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A METHOD OF OBTAINING A UNIFORM ELECTRIC FIELD

by

A. V. Dralle, Carnegie Institute of Technology, Pittsburg, Pa.

and D. L. Waidelich, Professor of Electrical Engineering,
University of Missouri

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A METHOD OF OBTAINING A UNIFORM ELECTRIC FIELD*

A. V. Dralle
Carnegie Institute of Technology
Pittsburgh, Pa.

and

D. L. Waidelich
University of Missouri
Columbia, Mo.

Abstract

Electric field meters are used for the measurement of the electric field at the surface of the earth, on airplanes and on satellites. For research work and calibration of the meters, a uniform electric field facility is necessary. The accessibility of the usual parallel plane configuration is very poor, so it was decided to investigate the possibilities of two and four-ring systems to produce very uniform electric fields. An analysis of these systems was made and the results are presented as tables and curves of the geometrical dimensions, the charge ratio, and size of the volume of uniformity.

Summary

Electric field meters[1] are useful in the determination of the electric field at the surface of the earth,[2] and in measuring the field on airplanes[3] and on satellites.[4] In research work on the field meters and in calibrating them it is necessary to have a facility for producing a uniform electrostatic field whose magnitude can be varied at will. The volume of uniformity should be sufficiently large to contain the instruments to be measured and, in addition, the distances from the conductors of the facility to the instruments should be large enough so that the charges on the instruments do not sensibly disturb the distribution of charges on the conductors.

Two parallel plane conductors make the easiest facility[5] to construct although fringing near the edges of the conductors contributes to the non-uniformity of the field. Access to the volume of uniformity is difficult because the plane conductors are in the way. It was decided to investigate the possibilities of using charged circular rings to produce a uniform electric field in an analogous way to that in which current loops are used to produce uniform magnetic fields.[6]

In producing the electric fields, a two-ring system would appear to be the simplest symmetrical system and still would offer a considerable amount of accessibility. If a more uniform electric field were needed or a larger volume of uniform field were required for a given maximum diameter of the rings, then the next possibility appeared to be the symmetrical four-ring system, although it has the disadvantage of possibly requiring two different potentials instead of the one needed for the two-ring system. The purpose of this paper is to present some theory and computed results for the two-ring and four-ring systems particularly in regard to the dimensions and charges required.

The potential and field components of the two-ring system were expressed in terms of an infinite series of Legendre functions. By symmetry, the odd terms of the series for the field components dropped out and by a suitable choice of the spacing and diameter of the rings, the squared term of the series could be made zero leaving a fourth order system. This is similar to the requirement for a Helmholtz two-coil system in uniform magnetic field work. The constant term in the series then gave the magnitude of the electric field at the center of the system, and from the fourth degree term or from the original expression for the field, the size of the volume of uniformity could be obtained for a given percent uniformity.

A similar analysis was made for the four-ring system, although in this case, three terms of the series could be made zero leading to an eighth order system of much higher uniformity. Two separate cases were studied, that in which the charges on both pairs of rings had the same sign and that with one pair having charge of opposite sign to that of the other pair. The parameters computed are shown as tables and as curves of the geometrical distances, magnitude of the charges and the volume of uniformity. The measurement and calculation of the capacitances of the rings are discussed so that the required charge ratio may be obtained.

Single Ring

The geometry for a single charged ring is shown in Figure 1. Let Q be the uniformly distributed charge on the ring. The potential V on the Z axis at the point A is

$$V = \frac{Q}{4\pi \epsilon d} \tag{1}$$

where ϵ is the permittivity of the surrounding medium. The inverse distance (d^{-1}) may be expressed as a series of Legendre polynomials[7] for $Z < B$:

*This work was performed at the Goddard Space Flight Center, in the 1965 Summer Workshop.

$$d^{-1} = \frac{1}{B}\sum_{n=0}^{\infty} \left(\frac{Z}{B}\right)^n P_n(\cos\theta) \tag{2}$$

and thus the potential on the axis for a single ring may be written:

$$V = \frac{Q}{4\pi\epsilon B} \sum_{n=0}^{\infty} \left(\frac{Z}{B}\right)^n P_n(\cos\theta) \tag{3}$$

On the axis, the electric field intensity in the axial direction may be obtained by use of the negative gradient of the potential

$$E_z = -\frac{\partial V}{\partial Z} = -\frac{Q}{4\pi\epsilon B} \sum_{n=0}^{\infty} \frac{nz^{n-1}}{B^n} P_n(\cos\theta) \tag{4}$$

Double-Ring System

The double-ring system is shown in Figure 2 with both rings identical in size and shape and made from conducting material. The charge (+Q) on one ring however is equal in magnitude but opposite in sign to that of the other ring. From Equation (4) the electric field intensity on the axis is

$$E_z = -\frac{Q}{2\pi\epsilon B} \sum_{n=1,3,5,\cdots}^{\infty} \frac{nz^{n-1}}{B^n} P_n(\cos\theta) \tag{5}$$

and the most uniform field at the origin will result when the n = 3 term of (5) is made zero. This results in

$$0 = P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{6}$$

where $x = \cos\theta$. Then

$$x = \cos\theta = \sqrt{\frac{3}{5}} = \frac{1}{5}\sqrt{15} = \frac{D}{B} = \frac{D}{\sqrt{D^2 + R^2}}$$

or

$$D = \frac{1}{2}\sqrt{6}\,R = 1.225\,R \tag{7}$$

This compares with the condition D = 0.5R employed as the condition for maximum uniformity near the origin for the magnetic field of a Helmholtz coil pair.

The electric field intensity at the origin is from (5)

$$E_z = -\frac{Q\cos\theta}{2\pi\epsilon B^2} = -\frac{QD}{2\pi\epsilon B^3} \tag{8}$$

By the use of (5) and (7) the electric field intensity along the axis is

$$E_z = -\frac{QD}{2\pi\epsilon B^3}\left[1 - 0.972\left(\frac{z}{D}\right)^4 + \cdots\right] \tag{9}$$

Thus for a one percent change in E_z along the axis

$$\left|0.972\left(\frac{Z}{D}\right)^4\right| = 0.01$$

or

$$|z| = 0.3182\,D$$

The above system might be termed a fourth order system since the first non-vanishing term in (9) beyond the constant term is of the fourth order in Z.

The charge on each ring could be calculated from the applied potential and the calculated[7] or measured capacitance. The rings should be sufficiently removed from any other conductors so the effect of the other conductors need not be considered.

Four-Ring System

If a uniformity greater than that obtained with a double-ring system is desired, a four ring system should be used. The geometry of such a system is shown in Figure 3 with the rings arranged symmetrically about the center. The charges on the No. 1 rings are assumed to be Q_1 in magnitude while those on the No. 2 rings are assumed to be Q_2. Then the axial field is

$$E_z = \sum_{n=0}^{\infty} a_{2n} z^{2n} \tag{10}$$

where

$$a_{2n} = \frac{2n+1}{2\pi\epsilon}\left[\frac{Q_1}{B_1^{2n+2}} P_{2n+1}(x_1) + \frac{Q_2}{B_2^{2n+2}} P_{2n+1}(x_2)\right]$$

$$x_1 = \cos\theta_1 = \frac{D_1}{B_1}$$

$$x_2 = \cos\theta_2 = \frac{D_2}{B_2}$$

An analysis was made as outlined in the Appendix in such a manner as to make the coefficients A_2, A_4, and A_6 zero thus resulting in an eighth order system. The three equations obtained by making A_2, A_4, and A_6 zero were solved for the ratios $q = (Q_1/Q_2)$ and $b = (B_1/B_2)$ and from these was obtained a function of $x = \cos\theta$

$$f(x) = \frac{P_3(x)\, P_7(x)}{\left[P_5(x)\right]^2} \qquad (11)$$

where the P's are Legendre polynomials as given in the Appendix. A sketch of f(x) from Equation (11) is shown in Figure 4 where the A's indicate the poles and zeros of f(x). The two poles, $A_2 = 0.53847$ and $A_5 = 0.90618$, are the roots of $P_5(x) = 0$. The four zeros of f(x) are $A_1 = 0.40585$, $A_3 = 0.74153$ and $A_6 = 0.94911$ which are the roots of $P_7(x) = 0$ and $A_4 = 0.77460$ which is the root of $P_3(x) = 0$.

A solution will result if $f(x_1) = f(x_2)$ where $0 < x_1 < x < 1$ and $b^2 > 0$. For positive values of q, the corresponding values of x_1 and x_2 are shown in Figure 5(a). Notice that x_1 ranges from A_1 to A_4 while x_2 ranges from A_4 to A_6. At $x_1 = A_7 = 0.758$, x_2 has its maximum value of 0.9494. Table I also presents some corresponding values of x_1 and x_2 for $q > 0$. If the x_2 corresponding to a given x_1 is desired to a greater accuracy than that obtainable from Figure 5(a) or by interpolation from Table I, the use of Equation (11) along with the aid of a computer will readily produce the required x_2 to the desired number of significant figures. For negative values of q, the corresponding values of x_1 and x_2 are shown in Figure 5(b). Notice that x_1 ranges the small amount from A_3 to A_7 while x_2 ranges a correspondingly small amount from A_7 to A_4. Table II also presents some corresponding values of x_1 and x_2 for $q < 0$.

Once the values of x_1 and x_2 are known, the charge ratio q and all of the dimensions relative to some specified dimension may be calculated. Thus in Tables I and II q may be obtained by the use of

$$b = \left[\frac{P_3(x_2)\, P_5(x_1)}{P_3(x_1)\, P_5(x_2)}\right]^{1/2} \qquad (12)$$

and

$$q = -b^4 \frac{P_3(x_2)}{P_3(x_1)} \qquad (13)$$

Similarly, the dimensions shown in Figure 3 may be calculated in terms of the radius of the No. 1 coil R :

$$\frac{R_2}{R_1} = \frac{1}{b}\sqrt{\frac{1 - x_2^2}{1 - x_1^2}} \qquad (14)$$

$$\frac{D_1}{R_1} = \frac{x_1}{\sqrt{1 - x_1^2}} \qquad (15)$$

$$\frac{D_2}{R_1} = \frac{1}{b}\,\frac{x_2}{\sqrt{1 - x_1^2}} \qquad (16)$$

Table I

Parameters for a Four-Ring System, $q > 0$

x_1	x_2	q	R_2	D_1	D_2	Remarks
0.40585	0.77460	0.0	∞	0.44	∞	End point
0.44	0.8417	0.0598	1.028	0.49	1.60	R_1 and R_2 almost equal
0.53	0.9025	1.004	0.5248	0.625	1.099	q almost equal to 1.0
0.64	0.9370	9.883	0.3100	0.833	0.832	D_1 and D_2 almost equal
0.758	0.9494	4270.	0.1169	1.162	0.3534	Maximum x_2
0.77460	0.94911	∞	0.0	1.22	0.0	End point

Table II

Parameters for a Four-Ring System, $q < 0$

x_1	x_2	q	R_2	D_1	D_2	Remarks
0.74153	0.77460	0.0	∞	1.10	∞	End point
0.758	0.758	-1.0	1.0	1.165	1.165	End point

In the Tables I and II, R_1 is arbitrarily set equal to unity and the corresponding dimensions R_2, D_1 and D_2 are calculated using (14), (15) and (16). Curves of R_2, D_1 and D_2 for the case of $q > 0$ are shown in Figure 6 and a curve of q is given in Figure 7. Similar curves for the case of $q < 0$ could be plotted. It was decided not to do this first because the range in x_1 is so small, and second the charges Q_1 and Q_2 oppose each other and thus produce a field at the center of the system which is much smaller than either charge would produce alone.

In a similar manner to that in which Equation (9) was obtained for the two-ring system, from Equation (10) for the four-ring system

$$E_z = A_0 \left[1 + U\left(\frac{z}{D_1}\right)^8 + \cdots\right] \qquad (17)$$

where

$$A_0 = \frac{Q_1 (q + b^2)}{2\pi\epsilon B_1^2\, q}$$

$$U = \frac{9x_1^8}{(q + b^2)}\left[q P_9(x_1) + b^{10} P_9(x_2)\right]$$

$$P_q(x) = \frac{1}{128}(12155\,x - 25740\,x + 18018\,x^5 - 4620\,x^3 + 315\,x)$$

When x_1 and the corresponding x_2 are known, then U may be calculated from Equation (17). For a uniform electric field the quantity $|U(z/D_1)^8|$ should be made small. If it were made 0.01 percent, for example, the distance Z along the axis in terms of D_1 could be calculated. Then approximately within a sphere of this radius Z, the field would have a uniformity of 0.01 percent or better.

Conclusions

An analysis has been presented of the case of two-ring and four-ring systems to produce uniform electric fields. As the results of the analysis, equations of the parameters and curves and tables of computed values are presented. These parameters include the charge ratio, the geometrical distances and the degree of uniformity of the central electric field of the systems.

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APPENDIX
OUTLINE OF FOUR-RING SOLUTION

The condition that $A_2 = 0$, $A_4 = 0$ and $A_6 = 0$ in Equation (10) leads to the following equations:

$$\frac{Q_1}{B_1^4} P_3(x_1) + \frac{Q_2}{B_2^4} P_3(x_2) = 0 \qquad \text{(A-1)}$$

$$\frac{Q_1}{B_1^6} P_5(x_1) + \frac{Q_2}{B_2^6} P_5(x_2) = 0 \qquad \text{(A-2)}$$

$$\frac{Q_1}{B_1^8} P_7(x_1) = \frac{Q_2}{B_2^8} P_7(x_2) = 0 \qquad \text{(A-3)}$$

where $$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

In (A-1), (A-2) and (A-3) put $q = (Q_1/Q_2)$ and $b = (B_1/B_2)$ and solve for q:

$$q = -b^4 \frac{P_3(x_2)}{P_3(x_1)} = -b^6 \frac{P_5(x_2)}{P_5(x_1)} = -b^8 \frac{P_7(x_2)}{P_7(x_1)} \qquad \text{(A-4)}$$

From (A-4)

$$b^2 = \frac{P_3(x_2)\, P_5(x_1)}{P_3(x_1)\, P_5(x_2)} = \frac{P_5(x_2)\, P_7(x_1)}{P_5(x_1)\, P_7(x_2)} \qquad \text{(A-5)}$$

From (A-5):

$$\frac{P_3(x_i)\, P_7(x_1)}{\left[P_5(x_1)\right]^2} = \frac{P_3(x_2)\, P_7(x_2)}{\left[P_5(x_2)\right]^2} \qquad \text{(A-6)}$$

Let $$f(x) = \frac{P_3(x)\, P_7(x)}{\left[P_5(x)\right]^2} \qquad \text{(A-7)}$$

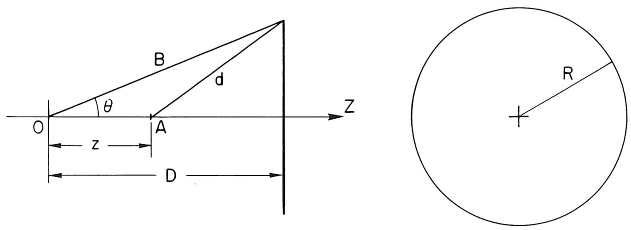

Figure 1 – A single charged ring

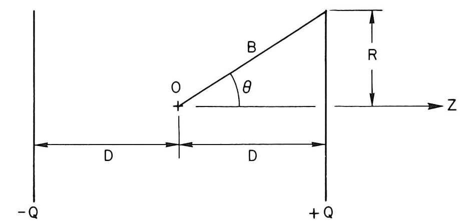

Figure 2 – A double-ring system

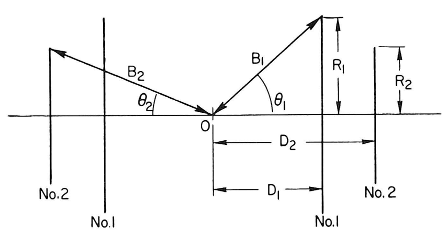

Figure 3 – A four-ring system

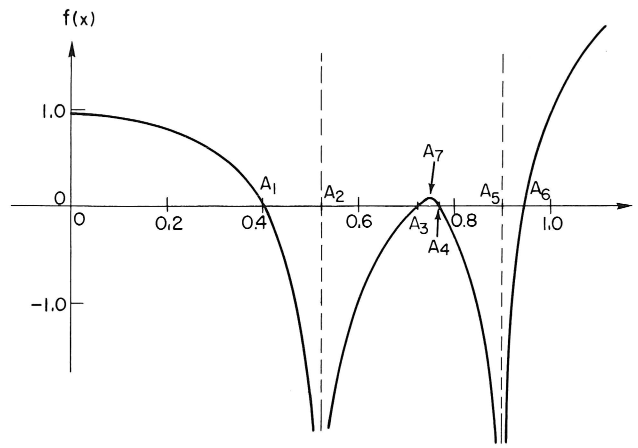

Figure 4 – Sketch of the function f(x)

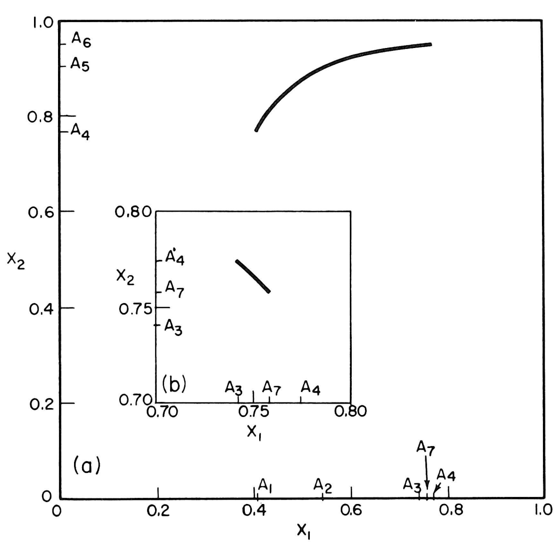

Figure 5 – (a) A plot of x_2 versus x_1 for $q > 0$
(b) A plot of x_2 versus x_1 for $q < 0$

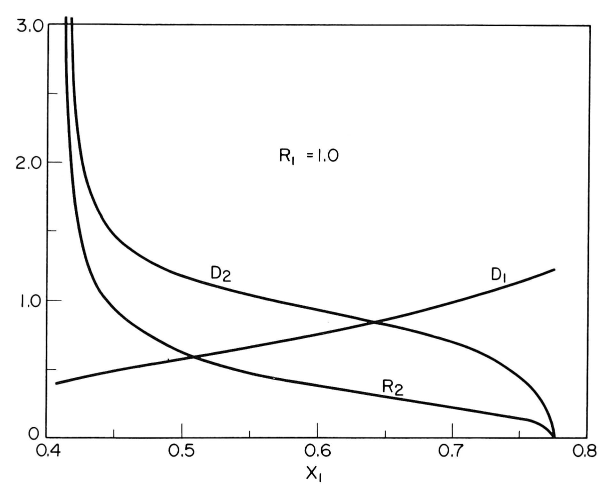

Figure 6 – The dimensions R_1, D_1 and R_2 as a function of x_1 for the case $q > 0$

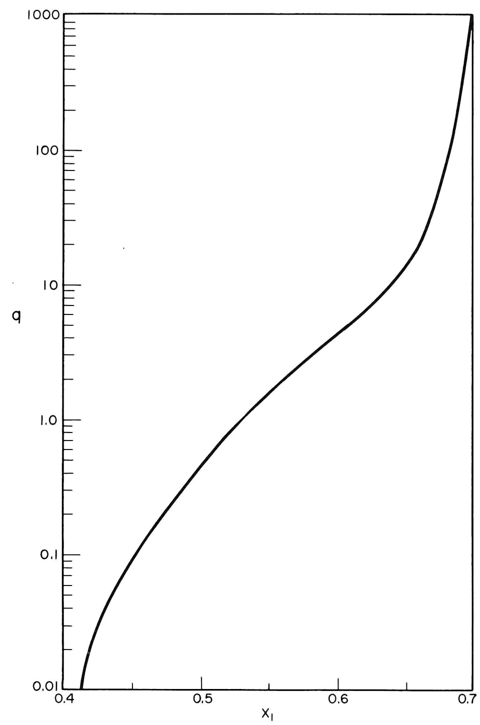

Figure 7 – q as a function of x_1 for the case $q > 0$

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