Sequences of Rank-1 Projections and Gabor Tight Fusion Frames

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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presented by Brian Tuomanen, a candidate for the degree of Doctor of Philosophy of Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

Professor Stephen Montgomery-Smith

Professor Jan Segert

Professor Nakhle Asmar

Professor Mihail Popescu

To the memory of Chase.

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Contents

SEQUENCES OF RANK-1 PROJECTIONS AND GABOR TIGHT FUSION FRAMES

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ABSTRACT

This dissertation provides new results in two different areas. The first concerns properties inherited by sequences of orthogonal rank-1 projections (ie, the outer product sequences such as $\{f_i f_i^*\}_{i=1}^M$) within the Hilbert space of symmetric operators ($sym(\mathcal{H}^{N\times N})$) from their inducing unit-norm vector sequences $\{f_i\}_{i=1}^M$ within a Hilbert space $\mathcal{H}^N$; notably, we show the cases where quantitative Riesz and frame bounds of $\{f_i\}_{i=1}^M$ are inherited by the induced projections $\{f_i f_i^*\}_{i=1}^M$. We then show that the family of unit norm frames which yield independent outer product sequences is open and dense (in a Euclidean-analytic sense) within the topological space $\otimes_{i=1}^M S_{N-1}$, where $M \leqslant dim\ sym(\mathcal{H}^{N\times N})$. We then give a full geometric characterization of the particular sequences that produce dependent sequences of projections.

The second part concerns a new method to construct so-called tight fusion frames. As tight frames are a very important topic within standard frame theory, tight fusion frames are similarly important; however, only trivial examples of tight fusion frames are hitherto known. Here we apply ideas from Gabor analysis to demonstrate a non-trivial construction of tight fusion frames. We then use this construction to further show their applicability in some cases for the retrieval of signals modulo phase.

Chapter 1

Introduction

1.0.1 A Statement of Authorship

This dissertation is based on two papers of the author. The first paper, *Riesz Outer Product Hilbert Space Frames: Quantitative Bounds, Topological Properties, and Full Geometric Characterization* ([31]), was initiated by the author in the Summer of 2012, and then finished with co-author Eric Pinkham over the course of 2013-2014. The second paper, *Gabor Tight Fusion Frames: Construction and Applications in Signal Retrieval Modulo Phase* ([32]), was written with co-author Mozhgan Mohammadpour over the course of Summer and Fall of 2015. These papers also attribute Pinkham's and Mohammadpour's advisors as authors, who provided editing and feedback for the drafts, but did not contribute any original material.

The author wishes to indicate that a significant portion of this dissertation is his original work; however, there are some pertinent discoveries of the co-authors that are being included for the sake of completion. There are also many "gray areas" where a theorem or example was co-developed.

While this is not an exhaustive lemma-by-lemma, theorem-by-theorem, proof-by-proof, example-by-example list, the author will now indicate which of the significant portions are due to each respective author. (Generally speaking, the less interesting small lemmas, remarks and examples that are not listed here were written by both authors.)

From *Riesz Outer Product Hilbert Space Frames: Quantitative Bounds, Topological Properties, and Full Geometric Characterization*:

- The so-called *Riesz Transference Theorem* of section 3.1 was originally discovered by the author in July of 2012; the original proof is included here. Pinkham later discovered a more terse proof, which is not included here but can be seen in our paper.

- The theory of optimal Frame-To-Riesz bound transference in section 3.2 was entirely developed by Pinkham; the examples that follow in that section were co-developed with him in the Summer of 2014.

- The characterization of the dependent sequences of rank-1 projections (outer-products), which ultimately culminates in theorem 4.14, is entirely the original work of the author. This answers a long-standing question in frame theory. Again, shorter proofs of the necessary theorems were later discovered by Pinkham, but they are not included here.

- The theory of topological density of rank-1 projections, as seen in section 4.1, was co-developed with Pinkham in the Summer of 2014.

From *Gabor Tight Fusion Frames: Construction and Applications in Signal Retrieval Modulo Phase*:

- The original idea of using Gabor Analysis as a tool to construct Tight Fusion Frames was conceived by Mohammadpour.

- The construction of Gabor Tight Fusion Frames, as in Proposition 6.1 and theorem 6.2, was co-developed with Mohammadpour in the Summer of 2015.
- The original idea to use Gabor Tight Fusion Frames for Signal Retrieval Modulo phase was conceived by the author. Theorem 6.8, which indicates the cases when this is possible, was co-developed with Mohammadpour in the Fall of 2015.
- Example 6.9 was originally given by Mohammadpour.

1.0.2 A Basic Background in Hilbert Space Frame Theory

At its root, this dissertation concerns sequences of vectors in Hilbert spaces. A (very broad) distinction of the sequences of interest is made into two classes: Riesz bases, and frames. A Riesz basis is a basis for a Hilbert space with quantitative lower and upper bounds that are known as *Riesz bounds*. Hilbert space frames are a similar notion where we have complete sequences with redundant vectors, with similar bounding properties.

Frames were first explicitly defined and introduced by Duffin and Schaefer in their study of nonharmonic Fourier series in 1952 [8], although some might contend that this field was originally founded by Dennis Gabor (the celebrated "Father of Holography") in his seminal 1946 paper [16], which laid the foundations for time-frequency analysis. The field of Frame Theory gained a renewed interest in the 1980's due to its applicability in wavelets and digital signal processing [13]. Frames, being overcomplete, allow for an infinite number of representations of a single signal; this of great value due to their robustness to noise [13] and erasures [14].

We precisely define Hilbert space frames and Riesz sequences below:

Definition 1.1. Let $\mathcal{H}$ be any separable Hilbert space. A countable sequence of vectors $\{\phi_i\}_{i \in I} \subset \mathcal{H}$ is a *frame* for $\mathcal{H}$ provided there exists $0 < A \leqslant B < \infty$ such that

$$A\|\psi\|^2 \leqslant \sum_{i \in I} |\langle \phi_i, \psi \rangle|^2 \leqslant B\|\psi\|^2$$

for all $\psi \in \mathcal{H}$. A and B are called the *lower* and *upper* frame bounds respectively.

Definition 1.2. Let $\mathcal{H}$ be any separable Hilbert space. A countable sequence of vectors $\{\phi_i\}_{i \in I} \subset \mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ provided there exists $0 < A \leqslant B < \infty$ such that for any sequence of scalars $(a_i)_{i \in I}$ we have:

$$A\|a\|^2 \leqslant \left\| \sum_{i \in I} a_i \phi_i \right\|^2 \leqslant B\|a\|^2$$

A and B are called the *lower* and *upper* Riesz bounds respectively.

This brings us to the following remark:

Remark 1.3. While $\mathcal{H}$ can denote a finite or infinite dimensional Hilbert space, we will specifically denote an N-dimensional Hilbert space as $\mathcal{H}^N$.

In the finite dimensional setting, a frame is just a spanning set; that is to say, the statement that $\{\phi_i\}_{i=1}^N$ is a frame for $\mathcal{H}^M$ is tautological with saying that $span\{\phi_i\}_{i=1}^N = \mathcal{H}^M$. For a proof, see [9].

It should be noted, that there are many frame bounds for a given frame. The largest lower frame bound and the smallest upper frame bound are the *optimal* frame bounds. We characterize several classes of frames of particular interest by their frame bounds. If $A = B$ the frame is said to be a *tight frame*, and if $A = B = 1$ it is a *Parseval frame*. These classes are particularly useful for reasons we will see below.

There are several important operators which go along with the study of frames.

Definition 1.4. Let $\Phi = \{\phi_i\}_{i=1}^M$ be a frame for $\mathcal{H}^N$.

1. The synthesis operator of Φ is

$$T : \ell_2^M \to \mathcal{H}^N \quad T : (a_i)_{i=1}^M \mapsto \sum_{i=1}^M a_i \phi_i.$$

   Its matrix representation is

$$T = \begin{bmatrix} | & | & & | \\ \phi_1 & \phi_2 & \cdots & \phi_M \\ | & | & & | \end{bmatrix}.$$

2. The *analysis operator* of Φ is the Hermitian adjoint of T,

$$T^* : \mathcal{H}^N \to \ell_2^M \quad T^* : \psi \mapsto (\langle \psi, \phi_i \rangle)_{i=1}^M.$$

3. The *frame operator* of Φ is $S = TT^*$ so that

$$S : \mathcal{H}^N \to \mathcal{H}^M \quad S : \psi \mapsto \sum_{i=1}^M \langle \psi, \phi_i \rangle \phi_i.$$

4. The *Gram matrix* of Φ is

$$G(\Phi) = T^*T = [\langle \phi_i, \phi_j]_{i,j=1}^M.$$

It follows that the non-zero eigenvalues of S and $G(\Phi)$ are equal and so the largest smallest non-zero eigenvalues of $G(\Phi)$ are the lower and upper frame bounds of Φ.

The frame operator exhibits great utility in understanding frame properties.

Theorem 1.5. *Let $\{\phi_i\}_{i=1}^M$ be a frame for $\mathcal{H}^N$. Then the frame operator S is self-adjoint, positive, and invertible. Furthermore, the largest and smallest eigenvalues of S are precisely the optimal upper and lower frame bounds of $\{\phi_i\}_{i=1}^M$ respectively.*

Reconstruction is carried out by

$$\psi = SS^{-1}\psi = \sum_{i=1}^{M} \langle \psi, \phi_i \rangle \, S^{-1}\phi_i = \sum_{i=1}^{M} \left\langle \psi, S^{-1}\phi_i \right\rangle \phi_i.$$

This provides useful representations of any vector in our Hilbert space through the frame operator. For applications, we want the frame operator to be as well conditioned as possible for stability of the representation. This means that frames which are close to being tight are more desirable than those with arbitrarily small lower frame bounds. Particularly useful frames for encoding and decoding as above are tight frames. Tight frames have the important property that their frame operator is a multiple of the identity and hence inverting them is trivial. This is especially useful when our space has very high dimension as is common in applications.

Again when dealing with finite dimensional vector spaces, these objects have a very simple characterization: a set is Riesz if and only if it is linearly independent. We will use *independent* and *Riesz* nearly interchangeably in this dissertation. We will use *Riesz* when we are particularly concerned with not only the independence but also the Riesz bounds.

Chapter 2

Preliminaries in Outer-Product Tensors

Before we can study the properties of projections and outer-products, we need to precisely define them:

Definition 2.1. For $\phi, \psi \in \mathcal{H}^N$, define the outer product of ϕ and ψ by $\phi\psi^*$ in terms of standard matrix multiplication. For any vector $\phi \in \mathcal{H}^N$, we define the *induced outer product of* ϕ as $\phi\phi^*$. Note that if ϕ is a unit norm vector, then this will be a rank one orthogonal projection; in this case, we may refer to this as the *induced projection of* ϕ.

Much of the following work will be in the space of $N \times N$ matrices over the real or complex fields. We will denote these spaces as $\mathcal{H}^{N\times N}$, and as needed clarifying the base field. In the case that we are restricting our attention to the symmetric or self-adjoint matrices we will use $\mathrm{sym}(\mathcal{H}^{N\times N})$; we may equivalently use notation such as $\mathrm{sym}(\mathcal{H})$ for brevity. To further simplify notation, given $S \in \mathcal{H}^{N\times N}$ we will use S^* for both the Hermitian adjoint and transpose understanding that the underlying field determines which is at play.

Remark 2.2. The ambient space of outer products is the space of self-adjoint matrices on $\mathcal{H}^N$. It has dimension $N(N+1)/2$ if $\mathcal{H}$ is real. If $\mathcal{H}$ is complex, the space of self-adjoint matrices does not form a *complex* vector space but instead a *real* vector space, as such it has dimension N^2.

For $\phi, \psi \in \mathcal{H}^N$ we will denote the i^{th} entry of ϕ by $\phi(i)$. For a matrix S we will denote the $(i,j)^{th}$ entry by $S[i,j]$.

We will equip these vector spaces with the Frobenius matrix inner product.

Definition 2.3. Let $S, T \in \mathcal{H}^{N\times N}$. The *Frobenius inner product* is

$$\langle S, T\rangle_F = \mathrm{Tr}(S^*T) = \sum_{i=1}^{N}\sum_{j=1}^{N} \overline{S[i,j]}T[i,j].$$

We may drop the subscript F when no confusion will arise.

For given $\phi, \psi \in \mathcal{H}^N$ we will use the usual ℓ_2 inner product

$$\langle \phi, \psi\rangle = \sum_{i=1}^{N} \phi(i)\overline{\psi(i)}.$$

Throughout this dissertation we will use I_N to be the $N \times N$ identity matrix and $1_N \in \mathcal{H}^N$ to be the vector 1_N to be the vector of all 1's.

2.1 Some Basic Calculations

We start with a simple calculation.

Lemma 2.4. *For any vectors $\phi_1, \phi_2 \in \mathcal{H}^N$ we have*

$$\langle \phi_1\phi_1^*, \phi_2\phi_2^*\rangle_F = |\langle \phi_1, \phi_2\rangle|^2.$$

Proof. We compute:

$$\langle \phi_1\phi_1^*, \phi_2\phi_2^*\rangle_F = Tr(\phi_2\phi_2^*\phi_1\phi_1^*)$$

$$= Tr(\phi_2 \langle \phi_2, \phi_1 \rangle \phi_1^*)$$

$$= Tr(\langle \phi_1, \phi_2 \rangle \langle \phi_2, \phi_1 \rangle)$$

$$= |\langle \phi_1, \phi_2 \rangle|^2.$$

∎

Corollary 2.1.1. $\phi_1 \perp \phi_2$ in $\mathcal{H}^N$ if and only if $\phi_1\phi_1^* \perp \phi_2\phi_2^*$ in $\mathrm{sym}(\mathcal{H}^{N\times N})$.

One of the main tools in examining the outer products of a collection of vectors will be the Gram matrices of our vectors. When dealing with a Riesz sequence, or a linearly independent collection of vectors, the Gram matrix will be positive-definite. Furthermore, the largest and smallest eigenvalues of this matrix represent the upper and lower Riesz bounds of our sequence respectively. In the case of redundant frames, the Gram matrix is singular. However, the largest and smallest non zero eigenvalues give the upper and lower frame bounds respectively. We will need the the Gram matrix of outer products.

Theorem 2.5. *Let $\{\phi_i\}_{i=1}^M$ be a sequence of vectors in $\mathcal{H}^N$. Then the Gram matrix of $\{\phi_i\phi_i^*\}_{i=1}^M$ is*

$$G = [|\langle \phi_i, \phi_j \rangle|^2].$$

Moreover,

1. *If $\{\phi_i\phi_i^*\}_{i=1}^M$ is a Riesz sequence, then the optimal Riesz bounds are the largest and smallest eigenvalue of G.*

2. *If $\{\phi_i\phi_i^*\}_{i=1}^M$ is a frame then the frame bounds are the largest and smallest non-zero eigenvalues of G.*

The Gram matrix of the induced outer products can be represented in terms of the Gram matrix of the original vectors by using the Hadamard product.

Definition 2.6. Given two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathcal{H}^{M \times N}$ the *Hadamard product* of A and B is

$$A \circ B = [a_{ij} b_{ij}].$$

The following is a well known theorem about Hadamard products, see [11] for example.

Theorem 2.7. *Let A and B be Hermitian with $A = [a_{ij}]$ positive semidefinite. Any eigenvalue $\lambda(A \circ B)$ of $A \circ B$ satisfies*

$$\begin{aligned} \lambda_{min}(A)\lambda_{min}(B) &\leqslant [\min_i a_{ii}]\lambda_{min}(B) \\ &\leqslant \lambda(A \circ B) \\ &\leqslant [\max_i a_{ii}]\lambda_{max}(B) \\ &\leqslant \lambda_{max}(A)\lambda_{max}(B). \end{aligned}$$

Remark 2.8. We see that if G is the Gramian of a frame $\{\phi_i\}_{i=1}^M$ for $\mathcal{H}^N$, then the Gramian of its induced outer-products $\{\phi_i \phi_i^*\}_{i=1}^M$ is $G \circ \overline{G}$.

2.2 Duals of Outer Products

Lemma 2.9. *Given a vector ϕ in $\mathcal{H}^N$ and operators T_1, T_2 acting on $\mathcal{H}^N$ with T_2 symmetric, we have*

(1) $T_1(\phi\phi^*) = (T_1\phi)\phi^*$.

(2) $T_1(\phi\phi^*)T_2 = (T_1\phi)(T_2\phi)^*$.

Proof. (1) We compute for $x \in \mathcal{H}^N$

$$\begin{aligned}T_1(\phi\phi^*)(x) &= T_1(\langle x, \phi\rangle\phi)\\ &= \langle x, \phi\rangle T(\phi)\\ &= (T_1\phi)\phi^*(x).\end{aligned}$$

(2) We compute for $x \in \mathcal{H}^N$

$$\begin{aligned}(\phi\phi^*)T_2(x) &= \langle T_2 x, \phi\rangle\phi\\ &= \langle x, T_2\phi\rangle\phi\\ &= \phi(T_2\phi)^*(x).\end{aligned}$$

∎

Proposition 2.10. *If $\{\phi_i\}_{i=1}^M$ is a Riesz sequence in $\mathcal{H}^N$ with biorthogonal vectors $\{\tilde{\phi}_i\}_{i=1}^M$, then the biorthogonal vectors for $\{\phi_i\phi_i^*\}_{i=1}^M$ are $\{P\tilde{\phi}_i\tilde{\phi}_i^*\}_{i=1}^M$ where P is the orthogonal projection onto the span of $\{\phi_i\phi_i^*\}_{i=1}^M$.*

Proof. We compute:

$$\left\langle \phi_i\phi_i^*, P\tilde{\phi}_j\tilde{\phi}_j^*\right\rangle_F = \left\langle P\phi_i\phi_i^*, \tilde{\phi}_j\tilde{\phi}_j^*\right\rangle_F = \left|\left\langle \phi_i, \tilde{\phi}_j\right\rangle\right|^2 = \delta_{ij}.$$

So the vectors $\{P\tilde{\phi}_i\tilde{\phi}_i^*\}_{i=1}^M$ are biorthogonal to $\{\phi_i\phi_i^*\}_{i=1}^M$. ∎

Remark 2.11. Projecting is necessary in the above proposition. For example, take $\{\phi_1, \phi_2\}$ to be a non-orthogonal Riesz basis for $\mathbb{R}^2$. Then $\tilde{\phi}_1 \perp \phi_2$ so take any $\psi_1 \perp \phi_2$ with norm 1 and scale $\tilde{\phi}_1$ so that $\left\langle \phi_1, \tilde{\phi}_1\right\rangle = 1$ i.e. $\tilde{\phi}_1 = \frac{1}{\langle \psi_1, \phi_1\rangle}\psi_1$. Then the Gram matrix of the induced outer products of $\{\phi_1, \phi_2, \tilde{\phi}_1\}$ is

$$\begin{bmatrix} 1 & |\langle \phi_1, \phi_2\rangle|^2 & 1\\ |\langle \phi_1, \phi_2\rangle|^2 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$$

which has determinant $-|\langle\phi_1, \phi_2\rangle|^4$. Since we have chosen $\phi_1 \not\perp \phi_2$ this matrix is invertible. Hence these outer products are Riesz. But then $\tilde{\phi}_1\tilde{\phi}_1^*$ is not in the span of the other two. Hence the projections are necessary.

Chapter 3

Transference of Riesz and Frame Bounds to Induced Riesz Sequences of Projections

In this section, we study the transference of bounds of Riesz bases and frames to their outer products; in other words, if we know that a frame or Riesz basis $\{\phi\}_{i\in I}$ has bounds of A and B within a Hilbert space $\mathcal{H}$, then what are the bounds for the corresponding sequence of induced rank-1 projections, $\{\phi\phi^*\}_{i\in I}$ within $sym(\mathcal{H}^{N\times N})$?

We start with a simple result that could be considered the cornerstone of this topic: the *Riesz transference theorem*. We first see a short example illustrating the underlying concept: that the Riesz bounds actually improve when we take the projections.

3.1 Transference of Riesz Bounds

Before we start our first theorem, we first need a small lemma:

Lemma 3.1. *Given operators $S = (b_{ij})_{i,j=1}^N$ and $T = (a_{ij})_{i,j=1}^N$ on $\mathcal{H}^N$ we have*

$$\langle T, S\rangle_F = \sum_{i,j=1}^N \overline{a_{ij}} b_{ij},$$

Moreover,

$$\|S\|_F^2 = \sum_{i,j=1}^N a_{ij}^2 = \sum_{i=1}^N \|R_i\|^2 = \sum_{i=1}^N \|C_i\|^2,$$

where R_i (resp. C_i) is the i^{th}-row vector of S (resp. i^{th}-column vector of S).

Proof. Note that

$$\begin{aligned}
\mathrm{Tr}(S^*T) &= \mathrm{Tr}\left(\begin{bmatrix} b_{11} & b_{21} & \cdots & b_{N1} \\ b_{12} & b_{22} & \cdots & b_{N2} \\ \vdots & \vdots & \cdots & \vdots \\ b_{1N} & b_{2N} & \cdots & b_{NN} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}\right) \\
&= \mathrm{Tr}\begin{bmatrix} \sum_{i=1}^N b_{i1}a_{i1} & * & \cdots & * \\ * & \sum_{i=1}^N b_{i2}a_{i2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \sum_{=1}^N b_{iN}a_{iN} \end{bmatrix} \\
&= \sum_{i,j=1}^N a_{ij}b_{ij}.
\end{aligned}$$

For the moreover part, we have $S^*S = (a_{ji})(a_{ij})$ has diagonal elements $\sum_{j=1}^N a_{ij}^2$ for $i = 1, 2, \ldots, N$. ∎

Theorem 3.2. ***(The Riesz Transference Theorem)*** *Let $\{\phi_i\}_{i=1}^N$ be a unit norm Riesz sequence in $\mathcal{H}^N$ with Riesz bounds A, B. Then $\{\phi_i\phi_i^*\}_{i=1}^N$ is also Riesz, also with bounds A, B.*

Proof. Given scalars $(a_i)_{i=1}^N$, we have that the (i,j)-entry of

$$S = \sum_{i=1}^N a_i\phi_i\phi_i^*,$$

is

$$\sum_{k=1}^N c_k\phi_k(i)\phi_k(j).$$

So the j^{th}-column vector is

$$C_j = \sum_{k=1}^N c_k\overline{\phi_k(i)}\phi_k.$$

So by our lemmas,

$$\begin{aligned}
\|S\|^2 &= \sum_{j=1}^{N} \|C_j\|^2 \\
&= \sum_{j=1}^{N} \| \sum_{k=1}^{N} c_k \overline{\phi_k(j)} \phi_k \|^2 \\
&\geqslant A \sum_{j=1}^{N} \sum_{k=1}^{N} |c_k|^2 |\phi_k(j)|^2 \\
&= A \sum_{k=1}^{N} |c_k|^2 \sum_{j=1}^{N} |\phi_k(j)|^2 \\
&= A \sum_{k=1}^{N} |c_k|^2.
\end{aligned}$$

The upper bound is done similarly. ∎

It may not be surprising that unit norm Riesz sequences produce Riesz outer products–but what is surprising is that *the same Riesz bounds hold!* That is, Riesz bounds cannot worsen when moving to the outer product space. A natural question to ask at this point is whether the Riesz bounds of the induced outer products can be better than the Riesz bounds of the original vectors. The answer is *yes.*

Example 3.3. Let $\phi_1 = [0,1]^T$, $\phi_2 = [\sqrt{\varepsilon}, \sqrt{1-\varepsilon}]^T$ for $0 < \varepsilon < 1/2$. Then $\{\phi_i\}_{i=1}^2$ is Riesz with Riesz bounds $1-\sqrt{\varepsilon}$ and $1+\sqrt{\varepsilon}$ while $\{\phi_i\phi_i^*\}_{i=1}^2$ is Riesz with bounds $1-\varepsilon$ and $1+\varepsilon$.

Proof. The Gram matrix of $\{\phi_1, \phi_2\}$ is

$$\begin{bmatrix} 1 & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & 1 \end{bmatrix}$$

while that of $\{\phi_1\phi_1^*, \phi_2\phi_2^*\}$ is

$$\begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}.$$

The eigenvalues of these matrices are as required. ∎

3.1.1 Non-Transference of Tight Frame Bounds

This is certainly an encouraging start–the logical question to ask is, in which cases do frame bounds carry over to their induced projections? We begin to answer this question cynically, showing that for *any tight, overcomplete frame* (i.e, tight frame whose cardinality is greater than the dimensionality of its ambient space), that its frame bounds will *never* carry over to its induced outer products within real Euclidean spaces.

Here, we only show the case of non-transference of real tight frames; the complex case is proven entirely analogously, and we omit this case without loss of generality.

We first remember the fact that $sym(\mathbb{R}^{N\times N})$ is an $N(N+1)/2$ dimensional real Hilbert space, and make a concrete connection between these spaces in the following remark:

Remark 3.4. In the proof of this lemma, for a given vector $v \in \mathbb{R}^N$, we use $v(k:\ell)$ (with $0 \leqslant k < \ell \leqslant N$) to mean the adridged sub-vector of v, with components k through ℓ; e.g., for $v = \begin{bmatrix} 10 \\ 20 \\ 30 \\ 40 \\ 50 \end{bmatrix}$, $v(2:4)$ would be $\begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}$.

Let $\phi \in \mathbb{R}^N$, and consider $\phi\phi^*$. We will now construct a vector $\varphi \in \mathbb{R}^{N(N+1)/2}$ to be the vector corresponding to $\phi\phi^* \in \mathrm{sym}(\mathbb{R}^{N\times N})$:

$$\phi \to \varphi_\phi = \begin{bmatrix} \phi \circ \phi \\ \sqrt{2}\phi(1)\phi(2:N) \\ \sqrt{2}\phi(2)\phi(3:N) \\ \vdots \\ \sqrt{2}\phi(N-1)\phi(N-3:N-2) \\ \sqrt{2}\phi(N)\phi(N-1) \end{bmatrix} \tag{3.1}$$

We include the "$\sqrt{2}$"s to cover for the repetition of the off-diagonal values in the symmetric matrix under the Frobenius inner product: for any $x, y \in \mathbb{R}^N$, we will have $\langle xx^*, yy^* \rangle_{Fr} = \langle \varphi_x, \varphi_y \rangle_{\ell^2}$. We see this is effectively a vectorization of a one-dimensional projection in $sym(\mathbb{R}^{N\times N})$.

Lemma 3.5. *If $\{\phi_i\}_{i=1}^M$ is a tight, overcomplete frame for $\mathbb{R}^N$ with $M > N$, then $\{\phi_i\phi_i^*\}_{i=1}^M$ is not tight within the space that it spans.*

Proof. Let $\{\phi_1, \ldots, \phi_M\}$ be a frame for $\mathbb{R}^N$, with corresponding outer products $\{\phi_i\phi_i^*\}_{i=1}^M$ that span sym$(\mathbb{R}^{N\times N})$. We denote the isomorphic vectors in $\mathbb{R}^{N(N+1)/2}$ in the sense of (3.4) as $\{\varphi_i\}_{i=1}^M$. We can write the synthesis matrix as:

$$[\varphi_1,\ \varphi_2, \ldots,\ \varphi_M]$$

We note that if we abridge this matrix to the first N rows, we have:

$$[\phi_1 \circ \phi_1\ \phi_2 \circ \phi_2\ \cdots \phi_M \circ \phi_M] \tag{3.2}$$

We know from [9] that a frame is tight if and only if the vectors comprising the rows of its synthesis matrix are equal-norm and orthogonal. Yet, the only possibility for (3.2) to be a matrix consisting of orthogonal rows is if each vector in $\{\phi_i \circ \phi_i\}_{i=1}^M$ has ℓ_0-norm of 1, since every element in each vector $\phi_i \circ \phi_i$ is non-negative. However, this will leave us with each ϕ_i being a multiple of some element in the standard orthonormal basis $\{e_i\}_{i=1}^N$ for $\mathbb{R}^N$. But then, that would mean that $M = N$, contradicting our assumption. The conclusion follows. ∎

3.2 Optimal Frame-to-Riesz Bound Transference

In the following section we will examine more closely the Riesz bounds of the induced outer products. Here, we give the "optimal" induced Riesz bounds in $sym(\mathcal{H})$ that may arise from a frame in $\mathcal{H}$, and sufficient conditions to achieve them.

The following is immediate by Lemma 2.4.

Proposition 3.6. *Let $\{\phi_i\}_{i=1}^M$ be vectors in $\mathcal{H}^N$. The sequence $\{\phi_i\phi_i^*\}_{i=1}^M$ is orthonormal if and only if $\{\phi_i\}_{i=1}^M$ is orthonormal.*

Since a redundant frame can not produce a Riesz sequence with tight Riesz bounds, one might ask how close we can get. Before computing the optimal Riesz bounds of a set of rank one projections we need to introduce the frame potential.

Definition 3.7. Let $\{\phi_i\}_{i=1}^M$ be a frame in $\mathcal{H}^N$. The frame potential is

$$\mathrm{FP}(\{\phi_i\}_{i=1}^M) = \sum_{i=1}^M \sum_{j=1}^M |\langle \phi_i, \phi_j \rangle|^2.$$

Proposition 3.8. *The frame potential of a unit norm tight frame with M elements in $\mathcal{H}^N$ is M^2/N, which is a minimum over all unit norm frames.*

See [4, 9] for a proof of the above result.

Theorem 3.9. *If $\{\phi_i\}_{i=1}^M$ is a unit norm frame for $\mathcal{H}^N$, then the upper Riesz bound of $\{\phi_i\phi_i^*\}_{i=1}^M$ is at least M/N. Moreover, we have equality if and only if $\{\phi_i\}_{i=1}^M$ is a unit norm tight frame.*

Proof. If $\{\phi_i\}_{i=1}^M$ is a unit norm frame whose outer products have Gram matrix G. Then

$$\begin{aligned}
\frac{M}{N} &\leqslant \frac{1}{M}\mathrm{FP}(\{\phi_i\}_{i=1}^M) \\
&= \frac{1}{M}\left\| \left(\sum_{i=1}^M |\langle \phi_i, \phi_j \rangle|^2 \right)_{j=1}^M \right\|_{\ell_1} \\
&\leqslant \frac{1}{M}\sqrt{M}\left\| \left(\sum_{i=1}^M |\langle \phi_i, \phi_j \rangle|^2 \right)_{j=1}^M \right\|_{\ell_2}
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \left(\frac{1}{\sqrt{M}} \sum_{i=1}^{M} |\langle \phi_i, \phi_j \rangle|^2 \right)_{j=1}^{M} \right\|_{\ell_2} \\
&= \left\| G \left(\frac{1}{\sqrt{M}}, \ldots, \frac{1}{\sqrt{M}} \right)^T \right\|_{\ell_2} \\
&\leqslant \|G\| \\
&= \lambda_1
\end{aligned}
$$

where λ_1 is the largest eigenvalue of G.

For the moreover part, if $\lambda_1 = \frac{M}{N}$ then we have that $\frac{M^2}{N} = FP(\{\phi_i\}_{i=1}^{M})$ so that $\{\phi_i\}_{i=1}^{M}$ is a unit norm tight frame. If on the other hand we have that $\{\phi_i\}_{i=1}^{M}$ is a unit norm tight frame, then

$$
\begin{aligned}
\frac{M}{N} &= \frac{1}{M} FP(\{\phi_i\}_{i=1}^{M}) \\
&= \frac{1}{M} \left\| \left(\sum_{i=1}^{M} |\langle \phi_i, \phi_j \rangle|^2 \right)_{j=1}^{M} \right\|_{\ell_1} \\
&= \frac{1}{M} \left\| \left(\frac{M}{N}, \ldots, \frac{M}{N} \right) \right\|_{\ell_1} \\
&= \frac{1}{M} \sqrt{M} \left\| \left(\frac{M}{N}, \ldots, \frac{M}{N} \right) \right\|_{\ell_2} \\
&= \left\| \frac{1}{\sqrt{M}} \left(\frac{M}{N}, \ldots, \frac{M}{N} \right) \right\|_{\ell_2} \\
&= \left\| G \left(\frac{1}{\sqrt{M}}, \ldots, \frac{1}{\sqrt{M}} \right) \right\|_{\ell_2} \\
&= \|G\| \\
&= \lambda_1.
\end{aligned}
$$

∎

Now we will compute the optimal lower Riesz bounds for outer product frames.

Theorem 3.10. *If $\{\phi_i\}_{i=1}^{M}$ is a unit norm frame for $\mathcal{H}^N$, then the lower Riesz bound* *of* $\{\phi_i\phi_i^*\}_{i=1}^M$ *is at most* $\dfrac{M(N-1)}{N(M-1)}$.

Proof. Let G be the Gram matrix of $\{\phi_i\phi_i^*\}_{i=1}^M$ with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_M$. Then $\mathrm{Tr}(G) = M$ gives

$$\sum_{i=2}^M \lambda_i = M - \lambda_1,$$

Also,

$$(M-1)\lambda_M \leqslant \sum_{i=2}^M \lambda_i,$$

and so

$$\lambda_M \leqslant \frac{\sum_{i=2}^M \lambda_i}{M-1}.$$

Finally, we have

$$\lambda_M \leqslant \frac{M-\lambda_1}{M-1} \leqslant \frac{M-\frac{M}{N}}{M-1} = \frac{M(N-1)}{N(M-1)}.$$

∎

In the next theorem, we see that the above bounds are sharp.

Theorem 3.11. *Let* $\{\phi_i\}_{i=1}^M$ *be a unit norm equiangular frame for* $\mathcal{H}^N$ *with* $M > N$ *and let* $c := |\langle\phi_i, \phi_j\rangle|^2$ *for* $i \neq j$. *Then* $\{\phi_i\phi_i^*\}_{i=1}^M$ *is a Riesz sequence whose Gram matrix has two distinct eigenvalues, both of which are non-zero:*

$$\lambda_1 = 1 + (M-1)c \text{ and } \lambda_i = 1 - c \text{ for all } i = 2, 3, \ldots, M.$$

Moreover, if $\{\phi_i\}_{i=1}^M$ *is also a tight frame, then* $c = \frac{M-N}{N(M-1)}$ *and* $\{\phi_i\phi_i^*\}_{i=1}^M$ *is a Riesz sequence with Riesz bounds* $\frac{M(N-1)}{N(M-1)}$, $\frac{M}{N}$.

Before proving the above result, we need a well known theorem (see e.g. [10]).

Theorem 3.12 (Sylvester's Determinant Theorem)**.** *Let S and T be matrices of size $M \times N$ and $N \times M$ respectively. Then*

$$\det(I_M + ST) = \det(I_N + TS).$$

Proof of Theorem 3.11. Let G be the Gram matrix for $\{\phi_i\phi_i^*\}_{i=1}^M$. Then

$$G[i,j] = \begin{cases} 1 & \text{if } i = j \\ c & \text{otherwise} \end{cases}$$

Then we can write $G = (1-c)I_M + c1_M1_M^*$ and expand using Sylvester's determinant theorem with $S = 1_M$ and $T = 1_M^*$:

$$\begin{aligned} \det\left((1-c)I_M + c1_M1_M^* - \lambda I\right) &= \det\left((1-c-\lambda)I_M + c1_M1_M^*\right) \\ &= (1-c-\lambda)^M \det\left(I_M + \frac{c}{1-c-\lambda}1_M1_M^*\right) \\ &= (1-c-\lambda)^M \det\left(I_1 + \frac{c}{1-c-\lambda}1_M^*1_M\right) \\ &= (1-c-\lambda)^{M-1}(1-c-\lambda+cM). \end{aligned}$$

Setting the above equal to zero and solving for λ we get the solutions $\lambda = 1-c$ occurring $(M-1)$-times and $\lambda = 1+(M-1)c$ occurring once.

If $c = 0$, then $\{\phi_i\phi_i^*\}_{i=1}^M$ are orthonormal and hence so are $\{\phi_i\}_{i=1}^M$ contradicting the assumption that $M > N$. If $c = 1$ then $\phi_i = \alpha_{ij}\phi_j$ with $|\alpha_{ij}| = 1$ for all i and j contradicting the fact that this is a frame. Hence, $0 < c < 1$ and the outer products are Riesz.

For the "moreover" part, we compute:

$$1 - c = 1 - \frac{M-N}{N(M-1)} = \frac{NM-N-M+N}{N(M-1)} = \frac{M(N-1)}{N(M-1)}$$

and

$$1 + (M-1)c = 1 + (M-1)\frac{M-N}{N(M-1)} = \frac{N+M-N}{N} = \frac{M}{N}.$$

∎

3.2.1 Sparsity and Vectorized Outer Products

Definition 3.13. Let $\phi \in \mathcal{H}^N$. Define the *vectorization* of $\phi\phi^*$ as the vector obtained by stacking the columns on top of each other. That is, the vectorization of $\phi\phi^*$ is

$$\begin{bmatrix} \phi(1)\overline{\phi} \\ \phi(2)\overline{\phi} \\ \vdots \\ \phi(N)\overline{\phi} \end{bmatrix}$$

where $\phi(k)$ is the kth entry of ϕ.

Proposition 3.14. *Let $\{\phi_i\}_{i=1}^M$ be a frame for $\mathcal{H}^N$ with no zero vectors. For $k = 1, \ldots, N$ define $I_k = \{i : \phi_i(k) \neq 0\}$. If $\{\phi_i\}_{i \in I_k}$ is independent for all k, then $\{\phi_i\phi_i^*\}_{i=1}^M$ is independent.*

Proof. Let $\{\phi_i\}_{i=1}^M$ be a frame with the properties as stated. Let C_i be the vectorization of $\phi_i\phi_i^*$. Now consider the synthesis operator of $\{C_i\}_{i=1}^M$:

$$\begin{bmatrix} \phi_1(1)\overline{\phi_1} & \phi_2(1)\overline{\phi_2} & \phi_3(1)\overline{\phi_3} & \cdots & \phi_M(1)\overline{\phi_M} \\ \phi_1(2)\overline{\phi_1} & \phi_2(2)\overline{\phi_2} & \phi_3(2)\overline{\phi_3} & \cdots & \phi_M(2)\overline{\phi_M} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1(N)\overline{\phi_1} & \phi_2(N)\overline{\phi_2} & \phi_3(N)\overline{\phi_3} & \cdots & \phi_M(N)\overline{\phi_M} \end{bmatrix}.$$

Notice that since $0 \notin \{\phi_i\}_{i=1}^M$ we have that each ϕ_i contains at least one nonzero entry, say $\phi_i(k) \neq 0$. Then since $\phi_i(k)\overline{\phi_i}$ is part of C_i we have that $C_i \neq 0$ for all i.

Now suppose that there exists scalars a_i (not all zero) such that

$$\sum_{i=1}^M a_i C_i = 0.$$

Then there is at least one l such that $a_l C_l \neq 0$. Then by hypothesis, there is a row k such that $\sum_i a_i \phi_i(k) \overline{\phi_i} = 0$ but $a_l \phi_l(k) \overline{\phi_l} \neq 0$. Then

$$\sum_{i \in I_k} a_i \phi_i(k) \overline{\phi_i} = 0$$

which contradicts that $\{\phi_i\}_{i \in I_k}$ is linearly independent.

∎

Remark 3.15. The conditions of the above proposition are fairly constrictive but, in certain cases, this can be useful. It will be used to verify a later example quickly.

Corollary 3.16. *Let $\{\phi_i\}_{i=1}^M$ be a frame for which every subset of size k is linearly independent. If the rows of the analysis operator are k-sparse then the induced outer products are linearly independent.*

3.3 Concrete Constructions of Riesz Bases of Outer Products

Up to now, we have provided no concrete constructions of Riesz outer product sequences. We rectify this with the following examples.

Example 3.17. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for $\mathbb{R}^N$ and define $\{E_{ij}\}$ as follows

$$E_{ij} = \begin{cases} e_i & \text{if } i = j \\ \frac{1}{\sqrt{2}}(e_i + e_j) & \text{if } j > i \end{cases}$$

for $i = 1, \cdots, N$ and $i \leqslant j$. Then $\{E_{ij} E_{ij}^*\}$ is a Riesz basis for the space of symmetric operators in $\mathrm{sym}(\mathbb{R}^{N \times N})$.

Proof. This follows immediately from Proposition 3.14 in the prior subsection. ∎

The following example provides an extension of the above to the complex case. It also provides a second (more intuitive) method of verifying that the above example is independent.

Example 3.18. Take E_{ij} as before, and add the following

$$E'_{ij} = \frac{1}{2}(e_i + \sqrt{-1}e_j)(e_i + \sqrt{-1}e_j)^*$$

for $j > i$. Then the resulting sequence is Riesz.

Proof. Note that E'_{ij} is a matrix with 1 in the (i,i) and (j,j) entry and $-\sqrt{-1}$ in the (i,j) entry and $\sqrt{-1}$ in the (j,i) entry. Then we know that $\sum_{i,j} a_{ij}E_{ij} + \sum_{i,j} a'_{ij}E'_{ij} = 0$ if and only if the real and complex parts are 0. We will do the real part and the complex part will follow immediately. E_{ij} with $i \neq j$ is the square matrix with 1's in the $(i,i), (i,j), (j,i)$, and (j,j) entry. Specifically, it is the only element in the sum for which the entries (i,j) and (j,i) could possibly be non-zero. Hence $a_{ij} = 0$ for all $i \neq j$. The remaining terms E_{ii} are orthonormal and hence $a_{ii} = 0$ for all i. Thus the real part is independent and the complex part follows by the same argument. ∎

We know that the optimal Riesz bounds for a Riesz basis of outer products are $(N+1)/(N+2)$ and $(N+1)/2$. Using unit norm tight frames we can always achieve the upper bound. The lower bound is then the problem. Here we give a class of unit norm tight frames which produce nice lower bounds as well.

Example 3.19. Let $\{\phi_i\}_{i=1}^{N+1}$ be the usual simplex equiangular tight frame for $\mathbb{R}^N$. Then consider the outer products

$$\Phi_{ij} = \left(\frac{\phi_i + \phi_j}{\|\phi_i + \phi_j\|}\right)\left(\frac{\phi_i + \phi_j}{\|\phi_i + \phi_j\|}\right)^*$$

for $j > i$. Then Φ_{ij} is Riesz provided $N \neq 3$ and has Riesz bounds $\frac{1}{2}$ and $\frac{N+1}{2}$ for $N \geqslant 7$.

Proof. Barg et al. showed in [1] that the frame

$$\frac{\phi_i + \phi_j}{\|\phi_i + \phi_j\|}$$

is a unit norm tight frame. Hence by Theorem 3.9 the upper Riesz bound of the induced outer products is

$$\frac{N(N+1)}{2}\frac{1}{N} = \frac{N+1}{2}.$$

For the lower bound, we can consider the simplex in $\mathbb{R}^N$ as $\left\{\frac{Pe_i}{\|Pe_i\|}\right\}_{i=1}^{N+1}$ for where $\{e_i\}_{i=1}^{N+1}$ is an orthonormal basis for $\mathbb{R}^{N+1}$, $P = I_{N+1} - ff^*$, and $f = \frac{1}{\sqrt{N+1}}\sum_{i=1}^{N+1} e_i$. Then we have

$$\begin{aligned}\phi_i &= \frac{Pe_i}{\|Pe_i\|}\\ &= \sqrt{\frac{N+1}{N}}\left(-\frac{1}{N+1}, \ldots, -\frac{1}{N+1}, 1 - \frac{1}{N+1}, -\frac{1}{N+1}, \ldots, -\frac{1}{N+1}\right)\end{aligned}$$

and

$$\begin{aligned}\langle \phi_i, \phi_j \rangle &= \frac{N+1}{N}\left(\frac{N-1}{(N+1)^2} - \frac{2}{N+1}\left(1 - \frac{1}{N+1}\right)\right)\\ &= -\frac{1}{N}.\end{aligned}$$

Now, $\|\phi_i + \phi_j\|^2 = 2\frac{N-1}{N}$ for $i \neq j$ and so we can compute the the Gram matrix of $\{\Phi_{ij}\}_{ij}$,

$$G_\Phi[ij, kl] = \langle \Phi_{ij}, \Phi_{kl} \rangle = \begin{cases} 1 & \text{if } i = j \text{ and } k = l \\ \frac{(N-3)^2}{4(N-1)^2} & \text{if } i = k \text{ or } i = l \text{ or } j = k \text{ or } j = l \\ \frac{4}{(N-1)^2} & \text{if no indices are equal} \end{cases}.$$

Consider the collection of unit norm vectors

$$E_{ij} = \frac{1}{2}(e_i + e_j)(e_i + e_j)^* \text{ for } j > i$$

and $\{e_i\}_{i=1}^{N+1}$ is an orthonormal basis for $\mathbb{R}^{N+1}$. Now its Gram matrix is

$$G_E[ij, kl] = \begin{cases} 1 & \text{if } i = j \text{ and } k = l \\ \frac{1}{4} & \text{if either } i = k \text{ or } i = l \text{ or } j = k \text{ or } j = l \\ 0 & \text{if no indices are equal} \end{cases} .$$

This gives us the decomposition

$$\begin{aligned} G_\Phi = &\left(1 - 4\left(\frac{(N-3)^2}{4(N-1)^2}\right)\right) I_{N(N+1)/2} \\ &+ 4\left(\frac{(N-3)^2}{4(N-1)^2} - \frac{4}{(N-1)^2}\right) G_E \\ &+ \frac{4}{(N-1)^2} 1_{N(N+1)/2} 1^*_{N(N+1)/2}. \end{aligned}$$

Some inequalities,

$$\frac{(N-3)^2}{4(N-1)^2} - \frac{4}{(N-1)^2} \geqslant 0$$

if $N \geqslant 7$ and

$$1 - 4\left(\frac{(N-3)^2}{4(N-1)^2}\right) > 0$$

if $N > 2$. The matrices $\left(1 - \frac{(N-3)^2}{(N-1)^2}\right) I_{N(N+1)/2}$ and $4\left(\frac{(N-3)^2}{4(N-1)^2} - \frac{4}{(N-1)^2}\right) G_E$ are positive-definite and $\frac{4}{(N-1)^2} 1_{N(N+1)/2} 1^*_{N(N+1)/2}$. is positive-semidefinite so

$$\begin{aligned} \lambda_{min}[G_\Phi] \geqslant\ & \lambda_{min}\left[\left(1 - \frac{(N-3)^2}{(N-1)^2}\right) I_{N(N+1)/2}\right] \\ &+ \lambda_{min}\left[4\left(\frac{(N-3)^2}{4(N-1)^2} - \frac{4}{(N-1)^2}\right) G_E\right] \\ &+ \lambda_{min}\left[\frac{4}{(N-1)^2} 1_{N(N+1)/2} 1^*_{N(N+1)/2}\right] \\ =\ & \left(1 - \frac{(N-3)^2}{(N-1)^2}\right) \end{aligned}$$

$$+4\left(\frac{(N-3)^2}{4(N-1)^2}-\frac{4}{(N-1)^2}\right)\lambda_{min}\left[G_E\right]+0 \tag{3.3}$$

We need to know $\lambda_{min}(G_E)$.

As in Example 3.17, we will break up the sum. Let $E_{ij}=\frac{1}{2}(e_i+e_j)(e_i+e_j)^*$. Then

$$\begin{aligned}\left\|\sum_{i=1}^{N+1}\sum_{j>i}a_{ij}E_{ij}\right\|^2 &= \frac{1}{4}\sum_{i=1}^{N+1}\left[\left|\sum_{j>i}a_{ij}+\sum_{j<i}a_{ji}\right|^2+2\sum_{j>i}|a_{ij}|^2\right]\\ &\geqslant \frac{1}{2}\sum_{j>i}|a_{ij}|^2\\ &= \frac{1}{2}\end{aligned}$$

for a_{ij} which square sum to 1.

Then (3.3) becomes

$$1-\frac{(N-3)^2}{4(N-1)^2}+2\left(\frac{(N-3)^2}{4(N-1)^2}-\frac{4}{(N-1)^2}\right)=\frac{N^2+2N-23}{2(N-1)^2}\geqslant\frac{1}{2}$$

for $N\geqslant 6$.

Since these inequalities only hold for $N\geqslant 7$, we have computed the lower Riesz bounds for $N=2,3,\ldots,6$ manually:

N	lower bound
2	3/4
3	0
4	5/36
5	3/8
6	63/100

∎

Remark 3.20. When $N=3$ we get another example of the strangeness of this problem. In this example we get that $\Phi_{14}=\Phi_{23}$ thus producing a dependent sequence.

Chapter 4

Full Topological and Geometric Characterization of Riesz Sequences of Rank-1 Projections

In this section we give topological and geometric conditions for independence (and dependence) of sequences of rank-1 projections in $sym(\mathcal{H}^{N\times N})$. We specifically show the precise necessary and sufficient conditions required on a set of vectors $\{\phi_i\}_{i=1}^M \subset \mathcal{H}^N$ so that $\{\phi_i\phi_i^*\}_{i=1}^M$ may be dependent or independent in $sym(\mathcal{H}^{N\times N})$, as well as give a point-set topological proof of the density of independent outer-products within the topological space $\otimes_{i=1}^M S_{N-1}$.

4.0.1 Some Necessary and Sufficient Conditions

This chapter heavily relies on the following theorem, which will be proven in Section 4.2.1.

Theorem 4.1. *Let T be a $N \times N$ positive semi-definite matrix. Let $\{e_i\}_{i=1}^N$ be the eigenvectors of T with the corresponding eigenvalues $\{\lambda_i\}_{i=1}^N$. Let $I_+ \subset \{1, \ldots, N\}$ be the index for the eigenvectors with positive eigenvalues, i.e., $i \in I_+ \Leftrightarrow \lambda_i > 0$.*

Let $\{a_i\}_{i\in I_+}$ be a sequence of scalars such that $\sum_{i\in I_+} |a_i|^2 = 1$. Then, for the vector $v = \sum_{i\in I_+} a_i\sqrt{\lambda_i}e_i$, we will have:

$$rank \begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} = rank\ T$$

Likewise, the converse is true: if we have rank $\begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} = rank\ T$, *then* $v = \sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i$ *for some collection of scalars indexed by* I_+, $\{a_i\}_{i \in I_+}$ *where* $\sum_{i \in I_+} |a_i|^2 = 1$.

Proposition 4.2. *Let* $\{\phi_i\}_{i=1}^M \subset \mathcal{H}^N$ *be unit norm, and add an additional unit norm vector* ϕ_{M+1}. *Assume the set of induced outer products is* $\{\phi_i \phi_i^*\}_{i=1}^M$ *is independent, and that* $M + 1 \leqslant \dim \mathrm{sym}(\mathcal{H}^{N \times N})$.

Let G_{op} *be the Gram matrix of the induced outer products for the original sequence, that is the Gram matrix of* $\{\phi_i \phi_i^*\}_{i=1}^M$, *and denote the eigenvectors of* G_{op} *as* $\{e'_i : 1 \leqslant i \leqslant M\}$ *and the associated eigenvalues* $\{\lambda'_i : 1 \leqslant i \leqslant M\}$.

We consider the analysis operator T *for* $\{\phi_i\}_{i=1}^M$ *acting on* ϕ_{M+1}. *This is*

$$T\phi_{M+1} = \begin{bmatrix} \langle \phi_{M+1}, \phi_1 \rangle \\ \langle \phi_{M+1}, \phi_2 \rangle \\ \vdots \\ \langle \phi_{M+1}, \phi_M \rangle \end{bmatrix}$$

Consider the following second order elliptic function:

$$f(x_1, x_2, \ldots, x_M) = \sum_{1 \leqslant i \leqslant M} \frac{|x_i|^2}{\lambda'_i} \tag{4.1}$$

Let $y_1 e'_1 + y_2 e'_2 + \cdots + y_M e'_M = T\phi_{M+1} \circ \overline{T\phi_{M+1}}$ *be the representation of* $T\phi_{M+1} \circ \overline{T\phi_{M+1}}$ *within* $\{e'_1, \ldots, e'_M\}$. *Then we will have that* $f(y_1, \ldots, y_M) = 1$ *if and only if* $\{\phi_i \phi_i^*\}_{i=1}^{M+1}$ *is a dependent set.*

Proof. This follows directly from Theorem 4.1 and the identity of $G_{op} = G \circ \overline{G}$. If we add the additional vector ϕ_{M+1} to our basis then the $(M+1)^{th}$ column of the Gram matrix for the outer products $\{\phi_i \phi_i^*\}_{i=1}^{M+1}$ is

$$\begin{bmatrix} T\phi_{M+1} \circ \overline{T\phi_{M+1}} \\ 1 \end{bmatrix},$$

while the $(M+1)^{th}$ row is $[(T\phi_{M+1} \circ \overline{T\phi_{M+1}})^* \quad 1]$. We know that the dimension spanned by a frame is exactly the rank of its Gram matrix; Theorem 4.1 implies that $T\phi_{M+1} \circ \overline{T\phi_{M+1}}$ must precisely meet the criteria of this proposition to have the condition that the rank of the Gram matrix does not increase, and thereby does not increase the dimension spanned by the set $\{\phi_i \phi_i^*\}_{i=1}^{M+1}$, i.e., this collection of outer products produces a dependent set. ∎

Remark 4.3. The previous theorem yields a quartic algebraic variety/manifold that will come in handy. Let $\{e_i'\}_{i=1}^{M}$ be as in the theorem. Consider the quartic equation for $v \in \mathcal{H}^M$:

$$\sum_{i=1}^{M} \frac{|\langle v \circ \overline{v}, e_i' \rangle|^2}{\lambda_i} = 1 \tag{4.2}$$

We use the notation $\mathcal{M}^4_{\{\phi_i\}_{i=1}^M}$ to signify this quartic manifold embedded in $\mathcal{H}^M$. Note that $v = T\phi_{M+1}$ satisfies this equation if and only if ϕ_{M+1} satisfies the criteria for the previous theorem. Thus, if we are to consider the fourth order algebraic variety for all $v \in \mathcal{H}^M$ that satisfy this equation, then the collection of all $T\phi_{M+1}$ such that ϕ_{M+1} satisfy the criteria for the previous theorem are contained entirely within this variety.

Without loss of generality, we order every frame in this section such that $\{\phi_1, \ldots, \phi_N\}$ is a basis for its Hilbert space $\mathcal{H}^N$, and $\{\phi_1, \ldots, \phi_{M_0}\}$ with $M_0 \leqslant M$ such that

$\{\phi_1\phi_1^*, \ldots, \phi_{M_0}\phi_{M_0}^*\}$ is an independent sequence within the induced set of outer products $\{\phi_i\phi_i^*\}_{i=1}^M$. Unless otherwise noted, we assume $M \leqslant \dim \operatorname{sym}(\mathcal{H}^{N\times N})$. By default, T will be the analysis operator for the frame $\{\phi_i\}_{i=1}^M$, while S_{N-1} is be the unit sphere in $\mathcal{H}^N$.

We start with some necessary lemmas.

Lemma 4.4. *Let S_{N-1} be the unit sphere in $\mathcal{H}^N$. TS_{N-1} is an ellipsoid embedded within $\mathcal{H}^M$ with a Euclidean surface of dimension $N-1$; moreover, T is injective from $S_{N-1} \mapsto TS_{N-1}$.*

Proof. By lemma 3.24 of [9], we know that T is injective on $\mathcal{H}^N$; limiting its domain to S_{N-1} retains injectivity. If we limit the codomain to the range of T, so that we have the mapping $T : \mathcal{H}^N \mapsto \mathit{Range}\ T$, then we have that TS_{N-1} is an ellipsoid in an N-dimensional subspace of $\mathcal{H}^M$ (see chapter 7 of [9]). If we expand the codomain to $\mathcal{H}^M$, we have an $N-1$ dimensional ellipsoidal manifold embedded in $\mathcal{H}^M$. ∎

Remark 4.5. We use the notation "T^{-1}" to indicate the inverse of the bijection $T\big|_{S^{N-1}}$, as above.

Lemma 4.6. *Let G be the Gram matrix of our frame. Arrange the eigenvalues of G so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N > 0$ and $\lambda_j = 0$ for $N < j \leqslant M$, and denote the corresponding eigenvectors with $\{e_i\}_{i=1}^M$. Then the ellipsoid TS_{N-1} is the set of vectors $v = v_1e_1 + \cdots v_Ne_N$ where $\sum_{i=1}^N \frac{|v_i|^2}{\lambda_1} = 1$.*

Proof. This again follows from the Lemma 4.6 and Theorem 4.1. ∎

Remark 4.7. For a given frame, we denote the quartic manifold given by the Gram matrix of outer products implicitly stated in theorem 4.2 and explicitly stated in the following remark 4.3 as $\mathcal{M}^4_{\{\phi_i\}_{i=1}^M}$; denote the second order (elliptic) manifold in lemmas 4.4 and 4.0.1 as $\mathcal{M}^2_{\{\phi_i\}_{i=1}^M}$.

4.1 Topological Properties of Independent Outer Product Sequences

We claim that "almost every" unit norm frame with a cardinality within a particular bound induces a set of independent outer products; here we will show what this means in a rigorous sense.

In this section, we will consider the family of unit norm frames with cardinality $M \leqslant \dim \operatorname{sym}(\mathcal{H}^{N\times N})$. We see that we can identify this family with the topological space $\bigotimes_{i=1}^M (S_{N-1})$. We will use the standard metric for frames, $d(\Phi,\Psi) = \sqrt{\sum_{i=1}^M \|\phi_i - \psi_i\|^2}$, which is compatible with the subspace topology of the Euclidean topology with regards to $\bigotimes_{i=1}^M (S_{N-1})$. Results of this kind are often done in frame theory using algebraic geometry which might give a slightly stronger result that the unit norm M-element frames which produce independent outer products form an open dense set in the Zariski topology in the family of all unit norm M-element frames. We have chosen not to do this because only a fraction of the field knows enough algebraic geometry to appreciate such results. Instead, we will give a direct, analytic construction for the density of of the frames giving independent outer products.

Lemma 4.8. *If $\{\phi_i\}_{i=1}^N$ is a Riesz sequence in $\mathbb{H}^N$ with Riesz bounds A, B and*

$$\sum_{i=1}^N \|\phi_i - \psi_i\|^2 < \varepsilon^2 < A,$$

then $\{\psi_i\}_{i=1}^N$ *is Riesz with Riesz bounds* $(\sqrt{A}-\varepsilon)^2, (\sqrt{B}+\varepsilon)^2$.

Proof. For any $\{a_i\}_{i=1}^N$ we compute:

$$\begin{aligned}
\|\sum_{i=1}^N a_i\psi_i\| &\leqslant \|\sum_{i=1}^N a_i\phi_i\| + \|\sum_{i=1}^N a_i(\psi_i-\phi_i)\| \\
&\leqslant B^{1/2}\left(\sum_{i=1}^N |a_i|^2\right)^{1/2} + \sum_{i=1}^N |a_i|\|\psi_i-\phi_i\| \\
&\leqslant B^{1/2}\left(\sum_{i=1}^N |a_i|^2\right)^{1/2} + \left(\sum_{i=1}^N |a_i|^2\right)^{1/2}\left(\sum_{i=1}^N \|\psi_i-\phi_i\|^2\right)^{1/2} \\
&\leqslant (B^{1/2}+\varepsilon)\left(\sum_{i=1}^N |a_i|^2\right)^{1/2}.
\end{aligned}$$

The stated upper Riesz bound is immediate from here. The lower Riesz bound follows similarly.

∎

Lemma 4.9. *If* $\|\phi\| = \|\psi\| = 1$, *then*

$$\|\phi\phi^* - \psi\psi^*\|_F^2 \leqslant 2\|\phi-\psi\|^2.$$

Proof. We compute

$$\begin{aligned}
\|\phi\phi^* - \psi\psi^*\|_F^2 &= \|\phi\phi^*\|_F^2 + \|\psi\psi^*\|_F^2 - 2\langle\phi\phi^*, \psi\psi^*\rangle_F \\
&= 1 + 1 - 2|\langle\phi,\psi\rangle|^2 \\
&= 2(1-|\langle\phi,\psi\rangle|^2) \\
&= 2(1-|\langle\phi,\psi\rangle|)(1+|\langle\phi,\psi\rangle|) \\
&= (2-2|\langle\phi,\psi\rangle|)(1+|\langle\phi,\psi\rangle|) \\
&\leqslant (2-2\mathrm{Re}\langle\phi,\psi\rangle)(1+|\langle\phi,\psi\rangle|)
\end{aligned}$$

$$= (\|\phi\|^2 + \|\psi\|^2 - 2\mathrm{Re}\langle \phi, \psi\rangle)(1 + |\langle \phi, \psi\rangle|)$$

$$= \|\phi - \psi\|^2 (1 + |\langle \phi, \psi\rangle|)$$

$$\leqslant 2\|\phi - \psi\|^2.$$

∎

Proposition 4.10. *Let $\{\phi_i\}_{i=1}^M$ are unit norm vectors in $\mathbb{H}^N$ with $\{\phi_i\phi_i^*\}_{i=1}^M$ a Riesz sequence having Riesz bounds A, B. Given $0 < \varepsilon < A/2$, choose a unit norm set of vectors $\{\psi_i\}_{i=1}^M$ so that*

$$\sum_{i=1}^M \|\phi_i - \psi_i\|^2 < \varepsilon < \frac{A}{2}.$$

Then $\{\psi_i\psi_i^*\}_{i=1}^M$ is Riesz with Riesz bounds

$$\left(\sqrt{A} - \sqrt{2\varepsilon}\right)^2 \text{ and } \left(\sqrt{B} + \sqrt{2\varepsilon}\right)^2.$$

Proof. Assume the hypotheses. It follows from our Lemma 4.9 that

$$\sum_{i=1}^M \|\phi_i\phi_i^* - \psi_i\psi_i^*\|_F^2 \leqslant 2\sum_{i=1}^M \|\phi_i - \psi_i\|^2 < 2\varepsilon$$

Now by Lemma 4.8 we have that $\{\psi_i\psi_i^*\}_{i=1}^M$ is Riesz with Riesz bounds

$$\left(\sqrt{A} - \sqrt{2\varepsilon}\right)^2, \left(\sqrt{B} + \sqrt{2\varepsilon}\right)^2.$$

∎

The above proposition says that the set of frames with cardinality $M \leqslant \dim \mathrm{sym}(\mathcal{H}^{N\times N})$ is open in $\bigotimes_{i=1}^M (S_{N-1})$. In the remainder of this section we will show that this set is also dense. While other authors have studied the density of outer products in terms of commutative algebra [2], here we show this fact constructively and quantitatively using only standard analytic and Euclidean topological notions.

Lemma 4.11. *Let S be an invertible operator and suppose $\{\phi_i\}_{i=1}^M$ are vectors in $\mathcal{H}^N$. Then $\{\phi_i\phi_i^*\}_{i=1}^M$ is independent if and only if $\{S\phi_i(S\phi_i)^*\}_{i=1}^M$ is independent.*

Proof. Let $\{a_i\}_{i=1}^M$ be scalars, not all zero. We have

$$0 = \sum_{i=1}^M a_i\phi_i\phi_i^*$$

if and only if

$$0 = S\left(\sum_{i=1}^M a_i\phi_i\phi_i^*\right)S^* = \sum_{i=1}^M a_i(S\phi_i)(S\phi_i)^*.$$

∎

Now we construct a large family of bases of outer products.

Lemma 4.12. *Given a unit norm vector $\psi \in \mathcal{H}^N$, $\varepsilon > 0$, there is a unit norm basis for $\operatorname{sym}(\mathcal{H}^{N\times N})$ consisting of outer products $\{\phi_i\phi_i^*\}_{i=1}^d$ with $d = \dim\operatorname{sym}(\mathcal{H}^{N\times N})$, such that $\|\phi_i - \psi\|^2 < \varepsilon$ for all $i = 1, \ldots, d$.*

Proof. First, we will assume that we have a unit norm basis $\{\psi_i\psi_i^*\}_{i=1}^d$ of $\operatorname{sym}(\mathcal{H}^{N\times N})$ with $\langle\psi, \psi_i\rangle > 0$ for all i and $\psi = e_1$ for an orthonormal basis $\{e_j\}_{j=1}^N$ of $\mathcal{H}^N$. We can see that such a basis exists by a unitary transformation of Example 3.17 or Example 3.18. Choose $\delta > 0$ with the following property: If

$$S = diag(1, \delta, \delta, \ldots, \delta),$$

then for all $i = 1, 2, \ldots, d$ we have

$$\sum_{j=2}^N |S\psi_i(j)|^2 = \delta^2\sum_{j=2}^N |\psi_i(j)|^2 \leqslant \frac{\varepsilon}{2}|\psi_i(1)|^2 \leqslant \frac{\varepsilon}{2}\|S\psi_i\|^2. \tag{4.3}$$

Let

$$\phi_i = \frac{S\psi_i}{\|S\psi_i\|} \text{ for all } i = 1, 2, \ldots, d,$$

and observe that $\|\phi_i\| = 1$ and Equation 4.3 imply

$$\phi_i(1) \geqslant 1 - \frac{\varepsilon}{2}.$$

Now we compute for all $i = 1, 2, \ldots, d$

$$\|\psi - \phi_i\|^2 = |1 - \phi_i(1)|^2 + \sum_{j=2}^{N} |\phi_i(j)|^2 \leqslant \epsilon.$$

Since $\{\psi_i \psi_i^*\}_{i=1}^d$ is linearly independent, by Lemma 4.11, the $\{\phi_i \phi_i^*\}_{i=1}^d$ are also independent.

For the general case, given ψ and $\{\psi_i\}_{i=1}^d$ with independent outer products, choose a vector ϕ so that $\langle \phi, \psi_i \rangle \neq 0$ for all $i = 1, 2, \ldots, d$. By replacing ϕ by $c_i \phi$ with $|c_i| = 1$ if necessary, we can assume these inner products are all strictly positive. By the above, we can find $\{\phi_i\}_{i=1}^d$ with their outer products independent and

$$\|\phi - \phi_i\|^2 < \varepsilon.$$

Choose a unitary operator U so that $U\phi = \psi$ and we have

$$\|\psi - U\phi_i\|^2 = \|U\phi - U\phi_i\|^2 = \|\phi - \phi_i\|^2 < \varepsilon.$$

This completes the proof. ∎

With the above lemmas we are ready to prove the following.

Theorem 4.13. *The set of all frames $\{\phi_i\}_{i=1}^M$ with $M \leqslant \dim \operatorname{sym}(\mathcal{H}^N)$ which produce independent outer products is open and dense in the family of M-element frames.*

Proof. This set was already shown to be open by Proposition 4.10. All that remains to show is that this set is also dense. We let $\{\phi_i\}_{i=1}^M$ be any sequence of unit-norm vectors, and we will construct a new sequence, $\{\phi_i'\}_{i=1}^M$ that is arbitrarily close to this whose corresponding outer products produce a basis for $sym(\mathcal{H}^{N\times N})$. Let $\phi'_1 = \phi_1$ and proceed by induction. Assume that we have a collection of vectors $\{\phi'_i\}_{i=1}^{M_0}$ such that $\|\phi'_i - \phi_i\| < \varepsilon/M$ for all $i = 1, \dots, M_0$ and $\{\phi'_i(\phi'_i)^*\}_{i=1}^{M_0}$ is independent. Then by Lemma 4.12 there exists a unit norm basis $\{\psi_i\}_{i=1}^{\dim \mathrm{sym}(\mathcal{H}^{N\times N})}$ such that $\|\phi_{M_0+1} - \psi_i\| < \varepsilon/M$ for all i. Since dim span $\{\psi_i\psi_i^*\}_{i=1}^{M_0} = M_0$, we can choose $\phi'_{M_0+1} = \psi_k$ such that $\psi_k\psi_k^* \notin \mathrm{span}(\{\phi'_i(\phi'_i)^*\}_{i=1}^{M_0})$. Then the set $\{\phi'_i\}_{i=1}^{M_0+1}$ induces independent outer products with $\|\phi'_i - \phi_i\| < \varepsilon/M$ for all i. By induction, we have obtained a set $\{\phi'_i\}_{i=1}^{M}$ such that

$$\sum_{i=1}^{M} \|\phi'_i - \phi_i\| < \varepsilon$$

and which induces independent outer products.

∎

4.1.1 A Characterization of All Frames That Yield Dependent Outer Products with Cardinality less than $\dim \mathrm{sym}(\mathcal{H}^{N\times N})$

Theorem 4.14. *Let $M < \dim \mathrm{sym}(\mathcal{H}^{N\times N})$. If $\{\phi_i\phi_i^*\}_{i=1}^{M}$ is independent, the set of vectors in S_{N-1} that will yield a dependent set of outer products will be $T^{-1}(\mathcal{M}^2_{\{\phi_i\}_{i=1}^M} \cap \mathcal{M}^4_{\{\phi_i\}_{i=1}^M})$, which will be compact in the Euclidean topology.*

Proof. Remembering the notation from remark 4.7, we see that $\mathcal{M}^2_{\{\phi_i\}_{i=1}^M} \cap \mathcal{M}^4_{\{\phi_i\}_{i=1}^M}$ are exactly the portion of the image of T that corresponds to the dependent outer products. Since the manifolds $\mathcal{M}^2_{\{\phi_i\}_{i=1}^M}$ and $\mathcal{M}^4_{\{\phi_i\}_{i=1}^M}$ are closed and bounded within a Euclidean space, they are compact and likewise their intersection $\mathcal{M}^2_{\{\phi_i\}_{i=1}^M} \cap \mathcal{M}^4_{\{\phi_i\}_{i=1}^M}$ is compact. By the injectivity of T on S_{N-1} and remark 4.5, we see that $T^{-1}(\mathcal{M}^2_{\{\phi_i\}_{i=1}^M} \cap \mathcal{M}^4_{\{\phi_i\}_{i=1}^M})$ forms a compact subset of S_{N-1}. ∎

4.1.2 A Geometric Result

While it is beyond the scope of this dissertation to fully analyze this, we find that carrying this on for frames with induced outer product sets of dimensionality equal to dim sym$(\mathcal{H}^{N\times N})$ yields a possibly interesting geometric result due to the loss of independence in the induced outer products.

Proposition 4.15. *Suppose that* $\{\phi_i\}_{i=1}^M$ *is a unit norm frame for* $\mathcal{H}^N$ *where* $\dim \operatorname{span}\{\phi_i\phi_i^*\}_{i=1}^M = \dim \operatorname{sym}(\mathcal{H}^{N\times N})$*. Then* $\mathcal{M}^2_{\{\phi_i\}_{i=1}^M} \subseteq \mathcal{M}^4_{\{\phi_i\}_{i=1}^M}$.

Proof. We already know that if we expand the frame $\{\phi_i\}_{i=1}^M$ to the point where *any* additional vector $v \in S_{N-1}$ induces a *dependent* outer product sequence $\{\phi_i\phi_i^*\}_{i=1}^M \cup \{vv^*\}$, we will have $Tv \in \mathcal{M}^4_{\{\phi_i\}_{i=1}^M}$. But this implies that $T^{-1}(\mathcal{M}^2_{\{\phi_i\}_{i=1}^M} \cap \mathcal{M}^4_{\{\phi_i\}_{i=1}^M}) = T^{-1}(\mathcal{M}^2_{\{\phi_i\}_{i=1}^M}) = S_{N-1}$. The conclusion follows. ∎

Remark 4.16. This gives us an instance where an elliptic manifold with a surface that is locally Euclidean of dimension $(N-1)$ embedded within $\mathcal{H}^M$, which is contained entirely within a fourth order manifold of dimension $(M-1)$ also embedded within the same $\mathcal{H}^M$, where $M > N$.

4.2 Expanding Positive Semi-Definite Matrices While Preserving Rank

4.2.1 Main Theorem on Positive Semi-Definite Matrices

Now we prove Theorem 4.1. We prove this Theorem in the form of two propositions ("forwards" and "converse"). Likewise, we prove several lemmas for each proposition.

4.2.2 Necessary Lemmas for "Forwards" Proposition

Lemma 4.17. *Let T be an $N \times N$ positive semi-definite matrix with eigenvector e_i and associated eigenvalue $\lambda_i > 0$. Then we will have*

$$rank \begin{bmatrix} T & \sqrt{\lambda_i} e_i \\ (\sqrt{\lambda_i} e_i)^* & 1 \end{bmatrix} = rank\ T$$

.

Proof. By the spectral theorem, we know that T has N eigenvectors $\{e_j\}_{j=1}^N$ with real-valued eigenvalues $\{\lambda_j\}_{j=1}^N$, and we have the representation $T = \sum_{j=1}^N \lambda_j P_j$, where P_j is the projection onto e_j. Since by the hypothesis $\lambda_i > 0$, we have $T\left((1/\sqrt{\lambda_i})e_i\right) = \sqrt{\lambda_i}e_i$ This means that $\sqrt{\lambda_i}e_i \in Range\ T$. Thus, $rank\ T = rank\ \begin{bmatrix} T \\ (\sqrt{\lambda_i} e_i)^* \end{bmatrix}$.

To complete the lemma, we show that the existence of a vector w such that $\begin{bmatrix} T \\ (\sqrt{\lambda_i} e_i)^* \end{bmatrix} w = \begin{bmatrix} \sqrt{\lambda_i} e_i \\ 1 \end{bmatrix}$. We set $w = \sqrt{\lambda_i}e_i$; this yields

$$\begin{bmatrix} T \\ (\sqrt{\lambda_i} e_i)^* \end{bmatrix} w = \begin{bmatrix} \sqrt{\lambda_i} e_i \\ 1 \end{bmatrix}$$

So we have that $\begin{bmatrix} \sqrt{\lambda_i} e_i \\ 1 \end{bmatrix} \in Range\ \begin{bmatrix} T \\ (\sqrt{\lambda_i} e_i)^* \end{bmatrix}$; this implies that

$$rank \begin{bmatrix} T & \sqrt{\lambda_i} e_i \\ (\sqrt{\lambda_i} e_i)^* & 1 \end{bmatrix} = rank \begin{bmatrix} T \\ (\sqrt{\lambda_i} e_i)^* \end{bmatrix}.$$

By our prior result we can conclude:

$$rank \begin{bmatrix} T & \sqrt{\lambda_i} e_i \\ (\sqrt{\lambda_i} e_i)^* & 1 \end{bmatrix} = rank\ T.$$

∎

Lemma 4.18. *Let T be an $N \times N$ positive semi-definite matrix, with distinct eigenvectors e_i and e_j with positive eigenvalues. Then, for any two scalars a, b such that $|a|^2 + |b|^2 = 1$, we will have:*

$$rank\ T = rank\ \begin{bmatrix} T & a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j \\ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)^* & 1 \end{bmatrix}$$

Proof. We proceed as in the prior theorem. First, we check that $rank\ T = rank\ [T\ \ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)]$. We see that $T\left((a/\sqrt{\lambda_i})e_i + (b/\sqrt{\lambda_j})e_j\right) = (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j) \in Range\ T$, which yields

$$rank\ T = rank\ [T\ \ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)] = rank\ \begin{bmatrix} T \\ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)^* \end{bmatrix}$$

We can now see that

$$\begin{bmatrix} T \\ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)^* \end{bmatrix} \left((a/\sqrt{\lambda_i})e_i + (b/\sqrt{\lambda_j})e_j\right) = \begin{bmatrix} (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j) \\ 1 \end{bmatrix}$$

which implies

$$\begin{bmatrix} (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j) \\ 1 \end{bmatrix} \in Range\ \begin{bmatrix} T \\ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)^* \end{bmatrix}$$

and so we have

$$rank \begin{bmatrix} T \\ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)^* \end{bmatrix} =$$
$$rank \begin{bmatrix} T & a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j \\ (a\sqrt{\lambda_i}e_i + b\sqrt{\lambda_j}e_j)^* & 1 \end{bmatrix}$$

the conclusion directly follows. ∎

4.2.3 First proposition

This is the "forwards" implication of theorem (4.1) ("⇒").

Proposition 4.19. *Let T be a $N \times N$ positive semi-definite matrix. Let $\{e_i\}_{i=1}^N$ be the eigenvectors of T with the corresponding eigenvalues $\{\lambda_i\}_{i=1}^N$. Let $I_+ \subset \{1, \dots, N\}$ be the index for the eigenvalues with positive eigenvectors, i.e., $i \in I_+ \Leftrightarrow \lambda_i > 0$.*

Let $\{a_i\}_{i \in I_+}$ be a sequence of scalars such that $\sum_{i \in I_+} |a_i|^2 = 1$. Then, for the vector $v = \sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i$, we will have:

$$rank \begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} = rank\ T$$

Proof. This is just an extension of lemma (4.18) to an arbitrary number of eigenvectors. Let $\{a_i\}_{i \in I_+}$ be a collection of scalars such that $\sum_{i \in I_+} |a_i|^2 = 1$. We first see that $T\left(\sum_{i \in I_+} a_i \frac{1}{\sqrt{\lambda_i}} e_i\right) = \sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i$. This means that $rank\ T = rank\ [T \quad \sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i] = rank \begin{bmatrix} T \\ \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right)^* \end{bmatrix}$.

We see that

$$\begin{bmatrix} T \\ \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right)^* \end{bmatrix} \left(\sum_{i \in I_+} a_i \frac{1}{\sqrt{\lambda_i}} e_i\right)$$

$$= \begin{bmatrix} T\left(\sum_{i \in I_+} a_i \frac{1}{\sqrt{\lambda_i}} e_i\right) \\ \left\langle \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right), \left(\sum_{i \in I_+} a_i \frac{1}{\sqrt{\lambda_i}} e_i\right) \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right) \\ \sum_{i \in I_+} a_i \overline{a_i} \frac{\sqrt{\lambda_i}}{\sqrt{\lambda_i}} \end{bmatrix} = \begin{bmatrix} \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right) \\ 1 \end{bmatrix}$$

This implies that the vector $\begin{bmatrix} \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right) \\ 1 \end{bmatrix}$ is within the range of the matrix $\begin{bmatrix} T \\ \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right)^* \end{bmatrix}$. This will give us

$$rank \begin{bmatrix} T \\ \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right)^* \end{bmatrix} = rank \begin{bmatrix} T & \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right) \\ \left(\sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i\right)^* & 1 \end{bmatrix}$$

The conclusion will follow. ∎

4.2.4 Necessary Lemmas for Converse Proposition

Observation 4.2.1. Let T be a positive semi-definite matrix on $\mathcal{H}^N$. By the spectral theorem, $T = \sum_{i=1}^{N} \lambda_i P_i$, where P_i is a projection onto the eigenvector e_i with the associated real eigenvalue λ_i.

We can partition $\mathcal{H}^N$ into two orthogonal subspaces, V_0 and V_+, where $V_+ = span\ \{e_i :\ \lambda_i > 0,\ 1 \leqslant i \leqslant N\ \}$, and $V_0 = span\ \{e_i :\ \lambda_i = 0,\ 1 \leqslant i \leqslant N\}$.

Notice the orthogonality of the eigenvectors transfers to these spaces: $\mathcal{H}^N = V_0 \oplus V_+$.)

Lemma 4.20. *Let T be a positive-semi-definite matrix on $\mathcal{H}^N$, and let V_0 be as in observation (4.2.1). Let $v \in \mathcal{H}^N$.*

If $P_{V_0} v \neq 0$, then rank $\begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} >$ rank T.

Proof. Since $\mathcal{H}^N = V_0 \oplus V_+$, we have that $V_0^{\perp} = V_+$.

We note that *ker* $T = V_0$, and *Range* $T = V_+$. If $v \in \mathcal{H}^N$, then $v = P_{V_+} v + P_{V_0} v$; if $P_{V_0} v \neq 0$, then $P_{V_0} v \notin$ *Range* T and hence $v \notin$ *Range* T. It follows that *rank* $[T\ \ v] >$ *rank* T, and that *rank* $\begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} \geqslant$ *rank* $[T\ \ v] >$ *rank* T. ∎

Lemma 4.21. *Let T be a positive semi-definite matrix on $\mathcal{H}^N$, and let V_0 be as in observation (4.2.1). Let $v \in \mathcal{H}^N$.*

If $P_{V_-} v \neq 0$, then rank $\begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} >$ rank T.

Proof. Since $P_{V_-}v \neq 0$, there must be some e_i, $\lambda_i < 0$, such that $c_i = \langle v, e_i \rangle \neq 0$. Let us first consider only the vector c_ie_i. c_ie_i is in the range of T; its preimage is $\{(c_i/\lambda_i)e_i + \nu : \nu \in Null\ T\}$. So we have that $rank \begin{bmatrix} T \\ (c_ie_i)^* \end{bmatrix} = rank\ T$.

We know $\begin{bmatrix} c_ie_i \\ 1 \end{bmatrix}$ is in the range of $\begin{bmatrix} T \\ (c_ie_i)^* \end{bmatrix}$. We proceed by contradiction. We know that any solution w for the following equation:

$$\begin{bmatrix} T \\ (c_ie_i)^* \end{bmatrix}(w) = \begin{bmatrix} T(w) \\ (c_ie_i)^*\, w \end{bmatrix} = \begin{bmatrix} c_ie_i \\ \langle w, c_ie_i \rangle \end{bmatrix}$$

is of the form $w = (c_i/\lambda_i)e_i + \nu$ for some $\nu \in Null\ T$. Yet, we see that in the $N+1^{th}$ slot in the above vector, we have $\langle (c_i/\lambda_i)e_i + \nu, c_ie_i \rangle = |c_i|^2/\lambda_i = 1$, i.e., $|c_i|^2 = \lambda_i < 0$. This is a contradiction.

This will suffice to show that for any eigenvector e_i with negative eigenvalue, if we let P_i be the one dimensional projection onto this vector and if $P_iv \neq 0$, then $rank \begin{bmatrix} T & P_iv \\ (P_iv)^* & 1 \end{bmatrix} > rank\ T$. It follows that if $P_{V_-}v \neq 0$, then $rank \begin{bmatrix} T & P_{V_-}v \\ (P_{V_-}v)^* & 1 \end{bmatrix} > rank\ T$. We extend this idea:

The preimage of $P_{V_-}v = \sum_{i \in I_-} c_ie_i$ with regards to T is $T^{-1}(P_{V_-}v) = \{\sum_{i \in I_-} \frac{c_i}{\lambda_i} e_i + \nu\ :\ \nu \in Null\ T\}$. Thus, for $\begin{bmatrix} P_{V_-}v \\ 1 \end{bmatrix}$ to be in the range of $\begin{bmatrix} T \\ (P_{V_-}v)^* \end{bmatrix}$, we must have

$$\begin{aligned} \begin{bmatrix} T \\ (P_{V_-}v)^* \end{bmatrix} (\sum_{i \in I_-} \frac{c_i}{\lambda_i} e_i + \nu) &= \begin{bmatrix} T(\sum_{i \in I_-} \frac{c_i}{\lambda_i} e_i + \nu) \\ (P_{V_-}v)^*(\sum_{i \in I_-} \frac{c_i}{\lambda_i} e_i + \nu) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i \in I_-} c_ie_i \\ \sum_{i \in I_-} |c_i|^2/\lambda_i \end{bmatrix} = \begin{bmatrix} P_{V_-}v \\ 1 \end{bmatrix} \end{aligned}$$

but this would mean that $|c_i|^2/\lambda_i = 1$, when $|c_i|^2/\lambda_i$ is a negative number.

∎

Lemma 4.22. *With the notation above, let* $\{a_i\}_{i\in I_+}$ *where* I_+ *is the index of eigenvectors with positive eigenvalues. Let*

$$v = \sum_{i\in I_+} a_i\sqrt{\lambda_i}e_i. \tag{4.4}$$

Assume that

$$rank \begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} = rank\ T$$

then $\sum_{i\in I_+} |a_i|^2 = 1$.

Proof. Let v be of the form as in (4.4). We assume that $rank \begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} = rank\ T$.

Then v is in the range of T, so $\begin{bmatrix} T \\ v^* \end{bmatrix}$ is of the same rank as T. The preimage of v is $T^{-1}(v) = \{\sum_{i\in I_+} \frac{a_i}{\sqrt{\lambda_i}}e_i + \nu\ :\ \nu \in Null\ T\}$. If we let ν be arbitrary, then

$$\begin{aligned}
\begin{bmatrix} T \\ v^* \end{bmatrix} \left(\sum_{i\in I_+} \frac{a_i}{\sqrt{\lambda_i}}e_i + \nu\right) &= \begin{bmatrix} T(\sum_{i\in I_+} \frac{a_i}{\sqrt{\lambda_i}}e_i + \nu) \\ \langle(\sum_{i\in I_+} \frac{a_i}{\sqrt{\lambda_i}}e_i + \nu), (\sum_{i\in I_+} a_i\sqrt{\lambda_i}e_i)\rangle \end{bmatrix} \\
&= \begin{bmatrix} v \\ \sum_{i\in I_+} |a_i|^2 \end{bmatrix}
\end{aligned}$$

This will force $\sum_{i\in I_+} |a_i|^2 = 1$. ∎

Corollary 4.23. *Let* e_i *be an eigenvector with positive eigenvalue. Then rank* $\begin{bmatrix} T & ce_i \\ (ce_i)^* & 1 \end{bmatrix} =$ *rank* T *if and only if* $|c| = \sqrt{\lambda_i}$.

Proof. (“$\Leftarrow$”) This is shown in the prior section.

(“$\Rightarrow$”) Apply lemma (4.22) with $a_i = 1$, and $a_j = 0$, for $j \neq i$. ∎

Proposition 4.24. *Let v be a vector such that rank $\begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix}$ = rank T for a positive semi-definite matrix T. Let $\{e_i\}_{i=1}^N$ be the eigenvectors for T with associated eigenvalues $\{\lambda_i\}_{i=1}^N$. We use $I_+ \subset \{1, \ldots, N\}$ as the index of the positive eigenvalues, i.e., $\lambda_i > 0 \Leftrightarrow i \in I_+$.*

Let $v \in \mathcal{H}^N$. If

$$rank \begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} = rank\ T$$

then $v \in span_{i \in I_+} e_i$, where $v = \sum_{i \in I_+} a_i \sqrt{\lambda_i} e_i$ for some collection of scalars $\{a_i\}_{i \in I_+}$ such that $\sum_{i \in I_+} |a_i|^2 = 1$.

Proof. We start with the assumption *rank* $\begin{bmatrix} T & v \\ v^* & 1 \end{bmatrix} = rank\ T$. By lemmas (4.20) and (4.21), we have $v \in span_{i \in I_+} e_i$. By lemma (4.22), we have the conclusion. ∎

4.2.5 Proof of theorem 4.1

Proof. ("⇒") This is shown by proposition (4.19).

("⇐") This is shown by proposition (4.24).

∎

Chapter 5

Overview of Gabor Analysis and Fusion Frames

5.1 Introduction

Fusion frame theory has recently garnered great interest among researchers who work in signal processing. Fusion frames extend the notion of a frame (i.e., an overcomplete set of vectors) within a Hilbert space $\mathcal{H}$ to a collection of subspaces $\{W_i\}_{i\in I}$ (with orthogonal projections $\{P_i\}_{i\in I}$) in $\mathcal{H}$ with an associated collection of weights $\{\nu_i\}_{i\in I}$. This concept was originally introduced by Gitta Kutyniok in [23].

A tight fusion frame is one such that we have the identity $\sum_{i\in I} P_i = CI_{N\times N}$, i.e., the sum of the projections is a multiple of the identity, with every weight set to 1. Such tight fusion frames are of interest for two reasons. First, they guarantee a very simple reconstruction of a signal; and second, tight fusion frames are robust against noise [21] and also remain robust against a one-erasure subspace when the rank of projections are equal to each other [27].

A fusion frame is defined as follows:

Definition 5.1. Consider a Hilbert space $\mathcal{H}$, with a collection of subspaces $\{W_i\}_{i\in I}$ and an associated set of positive weights $\{\nu_i\}_{i\in I}$. The associated orthogonal projections are likewise denoted as $P_i : \mathcal{H} \mapsto W_i$. Then we call $\{(W_i, \nu_i)\}$ a **fusion frame** if there are positive constants $0 < A \leqslant B < \infty$ such that for any $x \in \mathcal{H}$ we have the following:

$$A\|x\|^2 \leqslant \sum_{i\in I} \nu_i^2 \|P_i x\|^2 \leqslant B\|x\|^2$$

Definition 5.2. A **tight fusion frame** is a fusion frame as in 5.1 where $A = B$ and $\nu_i = 1$ for all $i \in I$. That is to say, we have the following for any $x \in \mathcal{H}$:

$$\sum_{i\in I} \|P_i x\|^2 = A\|x\|^2$$

Or, equivalently:

$$AI = \sum_{i=1}^{N} P_i$$

Now, consider an orthonormal basis for the range of P_i, that is $\{e_{i,\ell}\}_{i=1}^n$. We know that:

$$P_i x = \sum_{\ell=1}^{n} \langle x, e_{i,\ell}\rangle e_{i,\ell}$$

for all $\mathbf{x} \in \mathbb{C}^N$. Summing these equations over $i = 1, \cdots, N$ together

$$Ax = \sum_{i=1}^{N} P_i \mathbf{x} = \sum_{i=1}^{N} \sum_{\ell=1}^{n} \langle \mathbf{x}, e_{i,\ell}\rangle e_{i,\ell}$$

5.2 Necessary Background in Gabor Analysis

In this section, we provide a brief summary of Gabor frames which is used to construct our tight fusion frames. These concepts were originally introduced in Gabor's classic paper [16]; we will, however, be giving a modern treatment of those ideas here. (See, e.g., [6])

We index the components of a vector $\mathbf{x} \in \mathbb{C}^N$ by $\{0, 1, \cdots, N-1\}$, i.e., the cyclic group $\mathbb{Z}_N$. We will write $\mathbf{x}(k)$ instead of $\mathbf{x}_k$ to avoid algebraic operations on indices.

Gabor analysis concerns the interplay of the Fourier transform, translation operators, and modulation operators; we will review these concepts in this section. A reader who is familiar with these concepts may skip this section.

The discrete Fourier transform is fundamental for Gabor analysis. It is defined as

$$\mathcal{F}\mathbf{x}(m) = \hat{\mathbf{x}}(m) = \sum_{n=0}^{N-1} \mathbf{x}(n)\, e^{-2\pi i m \frac{n}{N}}.$$

The most important properties of the Fourier transform are the Fourier inversion formula and the Parseval formula [22]. The inversion formula shows that any $\mathbf{x}$ can be written as a linear combination of harmonics. This means the normalized harmonics $\{\frac{1}{\sqrt{N}} e^{2\pi i m \frac{(.)}{N}}\}_{m=0}^{N-1}$ form an orthonormal basis of $\mathbb{C}^N$:

$$\mathbf{x} = \frac{1}{N} \sum_{m=0}^{N-1} \hat{\mathbf{x}}(m)\, e^{2\pi i m \frac{n}{N}} \quad \mathbf{x} \in \mathbb{C}^N.$$

Moreover, the Parseval formula states:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{N} \langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^N,$$

which results in:

$$\sum_{n=0}^{N-1} |\mathbf{x}(n)|^2 = \frac{1}{N} \sum_{m=0}^{N-1} |\hat{\mathbf{x}}(m)|^2,$$

where $|\mathbf{x}(n)|^2$ quantifies the energy of the signal $\mathbf{x}$ at time n, and the Fourier coefficients $\hat{\mathbf{x}}(m)$ indicates that the harmonic $e^{2\pi i m \frac{(.)}{N}}$ contributes energy $\frac{1}{N}|\hat{\mathbf{x}}(m)|^2$ to $\mathbf{x}$.

We now define some necessary concepts in Gabor analysis that will be necessary in our construction. The cyclic translation operator $T : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$T\mathbf{x} = T\left(\mathbf{x}\left(0\right), \cdots, \mathbf{x}\left(N-1\right)\right)^t = \left(\mathbf{x}\left(N-1\right), \mathbf{x}\left(0\right), \cdots, \mathbf{x}\left(N-2\right)\right)^t.$$

The translation T_k is given by

$$T_k\mathbf{x}\left(n\right) = T^k\mathbf{x}\left(n\right) = \mathbf{x}\left(n-k\right).$$

The operator T_k alters the position of the entries of $\mathbf{x}$ modulo N. The modulation operator $M_\ell : \mathbf{C}^N \to \mathbb{C}^N$ is given by

$$M_\ell\mathbf{x} = \left(e^{-2\pi i\ell\frac{0}{N}}\mathbf{x}\left(0\right), e^{-2\pi i\ell\frac{1}{N}}\mathbf{x}\left(1\right), \cdots, e^{-2\pi i\ell\frac{N-1}{N}}\mathbf{x}\left(N-1\right)\right)^t.$$

Modulation operators are implemented as the pointwise product of the vector with harmonics $e^{-2\pi i\ell\frac{\cdot}{N}}$.

Translation and modulation operators are referred to as time-shift and frequency shift operators. The time-frequency shift operator $\pi\left(k,\ell\right)$ is the combination of translation operators and modulation operators:

$$\pi\left(k,\ell\right) : \mathbb{C}^N \to \mathbb{C}^N \quad \pi\left(k,\ell\right)\mathbf{x} = M_\ell T_k\mathbf{x}.$$

Hence, the short time-Fourier transform $V_\phi : \mathbb{C}^N \to \mathbb{C}^{N\times N}$ with respect to the window $\phi \in \mathbb{C}^N$ can be written as

$$V_\phi\mathbf{x}\left(k,\ell\right) = \left\langle \mathbf{x}, \pi\left(k,\ell\right)\phi\right\rangle = \sum_{n=0}^{N-1}\mathbf{x}\left(n\right)\overline{\phi\left(n-k\right)}e^{-2\pi i\ell\frac{n}{N}} \quad \mathbf{x} \in \mathbb{C}^N.$$

The short time-Fourier transform generally uses a window function ϕ, supported at neighborhood of zero that is translated by k. Hence, the pointwise product with $\mathbf{x}$

selects a portion of $\mathbf{x}$ centered at k, and this portion is analyzed using a Fourier transform. The inversion formula for the short time-Fourier transform is [22]

$$\begin{aligned}\mathbf{x}(n) &= \frac{1}{N\|\phi\|_2^2}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} V_\phi\mathbf{x}(k,\ell)\,\phi(n-k)\,e^{-2\pi i\ell\frac{n}{N}}\\ &= \frac{1}{N\|\phi\|_2^2}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}\langle\mathbf{x},\pi(k,\ell)\,\phi\rangle\pi(k,\ell)\,\phi(n) \quad \mathbf{x}\in\mathbb{C}^N.\end{aligned}$$

So it can be easily seen that for all $\phi \neq 0$, the system is a $N\|\phi\|^2$ tight Gabor frame.

Chapter 6

Gabor Tight Fusion Frames

6.1 A Tight Fusion Frame Construction For Finite Dimensional Signals

In this section, we show our method to construct Gabor tight fusion frames. The key idea is to start with a general approach for the construction of tight fusion frames, which has certain conditions that must be satisfied. We then show that these conditions are indeed satisfied using methods from the Gabor frame theory.

We begin by showing the following proposition, which is the generalization of our approach with certain conditions:

Proposition 6.1. *Consider a collection of frame sequences $\{\{f_{ij}\}_{i=1}^{L}\}_{j=1}^{M}$ within the finite dimensional Hilbert space $\mathbb{C}^N$, and denote $W_i := span\{f_{ij}\}_{j=1}^{M}$. Suppose there exists an index i_0 such that $\{f_{i_0 j}\}_{j=1}^{M}$ is a B-tight frame for W_{i_0} and also a set of coisometry operators $\{\mathcal{U}_i\}_{i=1}^{L}$ from $\mathbb{C}^N$ to $\mathbb{C}^N$ such that for each $j = 1, ..., M$, we have*

$$\{f_{ij}\}_{i=1}^{L} = \{\mathcal{U}_i f_{i_0 j}\}_{i=1}^{L}.$$

Furthermore, the set $\{f_{ij}\}_{i=1}^{L}$ is a A_j-tight frame in $\mathbb{C}^N$ for every $j = 1, \cdots, M$. Then we will have that $\{(W_i, 1)\}_{i=1}^{L}$ is a tight fusion frame.

Proof. Consider $\mathbf{x} \in W_i$. The set $\{\mathcal{U}_i f_{i_0 j}\}_{j=1}^{M}$ is a B-tight frame for W_i over $i =$

$1, \cdots, L$, because

$$\begin{aligned}\sum_{j=1}^{M} |\langle \mathbf{x}, \mathcal{U}_i f_{i_0 j}\rangle|^2 &= \sum_{j=1}^{M} |\langle \mathcal{U}_i^* \mathbf{x}, f_{i_0 j}\rangle|^2 \\ &= B\|\mathcal{U}_i^* \mathbf{x}\|^2 \\ &= B\|\mathbf{x}\|^2\end{aligned}$$

Hence we have, for any $x \in \mathbb{C}^N$:

$$\begin{aligned}\sum_{i=1}^{L} \|P_i x\|^2 &= \sum_{i=1}^{L} \frac{1}{B} \sum_{j=1}^{M} |\langle P_i \mathbf{x}, f_{ij}\rangle|^2 \\ &= \sum_{i=1}^{L} \frac{1}{B} \sum_{j=1}^{M} |\langle \mathbf{x}, f_{ij}\rangle|^2 \\ &= \frac{1}{B} \sum_{j=1}^{M} \sum_{i=1}^{L} |\langle \mathbf{x}, f_{ij}\rangle|^2 \\ &= \frac{1}{B} \sum_{j=1}^{M} A_j \|\mathbf{x}\|^2 \\ &= \frac{\sum_{j=1}^{M} A_j}{B} \|\mathbf{x}\|^2,\end{aligned}$$

where P_i is the orthogonal projection on W_i. The equality holds since $\{f_{ij}\}_{i=1}^{L}$ is a A-tight frame for $\mathbb{C}^N$ for $j = 1, \cdots, M$. ∎

In the following, we explain the method to construct tight fusion frame based on the Theorem 6.1 and Gabor frames on finite dimensional signals [22].

This theorem will form the cornerstone for our construction of Gabor tight fusion frames; however, we need some further concepts related to Gabor analysis and Gabor filters before we can continue with our construction.

6.1.1 Modeling Subspaces with Matrices

Every subspace W can be modeled by a matrix whose rows are an orthonormal basis for W. On the other hand, every subspace of dimension M can be represented by a matrix $N \times N$ whose first M rows are an orthonormal basis for W, since $\mathbb{C}^{N\times M}$ can be embeded in $\mathbb{C}^{N\times N}$. For example if the subspace W is generated by $\{\mathbf{e}_1, \cdots, \mathbf{e}_M\}$, then, the matrix associated to this subspace is as follows:

$$[\mathbf{e}_1, \cdots, \mathbf{e}_M, 0, \cdots, 0]^*$$

Moreover, a signal $\mathbf{x}$ of length N can be represenetd by a matrix of $N \times N$ since $\mathbb{C}^N$ can be embeded in $\mathbb{C}^{N\times N}$.

$$\tilde{X} = [\mathbf{x}, 0 \cdots, 0]^*$$

Based on the notation stated above, we define $\mathbb{C}^{N\times N}$-valued inner product on $\mathbb{C}^{N\times N}$ as follows:

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \mathbf{X}\mathbf{Y}^*$$

The translation and modulation operators for the space of complex valued square matrix of dimension N are defined as follows: Consider $l \in \mathbb{Z}_N$. The translation operator $\tilde{T}_\ell : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ is defined as follows:

$$\tilde{T}_\ell \left(\mathbf{e}_1, \cdots, \mathbf{e}_N\right)^* = \left(T_\ell \mathbf{e}_1, \cdots, T_\ell \mathbf{e}_N\right)^*$$

In fact the translation operator $\tilde{T}_\ell$ alters the position of each row of the matrix $\mathbf{X}$. The modulation operator $\tilde{M}_\ell : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ is given by

$$\tilde{M}_\ell \left(\mathbf{x}_1, \cdots, \mathbf{x}_N\right)^* = \left(M_\ell \mathbf{x}_1, \cdots, M_\ell \mathbf{x}_N\right)^*$$

Modulation operators are implemented as the pointwise product of each row of the matrix $\mathbf{X}$ with harmonics $e^{-2\pi i l \frac{\cdot}{N}}$. The translation and modulation operator on $\mathbb{C}^{N\times N}$ are unitary operators and the following properties can be concluded

$$\left(\tilde{T}_\ell\right)^* = \left(\tilde{T}_\ell\right)^{-1} = \tilde{T}_{N-l} \text{and} \left(\tilde{M}_\ell\right)^* = \left(\tilde{M}_\ell\right)^{-1} = \tilde{M}_{N-l}.$$

The circular convolution of two spaces $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{N\times N}$ is defined by the convolution of functions, which defined on the space $\mathbb{Z}_N \times \mathbb{Z}_N$ or can be written as:

$$\mathbf{X} * \mathbf{Y} = \left(\sum_{i=0}^{N-1} \mathbf{x}_i * \mathbf{y}_{0-i}, \cdots, \sum_{i=0}^{N-1} \mathbf{x}_i * \mathbf{y}_{N-1-i} \right)$$

Hence, if $\tilde{\mathbf{X}} = (\mathbf{x}, 0, \cdots, 0)$, the convolution of $\tilde{\mathbf{X}}$ and $\mathbf{Y}$ is given by

$$\tilde{\mathbf{X}} * \mathbf{Y} = (\mathbf{x} * \mathbf{y}_0, \cdots, \mathbf{x} * \mathbf{y}_{N-1})$$

Moreover, the circular involution or circular adjoint of $\mathbf{X} \in \mathbb{C}^{N\times N}$ is given by

$$\mathbf{X}^* = (\mathbf{x}_1^*, \cdots, \mathbf{x}_N^*)^*$$

where $\mathbf{x}_1, \cdots, \mathbf{x}_N \in \mathbb{C}^p$ and $\mathbf{x}_i^*(\ell) = \overline{\mathbf{x}(N-\ell)}$. Note that the complex linear space $\mathbb{C}^{N\times N}$ equipped with ℓ^1-norm, the circular convolution and involution defined above is a Banach $*$-algebra.

The unitary discrete Fourier transform of $\mathbf{X} \in \mathbb{C}^{N\times N}$ is defined by

$$\hat{\mathbf{X}} = (\mathcal{F}_N(\mathbf{x}_1), \cdots, \mathcal{F}_N(\mathbf{x}_N))$$

where $\mathbf{x}_1, \cdots, \mathbf{x}_N \in \mathbb{C}^N$ and the Fourier transform $\mathbf{x}_i$ is given by

$$\mathcal{F}_N(\mathbf{x}_i)(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}_i(k)\, \overline{\omega}_\ell(k) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}_i(k)\, e^{-2\pi i \ell \frac{k}{N}}$$

The Fourier transform is a unitary operator on the $\mathbb{C}^{N\times N}$ with the Frobenius norm. In fact, for all $\mathbf{X} \in \mathbb{C}^{N\times N}$:

$$\|\langle \hat{\mathbf{X}}, \hat{\mathbf{X}} \rangle\| = \|\langle \mathbf{X}, \mathbf{X} \rangle\|$$

We also have the following relationships.

$$\widehat{\tilde{T}_\ell \mathbf{X}} = \tilde{M}_\ell \hat{\mathbf{X}} \quad \widehat{\tilde{M}_\ell \mathbf{X}} = \tilde{T}_{N-\ell} \hat{\mathbf{X}} \quad \hat{\mathbf{X}}^* = \overline{\hat{\mathbf{X}}} \quad \widehat{\mathbf{X} * \mathbf{Y}} = \hat{\mathbf{X}}.\hat{\mathbf{Y}}$$

for $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{N\times N}$ and $\ell \in \mathbb{Z}_N$. The inverse Fourier formula for $\mathbf{X} \in \mathbb{C}^{N\times N}$ is given by

$$\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_N)^* = \left(\mathcal{F}_N^{-1}(\mathbf{x}_1), \cdots, \mathcal{F}_N^{-1}(\mathbf{x}_N)\right)^*$$

Translation operators are refered as time shift operators and modulation operators are refered as frequency shift operators. Time-frequency shift operators $\pi(k, l)$ combines translations by k and modulation by l.

$$\pi(k, \ell)\,\mathbf{X} = \tilde{M}_\ell \tilde{T}_k \mathbf{X}$$

The Gabor Fusion transform $V_\mathbf{Y}$ of a signal $\mathbf{x} \in \mathbb{C}^N$ with respect to the window $\mathbf{Y} \in \mathbb{C}^{N\times N}$ is given by

$$V_\mathbf{Y}\mathbf{x}(k, \ell) = \langle \mathbf{x}, \pi(k, \ell)\,\mathbf{Y}\rangle = \left(V_{\mathbf{y}_0}\mathbf{x}(k, \ell), \cdots, V_{\mathbf{y}_{N-1}}\mathbf{x}(k, \ell)\right)^* \tag{6.1}$$

Now consider $\mathbf{Y} \in \mathbb{C}^{N\times N}$ and $\Lambda \subset \{0, \cdots, N-1\} \times \{0, \cdots, N-1\}$. The set

$$(\mathbf{Y}, \Lambda) = \{\pi(k, \ell)\,\mathbf{Y}\}_{(k,\ell)\in\Lambda}$$

is called the Gabor Fusion System which is generated by $\mathbf{Y}$ and Λ. A Gabor Fusion System which spans $\mathbb{C}^N$ is a fusion frame and is referred to as a Gabor Fusion Frame.

6.2 Construction of Gabor Tight Fusion Frames

We now have all of the ingredients necessary for our construction; the next theorem explains the necessary conditions such that a set $\{\tilde{M}_\ell \tilde{T}_k \mathbf{Y}\}_{\ell=1,k=1}^{N,N}$ will be a tight fusion frame.

Theorem 6.2. *Assume $\mathbf{x} \in \mathbb{C}^N$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_M\}$ is a B-tight fusion frame for $W_{N,N} = span\{\mathbf{y}_1, \cdots, \mathbf{y}_M\}$. Consider also $W_{k,\ell} = span\{T_k M_\ell \mathbf{y}_j\}_{j=1}^M$ for $k, \ell = 1, \cdots, N$. Then, the set $\{W_{k,\ell}\}_{k=1,\ell=1}^{N,N}$ constitutes a $\frac{N\sum_{i=1}^N \|\mathbf{y_i}\|_2^2}{B}$ tight fusion frame.*

Proof. All that has to be done is to verify that $\left\{\{T_k M_\ell \mathbf{y}_i\}_{k,\ell=1}^{N}\right\}_{i=1}^{M}$ satisfies the criteria of proposition 6.1. First, for a given value of j, we have that $\{T_k M_\ell \mathbf{y_j}\}_{k,\ell=1}^{N}$ is a $A_j = N\|\mathbf{y}_j\|^2$ tight frame in $\mathbb{C}^N$ by the elementary Gabor theory (this can be seen the prior section). It should clear by its nature that the time-frequency shift operator $T_k M_\ell$ is a co-isometry for a set k, ℓ, since it was mentioned before T_k and M_ℓ are both unitary operators for every k, ℓ. Finally, we know by the assumption that $\{\mathbf{y}_j\}_{j=1}^{M}$ is B-tight on its ambient space $W_{0,0}$. Seeing that the conditions for the proposition are satisfied, we have the conclusion that $\{(W_{k,\ell}, 1)\}_{k,\ell=0}^{N-1}$ is a $\frac{N\|\mathbf{Y}\|_2^2}{B}$-tight fusion frame on $\mathbb{C}^N$. ∎

6.3 Gabor Fusion Frames and Signal Retrieval Modulo Phase

In this section, we are looking for some conditions such that the tight Gabor fusion frame allows for signal reconstruction modulo phase; we precisely define what this means in the following subsection.

6.3.1 Basic Background in Signal Retrieval Modulo Phase

Signal reconstruction modulo phase, (also known as *phaseless reconstruction* or *phase retrieval*) is a field that has gathered interest in the mathematical community in the last decade. Phaseless reconstruction is defined as the recovery of a signal modulo phase from the absolute values of fusion frame measurement coefficients arising from a fusion frame [19]. This is known to have applications to a disparate array of other scientific and applied disciplines, including X-ray crystallography [24], speech recognition[17, 28, 30], and quantum state tomography[29].

In the case of phase retrieval, the signal must be recovered from coefficients of dimension higher than one. Here, in the context of fusion frames, the problem is to recover $x \in \mathcal{H}_M$ "up to phase" from the measurements $\{\|P_i x\|\}_{i=1}^N$.

There are few publications about the phase retrievability of projections. The paper [20] used semidefinite programming to develop a reconstruction algorithm for when $\{W_i\}_{i=1}^N$ are equidimensional random subspaces. In [19] the authors characterized the phase retrieval fusion frames. Moreover, they show the relationship between the phase retrievality of fusion frames and the usual phase retrieval problem with families of measurement vectors.

Here a new method is demonstrated for the construction of tight fusion frames. There are hithero few examples of tight fusion frames except trivial ones made up of orthogonal subspaces, so this is a relevant and interesting advance. Moreover, there are few examples of phase retrieval fusion frames. Here, a condition that makes this structure allow phase retrieval is presented.

One can recover the signal modulo phase from fusion frame measurements. In this senario, consider we are given subspaces $\{W_i\}_{i=1}^N$ of M-dimensional Hilbert space $\mathcal{H}_M$ and orthogonal projections $P_i : \mathcal{H}_M \to W_i$. We want to recover any $\mathbf{x} \in \mathcal{H}_M$ (up to a global phase factor) from the fusion frame measurements $\{\|P_i \mathbf{x}\|\}_{i=1}^N$. To fix notation, denote $\mathbb{T} = \{c \in \mathbb{C}; |c| = 1\}$. The measurement process is then given by the map:

$$\mathcal{A} : \mathbb{C}^M/\mathbb{T} \to \mathbb{C}^N, \quad \mathcal{A}\mathbf{x}\,(n) = \|P_n \mathbf{x}\|$$

We say $\{W_i\}_{i=1}^N$ **allows phaseless reconstruction** or **allows phase retrieval** if $\mathcal{A}$ is injective; we call a frame (or fusion frame) with this property a **phase retrieval frame**. In the case where $\dim W_i = 1$ for $i = 1, \cdots, N$, the problem will be referred to as the classical phaseless reconstruction problem. In section 4, we will provide a novel structure of tight fusion frames where under particular conditions, will allow phaseless reconstruction.

6.3.2 A Brief Overview of Circulant Matrices

We will need to review a few key concepts of circulant matrices before we continue to the next section.

Definition 6.3. A circulant matrix is a matrix of the following form:

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{N-2} & & \ddots & \ddots & c_{N-1} \\ c_{N-1} & c_{N-2} & \dots & c_1 & c_0 \end{bmatrix}.$$

Remark 6.4. We denote the j^{th} division of unity as

$$\omega_j = \exp\left(\frac{2\pi i j}{N}\right)$$

We will need the following theorem; a proof is given in [33]

Theorem 6.5. *Let C be an $N \times N$ circulant matrix.*

Then $det(C) = \Pi_{j=0}^{N-1}\left(c_0 + c_1\omega_j + c_2\omega_j^2 + \cdots + c_{N-1}\omega_j^{N-1}\right)$.

Lemma 6.6. *Let C be a matrix as in 6.3 with $c_0, c_1, \dots, c_{n-1} = 1$ and $c_n, c_{n+1}, \dots, c_{N-1} = 0$ for some $0 < n < N$. Then C is singular if and only if there is some value j, $1 \leqslant j \leqslant N-1$, such that N divides into jn.*

Proof. By 6.5, we know that C is singular if and only if there is some j where $0 \leqslant j \leqslant N-1$ and $\sum_{k=0}^{N} c_k\omega_j^k = \sum_{k=0}^{n-1}\omega_j^k = 0$. We notice that for $j = 0$, we have $\sum_{k=0}^{n-1}\omega_0^k = \sum_{k=0}^{n-1} 1 = n$, so we will only consider the values $1 \leqslant j \leqslant N-1$.

Consider $\sum_{k=0}^{n-1} \omega_j^k$. The geometric series gives us that this is equal to $\frac{1-w_j^n}{1-w_j}$; this is zero if and only if $w_j^n = \exp\left(\frac{2\pi ijn}{N}\right) = 1$. But this will only happen exactly when $\frac{jn}{N}$ is an integer, that is to say, when N divides into jn.

∎

6.3.3 Phase Retrieval Properties of Gabor Tight Fusion Frames

To state these conditions, we provide some theorems should be necessary to explain the main result. The next lemma shows that if we add a vector to a phase retrieval frame, the new frame also allows so-called phase retrieval.

Lemma 6.7. *Let $\{\phi_i\}_{i=1}^N$ be a frame for $\mathbb{C}^N$ that allows phaseless reconstruction. If we add a vector ϕ_{N+1} to $\{\phi_i\}_{i=1}^N$, then $\{\phi_i\}_{i=1}^{N+1}$, this will also allow phaseless reconstruction.*

Proof. Consider that for $x_1, x_2 \in \mathbb{C}^N$, we have $\{|\langle x_1, \phi_i\rangle|\}_{i=1}^{N+1} = \{|\langle x_2, \phi_i\rangle|\}_{i=1}^{N+1}$. Hence, we have $\{|\langle x_1, \phi_i\rangle|\}_{i=1}^{N} = \{|\langle x_2, \phi_i\rangle|\}_{i=1}^{N}$. So, $x_1 = cx_2$ where $|c| = 1$ since $\{\phi_i\}_{i=1}^N$ allows phase retrieval for $\mathbb{C}^N$. Thus $\{\phi_i\}_{i=1}^{N+1}$ also allows phase retrieval. ∎

The prior lemma is important in the construction of phase retrieval frames. If we have a phase retrieval frame for $\mathbb{C}^N$, then we can construct a new frame that also allows phase retrieval by adding a vector to the frame vector set. On the other hand, to show the phase retrievability of a frame, it is enough to show that a subset of the frame vectors that spans the ambient space allows phaseless reconstruction.

In [18] the conditions on the window function such that the generated Gabor frame allows phaseless are given; we now present a method to produce a phase retrieval Gabor fusion frame. The following theorem demonstrates the relationship of the phase retrievability of the Gabor fusion frames and the phase retrievability of the frame vectors which spans subspaces.

Theorem 6.8. *Let* $\{e_i\}_{i=1}^N$ *be an orthonormal basis for* $\mathbb{C}^N$ *with the property* $T_k e_i = e_{i+k \ mod \ N}$. *Let* $\{e_i\}_{i=1}^n \subset \{e_i\}_{i=1}^N$ *span the* n*-dimensional subset* $W_{0,0} \subset \mathbb{C}^N$. *Moreover* $W_{k,\ell} = span \ \{T_k M_\ell e_i\}_{i=1}^n$ *for* $k, \ell = 0, 1, \cdots, N-1$. *If there exists an* i_0 *such that* $\{T_k M_\ell e_{i_0}\}_{k,\ell=0}^{N-1}$ *is a phase retrieval frame for* $\mathbb{C}^N$, *then* $\{W_{k,\ell}\}_{k,\ell=0}^{N-1}$ *is a phase retrieval fusion frame if and only if for all values* $1 \leqslant j \leqslant N-1$, *we have that* N *does not divide into* jn.

Proof. We assume that $\{T_k M_\ell \mathbf{e}_{i_0}\}_{k,\ell=0}^{N-1}$ is a phase retrieval frame. We show that the derived fusion frame inherits this property:

We consider some $\mathbf{x} \in \mathbb{C}^N$. Notice that we have $|\langle \mathbf{x}, T_k M_\ell \mathbf{e}_i\rangle|^2 = |\langle \mathbf{x}, M_\ell T_k e_i\rangle|^2 = |\langle \mathbf{x}, M_\ell \mathbf{e}_{i+k \ mod \ N}\rangle|^2$. This gives us the following:

$$\|P_{k,\ell}\mathbf{x}\|_2^2 = \sum_{i=1}^n |\langle \mathbf{x}, T_k M_\ell \mathbf{e}_i\rangle|^2 = \sum_{i=1}^n |\langle \mathbf{x}, M_\ell T_k \mathbf{e}_i\rangle|^2 = \sum_{i=1}^n |\langle \mathbf{x}, M_\ell \mathbf{e}_{i+k}\rangle|^2. \tag{6.2}$$

To show that there is an injective mapping from the fusion frame measurements, $\{\|P_{k,\ell}\mathbf{x}\|_2^2\}_{k,\ell=0}^{N-1}$, to the vector $\mathbf{x}$ modulo phase (i.e., the equivalence class $\{c\mathbf{x} : |c| = 1\}$), we can just show that we can derive the values of the original frame measurements $\{|\langle \mathbf{x}, T_k M_\ell \mathbf{e}_{i_0}\rangle|^2\}_{k,\ell=0}^{N-1}$ from the fusion frame measurements. We can see this in the following way:

For each $\ell \in \{0, \cdots, N-1\}$, consider the vector:

$$v_\ell = [|\langle \mathbf{x}, T_0 M_\ell \mathbf{e}_{i_0}\rangle|^2, |\langle \mathbf{x}, T_1 M_\ell \mathbf{e}_{i_0}\rangle|^2, \cdots, |\langle \mathbf{x}, T_{N-1} M_\ell \mathbf{e}_{i_0}\rangle|^2]^T$$

Now, consider the operator $S : \mathbb{R}^N \mapsto \mathbb{R}^N$, where S is the circulant matrix such that the j^{th} row is $T_{j-1}([1, \cdots, 1, 0, \cdots, 0])$, where the area of support in each row is n, and all nonzero values are 1:

$$S = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & \cdots & & & & & \cdots & & \\ 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 \\ \vdots & & & \cdots & & & & & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

By lemma 6.6, it can be seen that S is not singular.

Now consider the vector Sv_ℓ. We will get the following output, based on equation 6.2 with regard to the fusion frame measurements:

$$[\|P_{0,\ell}\mathbf{x}\|_2^2, \|P_{1,\ell}\mathbf{x}\|_2^2, \cdots, \|P_{N-1,\ell}\mathbf{x}\|_2^2]^T = Sv_\ell$$

This tells us that for a Gabor fusion frame to allow phase retrieval is tantamount to S being nonsingular, which we have already seen.

∎

We shall end with a brief example of a Gabor fusion frame that allows phase retrieval, as an application of the prior theorem:

Example 6.9. Consider the orthogonal unit vectors $e_1 = \mathbf{1}_{\{1,2,4\}}/\sqrt{3}$ and $e_2 = \mathbf{1}_{\{3\}}$ in the space $\mathbb{C}^7$. By the Proposition 2.2 in [18], $\{T_k M_l \mathbf{e}_1\}_{k,l=0}^6$ is a phase retrieval Gabor frame for $\mathbb{C}^7$. Suppose that $Y_{k,l} = span\ \{T_k M_l e_i\}_{i=1}^2$ for $k, l = 0, \cdots, 6$. Since e_1 and e_2 are orthogonal, they then comprise a tight frame for the subspace $W_{0,0}$. As a result we fullfill the requirements of the Theorem 6.8 and the Gabor fusion frame $\{\mathbf{Y}_{k,l}\}_{k,l=0}^6$ allows phase retrieval.

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