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# STABILITY OF LAMINAR FLOW IN CURVED CHANNELS

Chi-Shun Yih

*Associate Professor of Engineering Mechanics*

*University of Michigan*

W. M. Sangster

*Associate Professor of Civil Engineering*

*University of Missouri*

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# Stability of Laminar Flow in Curved Channels†

By CHIA-SHUN YIH‡ and W. M. SANGSTER§
Iowa Institute of Hydraulic Research, State University of Iowa

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ABSTRACT

Occurrence of secondary flow of a viscous fluid flowing under a pressure gradient in curved channels has often been explained by invoking the existence of boundary layers before the primary flow is established. In this paper it is shown that instability can occur even after the full establishment of the primary flow. Chandrasekhar's method is used in the analysis and the relationship between the stability parameter and the wave number of the disturbance for neutral stability is obtained.

## § 1. INTRODUCTION

WHEREAS the occurrence of spirals in fluid flow under a pressure gradient in curved channels is common knowledge, explanations of the phenomenon have been based upon the existence of boundary layers before the primary flow is fully established, and the fully developed two-dimensional flow has generally been believed to be free of such spirals (Goldstein 1938). In the present paper, this phenomenon will be analysed by Chandrasekhar's (1954) method under the assumption that the primary flow is two-dimensional and between two concentric circular cylinders of small spacing, and that it is already fully established. The object is to obtain definite information about the stability of such a flow against the formation of Taylor–Görtler vortices of various transverse wavelengths.

## § 2. VELOCITY DISTRIBUTION IN THE PRIMARY FLOW

Cylindrical coordinates $(r, \phi, z)$ will be used, with the $z$-axis coinciding with the axis of the cylinders. If the radius of the inner cylinder is $a$ and that of the outer one is $b$, the distribution of the velocity $V$ of the primary flow can be found by solving the Navier–Stokes equations and is given (Goldstein 1938) by

$$\frac{V}{K}=A\left(\frac{r}{b}-\frac{b}{r}\right)+\frac{r}{b}\ln\frac{r}{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which

$$A=\frac{a^2(\ln a-\ln b)}{b^2-a^2}, \qquad K=\frac{b}{2\mu}\frac{\partial P}{\partial\phi} \quad . \quad . \quad . \quad . \quad (2)$$

$\mu$ being the dynamic viscosity and $P$ the pressure for the primary flow.

† Communicated by S. Chandrasekhar, F.R.S.
‡ Now at University of Michigan
§ Now at University of Missouri

If the spacing $d$ of the cylinders (or $b-a$) is very small compared with the radii of the cylinders it can be shown from eqn. (1) that the velocity distribution of the primary flow is essentially the same as that for plane Poiseuille flow, i.e.

$$V=\frac{d^2}{2\mu a^2}\frac{\partial P}{\partial \phi}(\xi^2-\xi)=\frac{Kd^2}{a^2}(\xi^2-\xi) \qquad . \quad . \quad . \quad . \quad (3)$$

in which

$$\xi=\frac{r-a}{b}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The mean velocity is, to the same degree of approximation,

$$V_m=-\frac{d^2K}{6a^2}$$

so that the Reynolds number based on $V_m$ and $d$ is

$$R=\frac{V_m d}{\nu}=-\frac{d^3K}{6\nu a^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which $\nu$ is the kinematic viscosity. From eqn. (3) it follows that, with terms of higher order in $d$ neglected,

$$\frac{V}{r}=\frac{Kd^2}{a^3}(\xi^2-\xi) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$V'+\frac{V}{r}=\frac{Kd}{a^2}(2\xi-1) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which the prime indicates differentiation with respect to $r$. These quantities will be useful later in the investigation of stability.

## § 3. Formulation of the Stability Problem

To investigate the stability of the primary flow, one may superpose on it a time-dependent perturbation with (small) velocity components $(u, v, w)$ in the directions of $r$, $\phi$, and $z$, respectively, and with corresponding pressure $p$. If it is kept in mind that the primary flow is such that the Navier–Stokes equations and the equation of continuity are satisfied, and if terms of higher order than the first in the perturbation quantities are neglected, the Navier–Stokes equations are

$$\frac{\partial u}{\partial t}-\frac{2Vv}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 u-\frac{u}{r^2}\right), \qquad . \quad . \quad . \quad . \quad (8)$$

$$\frac{\partial v}{\partial t}+uV'+\frac{Vu}{r}=\nu\left(\nabla^2 v-\frac{v}{r^2}\right), \qquad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

in which $t$ and $\rho$ denote time and density respectively, and

$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\,.$$

The equation of continuity is

$$\frac{\partial(ur)}{\partial r}+\frac{\partial(wr)}{\partial z}=0. \quad \ldots\ldots\ldots \quad (11)$$

Now, following Taylor (1923), one may assume

$$(u,\, v,\, w)=\exp\,(\sigma t)[u_1(r)\cos kz,\ v_1(r)\cos kz,\ w_1(r)\sin kz]. \quad . \quad (12)$$

If $p$ is eliminated between eqns. (8) and (10), and if eqn. (12) is used, one has

$$\frac{\nu}{k}D\left(\nabla_1{}^2-k^2-\frac{\sigma}{\nu}\right)w_1=-\frac{2V}{r}v_1-\nu\left(\nabla_1{}^2-k^2-\frac{1}{r^2}-\frac{\sigma}{\nu}\right)u_1 \quad . \quad (13)$$

in which

$$D\equiv d/dr, \qquad \nabla_1{}^2=D^2+\frac{1}{r}D.$$

Furthermore eqn. (9) can now be written as

$$u_1\left(V'+\frac{V}{r}\right)=\nu\left(\nabla_1{}^2-k^2-\frac{1}{r^2}-\frac{\sigma}{\nu}\right)v_1 \quad \ldots \quad (14)$$

and the equation of continuity is

$$\frac{u_1}{r}+Du_1+kw_1=0. \quad \ldots\ldots\ldots \quad (15)$$

Although for neutral stability $\sigma$ may be purely imaginary, one may follow Taylor in taking it to be zero—at any rate if one is looking for the formation of Taylor–Görtler vortices. For neutral stability, therefore, eqn. (14) becomes

$$\nu L_1v_1=(V'+V/r)u_1 \quad \ldots\ldots\ldots \quad (16)$$

in which

$$L_1\equiv D^2+\frac{1}{r}D-k^2-\frac{1}{r^2}$$

and it can be readily shown that eqns. (13) and (15) yield

$$\frac{\nu}{k^2}L_1{}^2u_1=\frac{2V}{r}v_1. \quad \ldots\ldots\ldots \quad (17)$$

Since the spacing $d$ is much smaller than $a$ or $b$, and since the perturbation velocities change appreciably as $r$ varies between $a$ and $b$, and the wavelength $(2\pi/k)$ is comparable with $d$, the part

$$\frac{1}{r}D-\frac{1}{r^2}$$

in the operator $L_1$ is small compared with the rest of the terms. If now one denotes $kd$ by $\alpha$ and expresses $u_1$ and $v_1$ in terms of $V_m$, eqns. (16) and (17) become, in virtue of eqns. (5), (6), and (7),

$$Lv_1=\frac{Kd^3}{\nu a^2}u_1=-6R(2\xi-1)u_1, \quad \ldots\ldots \quad (18)$$

$$L^2u_1=\frac{2Kk^2d^6}{\nu a^3}(\xi^2-\xi)v_1=-12\alpha^2R\beta(\xi^2-\xi)v_1 \quad \ldots \quad (19)$$

in which

$$L = d^2/d\xi^2 - \alpha^2, \qquad \beta = d/a.$$

If one sets

$$u_1 = -12\alpha^2 R\beta W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

one has finally

$$L^2 W = (\xi^2 - \xi) v_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$L v_1 = S\alpha^2 (2\xi - 1) W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where

$$S = 72 R^2 \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

is the stability parameter. Equations (21) and (22) are to be solved with the boundary conditions

$$v_1(0) = v_1(1) = 0, \qquad W(0) = W(1) = 0, \qquad W'(0) = W'(1) = 0 \ . \quad (24)$$

the last two being a consequence of eqns. (15) and (20) and of the fact that $u_1$ and $w_1$ are zero at $\xi=0$ and $\xi=1$. The primes in the last two of eqns. (24) indicate differentiation with respect to $\xi$. The differential system consisting of eqns. (21), (22), and (24) represents an eigenvalue problem, the solution of which will result in a functional relationship between $S$ and $\alpha$ for neutral stability since $\sigma$ has been assumed to be zero to start with.

## § 4. Outline of the Method of Solution

Since the method of solution used in this paper is that of Chandrasekhar and is already described in detail elsewhere (Chandrasekhar 1954), only an outline of the essential steps will be presented here. First, one takes

$$v_1 = \sum_{n=1}^{\infty} A_n \sin n\pi\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

to satisfy the first two conditions in eqn. (24). Then this expression for $v_1$ is substituted into eqn. (21) which can be solved to yield the result

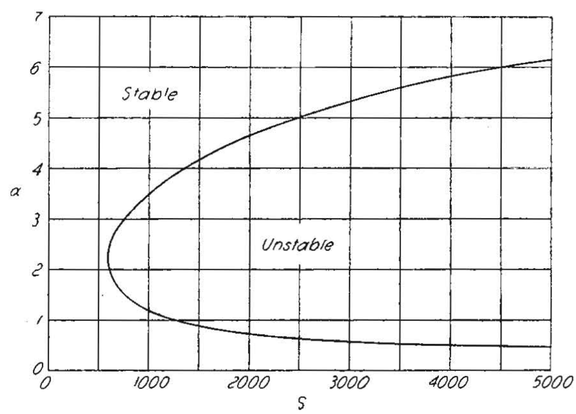
$$W = \sum_{m=1}^{\infty} \frac{A_m}{M^2} \left[ B_m \cosh \alpha\xi + C_m \sinh \alpha\xi + D_m \xi \cosh \alpha\xi + E_m \xi \sinh \alpha\xi + (\xi^2 - \xi) \sin m\pi\xi + \frac{4m\pi(2\xi-1)\cos m\pi\xi}{M} + \frac{4(M - 6m^2\pi^2)\sin m\pi\xi}{M^2} \right] \quad . \quad . \quad . \quad . \quad (26)$$

in which $M = m^2\pi^2 + \alpha^2$ and the constants $B$ to $D$ can be determined to satisfy the last four conditions in eqns. (24). With these constants so determined, eqn. (26) is then substituted into eqn. (22) and the right-hand side expressed in a Fourier sine series. Then coefficients of the individual harmonics at left and right are equated. This result in an infinite number of homogeneous linear equations involving the $A$'s. Finally the eliminant of these equations (or the condition that the $A$'s are not all zero) furnishes the desired relationship between $S$ and $\alpha$.

Although the eliminant is a determinant of infinitely many rows and columns, Chandrasekhar (1954) has shown that often it is sufficiently accurate to equate the first element (in the first column and first row) to zero, and that the result of the second approximation (by equating a two-by-two determinant to zero) does not differ much from that of the first. The same was found to be true in a previous calculation (for a slightly different primary flow) by the authors whenever the second approximation was tried for spot checking, so that the relationship between $S$ and $\alpha$ determined by the first approximation is believed to be sufficiently accurate. The ease with which this relationship was obtained is additional proof of the power of Chandrasekhar's method in solving problems of the present type.

## § 5. Results and Concluding Remarks

The desired relationship between $S$ and $\alpha$ for neutral stability of the primary flow is given graphically in the figure. It is seen from this




Relationship between the wave number $\alpha$ and the stability parameter $S$ for neutral stability.

figure that established laminar flow under a pressure gradient in a narrow curved channel is unstable for values of the parameter $S$ greater than the approximate critical value 600, which corresponds to a wavelength of the Taylor–Görtler vortices equal to $2\pi/2{\cdot}25$ or $2{\cdot}79$ times the width of the channel. Since the flow tends to be unstable as $S$ is increased, for the same Reynolds number and the same (small) width of the channel it does so as the curvature is increased, as can be seen from the definition of $S$. Thus centrifugal force is as much a cause of instability for the established flow as it is for the unestablished one, and the occurrence of secondary flow cannot be considered to be exclusively a consequence of boundary-layer development.

Attention might also be called to the fact that, since centrifugal force can be considered as a form of gravitation (Mach, Einstein), the non-uniformity of it is a kind of stratification—a stratification not of density, for the density is homogeneous, but of specific weight in the general sense. Secondary flow occurs if this stratification is not stable, and the mechanics of it is not unlike the falling of a heavier fluid in a lighter one, particularly if the boundary conditions are alike, and diffusion takes place to render the variation of density and that of centrifugal acceleration strictly comparable. This point of view is substantiated elsewhere by the equality of the Rayleigh number for thermal instability and the Taylor number for rotational instability (both being 1708), as first pointed out by Low (1929).

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## References

Goldstein, S. (Editor), 1938, *Modern Developments in Fluid Dynamics* (Oxford), vol. I, pp. 84–87, 315–316.
Chandrasekhar, S., 1954, *Mathematika*, **1**, 5.
Taylor, Sir Geoffrey, 1923, *Phil. Trans. Roy. Soc.* A, **223**, 289.
Low, A. R., 1929, *Proc. Roy. Soc.* A, **125**, 180.

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