}_2$ by

$\mathrm{ad}\,\mathcal{F}_2$, i.e. $\mathrm{ad}\,\mathcal{F}_2(X) = [\mathcal{F}_2, X]$. Further, we denote the $k-$fold action by $\mathrm{ad}^k_{\mathcal{F}_2}$.

By a standard Lie theory result, the left-hand side is equal to

$$
\begin{aligned}
e^{u\mathcal{F}_2}\mathcal{E}_{-\nu}(A)e^{-u\mathcal{F}_2} &= e^{u\,\mathrm{ad}\,\mathcal{F}_2}\mathcal{E}_{-\nu}(A) = \sum_{k=0}^{\infty} \frac{u^k}{k!}\,\mathrm{ad}^k_{\mathcal{F}_2}\,\mathcal{E}_{-\nu}(A) \\
&= \sum_{k=0}^{\infty} \frac{u^k}{k!} \sum_{l\in\mathbb{Z}'} \left(\frac{(l+\nu)^2 - l^2}{2}\right)^k e^{A(l+\frac{\nu}{2})} E_{l+\nu,l} \\
&= \sum_{l\in\mathbb{Z}'} \sum_{k=0}^{\infty} \frac{u^k(l\nu + \frac{\nu^2}{2})^k}{k!} e^{A(l+\frac{\nu}{2})} E_{l+\nu,l} \\
&= \sum_{l\in\mathbb{Z}'} e^{u\nu(l+\frac{\nu}{2})} e^{A(l+\frac{\nu}{2})} E_{l+\nu,l} = \sum_{l\in\mathbb{Z}'} e^{(A+u\nu)(l+\frac{\nu}{2})} E_{l+\nu,l}\,,
\end{aligned}
$$

which coincides with the right-hand side by definition. □

Lemma 3.3.9.

$$
\begin{aligned}
\mathcal{D}^{(h)}(u)\mathcal{E}_{-\nu}(A)\mathcal{D}^{(h)}(u)^{-1} &= \sum_{v=0}^{\infty} \frac{(v+\nu-1)!}{(\nu-1)!}[z^v]\mathcal{S}(uz)^{\nu-1}\mathcal{E}_{-\nu}(A+uz)\,; \\
\mathcal{D}^{(\sigma)}(u)\mathcal{E}_{-\nu}(A)\mathcal{D}^{(\sigma)}(u)^{-1} &= \sum_{v=0}^{\nu} \frac{\nu!}{(\nu-v)!}[z^v]\mathcal{S}(uz)^{-\nu-1}\mathcal{E}_{-\nu}(A+uz)\,.
\end{aligned}
$$

Proof. It is again a straightforward computation. It can obtained by modifying slightly the proofs of [KLS16, Lemmata 4.1 and 4.3]. □

By the previous discussion, we can express a generation function for the triply mixed Hurwitz numbers as follows:

$$
\begin{aligned}
&\sum_{p,q,r=0}^{\infty} \frac{h^{(2),\leq,<}_{p,q,r;\mu,\nu}}{p!} X^p Y^q Z^r = \\
&\frac{1}{\prod \mu_i \prod \nu_j} \left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) e^{X\mathcal{F}_2} \mathcal{D}^{(h)}(Y)\mathcal{D}^{(\sigma)}(Z) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(0) \right\rangle\,.
\end{aligned}
$$

In the same way as before, we can rearrange the correlator as

$$\frac{\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} e^{X\mathcal{F}_2} \mathcal{D}^{(h)}(Y) \mathcal{D}^{(\sigma)}(Z) \mathcal{E}_{-\nu_j}(0) \mathcal{D}^{(\sigma)}(Z)^{-1} \mathcal{D}^{(h)}(Y)^{-1} e^{-X\mathcal{F}_2} \right\rangle}{\prod \mu_i \prod \nu_j}.$$

By lemma 3.3.9, we can express these numbers as a linear combination of correlators purely in terms of the $\mathcal{E}$-operators and we obtain the following lemma generalising 2.2.30 and lemma 2.2.31

Lemma 3.3.10. *Let g be a non-negative number, μ and ν partitions of the same positive integer, p, q, r non-negative integers, such that $p + q + r = b$. The triply mixed Hurwitz number corresponding to these data can be computed as*

$$\begin{aligned} h^{(2),\leq,<}_{p,q,r;\mu,\nu} = p![X^p Y^q Z^r] \sum_{v,w \in \mathbb{N}^n} \prod_{j=1}^{n} \frac{(\nu_j + v_j - 1)!}{\mu_j(\nu_j - w_j)!} \cdot \\ [\underline{y}^v \underline{z}^w] \prod_{j=1}^{n} \frac{\mathcal{S}(Yy_j)^{\nu_j - 1}}{\mathcal{S}(Zz_j)^{\nu_j + 1}} \left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(X\nu_j + Yy_j + Zz_j) \right\rangle \end{aligned}.$$

Let us analyse this expression. First, the variable u has been omitted and is replaced by three variables, X, Y, and Z, that count one kind of ramification each. Furthermore, Y always occurs together with a y_j, and similarly for Z. Hence, the parameter q on the left-hand side corresponds to $\sum_{i=1}^{n} v_n$ on the right-hand side and similarly r corresponds to $\sum_{j=1}^{n} w_n$.

Theorem 3.3.11 (Piecewise polynomiality for triply mixed Hurwitz)**.** *Let p, q, r be non-negative integers and let m, n be positive integers such that $(g, n+m) \neq (0, 2)$, where $p + q + r = 2g - 2 + m + n$. Let $\mathfrak{c}$ be a chamber of the hyperplane arrangement $\mathcal{W}(m, n)$. Then there exist polynomials $P^{\mathfrak{c};(2),\leq,<}_{p,q,r}$ of degree $4g - 3 + m + n$ in $m + n$ variables such that*

$$h^{(2),\leq,<}_{p,q,r;\mu,\nu} = P^{\mathfrak{c};(2),\leq,<}_{p,q,r}(\mu, \nu)$$

for all $(\mu, \nu) \in \mathfrak{c}$.

Remark 3.3.12. Notice that theorem 3.3.5 is a special case of this theorem, obtained by setting r and either p or q to zero. Likewise, the mixed cases of two out of the three kinds of Hurwitz number can be obtained by setting the third parameter to zero. In particular, we recover theorem 2.2.9 by setting $p = 0$.

Proof. In lemma 3.3.10, let us first look at a single factor

$$[X^p \vec{y}^v \vec{z}^w] \prod_{j=1}^{n} \frac{\mathcal{S}(y_j)^{\nu_j - 1}}{\mathcal{S}(z_j)^{\nu_j + 1}} \Big\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(X\nu_j + y_j + z_j) \Big\rangle ,$$

where $|v| = q$ and $|w| = r$.

Because $\mathcal{S}(z)$ is an even analytic function with constant term 1 and non-zero coefficient of z^2, the coefficient of z^{2t} in both $\mathcal{S}(z)^{\nu-1}$ and $\mathcal{S}(z)^{-\nu-1}$ is a polynomial in ν of degree t. On the other hand, the commutations produce factors where every factor of y or z brings a linear polynomial in ν and μ and every factor of X brings a quadratic polynomial. As the final correlator of the commutation pattern still gives a factor $\varsigma(X\nu_{[n]} + y_{[n]} + z_{[n]})^{-1}$, this complete factor gives a polynomial in μ and ν of degree $2p + q + r + 1$.

The correlator can be calculated using Johnson's algorithm, where the set of commutation patterns is fixed by the chamber $\mathfrak{c}$. Every commutation gives a factor of ς with a certain argument linear or quadratic in μ and ν, until we end up with

$$\Big\langle \mathcal{E}_a(X\nu_I + y_I + z_I)\mathcal{E}_{-a}(X\nu_{[n]\setminus I} + y_{[n]\setminus I} + z_{[n]\setminus I}) \Big\rangle$$

for some $a \geq 0$ and $I \subset [n]$. By the commutation rules, this is equal to

$$\varsigma\big(a(X\nu_{[n]} + y_{[n]} + z_{[n]})\big) \Big\langle \mathcal{E}_0(X\nu_{[n]} + y_{[n]} + z_{[n]}) \Big\rangle = \frac{\varsigma\big(a(X\nu_{[n]} + y_{[n]} + z_{[n]})\big)}{\varsigma(X\nu_{[n]} + y_{[n]} + z_{[n]})} .$$

The possible pole coming from the denominator is cancelled by the numerator, so this entire term is polynomial in μ and ν.

Furthermore, this polynomial is divisible by μ_i, as the operator $\mathcal{E}_{\mu_i}(0)$ must be commuted with some negative-energy operator $\mathcal{E}_{-a}(x)$, producing a factor $\varsigma(\mu_i x)$.

Also, the factor $\frac{(\nu_j + v_j - 1)!}{(\nu_j - w_j)!}$ is polynomial in ν – of degree $v_j + w_j - 1$ – unless $v_j = w_j = 0$, in which case it is $\frac{1}{\nu_j}$. However, in this case we have the operator

$\mathcal{E}_{-\nu_j}(X\nu_j)$, which must commute to the left, and will always yield some factor $\varsigma(\nu_j x)$ in the commutator. Hence, the entire term

$$p![X^p]\prod_{j=1}^{n}\frac{(\nu_j+v_j-1)!}{\mu_j(\nu_j-w_j)!}[\vec{y}^v\vec{z}^w]\prod_{j=1}^{n}\frac{\mathcal{S}(y_j)^{\nu_j-1}}{\mathcal{S}(z_j)^{\nu_j+1}}\left\langle\prod_{i=1}^{m}\mathcal{E}_{\mu_i}(0)\prod_{j=1}^{n}\mathcal{E}_{-\nu_j}(X\nu_j+y_j+z_j)\right\rangle$$

is polynomial in μ and ν.

To calculate the coefficient of $X^pY^qZ^r$ in lemma 3.3.10, we take a *finite* sum over such polynomials, where the number of summand is independent of μ and ν, as the sum runs over non-negative $\{v_i, w_i \mid 1 \le i \le n\}$ such that $\sum_i v_i = q$ and $\sum_i w_i = r$.

The maximal degree of this polynomial is then

$$(2p+q+r+1)+\sum_{j=1}^{n}(v_j+w_j-1)-m = 2(p+q+r)+1-n-m = 4g-3+m+n\,,$$

which proves the theorem. □

The lower bound of the polynomial corresponds to the power of X that we choose in the polynomial $h^{<,\le,(2)}_{g;\mu,\nu}(X,Y,Z)$, since the powers of X do not come from any other expansion.

3.3.5 Wall-crossing formulae

In the previous sections, we have given an explicit way of computing polynomials representing strictly and monotone and simple Hurwitz numbers, or any mix of the three, within a chamber of the hyperplane arrangement. In this section, we show how these different polynomials are connected via wall-crossing formulas, expressing the difference between generating functions in adjacent chambers recursively as a product of two generating functions of Hurwitz numbers of similar kind.

Wall-crossing formulae for dessins d'enfant and monotone Hurwitz numbers

In this section, we study the wall-crossing behaviour of the Hurwitz numbers $h^{\leq}_{g;\mu,\nu}$ and $h^{<}_{g;\mu,\nu}$. We write $h^{\bullet}_{g;\mu,\nu}$ in the following to mean either of them, and similarly for related quantities. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two chambers in the hyperplane arrangement given by $\mathcal{W}$ that are separated by the wall $\delta := \mu_I - \nu_J = 0$. Without loss of generality, we assume that $\delta > 0$ on $\mathfrak{c}_2$ and $\delta < 0$ on $\mathfrak{c}_1$. Let $p^{\mathfrak{c}_i}_{g;\mu,\nu}$ be the polynomial expressing $h^{\bullet}_{g;\mu,\nu}$ in $\mathfrak{c}_i$. The goal of this section is to compute the wall-crossing at $\delta = 0$ between $\mathfrak{c}_1$ and $\mathfrak{c}_2$

$$WC_\delta := p^{\mathfrak{c}_2}_{g;\mu,\nu} - p^{\mathfrak{c}_1}_{g;\mu,\nu} \in \mathbb{Q}[\mu, \nu] \,.$$

Our approach to the wall-crossing is motivated by the expression of $h^{\bullet}_{g;\mu,\nu}$ in theorem 3.3.4.

Notation 3.3.13. For a partition μ, a subset $I \subset \{1, \dots, m\}$, and a wall $\delta = 0$, we denote the partition $(\mu_i)_{i\in I}$ by μ^I and the partition (μ, δ) by $\mu + \delta$, whereas the notation μ_I is still reserved for $\sum_{i\in I} \mu_i$. Moreover, for a collection of variables $\underline{u} = u_1, \dots, u_n$ and a subset $J \subset \{1, \dots, n\}$, we denote the collection $(u_j)_{j\in J}$ by u^J.

Definition 3.3.14. Let μ, ν be ordered partitions of the same natural number. We define the *refined monotone generating series* as

$$\mathcal{H}^{\leq}_{\mu,\nu}(\underline{u}, \underline{z}) = \sum_{v_1,\dots,v_n=0}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{(v_j + \nu_j - 1)!}{(\nu_j - 1)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j - 1} \frac{\left\langle \prod_{i=1}^m \mathcal{E}_{\mu_i}(0) \prod_{j=1}^n \mathcal{E}_{-\nu_j}(z_j) \right\rangle}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j} \,. \tag{3.20}$$

Similarly, we define the *refined Grothendieck dessins d'enfant generating series* as

$$\mathcal{H}^{<}_{\mu,\nu}(\underline{u}, \underline{z}) = \sum_{\substack{v_1,\dots,v_n=0 \\ 0\leq v_i \leq \nu_i}}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{\nu_j!}{(\nu_j - v_j)!}$$

$$\prod_{j=1}^{n} \mathcal{S}(z_j)^{-\nu_j-1} \frac{\left\langle \prod_{i=1}^m \mathcal{E}_{\mu_i}(0) \prod_{j=1}^n \mathcal{E}_{-\nu_j}(z_j) \right\rangle}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j}\,.$$

The following lemma follows from equation (3.19).

Lemma 3.3.15. *Let g be a non-negative integer, μ, ν ordered partitions of the same natural number and $b = 2g - 2 + \ell(\mu) + \ell(\nu)$. Then*

$$h^{\bullet}_{g;\mu,\nu} = \sum_{\substack{v_1,\dots,v_n \in \mathbb{Z}_{\geq 0} \\ |\vec{v}|=b}} [z_1^{v_1} \cdots z_n^{v_n}][u_1^{v_1} \cdots u_n^{v_n}] \mathcal{H}^{\bullet}_{\mu,\nu}(\underline{u}, \underline{z})\,.$$

By theorem 3.3.5, the polynomial expressing

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(z_j) \right\rangle \tag{3.21}$$

in equation (3.20) only depends on the chamber $\mathfrak{c}$ given by $\mathcal{W}$, which motivates the following definition.

Definition 3.3.16. Let $\mathfrak{c}$ be a chamber induced by the hyperplane arrangement $\mathcal{W}$ and denote by $q^{\mathfrak{c}}(\underline{z})$ the polynomial expressing equation (3.21) in $\mathfrak{c}$. Then we define

$$\mathcal{H}^{\leq}_{\mu,\nu}(\mathfrak{c}, \underline{u}, \underline{z}) = \sum_{v_1,\dots,v_n=0}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{(v_j + \nu_j - 1)!}{(\nu_j - 1)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j - 1} \frac{q^{\mathfrak{c}}(\underline{z})}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j}. \tag{3.22}$$

and

$$\mathcal{H}^{<}_{\mu,\nu}(\mathfrak{c}, \underline{u}, \underline{z}) = \sum_{\substack{v_1,\dots,v_n=0 \\ 0 \leq v_i \leq \nu_i}}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{\nu_j!}{(\nu_j - v_j)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{-\nu_j - 1} \frac{q^{\mathfrak{c}}(\underline{z})}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j}.$$

Let $\delta = \mu_I - \nu_J$ for some fixed $I \subset \{1, \dots, m\}, J \subset \{1, \dots, n\}$. This defines a wall in $\mathcal{W}$ by $\delta = 0$. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be chambers which are seperated by $\delta = 0$ and contain $\delta = 0$ as a codimension one subspace. Then we define the wall-crossings by

$$\mathcal{WC}^\bullet_\delta(\underline{u},\underline{z}) = \mathcal{H}^\bullet_{\mu,\nu}(\mathfrak{c}_2,\underline{u},\underline{z}) - \mathcal{H}^\bullet_{\mu,\nu}(\mathfrak{c}_1,\underline{u},\underline{z})\,. \tag{3.23}$$

The following lemma follows from lemma 3.3.15.

Lemma 3.3.17. *Let g be a non-negative integer, μ,ν ordered partitions of the same natural number and $b = 2g-2+\ell(\mu)+\ell(\nu)$. Then*

$$WC^\bullet_\delta = \sum_{\substack{v_1,\dots,v_n\in\mathbb{Z}_{\geq 0}\\ |\vec{v}|=b}} [z_1^{v_1}\cdots z_n^{v_n}][u_1^{v_1}\cdots u_n^{v_n}]\mathcal{WC}^\bullet_\delta(\underline{u},\underline{z})\,.$$

The first main result of this subsection is the following theorem.

Theorem 3.3.18. *Let μ,ν be ordered partitions of the same positive integer and let $\delta = \sum_{i\in I}\mu_i - \sum_{j\in J}\nu_j$. Then we have the following recursive structures*

$$\begin{aligned}\mathcal{WC}^\leq_\delta(\underline{u},\underline{z}) =& \delta^2\frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_J)\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}\\ &[(u')^0]\mathcal{H}^\leq_{\mu^I,\nu^J+\delta}(\underline{u}^J,u',\underline{z}^J,0)\mathcal{H}^\leq_{\mu^{I^c}+\delta,\nu^{J^c}}(\underline{u}^{J^c},\underline{z}^{J^c})\end{aligned}$$

and

$$\begin{aligned}\mathcal{WC}^<_\delta(\underline{u},\underline{z}) =& \delta^2\frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_J)\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}[(u')^0]\\ &\mathcal{H}^<_{\mu^I,\nu^J+\delta}(\underline{u}^J,u',\underline{z}^J,0)\mathcal{H}^<_{\mu^{I^c}+\delta,\nu^{J^c}}(\underline{u}^{J^c},\underline{z}^{J^c})\,.\end{aligned}$$

Here, the argument 0 is the z-variable related to δ in $\mathcal{H}_{\mu^I,\nu^J+\delta}$.

Proof. Both formulae are derived by similar calculations, so we only prove the recursive structure for $\mathcal{WC}^\leq_\delta$. The strategy of the proof consists of comparing the generating series $\mathcal{WC}^\leq_\delta$ and $\mathcal{H}^\leq_{\mu^I,\nu^J+\delta}(\underline{u}^J,\underline{z}^J,z')\mathcal{H}^\leq_{\mu^{I^c}+\delta,\nu^J}(\underline{u}^{J^c},\underline{z}^{J^c})$, using Johnson's algorithm. We start by studying $\mathcal{WC}^\leq_\delta$. Substituting equation (3.22) into equation (3.23), we obtain

$$\mathcal{WC}^\leq_\delta = \sum_{v_1,\dots,v_n=0}^{\infty} u_1^{v_1}\cdots u_n^{v_n}\prod_{j=1}^n\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}\prod_{j=1}^n\mathcal{S}(z_j)^{\nu_j-1}\frac{q^{\mathfrak{c}_2}(\underline{z})-q^{\mathfrak{c}_1}(\underline{z})}{\prod_{i=1}^m\mu_i\prod_{j=1}^n\nu_j}\,. \tag{3.24}$$

Let us compute the difference $q^{\mathfrak{c}_2}(z_1, \ldots, z_n) - q^{\mathfrak{c}_1}(z_1, \ldots, z_n)$. This quantity is almost the same as the one appearing in the proof of the wall-crossing formula for double Hurwitz numbers in [Joh15, Section 4.2]. We follow the idea of that proof, making the required adjustments. The main difference is that the vacuum expection

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(z_j) \right\rangle$$

we consider depends on several variables z_j (one for each entry of ν), whereas the vacuum expectation in [Joh15]

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(\nu_j z) \right\rangle$$

only depends on one variable z.

Let us first observe that every commutation pattern in which no operator of energy δ is produced is a summand in both $q^{\mathfrak{c}_2}$ and $q^{\mathfrak{c}_1}$, and therefore contributes trivially to their difference. Thus, it is sufficient to compute the contribution of those commutation patterns producing δ energy operators. Let us choose the following ordering of operators in the vacuum expectation

$$\left\langle \prod_{i \notin I} \mathcal{E}_{\mu_i}(0) \prod_{i \in I} \mathcal{E}_{\mu_i}(0) \prod_{j \in J} \mathcal{E}_{-\nu_j}(z_j) \prod_{j \notin J} \mathcal{E}_{-\nu_j}(z_J) \right\rangle. \tag{3.25}$$

If a commutation pattern produces an operator of energy δ, the first vacuum expectation containing that operator must be

$$\left\langle \prod_{i \notin I} \mathcal{E}_{\mu_i}(0) \mathcal{E}_{\delta}(z_J) \prod_{j \notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle. \tag{3.26}$$

Let T_1 be the product of ς-factors the algorithm produces up to equation (3.26). Let us observe that, up until equation (3.26), the algorithm runs identically on $\mathfrak{c}_1$ and $\mathfrak{c}_2$. Therefore, T_1 divides $q^{\mathfrak{c}_2} - q^{\mathfrak{c}_1}$. In order to compute the quantity T_1, we consider the vacuum expectation

$$\left\langle \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0) \right\rangle$$

inside the chamber $\mathfrak{c}_1$. We claim that the operator $\mathcal{E}_{-\delta}(0)$ cannot be involved in a commutator leading to a non-zero vacuum expectation until the very last commutator. Clearly, the commutator with any negative energy operator is equal to zero. Suppose therefore that $\mathcal{E}_{-\delta}(0)$ is involved in the commutator with some operator

$$\mathcal{E}_{\mu_K-\nu_L}(z_L)\,,$$

for subsets $K \subset I$ and $L \subset J$, where at least one is a proper subset. Because we are inside a chamber, we have $\mu_K - \nu_L \neq 0$. Hence we assume $\mu_K - \nu_L > 0$. Since we also assumed that the vacuum expectation does not vanish, the commutator must have negative energy. Hence $\mu_K - \nu_L - \delta < 0$, which implies

$$\delta > \mu_K - \nu_L > 0\,.$$

This provides a lower bound for δ, contradicting the fact that the chamber $\mathfrak{c}_1$ borders $\delta = 0$. We showed that every commutation pattern contributing nontrivially commutes $\mathcal{E}_{-\delta}(0)$ at the very end. Thus all the other commutators must be computed first. Therefore we can compute

$$\begin{aligned}&\left\langle \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0) \right\rangle = \\ &T_1\Big\langle \mathcal{E}_{\delta}(z_J)\mathcal{E}_{-\delta}(0)\Big\rangle = T_1\varsigma(\delta z_J)\Big\langle \mathcal{E}_0(z_J)\Big\rangle = T_1\frac{\varsigma(\delta z_J)}{\varsigma(z_J)}\,.\end{aligned}$$

Re-arranging the equation, we obtain

$$T_1 = \frac{\varsigma(z_J)}{\varsigma(\delta z_J)}\left\langle \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0) \right\rangle.$$

The quantity in equation (3.25) is therefore

$$\left\langle \prod_{i\notin I} \mathcal{E}_{\mu_i}(0) \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j) \prod_{j\notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle =$$
$$T_1 \left\langle \prod_{i\notin I} \mathcal{E}_{\mu_i}(0) \mathcal{E}_{\delta}(z_J) \prod_{j\notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle =$$
$$\frac{\varsigma(z_J)}{\varsigma(\delta z_J)} \left\langle \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j) \mathcal{E}_{-\delta}(0) \right\rangle \left\langle \prod_{i\notin I} \mathcal{E}_{\mu_i}(0) \mathcal{E}_{\delta}(z_J) \prod_{j\notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle .$$

We will compare the last factor containing an operator of energy δ with the vacuum expectation

$$\left\langle \mathcal{E}_{\delta}(0) \prod_{i\notin I} \mathcal{E}_{\mu_i}(0) \prod_{j\notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle . \tag{3.27}$$

Let T_2 be the series denoting the difference of the vacuum expectation equation (3.26) on $\mathfrak{c}_2$ and $\mathfrak{c}_1$. Applying Johnson's algorithm to equation (3.26), the operator of energy δ would be commuted into different directions in the very first step. In order to compare the contributions in each chamber, we commute $\mathcal{E}_{\delta}(z_J)$ to the left in both chambers, even though it has positive energy on $\mathfrak{c}_2$. If this operator is involved in a cancelling term as we move to the left, the algorithm will run as usual in both chambers after this commutator: after the cancellation, we will have an operator $\mathcal{E}_{\mu_{K\sqcup I}-\nu_{L\sqcup J}}(z_{L\sqcup J})$, where at least one the subsets K and L is non-empty. All contributions up to the cancellation coincide in both chambers (since we chose to commute $\mathcal{E}_{\delta}(z_J)$ to the left) and by the above argument above so do the contributions after the cancellation. Therefore, we have the same contributions in both chambers with the same sign and they cancel in the wall-crossing. The key observation in computing the difference between $\mathfrak{c}_2$ and $\mathfrak{c}_1$ is that, whenever $\mathcal{E}_{\delta}(z_J)$ reaches the far left, the vacuum expectation vanishes on $\mathfrak{c}_1$ but not on $\mathfrak{c}_2$. Thus, we obtain

$$T_2 = \left\langle \mathcal{E}_{\delta}(z_J) \prod_{i\notin I} \mathcal{E}_{\mu_i}(0) \prod_{j\notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle . \tag{3.28}$$

Comparing equation (3.27) and equation (3.28), the only difference is the operator on the far left. By a similar argument as in our computation of T_1, this vacuum expectation vanishes whenever the operator in $\mathcal{E}_\delta(z_J)$ is not only involved in the last commutation. Thus, the last step of the algorithm for equation (3.28) ends with

$$\left\langle \mathcal{E}_\delta(z_J)\mathcal{E}_{-\delta}(z_{J^c}) \right\rangle = \frac{\varsigma(\delta z_{[n]})}{\varsigma(z_{[n]})}$$

instead of the last step for equation (3.27), which ends with

$$\left\langle \mathcal{E}_\delta(0)\mathcal{E}_{-\delta}(z_{J^c}) \right\rangle = \frac{\varsigma(\delta z_{J^c})}{\varsigma(z_{J^c})}\,.$$

Therefore the following equality holds for T_2:

$$T_2 = \frac{\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}\left\langle \mathcal{E}_\delta(0)\prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\right\rangle .$$

Substituting $q^{c_2}(z_1,\ldots,z_n) - q^{c_1}(z_1,\ldots,z_n) = T_1T_2$ into equation (3.24), we obtain

$$\mathcal{WC}_\delta^{\leq} = \sum_{v_1,\ldots,v_n=0}^{\infty} u_1^{v_1}\cdots u_n^{v_n}\prod_{j=1}^{n}\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}\prod_{j=1}^{n}\mathcal{S}(z_j)^{\nu_j-1}\frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta(z_J))\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}$$
$$\frac{\left\langle \prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0)\right\rangle\left\langle \mathcal{E}_\delta(0)\prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\right\rangle}{\prod_{i=1}^{m}\mu_i\prod_{j=1}^{n}\nu_j}\,.$$

Comparing this extension to the following extension of the product

$$\mathcal{H}_{\mu^I,\nu^J+\delta}\mathcal{H}_{\mu^{I^c},\nu^{J^c}+\delta},$$

we obtain

$$\mathcal{H}^{\leq}_{\mu^I,\nu^J+\delta}(\underline{u}^J,u',\underline{z}^J,0)\mathcal{H}^{\leq}_{\mu^{I^c}+\delta,\nu^J}(\underline{u}^{J^c},\underline{z}^{J^c}) = \sum_{v_1,\dots,v_n,v'=0}^{\infty} u_1^{v_1}\cdots u_n^{v_n}u'^{v'}$$

$$\prod_{j=1}^{n}\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}\frac{(v'+\delta-1)!}{(\delta-1)!}\prod_{j=1}^{n}\mathcal{S}(z_j)^{\nu_j-1}\mathcal{S}(0)^{\delta-1}$$

$$\frac{\left\langle \prod_{i\in I}\mathcal{E}_{\mu_i}(0)\prod_{j\in J}\mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0)\right\rangle \left\langle \mathcal{E}_{\delta}(0)\prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\right\rangle}{\prod_{i=1}^{m}\mu_i\prod_{j=1}^{n}\nu_j\delta^2}$$

and we see immediately that

$$\mathcal{W}C^{\leq}_{\delta}(\underline{u},\underline{z}) = \delta^2\frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_J)\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}[u'^0]\mathcal{H}^{\leq}_{\mu^I,\nu^J+\delta}(\underline{u}^J,u',\underline{z}^J,0)\mathcal{H}^{\leq}_{\mu^{I^c}+\delta,\nu^J}(\underline{u}^{J^c},\underline{z}^{J^c}),$$

as desired. □

Wall-crossing formulae for triply mixed Hurwitz numbers

In this subsection we deal with the triply mixed Hurwitz numbers. The procedure is very similar to one in the previous subsection, so we only outline the main steps and give the results. We begin by defining the refined generating series for triply mixed Hurwitz numbers.

Definition 3.3.19. Let p, q and r be non-negative integers and let μ and ν be partitions as before. We define the *refined triply mixed generating series* as

$$\mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\underline{t},\underline{u},X,\underline{y},\underline{z}) := \sum_{\substack{v_1,\dots,v_n=0\\ w_1,\dots,w_n=0\\ 0\leq w_i\leq v_i}}^{\infty} t_1^{v_1}\cdots t_n^{v_n}u_1^{w_1}\cdots u_n^{w_n}\prod_{j=1}^{n}\frac{(v_j+\nu_j-1)!}{(\nu_j-w_j)!}\prod_{j=1}^{n}$$

$$\frac{\mathcal{S}(y_j)^{\nu_j-1}}{\mathcal{S}(z_j)^{\nu_j+1}}\qquad\frac{\left\langle\prod_{i=1}^{m}\mathcal{E}_{\mu_i}(0)\prod_{j=1}^{n}\mathcal{E}_{-\nu_j}(X\nu_j+y_j+z_j)\right\rangle}{\prod_{i=1}^{m}\mu_i}.$$

Moreover, let c be induced by the hyperplane arrangement $\mathcal{W}$ and denote by

$q^{\mathfrak{c}}(X, \underline{y}, \underline{z})$ the polynomial expressing

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(X\nu_j + y_j + z_j) \right\rangle$$

in the chamber $\mathfrak{c}$. Then we define

$$\mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c}, \underline{t}, \underline{u}, X, \underline{y}, \underline{z}) := \sum_{\substack{v_1,\ldots,v_n=0 \\ w_1,\ldots,w_n=0 \\ 0\leq w_i\leq v_i}}^{\infty} t_1^{v_1}\cdots t_n^{v_n} u_1^{w_1}\cdots u_n^{w_n} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{(\nu_j-w_j)!}$$
$$\prod_{j=1}^{n} \frac{\mathcal{S}(y_j)^{\nu_j-1}}{\mathcal{S}(z_j)^{\nu_j+1}} \frac{q^{\mathfrak{c}}(X, \underline{y}, \underline{z})}{\prod_{i=1}^{m}\mu_i}.$$

Let $\delta = \sum_{i\in I}\mu_i - \sum_{j\in J}\nu_j = 0$ define a wall in $\mathcal{W}$ and let $\mathfrak{c}_1$, $\mathfrak{c}_2$ be chambers separated by this wall. Define

$$\mathcal{WC}^{(2),\leq,<}_{\delta}(\underline{t}, \underline{u}, X, \underline{y}, \underline{z}) := \mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c}_2, \underline{t}, \underline{u}, X, \underline{y}, \underline{z}) - \mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c}_1, \underline{t}, \underline{u}, X, \underline{y}, \underline{z}).$$

As in the previous subsection, we have the following lemma

Lemma 3.3.20. *Let g, p, q and r be a non-negative integer, μ, ν ordered partitions as before and let $b = 2g-2+m+n = p+q+r$, then*

$$\frac{1}{p!}h^{(2),\leq,<}_{p,q,r;\mu,\nu} = \sum_{\substack{v_1,\ldots,v_n=0 \\ w_1,\ldots,w_n=0 \\ |\underline{v}|=p, |\underline{w}|=q, 0\leq w_i\leq v_i}} \left[X^p y_1^{v_1}\cdots y_n^{v_n} z_1^{w_n}\cdots z_n^{w_n}\right]\left[t_1^{v_1}\cdots t_n^{v_n} u_1^{w_1}\cdots u_n^{w_n}\right]$$
$$\mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c}, \underline{t}, \underline{u}, X, \underline{y}, \underline{z})$$

and for a wall δ seperating $\mathfrak{c}_2$ and $\mathfrak{c}_1$, we obtain

$$\frac{1}{p!}WC^{(2),\leq,<}_{\delta} = \sum_{\substack{v_1,\ldots,v_n=0 \\ w_1,\ldots,w_n=0 \\ |\underline{v}|=p, |\underline{w}|=q, 0\leq w_i\leq v_i}} \left[X^p y_1^{v_1}\cdots y_n^{v_n} z_1^{w_n}\cdots z_n^{w_n}\right]\left[t_1^{v_1}\cdots t_n^{v_n} u_1^{w_1}\cdots u_n^{w_n}\right]$$
$$\mathcal{WC}^{(2),\leq,<}_{\delta}(\underline{t}, \underline{u}, X, \underline{y}, \underline{z}).$$

By a similar calculation as in the proof of theorem 3.3.18, we get the following result.

Theorem 3.3.21. *Let μ, ν be ordered partitions of the same positive integer and let $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$. Then*

$$\mathcal{WC}_{\delta}^{(2),\leq,<}(\underline{t}, \underline{u}, X, \underline{y}, \underline{z}) = \delta^2 \frac{\varsigma(A_J + X\delta)\varsigma(A_{J^c})\varsigma(\delta A_{[n]})}{\varsigma\big(\delta(A_J + X\delta)\big)\varsigma(\delta A_{J^c})\varsigma(A_{[n]})}$$
$$\big[(t')^0(u')^0\big]\, \mathcal{H}^{(2),\leq,<}_{\mu^I, \nu^J + \delta}(\underline{t}^J, t', \underline{u}^J, u', X, \underline{y}^J, 0, \underline{z}^J, 0)$$
$$\mathcal{H}^{(2),\leq,<}_{\mu_{I^c} + \delta, \nu_{J^c}}(\underline{t}^{J^c}, \underline{u}^{J^c}, X, \underline{y}^{J^c}, \underline{z}^{J^c}),$$

where

$$A_J = \sum_{j \in J} X\nu_j + y_j + z_j$$

and the zero arguments in the first $\mathcal{H}$ are the y and z variables corresponding to the part δ of the partition $\nu^J + \delta$.

So also in the general, mixed, case, the wall-crossing generating function can be related to a product of two Hurwitz generating functions of lower degree.

3.4 Outlook: Tropical correspondence theorem for pillowcase covers

In this section, we introduce the notion of tropical pillowcase covers and prove a correspondence theorem for the enumerative problems introduced in section 2.2.3. We note that a parallel discussion for orientation-reversing involutions can be found in [MR15]. One of the key components is the following reformulation of the enumerative problem posed in [EO06]. One of the key ingredients is the following reformulation of the enumerative problem posed in [EO06]. This new formulation is accessible by tropical geometry which we will use in the proof of theorem 3.4.6 using the methods of [BBM11] and [MR15].

Let $\pi : \mathfrak{C} \to \mathfrak{P}$ be a pillowcase cover of type (μ, ν, d) by which we mean π is a degree $2d$ cover which ramifies with profile $(\nu, 2^{d-|\nu|/2})$ over $0 \in \mathfrak{P}$, (2^d) over the other three orbifold points and $(\mu_i, 1^{2d-\mu_i})$ over some $\ell(\mu)$ given points. Instead of working with orbifolds we now make the transition to torus covers which satisfy certain symmetries. Considering the quotient map $\Pi : T \to \mathfrak{P}$, we obtain the following diagram

$$\begin{tikzcd} & \mathfrak{C} \arrow[d, "\pi"] \\ T \arrow[r, "\Pi"] & \mathfrak{P} \end{tikzcd}$$

which we complete with the fiber product $S = T \times_{\mathfrak{P}} \mathfrak{C}$ and obtain

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{C} \arrow[d, "\pi"] \\ T \arrow[r, "\Pi"] & \mathfrak{P} \end{tikzcd}$$

Using this, we formulate another enumerative problem which is equivalent to the one posed in [EO06]. We count degree $2d$ covers $\alpha : S \to T$, such that

1. S is a curve,
2. α ramifies with profile $(\nu, 2^{d-|\nu|/2})$ over $\Pi^{-1}(0)$,
3. α ramifies with profile (2^d) over the preimages of the other three orbifold points of $\mathfrak{P}$,
4. α ramifies with profile $(\mu_i, 1^{2d-\mu_i})$ over $2\ell(\mu)$ given points $q_1, \ldots, q_{\ell(\mu)}, \tilde{q}_1, \ldots, \tilde{q}_{\ell(\mu)}$,
5. $\Pi(q_i) = \Pi(\tilde{q}_i)$ for $i = 1, \ldots, \ell(\mu)$ and
6. there exists an orientation preserving involution $i_S : S \to S$, such that the

diagram

$$\begin{tikzcd}
S \arrow[r, "i_S"] \arrow[d, "\alpha"] & S \arrow[d, "\alpha"] \\
T \arrow[r, "\pm"] & T
\end{tikzcd}$$

commutes.

We call such covers *involuted torus covers of type* (d, μ, ν, i_S). Moreover, we call two involuted Hurwitz covers $\alpha : S_1 \to T$ and $\alpha' : S_2 \to T$ of respective type (μ, ν, d, i_{S_1}) and (μ, ν, d, i_{S_2}) isomorphic or equivalent if the following diagram commutes

$$\begin{tikzcd}
S_1 \arrow[rr, "i_{S_1}"] \arrow[dr, "h"] \arrow[dd, "\alpha"] & & S_1 \arrow[dr, "h"] \arrow[dd, "\alpha"] & \\
& S_2 \arrow[rr, "i_{S_2}"] \arrow[dd, "\alpha'"] & & S_2 \arrow[dd, "\alpha'"] \\
T \arrow[rr, "\pm"] \arrow[dr, "\mathrm{id}"] & & T \arrow[dr, "\mathrm{id}"] & \\
& T \arrow[rr, "\pm"] & & T
\end{tikzcd}$$

In the following we will remember the involution of the source surface as well and thus speak of involuted torus covers $\alpha : (S, i_S) \to (T, \pm)$ of type (μ, ν, d) with the same notion of isomorphisms. From the previous discussion we deduce the following equalitiy

$$[q^d]Z(\mu, \nu; q) = \sum \frac{1}{|\mathrm{Aut}(\alpha)|},$$

where we sum over all involuted torus covers α of type (μ, ν, d).

3.4.1 Tropical pillowcase covers

We begin with the same data as in section 2.2.3. Let μ be a partition and ν a partition of an even number into odd parts. Furthermore, we define the tropical analogue of the pillowcase orbifold.

Definition 3.4.1. We define the *tropical pillowcase orbifold* to be the graph depicted in figure 3.12 with two 4–valent vertices and two paths between them consisting of $\ell(\mu)$ 3–valent vertices each, together with its natural involution i_P. The genus at each vertex is 0 and (as convention) we set the length of each edge to be 1. We denote the tropical pillowcase orbifold by $\mathbb{T}\mathfrak{P}$. We call the tropical curve we obtain from $\mathbb{T}\mathfrak{P}$ by forgetting the involution the *underlying tropical curve* $\mathbb{T}P$.

We now define the tropical analogue of the enumerative problem posed in section 2.2.3.

Definition 3.4.2. Let C be a tropical curve and $i_C : C \to C$ a tropical morphism which is an involution, i.e. $i_C^2 = id$. We call a tropical morphism $h : C \to \mathbb{T}P$ between C and $\mathbb{T}P$ a *tropical pillowcase cover of type* (μ, ν, d, i_C) if

1. $\deg(h) = 2d$,
2. h ramifies with profile $(\nu, 2, \ldots, 2)$ over v_1,
3. h ramifies with profile $(2, \ldots, 2)$ over v_2, v_3, v_4
4. h ramifies with profile $(\mu_i, 1, \ldots, 1)$ over w_i and $\tilde{w}_i$,
5. h is unramified everywhere else,
6. the diagram

$$\begin{tikzcd} C \arrow[r, "i_C"] \arrow[d, "h"] & C \arrow[d, "h"] \\ \mathbb{T}P \arrow[r, "i_P"] & \mathbb{T}P \end{tikzcd}$$

   commutes.

We say $h : (C, i_C) \to \mathbb{T}\mathfrak{P}$ is a *tropical pillowcase cover of type* (μ, ν, d), if $h : C \to \mathbb{T}P$ is a tropical pillowcase cover of type (μ, ν, d, i_C). We call two tropical pillowcase covers $h : (C_1, i_{C_1}) \to \mathbb{T}\mathfrak{P}$ and $h' : (C_2, i_{C_2}) \to \mathbb{T}\mathfrak{P}$ of type (μ, ν, d) isomorphic if there exists an isomorphism of tropical curves $g : C_1 \to C_2$, such that the following diagramm commutes:

$$\begin{tikzcd}
C_1 \arrow[rr, "i_{C_1}"] \arrow[dr, "g"'] \arrow[dd, "h"'] & & C_1 \arrow[dr, "g"] \arrow[dd, "h"' near start] & \\
& C_2 \arrow[rr, crossing over, "i_{C_2}" near start] & & C_2 \arrow[dd, "h'"] \\
\mathbb{T}P \arrow[rr, "i_P" near end] \arrow[dr, "\mathrm{id}"'] & & \mathbb{T}P \arrow[dr, "\mathrm{id}"'] & \\
& \mathbb{T}P \arrow[rr, "i_P"] \arrow[from=uu, crossing over, "h'" near start] & & \mathbb{T}P
\end{tikzcd}$$

As usual, we count every tropical pillowcase cover with a certain multiplicity. In order to define this multiplicity, we introduce the notion of *involuted Hurwitz covers* and *involuted Hurwitz numbers.*

Definition 3.4.3. Let $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ be partitions of the same positive integer d, let $q = (q_1, q_2, q_3, q_4)$ be a collection of points on $\mathbb{P}^1$, such that we have their cross-ratio $(q_1, q_2; q_3, q_4) = -1$ (e.g. $(q_1, q_2, q_3, q_4) = (0, 1, \infty, \frac{1}{2})$). We call $f : S \to \mathbb{P}^1$ an *involuted Hurwitz cover of type* (μ, g) if

1. S is of genus g,
2. f ramifies with profile μ_i over q_i,
3. there exists an orientation preserving involution $i_S : S \to S$, such that for the linear automorphism $L_q : \mathbb{P}^1 \to \mathbb{P}^1$ which fixes q_1 and q_2, maps q_3 to q_4 and q_4 to q_3 the diagram

$$\begin{tikzcd}
S \arrow[r, "i_S"] \arrow[d, "f"] & S \arrow[d, "f"] \\
\mathbb{P}^1 \arrow[r, "L_q"] & \mathbb{P}^1
\end{tikzcd}$$

commutes.

We call two involuted Hurwitz covers $f : S_1 \to \mathbb{P}^1$ and $g : S_2 \to \mathbb{P}^1$ of type (μ, g) isomorphic or equivalent, if there exists an isomorphism $h : S_1 \to S_2$, such that the following diagram commutes

$$\begin{tikzcd}
S_1 \arrow[rr, "i_{S_1}"] \arrow[dr, "h"] \arrow[dd, "f"'] & & S_1 \arrow[dr, "h"] \arrow[dd, "f" near start] & \\
& S_2 \arrow[rr, "i_{S_2}" near start] \arrow[dd, "g" near start] & & S_2 \arrow[dd, "g"] \\
\mathbb{P}^1 \arrow[rr, "L_q" near end] \arrow[dr, "\mathrm{id}"'] & & \mathbb{P}^1 \arrow[dr, "\mathrm{id}"'] & \\
& \mathbb{P}^1 \arrow[rr, "L_q"] & & \mathbb{P}^1
\end{tikzcd}$$

and we define the involuted Hurwitz numbers of type (g, μ)

$$h_g^{in}(\mu) = \sum \frac{1}{|\mathrm{Aut}(f)|},$$

where we sum over all equivalence classes of involuted Hurwitz covers of type (g, μ).

Remark 3.4.4. The cross-ratio condition in definition 3.4.3 ensures the existence of the linear automorphism L_q. Moreover, the third condition in the definition of involuted Hurwitz covers implies that $\mu_3 = \mu_4$.

Now, we define the multiplicity with which we count every tropical pillowcase cover. In order to give this definition, we introduce the following notation: For a tropical pillowcase cover $h : (C, i_C) \to \mathbb{T}\mathfrak{P}$, we consider the tropical morphism $h : C \to \mathbb{T}P$. Let v be a vertex of C, such that $h(v)$ is adjacent to the edges $e_1, \dots,$ e_{k_v} of $\mathbb{T}\mathfrak{P}$. Let μ_i be the profile of h over e_i at v and we define $\mu_v = (\mu_1, \dots, \mu_{k_v})$. In particular, we denote by u_1 the 4−valent vertex adjacent to v_1 and v_2, and by u_2

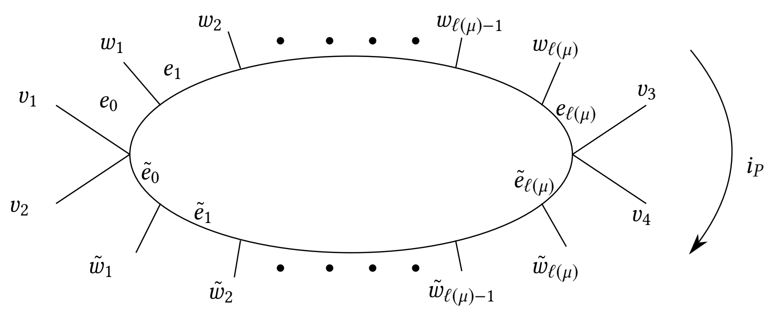


Figure 3.12: The tropical pillowcase orbifold with involution indicated by i_P.

the 4-valent vertex adjacent to v_3 and v_4. Then, for every $u' \in h^{-1}(u_1)$ we define

$$\mu_{u'} = (\mu_{(u_1 v_1)}, \mu_{(u_1 v_2)}, \mu_{e_0}, \mu_{\tilde{e}_0})$$

and for every $u'' \in h^{-1}(u_2)$ we define

$$\mu_{u''} = (\mu_{(u_2 v_3)}, \mu_{(u_2 v_4)}, \mu_{e_{\ell(\mu)}}, \mu_{\tilde{e}_{\ell(\mu)}}).$$

Definition 3.4.5. Let $h : (C, i_C) \to \mathbb{T}\mathfrak{P}$ be a tropical pillowcase cover, x_i be the trivalent vertex adjacent to w_i, $E^+ := \{e_0, \ldots, e_{\ell(\mu)}\}$ and $V^+ := \{x_1, \ldots, x_{\ell(\mu)}\}$. We then define the multiplicity of h by

$$\begin{aligned} m_{\mathbb{T}\mathfrak{P}}(h) =& \frac{1}{|\mathrm{Aut}(h)|} \prod_{e \in E^+} \omega(e) \left(\prod_{v \in V^+} \prod_{v' \in h^{-1}(v)} h_{g(v')}(\mu_{v'}, \mathbb{P}^1) \right) \\ & \prod_{u' \in h^{-1}(u_1)} h^{in}_{g(u')}(\mu_{u'}, \mathbb{P}^1) \prod_{u'' \in h^{-1}(u_2)} h^{in}_{g(u'')}(\mu_{u''}, \mathbb{P}^1). \end{aligned}$$

Theorem 3.4.6. *Let μ be a partition and ν a partition of an even number into odd parts. Then we can compute the coefficients of $Z(\mu, \nu; q)$ in terms of tropical pillowcase covers*

$$[q^d] Z(\mu, \nu; q) = \sum m_{\mathbb{T}\mathfrak{P}}(h),$$

where we sum over all equivalence classes of tropical pillowcase cover $h : (C, i_C) \to \mathbb{T}\mathfrak{P}$ of type (μ, ν, d).

Proof. We now tropicalise involuted torus covers of type (μ, ν, d) to tropical pillowcase covers of type (μ, ν, d) and analyse how many involuted torus covers tropicalise to the same tropical orbifold cover. First, we remember how to associate a surface to the underlying tropical curve of $\mathbb{T}\mathfrak{P}$ as discussed prior to theorem 2.2.19. This yields a torus which for each edge e of $\mathbb{T}P$ contains an embedded circle obtained from the gluing of the boundary circles in the construction. For an involuted torus cover $\alpha : (S, i_S) \to (T, \pm)$ of type (μ, ν, d), the preimages of these circles are disjoint unions of circles in S. In the flavour of [MR15], we call a *component* of S the closure of a connected component of the surface minus the embedded circles. This yields a graph C which consists of a vertex v for each connected component S and an edge e between each two vertices whose respective components are adjacent.

Thus, we obtain a map between graphs $\mathbb{T}\alpha : C \to \mathbb{T}P$. In order to make C a tropical curve, we assign to each vertex a genus which is the genus of the respective component. We determine the weights of $\mathbb{T}\alpha$ at each edge of C. Let e be an edge of C mapping to the edge $\mathbb{T}\alpha(e)$ of $\mathbb{T}P$. Moreover, let B_e be the embedded circle of S corresponding to e and $B_{\mathbb{T}\alpha(e)}$ the embedded circle of T corresponding to $\mathbb{T}\alpha(e)$ then we define

$$\omega_{e,\mathbb{T}\alpha} = \deg(\alpha : B_e \to B_{\mathbb{T}\alpha(e)}).$$

Moreover, we obtain a metric structure on C by requiring $\mathbb{T}\alpha$ to be a tropical morphism.

We call $\mathbb{T}\alpha$ the *tropicalisation of* $\alpha : S \to T$. The involution $i_S : S \to S$ induces an involution on the underlying graph structure $i_C : C \to C$, which is compatible with the tropical morphism $\mathbb{T}\alpha$ in the natural sense. Thus we obtain that $(\mathbb{T}\alpha, i_C)$ is a tropical pillowcase cover of type (μ, ν, d). In order to complete the proof of our theorem, we analyse how many involuted torus covers tropicalise to a fixed tropical pillowcase cover. In order to do this, we determine the number of choices we have in regluing an involuted torus cover of type (μ, ν, d) from a tropical pillowcase cover of type (μ, ν, d):

Let $\phi : (C, i_C) \longrightarrow \mathbb{T}\mathfrak{P}$ be a tropical pillowcase cover of type (μ, ν, d). For each vertex $x \in \phi^{-1}(x_i)$, the corresponding component S_x maps to S_{x_i}. This is a covering map of a surface with embedded boundary circles by a surface with embdedded boundary circles. This translates to Hurwitz covers as follows: For each embedded boundary circle, we glue in a circle with a marked point. These points are the branch points in the target surface and the ramification points in the source surface. Furthermore, the local degree of the map at each ramification point is the degree of the covering map at the corresponding surface before gluing. In this way, we obtain a Hurwitz cover between the smoothened surfaces $\tilde{S}_x$ and $\tilde{S}_{x_i}$. The number of such covers is exactly $h_{g(x)}(\mu_x, \mathbb{P}^1)$ which yields the factor

$$\left(\prod_{v \in V^+} \prod_{v' \in h^{-1}(v)} h_{g(v')}(\mu_{v'}, \mathbb{P}^1)\right).$$

Furthermore, we note that it is possible to glue two unramified circle coverings of degree D in D many ways, which yields a factor of $\omega_{e,\phi}$ for each edge. This yields the factor for the edges in $e \in E^+$

$$\prod_{e \in E^+} \omega(e).$$

Since we want to obtain involuted torus covers $\pi : S \longrightarrow \mathfrak{P}$, such that the involution i_S tropicalises to i_C the choices for V^+ and E^+ already determine the choices for $i_C(E^+)$ and $i_C(V^+)$. The only thing left to understand is the contribution of the two 4−valent vertices, since the the factor $\frac{1}{\mathrm{Aut}(h)}$ corresponds to the automorphisms of π.

For each vertex $u \in \phi^{-1}(u_1)$ the corresponding cover $\tilde{S}_u \longrightarrow \tilde{S}_{u_1}$ needs to be compatible with the symmetry induced by the involution i_P mapping e_0 to $\tilde{e}_0$. This is exactly the data encoded in the cross-ratio and involution condition in the definition of involuted Hurwitz covers. The same is true for $u \in \phi^{-1}(u_2)$ and we obtain the factor

$$\prod_{u' \in h^{-1}(u_1)} h^{in}_{g(u')}(\mu_{u'}, \mathbb{P}^1) \prod_{u'' \in h^{-1}(u_2)} h^{in}_{g(u'')}(\mu_{u''}, \mathbb{P}^1)$$

□

Remark 3.4.7. In order to obtain the connected algebro-geometric enumeration, we consider tropical pillowcase covers $h : (C, i_C) \to \mathbb{T}\mathfrak{P}$, such that the graph ${}^{C}\!/_{i_C}$, obtained from C by identifying along $i_C : C \to C$, is connected.

CHAPTER 4

MUSTAFIN VARIETIES, MODULI SPACES AND TROPICAL GEOMETRY

4.1 Introduction

Mustafin varieties are flat degenerations of projective spaces induced by choosing an n-tuple of lattices in the Bruhat-Tits building $\mathfrak{B}_d$ associated to $\mathrm{PGL}(V)$ over a non-archimedean field K (see section 2.3). These objects were introduced by Mustafin in [Mus78] in order to generalise Mumford's groundbreaking work on the uniformisation of curves to higher dimensions [Mum72]. Since then they were repeatedly studied under the name *Deligne schemes* (see e.g. [Fal01; KT06]). By studying degenerations of projective spaces, this gives a framework for the study of degenerations of projective subvarieties. In his original work, Mustafin studied the case of so-called *convex point configurations* in $\mathfrak{B}_d$ as defined in definition 2.3.1. An approach to study arbitrary point configurations was developed in [CHSW11], where the total space of this type of degenerations was named *Mustafin variety* for the first time. There it was proved that if the lattices in the point configuration have diagonal form with respect to a common basis (i.e. they lie in the same apartment), the corresponding Mustafin variety is essentially a toric degeneration given by mixed subdivisions of a scaled simplex. These mixed subdivision are beautiful combinatorial objects that are known to be equivalent to tropical polytopes and triangulations of products of simplices. For point configurations that do not obey this property some first structural results were proved in [CHSW11].

(Pre)linked Grassmannians

In this thesis, we uncover a connection between Mustafin varieties and so-called (pre)linked Grassmannians. *Linked Grassmannians* were introduced in [Oss06] in the context of limit linear series. Osserman introduced a new theory of limit linear series that in a certain sense compactifies the Eisenbud-Harris limit linear series theory. A generalisation of this notion was introduced in [Oss14], as so-called *prelinked Grassmannians*. (Pre)linked Grassmannians are degenerations of Grassmannians induced by a graph with additional data at the vertices and edges. If this graph is just a path, we call these objects linked Grassmannians and they were proved to be flat with reduced fibers in [HO08]. The focal objects in studying (pre)linked Grassmannians are so-called *simple points*.

4.1.1 Outline of Results

In chapter 4 we conduct a study of Mustafin varieties from new perspectives, where we focus on their special fibers (see definition 2.3.7) when the residue field k is algebraically closed. The main tool in this paper is the study of closures images of rational maps of the form

$$f : \mathbb{P}(W) \dashrightarrow \mathbb{P}\left({}^{W}\!/_{W_1}\right) \times \cdots \times \mathbb{P}\left({}^{W}\!/_{W_n}\right),$$

where W is a vector space over k of dimension d and $(W_i)_{i\in[n]}$ is a tuple of sub-vector spaces $W_i \subset W$, such that $\bigcap W_i = \langle 0 \rangle$ (see [Li17], section 2.3.2). We denote the closure of the above map by $X(W, W_1, \ldots, W_n)$.

We proceed as follows: Let Γ be a point configuration in the Bruhat-Tits building $\mathfrak{B}_d$ associated to $\mathrm{PGL}(V)$. We consider the convex hull $\mathrm{conv}(\Gamma)$ (see Definition 2.3.1), which is a set of lattices. To each lattice class $[L] \in \mathrm{conv}(\Gamma)$, we associate a variety $X_{\Gamma,[L]}$ of the form $X(k^d, W_1, \ldots, W_n)$ for some W_i depending on $[L]$ and Γ (see Construction 4.2.1). Then we define two varieties in $\left(\mathbb{P}_k^{d-1}\right)^n$ as follows

$$\widetilde{\mathcal{M}}(\Gamma) = \bigcup_{[L]\in \mathrm{conv}(\Gamma)} X_{[L]} \text{ and } \widetilde{\mathcal{M}}^r(\Gamma) = \bigcup_{[L]\in V(\mathrm{conv}(\Gamma))} X_{[L]},$$

where $V(\Gamma)$ is the set of polyhedral vertices of $\mathrm{conv}(\Gamma)$. The following is one of the main results of this chapter.

Theorem 4.1.1. *The irreducible components of Mustafin varieties are related to images of rational maps as follows:*

1. *If Γ is an arbitrary point configuration, we have*

$$\mathcal{M}(\Gamma)_k = \widetilde{\mathcal{M}}(\Gamma).$$

2. *If Γ is a point configuration in one apartment, we have*

$$\mathcal{M}(\Gamma)_k = \widetilde{\mathcal{M}}^r(\Gamma).$$

In each case, it is easy to see that the right hand side is contained in the special fiber of the Mustafin variety. For the other direction, we have to identify those lattice points that actually contribute an irreducible component. This is done by means of tropical intersection theory and multidegrees. Using this description we get a complete classification of the irreducible components of special fibers of Mustafin varieties as each variety of the form $X(k^d, W_1, \ldots, W_n)$ occurs as an irreducible component. Thus, we obtain the following application of Theorem 4.1.1:

Theorem 4.1.2. *The varieties $X(k^d; W_1, \ldots, W_d)$ classify all irreducible components of special fibers of Mustafin varieties, i.e.*

1. *any irreducible component of the special fibers of a Mustafin varieties is a variety of the form $X(k^d; W_1, \ldots, W_d)$, and*
2. *every variety $X(k^d; W_1, \ldots, W_n)$ appears as an irreducible component of the special fiber $\mathcal{M}(\Gamma)_k$ for some Γ.*

As mentioned before, we show that Mustafin varieties are closely related to (pre)linked Grassmannians.

For a convex point configuration Γ we associate a (pre)linked Grassmannian as a scheme $LG(1, \Gamma)$ over R. In [Fal01], Faltings introduced a moduli functor for Mustafin varieties. Using this result, we can interpret Mustafin varieties as the moduli space for the linked Grassmannian problem. Furthermore, this proves that the class of linked Grassmannians induced by the data Γ is flat with reduced fibers. Moreover, using a connection between Mustafin varieties and the simple points of a linked Grassmannian, the following theorem is a direct application of Theorem 4.1.1.

Theorem 4.1.3. *The locus of simple points of* $\mathrm{LG}(1, \Gamma)$ *is dense in every fiber over* R*.*

4.2 Special fibers of Mustafin varieties

The goal of this section is to prove Theorem 4.1.1 and Theorem 4.1.2.

4.2.1 Constructing the varieties

We begin by constructing the varieties $\widetilde{\mathcal{M}}(\Gamma)$ and $\widetilde{\mathcal{M}}^r(\Gamma)$.

Construction 4.2.1. Let $\Gamma = \{L_1, \ldots, L_n\}$ be a point configuration in $\mathfrak{B}_d^0$ as in section 2.3.1. Let $[L]$ be a homothety class in $\operatorname{conv}(\Gamma)$. We choose $L \in [L]$ to be the reference lattice in our choice of coordinates in Subsection 2.3.3. We describe maps between $[L]$ and $[L_i]$. Pick an apartment A_i, corresponding to the basis $e_1^i, \ldots, e_d^i$, such that $[L]$ and $[L_i]$ are contained in A, i.e.:

$$L = \pi^{m_1} R e_1^i + \cdots + \pi^{m_d} R e_d^i$$

and

$$L_i = \pi^{n_1^i} R e_1^i + \cdots + \pi^{n_d^i} R e_d^i.$$

(This is always possible, see e.g. the discussion prior to proposition 6.111 in [AB08].) There is a canonical map g_i, such that $g_i[L] = [L_i]$, which up to homothety (i.e. multiplication by a scalar) is given by the matrix

$$h_i = \begin{pmatrix} \pi^{n_1^i - m_1} & & \\ & \ddots & \\ & & \pi^{n_d^i - m_d} \end{pmatrix}$$

in the basis $e_1^i, \ldots, e_d^i$. Thus, in our coordinates the Mustafin variety $\mathcal{M}(\Gamma)$ is given by the closure of the image of

$$f_{\Gamma,[L]} : \mathbb{P}(V) \xrightarrow{(g_1^{-1} \times \cdots \times g_n^{-1}) \circ \Delta} \mathbb{P}(L)^n.$$

We choose a set of generators $e_1^{[L]}, \ldots, e_d^{[L]}$ for L and we denote the transformation matrix representing g_i with respect to $e_1^{[L]}, \ldots, e_d^{[L]}$ by G_i'. We define an invertible matrix G_i by $G_i^{-1} = \pi^{s_i} G_i'$, where $s_i = -\min_{m,n=1,\ldots,d}\{\operatorname{val}\left((G_i^{-1})_{m,n}\right)\}$. This induces a rational map over the special fiber.

$$\widetilde{f_{\Gamma,[L]}} : \mathbb{P}_k^{d-1} \dashrightarrow (\mathbb{P}_k^{d-1})^n,$$

given by $(\widetilde{g_1^{-1}} \times \cdots \times \widetilde{g_1^{-1}}) \circ \Delta$, where $\widetilde{g_i^{-1}}$ is obtained from G_i^{-1} by the entry-wise residue map. Denoting the varieties $X_{\Gamma,[L]} = \overline{\operatorname{Im}(\widetilde{f_{\Gamma,[L]}})}$, we define our desired varieties as the union of the closures of the images of these rational maps:

$$\widetilde{\mathcal{M}}(\Gamma) = \bigcup_{[L]\in \mathrm{conv}(\Gamma)} X_{\Gamma,[L]} \quad \text{and} \quad \widetilde{\mathcal{M}}^r(\Gamma) = \bigcup_{[L]\in V(\mathrm{conv}(\Gamma))} X_{\Gamma,[L]},$$

where $V(\mathrm{conv}(\Gamma)$ is the set of polyhedreal vertices of Γ.

Remark 4.2.2. The expression

$$\widetilde{\mathcal{M}}(\Gamma) = \bigcup_{[L]\in \mathrm{conv}(\Gamma)} X_{\Gamma,[L]} \quad \text{and} \quad \widetilde{\mathcal{M}}^r(\Gamma) = \bigcup_{[L]\in V(\mathrm{conv}(\Gamma))} X_{\Gamma,[L]}$$

include a slight abuse of notation as $X_{\Gamma,[L]}$ is contained in the special fiber of $\mathbb{P}(L)^n$ and $X_{\Gamma,[L']}$ is contained in the special fiber of $\mathbb{P}(L')^n$. In order to take the union , we observe that the isomorphism $\mathbb{P}(L')^n \to \mathbb{P}(L)^n$ induces an isomorphism of special fibers and thus maps $X_{\Gamma,[L']}$ into the special fiber of $\mathbb{P}(L)^n$, where we can take the union.

When Γ is a point configuration in one apartment induced by $e_1, \ldots, e_d$, we can choose a set of generators $e_1^{[L]}, \ldots, e_d^{[L]}$, such that the matrices g_i coincide with the matrices h_i up to homothety. The maps $\widetilde{g_i^{-1}}$ are then given by

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix},$$

where

$$a_j = 1, \text{ if } n_j^i - m_j = \min_{k=1,\ldots,d} (n_k^i - m_k)$$

and $a_j = 0$ else. (see example 4.2.3).

The key observation for the proof of Theorem 4.1.1 is that the maps $\widetilde{f_{\Gamma,[L]}}$

factorise as follows:

$$\begin{tikzcd}
\mathbb{P}^{d-1}_k \arrow[r, dashed] \arrow[dr, dashed, "\widetilde{f_{\Gamma,[L]}}"] & \mathbb{P}\left({k^d}/{\mathrm{Ker}(\widetilde{g_1^{-1}})}\right) \times \cdots \times \mathbb{P}\left({k^d}/{\mathrm{Ker}(\widetilde{g_n^{-1}})}\right) \arrow[d, hook] \\
& \left(\mathbb{P}^{d-1}_k\right)^n
\end{tikzcd} \tag{4.1}$$

Thus, each variety $\overline{\mathrm{Im}(\widetilde{f_L})}$ is a variety of the form $X(k^d, W_1, \ldots, W_n)$, where $W_i = \ker(\widetilde{g_i^{-1}})$. Note that $\bigcap_{i=1}^n W_i$ might not be trivial. However, the components of $\mathcal{M}(\Gamma)_k$ are equidimensional. The vertices with $\bigcap_{i=1}^n W_i \neq \langle 0 \rangle$ contribute varieties of lower dimension and thus are contained in an irreducible component by lemma 4.2.4. This irreducible component is contributed by a vertex satisfying $\bigcap_{i=1}^n W_i = \langle 0 \rangle$.

Before we prove Theorem 4.1.1, we illustrate our construction in the following example:

Example 4.2.3. By lemma 2.3.2, there is a natural correspondence between lattice points in ${\mathbb{Z}^d}/{\mathbb{Z}\mathbf{1}}$ and lattices in V over R. In example 2.2 in [CHSW11], the special fiber of the Mustafin variety corresponding to the vertices in example 2.1.12 was computed to be the union of the following irreducible components

$$\mathbb{P}^2 \times pt \times pt,\ pt \times \mathbb{P}^2 \times pt,\ pt \times pt \times \mathbb{P}^2,\ \mathbb{P}^1 \times \mathbb{P}^1 \times pt,\ \mathbb{P}^1 \times pt \times \mathbb{P}^1,\ pt \times \mathbb{P}^1 \times \mathbb{P}^1$$

Our construction yields the same variety as seen in the following computations. For $v = v_3$, we obtain the map

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi^2 & 0 \\ 0 & 0 & \pi^4 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi^2 \end{pmatrix} \times \mathrm{id}$$

Modding out π, we immediately see that

$$\mathrm{Im}(\widetilde{f_{\Gamma,[L]}}) = pt \times pt \times \mathbb{P}^2,$$

where $[L]$ is the lattice class corresponding to v. Analogously, we obtain $pt\times\mathbb{P}^2\times pt$ and $\mathbb{P}^2 \times pt \times pt$ for $v = v_2$ and $v = v_1$ respectively.
For $v = (0, -1, -4)$, we obtain the map

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi^2 \end{pmatrix} \times \begin{pmatrix} \pi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} \times \begin{pmatrix} \pi^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Modding out π, we immediately see that

$$\mathrm{Im}(f_{\Gamma,[L]}) = \mathbb{P}^1 \times pt \times \mathbb{P}^1,$$

where once again $[L]$ is the lattice class corresponding to v. Analogously, we obtain $\mathbb{P}^1 \times \mathbb{P}^1 \times pt$ and $pt \times \mathbb{P}^1 \times \mathbb{P}^1$ for $v = (0, -1, -3)$ and $v = (0, -2, -4)$ respectively.

The following lemma implies the easier containment for the equalities of Theorem 4.1.1.

Lemma 4.2.4. *For any lattice* $[L]$ *in* $\mathrm{conv}(\Gamma)$*, the following relation holds:*

$$\overline{\mathrm{Im}(\widetilde{f_{\Gamma,[L]}})} \subset \mathcal{M}(\Gamma)_k.$$

Moreover, $\overline{\mathrm{Im}(\widetilde{f_{\Gamma,[L]}})}$ *is an irreducible component of* $\mathcal{M}(\Gamma)_k$ *if and only if*

$$\dim(\overline{\mathrm{Im}(\widetilde{f_{\Gamma,[L]}})}) = \dim(\mathcal{M}(\Gamma)_k).$$

Proof. To prove the first statement, denote the ideal in $R[x_{ij}]$ defining $\mathcal{M}(\Gamma)$ in $\mathbb{P}(L)^n$ by $I(\Gamma, [L])$ and the ideal in $k[x_{ij}]$ defining $\mathcal{M}(\Gamma)_k$ in the special fiber of $\mathbb{P}(L)^n$ by $I(\Gamma, [L])_k$. Let $v \in \mathbb{P}_k^{d-1} \hookrightarrow \mathbb{P}_K^{d-1}$, $w = \widetilde{f_{\Gamma,[L]}}(v)$ and let $g \in I(\Gamma, [L])_k$, we want to prove that $g(w) = 0$. We choose a polynomial $g' \in I(\Gamma, [L])$, which specialises to g and define $W' = f_{\Gamma,[L]}(v) \in \mathcal{M}(\Gamma) \subset \mathbb{P}(L)^n$. We see immediately that $g'(W') = 0$. Moreover, since we chose the component-wise maps G_i^{-1}, such that $G_i^{-1} \in \mathrm{Mat}(R, d \times d)$ and such that there are elements in G_i^{-1} of valuation 0, we see that the constant term of $g'(W')$ is given by $g(W)$. However, since $g'(w') = 0$, the constant term vanishes as well and thus $g(w) = 0$. We obtain $\widetilde{f_{\Gamma,[L]}}(v) \in \mathcal{M}(\Gamma)_k$ and thus $\widetilde{f_{\Gamma,[L]}} \subset \mathcal{M}(\Gamma)_k$ as desired. By definition $\overline{\mathrm{Im}(\widetilde{f_{\Gamma,L}})}$ is reduced and irreducible and by [CHSW11] the same is true for the irreducible components of $\mathcal{M}(\Gamma)_k$. Therefore, the second statement follows. □

4.2.2 A basic case: Two lattices

In this section we use results from [EO13] to prove Theorem 4.1.1 (2) if Γ consists of two lattices. That is, we show $\widetilde{\mathcal{M}}^r(\Gamma) = \mathcal{M}(\Gamma)_k$, demonstrating some of the basic ideas for the proof of Theorem 4.1.1 (2) in subsection 4.2.4. By lemma 4.2.4, the variety $\widetilde{\mathcal{M}}^r(\Gamma)$ is contained in $\mathcal{M}(\Gamma)_k$. Since both schemes are reduced, we can deduce equality if their bivariate Hilbert polynomials coincide.
Each pair of two lattices is contained in a common apartment. Moreover, their convex hull forms a path between the two lattices we started with. Using the maps over the special fiber as in construction 4.2.1, we obtain a complex as follows:

$$B_1 \rightleftarrows \cdots \rightleftarrows B_n,$$

where each $B_i = k^d$ (and each B_i corresponds to a polyhedral vertex). We use the notation of the B_is in order to keep track of the position. This is the same situation as in [EO13]:

Definition 4.2.5 ([EO13])**.** Let $f_i : \mathbb{P}(B_i) \dashrightarrow \mathbb{P}(B_1) \times \mathbb{P}(B_n)$ be the induced map obtained by composing the maps above along the shortest path to the extremal vertices. We define $\bigcup_{i=1}^n \overline{\mathrm{Im}(f_i)}$ to be the *associated Esteves-Osserman variety.*

Remark 4.2.6. A similar discussion concerning the connection between more general Esteves-Osserman varieties and linked Grassmannian can be found in [HL18]. However, since this work is still in preparation, we decided to include the arguments needed for our case.

In [EO13], it is proved that $\widetilde{\mathcal{M}}(\Gamma)$ has the same bivariate Hilbert polynomial as $\mathbb{P}^{d-1}$ if the maps in the complex fulfil the following exactness condition:

$$\ker(B_i \to B_{i+1}) = \mathrm{Im}(B_{i+1} \to B_i)$$
$$\ker(B_{i+1} \to B_i) = \mathrm{Im}(B_i \to B_{i+1})$$
$$\mathrm{Im}(B_{i-1} \to B_i) \cap \ker(B_i \to B_{i+1}) = \langle 0 \rangle$$
$$\mathrm{Im}(B_{i+1} \to B_i) \cap \ker(B_i \to B_{i-1}) = \langle 0 \rangle$$

Lemma 4.2.7. *Let $\Gamma = \{L_1, L_2\}$ and apply construction 4.2.1 to obtain a complex as follows:*

$$B_1 \rightleftarrows \cdots \rightleftarrows B_n,$$

where $B_i = k^d$ for all $i \in \{1, \ldots, n\}$. Then this complex fulfils the exactness conditions above.

Proof. The two vertices lie in a common apartment. We see immediately that over R the maps

$$A_i \to A_{i+1}$$

are given by diagonal matrices

$$A = \mathrm{diag}(a_1, \ldots, a_d),$$

where $a_j = 1$ for $j \in J_i$ and $a_j = \pi$ for $j \in J_i^c$, where J_i is some subset of $[d]$. Moreover, over R the maps

$$A_{i+1} \to A_i$$

are given by diagonal matrices

$$B = \mathrm{diag}(b_1, \ldots, b_d),$$

where $b_j = \pi$ for $j \in J_i$ and $b_j = 1$ for $j \in J_i^c$. Thus, the first two conditions follow immediately after modding out π.

The third and fourth condition follow from the fact that $J_i \subset J_{i+1}$ which is a consequence of the structure of the tropical convex hull (see lemma 2.1.10). $\square$

Thus, we obtain Theorem 4.1.1 restricted to the case where $n = 2$.

4.2.3 Proof of Theorem 4.1.1 (1)

This subsection is devoted to proving Theorem 4.1.1 (1). The strategy of the proof is to show that

$$\mathcal{M}(\Gamma)_k = \widetilde{\mathcal{M}}(\Gamma)$$

for convex point configurations Γ and recover the general case by using lemma 2.4 of [CHSW11].

Let Γ be a convex point configuration. We need the following corollary, which was proved in [CHSW11], which is essentially a consequence of proposition 2.3.8 (5).

Corollary 4.2.8. *Let $\Gamma = \{L_1, \ldots, L_n\}$ be an arbitrary point configuration. For every $i \in [n]$, the special fiber $\mathcal{M}(\Gamma)_k$ has a unique irreducible component which maps birationally onto $\mathbb{P}(L_i)_k$ under the projection $\mathbb{P}(L_1)_k \times \cdots \times \mathbb{P}(L_n)_k \to \mathbb{P}(L_i)_k$.*

Moreover, by [Fal01], the special fiber consists of n irreducible components for convex point configurations Γ. Thus, in order to prove

$$\mathcal{M}(\Gamma)_k = \widetilde{\mathcal{M}}(\Gamma)$$

we need to find the unique component C_i for each $i \in [n]$ with the properties as in corollary 4.2.8 in $\widetilde{\mathcal{M}}(\Gamma)$. Since there are only n components those are all components and the equality is established.

Let $[L] \in \Gamma$ and consider the map $\widetilde{f_{\Gamma,[L]}}$ in construction 4.2.1

$$\widetilde{f_{\Gamma,[L]}} : \mathbb{P}_k^{d-1} \to \prod_{L' \in \Gamma} \mathbb{P}_k^{d-1}.$$

By construction, the map to the factor corresponding to $[L]$ in $\prod_{L' \in \Gamma} \mathbb{P}_k^{d-1}$ is the identity map. Thus, by lemma 4.2.4 the closure of the image will be a component of

$\mathcal{M}(\Gamma)_k$. Moreover, since the map to the factor corresponding to $[L]$ is the identity map, the respective component maps birationally to the factor $\mathbb{P}(L)_k$ under the natural projection. This completes the proof of Theorem 4.1.1 for convex point configurations.

Now, let Γ be an arbitrary point configuration. By lemma 2.4 of [CHSW11], we can compare the special fibers of $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\mathrm{conv}(\Gamma))$: Let C be an irreducible component of $\mathcal{M}(\Gamma)_k$, then there exists a unique irreducible component $\tilde{C}$ of $\mathcal{M}(\mathrm{conv}(\Gamma))_k$ which projects onto C under

$$p_\Gamma : \prod_{L \in \mathrm{conv}(\Gamma)} \mathbb{P}(L)_k \to \prod_{L \in \Gamma} \mathbb{P}(L)_k.$$

The key idea in proving Theorem 4.1.1 (1) for arbitrary point configurations is to observe that construction 4.2.1 commutes with projections, i.e. for $\Gamma' = \mathrm{conv}(\Gamma)$ the following diagramm commutes on an open set

$$\begin{tikzcd} \mathbb{P}_k^{d-1} \arrow[r, dashed, "\widetilde{f_{\Gamma',[L]}}"] \arrow[dr, dashed] & \prod_{L \in \Gamma'} \mathbb{P}_k^{d-1} \arrow[d, "p_\Gamma"] \\ & \left(\mathbb{P}_k^{d-1}\right)^n \end{tikzcd}$$

We have proved that

$$\mathcal{M}(\Gamma')_k = \bigcup_{L \in \Gamma'} \overline{\mathrm{Im}(\widetilde{f_{\Gamma',L}})}.$$

To see that

$$\mathcal{M}(\Gamma)_k = \bigcup_{L \in \Gamma'} \overline{\mathrm{Im}(\widetilde{f_{\Gamma,L}})}.$$

we fix an irreducible component $C \in \mathcal{M}(\Gamma)_k$ and denote the component which projects onto it by $\widetilde{C}$. By the previous discussion, there exists a unique lattice $L_C \in \Gamma'$, such that $\widetilde{C} = \overline{\mathrm{Im}(\widetilde{f_{\Gamma',[L_C]}})}$. By the commutative diagram, we see that

$C = p_\Gamma(\widetilde{C}) = \overline{\mathrm{Im}(f_{\Gamma,[L_C]})}$. Thus every component C is contained in $\widetilde{\mathcal{M}}(\Gamma)$ and we obtain

$$\mathcal{M}(\Gamma)_k = \bigcup_{C \in C_\Gamma} C \subset \widetilde{\mathcal{M}}(\Gamma) \subset \mathcal{M}(\Gamma)_k,$$

where C_Γ is the set of irreducible components of $\mathcal{M}(\Gamma)_k$. We deduce

$$\mathcal{M}(\Gamma)_k = \widetilde{\mathcal{M}}(\Gamma)$$

for arbitrary point configurations Γ, which completes the proof. □

4.2.4 Proof of Theorem 4.1.1 (2)

In this subsection, we prove the equality of the variety $\widetilde{\mathcal{M}}^r(\Gamma)$ defined in construction 4.2.1 and the special fiber of the Mustafin variety $\mathcal{M}(\Gamma)$ defined in definition 2.3.7. We have already seen that $\widetilde{\mathcal{M}}^r(\Gamma) \subset \mathcal{M}(\Gamma)_k$ in lemma 4.2.4. To see the other direction, we compare the multidegrees. Since $\mathcal{M}(\Gamma)_k$ is a flat degeneration of the diagonal of $(\mathbb{P}^{d-1})^n$, we know (see e.g. [CS10]) that

$$\mathrm{multDeg}(\mathcal{M}(\Gamma)_k) = \left\{ (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0} : \sum_{i=1}^n m_i = d-1 \right\}. \qquad (4.1)$$

Moreover, the multidegree function takes value 1 at each multidegree. It is easy to see that $\mathrm{multDeg}(\mathcal{M}(\Gamma)_k) = \binom{n+d-2}{d-1}$
If we prove that the same holds true for the multidegree function of $\widetilde{\mathcal{M}}^r(\Gamma)$, we can deduce equality. Thus, we have to prove two statements:

- The special fiber of the Mustafin variety and the variety we constructed have the same multidegrees: $\mathrm{multDeg}(\widetilde{\mathcal{M}}^r(\Gamma)) = \mathrm{multDeg}(\mathcal{M}(\Gamma)_k)$, where we know the right hand side of the equation.

- The multidegree function takes value one at each

$$(m_1, \ldots, m_n) \in \mathrm{multDeg}(\widetilde{\mathcal{M}}^r(\Gamma))$$

.

The basic idea is as follows: For every tuple $(m_1, \ldots, m_n)$ as above, we find a vertex in tconv(Γ) whose associated variety contributes multidegree $(m_1, \ldots, m_n)$. We use basic tropical intersection theory. By our method, it is immediate that there is only one vertex that can contribute a variety of this multidegree.

We begin by treating the case of lattices being in tropical general position. The case of arbitrary point configurations will follow using stable intersection.

Point configurations in tropical general position

For a point configuration Γ in tropical general position, the number of polyhedral vertices in the respective tropical convex hull is $\binom{n+d-2}{d-1}$ as proved in [DS04]. This coincides with the cardinality of $\mathrm{multDeg}(\mathcal{M}(\Gamma)_k)$. Therefore, the natural candidates to for the correct multidegrees are those vertices.

We begin by describing the lattices corresponding to the lattice points of the polyhedral complex associated to tconv(Γ) and start by defining a map

$$\begin{aligned} C : V\left(\mathrm{tconv}(\Gamma)) \cap {}^{\mathbb{Z}^{d-1}}\!\big/\!_{\mathbb{Z}\mathbf{1}}\right) &\longrightarrow \left\{(m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0} \middle| \sum_{i=1}^{n} m_i = d-1\right\} \\ v &\longmapsto (C(v)_1, \ldots, C(v)_n), \end{aligned} \tag{4.2}$$

where $C(v)_i$ is the largest k, such that v lies in the codimension k-skeleton of the hyperplane rooted at at the lattice $L_i \in \Gamma$. It is not obvious why this map is well defined. This is where tropical intersection theory comes into play.

As mentioned before the number of vertices of tconv(Γ) coincides with the number of tuples $(m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}$. If we construct one pre-image for each such tuple under the map C in equation (4.2), we are done. Let $(m_1, \ldots, m_n)$ be such a tuple. We identify a vertex in tconv(Γ) as follows: Let $\Gamma = \{L_1, \ldots, L_n\}$ and $v_1, \ldots, v_n$ the corresponding vertices in the tropical torus. We intersect the standard max-hyperplanes H_i at v_i

$$H_1^{m_1} \cap_{st} \cdots \cap_{st} H_n^{m_n} = H_1^{m_1} \cap \cdots \cap H_n^{m_n} = \{v\}$$

as in proposition 2.1.21. This intersection point v is vertex of $\mathrm{tconv}(\Gamma)$ and thus C is well-defined. In fact we have proved that the map C is bijective.

Now, we study the maps $\widetilde{f_{\Gamma,[L]}} = (\widetilde{g_1^{-1}}, \ldots, \widetilde{g_n^{-1}})$ as in construction 4.2.1, in particular the associated numerical data

$$d_I = \dim \bigcap_{i \in I} \ker \widetilde{g_i^{-1}}.$$

Lemma 4.2.9. *Let $v \in \mathrm{tconv}(v_1, \ldots, v_n)$ such that v is contained in the codimension m_i skeleton at v_i, but not in the codimension $m_i + 1$ skeleton, in the tropical torus. Let L be a lattice in the class corresponding to v, then*

$$\dim \mathrm{Ker}(\widetilde{g_i^{-1}}) = d - 1 - m_i,$$

where $\widetilde{g_1^{-1}}$ is the map to the $i-th$ factor in construction 4.2.1.

Proof. We look at $v_i - v = (v_{i0} - v_1, \ldots, v_{id} - v_d)$. Since v is contained in the codimension m_i skeleton at v_i, but not in the codimension $m_i + 1$ skeleton, the minimum of $v_{i0} - v_1, \ldots, v_{id} - v_d$ is attained $m_i + 1$ times. As described in construction 4.2.1, the map $\widetilde{g_1^{-1}}$ over the special fiber is given by

$$D = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix},$$

where $a_j = 1$, if the minimum is attained at $v_{ij} - v_1$ and $a_j = 0$ else. Thus, the dimension of the kernel of the map is given by $d - (m_i + 1)$. ◻

Lemma 4.2.10. *Let v be a vertex of $\mathrm{tconv}(v_1, \ldots, v_n)$, where the v_i are in tropical general position. Let $[L]$ be the lattice class corresponding to v. For any $I \subset \{1, \ldots,$*

$n\}$, we have

$$d_I = \bigcap_{i\in I} \ker(\widetilde{g_1^{-1}})_i = d - 1 - \sum_{i\in I} m_i,$$

where $\widetilde{g_1^{-1}}$ is the map to the i−th factor in construction 4.2.1.

Proof. Let v be the intersection point of the max-hyperplanes at $v_1, \dots, v_n$ as before. Moreover, let C_i be the codimension m_i cone of the hyperplane at v_i in which v is contained.
Let H_I be the intersection of the C_i for $i \in I$. We can describe each C_i as follows:

$$\begin{aligned} C_i = \{w \in {}^{\mathbb{R}^n}\!/_{\mathbb{R}\mathbf{1}} :\ & \text{The minimum of } v_{i1} - w_1, \dots, v_{id} - w_d \\ & \text{is attained exactly at } w_j - v_{ij} \text{ for } j \in J_i\}, \end{aligned}$$

for some $J_i \subset \{1, \dots, d\}$ such that $|J_i| = m_i + 1$. Since $v_1, \dots, v_n$ are in general position, $(v_i)_{i\in I}$ are in general position as well. Moreover, with the same arguments as before, the codimension m_i skeleta at v_i for $i \in I$ intersect in codimension $\sum_{i\in I} m_i$. Thus, we have

$$\left|\bigcup_{i\in I} J_i\right| = \sum_{i\in I} m_i + 1.$$

Now the kernel of the map $\widetilde{f_{\Gamma,[L]}}_i$ is generated by the basis vectors e_j, such that $j \notin J_i$. Thus, the intersection of the kernels is generated by those basis vectors e_j such that $j \notin J_i$ for all i, that is

$$d_I = \left|\left(\bigcup_{i\in I} J_i\right)^c\right| = d - \left(\sum_{i\in I} m_i + 1\right).$$

Thus the lemma follows. □

Example 4.2.11. Let v_1, v_2 and v_3 as in example 2.1.12. We want to find the vertex which contributes the multidegree $(1, 0, 1)$. In order to do this, we intersect the codimension 1 skeleton at v_1, the codimension 0 skeleton at v_2 and the codimension 1 skeleton at v_3 as illustrated in Figure 4.1. We obtain the vertex $(0, -1, -4)$, which by example 4.2.3 contributes the variety $\mathbb{P}^1 \times pt \times \mathbb{P}^1$, which has multidegree

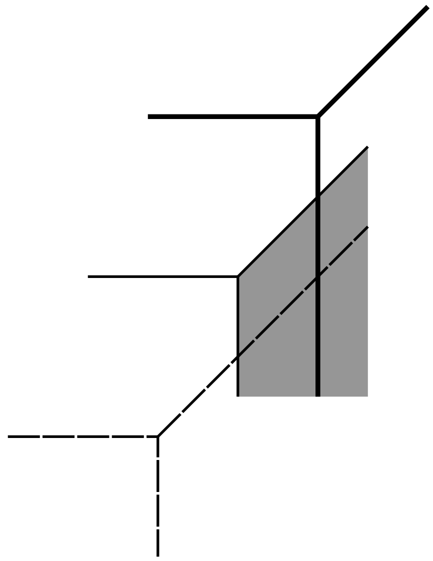


Figure 4.1: Illustration of the intersections of the respective skeleta.

$(1, 0, 1)$.

We are now ready to prove Theorem 4.1.1 for the case that Γ is in tropical general position.

Proof of Theorem 4.1.1 (2) for Γ in tropical general position. Fix a multidegree

$$(m_1, \ldots, m_n) \in \mathrm{multDeg}(\mathcal{M}(\Gamma)_k)$$

(see Equation (4.1)). We pick the vertex v with $C(v) = (m_1, \ldots, m_n)$ (where the map C is defined in Equation 4.2) and let $[L]$ be the lattice class corresponding to v. We want to prove that

$$\mathrm{multDeg}(\overline{\widetilde{\mathrm{Im}(f_{\Gamma,[L]})}}) = \{(m_1, \ldots, m_n)\}.$$

By Theorem 2.3.5, we need to show that the tuple $(m_1, \ldots, m_n)$ is the only one with $\sum m_i = d-1$ that fulfils the inequalities

$$d - \sum_{i \in I} m_i > d_I$$

for all $I \subset \{1, \ldots, n\}$ and that there is no tuple $(n_1, \ldots, n_n)$ with $\sum n_i > d-1$ that fulfils these inequalities. The first claim follows immediately from lemma 4.2.9 and 4.2.10. For the second claim note that if such a tuple existed, the dimension of $\mathrm{Im}\,\widetilde{f_{\Gamma,[L]}}$ would be bigger than $d-1$ which is contradiction to lemma 4.2.4. This proves that

$$\mathrm{multDeg}(\widetilde{\mathcal{M}}^r(\Gamma)) = \mathrm{multDeg}(\mathcal{M}(\Gamma)_k).$$

To see that the multidegree function takes value 1 at each element in

$$\mathrm{multDeg}(\widetilde{\mathcal{M}}^r(\Gamma))$$

we use Theorem 2.3.5: For every variety $X = X(V, V_1, \ldots, V_n)$ the multidegree function takes value 1 at each element of $\mathrm{multDeg}(X)$. Thus we need to prove that for each multidegree, there is exactly one variety contributing it. However, this is true by the arguments we needed for the set-theoric equality of the multidegrees.

$\square$

Point configurations in arbitrary position

We now use the theory of stable intersection to deduce Theorem 4.1.1 (2) for arbitrary point configurations.

Proof of Theorem 4.1.1 (2) for Γ in arbitrary position. Let Γ be a point configuration in arbitrary position. We consider a slight perturbation of the vertices in Γ to obtain a point configuration $\widetilde{\Gamma}$ in general position. Taking the limit $\widetilde{\Gamma} \to \Gamma$, we see that every vertex v of the polyhedral complex $\mathrm{tconv}(\Gamma)$ is the limit of several vertices $w_1, \ldots, w_l$ in $\mathrm{tconv}(\widetilde{\Gamma})$ and vice versa. Let $[L], [L_1'], \ldots, [L_l']$ be the lattice classes corresponding to $v, w_1, \ldots, w_l$.

Remark 4.2.12. We note, that we can also use section 4.2.1 for lattice classes with real exponents byy just considering the linear maps $g_i : L \longrightarrow L_i$ and modding out t. Our analysis of point configurations in general position applies for those cases with real exponents as well.

We claim that

$$\operatorname{multDeg}(\widetilde{f_{\Gamma,[L]}}) = \{C(w_1), \ldots, C(w_l)\}, \tag{4.3}$$

then we have

$$\begin{aligned}\operatorname{multDeg}(\widetilde{\mathcal{M}}^r(\Gamma)) &= \operatorname{multDeg}(\widetilde{\mathcal{M}}^r(\widetilde{\Gamma})) \\ &= \operatorname{multDeg}(\Delta(\mathbb{P}(V)) = \operatorname{multDeg}(\mathcal{M}(\Gamma)_k)\end{aligned}$$

and together with the same argument about the multidegree function as in the tropical general position case, the theorem follows. Note that

$$\operatorname{multDeg}(\widetilde{\mathcal{M}}^r(\Gamma)) = \bigcup_{[L]\in \operatorname{conv}(\Gamma)} \operatorname{multDeg}(\widetilde{f_{\Gamma,[L]}}), \tag{4.4}$$

where each $\bigcup_{[L]\in\operatorname{conv}(\Gamma)} \operatorname{multDeg}(\widetilde{f_{\Gamma,[L]}})$ is a set of multidegrees as in Equation (4.3) and the union is disjoint.
To prove $\operatorname{multDeg}(\widetilde{f_{\Gamma,[L]}}) = \{C(w_1), \ldots, C(w_n)\}$, we observe that

$$\operatorname{Ker}\left(\widetilde{f_{\Gamma,[L]}}\right)_i \subset \operatorname{Ker}\left(\widetilde{f_{\widetilde{\Gamma},[L_j']}}\right)_i$$

for all i and j and there exists j_i, such that

$$\operatorname{Ker}\left(\widetilde{f_{\Gamma,[L]}}\right)_i = \operatorname{Ker}\left(\widetilde{f_{\widetilde{\Gamma},[L_{j_i}']}}\right)_i.$$

This follows from the following consideration: Let $\Gamma = \{L_1, \ldots, L_n\}$ and let v_i be the vertex corresponding to L_i. For each i and j, there exists j_i, such that there is a homeomorphism between $\operatorname{tconv}(v, v_i)$ and $\operatorname{tconv}(w_{j_i}, v_i)$. For $j \neq j_i$, the natural map from $\operatorname{tconv}(w_{j_i}, v_i)$ to $\operatorname{tconv}(v, v_i)$ is surjective but shrinks edges. This translates to the relation between the kernels described above.

We denote by d_I^v and $d_I^{w_i}$ the dimensions of the intersections of the kernels with respect to $\widetilde{f_{\Gamma,[L]}}$ and $\widetilde{f_{\Gamma,[L_i']}}$ respectively. We see that $d_I^v \leq d_I^{w_i}$. Now, let $C(w_i) = (m_1^i, \ldots, m_n^i)$. This tuple satisfies the equations $d - \sum_{j\in I} m_j^i > d_I^v$. We still have to prove that $M(h) = \emptyset$ for $h > |C(w_i)|$: If $M(h) \neq \emptyset$ for some $h > |C(w_i)|$, then by Theorem 2.3.5 $\dim(\overline{\mathrm{Im}(f_L)}) > d$. However, this is a contradiction to $\overline{\mathrm{Im}(\widetilde{f_{\Gamma,[L]}})}$ being contained in the special fiber of the Mustafin variety. Thus we obtain set-theoric equality of the multidegrees. As mentioned before, the multidegree functions coincide by the same arguments as in the tropical general position case and the fact that the union in Equation (4.4) is disjoint. □

Example 4.2.13. Let $v_1 = (0,0,0)$ and $v_2 = (0,1,1)$. Let $\Gamma = \{L_1, L_2\}$ be the point configuration with L_i being the lattice corresponding to v_i. We compute

$$I(\mathcal{M}(\Gamma)_k) = \langle x_{22}, x_{32}\rangle \cap \langle x_{11}, x_{21}x_{32} - x_{31}x_{22}\rangle,$$

where x_{1i}, x_{2i}, x_{3i} are the coordinates in the $i-$th factor. We see that v_1 contributes the variety corresponding to $\langle x_{22}, x_{32}\rangle$ of multidegree $\{(2,0)\}$ and that v_2 contributes the variety corresponding to $\langle x_{11}, x_{21}x_{32} - x_{31}x_{22}\rangle$ of multidegree $\{(1,1),(0,2)\}$. We want to see these multidegrees in the combinatorics of the tropical convex hull. In order to do this, we pertubate the vertex v_2 as illustrated in Figure 4.2, to obtain the point configuration Γ' corresponding to $v_1' = (0,0,0)$, $v_2' = (0,1,2)$. We see that v_1 is the limit of v_1' and v_2 is the limit of v_2' and $(0,1,1)$ by reversing the pertubation. We note that v_1' contributes a variety to $\mathcal{M}(\Gamma')_k$ of multidegree $\{(2,0)\}$, $(0,1,1)$ contributes a variety of multidegree $\{(1,1)\}$ and v_2' contributes a variety of multidegree $\{(0,2)\}$. Then we see as in the proof that the multidegree of a vertex v in $\mathrm{conv}(\Gamma)$ is given by the union of the multidegrees of the vertices whose limit is v by reversing the pertubation.

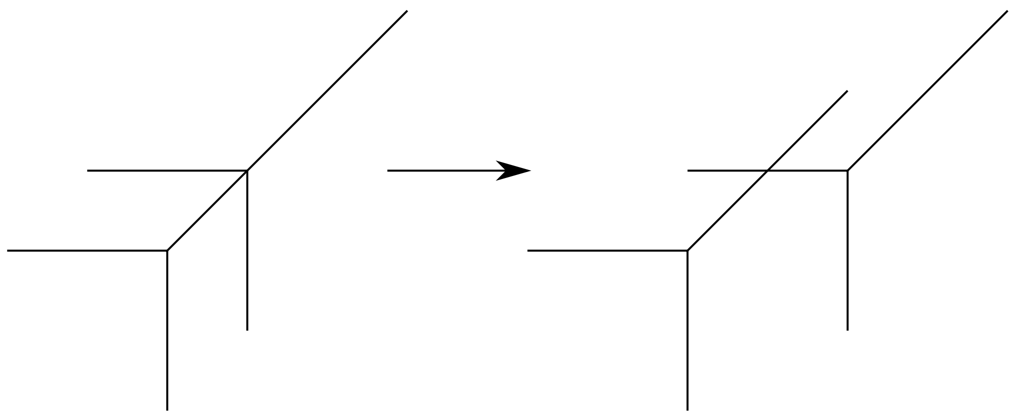


Figure 4.2: Pertubating the vertex v_2.

4.2.5 Classification of the irreducible components of special fibers of Mustafin varieties

We begin this subsection by proving Theorem 4.1.2.

Proof of Theorem 4.1.2. The first part of the Theorem follows from Theorem 4.1.1 and the map in Equation (4.1). To see the second part, fix a variety $X(k^d; W_1, \ldots, W_n)$, where $\bigcap_{i=1}^n W_i = \langle 0 \rangle$ and fix linear maps $g_i : k^d \to k^d$, such that $\ker(g_i) = W_i$. Fix a reference lattice L and choose invertible $d \times d$ matrices h_i over K, such that we obtain g_i from h_i by setting $t = 0$. Finally, let $\Gamma = \{h_1^{-1}L, \ldots, h_n^{-1}L\}$. By lemma 4.2.4 $\overline{\mathrm{Im}(\widetilde{f_{[L],\Gamma}})}$ is contained in $\mathcal{M}(\Gamma)_k$. Moreover, since $\bigcap_{i=1}^n W_i = \langle 0 \rangle$ we obtain

$$\dim(\overline{\mathrm{Im}(\widetilde{f_{[L],\Gamma}})}) = d - 1$$

and thus by lemma 4.2.4 $X(k^d, W_1, \ldots, W_n)$ is an irreducible component as desired. □

Another question raised in [CHSW11] is how many irreducible components the special fiber of a Mustafin variety has. In the one apartment case, this count is inherent in the combinatorial data inherent in the tropical convex hull, which was observed in Theorem 4.4 in [CHSW11] (and also follows from Theorem 4.1.1

(2)). The next remark comments on the number of irreducible components for arbitrary point configurations.

Remark 4.2.14. We note that Theorem 4.1.1 yields a linear algebra algorithm for counting the number of irreducible components of $\mathcal{M}(\Gamma)_k$ for arbitrary point configuration. Namely, computing the convex hull $\mathrm{conv}(\Gamma)$ and computing the number of lattice classes $[L] \in \mathrm{conv}(\Gamma)$, such that $\dim(\mathrm{Im}\left(\widetilde{f_{\Gamma,[L]}}\right) = d-1$, which only depends on the numerical data given by the d_I. This number coincides with the number of irreducible components. In the one apartment case, we can express the number of irreducible components as combinatorial data inherent in the tropical convex hull. We believe that a similar description is possible for arbitrary point configurations using the tropical description of convex hulls in a Bruhat-Tits building given in [JSY07]. Computing the tropical convex hull in the Bergmann fan in the associated matroid, we can perform similar intersection products in the respective tropical linear space. More precisely, the tropical linear space will be of dimension $d-1$. Thus, for a fixed multi-degree $(m_1, \ldots, m_n)$, intersecting the linear space with the codimension m_i skeleta at the $i-$th vertices, we will obtain exactly one polyhedral vertex in the associated tropical convex hull. However, proving that the variety associated to the polyhedral vertex contributes in fact an irreducible component with multi-degree $(m_1, \ldots, m_n)$ is not as simple as in the one apartment case. More precisely, the first step in computing the dimension of the intersections of the corresponding kernels requires knowing the basis vectors of each lattice, which in general is not as evident from the combinatorial structure as in the one apartment case.

4.3 Mustafin varieties and linked Grassmannians

In this section, we relate Mustafin varieties to the theory of (pre)linked Grassmannians (see section 2.3.4). Let $\Gamma \subset \mathfrak{B}_d^0$ be a convex point configuration in one apartment. We associate a prelinked Grassmannian $\mathrm{LG}(r, \Gamma)$ of rank r subbundles as follows:

1. The base scheme is R.

2. We associate a graph $G = G(\Gamma) = (V(\Gamma), E(\Gamma))$ to Γ.

2a. The vertex set $V(\Gamma)$ of the graph G is given by the set of lattice classes in $\mathrm{conv}(\Gamma) = \Gamma$.

2b. The edge set is given by

$$E(\Gamma) = \{([L_i], [L_j]) : \ [L_i] \text{ is adjacent to } [L_j] \text{ in the building}\}.$$

3. The vector bundle at each vertex v is the corresponding lattice class $[L]$.

4. The maps between adjacent vertices v, w are given as follows: Let $[L]$ and $[M]$ be the lattice classes corresponding to v and v' respectively. Since those lattice classes are adjacent, there are representatives $L' \in [L]$ and $M' \in [M]$, such that $\pi M' \subset L' \subset M'$. The natural inclusion maps $\tilde{f} : \pi M' \longrightarrow L$ and $\tilde{g} : L' \longrightarrow M'$ induce maps $f : [M] \rightarrow [L]$ and $g : [L] \rightarrow [M]$. Those are the maps between the vertices.

Remark 4.3.1. We note the following two subtleties of the above situation:

1. In definition 2.3.19, we required a vector bundle at each vertex, whereas in the situation above, we have homothety classes at each vertex. However, the construction does not depend on the representatives as long as the lattices themselves are adjacent.

2. The map between adjacent vertices is the same as the corresponding map between $[L_i]$ and $[L_j]$ (with $[L_i]$ as the reference lattice) in construction 4.2.1.

Example 4.3.2. We pick an apartment A corresponding to the basis $e_1, \ldots, e_d$. Now we take $[L_1], [L_2], [L_3]$ to be the lattice classes corresponding to $(0, 0, 0), (0, -1, 0), (0, -2, -1)$ in the tropical torus corresponding to A. This is a tropically convex set, illustrated in Figure 4.3. The linked Grassmannian parametrises rank

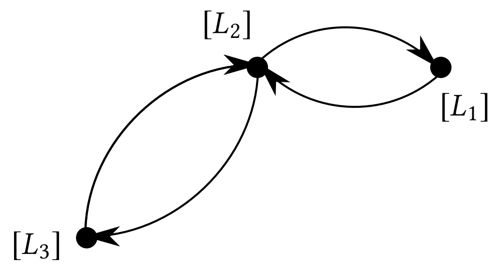


Figure 4.3: Graph associated to $(0, 0, 0), (0, -1, 0), (0, -2, -1)$.

1 sub-bundles of $[L_1], [L_2], [L_3]$. Picking $L_1 = Re_1 + \cdots + Re_d$ as the reference lattice, the image of $[L_2]$ in $[L_1]$ is isomorphic to the sublattice of $[L_1]$ described by the image of the linear map $[L_1] \to [L_1]$ over R given by the following matrix over

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}.$$

We will use this description in the proof of Theorem 4.1.3 by choosing coordinates on $\mathrm{LG}(1, \Gamma)$ similar to the coordinates on $\mathcal{M}(\Gamma)$.

For the rest of the chapter, let $\mathrm{LG}(r, \Gamma)$ be the prelinked Grassmannian of rank r associated to this data. Moreover, let $\mathrm{Simp}(\Gamma)$ be the locus of simple points in the special fiber $\mathrm{LG}(1, \Gamma)_k$.

4.3.1 Linked Grassmannians and Faltings' functor

In [Fal01], a moduli functor description for $\mathcal{M}(\Gamma)$ was introduced. We begin by relating this functor to the linked Grassmannian problem. We note that Faltings' notion of projective space is the dual construction to the one used here, i.e. in [Fal01] $\mathbb{P}(V)$ parametrises hyperplanes instead of lines. This yields the notion of quotient bundles in *loc. cit.* instead of sub-bundles, which we use here:

Theorem 4.3.3 ([Fal01])**.** *Let Γ be a convex point configuration. Then $\mathcal{M}(\Gamma)$ represents the following functor: To every $R-$scheme S we associated the set of all tuples* *of line bundles* $(l(L))_{L\in\Gamma}$*, such that each inclusion* $L_i \subset L_j$ *(for* $L_i, L_j \subset \Gamma$*) maps* $l(L_i)$ *to* $l(L_j)$*.*

Since the maps along the tropical convex hull in the linked Grassmannian we associated to Γ at the beginning of this section coincide with the inclusion maps, we obtain the following result as a corollary, interpreting Mustafin varieties as a moduli space in limit linear series theory:

Theorem 4.3.4. *Let* Γ *be a convex point configuration, then* $\mathcal{M}(\Gamma)$ *represents the linked Grassmannian functor, i.e.*

$$\mathcal{M}(\Gamma) = \mathrm{LG}(1, \Gamma)$$

as schemes.

Moreover, we have thus found a class of reduced and flat linked Grassmannians.

Corollary 4.3.5. *Let* Γ *be an arbitrary point configuration. The associated linked Grassmannians* $\mathrm{LG}(1, \Gamma)$ *is a flat scheme with reduced fibers.*

Most results on (pre)linked Grassmannians required the study of the simple points as elaborated in Section 2.3.4. Using Theorem 4.1.1, we can relate the simple points to Mustafin varieties. We note that it is clear that the simple points are dense in the generic fiber, since the maps between vertices over the generic fiber are isomorphisms. In the next subsection, we prove that the simple points are dense in the special fiber as well, whenever k is algebraically closed.

4.3.2 Mustafin varieties and simple points

A motivating example: Two lattices

A first example for Theorem 4.3.4 was given in Warning A.16 in [Oss06]. We pick $S = \mathrm{Spec}(k), n = d = 2, r = 1$. (Note that this corresponds to picking any two adjacent lattices in dimension 2 as each two adjacent lattices will yield the same maps.) Thus the vector space associated to v_1 and v_2 is k^2 with maps $f_{(v_1,v_2)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f_{(v_2,v_1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus the linked Grassmannian consists of points $(V_1 = \left\langle \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} \right\rangle, V_2 = \left\langle \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \right\rangle)$, such that $f_1(V_1) \subset V_2$ and vice versa. The condition of being linked is given by $X_0 Y_1 = 0$. In fact $\mathrm{LG}(1,\Gamma)_k$ is scheme-theoretically cut out by this equation in $\mathrm{P}^1 \times \mathrm{P}^1$, which translates to a pair of $\mathbb{P}^1$'s attached at $X_0 = Y_1 = 0$ (which is the only not exact point).

Given Γ as mentioned above, we can compute the special fiber of the Mustafin variety, which by Theorem 4.1.1 is given by $\overline{\mathrm{Im}(f_1)} \cup \overline{\mathrm{Im}(f_2)}$ for $f_{(v_1,v_2)} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and $f_{(v_2,v_1)} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, by Theorem 4.1.1, $\mathcal{M}(\Gamma)_k$ is a pair of $\mathbb{P}^1$s attached at $X_0 = Y_1 = 0$ as well and we can conclude $\mathrm{LG}(1,\Gamma)_k = \mathcal{M}(\Gamma)_k$ in this case.

Special fibers of Mustafin Varieties and prelinked Grassmannians

The key step in proving Theorem 4.1.3 is the following proposition.

Proposition 4.3.6. *Let Γ be a convex point configuration. Then the closure of the locus of simple points in $\mathrm{LG}(1,\Gamma)_k$ is set-theoretically given by the special fiber of the Mustafin varieties:*

$$\overline{\mathrm{Simp}(\Gamma)} = \mathcal{M}(\Gamma)_k.$$

Proof. The statement is about the special fiber, thus we base change to $\mathrm{Spec}(k)$. Moreover, we can choose coordinates on the linked Grassmannian as for Mustafin varieties by performing a linear coordinate change

$$\mathrm{LG}(1,\Gamma) \subset \mathbb{P}(L_1) \times \cdots \times \mathbb{P}(L_n) \xrightarrow{g_1 \times \cdots \times g_n} \mathbb{P}(L) \times \cdots \times \mathbb{P}(L)$$

for a reference lattice L and linear maps g_i, such that $g_i L = L_i$. We observe that the maps between adjacent lattices over k coincide with the respective maps in construction 4.2.1. First, we have to check that $G = G(\Gamma)$ together with the associated data at the vertices and edges actually satisfies the conditions in definition 2.3.19. The only thing to check is the condition on the paths. Let $\widetilde{g_P}$ be the map obtained by the composition of maps corresponding to edges along the path P. We make the following claim from which the required conditions follow immidiately:

Lemma 4.3.7. *Let P be a path and P' a minimal path between $[L]$ and $[L']$. Then we have*

$$g_P = g_{P'} \text{ or } g_P \equiv 0.$$

Proof. In order to prove this claim, we consider the maps before base changing. As maps over R, we observe that there exists $n \in \mathbb{N}$, such that

$$t^n g_P = g_{P'}.$$

If $n = 0$, we obtain $g_P = g_{P'}$ over the special fiber and $g_P \equiv 0$ if $n > 0$. Therefore, $G = G(\Gamma)$ satisfies the conditions in definition 2.3.19. □

In order to link the simple points to the special fiber, we need the following lemma.

Lemma 4.3.8. *The tropical convex hull between two lattice classes $[L]$ and $[L']$ is a minimal path.*

In the rank 1 case, simple points translate to the following situation: Let $(\mathcal{F}_v)_{v \in V(G)}$ be a simple point. Then there exists a $v' \in V(G)$, such that taking minimal paths $P_{v',v}$ from v' to $v \in V(G)$ for each v, we obtain $\mathcal{F}_v = f_{P_{v',v}}(\mathcal{F}'_v)$ and we say $(\mathcal{F}_v)_{v\in V(G)}$ is *rooted* at v'. Thus we can classify the set $\mathrm{Simp}(\Gamma)_{v'}$ of all simple points rooted at v' as the image of the following rational map

$$g_{v'} : \mathbb{P}_k^{d-1} \xrightarrow{(f_{P_{v',v}})_{v\in V(G)}} \left(\mathbb{P}_k^{d-1}\right)^{|\Gamma|},$$

since $\mathbb{P}_k^{d-1}$ parametrises one dimensional subvectorspaces of k^d. Moreover, we see immediately that

$$\mathrm{Simp}(\Gamma) = \bigcup_{v\in V(G)} \mathrm{Simp}(\Gamma)_{v'}.$$

It is easy to see, that the map $g_{v'}$ coincides with the map $\widetilde{f_{\Gamma,[L']}}$ constructed in construction 4.2.1, where $[L']$ is the lattice class corresponding to v'. Therefore, we obtain

$$\mathrm{Simp}(\Gamma) = \bigcup_{v\in V(\Gamma)} \mathrm{Simp}(\Gamma)_v = \bigcup_{[L]\in\mathrm{conv}(\Gamma)} \mathrm{Im}(\widetilde{f_{\Gamma,[L]}})$$

and by taking the closure and applying Theorem 4.1.1 we obtain

$$\overline{\mathrm{Simp}(\Gamma)} = \mathcal{M}(\Gamma)_k$$

as desired. □

Since $\mathcal{M}(\Gamma) = \mathrm{LG}(1,\Gamma)$ by Theorem 4.3.4, we see that $\mathrm{LG}(1,\Gamma)_k = \mathcal{M}(\Gamma)_k = \overline{\mathrm{Simp}(\Gamma)}$, which proves Theorem 4.1.3.

CHAPTER 5

TROPICALISED QUARTICS AND CANONICAL EMBEDDINGS FOR ABSTRACT TROPICAL CURVES OF GENUS 3

5.1 Introduction

One problem one runs into when working with tropical geometry is that the naive tropicalisation depends on the chosen embedding of the algebraic variety. Berkovich analytic geometry offers a way to overcome this problem: Berkovich spaces, which can be viewed as the inverse limit of all tropicalisation [Pay09], do not depend on an embedding and capture the maximal geometric information a tropicalisation can possibly have. The price to pay is that Berkovich analytic spaces are complicated objects, while in tropical geometry we hope to replace an algebraic variety with an easier combinatorial object in order to study the algebraic variety by means of combinatorics. A Berkovich analytic curve is an infinitely branched metric graph. A faithful tropicalisation can be viewed as a hands-on compromise between a naive tropicalisation and a Berkovich analytification [BPR16]. For curves, it offers a way to avoid infinite graphs by working with a concrete tropicalisation, which is combinatorially much easier to digest. On the other hand, it is a tropicalisation which captures the wanted geometric properties.

For further applications of tropical geometry in algebraic geometry, the question how to capture geometric properties in a tropicalisation is foundational, and consequently the question how to construct faithful tropicalisations has attracted lots of attention in the last years [CM16b; CHW14; DP16; KY16; Wag17; CM18].

Modifications are intimately related with faithful tropicalisations. They can be viewed as an intrinsic combinatorial way to overcome the problem that tropicalisation depends on the embedding. For example, in the case of a Bergman fan of a matroid, i.e. for tropical linear spaces, modification along linear divisors corresponds to deletion [Sha13]. In several occassions, modifications and re-embeddings have already been used to construct faithful tropicalisations [CM16b; LM17; CM18]. The tropicalisation of a re-embedded variety on a modification is also referred to as tropical refinement [IMS09; MMS12].

In this chapter, we propose modifications as a tool to understand the moduli of plane curves from a tropical perspective (see section 5.1.1). We make this precise for the following problem: It is well-known that any smooth projective non-hyperelliptic curve of genus 3 can be embedded as a smooth quartic in the plane. Moreover the embedding is given by the complete canonical linear system, and any smooth plane quartic is obtained this way. In [BJMS15], Brodsky, Joswig, Morrison and Sturmfels observed that the naive tropical analogue of the above statement does not hold. Namely, not any non-hyperelliptic curve of genus 3 admits an embedding as a tropical quartic in $\mathbb{R}^2$. More precisely, they computed the locus of those genus 3 curves which admit an embedding into the tropical plane, which is far from being the union of all open top-dimensional cones in $M_3^{\mathrm{trop}} \setminus M_{\mathrm{hyp}}^{\mathrm{trop}}$ (see [BJMS15, Theorem 5.1]).

In fact, whole top-dimensional cone of M_3^{trop} (corresponding to hyperelliptic, but not realisably hyperelliptic tropical curves) is not in the locus. With this computational project — which offers interesting perspectives on computations with secondary fans and tropical moduli spaces beyond this result — they thus uncovered a discrepancy between the algebraic and tropical world, which as commonly suspected arises due to the use of naive tropicalisation of plane curves.

In this chapter, we propose a natural setting in which the tropical analogue of the above statement does hold true. Before presenting our setting, let us mention that Payne's theorem [Pay09] and basic deformation theory (cf. [Tyo12]) suggests that *any* (even hyperelliptic) tropical curve of genus 3 can be realised faithfully as a quartic in some tropical model of the plane. However, since abstract tropical models of the plane are very complicated such a tautological tropical analogue is not very satisfactory. In this paper we consider the class of tropical planes, which are tropicalisations of linear subplanes of projective spaces. Thus, embeddings of genus 3 curves as quartics in such planes correspond to different choices of spanning systems of canonical sections, and on the tropical side, it is natural to ask: *What tropical curves $\Gamma \in M_3^{\mathrm{trop}}$ admit embeddings in $\mathbb{R}^n$ given by an n-tuple of canonical sections?* Notice that the embedded images of such curves can be obtained as *linear modifications* of tropical quartics in the usual tropical plane $\mathbb{R}^2$.

5.1.1 Outline of Results

To formulate the results we recall the following notions: (1) an (embedded) hyperelliptic tropical curve is called *realisably hyperelliptic* if it is the tropicalisation of an algebraic hyperelliptic curve (see definition 2.4.2); (2) a tropical curve is called *maximal* if it belongs to the open interior of a cone of maximal dimension in the moduli space of tropical curves (see definition 2.4.1); (3) a realisable tropical plane is the tropicalisation of a plane (see section 2.1.2). Our first main result is the following:

Theorem 5.1.1. *Any maximal tropical curve of genus* 3*, which is not realisably* *hyperelliptic, can be embedded as a faithfully tropicalised quartic in a realisable tropical plane (after attaching unbounded edges appropriately).*

Our proof of this result is constructive. Namely, for a given maximal not realisably hyperelliptic tropical curve Γ of genus 3, we explicitly construct a map from Γ to $\mathbb{R}^2$ and a series of linear modifications yielding an embedding of Γ into the modified plane as a faithful tropicalised quartic. Our construction can be viewed as a *canonical embedding* of Γ into $\mathbb{R}^n$ via an n-tuple of canonical divisors. In addition, it provides a way to realise algebraically a curve of genus 3 together with an n-tuple of canonical divisors. Examples for all our constructions can be found on the webpage `software.mis.mpg.de`.

Our second main result shows that the remaining tropical curves admit no faithful embedding as above.

Theorem 5.1.2. *No realisably hyperelliptic maximal tropical curve of genus* 3 *can be embedded faithfully and realisably into a realisable tropical plane.*

The proof of this theorem is based on the theory of tropical divisors (cf. [HMY12; Ami13]), and uses the recent description of the locus of realisable sections of the tropical canonical divisor of Möller, Ulirsch, and Werner [MUW17]. We shall emphasise that although for some curves there are purely tropical obstructions to the existence of an embedding as a faithfully tropicalised quartic, there also exist hyperelliptic tropical curves that can be embedded in a linearly-modified tropical plane, but which do not come as tropicalisations of plane quartics.

Here, we restrict our study to maximal tropical curves. It would be interesting to extend this study to lower-dimensional cones, and our methods are suitable for doing so. We leave this task for further research.

5.2 Constructing faithful tropicalised quartics

5.2.1 Modification techniques

In this subsection, we present different scenarios which allow us to unfold certain edges using suitable linear modifications. We will employ them in the constructive proof of Theorem 5.1.1 (see Section 5.2.2).

Enlarging edges from trapezoids

Let $V(q)$, $q = \sum_{i+j\leq 4} a_{i,j}x^i y^j \in K[x,y]$, be a plane curve whose Newton subdivision contains a trapezoid with a simplex adjacent to its shorter edge as in Figure 5.1. For each of the three types, we will introduce conditions under which the shorter edge hides an edge of prescribed length.

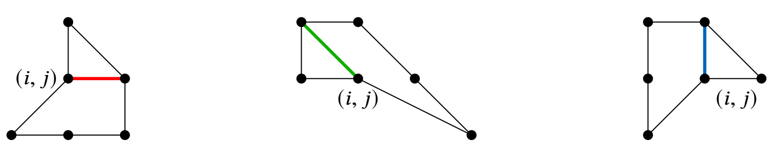


Figure 5.1: Trapezoids which allow to enlarge edge lengths.

Proposition 5.2.1. *Suppose the Newton subdivision of $V(q)$ contains a trapezoid and triangle as on the left in Figure 5.1. Let f' be the length of the edge E dual to the red edge. Without restriction, assume that the upper vertex of E is at $(0,0)$, and the valuations of the coefficients corresponding to the triangle dual to $(0,0)$ are 0 (see Figure 5.2 top).*

Let $f > f'$ and suppose the coefficients a_{ij} of q satisfy the following conditions for some l_1, l_2 with $l_2 < f'$ and $f = 2l_1 + l_2 + f'$:

$$\operatorname{val}\Big(\sum_{k\geq i-1}(-1)^k a_{k,j-1}\Big) \geq f+1, \qquad \operatorname{val}\Big(\sum_{k\geq i}(-1)^k a_{k,j}\Big) = l_1 + l_2,$$

$$\operatorname{val}\Big(\sum_{k\geq i}(-1)^k a_{k,j+1}\Big) = 0, \qquad \operatorname{val}\Big(\sum_{k\geq i-1}(-1)^{k-1}\cdot k\cdot a_{k,j-1}\Big) = l_1 + f',$$

$$\operatorname{val}\Big(\sum_{k\geq i+1}(-1)^{k-1}\cdot k\cdot a_{k,j}\Big)=0,\quad \operatorname{val}\Big(\sum_{k\geq i-1}(-1)^{k-2}\cdot\binom{k}{2}\cdot a_{k,j-1}\Big)=f'.$$

Then the modification $x = z - 1$ unfolds an edge of length f from E (see Figure 5.2 bottom).

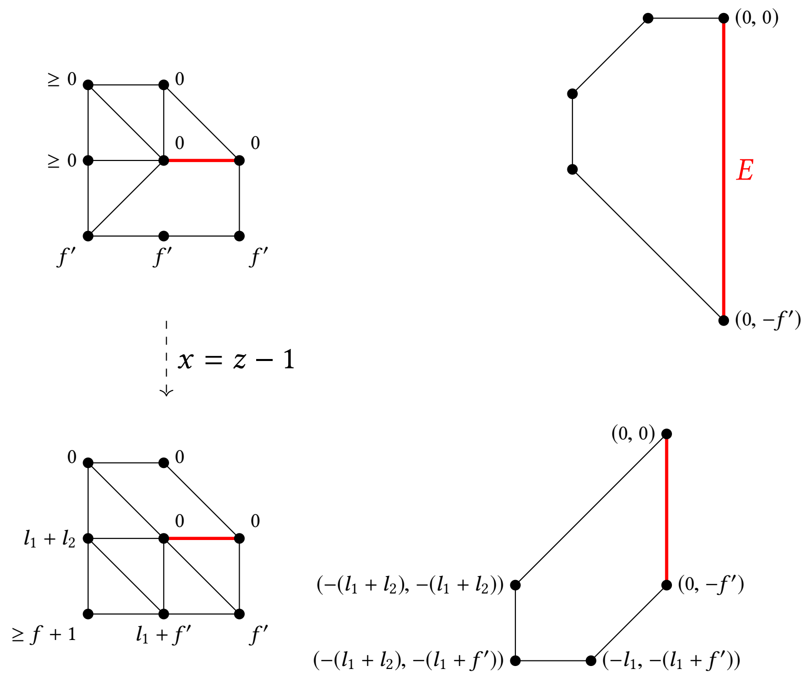


Figure 5.2: Unfolding the shorter edge of the left trapezoid in Figure 5.1.

Proof. We add the equation $x = z - 1$ to q and project to the yz-coordinates to study the newly attached part of the re-embedded curve $\operatorname{Trop}(V(q, x-z+1))$. The conditions given above are precisely the conditions on the y^{j-1}-, y^j-, y^{j+1}-, $y^{j-1}z$-, y^jz- and $y^{j-1}z^2$-terms in $q(z-1, y)$ as imposed in the lower left of Figure 5.2. The sums above are taken over all terms of q which “feed” to the term in question. The valuations determine the positions of the vertices of $\operatorname{Trop}(V(q(z-1, y)))$, as in the lower right of Figure 5.2. By adding the lengths of the edges projecting to E, it follows that the length f is unfolded. □

Remark 5.2.2. Note that the conditions in Proposition 5.2.1 imply cancellation behaviour for the coefficients, which requires more than one coefficient of the same valuation. This explains our requirement that the subdivision must contain a trapezoid.

Proposition 5.2.3. *Suppose the Newton subdivision of $V(q)$ contains a trapezoid and triangle as in the middle of Figure 5.1. Let d' be the length of the edge E dual to the red edge. Without restriction, assume that the upper vertex of E is at $(0, 0)$, and the valuations of the coefficients corresponding to the triangle dual to $(0, 0)$ are 0.*

Let $d > d'$ and suppose the coefficients a_{ij} of q satisfy the following conditions for some l_1, l_2 with $l_2 < d'$ and $f = 2l_1 + l_2 + d'$:

$$\begin{aligned}
&\operatorname{val}\Big(\sum_{k\geq 0}(-1)^k a_{i+j-1-k,k}\Big) = 0, && \operatorname{val}\Big(\sum_{k\geq 0}(-1)^k a_{i+j-k,k}\Big) = l_1 + l_2,\\
&\operatorname{val}\Big(\sum_{k\geq 0}(-1)^k a_{i+j+1-k,k}\Big) \geq d + 1, && \operatorname{val}\Big(\sum_{k\geq 0}(-1)^{k-1}\cdot k\cdot a_{i+j-k,k}\Big) = 0,\\
&\operatorname{val}\Big(\sum_{k\geq 0}(-1)^{k-1}\cdot k\cdot a_{i+j+1-k,k}\Big) = l_1 + d', && \operatorname{val}\Big(\sum_{k\geq 0}(-1)^{k-2}\tbinom{k}{2}\cdot a_{i+j+1-k,k}\Big) = d'.
\end{aligned}$$

Then the modification $y = z - x$ unfolds an edge of length d from E.

Proof. We add the equation $y = z - x$ and project to the yz-coordinates to study the newly attached part of the re-embedded curve $\operatorname{Trop}(V(q, y - z + x))$ in terms of $\operatorname{Trop}(V(q(x, z - x))$. The conditions given above are conditions on the x^{i+j-1}-, x^{i+j}-, x^{i+j+1}-, $x^{i+j-1}z$-, $x^{i+j}z$- and $x^{i+j+1}z^2$-terms of $q(x, z - x)$ which determine the position of the relevant vertices of $\operatorname{Trop}(V(q(x, z - x))$. The positions imply that the length d is unfolded from the edge E. ☐

Proposition 5.2.4. *Suppose the Newton subdivision of $V(q)$ contains a trapezoid and triangle as on the right in Figure 5.1. Let e' be the length of the edge E dual to the red edge. Without restriction, assume that the upper vertex of E is at $(0, 0)$, and the valuations of the coefficients corresponding to the triangle dual to $(0, 0)$ are 0.*

Let $e > e'$ and suppose the coefficients a_{ij} of q satisfy the following conditions *for some l_1, l_2 with $l_2 < e'$ and $f = 2l_1 + l_2 + e'$:*

$$\begin{aligned}
&\mathrm{val}\Big(\sum_{k\geq j-1}(-1)^k a_{i-1,k}\Big) \geq e+1, &&\mathrm{val}\Big(\sum_{k\geq j}(-1)^k a_{i,k}\Big) = l_1 + l_2,\\
&\mathrm{val}\Big(\sum_{k\geq j+1}(-1)^k a_{i+1,k}\Big) = 0, &&\mathrm{val}\Big(\sum_{k\geq j-1}(-1)^{k-1}\cdot k\cdot a_{i-1,k}\Big) = l_1 + e',\\
&\mathrm{val}\Big(\sum_{k\geq j}(-1)^{k-1}\cdot k\cdot a_{i,k}\Big) = 0, &&\mathrm{val}\Big(\sum_{k\geq j+1}(-1)^{k-2}\cdot \binom{k}{2}\cdot a_{i-1,k}\Big) = e'.
\end{aligned}$$

Then the modification $y = z - 1$ unfolds an edge of length d from E.

Proof. We add the equation $y = z - 1$ and project to the xz-coordinates to study the newly attached part of the re-embedded curve $\mathrm{Trop}(V(q, y - z + 1))$ in terms of $\mathrm{Trop}(V(q(x, z-1)))$. The conditions given above are conditions on the x^{i-1}-, x^i-, x^{i+1}-, $x^{i-1}z$-, x^iz-, and $x^{i-1}z^2$-terms of $q(x, z-1)$ which determine the position of the relevant vertices of $\mathrm{Trop}(V(q(x, z-1)))$. The positions imply that the length e is unfolded from the edge E. □

Unfolding lollis from weight 2 edges

Let $V(q)$, $q = \sum_{i+j\leq 4} a_{i,j}x^iy^j \in \mathbb{C}[x, y]$, be a plane curve whose Newton subdivision contains either the triangle with vertices $(0, 2)$, $(2, 2)$ and $(0, 4)$ or the triangle with vertices $(2, 0)$, $(2, 2)$ and $(4, 0)$ or the triangle with vertices $(2, 0)$, $(0, 2)$ and $(0, 0)$. We will introduce conditions under which the corresponding bounded weight 2 edges hide a lolli of prescribed edge and cycle length. For the sake of brevity, we consider the first case only.

Let V be the dual vertex of the triangle V and let E be the weight 2 edge of $\mathrm{Trop}(V(q))$ dual to $E^\vee := \mathrm{conv}((0, 2), (2, 2))$. We call V' the other vertex of E (see Figure 5.3 top). Without restriction, we can assume that the vertex V' is at $(0, 0)$, that the valuations of the coefficients of y^2, xy^2 and x^2y^2 are 0, and that the coefficient of x^2y^2 is 1.

Proposition 5.2.5. *For a tropicalised quartic as above, assume that*

1. *q restricted to $E^\vee$ is the square of y times a linear form L in x up to order $2a$,*

2. *the t^{2a}-contribution of $q|_{E^\vee}$ is not divisible by the t^0-contribution of L,*

3. *the length of E is $3a + \frac{b}{2}$,*

4. *the sum of the t^0-terms of q corresponding to monomials $y \cdot x^i$ for some i is not divisible by the t^0-contribution of L.*

Then we can use one linear modification to unfold a lolli with edge length a and cycle length b from the edge E (see Figure 5.3 bottom right).

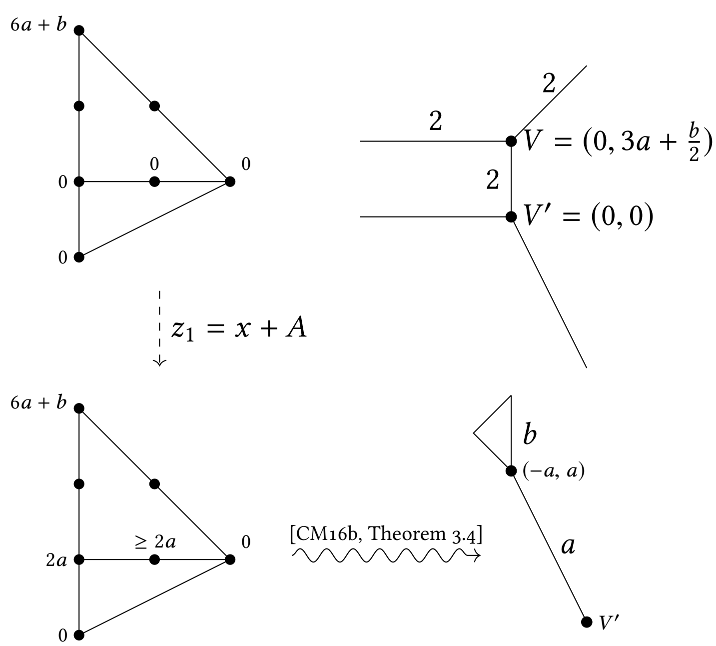


Figure 5.3: Unfolding a lolli from weight 2 edges.

Proof. First, we unfold a bridge edge and a loop in two steps. Then we combine the two steps to one. By condition (1), q restricted to $E^\vee$ equals

$$(y \cdot (x + A))^2 + t^{2a} \cdot y^2 \cdot (\alpha x + \beta) + O(t^{2a+1}),$$

where $A = A_0 + A_1 t + A_2 t^2 + \ldots + A_{2a-1} t^{2a-1} \in K$ is of valuation 0, and $\alpha, \beta \in \mathbb{C}$ satisfy either $\alpha = 0, \beta \neq 0$ or $0 \neq \frac{\beta}{\alpha} \neq A_0$ by condition (2). By assumption (3), V is at $(0, 6a + b)$, and the valuation of the y^4-term is $6a + b$.

We start by adding the equation $z_1 = x + A$. As described in Section 2.4.2, we can make the new part of $\mathrm{Trop}(V(q, z_1 - x - A))$ visible using the projection π_{yz_1} defined by $\tilde{q} := q(z_1 - A, y)$.

Condition (4) ensures that there is no cancellation in $\tilde{q}$ involving the terms $y \cdot x^i$. Thus, the y-term of $\tilde{q}$ has valuation 0. For the y^2- and $z_1 y^2$-term however, there is cancellation. The respective terms of $\tilde{q}$ are:

$$(y \cdot z_1)^2 + t^{2a} \cdot y^2 \cdot (\alpha(z_1 - A) + \beta) + O(t^{2a+1}),$$

By condition (2), the y^2-term has valuation $2a$, and the $y^2 z_1$ term has valuation bigger or equal to $2a$.

The Newton subdivision of newly attached part is depicted below in the lower right of Figure 5.3. Notice that the bounded edge of weight 2 is dual to the edge with initial $y^2 z_1^2 + t^{2a} \cdot (-A_0\alpha + \beta) \cdot y^2 = y^2 \cdot (z_1 - t^a \cdot \sqrt{-A_0\alpha + \beta})(z_1 + t^a \cdot \sqrt{-A_0\alpha + \beta})$. As this is not a square, it follows from Theorem 3.4 in [CM16b] that we can use the linear modification induced by the equation $z_2 = z_1 - t^a \cdot \sqrt{-A_0\alpha + \beta}$ to unfold a cycle of length $2 \cdot \frac{b}{2} = b$. This cycle is attached to the vertex $(0, 0)$ via an edge of length a.

The variable z_1 is not needed to produce a faithful tropicalisation, we can eliminate it and combine the two steps to one. That is, we only add one equation, namely $z_2 = x + A - t^a \cdot \sqrt{-A_0\alpha + \beta}$. □

5.2.2 Proof of Theorem 5.1.1

This sections forms a constructive proof of Theorem 5.1.1. Examples for all constructions can be found on `software.mis.mpg.de`.

For a maximal, not realisably hyperelliptic abstract tropical curve C of genus 3 we explicitly construct a faithful tropicalised quartic in a realisable model of the tropical plane such that its image under $\mathrm{ft}^{\mathrm{trop}}$ (resp. its minimal skeleton) is C. We obtain the model of the tropical plane and the tropicalised quartic by a series of linear modifications, starting with a plane quartic defined over K. We built on [BJMS15, Theorem 5.1]: we only construct faithful tropicalised quartics in a realisable model of the tropical plane for abstract tropical curves which are not embeddable in $\mathbb{R}^2$ as tropical quartics.

Type

Let a, b, c, d, e, f be the edge lengths of an abstract curve C as in Figure 5.4. Due to symmetry, we may assume $a \leq b$, $c \leq d$, $e \leq f$ and $a \leq e$. By [BJMS15, Theorem 5.1], C is embeddable as a quartic in $\mathbb{R}^2$ if $c + e \leq d$. Additionally, if the inequality holds as an equality and $a = b$, then necessarily $a < e < f$.

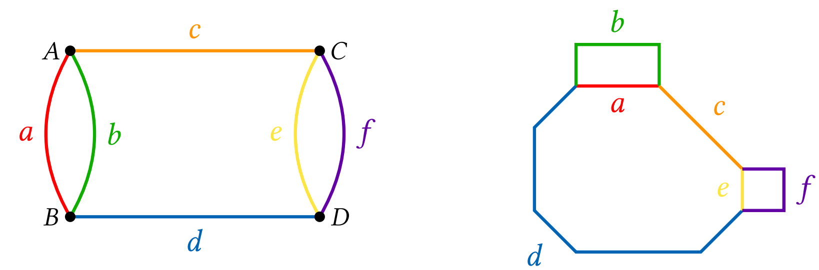


Figure 5.4: An abstract and plane tropical curve of type .

To show that any curve C is embeddable as a quartic in a tropical plane, let a, b, c, d, e, f be any lengths satisfying our assumptions due to symmetry. Note that curves of type with $c = d$ are realisably hyperelliptic, which is why they are excluded in our considerations. Hence we will focus on the case that $e \geq d - c > 0$. Let $e' := d - c > 0$ and pick $l_1, l_2 \geq 0$ such that $2l_1 + l_2 + e' = e$. Consider the quartic polynomial $g = \sum_{i,j} a_{ij} x^i y^j$ where

- all a_{ij} except a_{11}, a_{10}, a_{00} are of the form $t^{\lambda_{ij}}$ for suitable valuations λ_{ij} such that the tropical curve of g is as depicted in Figure 5.5, provided the valuations of a_{00}, a_{10}, a_{11} are 0 (this is possibly by [BJMS15, Theorem 5.1]),

- $a_{11} = 1 + a_{01} - a_{31} - t^{l_1+l_2}$,
- $a_{10} = 2 - 3a_{30} + 4a_{40} + t^{l_1}$,
- $a_{00} = 1 - 2a_{30} + 3a_{40} + t^{l_1} + t^{2l_1+l_2+1}$.

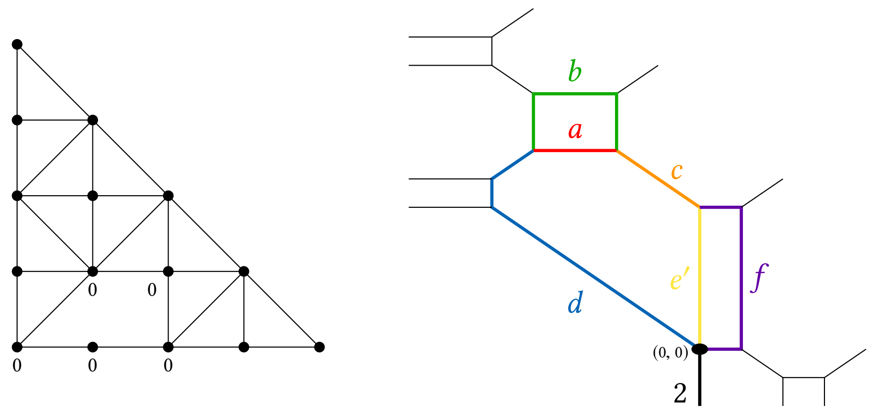


Figure 5.5: A degenerate tropical plane quartic of type whose edge e' can be prolonged.

This is a tilted version of Proposition 5.2.4, i.e. the modification $x = z - 1$ will reveal an bounded edge of length e..

Type

Let a, b, c, d, e, f be the edge lengths of an abstract tropical curve C as in Figure 5.6. Due to symmetry, we may assume $b \leq c \leq a$. By [BJMS15, Theorem 5.1], C is embeddable as a quartic in $\mathbb{R}^2$ if $\max(d, e) \leq a$, $\max(e, f) \leq b$ and $\max(d, f) \leq c$, such that

- at most two of the inequalities may hold as equalities,
- if two hold as equalities, then either d, e, f are distinct and the edge a, b or c that connects the shortest two of d, e, f attains equality or $\max(d, e, f)$ is attained exactly twice and the edge connecting the longest two does not attain equality.

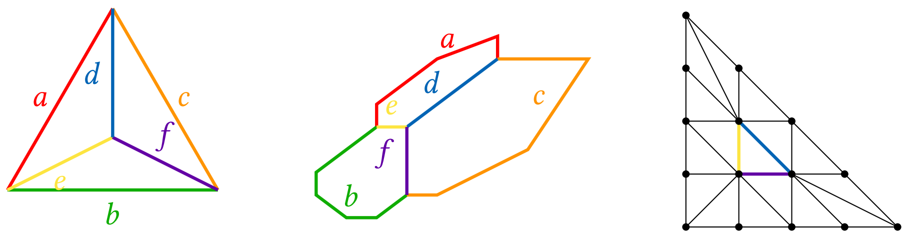


Figure 5.6: An abstract and plane tropical curve of type .

To show that any curve C is embeddable as a quartic in a tropical plane, let a, b, c, d, e, f be any lengths satisfying our assumptions due to symmetry. Pick $d' \le d, e' \le e, f' \le f$ such that a, b, c, d', e', f' is realisable as a plane tropical quartic. We distinguish between the following cases:

$(===)$ $d' = d, e' = e, f' = f,$	$(<<=)$ $d' < d, e' < e, f' = f,$
$(<==)$ $d' < d, e' = e, f' = f,$	$(<=<)$ $d' < d, e' = e, f' < f,$
$(=<=)$ $d' = d, e' < e, f' = f,$	$(=<<)$ $d' = d, e' < e, f' < f,$
$(==<)$ $d' = d, e' = e, f' < f,$	$(<<<)$ $d' < d, e' < e, f' < f,$

In the case $(===)$ the curve C is even embeddable as a quartic in $\mathbb{R}^2$. Cases $(+--)$, $(-+-)$ and $(--+)$ can be obtained via modifications from the subdivisions in Figure 5.7, if the coefficients are chosen according to Propositions 5.2.1, 5.2.4 and 5.2.3.

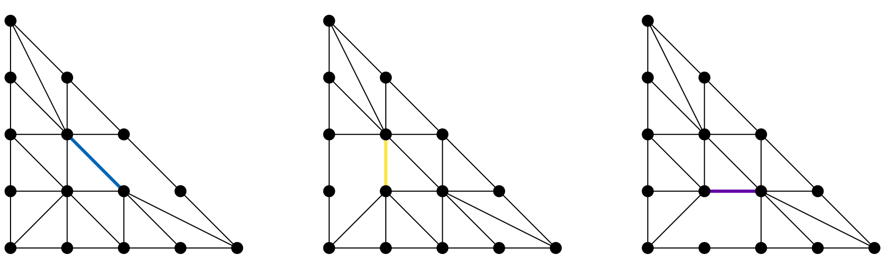


Figure 5.7: Subdivisions of type that allow to prolong an edge.

For the subdivision depicted on the left, consider a quartic $g = \sum_{i,j} a_{ij} x^i y^j$

where all a_{ij} except a_{11}, a_{10}, a_{00} are of the form $t^{\lambda_{ij}}$ for suitable valuations λ_{ij} and

$$\begin{aligned} a_{21} =& a_{12} + a_{30} - a_{03} - t^{l_1+l_2}, \quad a_{31} = 2a_{40} - t^{d'+l_1} - 2t^{2l_1+l_2+d'+1} + a_{13} - 2a_{04}, \\ a_{22} =& a_{40} - t^{d'+l_1} - t^{2l_1+l_2+d'+1} + 2a_{13} - 3a_{04}. \end{aligned}$$

Then the coefficients satisfy the equations in Proposition 5.2.3, and C is embeddable in the tropical curve given by $\langle g, y - z + x \rangle$ in the tropical plane given by $y - z + x$.

For the subdivision in the middle of Figure 5.7, consider a quartic

$$g = \sum_{i,j} a_{ij} x^i y^j$$

where all a_{ij} except a_{01}, a_{02}, a_{12} are of the form $t^{\lambda_{ij}}$ for suitable valuations λ_{ij} and

$$\begin{aligned} a_{02} =& a_{00} - t^{l_1+e'} - t^{2l_1+l_2+e'+1} + 2a_{03} - 3a_{04}, \quad a_{12} = -a_{10} + a_{11} + a_{13} + t^{l_1+l_2}, \\ a_{01} =& 2a_{00} - t^{l_1+e'} - 2t^{2l_1+l_2+e'+1} + a_{03} - 2a_{04}. \end{aligned}$$

Then the coefficients satisfy the equations in Proposition 5.2.1, and C is embeddable in the tropical curve given by $\langle g, x - z + 1 \rangle$ in the tropical plane given by $x - z + 1$.

For the subdivision on the right of Figure 5.7, consider a quartic $g = \sum_{i,j} a_{ij} x^i y^j$ where all a_{ij} except a_{10}, a_{20}, a_{11} are are of the form $t^{\lambda_{ij}}$ for suitable valuations λ_{ij} and

$$\begin{aligned} a_{10} =& 2a_{00} - t^{l_1+f'} - 2t^{2l_1+l_2+f'+1} + a_{30} - 2a_{40}, \quad a_{11} = a_{01} + a_{21} - a_{31} - t^{l_1+l_2}, \\ a_{20} =& a_{00} - t^{l_1+f'} - t^{2l_1+l_2+f'+1} + 2a_{30} - 3a_{40}. \end{aligned}$$

Then the coefficients satisfy the equations of Proposition 5.2.4, and C is embeddable in the tropical curve given by $\langle g, y - z + 1 \rangle$ in the tropical plane given by $y - z + 1$.

Cases $(<<=), (<=<), (=<<)$ can be obtained via two independent modifications from the subdivisions Figure 5.8, combining the two respective choices for the coefficients from above.

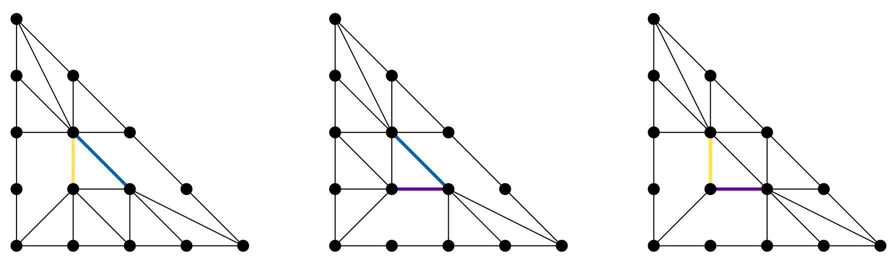


Figure 5.8: Subdivisions of type that allow to prolong two edges.

Finally, we consider the case (<<<). The previous cases were covered with up to two modifications which were independent of each other. This is no longer possible in the current case, in which we will start with a subdivision as in the center of Figure 5.9 and require three modifications. The central coefficients a_{11}, a_{12} and a_{21} are critical in the sense that they play a role for each of the three pentagons which we use to prolong edges.

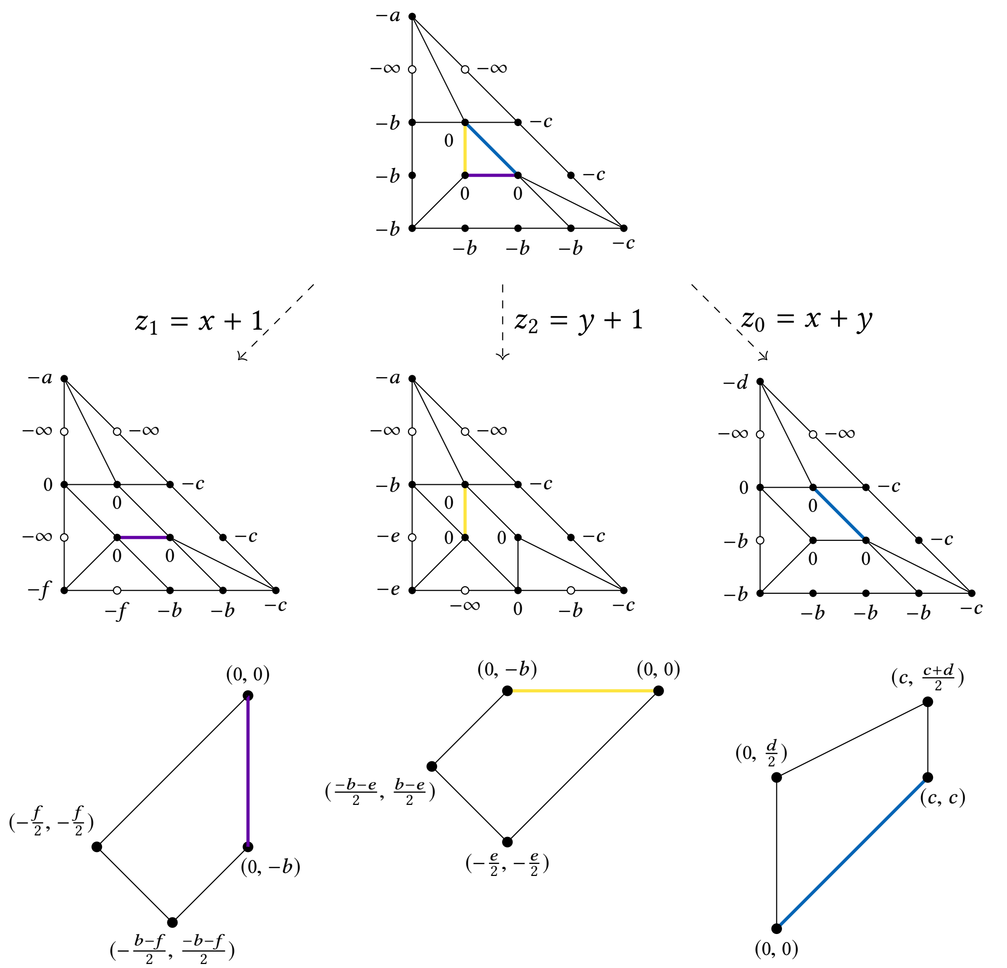


Figure 5.9: A subdivision that allows us to prolong three edges.

Recall that, we assumed $b \le c \le a$. For the current case, we may also assume $d' = c, e' = b, f' = b$. Consider a quartic $g = \sum_{i,j} a_{ij} x^i y^j$ where $a_{03} = a_{13} = 0$ and the remaining a_{ij} are chosen as follows:

$$a_{00} = -t^e - 3t^d - t^f + 3t^a + 2t^b, \qquad a_{01} = -2t^e - 6t^d - 2t^f + 4t^a + 4t^b,$$
$$a_{10} = -4t^d - 2t^f - 2t^c + 5t^b, \qquad a_{02} = -3t^d - t^f + 2t^b,$$
$$a_{11} = -4t^d - 2t^f - 2t^c + 6t^b + 1, \qquad a_{20} = 3t^e - 8t^a - 3t^c + 4t^b,$$
$$a_{12} = t^b + 1, \quad a_{21} = 2t^e - 4t^a + 2t^b + 1, \quad a_{30} = 2t^e - 4t^a + t^b, \quad a_{04} = t^a,$$

$$a_{31} = -2t^d + 2t^c, \quad a_{22} = -2t^a + t^c, \qquad a_{40} = -t^d + t^a + t^c.$$

Then the modification $z_1 = x + 1$ yields the following coefficients a'_{ij} in $g(z_1 - 1, y) = \sum_{i,j} a'_{ij} z_1^i y^j$:

$$\begin{aligned}
a'_{00} &= a_{00} - a_{10} + a_{20} - a_{30} + a_{40} = t^f, \\
a'_{10} &= a_{10} - 2a_{20} + 3a_{30} - 4a_{40} = -2t^f, \\
a'_{20} &= a_{20} - 3a_{30} + 6a_{40} = t^b + 3t^c + 10t^a - 6t^d - 3t^e \\
a'_{01} &= a_{01} - a_{11} + a_{21} - a_{31} = 0.
\end{aligned}$$

Moreover, $\mathrm{val}(a'_{02}) = \mathrm{val}(a'_{12}) = \mathrm{val}(a'_{11}) = \mathrm{val}(a'_{21}) = 0$. Thus, the Newton subdivision is as in the left branch of Figure 5.9, and the length of the prolonged edge is as desired:

$$f/2 + b/2 + (f - b)/2 = f.$$

Furthermore, the modifications $z_2 = y + 1$ resp. $z_0 = x + y$ will yield edges of desired lengths d resp. e in the tropical curves given by $q(x, z_2 - 1)$ resp. $q(z_3 - y, y)$, see the bottom resp. top branches of Figure 5.9.

Type

Let a, b, c, d, e, f be the edge lengths of an abstract tropical curve C as in Figure 5.10. Due to symmetry, we may assume $a \geq c \geq e$. By [BJMS15, Theorem 5.1], no curve of type is embeddable as a quartic in $\mathbb{R}^2$.

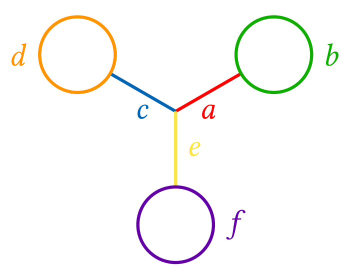


Figure 5.10: An abstract tropical curve of type .

To show that any curve C is embeddable as a quartic in a tropical plane, let a, b, c, d, e, f be any lengths satisfying our assumptions due to symmetry. Consider the following plane quartic, whose Newton subdivision and tropical curve is depicted in Figure 5.11:

$$p = t^{6a+b} \cdot y^4 + t^{6c+d} \cdot x^4 + t^{6e+f} + x^2y^2 + 2 \cdot xy^2 + (1 - t^{2a})y^2 + 2 \cdot x^2y \\ + (1 - t^{2c}) \cdot x^2 - 2 \cdot (1 + t^{2e}) \cdot xy.$$

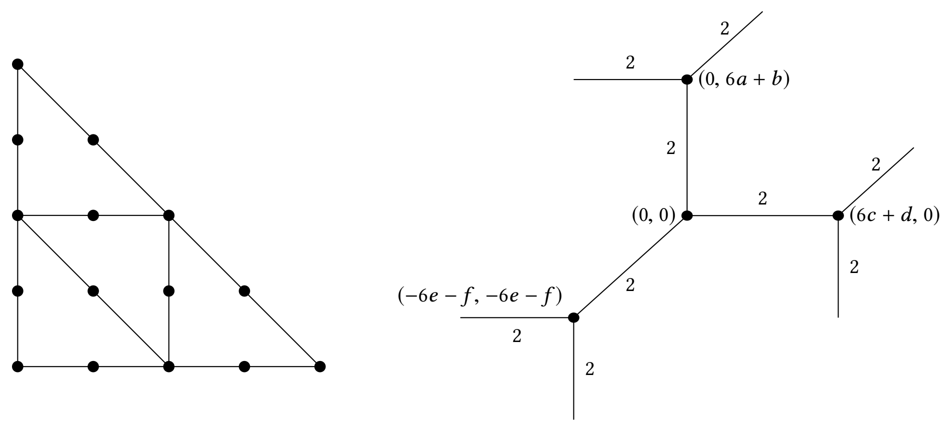


Figure 5.11: The Newton subdivision and plane tropical quartic from which we can unfold three lollis.

For the polygon $\mathrm{conv}\{(0, 2), (2, 2), (0, 4)\}$, we can read off immediately that the requirements of Proposition 5.2.5 are satisfied: p restricted to the corresponding weight 2 edge equals

$$x^2y^2 + 2xy^2 + (1 - t^{2a}) \cdot y^2 = (y(x + 1))^2 + y^2 \cdot t^{2a} \cdot (-1).$$

This is a square of y times the linear form $L = x + 1$ up to order t^{2a}, and the t^{2a}-contribution, $-y^2$, is not divisible by $x + 1$. Thus conditions (1) and (2) of Proposition 5.2.5 are satisfied. The vertex V is at $(0, 6a + b)$ as requested in condition (3). Finally, the t^0-terms of xy and x^2y are $2 \cdot (-1 + x)xy$ which is again not divisible by $x + 1$, so (4) is satisfied. The reasoning for the polygon conv$\{(2,0),(2,2),(4,0)\}$ is symmetric. For the polygon conv$\{(2,0),(0,2),(0,0)\}$, we have to divide by $(1-t^{2a}) = \sum_{k\geq 0}(t^{2a})^k$ first to express the polynomial in the form used in Proposition 5.2.5. To simplify the computation, we assume $e \leq c \leq a$. If $e < c$, p restricted to the corresponding weight 2 edge equals

$$y^2 + x^2 - 2\cdot(1+t^{2e})\cdot xy + O(t^{2c}) = (x-y)^2 + t^{2e}\cdot x\cdot(-2y) + O(t^{2c}).$$

If $e = c < a$, we have

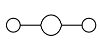
$$y^2 + (1-t^{2e})x^2 - 2\cdot(1+t^{2e})\cdot xy + O(t^{2a}) = (x-y)^2 + t^{2e}\cdot x\cdot(-2y-x) + O(t^{2a}).$$

If $e = c = a$, we have

$$\begin{aligned} y^2 + x^2 - 2\cdot(1+t^{2e})/(1-t^{2e})\cdot xy + O(t^{2a+1}) &= y^2 + x^2 - 2\cdot(1+2t^{2e})\cdot xy \\ + O(t^{2a+1}) &= (x-y)^2 + t^{2e}\cdot x\cdot(-4y) + O(t^{2a+1}). \end{aligned}$$

In any case, the t^{2e}-contribution is not divisible by $x-y$. The t^0-contribution of p restricted to conv$\{(1,2),(2,1)\}$ equals $2\cdot xy\cdot(y+x)$ which is not divisible by $x-y$.

With three linear modifications in total, we arrive at a model of the tropical plane and a tropicalised quartic defined by p and the additional linear equations which is faithful on the skeleton and realises the desired abstract curve.

Type 

Let a,b,c,d,e,f be the edge lengths of a curve C as in Figure 5.12. Due to symmetry, we may assume $c \leq d$. By [BJMS15, Theorem 5.1], C is embeddable as quartic in $\mathbb{R}^2$ if $c < d \leq 2c$.

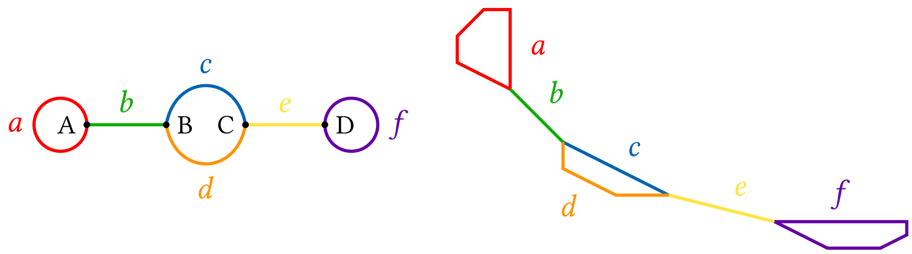


Figure 5.12: An abstract and plane tropical curve of type .

To show that any curve C is embeddable as a quartic in a tropical plane, let a, b, c, d, e, f be any lengths satisfying our assumptions due to symmetry. Note that curves with $c = d$ are realisably hyperelliptic, which is why they are excluded in our considerations. To construct curves satisfying $d > 2c$, consider the following quartic polynomial whose Newton subdivision and tropical curve as depicted in Figure 5.13:

$$
\begin{aligned}
p = {} & t^{a+6b+2c} \cdot x^4 + t^{6e+f+2c} \cdot y^4 + x^2y^2 + 2 \cdot xy^2 + (1 - t^{2b})y^2 + 2 \cdot x^2y \\
& + (1 - t^{2e}) \cdot x^2 + t^{-c}xy + t^{\frac{-d+4c-1}{2}} y + t^{\frac{-d+4c-1}{2}} x + t^{-d+3c}.
\end{aligned}
$$

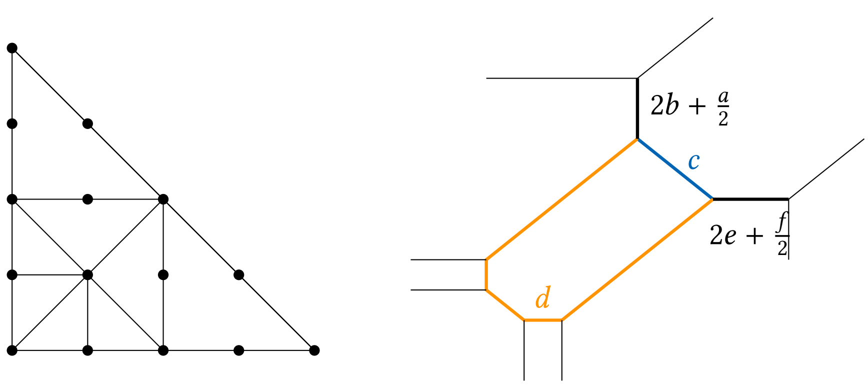


Figure 5.13: A tropicalised quartic in $\mathbb{R}^2$ from which we can unfold two lollis to obtain a curve of type .

Type

Let (a, b, c, d, e, f) be the edge lengths of a curve C as in Figure 5.14. Due to symmetry, we may assume $a \geq b$ and $c \leq d$. By [BJMS15, Theorem 5.1], C is embeddable as a quartic in $\mathbb{R}^2$ if $b + c < d < b + 3c$.

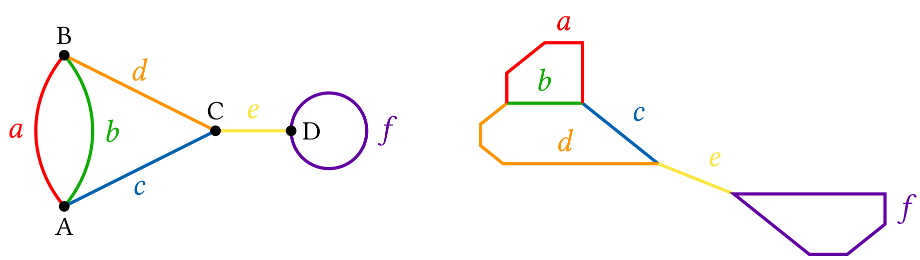


Figure 5.14: An abstract and plane tropical curve of type .

To show that any curve C is embeddable as a quartic in a tropical plane, let a, b, c, d, e, f be any lengths satisfying our assumptions due to symmetry. Note that curves with $c = d$ are realisably hyperelliptic, which is why they are excluded in our considerations. We distinguish between three cases

1. $d \geq b + 3c$ and $a > b$,

2. $d \geq b + 3c$ and $a = b$,

3. $c < d \leq b + c$.

If $d \geq b + 3c$ and $a > b$, we can consider a tropical plane quartic as in Figure 5.15, where the coefficients responsible for the bounded edge of weight 2 are chosen according to Proposition 5.2.5. This allows us to unfold a lolli with stick e and cycle length f.

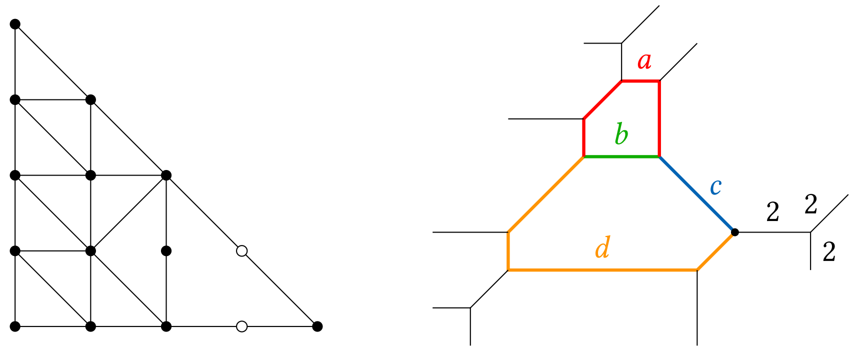


Figure 5.15: A curve of type with $d \geq b + 3c$, $a < b$, and a hidden lolli with edge lengths e, f.

If $d \geq b + 3c$ and $a = b$, we can further degenerate the picture above, letting the edges a and b form a double edge of weight 2 which we can unfold for suitably chosen coefficients using [CM16b, Theorem 3.4].

Now assume $c < d \leq b + c$. Picking b' such that $b' + c = d < b' + 3c$, we can consider a tropical plane quartic as in Figure 5.16, where the coefficients are chosen according to Proposition 5.2.4. This allows us to unfold an edge of length b.

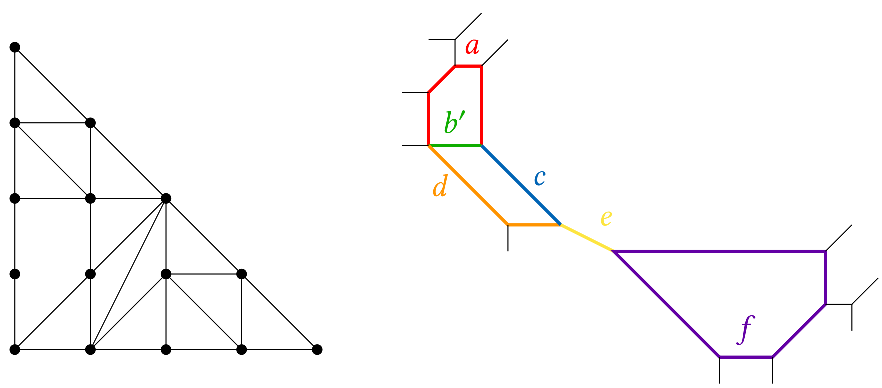


Figure 5.16: A curve of type with $c < d \leq b + c$.

5.3 Obstructions for realisably hyperelliptic

curves

In this section, we prove Theorem 5.1.2.

Lemma 5.3.1. *Assume a faithfully tropicalised quartic C in a realisable model of the tropical plane satisfies* $\mathrm{ft}^{\mathrm{trop}}(C) = \Gamma \in M_3^{\mathrm{trop}}$*. Then the tropical canonical divisor of* Γ *is very ample, and its sections obtained by intersecting C with a hyperplane are simultaneously realisable.*

Proof. Since C is a tropicalised quartic in a realisable model of the tropical plane, there exists a smooth algebraic curve $\mathcal{C}$ of genus 3 and an embedding $\mathcal{C} \to (K^*)^2 \to (K^*)^n$ tropicalising to C in the tropical plane. Here, the second arrow is a linear embedding. After compactifying, the first map yields $\overline{\mathcal{C}} \to \mathbb{P}^2$, which is necessarily a canonical embedding. Composing with the map $(K^*)^2 \to (K^*)^n$ amounts to picking n generators of the canonical linear system. The n coordinate functions of $C \subset \mathbb{R}^n$ are thus tropicalisations of sections of the canonical divisor. In particular, the tropical canonical divisor must be very ample, since by faithfulness the map $\mathrm{ft}(C) \to C$ is injective and thus any two points are separated by a section. Also, the sections are realisable by sections of the canonical divisor of $\mathcal{C}$, so they are simultaneously realisable. ◻

Let Γ be a trivalent tropical curve and $\chi\colon \Gamma \to \mathbb{R}$ be a piece-wise linear function such that

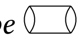
$$K_\Gamma + div(\chi) \geq 0. \tag{5.1}$$

The following simple observations will be useful in the proofs below:

(a) χ restricted to any edge is convex, and hence achieves no maximum in the inner points of the edge unless it is constant on this edge;

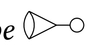
(b) for any vertex where χ achieves its global maximum M_χ, the slopes of χ vanish along at least two of the attached edges due to condition (5.1). Moreover, χ must be constant on any such edge by convexity.

For a piece-wise linear function $\chi\colon \Gamma \longrightarrow \mathbb{R}$, we denote the slope of χ at a vertex V along an attached edge E by $\frac{\partial\chi}{\partial E}(V)$.

Lemma 5.3.2. *The canonical divisor of an abstract tropical curve* Γ *of type*  *, where the edges forming the* 2*-cut have equal lengths, is not very ample.*

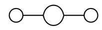
Proof. Denote the vertices of Γ by A, B, C, D and the edges by $E_a, E_b, E_c, E_d, E_e, E_f$ as in Figure 5.4. The length of E_i is i, and we have $c = d$. Let $\chi\colon \Gamma \longrightarrow \mathbb{R}$ be a piece-wise linear function satisfying (5.1). We shall show that $\chi(A) = \chi(B)$ and $\chi(C) = \chi(D)$, and hence the canonical linear system does not distinguish points, and as a result is not very ample.

Assume without loss of generality that $\chi(A) = M_\chi$. Then $\chi(B) = \chi(A)$ by (b). If χ is constant on either E_c or E_d then by the same argument $\chi(C) = M_\chi = \chi(D)$. Otherwise $\frac{\partial\chi}{\partial E_c}(A) = \frac{\partial\chi}{\partial E_d}(B) = -1$ by (b) and (5.1). Assume to the contrary that $\chi(C) \neq \chi(D)$, say $\chi(C) > \chi(D)$. By (a), $\frac{\partial\chi}{\partial E_e}(C), \frac{\partial\chi}{\partial E_f}(C) < 0$, and hence $\frac{\partial\chi}{\partial E_c}(C) \geq 1$ by condition (5.1). But χ is convex on E_c and hence linear. Thus, by convexity of χ on E_d, we obtain $\chi(D) \geq \chi(B) - d = \chi(A) - c = \chi(C)$, which is a contradiction. $\square$

Lemma 5.3.3. *The canonical divisor of an abstract tropical curve* Γ *of type*  *, where the edges forming the* 2*-cut have equal lengths, is not very ample.*

Proof. The argument is similar to the previous proof. Let Γ have vertices A, B, C and D and edges E_i of length i for $i = a, \ldots, f$ as in Figure 5.14.

Let $\chi\colon \Gamma \longrightarrow \mathbb{R}$ be a piece-wise linear function satisfying (5.1). We shall show that $\chi(A) = \chi(B)$, and hence the canonical system is not very ample. First, notice that by convexity, the slopes of χ along the loop at D are non-positive, and hence $\frac{\partial\chi}{\partial E_e}(D) \geq -1$. Thus, again by convexity, $\frac{\partial\chi}{\partial E_e}(C) \leq 1$. Second, assume to the contrary that $\chi(A) \neq \chi(B)$, say $\chi(A) < \chi(B)$. Then $\frac{\partial\chi}{\partial E_a}(B), \frac{\partial\chi}{\partial E_b}(B) < 0$ by (a), and hence $\frac{\partial\chi}{\partial E_d}(B) \geq 1$. By convexity this implies $\chi(C) > \chi(B) > \chi(A)$. Thus, again by convexity, $\frac{\partial\chi}{\partial E_d}(C), \frac{\partial\chi}{\partial E_c}(C) \leq -1$, and hence $\frac{\partial\chi}{\partial E_d}(C) = \frac{\partial\chi}{\partial E_c}(C) = -1$ since the sum of the three slopes at C must be at least -1. We conclude that χ is linear along E_d. Finally, $\chi(B) = \chi(C) - d = \chi(C) - c \leq \chi(A)$ by convexity of χ restricted to E_c, which is a contradiction. $\square$

Remark 5.3.4. For Γ of type , the canonical divisor is very ample, even if the lengths of the banana edges are equal. It is clear that we can separate points which are not on the banana edges, of the same distance to the vertices. For such two points, we can use functions as depicted in Figure 5.17.

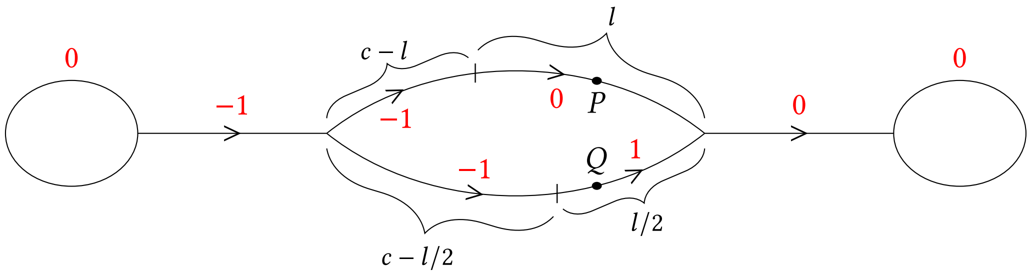
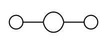


Figure 5.17: A function in the linear system of the canonical divisor, separating the two points P and Q. The red numbers indicate the slope along an edge.

Notice that [Ami13, Theorem 10] classifies all graphs with a not very ample canonical divisor, but this result seems to contain a small gap. For that reason, we include the arguments used in genus 3 here.

Lemma 5.3.5. *Let* $\Gamma =$  *with the lengths of the banana edges equal. Then any realisable canonical divisor* $K_\Gamma + div(\chi)$ *satisfies* $\chi|_{E_c} = \chi|_{E_d}$. *In particular, points on the banana edges cannot be separated with realisable sections.*

Proof. Without loss of generality $\chi(B) \geq \chi(C)$. Thus, $\frac{\partial\chi}{\partial E_c}(B), \frac{\partial\chi}{\partial E_d}(B) \leq 0$ by convexity. If one of the slopes vanishes then χ is constant on both edges by convexity and condition (ii) in [MUW17, Theorem 6.3]. Assume now that both slopes are negative. An argument identical to the one used in the proof of Lemma 5.3.3, shows that $\frac{\partial\chi}{\partial E_c}(B) + \frac{\partial\chi}{\partial E_d}(B) \geq -2$, and hence $\frac{\partial\chi}{\partial E_c}(B) = \frac{\partial\chi}{\partial E_d}(B) = -1$. Similarly,

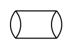
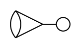
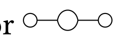
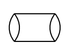
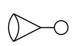
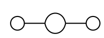
$$\frac{\partial\chi}{\partial E_c}(C) + \frac{\partial\chi}{\partial E_d}(C) \geq -2. \tag{5.2}$$

Let us identify $E_c \simeq [0, c]$ and $E_d \simeq [0, d]$ such that B is identified with 0. Then there exist $0 = t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 = c$ and $0 = s_0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 = d$ such that $\frac{\partial\chi}{\partial x}|_{(t_i,t_{i+1})} = \frac{\partial\chi}{\partial x}|_{(s_i,s_{i+1})} = i-1$. Thus, $\chi(C) = 2t_4 - t_1 - t_2 - t_3 = 2s_4 - s_1 - s_2 - s_3$, and since $c = d$, we obtain

$$t_1 + t_2 + t_3 = s_1 + s_2 + s_3. \tag{5.3}$$

By the symmetry of Γ we may assume that $t_3 \leq s_3$. If $t_3 < s_3$ then $\frac{\partial \chi}{\partial E_c}(C) = -2$ and $\frac{\partial \chi}{\partial E_d}(C) \geq 0$ by (5.2). This implies that $s_2 = s_3 = s_4 = d$, and hence $t_1 > s_1$ by (5.3). But this contradicts condition (ii) in [MUW17, Theorem 6.3]. Thus, $t_3 = s_3$.

Similarly we may assume that $t_1 \leq s_1$. If $t_1 < s_1$ then $t_2 > s_2 \geq s_1 > t_1$ by (5.3), which again contradicts condition (ii) in [MUW17, Theorem 6.3]. Thus, $t_1 = s_1$, and hence also $t_2 = s_2$ by (5.3), which completes the proof. ◻

Proof of Theorem 5.1.2. Let Γ be realisably hyperelliptic. Then, as described in Section 2.4.1, Γ is of type  ,  or  , where in each case the edges forming the 2-cut have equal lengths. Assume to the contrary that there exists a faithfully tropicalised quartic C in a realisable model of the tropical plane with $\mathrm{ft}(C) = \Gamma$. Then the canonical divisor on Γ is very ample by Lemma 5.3.1. If Γ is either of type  or  we get a contradiction to Lemmas 5.3.2 and 5.3.3. Thus, Γ must be of type  . Although in this case the tropical canonical divisor on Γ is tropically very ample, it admits no realisable sections separating the two banana edges by Lemma 5.3.5. We again obtain a contradiction, which completes the proof. ◻

Remark 5.3.6. Our constructive proof of Theorem 5.1.1 can be viewed as a way to construct simultaneous lifts of sections of the canonical divisor.

Remark 5.3.7. As a side-product, we can list all maximal abstract tropical curves of genus 3 for which the canonical divisor is not very ample. In [HMY12] , the question how to characterise an abstract tropical curve whose canonical divisor is not very ample was posed. In [Ami13], a list classifying all such tropical curves is presented, but there is a small gap in the classification, which is why we repeat the arguments in genus 3 here.

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