Model Decomposition of Timed Event Graphs under Partial Synchronization in Dioids

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Abstract: Timed Event Graphs (TEGs) are a graphical model for decision free and time-invariant Discrete Event Systems (DESs). To express systems with time-variant behaviors, a new form of synchronization, called partial synchronization (PS), has been introduced for TEGs. Unlike exact synchronization, where two transitions t_1, t_2 can only fire if both transitions are simultaneously enabled, PS of transition t_1 by transition t_2 means that t_1 can fire only when transition t_2 fires, but t_1 does not influence the firing of t_2. Under some assumptions, we can show that the dynamic behavior of a TEG under PS can be decomposed into a time-variant and a time-invariant part. The time-invariant part can be interpreted as a standard TEG. Moreover, it is shown that the tools introduced for standard TEGs can be used to analyze the overall system.

Keywords: Timed Event Graphs, Discrete Event Systems, synchronization, max-plus algebra.

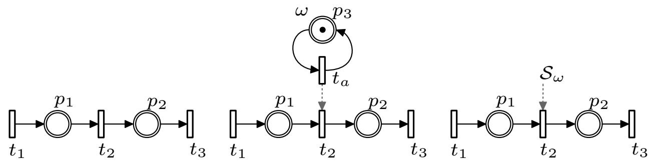
1. INTRODUCTION AND MOTIVATION

TEGs are a subclass of timed Petri nets where each place has exactly one input and one output transition and all arcs have weight 1. If an earliest functioning rule is adopted, the behavior of a TEG can be modeled by linear equations in a specific algebraic structure called dioids. Based on such dioids, a general theory has been developed for performance evaluation and control of TEGs, e.g. Baccelli et al. (1992); Heidergott et al. (2005). Timed Event Graphs under Partial Synchronization (TEGsPS) are an extension of TEGs introduced in David-Henriet et al. (2014). A similar extension was introduced in De Schutter and van den Boom (2003), where TEGs with hard and soft synchronization are studied. TEGsPS can express some time-variant phenomena which cannot be expressed by standard TEGs. For instance, partial synchronization (PS) is useful to model systems where particular events can only occur in a specific time window. E.g., at an intersection, a vehicle can only cross when the traffic light is green. Clearly this describes a time-variant behavior, since the vehicle is delayed by a time that depends on its time of arrival at the intersection. In David-Henriet et al. (2014, 2015, 2016), first results have been obtained for performance evaluation and controller synthesis for TEGsPS. This includes output reference control and model predictive control.
In this paper we investigate TEGsPS where partial synchronization is periodic. We show that under this assumption the dynamic behavior of TEGsPS can be modeled in a specific dioid called $\mathcal{T}[\![\gamma]\!]$. A specific time-variant operator is introduced to take PSs into account. Similarly to transfer functions for standard TEGs in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, the transfer behavior of TEGsPS is described by ultimately cyclic series in the dioid $\mathcal{T}[\![\gamma]\!]$. These corresponding transfer functions are useful, for instance, for computing the output for a given input of a system, for system composition and for control synthesis. Moreover, we show that operations on ultimately cyclic series in $\mathcal{T}[\![\gamma]\!]$ can be reduced to operations between matrices in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Therefore many tools developed for TEGs in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ can be applied to the more general class of TEGsPS.
This paper is organized as follows: Section 2 summarizes the necessary facts on TEGsPS and dioid theory. In Section 3, modeling of TEGsPS in the dioid $\mathcal{T}[\![\gamma]\!]$ is introduced. Section 4 discusses a decomposition method for elements in $\mathcal{T}[\![\gamma]\!]$ and provides tools to handle operations on ultimately cyclic series in $\mathcal{T}[\![\gamma]\!]$.

2. TIMED EVENT GRAPHS AND DIOIDS

2.1 Timed Event Graphs

In the following, we briefly recall the necessary facts on TEGs. For details, see Baccelli et al. (1992); Heidergott et al. (2005). A TEG consists of a set of places $P = \{p_1, \cdots, p_n\}$, a set of transitions $T = \{t_1, \cdots, t_m\}$ and a set of arcs $A \subseteq (P \times T) \cup (T \times P)$, all with weight 1. p_i is an upstream place of transition t_j (and t_j is a downstream transition of place p_i), if $(p_i, t_j) \in A$. Conversely, p_i is a downstream place of transition t_j (and t_j is an upstream transition of place p_i), if $(t_j, p_i) \in A$. For TEGs, each place p_i has exactly one upstream transition and exactly one downstream transition. Moreover, each place p_i exhibits an initial marking $(\boldsymbol{\mathcal{M}}_0)_i \in \mathbb{N}_0$ and a holding time $(\boldsymbol{\phi})_i \in \mathbb{N}_0$. A transition t_j is said to be enabled, if the marking in every upstream place is at least 1. When t_j fires, the marking $(\boldsymbol{\mathcal{M}})_i$ in every upstream place p_i is reduced by 1 and the marking $(\boldsymbol{\mathcal{M}})_o$ in every downstream place p_o is increased by 1. The holding time $(\boldsymbol{\phi})_i$ is the time a token must remain in place p_i before it contributes to the firing of the downstream transition of



(a) standard TEG. (b) TEG with PS. (c) PS by signal $\mathcal{S}_\omega$.

Fig. 1. (a) standard TEG. (b) PS of t_2 by t_a, triggered every ω time units. (c) equivalent PS expressed by a signal $\mathcal{S}_\omega$.

p_i. The set T of transitions is partitioned into input transitions, i.e., transitions without upstream places, output transitions, i.e., transitions without downstream places and internal transitions, i.e., transitions with both upstream and downstream places. We say that a TEG is operating under the earliest functioning rule, if all internal and output transitions are fired as soon as they are enabled.

2.2 *Timed Event Graph under Partial Synchronization*

TEGsPS provide a suitable model for some time-variant discrete event systems. In the following, we give a brief introduction. For further information the reader is invited to consult David-Henriet et al. (2014). Considering the TEG in Fig. 1a, assuming the earliest functioning rule, incoming tokens in place p_1 are immediately transferred to place p_2 by the firing of transition t_2. In contrast, Fig. 1b illustrates a TEG with PS of transition t_2 by transition t_a. This means that t_2 can only fire if t_a fires, but the firing of t_a does not depend on t_2. In this example, place p_3 and transition t_a, together with the corresponding arcs, constitute an autonomous TEG. Under the earliest functioning rule, the firing of transition t_a generates a periodic signal $\mathcal{S}_\omega$ with a period $\omega \in \mathbb{N}$. Therefore, the PS of t_2 by t_a can also be described by a predefined signal $\mathcal{S}_\omega$: $\mathbb{Z} \mapsto \{0, 1\}$, enabling the firing of t_2 at times t where $\mathcal{S}_\omega(t) = 1$. The signal $\mathcal{S}_\omega(t) = 1$ if $t \in \{j\omega \text{ with } j \in \mathbb{Z}\}$ and 0 otherwise.

Definition 1. A Timed Event Graph under Partial Synchronization is a TEG where internal and output transitions are subject to partial synchronization. ◁

Note that the assumption that only internal and output transitions are subject to PS is not restrictive since we can always add new input transitions and extend the set of internal transitions by the former input transitions. In David-Henriet et al. (2015), ultimately periodic signals are considered for PS transitions. It was shown that the behavior of a Timed Event Graph under Partial Synchronization (TEGPS) can be described by recursive equations in state space form. In this work, we focus on (immediately) periodic signals for PS of a transitions.

Definition 2. A periodic signal $\mathcal{S} : \mathbb{Z} \to \{0, 1\}$ is defined by a string $\{n_0, n_1, \cdots, n_I\} \in \mathbb{N}_0$ and a period $\omega \in \mathbb{N}$, such that $\forall j \in \mathbb{Z}$

$$\mathcal{S}(t) = \begin{cases} 1 & \text{if } t \in \{n_0 + \omega j,\ n_1 + \omega j,\ \cdots,\ n_I + \omega j\}, \\ 0 & \text{otherwise,} \end{cases}$$

where the string $\{n_0, n_1, \cdots, n_I\}$ is strictly ordered, i.e., $\forall i \in \{1, \cdots, I\},\ n_{i-1} < n_i$, and $n_I < \omega$. ◁

Example 1. The signal

$$\mathcal{S}_1(t) = \begin{cases} 1 & \text{if } t \in \{\cdots, -3, 0, 1, 4, 5, 8, 9, \cdots\}, \\ 0 & \text{otherwise,} \end{cases}$$

is a periodic signal with a period $\omega = 4$ and a string $\{0, 1\}$. Therefore $\forall j \in \mathbb{Z}$,

$$\mathcal{S}_1(t) = \begin{cases} 1 & \text{if } t \in \{0 + 4j, 1 + 4j\}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

◁

In the following, we only consider PS of transitions by periodic signals as given in Definition 2. We call such a PS a *periodic* PS. To consider only *periodic* PS allows us to define a dioid of operators to describe the behavior of TEGsPS. In particular we can show that the transfer behavior of a TEGPS is described by a rational power series of an ultimately cyclic form. Let us note that focusing on periodic signals for a PS of a transition is not overly restrictive as periodic schedules are common in many applications.

2.3 *Dioid Theory*

A dioid $\mathcal{D}$ is an algebraic structure with two binary operations, $\oplus$ (addition) and $\otimes$ (multiplication). Addition is commutative, associative and idempotent (i.e. $\forall a \in \mathcal{D},\ a \oplus a = a$). The neutral element for addition, denoted by ε, is absorbing for multiplication (i.e., $\forall a \in \mathcal{D},\ a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$). Multiplication is associative, distributive over addition and has a neutral element denoted by e. The element e (resp, ε) is called unit (resp. zero) element of the dioid. Both operations can be extended to the matrix case. For matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{D}^{m \times n}$, $\boldsymbol{C} \in \mathcal{D}^{n \times q}$ and a scalar $\lambda \in \mathcal{D}$, matrix addition and multiplication are defined by

$$(\boldsymbol{A} \oplus \boldsymbol{B})_{i,j} := (\boldsymbol{A})_{i,j} \oplus (\boldsymbol{B})_{i,j}, \quad (\lambda \otimes \boldsymbol{A})_{i,j} := \lambda \otimes (\boldsymbol{A})_{i,j},$$

$$(\boldsymbol{A} \otimes \boldsymbol{C})_{i,j} := \bigoplus_{k=1}^{n} \left((\boldsymbol{A})_{i,k} \otimes (\boldsymbol{C})_{k,j} \right).$$

The identity matrix, denoted by $\boldsymbol{I}$, is a matrix with elements e on the diagonal and ε otherwise. Note that, as in conventional algebra, the multiplication symbol $\otimes$ is often omitted. A dioid $\mathcal{D}$ is said to be complete if it is closed for infinite sums and if multiplication distributes over infinite sums. A complete dioid is a partially ordered set, with a canonical order $\succeq$ defined by $a \oplus b = a \Leftrightarrow a \succeq b$. Moreover, in a complete dioid, the Kleene star of an element $a \in \mathcal{D}$, denoted a^*, is defined by $a^* = \bigoplus_{i=0}^{\infty} a^i$ with $a^0 = \text{e}$ and $a^{i+1} = a \otimes a^i$.

Theorem 1. (Baccelli et al. (1992)). In a complete dioid $\mathcal{D}$, $x = a^* b$ is the least solution of the implicit equation $x = ax \oplus b$. ◁

Let $\mathcal{C} \subseteq \mathcal{D}$ then $(\mathcal{C}, \oplus, \otimes)$ is a subdioid of $(\mathcal{D}, \oplus, \otimes)$ if e and ε are in $\mathcal{C}$ and $\mathcal{C}$ is closed for $\oplus$ and $\otimes$.

3. MODELING OF TEGS UNDER PS IN THE DIOID $\mathcal{T}[\![\gamma]\!]$

To model TEGsPS, a dater function $x_i : \mathbb{Z} \to \mathbb{Z}_{max} = \{\mathbb{Z} \cup \pm\infty\}$ is associated to each transition t_i. $x_i(k)$ gives the date when the transition fires the k^{th} time. Naturally, dater functions are nondecreasing functions, i.e., $x_i(k+1) \geq x_i(k)$. The set of dater functions is denoted by Σ, and on Σ addition and multiplication by a constant are defined as follows:

$$x, y \in \Sigma,\ (x \oplus y)(k) := \max(x(k), y(k)),$$
$$\lambda \in \mathbb{Z}_{max},\ (\lambda \otimes x)(k) := \lambda + x(k).$$

The $\oplus$ operation induces an order relation on Σ, i.e. $\forall x, y \in \Sigma,\ x \preceq y \Leftrightarrow x \oplus y = y$. An operator $\rho : \Sigma \to \Sigma$ is linear if (a) $\forall x, y \in \Sigma : \rho(x \oplus y) = \rho(x) \oplus \rho(y)$ and (b)

$\lambda \otimes \rho(x) = \rho(\lambda \otimes x)$. An operator is additive if (a) is satisfied. The set of additive operators on Σ is denoted $\mathcal{O}$.

Proposition 1. (Cottenceau et al. (2014)). The set $\mathcal{O}$ equipped with addition and multiplication: $x \in \Sigma, \forall \rho_1, \rho_2 \in \mathcal{O}$,

$$(\rho_1 \oplus \rho_2)(x) := \rho_1(x) \oplus \rho_2(x),\ (\rho_1 \otimes \rho_2)(x) := \rho_1(\rho_2(x)),$$

is a noncommutative complete dioid. The identity operator (unit element) is denoted by $\mathrm{e} : \forall x \in \Sigma,\ (\mathrm{e}(x))(k) = x(k)$, and the zero operator (zero element) is denoted by $\varepsilon : \forall x \in \Sigma,\ (\varepsilon(x))(k) = -\infty$.

To simplify notation, we write ρx instead of $\rho(x)$ from now on.

Definition 3. (Basic operators in TEGsPS). Dynamic phenomena arising in TEGsPS can be described by the following additive basic operators in $\mathcal{O}$:

$$\varsigma \in \mathbb{Z},\ \delta^{\varsigma} : \forall x \in \Sigma,\ (\delta^{\varsigma} x)(k) = x(k) + \varsigma, \tag{2}$$

$$\nu \in \mathbb{Z},\ \gamma^{\nu} : \forall x \in \Sigma,\ (\gamma^{\nu} x)(k) = x(k - \nu), \tag{3}$$

$$\omega, \varpi \in \mathbb{N},\ \Delta_{\omega|\varpi} : \forall x \in \Sigma,\ (\Delta_{\omega|\varpi} x)(k) = \lceil x(k)/\varpi \rceil \omega, \tag{4}$$

where $\lceil a \rceil$ is the smallest integer greater than or equal to a. ◁

The time- and event-shift operator δ and γ are used to model the dynamic behavior of standard TEGs, e.g., Baccelli et al. (1992). In addition we introduce the $\Delta_{\omega|\varpi}$ operator to consider phenomena caused by PS.

Proposition 2. The basic operators satisfy the following relations

$$\gamma^{\nu} \oplus \gamma^{\nu'} = \gamma^{\min(\nu,\nu')}, \qquad \delta^{\tau} \oplus \delta^{\tau'} = \delta^{\max(\tau,\tau')}, \tag{5}$$

$$\gamma^{\nu} \otimes \gamma^{\nu'} = \gamma^{\nu+\nu'}, \qquad \delta^{\tau} \otimes \delta^{\tau'} = \delta^{\tau+\tau'}, \tag{6}$$

$$\Delta_{\omega|\varpi} \otimes \delta^{\varpi} = \delta^{\omega} \otimes \Delta_{\omega|\varpi}. \tag{7}$$

Proof. See Baccelli et al. (1992) for (5), (6). For the proof of (7), recall (2) and (4),

$$\begin{aligned}(\Delta_{\omega|\varpi}\delta^{\varpi} x)(k) &= \left\lceil \frac{x(k) + \varpi}{\varpi} \right\rceil \omega = \left\lceil \frac{x(k)}{\varpi} + 1 \right\rceil \omega, \\ &= \left\lceil \frac{x(k)}{\varpi} \right\rceil \omega + \omega = (\delta^{\omega}\Delta_{\omega|\varpi} x)(k).\end{aligned}$$

□

Remark 1. (7) implies that for $-b < \tau \leq 0$, $\Delta_{\omega|b}\delta^{\tau}\Delta_{b|\varpi} = \Delta_{\omega|\varpi}$, since,

$$\begin{aligned}(\Delta_{\omega|b}\delta^{\tau}\Delta_{b|\varpi} x)(k) &= \left\lceil \frac{\lceil x(k)/\varpi \rceil b + \tau}{b} \right\rceil \omega \\ &= \left\lceil \left\lceil \frac{x(k)}{\varpi} \right\rceil + \frac{\tau}{b} \right\rceil \omega \\ &= \left\lceil \frac{x(k)}{\varpi} \right\rceil \omega \quad \text{since } -1 < \tau/b \leq 0, \\ &= (\Delta_{\omega|\varpi} x)(k).\end{aligned}$$

◁

3.1 Dioid of Time Operators $\mathcal{T}$

In the following, we introduce a dioid of specific time operators in order to model the time-variant behavior of *periodic* PS.

Definition 4. (Dioid of T-operators $\mathcal{T}$). We denote by $\mathcal{T}$ the dioid of operators obtained by addition and composition of operators in $(\varepsilon, \mathrm{e}, \delta^{\varsigma}, \Delta_{\omega|\varpi})$ with $\varsigma \in \mathbb{Z}$, and $\omega, \varpi \in \mathbb{N}$. The elements of $\mathcal{T}$ are called T-operators (T is for time). ◁

For example, $\delta^3\Delta_{4|4}\delta^1\Delta_{3|2} \in \mathcal{T}$. Since a T-operator only describes a time relation in a system, e.g. a delay, we can associate a function $\mathcal{R}_v : \mathbb{Z}_{max} \to \mathbb{Z}_{max}$ to a T-operator v. This function when evaluated on t is obtained by replacing $x(k)$ by t in the expression of $v(x)(k)$. For example, $((\Delta_{3|4}\delta^1 \oplus \delta^2\Delta_{3|3})x)(k) = \max(\lceil (x(k)+1)/4 \rceil 3, 2 + \lceil x(k)/3 \rceil 3)$ and therefore $\mathcal{R}_{\Delta_{3|4}\delta^1 \oplus \delta^2\Delta_{3|3}}(t) = \max(\lceil (t+1)/4 \rceil 3, 2 + \lceil t/3 \rceil 3)$. We denote by $\mathscr{R}$ the set of functions generated by all operators in $\mathcal{T}$. Clearly, there is an isomorphism between the set of T-operators and the set $\mathscr{R}$. The order relation over the dioid $\mathcal{T}$ corresponds to the order induced by the $\max$ operation on $\mathscr{R}$. For $v_1, v_2 \in \mathcal{T}$,

$$\begin{aligned}&v_1 \succeq v_2 \Leftrightarrow v_1 \oplus v_2 = v_1 \Leftrightarrow v_1 x \oplus v_2 x = v_1 x,\ \ \forall x \in \Sigma, \\ &\Leftrightarrow (v_1 x)(k) \oplus (v_2 x)(k) = (v_1 x)(k),\ \ \forall x \in \Sigma,\ \forall k \in \mathbb{Z}, \\ &\Leftrightarrow \mathcal{R}_{v_1}(t) \geq \mathcal{R}_{v_2}(t),\ \ \forall t \in \mathbb{Z}_{max}.\end{aligned} \tag{8}$$

Definition 5. (Periodic T-operators). A T-operator $v \in \mathcal{T}$ is said to be periodic if its corresponding function $\mathcal{R}_v$ is quasi-periodic, i.e., $\exists \omega \in \mathbb{N}$ such that $\forall t \in \mathbb{Z}_{max}$, $\mathcal{R}_v(t + \omega) = \omega + \mathcal{R}_v(t)$. ◁

The set of periodic operators, denoted by $\mathcal{T}_{per}$, is a subdioid of $\mathcal{T}$.

Proposition 3. (Canonical form of a periodic T-operator). A periodic T-operator $v \in \mathcal{T}_{per}$ with period ω has a canonical form given by a finite sum $\bigoplus_{i=1}^{I} \delta^{\tau_i}\Delta_{\omega|\omega}\delta^{\tau_i'}$. Moreover the sum is strictly ordered such that $\forall i \in \{1, \cdots, I-1\}$, $\tau_i < \tau_{i+1}$.

Proof. We first show that a periodic T-operator $v \in \mathcal{T}_{per}$ with period ω can be represented as

$$v = \bigoplus_{i=0}^{\omega-1} \delta^{\mathcal{R}_v(-i)}\Delta_{\omega|\omega}\delta^{i-\omega+1}. \tag{9}$$

For this, we consider the operator $\mathsf{w} = \bigoplus_{i=0}^{\omega-1} \mathsf{w}_i$ with $\mathsf{w}_i = \delta^{\mathcal{R}_v(-i)}\Delta_{\omega|\omega}\delta^{i-\omega+1}$. The function $\mathcal{R}_{\mathsf{w}_i}$ associated to w_i is

$$\mathcal{R}_{\mathsf{w}_i}(t) = \mathcal{R}_v(-i) + \left\lceil \frac{t + i - \omega + 1}{\omega} \right\rceil \omega.$$

Therefore $\mathcal{R}_{\mathsf{w}}$ is

$$\begin{aligned}&\mathcal{R}_{\mathsf{w}}(t) = \max\Big(\mathcal{R}_v(0) + \left\lceil \frac{t + 1 - \omega}{\omega} \right\rceil \omega, \\ &\mathcal{R}_v(-1) + \left\lceil \frac{t + 2 - \omega}{\omega} \right\rceil \omega, \cdots, \mathcal{R}_v(1-\omega) + \left\lceil \frac{t}{\omega} \right\rceil \omega\Big),\end{aligned} \tag{10}$$

which clearly has period ω. To prove that v can be represented as (9) we have to show that $\mathcal{R}_{\mathsf{w}}(t) = \mathcal{R}_v(t)$. Because $\mathcal{R}_{\mathsf{w}}$ and $\mathcal{R}_v$ are quasi ω-periodic functions it is sufficient to check that $\mathcal{R}_{\mathsf{w}}(t) = \mathcal{R}_v(t)$ for $t = \{1-\omega, \cdots, 0\}$. Let us remark that $\mathcal{R}_v$ is nondecreasing and thus, $\cdots \leq \mathcal{R}_v(0) - \omega \leq \mathcal{R}_v(1-\omega) \leq \cdots \leq \mathcal{R}_v(0) \leq \mathcal{R}_v(1-\omega) + \omega \leq \cdots$. We evaluate (10) for $t = 0$, this leads to

$$\begin{aligned}\mathcal{R}_{\mathsf{w}}(0) =& \max\Big(\mathcal{R}_v(0) + \left\lceil \frac{1 - \omega}{\omega} \right\rceil \omega, \\ &\mathcal{R}_v(-1) + \left\lceil \frac{2 - \omega}{\omega} \right\rceil \omega, \cdots, \mathcal{R}_v(1-\omega) + \left\lceil \frac{0}{\omega} \right\rceil \omega\Big) \\ =& \max\big(\mathcal{R}_v(0), \mathcal{R}_v(-1), \cdots, \mathcal{R}_v(1-\omega)\big) \\ =& \mathcal{R}_v(0).\end{aligned}$$

Similarly we can show that for $t \in \{1-\omega, \cdots, -1\}$, $\mathcal{R}_{\mathsf{w}}(t) = \mathcal{R}_v(t)$. For this, recall (10)

$$\begin{aligned}&\mathcal{R}_{\mathsf{w}}(t) = \max\Big(\mathcal{R}_v(0) + \left\lceil \frac{t + 1 - \omega}{\omega} \right\rceil \omega, \\ &\mathcal{R}_v(-1) + \left\lceil \frac{t + 2 - \omega}{\omega} \right\rceil \omega, \cdots, \mathcal{R}_v(1-\omega) + \left\lceil \frac{t}{\omega} \right\rceil \omega\Big).\end{aligned}$$

For $1 \leq j \leq \omega$ and $1 - \omega \leq t \leq -1$ observe that,

$$\left\lceil \frac{t + j - \omega}{\omega} \right\rceil \omega = \begin{cases} -\omega, & \text{for } t + j < 0 \\ 0, & \text{for } t + j \geq 0, \end{cases}$$

therefore,

$$\begin{aligned}\mathcal{R}_{\mathsf{w}}(t) = \max\big(&\mathcal{R}_v(0)-\omega,\cdots,\mathcal{R}_v(t+1)-\omega,\mathcal{R}_v(t),\cdots\\ &\cdots,\mathcal{R}_v(1-\omega)\big),\\ = &\mathcal{R}_v(t),\end{aligned}$$

and $v = \mathsf{w} = \bigoplus_{i=0}^{\omega-1}\mathsf{w}_i = \bigoplus_{i=0}^{\omega-1}\delta^{\mathcal{R}_v(-i)}\Delta_{\omega|\omega}\delta^{i-\omega+1}$. The canonical representation is the one obtained by removing redundant w_i according to the order relation given in (8). □

Remark 2. Clearly periodic operator $v \in \mathcal{T}_{per}$ with period ω can be represented with a multiple period $n\omega$ as follows,

$$v = \bigoplus_{i=0}^{n\omega-1}\delta^{\mathcal{R}_v(-i)}\Delta_{n\omega|n\omega}\delta^{i-n\omega+1}.$$

◁

Remark 3. In particular the periodic operator $\Delta_{\omega|\omega}$ with period ω is represented with period $n\omega$ by the sum

$$\Delta_{\omega|\omega} = \bigoplus_{i=0}^{n-1}\delta^{-i\omega}\Delta_{n\omega|n\omega}\delta^{-(n-1-i)\omega}.$$

◁

Example 2. The identity operator $\mathrm{e} = \Delta_{1|1}$ with period 1 can be represented with period 3 as follows, $\mathrm{e} = \Delta_{3|3}\delta^{-2} \oplus \delta^{-1}\Delta_{3|3}\delta^{-1} \oplus \delta^{-2}\Delta_{3|3}$, see Fig. 2. ◁

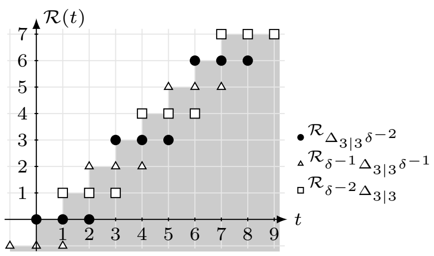


Fig. 2. $\mathcal{R}_{\mathrm{e}}(t) = \max(\mathcal{R}_{\Delta_{3|3}\delta^{-2}}(t), \mathcal{R}_{\delta^{-1}\Delta_{3|3}\delta^{-1}}(t), \mathcal{R}_{\delta^{-2}\Delta_{3|3}}(t))$.

The time-variant behavior caused by a *periodic* PS of a transition can be modeled in the dioid $\mathcal{T}$. For this, recall the definition of a periodic signal $\mathcal{S}$ (Definition 2). We associate with a periodic signal $\mathcal{S}$ a function $\mathcal{R}_S : \mathbb{Z}_{max} \to \mathbb{Z}_{max}$. This function $\mathcal{R}_S(t)$ is defined by, $\forall j \in \mathbb{Z}$,

$$\mathcal{R}_S(t) = \begin{cases} -\infty & \text{if } t=-\infty\\ n_0+\omega j & \text{if } (n_I-\omega)+\omega j < t \le n_0+\omega j,\\ n_1+\omega j & \text{if } n_0+\omega j < t \le n_1+\omega j,\\ \vdots & \\ n_I+\omega j & \text{if } n_{I-1}+\omega j < t \le n_I+\omega j,\\ \infty & \text{if } t=\infty.\end{cases} \tag{11}$$

Example 3. The function $\mathcal{R}_{S_1}(t)$ associated to the signal S_1 given in Example 1 is

$$\mathcal{R}_{S_1}(t) = \begin{cases} -\infty & \text{if } t=-\infty\\ 0+4j & \text{if } -3+4j < t \le 0+4j,\\ 1+4j & \text{if } 0+4j < t \le 1+4j,\\ \infty & \text{if } t=\infty.\end{cases}$$

◁

The value of $\mathcal{R}_S(t)$ can be interpreted as the next time when the signal $\mathcal{S}$ enables the firing of the corresponding transition.

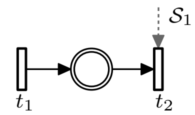


Fig. 3. Simple TEGPS with a *periodic* PS of t_2.

Clearly, an ω-periodic signal $\mathcal{S}$ leads to a corresponding function $\mathcal{R}_S(t)$ which satisfies $\forall t \in \mathbb{Z}_{max}, \mathcal{R}_S(t+\omega) = \omega + \mathcal{R}_S(t)$. To prove that a *periodic* PS of a transition (i.e. the PS is specified by a periodic signal $\mathcal{S}$) admits an operator representation in the dioid $\mathcal{T}$, we must show the existence of an operator $v \in \mathcal{T}$ such that $\mathcal{R}_v = \mathcal{R}_S$.

Proposition 4. A *periodic* partial synchronization of a transition by signal $\mathcal{S}$ (from Definition 2) has an operator representation given by

$$\begin{aligned} v = &\delta^{n_0}\Delta_{\omega|\omega}\delta^{-n_I} \oplus \delta^{n_1-\omega}\Delta_{\omega|\omega}\delta^{-n_0} \oplus \cdots\\ &\cdots \oplus \delta^{n_I-\omega}\Delta_{\omega|\omega}\delta^{-n_{(I-1)}}.\end{aligned} \tag{12}$$

Proof. Recall that a periodic signal $\mathcal{S}$ corresponds to a quasi periodic function $\mathcal{R}_S$, see (11), and the isomorphism between the function $\mathcal{R}_v$ and the T-operator v. It remains to show that $\mathcal{R}_v = \mathcal{R}_S$. The function $\mathcal{R}_v$ is given by

$$\begin{aligned}\mathcal{R}_v(t) = \max\Big(&n_0 + \Big\lceil\frac{t-n_I}{\omega}\Big\rceil\omega,\ n_1-\omega+\Big\lceil\frac{t-n_0}{\omega}\Big\rceil\omega,\cdots\\ &,n_I-\omega+\Big\lceil\frac{t-n_{(I-1)}}{\omega}\Big\rceil\omega\Big).\end{aligned} \tag{13}$$

To show equality, we evaluate $\mathcal{R}_v(t)$ for intervals defined in (11). E.g., for $(n_I-\omega)+\omega j < t \le n_0+\omega j$, observe that

$$\Big\lceil\frac{t-n_i}{\omega}\Big\rceil = j, \quad i=0,\cdots I,$$

hence

$$\begin{aligned}\mathcal{R}_v(t) &= \max\big(n_0+j\omega, n_1-\omega+j\omega,\cdots,n_I-\omega+j\omega\big)\\ &= n_0+j\omega.\end{aligned}$$

Second, for $(n_0+\omega j) < t \le n_1+\omega j$, we have

$$\Big\lceil\frac{t-n_i}{\omega}\Big\rceil = \begin{cases} j+1, & \text{for } i=0\\ j, & \text{for } i=1,\cdots,I,\end{cases}$$

hence

$$\begin{aligned}\mathcal{R}_v(t) = \max\big(&n_0+j\omega, n_1+j\omega, n_2-\omega+j\omega,\cdots\\ &\cdots,n_I-\omega+j\omega\big)\\ = &n_1+j\omega.\end{aligned}$$

By going through the remaining intervals defined in (11) we establish

$$\mathcal{R}_v(t) = \mathcal{R}_S(t), \quad \forall t \in \mathbb{Z}_{max}.$$

□

Example 4. Consider the TEGPS shown in Fig. 3, where the signal $\mathcal{S}_1$ is given in (1) (Example 1) and dater functions $x_1(k)$ (resp. $x_2(k)$) is associated with transition t_1 (resp. t_2). According to Prop. 4, the behavior of the *periodic* PS of transition t_2 is modeled by the following operator:

$$v_{\mathcal{S}_1} = \delta^0\Delta_{4|4}\delta^{-1} \oplus \delta^{-3}\Delta_{4|4}\delta^{-0} = \delta^{-3}\Delta_{4|4} \oplus \Delta_{4|4}\delta^{-1}.$$

This operator describes the firing relation between t_1 and t_2, i.e. $x_2 = (\delta^{-3}\Delta_{4|4} \oplus \Delta_{4|4}\delta^{-1})x_1$. Therefore, $x_2(k) = \max(-3+\lceil x_1(k)/4\rceil 4, \lceil (x_1(k)-1)/4\rceil 4)$. ◁

Remark 4. Due to the influence of the PS, this firing relation between t_1 and t_2 is time-variant. For instance, if the k-th firing of t_1 is at time instance $x_1(k) = 1$, then the k-th firing of t_2 is at $x_2(k) = 1$, i.e., we have no delay. In contrast, if the k-th firing of t_1 is at time instant $x_1(k) = 2$, then the k-th firing of t_2 is at $x_2(k) = 4$, and the delay is 2. ◁

3.2 Dioid $\mathcal{T}[\![\gamma]\!]$

Since the γ operator commutes with all T-operators, i.e. $\forall v \in \mathcal{T},\ v\gamma = \gamma v$, we can define the dioid $\mathcal{T}[\![\gamma]\!]$ as follows.

Definition 6. (Dioid $\mathcal{T}[\![\gamma]\!]$) We denote by $\mathcal{T}[\![\gamma]\!]$ the quotient dioid in the set of formal power series in one variable γ with exponents in $\mathbb{Z}$ and coefficients in the noncommutative complete dioid $\mathcal{T}$ induced by the equivalence relation, $\forall s \in \mathcal{T}[\![\gamma]\!]$,

$$s = s(\gamma^1)^*. \tag{14}$$

◁

Note that we can interpret elements in $\mathcal{T}[\![\gamma]\!]$ as nondecreasing functions $s : \mathbb{Z} \to \mathcal{T}$, where $s(\nu)$ refers to the coefficient of γ^ν.

Definition 7. Let $s_1, s_2 \in \mathcal{T}[\![\gamma]\!]$, then addition and multiplication are defined by

$$s_1 \oplus s_2 := \bigoplus_{\nu\in\mathbb{Z}} (s_1(\nu) \oplus s_2(\nu))\gamma^\nu,$$

$$s_1 \otimes s_2 := \bigoplus_{\nu\in\mathbb{Z}} \left(\bigoplus_{n+n'=\nu} (s_1(n) \otimes s_2(n')) \right) \gamma^\nu.$$

◁

As before, $\oplus$ defines an order on $\mathcal{T}[\![\gamma]\!]$, i.e., $a, b \in \mathcal{T}[\![\gamma]\!] :\ a \oplus b = b \Leftrightarrow a \preceq b$. A monomial in $\mathcal{T}[\![\gamma]\!]$ is defined by $v\gamma^\nu$, where $v \in \mathcal{T}$. A polynomial is a finite sum of monomials, i.e., $\bigoplus_i v_i\gamma^{\nu_i}$.

Definition 8. (Ultimately Cyclic Series of $\mathcal{T}[\![\gamma]\!]$): A series $s \in \mathcal{T}[\![\gamma]\!]$ is said to be ultimately cyclic if it can be written as $s = p \oplus q(\gamma^\nu\delta^\tau)^*$, where $\nu, \tau \in \mathbb{N}$ and p, q are polynomials in $\mathcal{T}[\![\gamma]\!]$. ◁

3.3 Subdioids of $\mathcal{T}[\![\gamma]\!]$

We denote by $\mathcal{T}_{per}[\![\gamma]\!]$ the subdioid of $\mathcal{T}[\![\gamma]\!]$, obtained by restricting the coefficients v to periodic operators, i.e. $v \in \mathcal{T}_{per}$.

An other important subdioid of $\mathcal{T}[\![\gamma]\!]$ is the dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. This dioid is obtained by restricting the coefficients v to the $(\varepsilon, \delta^\tau)$ operators, i.e., an element in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is written as $\bigoplus_i \delta^{\tau_i}\gamma^{n_i}$ with $\tau_i, n_i \in \mathbb{Z}$. This dioid has been extensively studied, e.g. Gaubert and Klimann (1991); Baccelli et al. (1992). The product of two monomials $\gamma^{n_1}\delta^{t_1}, \gamma^{n_2}\delta^{t_2} \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ is obtained by,

$$\gamma^{n_1}\delta^{t_1} \otimes \gamma^{n_2}\delta^{t_2} = \gamma^{n_1+n_2}\delta^{t_1+t_2}.$$

Moreover, we have the following order relation for monomials

$$\gamma^{n_1}\delta^{t_1} \preceq \gamma^{n_2}\delta^{t_2} \Leftrightarrow n_1 \geq n_2 \text{ and } t_1 \leq t_2.$$

A comprehensive representation of calculations with series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ can be found in Baccelli et al. (1992). It is well known that the input-output behavior of a standard TEG can be described by a transfer function matrix composed of ultimately cyclic series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Moreover, based on $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$, methods for performance evaluation and controller synthesis have been introduced for TEGs, e.g. Gaubert and Klimann (1991); Maia et al. (2003); Hardouin et al. (2017). In Hardouin et al. (2009), software tools have been made available for computations in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. The dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ plays a key role in this paper. In particular in Section 4, we show that all relevant operations on ultimately cyclic series $s \in \mathcal{T}_{per}[\![\gamma]\!]$ can be reduced to operations on matrices in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. We can therefore use the existing tools for $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ to study TEGsPS.

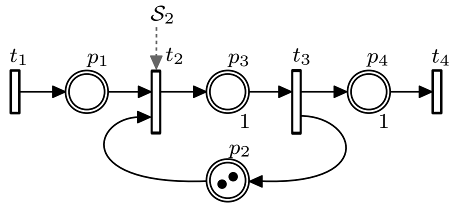


Fig. 4. Example of a TEGPS.

3.4 Modeling of TEGsPS in $\mathcal{T}_{per}[\![\gamma]\!]$

A TEGPS operating under the earliest functioning rule, admits a state-space representation in $\mathcal{T}_{per}[\![\gamma]\!]$,

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \tag{15}$$

where $\boldsymbol{x}$ (resp. $\boldsymbol{u}, \boldsymbol{y}$) refers to the vector of dater functions of internal (resp. input, output) transitions. The matrices $\boldsymbol{A} \in \mathcal{T}_{per}[\![\gamma]\!]^{n\times n}$, $\boldsymbol{B} \in \mathcal{T}_{per}[\![\gamma]\!]^{n\times g}$ and $\boldsymbol{C} \in \mathcal{T}_{per}[\![\gamma]\!]^{p\times n}$ describe the influence of transitions on each other, encoded by operators in $\mathcal{T}_{per}[\![\gamma]\!]$. Let us consider a path constituted by the arcs (t_j, p_i) and (p_i, t_o). The influence of transition t_j on transition t_o is coded as an operator

$$v_{t_o}\delta^{(\boldsymbol{\phi})_i}\gamma^{(\boldsymbol{\mathcal{M}}_0)_i}$$

where v_{t_o} is the operator representation of the signal $\mathcal{S}_o$ corresponding to the PS of t_o (see Example 4), $(\boldsymbol{\phi})_i$ is the holding time of place p_i and $(\boldsymbol{\mathcal{M}}_0)_i$ is the initial marking of p_i.

Example 5. Consider the TEGPS in Fig. 4 with PS of transition t_2 by the signal

$$\mathcal{S}_2(t) = \begin{cases} 1 & \text{if } t \in \{1 + 2j\}, \\ 0 & \text{otherwise.} \end{cases}$$

As $\omega = 2$, $I = 0$, $n_0 = 1$ according to Prop. 4 $v_{\mathcal{S}_2} = v_{t_2} = \delta^1\Delta_{2|2}\delta^{-1}$. For the path $(t_3, p_2)(p_2, t_2)$, the influence of t_3 on transition t_2 corresponds to an operator representation $v_{t_2}\delta^0\gamma^2 = v_{t_2}\gamma^2 = \delta^1\Delta_{2|2}\delta^{-1}\gamma^2$. Moreover, by assigning a dater function $u(t)$ (resp. $x_1(k), x_2(k), y(k)$) to transition t_1 (resp. t_2, t_3, t_4), the earliest functioning of the TEGPS is described in state space form $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{B}\boldsymbol{u}$; $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} \varepsilon & \delta^1\Delta_{2|2}\delta^{-1}\gamma^2 \\ \delta^1 & \varepsilon \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} \delta^1\Delta_{2|2}\delta^{-1} \\ \varepsilon \end{bmatrix},\ \boldsymbol{C} = \begin{bmatrix} \varepsilon & \delta^1 \end{bmatrix}.$$

◁

According to Theorem 1, the least solution of the equation $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{B}\boldsymbol{u}$ is $\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u}$. Therefore, the transfer function matrix $\boldsymbol{H}$ of a TEGPS can be obtained by $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{u} = \boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\boldsymbol{u}$. In order to compute this transfer function matrix, we have to perform addition, multiplication and the Kleene star operation of series in $\mathcal{T}_{per}[\![\gamma]\!]$. In the next section, we show how these operations between series in $\mathcal{T}_{per}[\![\gamma]\!]$ can be reduced to operations between matrices in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

4. CORE REPRESENTATION OF A SERIES IN $\mathcal{T}_{PER}[\![\gamma]\!]$

In this section, we propose a specific decomposition of series in $\mathcal{T}_{per}[\![\gamma]\!]$. We show that series $s \in \mathcal{T}_{per}[\![\gamma]\!]$ with period ω can always be represented as $s = \mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega$ where $\boldsymbol{Q}$ is a square matrix in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ of size $\omega \times \omega$, $\mathbf{m}_\omega$ is a row vector defined as

$$\mathbf{m}_\omega := \begin{bmatrix} \Delta_{\omega|1} & \delta^{-1}\Delta_{\omega|1} & \cdots & \delta^{1-\omega}\Delta_{\omega|1} \end{bmatrix}$$

and $\mathbf{b}_\omega$ is a column vector defined as

$$\mathbf{b}_\omega := \begin{bmatrix} \Delta_{1|\omega}\delta^{1-\omega} & \cdots & \Delta_{1|\omega}\delta^{-1} & \Delta_{1|\omega} \end{bmatrix}^T.$$

It is important to note that in this form the core matrix $\boldsymbol{Q}$ is a matrix in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. We first demonstrate how to obtain this form on a small example and then provide a formal method.

Example 6. Consider the following series in $\mathcal{T}_{per}[\![\gamma]\!]$,

$$s=\Delta_{2|2}\oplus\delta^1\Delta_{2|2}\delta^{-1}\oplus\delta^2\Delta_{2|2}\gamma^2(\delta^2\gamma^2)^*.$$

Because of $\Delta_{\omega|\varpi}=\Delta_{\omega|b}\Delta_{b|\varpi}$ (Remark 1) and $\delta^\omega\Delta_{\omega|\varpi}=\Delta_{\omega|\varpi}\delta^\varpi$ (7) this series can be rewritten as

$$s=\Delta_{2|1}\underbrace{\mathrm{e}}_{M_1}\Delta_{1|2}\oplus\delta^{-1}\Delta_{2|1}\underbrace{\delta^1}_{M_2}\Delta_{1|2}\delta^{-1}\oplus\Delta_{2|1}\underbrace{\delta^1\gamma^2(\delta^1\gamma^2)^*}_{S_1}\Delta_{1|2}.$$

Clearly M_1,M_2 and S_1 are elements in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. We now can rewrite s in the core representation,

$$s=\underbrace{\begin{bmatrix}\Delta_{2|1} & \delta^{-1}\Delta_{2|1}\end{bmatrix}}_{\mathbf{m}_2}\underbrace{\begin{bmatrix}\varepsilon & \mathrm{e}\oplus\delta^1\gamma^2(\delta^1\gamma^2)^*\\ \delta^1 & \varepsilon\end{bmatrix}}_{\boldsymbol{Q}}\underbrace{\begin{bmatrix}\Delta_{1|2}\delta^{-1}\\ \Delta_{1|2}\end{bmatrix}}_{\mathbf{b}_2},$$

which is in the required form. ◁

The algorithm to obtain the core form for an arbitrary ultimately cyclic series $s\in\mathcal{T}_{per}[\![\gamma]\!]$ is as follows. The ultimately cyclic series $s=\bigoplus_{i=1}^I v_i\gamma^{n_i}\oplus\bigoplus_{j=1}^J v'_j\gamma^{n'_j}(\delta^\tau\gamma^\nu)^*\in\mathcal{T}_{per}[\![\gamma]\!]$ is written such that all coefficients v_i and v'_j are represented with their least common period (Remark 2), i.e.,

$$s=\bigoplus_{l=1}^{L}\delta^{t_l}\Delta_{\omega|\omega}\delta^{t'_l}\gamma^{n_l}\oplus\bigoplus_{k=1}^{K}\delta^{\xi_k}\Delta_{\omega|\omega}\delta^{\xi'_k}\gamma^{n'_k}(\delta^\tau\gamma^\nu)^*.$$

Recall that $\Delta_{\omega|\varpi}=\Delta_{\omega|b}\Delta_{b|\varpi}$ (Remark 1) therefore,

$$s=\bigoplus_{l=1}^{L}\delta^{t_l}\Delta_{\omega|1}\Delta_{1|\omega}\delta^{t'_l}\gamma^{n_l}\oplus\bigoplus_{k=1}^{K}\delta^{\xi_k}\Delta_{\omega|1}\Delta_{1|\omega}\delta^{\xi'_k}\gamma^{n'_k}(\delta^\tau\gamma^\nu)^*.$$

Note that the δ^ω operator commutes with $\Delta_{\omega|\omega}$, i.e., $\delta^\omega\Delta_{\omega|\omega}=\Delta_{\omega|\omega}\delta^\omega$ (7). Moreover, we can always represent an ultimately cyclic series $s\in\mathcal{T}[\![\gamma]\!]$ such that τ is a multiple of ω, i.e., we can extend $(\gamma^\nu\delta^{\tilde{\tau}})^*$ such that, $\tau=\tilde{\tau}l=lcm(\tilde{\tau},\omega)$. Hence,

$$\begin{aligned}(\gamma^\nu\delta^{\tilde{\tau}})^*&=(\mathrm{e}\oplus\gamma^\nu\delta^{\tilde{\tau}}\oplus\cdots\oplus\gamma^{(l-1)\nu}\delta^{(l-1)\tilde{\tau}})(\gamma^{l\nu}\delta^{l\tilde{\tau}})^*,\\&=(\mathrm{e}\oplus\gamma^\nu\delta^{\tilde{\tau}}\oplus\cdots\oplus\gamma^{(l-1)\nu}\delta^{(l-1)\tilde{\tau}})(\gamma^{l\nu}\delta^{\tau})^*.\end{aligned}$$

Therefore, in the following we assume $\tau/\omega\in\mathbb{N}$, thus $\Delta_{1|\omega}(\delta^\tau\gamma^\nu)^*=(\delta^{\tau/\omega}\gamma^\nu)^*\Delta_{1|\omega}$. This leads to

$$s=\bigoplus_{l=1}^{L}\delta^{\tilde{t}_l}\Delta_{\omega|1}\underbrace{\delta^{\hat{t}_l}\gamma^{n_l}}_{M_l}\Delta_{1|\omega}\delta^{\tilde{t}'_l}\oplus\bigoplus_{k=1}^{K}\delta^{\tilde{\xi}_k}\Delta_{\omega|1}\underbrace{\delta^{\hat{\xi}_k}\gamma^{n'_k}(\delta^{\tau/\omega}\gamma^\nu)^*}_{S_k}\Delta_{1|\omega}\delta^{\tilde{\xi}'_k},$$

with $-\omega<\tilde{t}_l,\tilde{t}'_l,\tilde{\xi}_k,\tilde{\xi}'_k\leq 0$. In this representation M_l are monomials and S_k are series in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$. Moreover, the entries of the $\mathbf{b}_\omega$-vector appear on the right and the entries of the $\mathbf{m}_\omega$-vector appear on the left of monomial M_l (resp. series S_k). For a given s we denote the set of monomials by $\mathscr{M}=\{M_1,\cdots,M_L\}$ and the set of series by $\mathscr{S}=\{S_1,\cdots,S_K\}$. Furthermore, the subsets $\mathscr{M}_{i,j}$ (resp. $\mathscr{S}_{i,j}$) are defined as $\forall i,j\in\{0,\cdots,\omega-1\}$,

$$\mathscr{M}_{i,j}:=\{M_l\in\mathscr{M}\,|\,\delta^{-i}\Delta_{\omega|1}M_l\Delta_{1|\omega}\delta^{-j}\in\bigoplus_{l=1}^{L}\delta^{\tilde{t}_l}\Delta_{\omega|1}M_l\Delta_{1|\omega}\delta^{\tilde{t}'_l}\},$$

$$\mathscr{S}_{i,j}:=\{S_k\in\mathscr{S}\,|\,\delta^{-i}\Delta_{\omega|1}S_k\Delta_{1|\omega}\delta^{-j}\in\bigoplus_{k=1}^{K}\delta^{\tilde{\xi}_k}\Delta_{\omega|1}S_k\Delta_{1|\omega}\delta^{\tilde{\xi}'_k}\}.$$

The element $(\boldsymbol{Q})_{i+1,\omega-j}$ of the core matrix is then obtained by

$$(\boldsymbol{Q})_{i+1,\omega-j}=\bigoplus_{M\in\mathscr{M}_{i,j}}M\oplus\bigoplus_{S\in\mathscr{S}_{i,j}}S.$$

Hence, any ultimately cyclic series $s\in\mathcal{T}_{per}[\![\gamma]\!]$ can be expressed by $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega$. Let us note that the core $\boldsymbol{Q}$ of a series $s\in\mathcal{T}_{per}[\![\gamma]\!]$ is not unique, in other words we can express the same series with different cores, i.e., we may have $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega=\mathbf{m}_\omega\tilde{\boldsymbol{Q}}\mathbf{b}_\omega$ with $\boldsymbol{Q},\tilde{\boldsymbol{Q}}\in\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\omega\times\omega}$ but $\boldsymbol{Q}\neq\tilde{\boldsymbol{Q}}$. We illustrate this in the following example.

Example 7. Recall the series $s=\Delta_{2|2}\oplus\delta^1\Delta_{2|2}\delta^{-1}\oplus\delta^2\gamma^2(\delta^2\gamma^2)^*\Delta_{2|2}$ given in Example 6. s can be expressed by $\mathbf{m}_2\tilde{\boldsymbol{Q}}\mathbf{b}_2$ where,

$$\tilde{\boldsymbol{Q}}=\begin{bmatrix}\mathrm{e} & \mathrm{e}\oplus\delta^1\gamma^2(\delta^1\gamma^2)^*\\ \delta^1 & \varepsilon\end{bmatrix}.$$

Clearly $\tilde{\boldsymbol{Q}}\succeq\boldsymbol{Q}$ and $\tilde{\boldsymbol{Q}}\neq\boldsymbol{Q}$ (see Example 6). However, $\tilde{\boldsymbol{Q}}$ is a valid core of s since:

$$\mathbf{m}_2\boldsymbol{Q}_2\mathbf{b}_2=\mathbf{m}_2\begin{bmatrix}\Delta_{1|2}\delta^{-1}\oplus\Delta_{1|2}\oplus\delta^1\gamma^2(\delta^1\gamma^2)^*\Delta_{1|2}\\ \delta^1\Delta_{1|2}\delta^{-1}\end{bmatrix},$$

recall $\Delta_{1|2}\delta^{-1}\oplus\Delta_{1|2}=\Delta_{1|2}(\delta^{-1}\oplus\delta^0)=\Delta_{1|2}$, (5),

$$\begin{aligned}&=\begin{bmatrix}\Delta_{2|1} & \delta^{-1}\Delta_{2|1}\end{bmatrix}\begin{bmatrix}\Delta_{1|2}\oplus\delta^1\gamma^2(\delta^1\gamma^2)^*\Delta_{1|2}\\ \delta^1\Delta_{1|2}\delta^{-1}\end{bmatrix}\\&=\Delta_{2|1}\Delta_{1|2}\oplus\Delta_{2|1}\delta^1\gamma^2(\delta^1\gamma^2)^*\Delta_{1|2}\oplus\delta^{-1}\Delta_{2|1}\delta^1\Delta_{1|2}\delta^{-1}\\&=\Delta_{2|2}\oplus\delta^1\Delta_{2|2}\delta^{-1}\oplus\delta^2\gamma^2(\delta^2\gamma^2)^*\Delta_{2|2}.\end{aligned}$$

◁

To show how the core form can be used to perform basic operations between ultimately cyclic series in $\mathcal{T}_{per}[\![\gamma]\!]$ we first elaborate some properties of the $\mathbf{m}_\omega$-vector and $\mathbf{b}_\omega$-vector. The scalar product $\mathbf{m}_\omega\mathbf{b}_\omega$ of these two vectors is the identity e, because of Remark 3 and the fact that $\Delta_{\omega|1}\Delta_{1|\omega}=\Delta_{\omega|\omega}$ (Remark 1),

$$\begin{aligned}\mathbf{m}_\omega\otimes\mathbf{b}_\omega&=\delta^0\Delta_{\omega|1}\Delta_{1|\omega}\delta^{1-\omega}\oplus\cdots\oplus\delta^{1-\omega}\Delta_{\omega|1}\Delta_{1|\omega}\delta^0\\&=\delta^0\Delta_{\omega|\omega}\delta^{1-\omega}\oplus\cdots\oplus\delta^{1-\omega}\Delta_{\omega|\omega}\delta^0=\mathrm{e},\end{aligned}\tag{16}$$

as discussed in Example 2. The dyadic product $\mathbf{b}_\omega\otimes\mathbf{m}_\omega$ is a square matrix in $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ denoted by $\mathbf{N}$. For $i,j\in\{1,\cdots,\omega\}$, the entry $(\mathbf{b}_\omega\otimes\mathbf{m}_\omega)_{i,j}$ is given by,

$$(\mathbf{N})_{i,j}=(\mathbf{b}_\omega\otimes\mathbf{m}_\omega)_{i,j}=\Delta_{1|\omega}\delta^{(i-j)+(1-\omega)}\Delta_{\omega|1}.$$

Then, because of $\Delta_{1|\omega}\delta^{-\omega}=\delta^{-1}\Delta_{1|\omega}$ and $\Delta_{1|\omega}\delta^{-n}\Delta_{\omega|1}=\Delta_{1|1}=\mathrm{e}$ for $-\omega<n\leq 0$, see Remark 1,

$$\mathbf{N}=\mathbf{b}_\omega\otimes\mathbf{m}_\omega=\begin{bmatrix}\mathrm{e} & \delta^{-1} & \cdots & \delta^{-1}\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \delta^{-1}\\ \mathrm{e} & \cdots & \cdots & \mathrm{e}\end{bmatrix}.\tag{17}$$

Proposition 5. For the $\mathbf{N}$ matrix the following relations hold $\mathbf{N}\otimes\mathbf{N}=\mathbf{N}$; $\mathbf{N}\otimes\mathbf{b}_\omega=\mathbf{b}_\omega$; $\mathbf{m}_\omega\otimes\mathbf{N}=\mathbf{m}_\omega$.

Proof.

$$\mathbf{N}\otimes\mathbf{N}=\mathbf{b}_\omega\otimes\mathbf{m}_\omega\otimes\mathbf{b}_\omega\otimes\mathbf{m}_\omega=\mathbf{b}_\omega\otimes e\otimes\mathbf{m}_\omega=\mathbf{N},$$
$$\mathbf{N}\otimes\mathbf{b}_\omega=\mathbf{b}_\omega\otimes\mathbf{m}_\omega\otimes\mathbf{b}_\omega=\mathbf{b}_\omega\otimes e=\mathbf{b}_\omega,$$
$$\mathbf{m}_\omega\otimes\mathbf{N}=\mathbf{m}_\omega\otimes\mathbf{b}_\omega\otimes\mathbf{m}_\omega=e\otimes\mathbf{m}_\omega=\mathbf{m}_\omega.$$

□

4.1 Operations between Core Matrices

To perform addition and multiplication of two ultimately cyclic series $s_1=\mathbf{m}_{\omega_1}\boldsymbol{Q}_1\mathbf{b}_{\omega_1}$, $s_2=\mathbf{m}_{\omega_2}\boldsymbol{Q}_2\mathbf{b}_{\omega_2}\in\mathcal{T}_{per}[\![\gamma]\!]$ in the core form, it is necessary to express the core matrices $\boldsymbol{Q}_1\in\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\omega_1\times\omega_1}$ and $\boldsymbol{Q}_2\in\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{\omega_2\times\omega_2}$ with equal dimensions. This is possible by expressing both series with their least common period $\omega=lcm(\omega_1,\omega_2)$, see the following proposition.

Proposition 6. A series $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega\in\mathcal{T}_{per}[\![\gamma]\!]$ can be expressed with a multiple period $n\omega$ by extending the core matrix $\boldsymbol{Q}$, i.e., $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega=\mathbf{m}_{n\omega}\boldsymbol{Q}'\mathbf{b}_{n\omega}$, where $\boldsymbol{Q}'\in\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]^{n\omega\times n\omega}$ and is given by

$$\boldsymbol{Q}'=\begin{bmatrix}\Delta_{1|n}\delta^{1-n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\Delta_{n|1} & \cdots & \Delta_{1|n}\delta^{1-n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\delta^{1-n}\Delta_{n|1}\\ \vdots & & \vdots\\ \Delta_{1|n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\Delta_{n|1} & \cdots & \Delta_{1|n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\delta^{1-n}\Delta_{n|1}\end{bmatrix}.$$

Proof. See Appendix A. □

Proposition 7. (Sum of series).
Let $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega$, $s'=\mathbf{m}_\omega\boldsymbol{Q}'\mathbf{b}_\omega\in\mathcal{T}_{per}[\![\gamma]\!]$, the sum $s\oplus s'=\mathbf{m}_\omega\boldsymbol{Q}''\mathbf{b}_\omega$, where $\boldsymbol{Q}''=\boldsymbol{Q}\oplus\boldsymbol{Q}'$.

Proof. We have, $\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega\oplus\mathbf{m}_\omega\boldsymbol{Q}'\mathbf{b}_\omega=\mathbf{m}_\omega(\boldsymbol{Q}\oplus\boldsymbol{Q}')\mathbf{b}_\omega$. □

Proposition 8. (Product of series).
Let $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega$, $s'=\mathbf{m}_\omega\boldsymbol{Q}'\mathbf{b}_\omega\in\mathcal{T}_{per}[\![\gamma]\!]$, the product $s\otimes s'=\mathbf{m}_\omega\boldsymbol{Q}''\mathbf{b}_\omega$, where $\boldsymbol{Q}''=\boldsymbol{Q}\mathbf{N}\boldsymbol{Q}'$.

Proof. Recall that $\mathbf{b}_\omega\mathbf{m}_\omega=\mathbf{N}$, therefore $\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega\mathbf{m}_\omega\boldsymbol{Q}'\mathbf{b}_\omega=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{N}\boldsymbol{Q}'\mathbf{b}_\omega$. □

Proposition 9. (Kleene star of series). Let $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega\in\mathcal{T}_{per}[\![\gamma]\!]$. Then, s^* can be obtained by

$$s^*=\mathbf{m}_\omega(\boldsymbol{Q}\mathbf{N})^*\mathbf{b}_\omega. \tag{18}$$

Proof. Recall that $\mathbf{b}_\omega=\mathbf{N}\mathbf{b}_\omega$ and therefore $s=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{b}_\omega=\mathbf{m}_\omega\boldsymbol{Q}\mathbf{N}\mathbf{b}_\omega$. In the core form, the Kleene star of a series $s\in\mathcal{T}_{per}[\![\gamma]\!]$ can be written as

$$s^*=e\oplus\mathbf{m}_\omega\boldsymbol{Q}\mathbf{N}\mathbf{b}_\omega\oplus\mathbf{m}_\omega\boldsymbol{Q}\mathbf{N}\mathbf{b}_\omega\mathbf{m}_\omega\boldsymbol{Q}\mathbf{N}\mathbf{b}_\omega\oplus\cdots$$

Recall that $\boldsymbol{Q}$ is a square matrix, $e=\mathbf{m}_\omega\mathbf{b}_\omega$ (16), $\mathbf{N}=\mathbf{b}_\omega\mathbf{m}_\omega$ (17) and $\mathbf{N}\mathbf{N}=\mathbf{N}$ (Prop. 5),

$$\begin{aligned}s^*&=\mathbf{m}_\omega\mathbf{b}_\omega\oplus\mathbf{m}_\omega\boldsymbol{Q}\mathbf{N}\mathbf{b}_\omega\oplus\mathbf{m}_\omega\boldsymbol{Q}\mathbf{N}\boldsymbol{Q}\mathbf{N}\mathbf{b}_\omega\oplus\cdots\\&=\mathbf{m}_\omega(\boldsymbol{I}\oplus\boldsymbol{Q}\mathbf{N}\oplus(\boldsymbol{Q}\mathbf{N})^2\oplus\cdots)\mathbf{b}_\omega\\&=\mathbf{m}_\omega(\boldsymbol{Q}\mathbf{N})^*\mathbf{b}_\omega.\end{aligned}$$

□

Due to Prop. 7, Prop. 8 and Prop. 9, it is clear that the sum, product and Kleene star operation of ultimately cyclic series in $\mathcal{T}_{per}[\![\gamma]\!]$ can be computed based on the core matrices $\boldsymbol{Q}\in\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ of the series. Finally let us note that this core-form of series $s\in\mathcal{T}_{per}[\![\gamma]\!]$ is similar to the core-form of series $s\in\mathcal{E}_{per}[\![\delta]\!]$ see, Trunk et al. (2017). The dioid $\mathcal{T}_{per}[\![\gamma]\!]$ with periodic time-operators can be seen as the counter part of the dioid $\mathcal{E}_{per}[\![\delta]\!]$, introduced in Cottenceau et al. (2014), with periodic event-operators. The dioid $\mathcal{E}_{per}[\![\delta]\!]$ is useful to obtain transfer function matrices for Weight-Balanced Timed Event Graphs (WBTEGs).

4.2 Transfer Functions of a TEGPS in $\mathcal{T}_{per}[\![\gamma]\!]$

Recall the state space form of a TEGPS, $\boldsymbol{x}=\boldsymbol{A}\boldsymbol{x}\oplus\boldsymbol{B}\boldsymbol{u}$; $\boldsymbol{y}=\boldsymbol{C}\boldsymbol{x}$, Section 3.4, (15).

Theorem 2. For a g-input p-output TEGPS with *periodic* PSs (Definition 2) the transfer function matrix is given by $\boldsymbol{H}=\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}\in\mathcal{T}_{per}[\![\gamma]\!]^{p\times g}$. Moreover, the entries of the transfer function matrix $\boldsymbol{H}$ are ultimately cyclic series in $\mathcal{T}_{per}[\![\gamma]\!]$. ◁

Proof. A *periodic* PS of a transition by a periodic signal refers to a periodic T-operator Prop. 4. As every monomial/polynomial in $\mathcal{T}_{per}[\![\gamma]\!]$ is a specific ultimately cyclic series, the entries of the $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ matrices are composed of ultimately cyclic series in $\mathcal{T}_{per}[\![\gamma]\!]$. The sum (resp. product, Kleene star) of ultimately cyclic series in $\mathcal{T}_{per}[\![\gamma]\!]$ are again ultimately cyclic series in $\mathcal{T}_{per}[\![\gamma]\!]$, see Prop. 7 (resp. Prop. 8, Prop. 9). Thus the transfer matrix $\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}$ is also composed of ultimately cyclic series in $\mathcal{T}_{per}[\![\gamma]\!]$. □

Example 8. Consider the TEGPS in Example 5 the transfer function is obtained by

$$\begin{aligned}h=\boldsymbol{C}\boldsymbol{A}^*\boldsymbol{B}&=\begin{bmatrix}\varepsilon & \delta^1\end{bmatrix}\begin{bmatrix}\varepsilon & \delta^1\Delta_{2|2}\delta^{-1}\gamma^2\\ \delta^1 & \varepsilon\end{bmatrix}^*\begin{bmatrix}\delta^1\Delta_{2|2}\delta^{-1}\\ \varepsilon\end{bmatrix}\\&=\delta^1(\boldsymbol{A}^*)_{2,1}\delta^1\Delta_{2|2}\delta^{-1},\end{aligned}$$

where $(\boldsymbol{A}^*)_{2,1}=(\delta^2\Delta_{2|2}\delta^{-1}\gamma^2)^*\delta^1$. To express h as an ultimately cyclic series we rewrite $(\boldsymbol{A}^*)_{2,1}$ in the core-form and compute the Kleene star with the toolbox minmaxGD Hardouin et al. (2009). Recall Prop. 9, therefore

$$\begin{aligned}(\boldsymbol{A}^*)_{2,1}&=\Big(\mathbf{m}_2\begin{bmatrix}\gamma^2\delta^1 & \varepsilon\\ \varepsilon & \varepsilon\end{bmatrix}\mathbf{b}_2\Big)^*\delta^1=\mathbf{m}_2\Big(\begin{bmatrix}\gamma^2\delta^1 & \varepsilon\\ \varepsilon & \varepsilon\end{bmatrix}\mathbf{N}\Big)^*\mathbf{b}_2\delta^1\\&=\mathbf{m}_2\begin{bmatrix}(\gamma^2\delta^1)^* & \gamma^2(\gamma^2\delta^1)^*\\ \varepsilon & e\end{bmatrix}\mathbf{b}_2\delta^1.\end{aligned}$$

After multiplication we obtain, $h=\delta^3\Delta_{2|2}\delta^{-1}(\gamma^2\delta^2)^*\oplus\delta^2\gamma^2\Delta_{2|2}\delta^{-1}(\gamma^2\delta^2)^*\oplus\delta^2\Delta_{2|2}\delta^{-1}=\delta^3\Delta_{2|2}\delta^{-1}(\gamma^2\delta^2)^*$. For the given TEGPS, this transfer function is useful to compute the output y to a given input u, where y and u represent the dater functions associated with the transitions t_1 and t_4. For instance consider the input dater function

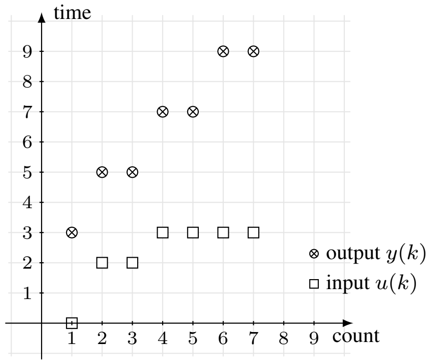
$$u(k)=\begin{cases}-\infty & \text{for } k<1;\\ 0 & \text{for } k=1;\\ 2 & \text{for } k=2,3;\\ 3 & \text{for } k=4,5,6,7;\\ \infty & \text{for } k\geq 8.\end{cases}$$

The output dater function to this input is given by

$$\begin{aligned}y(k)&=((\delta^3\Delta_{2|2}\delta^{-1}(\gamma^2\delta^2)^*)u)(k)\\&=((\delta^3\Delta_{2|2}\delta^{-1}u)(k)\oplus(\delta^5\Delta_{2|2}\delta^{-1}\gamma^2u)(k)\\&\quad\oplus(\delta^7\Delta_{2|2}\delta^{-1}\gamma^4u)(k)\oplus\cdots)\\&=\max\Big(3+\Big\lceil\frac{u(k)-1}{2}\Big\rceil 2, 5+\Big\lceil\frac{u(k-2)-1}{2}\Big\rceil 2,\\&\quad 7+\Big\lceil\frac{u(k-4)-1}{2}\Big\rceil 2,\cdots\Big).\end{aligned}$$

Therefore,

$$y(k) = \begin{cases} -\infty & \text{for } k < 1; \\ 3 & \text{for } k = 1; \\ 5 & \text{for } k = 2, 3; \\ 7 & \text{for } k = 4, 5; \\ 9 & \text{for } k = 6, 7; \\ \infty & \text{for } k \geq 8. \end{cases}$$



$\triangleleft$

Fig. 5. System response y to the input u.

5. CONCLUSION

In this paper, we introduce algebraic tools to obtain transfer function matrices for a subclass of TEGsPS, where the partial synchronization of transitions are characterized by periodic signals. We introduce a dioid called $\mathcal{T}_{per}[\![\gamma]\!]$ with periodic-time operators and show that all relevant operations on ultimately cyclic series $s \in \mathcal{T}_{per}[\![\gamma]\!]$ in this dioid can be reduced to operations on matrices in the subdioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. An advantage of this approach is that the already existing tools for standard TEGs in the dioid $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ can be applied to the more general class of TEGsPS. Based on these results, many control concepts already introduced for TEGs, such as model-reference control Maia et al. (2003), observer based control Hardouin et al. (2017), etc. can be generalized for TEGsPS. This will be the topic of further work.

Appendix A. PROOFS

Proof. of Prop. 6. A series $s \in \mathcal{T}_{per}[\![\gamma]\!]$ with period ω can be written as,

$$s = \mathbf{m}_\omega \boldsymbol{Q} \mathbf{b}_\omega = \mathbf{m}_{n\omega} \underbrace{\mathbf{b}_{n\omega} \mathbf{m}_\omega \boldsymbol{Q} \mathbf{b}_\omega \mathbf{m}_{n\omega}}_{\hat{\boldsymbol{Q}}'} \mathbf{b}_{n\omega}.$$

Since, $\Delta_{1|n\omega}\delta^{1-n\omega} = \Delta_{1|n}\Delta_{1|\omega}\delta^{-\omega(n-1)}\delta^{1-\omega}$ $= \Delta_{1|n}\delta^{1-n}\Delta_{1|\omega}\delta^{1-\omega}$ then

$$\mathbf{b}_{n\omega} = \begin{bmatrix} \begin{bmatrix} \Delta_{1|n}\delta^{1-n}\Delta_{1|\omega}\delta^{1-\omega} \\ \vdots \\ \Delta_{1|n}\delta^{1-n}\Delta_{1|\omega} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \Delta_{1|n}\Delta_{1|\omega}\delta^{1-\omega} \\ \vdots \\ \Delta_{1|n}\Delta_{1|\omega} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \Delta_{1|n}\delta^{1-n}\mathbf{b}_\omega \\ \vdots \\ \Delta_{1|n}\mathbf{b}_\omega \end{bmatrix}.$$

This leads to

$$\mathbf{b}_{n\omega}\mathbf{m}_\omega = \begin{bmatrix} \Delta_{1|n}\delta^{1-n}\mathbf{N} \\ \vdots \\ \Delta_{1|n}\mathbf{N} \end{bmatrix}.$$

Respectively $\mathbf{b}_\omega \mathbf{m}_{n\omega} = \begin{bmatrix} \mathbf{N}\Delta_{n|1} & \cdots & \mathbf{N}\delta^{1-n}\Delta_{n|1} \end{bmatrix}$. Finally we obtain

$$\begin{aligned} \boldsymbol{Q}' &= \begin{bmatrix} \Delta_{1|n}\delta^{1-n}\mathbf{N} \\ \vdots \\ \Delta_{1|n}\mathbf{N} \end{bmatrix} \boldsymbol{Q} \begin{bmatrix} \mathbf{N}\Delta_{n|1} & \cdots & \mathbf{N}\delta^{1-n}\Delta_{n|1} \end{bmatrix}, \\ &= \begin{bmatrix} \Delta_{1|n}\delta^{1-n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\Delta_{n|1} & \cdots & \Delta_{1|n}\delta^{1-n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\delta^{1-n}\Delta_{n|1} \\ \vdots & & \vdots \\ \Delta_{1|n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\Delta_{n|1} & \cdots & \Delta_{1|n}\mathbf{N}\boldsymbol{Q}\mathbf{N}\delta^{1-n}\Delta_{n|1} \end{bmatrix}. \end{aligned}$$

The extended core is a matrix in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, since $\Delta_{1|n}\delta^{\tau}\Delta_{n|1}$ $= \delta^{\lceil \tau/n \rceil n}$, see Remark 1. $\square$

REFERENCES

Baccelli, F., Cohen, G., Olsder, G., and Quadrat, J. (1992). *Synchronization and Linearity: An Algebra for Discrete Event Systems*. John Wiley and Sons, New York.

Cottenceau, B., Hardouin, L., and Boimond, J.L. (2014). Modeling and control of weight-balanced timed event graphs in dioids. *IEEE Trans. Autom. Control*, 59(5), 1219–1231. doi: 10.1109/TAC.2013.2294822.

David-Henriet, X., Raisch, J., Hardouin, L., and Cottenceau, B. (2014). Modeling and control for max-plus systems with partial synchronization. In *Proceedings of the 12th IFAC-IEEE International Workshop on Discrete Event Systems (WODES), Paris, France*, 105–110.

David-Henriet, X., Raisch, J., Hardouin, L., and Cottenceau, B. (2015). Modeling and control for (max, +)-linear systems with set-based constraints. In *Automation Science and Engineering (CASE), 2015 IEEE International Conference on*, 1369–1374. doi:10.1109/CoASE.2015.7294289.

David-Henriet, X., Hardouin, L., Raisch, J., and Cottenceau, B. (2016). Model predictive control for discrete event systems with partial synchronization. *Automatica*, 70, 9 – 13.

De Schutter, B. and van den Boom, T.J.J. (2003). MPC for discrete-event systems with soft and hard synchronization constraints. *International Journal of Control*, 76(1), 82–94. doi:10.1080/0020717021000049188.

Gaubert, S. and Klimann, C. (1991). Rational computation in dioid algebra and its application to performance evaluation of discrete event systems. In *Algebraic computing in control*, 241–252. Springer.

Hardouin, L., Shang, Y., Maia, C.A., and Cottenceau, B. (2017). Observer-based controllers for max-plus linear systems. *IEEE Transactions on Automatic Control*, 62(5), 2153–2165. doi:10.1109/TAC.2016.2604562.

Hardouin, L., Le Corronc, E., and Cottenceau, B. (2009). Minmaxgd a Software Tools to Handle Series in (max,+) Algebra. In *SIAM Conference on Computational Science and Engineering*. Miami, USA.

Heidergott, B., Olsder, G., and van der Woude, J. (2005). *Max Plus at Work : Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and Its Applications (Princeton Series in Applied Mathematics)*. Princeton University Press.

Maia, C.A., Hardouin, L., Santos-Mendes, R., and Cottenceau, B. (2003). Optimal closed-loop control of timed event graphs in dioids. *IEEE Trans. Autom. Control*, 48(12), 2284–2287.

Trunk, J., Cottenceau, B., Hardouin, L., and Raisch, J. (2017). Model decomposition of weight-balanced timed event graphs in dioids: Application to control synthesis. In *Proceedings of the 20th IFAC World Congress*, 13995–14002.