On Some Contributions to Size-biased Probability Distributions

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CERTIFICATE

This is to certify that the work embodied in this thesis entitled, **"On Some Contributions to Size Biased Probability Distributions"** is the original work carried out by **Mr. Javaid Ahmad Reshi**, under our supervision and is suitable for the award of the degree of ***Doctor of Philosophy in Statistics***.

The thesis has reached the standard fulfilling the requirements of regulations relating to the degree. The results contained have not been submitted earlier to this or any other University or Institute for the award of degree or diploma.

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ABSTRACT

Statistical distributions and models are used in many applied areas such as economics, engineering, social, health and biological sciences. In this era of inexpensive and faster personnel computers, practitioners of statistics and scientists in various disciplines have no difficulty in fitting a probability model to describe the distributions of a real-life data set. Traditional enviromentric theory and practice have been occupied with randomization and replication. But in environmental and ecological work, observations also fall in the non-experimental, non-replicated and non-random catogries.The problems of model specification and data interpretation then acquire special importance and great concern. The theory of weighted distributions provides a unifying approach for these problems. Weighted distributions take into account the method of ascertainment, by adjusting the probabilities of actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Failure to make such adjustments can lead to incorrect conclusions. The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weight function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. Size-biased distributions are a special case of the more general form known as weighted distributions. These distributions arise in practice when observations from a sample are recorded with unequal probability. In Bayesian Statistics, the posterior distribution summarizes the current state of knowledge about all the uncertain quantities including unobservable parameters. In this thesis, the efforts have been made to study the areas. The thesis is divided into five chapters:

Chapter 1 is devoted to the introduction and genesis of the probability Distributions, Definitions, pre-requisites and other preliminaries for the use in the subsequent chapter. An extensive survey of the literature available on the topic has also been reviewed. The first chapter provides a strong basis for the development of the rest of the chapters.

Chapter 2 deals with the introduction to Gamma, Beta and Exponential distributions. We have proposed a new class of Size-biased classical Gamma, beta and exponential distributions. The

structural and characterizing properties including moments, moment generating function, characteristic function, Shannon's entropy, Fisher's information matrix etc. have been derived and studied. Some important theorems are derived and the relation with other related distributions are identified. Also, a likelihood ratio tests for size-biasedness is conducted. A new moment method of estimation has been proposed to estimate the parameters of the new models. The estimation of parameters of new models has been obtained by employing the methods of moments, maximum likelihood and Bayesian method of estimation. The Bayesian estimation of Size biased Gamma and exponential Distributions have been obtained by using squared-error and Al-Bayyati's (2002) new loss function under different priors. The survival functions of size biased gamma and exponential distributions have been derived under Jaffrey's and extension of Jaffrey's prior. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator, when sample sizes are assumed to be low, median and high.

Generalized Gamma distribution has been considered in **Chapter 3,** we provide a new class of Size-biased Generalized Gamma distribution. The Structural properties of the new model including moments, hazard function, reverse hazard functions, coefficient of variation, mode, moment generating function, characteristic function, Shannon's entropy, Generalized entropy, entropy estimation, AIC and BIC and Fisher's information matrix has been obtained. Some important theorems have been derived and the relation with other related distributions are identified. Also; a likelihood ratio test for size-biasedness is conducted. A new moment method of estimation has been proposed to estimate the parameters of the new model. The estimation of parameters of new model has been obtained by employing the methods of moments, maximum likelihood and Bayesian method of estimation. The Bayesian estimation of parameters of Size-biased Generalized Gamma distribution are obtained by using squared-error and Al-Bayyati's new loss function under different priors. The survival functions of size biased Generalized Gamma distribution have been derived and studied under Jaffrey's and extension of Jaffrey's prior. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator, when sample sizes are assumed to be low, median and high. Also, it has been observed that Bayes' estimator provides better results and estimates as compared to classical estimators. In this chapter, the AIC and BIC values of exponential model are smaller as compared to Size biased exponential and Size biased Gamma model, so exponential model is more preferable than the other models for the real data in hand.

Chapter 4 Completely devoted to introduction to Generalized Beta distribution of first and second kind. We have considered a new class of Weighted Generalized Beta distribution of first kind, Size-biased Generalized Beta distribution of first kind and Size-biased Generalized Beta distribution of second kind. The Structural properties of these new models including mean, variance, coefficient of variation, mode and harmonic mean has been derived. Also, a likelihood ratio test for size-biasedness is conducted. A new moment method of estimation has been proposed to estimate the parameters of these new models. The estimation of parameters of these new models has been obtained by employing the method of moments. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator, when sample sizes are assumed to be low, median and high. Also, it has been observed that Bayes' estimator under squared error and Al-Bayyati's loss function provides better results and estimates as compared to classical estimators.

Generalized Rayleigh distribution has been studied in **Chapter 5.** We have proposed a new class of Size-biased Generalized Rayleigh distribution. The Structural properties of the new model including moments, hazard function, reverse hazard functions, coefficient of variation, mode, moment generating function, characteristic function, Shannon's entropy and Fisher's information matrix have been derived. Also, a likelihood ratio test for size-biasedness is conducted. A new moment method of estimation has been proposed to estimate the parameters of the new model. The estimation of parameters of new model has been obtained by employing the methods of moments, maximum likelihood and Bayesian method of estimation. The Bayesian estimation of parameters of Size-biased Generalized Rayleigh distribution are obtained by using squared-error and Al-Bayyati's (2002) new loss function under different priors. A simulation study has been performed to compare the Bayes' estimators with the MLE estimator, when sample sizes are assumed to be low, median and high. Also, it has been observed that Bayes' estimator under squared error and Al-Bayyati's loss function provides better results and estimates as compared to classical estimators.

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PARENTS

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In the name of Allah Most Gracious, Most Merciful. All bounties Are in the hand of Allah: He grants them To whom He pleases And Allah cares for all, And He knows all things. For His Mercy He specially chooses Whom he pleases: For Allah is the Lord of bounties unbounded.

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Javaid Ahmad Reshi

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CHAPTER – 1

INTRODUCTION

1.1 Introduction

Uncertainty plays an important role in our lives. A satisfactory description of uncertainty is by means of probability. The probability is a powerful tool of maintaining, understanding, and controlling this important concept in our decision making. Statistical distributions and models are used in many applied areas such as economics, engineering, social, health and biological sciences. In this era of inexpensive and faster personnel computers, practitioners of statistics and scientists in various disciplines have no difficulty in fitting a probability model to describe the distributions of a real-life data set. Indeed, statistical distributions are used to model a wide range of practical problems. Successful applications of these probability models require a thorough understanding of the theory and familiarity with the practical situations where some distributions can be postulated.

In modern life, it has become a fashion to describe phenomena in quantitative terms for its investigation. In scientific researchers we prefer studying the results of the investigations in quantitative rather than qualitative terms. Indeed, the degree of dependence on quantitative methods has come to be regarded as the measure of the maturity of any science. In studying the phenomenon, a statistician builds up a suitable model for the probability law that is actually in operation. One has to decide whether the underlying distribution should approximately be regarded as discrete or continuous or of a mixed type. One of the twin functions of statistical theory is to suggest probability models that may be appropriate for different types of situations and to classify them into broad groups. A mathematical model or a function which associates different probabilities with the various outcomes of a random experiment which are eventually quantified in that a real value of each outcome is assigned is known as "probability distribution function".

1.2 Some basic Probability Distributions

The probability distributions form a basic and promising field of study in the domain of statistics. These distributions provide a simple and rational concept of stochastic models. In fact, the moment one gets into a stochastic problem where nothing more than counting is involved, one is dealing with discrete distributions.

The field of discrete distributions has been found to have tremendous potential for wider and deeper exploration. The last fifty years or so have seen a vast amount of literature appearing in this field. A large number of discrete distributions have been evolved in recent past and in a very short span of time the number of these discrete distributions has gone very high. The obvious reason for this is that a distribution stands on some stipulated assumptions and any variation in these stipulated assumptions leads to a different distribution to represent a different situation. A distribution essentially needs revision and modification depending upon the nature of change in the situation and this gives rise to a new distribution.

In almost all the basic discrete distributions, two important assumptions are made. (i) The trials are independent and (ii) the probability of success at each trial is constant. It has been observed that the probability of occurrence of an event does not always remain constant and the trails are not independent. This opens a direction for the generalization of classical discrete distributions and many researchers have obtained different generalizations of these distributions dropping the assumption of independence of trials and constant probability of success at each trial.

Presently, there exist a large number of generalizations of basic discrete distributions in the statistical literature. These distributions are classified as compound, mixed, modified, contagious or generalized distributions. A good account of these distributions is available in Johnson and Kotz (1969) and Johnson, Kotz and Kemp (1992).

In case of continuous distributions, when we deal with variables like heights and weights, we find that such variables can take a non-enumerable infinite set of values or more precisely they can take any values in the given interval $a \leq x \leq b$ of the arithmetic continuum. Such variates are called continuous variates and their probability distributions are accordingly known as continuous probability distributions. Mathematicians also call such a distribution absolutely continuous, since its cumulative distribution function is absolutely continuous with respect to the Lebesgue measure λ. Formally, if X is a continuous random variable, then it has a probability density function $f(x)$, and therefore its probability of falling into a given interval, say $[a, b]$ is given by the integral

$$\Pr(a \leq X \leq b) = \int_a^b f(x)dx \tag{1.2}$$

In particular, the probability for X to take any single value a is zero, because an integral with coinciding upper and lower limits is always equal to zero. The definition states that a continuous probability distribution must possess a density, or equivalently, its cumulative distribution function is absolutely continuous. This requirement is stronger than simple continuity of the cumulative distribution function, and there is neither a special class of distributions, singular distributions, which are neither continuous nor discrete nor a mixture of those. We discuss below some important continuous probability distributions.

1.2.1 Normal Distribution

The normal distribution plays a very important role in the statistical theory as well as methods. The names of the great mathematician such as Gauss, Laplace, Legendre & others are associated with the discovery & use of the distribution of errors of measurement. The earliest published derivation of the normal distribution was an approximation to a binomial distribution by De-Morvie (1738). Laplace (1774) obtained the normal distribution as an approximation to hyper-geometric distribution and advocated tabulation of the probability integral $\Phi(x)$ .The work of Gauss (1809, 1816) respectively established techniques based on the normal distribution which became standard methods used during the nineteenth century. Davis (1952) has shown that the normal distributions give quite a good fit for the failure time data. Bazovsky (1961) discussed the use of the normal distribution in life testing & reliability problems.

Mathematically, a random variable X is said to have normal distribution with location parameter μ and scale parameter σ if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} ; -\infty < x < \infty \; ; \; -\infty < \mu < \infty, \sigma > 0 \qquad (1.2.1)$$

Some of its important properties are discussed below:

(i) The normal distribution curve is bell shaped &symmetrical about the line $x = \mu$

(ii) The mean, median and mode of the normal distribution coincide.

(iii) The area under the normal curve within its range $-\infty to \infty$ in always unity i.e.

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = 1 \qquad (1.2.2)$$

(iv) All odd order moments of the normal distribution are zero.

(v) The first raw moments, i.e. mean $= \mu, \mu_2 = \sigma^2, \mu_3 = 0, \mu_4 = 3\sigma^2, = 3\mu_2{}^2$.

(vi) The maximum probability say, $Max\left[f(x)\right]$ of the normal curve occurs at the $x = \mu$ whereas $Max\left[f(x)\right] = \frac{1}{\sigma\sqrt{2\pi}}$. As σ increase $f(x)$ decreases and the curve becomes more and more flat and vice-versa.

(vii) Moment generating function of the normal distribution $N(x : \mu, \sigma^2)$ is

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(tx) \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} dx = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\} \qquad (1.2.3)$$

(viii) Characteristic function of normal distribution $N(x : \mu, \sigma^2)$ is

$$\Phi_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(itx) \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\} dx = \exp\left\{i\mu t + \frac{1}{2}\sigma^2 t^2\right\} \qquad (1.2.4)$$

(ix) The point of inflexion of the normal curve is given by $x = \mu \pm \sigma$ and at this point.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}}$$

(x) Most of the discrete distributions such as binomial, Poisson, etc tend to normal distribution as n increase i.e. $n \to \infty$.

(xi) Many variables which are not normally distributed can be normalized through suitable transformations.

1.2.2 Gamma Distribution

Gamma distribution has been quite extensively used as a lifetime model, though not censored. The gamma distribution is most widely used model for precipitation data. It fits a wide variety of lifetime data adequately, besides failure process models that lead to it. Inference for gamma model has been considered by Engelhard and Bain (1978), Choa and Glaser (1978), Prentice (2002) and Lawless (2003) have made significant contributions.

A random variable X is said to have a two parameter Gamma distribution if its p.d.f is given by

$$f(x;\alpha,\beta)=\frac{1}{\Gamma(\beta)}\frac{x^{\beta-1}}{\alpha^{\beta}}\exp\left\{\frac{-x}{\alpha}\right\}\ ; x>0;\ \alpha,\beta>0 \qquad (1.2.5)$$

where α is a scale parameter and β is sometimes called the index or shape parameter. $\Gamma(\beta)$ is the well known gamma function. For $\beta=1$, the gamma distribution reduces to the one parameter exponential distribution with parameter α and its pdf

$$f(x)=\frac{1}{\alpha}\exp\left\{\frac{-x}{\alpha}\right\}\ ; x>0;\ \alpha>0 \qquad (1.2.6)$$

For α =1, the distribution is called the one parameter gamma distribution with pdf

$$f(x;\beta)=\frac{x^{\beta-1}}{\Gamma(\beta)}\exp\{-x\}\ ; x>0;\ \beta>0 \qquad (1.2.7)$$

The moments of gamma distribution are

$$E(X)=\mu_1'=\alpha\beta \text{ and } V(X)=\mu_2'=\alpha\beta^2 \qquad (1.2.8)$$

The moment generating function of gamma distribution is

$$M_x(t)=(1-\alpha t)^{-\beta} \qquad (1.2.9)$$

Gamma distribution does not fit a wide variety of lifetime data adequately; however, there are failure process models that lead to it. It also arises in some situations involving the exponential distribution, because of the well known results that the sums of independently and identically distributed exponential random variables have a gamma distribution.

1.2.3 Exponential Distribution

The exponential distribution occurs when describing the lengths of the inter-arrival times in a homogeneous Poisson process. Exponential variables can also be used to model situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand, or between road kills on a given road. In queuing theory, the service times of agents in a system (e.g. how long it takes for a bank teller etc. to serve a customer) are often modeled as exponentially distributed variables. Reliability theory and reliability engineering also make extensive use of the exponential distribution. Because of the memory less property of this distribution, it is well-suited to the model that a constant hazard rate portion of the bathtub curve used in reliability theory.

A random variable X has an exponential distribution with parameter $\theta(\theta > 0)$ if its probability density function is of the form

$$f(x) = \theta \exp(-\theta x); x \geq 0; \theta > 0 \tag{1.2.10}$$

with mean $\frac{1}{\theta}$ and variance $\frac{1}{\theta^2}$ respectively.

The distribution is often written using the parameterization $\theta = \frac{1}{\lambda}$, in which the pdf becomes

$$f(x) = \frac{1}{\theta} \exp\left(\frac{-x}{\theta}\right), x \geq 0 \tag{1.2.11}$$

The parameter θ is called rate parameter with mean θ and variance θ^2 respectively.

The most important properties of the exponential distribution is the memory less property i.e., probability of its surviving an additional h hours is exactly the same as the probability of surviving h hours of a new item.

$$P(X \leq (x + y \mid X > x)) = P(X \leq y) \tag{1.2.12}$$

where X is the time we need to wait before a certain events occurs. This property says that events happens during a time interval of length y is independent of how much time has already elapsed (x) without the event happening.

The pdf of two parameter exponential distribution is given by

$$f(x;\mu,\theta)=\frac{1}{\theta}\exp\left[\frac{-(x-\mu)}{\theta}\right]\ ;-\infty<\mu<x<\infty,\theta>0 \qquad (1.2.13)$$

1.2.4 Erlang Distribution

The Erlang distribution is a continuous probability distribution with wide applicability primarily due to its relation to the exponential and Gamma distributions. The queuing theory had its origin in 1909, when Erlang (1878-1929) published his fundamental paper relating to the study of congestion in telephone traffic (Brockmeyer et. al. (1948)). The literature on the theory of queues and on the diverse field of its applications has grown tremendously over the years. The analysis for such an Erlangian queue is now folklore in the queuing literature. The Erlang distribution is the distribution of sum of exponential varieties. This distribution can be expressed as a waiting time and message length in telephone traffic. If the duration of individual calls are exponentially distributed then the duration of succession of calls follows Erlang distribution. The Erlang variate becomes Gamma variate when its shape parameter is an integer (Evans et. al. (2000)).Harischandra and Subba Rao (1988) discussed some problems of classical inference for the Erlangian queue. Bhattacharyya and Singh (1994) obtained Bayes' estimator for the Erlangian queue under two prior densities. Wiper (1998) studied Er/*M*/1 and Er/*M*/*c*m queues under Bayesian setup and estimated equilibrium probabilities of the queue size and waiting time distributions using conditional Monte-Carlo simulation methods. Jain (2001) discussed the problem of the change point for the inter arrival time distribution in the context of exponential families for the Ek/GIc queuing system and obtained Bayes' estimates of the posterior probabilities and the positions of change from the Erlang distribution. Nair et al. (2003) studied Erlang distribution as a model for ocean wave periods and obtained different characteristics of this distribution under classical set up. Suri et al (2009) used Erlang distribution to design a simulator for time estimation of project management process. The distribution is also used in the fields of stochastic processes and bio-mathematics. The probability density function of Erlang distribution is given as:

$$f(x;k,\lambda)=\frac{\lambda^k x^{k-1}e^{-\lambda x}}{\Gamma(k)}\ for\ x,\lambda\geq 0 \qquad (1.2.14)$$

The parameter k is called the shape parameter and the parameter λ is called the rate parameter. An alternative, but equivalent, parameterization (gamma distribution) uses the scale parameter μ which is the reciprocal of the rate parameter (i.e, $\mu = \frac{1}{\lambda}$):

$$f(x;k,\mu) = \frac{x^{k-1}e^{-\frac{x}{\mu}}}{\mu^k \Gamma(k)} \quad for\, x, \mu \geq 0 \qquad (1.2.15)$$

When the scale parameter μ equals 2, the distribution simplifies to the chi-square distribution with *2k* degrees of freedom. It can therefore be regarded as a generalized-chi-squared distribution, for even degrees of freedom.

The cumulative distribution function of the Erlang distribution is:

$$F(x;k,\lambda) = \frac{\gamma(k, \lambda x)}{(k-1)!} \qquad (1.2.16)$$

where $\gamma(.)$ is the lower incomplete gamma function. The CDF may also be expressed as

$$F(x;k,\lambda) = 1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\lambda x} (\lambda x)^n \qquad (1.2.17)$$

The mean and variance of Erlang distribution are given by:

$$\mu_1' = \frac{k}{\lambda} \qquad (1.2.18)$$

$$\mu_2 = \frac{k}{\lambda^2} \qquad (1.2.19)$$

When the shape parameter k equals 1, the distribution simplifies to the exponential distribution. The Erlang distribution is a special case of the Gamma distribution where the shape parameter k is an integer. In the Gamma distribution, this parameter is not restricted to the integers.

1.2.5 Rayleigh distribution

The Rayleigh distribution is often used in physics related fields to model processes such as sound and light radiation, wave heights, and wind speed, as well as in communication theory to describe hourly median and instantaneous peak power of received radio signals. It has been used to model the frequency of different wind speeds over a year at wind turbine sites and daily average wind speed. The Rayleigh distribution (RD) is considered to be a very useful life distribution. Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. One major application of this model is used in analyzing wind speed data. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2.This model was first introduced by Rayleigh (1980), Siddiqui (1962) discussed the origin and properties of the Rayleigh distribution. Inference for model Rayleigh model has been considered by Sinha and Howlader (1993) and Abd Elfattah *et al*. (2006).Ahmed *et.al* (2013) estimates the parameter of Rayleigh distribution using R-software.

The probability density function of Rayleigh distribution is given as:

$$f(x;\theta) = \frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right) \qquad for\ x \geq 0, \theta > 0 \qquad (1.2.20)$$

As an example of how it arises, the wind speed will have a Rayleigh distribution if the components of the two-dimensional wind velocity vector are uncorrelated and normally distributed with equal variance.

1.2.6 Beta Distribution

The Beta function was introduced by Leonhard Euler. The "problem in the doctrine of chances" that Bayes' treated produced a beta distribution for the posterior density of the probability of a success in Bernoulli trials. In early literature, the distribution was commonly referred to by its designation in the Pearson family of curves. Pearson is the English mathematician responsible for creating mathematical statistics. He created the Pearson Type I method which is a generalization of the Beta distribution to help model visibly skewed observations. The differences between the two are trivial and they can be set equal given proper parameters. The Beta distribution was standardized by Corrado Gini (1911) "*A First Course in Mathematical Statistics*, (1946)". Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them.

1.2.6.1 Beta Distribution of First Kind

A random variable *X* is said to have a Beta distribution of first kind with parameters a and b (a, b > 0) if its probability density function is given as:

$$f(x;a,b) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \;;0 < x < 1, a > 0 \, and \; b > 0 \tag{1.2.21}$$

where $\beta(a,b) = \int_o^1 t^{a-1}(1-t)^{b-1}dt$ denotes the beta function

The cumulative distribution function often called incomplete beta function is given by

$$F(X) = \begin{cases} 0, & x < 0 \\ \int_0^x \frac{1`}{B(a,b)} x^{a-1}(1-x)^{b-1}dx, & 0 < x < 1 \\ 1, & x > 1 \end{cases} \tag{1.2.22}$$

1.2.6.2 Beta Distribution of second Kind

A continuous random variable X is said to have a beta distribution of second kind with parameters a and b if probability density function (pdf) is:

$$f(x;a,b) = \frac{1}{\beta(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}} \tag{1.2.23}$$

$for 0 < x < 1, a > 0 \, and \; b > 0$

where $\beta(a,b) = \int_o^1 t^{a-1}(1-t)^{b-1}dt$ denotes the beta function.

1.2.7 Weibull Distribution

The Weibull distribution is one of the important distributions in reliability theory. It is the distribution that received maximum attention in the past few decades. The distribution is named after Waloddi Weibull (1939), a Swedish physicist represent the distribution of the breaking strength of materials. Numerous articles have been written demonstrating applications of the Weibull distribution in various sciences. It is widely used to analyze the cumulative loss of performance of a complex system in systems engineering. In general, it can be used to describe the data on waiting time futile an event occurs. Although a Weibull distribution may be a good choice to describe the data on lifetimes or strength data but in some practical situations Weibull distribution does not provide a reasonable parametric fit where the underlying failure rates are non constant, in that situation compounding procedure provides us the way out.

Mathematically, a random variable *X* is said to have a Weibull distribution if its pdf is of the form

$$f(x;\beta,\alpha)=\alpha\beta^{\alpha}x^{\alpha-1}e^{(-\beta x)^{\alpha}}\;;\;\; x>0,\alpha>0,\beta>0 \qquad (1.2.\,24)$$

where α and β are the parameters of the distribution.

The mean and variance of the Weibull distribution (1.2.24) are given as

$$\mu_1' = \frac{1}{\beta}\Gamma(1+\frac{1}{\alpha}) \qquad (1.2.25)$$

$$\mu_2 = \frac{1}{\beta^2}\left\{\Gamma\left(1+\frac{2}{\alpha}\right)-\left[\Gamma(1+\frac{1}{\alpha})\right]^2\right\} \qquad (1.2.26)$$

1.2.8 Generalized Beta Distribution (GBD) of first kind

A random variable *X* is said to have a generalized beta distribution of first kind and its probability density function is given by

$$f(x;a,b,p,q)=\frac{a}{b^{ap}\beta(p,q)}x^{ap-1}\left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1} \quad for\; x>0$$
$$= \; 0, otherwise \qquad (1.2.27)$$

where a,p,q are shape parameters and b is a scale parameter, $\beta(p,q)=\frac{\Gamma p\Gamma q}{\Gamma p+q}$ is a beta function, a,b,p,q are positive real values.

The kth moment of generalized beta distribution of first kind is given by McDonald (1995):

$$E(X^k)=\frac{b^k\beta\left(p+\frac{k}{a},q\right)}{\beta(p,q)} \qquad (1.2.28)$$

And $E(X)=\frac{b\beta\left(p+\frac{1}{a},q\right)}{\beta(p,q)}$ (1.2.29)

1.2.9 Generalized Beta Distribution of second kind

The probability density function (pdf) of the generalized beta distribution of second kind (GBD2) is given by:

$$f(x;a,b,p,q) = \frac{ax^{ap-1}}{b^{ap}\beta(p,q)\left[1+\left(\frac{x}{b}\right)^{a}\right]^{p+q}} \qquad for\ x>0$$

$$= \ 0, otherwise \tag{1.2.30}$$

Where a, p, q are shape parameters and b is a scale parameter, $\beta(p,q) = \frac{\Gamma p \Gamma q}{\Gamma p+q}$ is a beta function, a,b,p,q are positive real values.

The rth moment of generalized beta distribution of second kind is given by:

$$E(X^r) = \frac{b^r \Gamma\left(p+\frac{r}{a}\right)\Gamma\left(q-\frac{r}{a}\right)}{\Gamma p \ \Gamma q} \tag{1.2.31}$$

and
$$E(X \ ) = \frac{b \ \Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)}{\Gamma p \ \Gamma q} \tag{1.2.32}$$

1.2.10 Generalized Rayleigh Distribution

The probability distribution of Generalized Rayleigh distribution is given as:

$$f(x;\theta,k) = \frac{k}{\theta^{\frac{1}{k}}\Gamma\left(\frac{1}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right) \qquad for\ x \geq 0, k\ and\ \theta > 0$$

$$= 0, otherwise \tag{1.2.33}$$

The generalized Rayleigh distribution (GRD) is considered to be a very useful life-time distribution. Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. Surles and Padgett (2005) introduced two-parameter Burr Type X distribution and correctly named as the generalized Rayleigh distribution. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution, originally proposed by Mudholkar and Srivastava (1993).

Its mean and variance are given by:

$$\mu = \frac{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)} \tag{1.2.34}$$

$$\mu_2' = \frac{\theta^{\frac{2}{k}}\Gamma\left(\frac{3}{k}\right)}{\Gamma\left(\frac{1}{k}\right)} \tag{1.2.35}$$

1.2.12 Amoroso distribution

The Amoroso (generalized gamma, Stacy-Mihram) distribution is a four parameter, continuous, univariate, unimodal probability density, with semi infinite range. The functional form in the most straightforward parameterization is:

$$Amoroso(x/a,\alpha,\beta,\theta) = \frac{1}{\Gamma(\alpha)}\left|\frac{\beta}{\theta}\right|\left(\frac{x-a}{\theta}\right)^{\alpha\beta-1} e^{\left\{-\left(\frac{x-a}{\theta}\right)^{b}\right\}} \tag{1.2.36}$$

$For\, x, a, \alpha, \beta, \theta \; in R, \alpha > 0$

Support $x \geq a, if\; \theta > 0, x \leq a\, if\; \theta < 0$

This distribution was originally developed to model lifetimes by Amoroso (1925). It occurs as the Weibullization of the standard gamma distribution and, with integer α, in extreme value statistics. The Amoroso distribution is itself a limiting form of various more general distributions, most notable the generalized beta and generalized beta prime distributions.

A useful and important property of the Amoroso distribution is that many common and interesting probability distributions are special cases or limiting forms. This provides a convenient method for systemizing a significant fraction of the probability distributions that are encountered in practice, provides a consistent parameterization for those distributions and the need to enumerate the properties (mean, mode, variance, entropy and so on) of each and every specialization.

1.2.13 Generalized inverse Gaussian distribution

In probability theory and statistics, the generalized inverse Gaussian distribution (GIG) is a three-parameter family of continuous probability distribution with probability density function

$$f(x)=\frac{\left(\frac{a}{b}\right)^{\frac{p}{2}}}{2k_p\left(\sqrt{ab}\right)}x^{(p-1)}e^{-\frac{\left(\frac{ax+b}{x}\right)}{2}} \quad ;x>0 \tag{1.2.37}$$

Where k_p is a modified Bessel function of the second kind, $a > 0$, $b > 0$ and p is a real parameter. It is used extensively in Geo-statistics, statistical linguistics, finance, etc. This distribution was first proposed by Étienne Halphen (1993).It was rediscovered and popularized by Ole Barndorff-Nielsen, who called it the generalized inverse Gaussian distribution. It is also known as the Sichel distribution, after Herbert Sichel. The inverse Gaussian and Gamma distributions are special cases of the generalized inverse Gaussian distribution for p = -1/2 and b = 0, respectively. The mean and variance of generalized inverse Gaussian distribution are given as:

$$E(X)=\frac{\sqrt{b}k_{p+1}\left(\sqrt{ab}\right)}{\sqrt{a}\,k_p\left(\sqrt{ab}\right)} \tag{1.2.38}$$

$$V(X)=\left(\frac{b}{a}\right)\left[\frac{k_{p+2}\left(\sqrt{ab}\right)}{k_p\left(\sqrt{ab}\right)}-\left(\frac{k_{p+1}\left(\sqrt{ab}\right)}{k_p\left(\sqrt{ab}\right)}\right)^2\right] \tag{1.2.39}$$

1.3 Introduction to Size biased Probability Distributions

Traditional enviromentric theory and practice have been occupied with randomization and replication. But in environmental and ecological work, observations also fall in the non-experimental, non-replicated and non-random catogaries. The problems of model specification and data interpretation then acquire special importance and great concern. The theory of weighted distributions provides a unifying approach for these problems. Weighted distributions take into account the method of ascertainment, by adjusting the probabilities of actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Failure to make such adjustments can lead to incorrect conclusions.

The concept of weighted distributions can be traced to the work of Fisher (1934), in connection with his studies on how methods of ascertainment can influence the form of distribution of recorded observations. Later it was introduced and formulated in general terms by Rao (1965), in connection with modeling statistical data where the usual practice of using standard distributions for the purpose was not found to be appropriate. In Rao's paper (1965), he identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distributions. This may occur due to non-observability of some events or damage caused to the original observation resulting in a reduced value or adoption of a sampling procedure which gives unequal chances to the units in the original. The concept of weighted distributions can be traced to the study of the effect of methods of ascertainment upon estimation of frequencies by Fisher (1934). In extending the basic ideas of Fisher, Rao [(1934, 1965)] saw the need for a unifying concept and identified various sampling situations that can be modeled by what he called weighted distributions. The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weighted function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to measure of the unit size, resulting distribution is called size-biased. Size biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher (1934) to model ascertainment bias, these were later formalized in a unifying theory by Rao (1965). These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non –experimental, non –replicated and non –random categories. Van Deusen (1998) arrived at size biased distribution theory independently and applied it to fitting distributions of diameter at breast height (DBH), data arising from horizontal point sampling (HPS) (Grosenbaugh) inventories. Subsequently, Lappi and Bailey (1987) used weighted distributions to analyze HPS diameter increment data. In ecology, Dennis and Patil (1984) used stochastic differential equations to arrive at a weighted gamma distribution as the stationary probability density function (PDF) for the stochastic population model with predation effects. Gove (2003) reviewed some of the more recent results on size-biased distributions pertaining to parameter estimation in forestry. Warren (1975) was the first to apply the size biased distributions in connection with sampling wood cells. More recently; these distributions were used to recover the distribution of canopy heights from airborne laser scanner measurements. In fisheries, Taillie et al (1995) modeled populations of fish stocks using weights. Most of the statistical applications of weighted distributions, especially to the analysis of data relating to human populations and ecology, can be found in Patil and Rao (1997, 1978). In a series of papers with co-authors, Patil ((1988), (1993), (1981), (1991) and (1996)) has pursued weighted distributions for purposes of encountered data analysis, equilibrium population analysis subject to harvesting and predation, meta analysis incorporating publication bias and heterogeneity, modeling clustering and extraneous variation, etc. Sandal C.E (1964) derived the method of estimation of parameters of the gamma distribution. Ahmed *et al* (2013b) discussed the size-biased Generalized Gamma distribution with its structural properties, estimates the parameters of new model by using new moment method of estimation and its characterizations and also discussed some important information measures of size biased generalized gamma distribution(discussed in chapter 3th). The usefulness and applications of weighted distributions to biased samples in various areas including medicine, ecology, reliability, and branching processes can be seen in Patil and Rao (1978), Gupta and Kirmani(1990),Gupta and Keating (1985),Oluyede (1999) and in references there in. Reshi *et al* (2013a) estimates the parameter of the size biased generalized Rayleigh distribution under the extended Jeffrey's prior assuming two different loss functions (discussed in chapter 5th). Within the context of cell kinetics and the early detection of disease, Zelen (1974) introduced weighted distributions to represent what he broadly perceived as length-biased sampling (introduced earlier in Cox, D.R. (1962)). For additional and important results on weighted distributions Rao (1997), Patil and Ord

(1997), Zelen and Feinleib (1969), Application examples for weighted distributions see El-Shaarawi and Walter (2002). Jing (2010) introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution, theoretical properties of them, Castillo and Perez-Casany (1998) introduced new exponential families, that come from the concept of weighted distribution, that include and generalize the Poisson distribution, Shaban and Boudrissa (2000) have shown that the length biased version of the Weibull distribution known as Weibull Length-biased (WLB) distribution is unimodal throughout examining its shape, with other properties, Das and Roy (2011) discussed the length-biased Weighted Generalized Rayleigh distribution with its properties, also they have develop the length-biased from of the weighted Weibull distribution. Patil and Ord (1976) introduced the concept of size-biased sampling and weighted distributions by identifying some of the situations where the underlying models retain their form. Other contributions have been made by (Oluyede and George (2000), Ghitany and Al-Mutairi (2008), Oluyede and Terbeche (2007). The statistical interpretation of length-biased distributions was originally identified by Cox (1962) in the context of renewal theory. Length-biased sampling situations may occur in clinical trials, reliability theory, and survival analysis and population studies, where a proper sampling frame is absent. In such situations, items are sampled at a rate proportional to their length, so that larger values of the quantity being measured are sampled with higher probabilities. Numerous works on various aspects of length-biased sampling are available in literature which include family size and sex ratio , wild life population and line transect sampling , analysis of family data, cell cycle analysis , efficacy of early screening for disease, aerial survey and visibility bias Patil and Rao (1978).

1.3.1 Materials and Methods

Suppose X is a non-negative random variable with its natural probability density function $f(x;\theta)$, where the natural parameter is θ. Suppose a realization x of X under $f(x;\theta)$ enters the investigator's record with probability proportional to $w(x;\beta)$, so that the weight function $w(x;\beta)$ is a non-negative function with the parameter β representing the recording mechanism. Clearly, the recorded x is not an observation on X, but on the random variable X_w, having a pdf

$$f_w\ (x;\theta,\beta) = \frac{w(x;\beta)\ \ f(x;\theta)}{\int_0^\infty w(x) f(x)}; \ x > 0 \tag{1.3.1}$$

Assuming that $E(X) = \int_0^\infty w(x) f(x)\ dx$ i,e the first moment of w(x) exists.

By taking weight w(x) = x we obtain length biased distribution. Where w is the normalizing factor obtained to make the total probability equal to unity by choosing $w = E[w(x,\beta)]$.The variable X_W is called weighted version of X, and its distribution is related to that of X and is called the weighted distribution with weight function w.For example, when $w(x;\beta) = x$, in that case $w = \mu$ is called the size-biased version of X.The distribution of X^* is called the size-biased distribution with pdf

$$f^*(x;\theta) = \frac{x\ f(x;\theta)}{\mu} \tag{1.3.2}$$

where $\mu = E(X.)$The pdf $f^*(x;\theta)$ is called length biased or size- biased version of $f(x;\theta)$, and the corresponding observational mechanism is called length or size- biased sampling. Weighted distributions have seen much use as a tool in the selection of appropriate models for observed data drawn without a proper frame. In many situations the model given above is appropriate, and the statistical problems that arise are the determination of a suitable weighted function, $w(x;\beta)$ and drawing inferences on θ. Appropriate statistical modeling helps accomplish unbiased inference in spite of the biased data and, at times, even provides a more informative and economic setup.

These weight functions are also useful for modeling through the identities connecting the original and weighted random variables. Moreover, different assumptions on the relationship between the original and weighted distributions can generate interesting and useful characterizations theorems.

1.3.1 Binomial Distribution

The distribution was derived by James Bernoulli in his treatise Ars Conjectandi, published in (1713).The distribution arises when 'n' independent trials are made with a constant probability 'p' of success at each trial, each trial resulting either in a success or in a failure. In this case the random variable X is the number of successes and its distribution is given by

$$P[X = x] = p(x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, \cdots n \qquad (1.3.3)$$

where q + p =1, p > 0, q > 0 and n is a positive integer.

The probabilities in (1.3.3) are terms of binomial expansion of $(q + p)^n$ hence the name binomial distribution. When n = 1, binomial distribution (1.3.3) reduces to Bernoulli distribution.

1.3.1.1 Size-Biased Binomial Distribution (SBBD)

A size-biased binomial distribution(SBBD), a particular case of the weighted binomial distribution, taking weights as the variate value can be obtained from(1.3.3) as

$$\sum_{x=0}^{n} x.P[X = x] = np$$

$$\frac{1}{np} \sum_{x=1}^{n} x.P[X = x] = 1$$

$$P[x = x] = \binom{n-1}{x-1} p^{x-1} q^{n-x} \; ; x = 1,2..... \qquad (1.3.4)$$

The mean and variance of size-biased distribution (1.3.4) are given as

$$\mu_1' = (\text{mean}) = np + q \qquad (1.3.5)$$

$$\mu_2 = (\text{variance}) = (n-1)pq \qquad (1.3.6)$$

1.3.2 Poisson Distribution

There are many situations where the number of independent trials is infinitely large but the probability of success is so small that the expected number of success is of moderate size. For such situations the distribution is the Poisson Distribution (PD) which was obtained by S.D Poisson (1837) as a limiting case of the Binomial distribution. He took n, the number of independent trials, very large tending to infinity and p sufficiently small such that np = λ_1, where λ_1 is a finite number and obtained the probability mass function (pmf) of the distribution as

$$P[X = x] = p(x) = \frac{e^{-\lambda_1}(\lambda_1)^x}{x!} \text{ ; } x = 0, 1, 2, \cdots \text{ and } \lambda_1 > 0 \qquad (1.3.7)$$

For some reasons Johnson and David (1952) preferred to give credit to De Moivre (1718) rather than to S.D. Poisson for discovering of Poisson distribution. The distribution is so important among the discrete distributions that even Fisher, once remarked 'Among discontinuous distributions', the Poisson series is of the first importance. Johnson, Kotz and Kemp (1992) have discussed the genesis of Poisson distribution in detail. Munir Ahmad and Ayesha Roohi (2004) have discussed the characterization of the Poisson distribution. The Poisson distribution has been described as playing a "similar role with respect to discrete distribution to that of the normal for absolutely continuous distribution. The unique property of Poisson distribution in discrete distributions is the equality of mean and variance.

$$\mu_1' = (\text{mean}) = \lambda_1 \qquad (1.3.8)$$

$$\mu_2 = (\text{variance}) = \lambda_1 \qquad (1.3.9)$$

1.3.2.1 Size- Biased Poisson distribution (SBPD)

A size-biased Poisson distribution can be obtained from (1.3.7) as

$$\sum_{x=0}^{\infty} x\, P[X = x] = \lambda_1$$

$$\frac{1}{\lambda_1}\sum_{x=1}^{\infty} x.P[X = x] = 1$$

$$P[X = x] = \frac{\lambda_1^{\,x-1} e^{-\lambda_1}}{(x-1)!}; \; x = 1,2..... \qquad (1.3.10)$$

1.3.3 Negative Binomial Distribution

The Negative Binomial Distribution provides a good fit to the situations where the mean is always less than the variance. Montmort (1914) gave the derivative of this distribution in, (although Pascal (1679) discussed a special form of the distribution much earlier.

Pascal obtained the distribution for getting ‘r’ successes and ‘x’ failures in exactly (r + x) independent trials, where x is the value of a random variable. The pmf of the distribution is given by

$$P(X=x)=\binom{x+r-1}{r-1} p^r q^x = \binom{-r}{x} p^r (-q)^x ;\ x=0,\ 1,\ 2\ldots. \qquad (1.3.11)$$

where parameters satisfy

$q = 1 - p$ and $0 < p < 1$ and $r = 1, 2, 3 \ldots$

1.3.3.1 Size-Biased Negative Binomial Distribution (SBNBD)

A size-biased negative binomial distribution can be obtained from (1.3.11) as

$$\sum_{x=0}^{n} x.P[X=x]=rp$$

$$\frac{1}{rp}\sum_{x=1}^{\infty} x.P[X=x]=1$$

$$P[X=x]=\binom{x+r-1}{x-1} p^{r-1} q^x ;\ x=1,2,\ldots. \qquad (1.3.12)$$

The mean and variance of size-biased distribution (1.3.12) are given as

$$\mu_1' = (mean) = \frac{rp}{q}+\frac{1}{q} \qquad (1.3.13)$$

$$\mu_2 = (\text{variance}) = \frac{rp}{q^2}+\frac{p}{q^2} \qquad (1.3.14)$$

1.3.4 Logarithmic Series Distribution

The Logarithmic Series Distribution is one of the most widely used basic distributions. It is the importance and superb potentiality of the LSD in describing the empirical situations that have attracted the attention of the statisticians which have caused the manifold developments of the distribution during a very short span of time. The LSD is wider in scope and simple in nature. Therefore, it finds its key place as a promising distribution in describing important phenomena of various fields. The distribution was first obtained by the Fisher (1943) as a limiting case of the zero-truncated negative binomial distribution.

The probability mass function of LSD is given by

$$P[X = x] = p(x) = -\frac{1}{[\log(1-\alpha)]}\frac{\alpha^x}{x} \text{ and } 0 < \alpha < 1 \text{ ; } x = 1, 2\cdots \quad (1.3.15)$$

$$P[X = x] = p(x) = \frac{\theta\,\alpha^x}{x} \quad (1.3.16)$$

$$\text{where } \theta = -\frac{1}{[\log(1-\alpha)]} \quad (1.3.17)$$

The important structural properties of the distribution have been described by Patil and Taillie (1989). Johnson and Kotz (1970), Johnson, Kotz and Kemp (1992) have presented a brief description of the most of the important features of the distribution. Katti and Rao (1965) have used mixture of LSD (log Poisson) and its truncated form (log zero-Poisson) to represent a wide variety of data. They have also provided with the tables for the probabilities to make its use easy. Though the LSD gives very satisfactory fit in all these situations.

1.3.4.1 Size-Biased Logarithmic Series Distribution

A size-biased logarithmic series distribution can be obtained from (1.3.16) as

$$\sum_{x=1}^{\infty} x.P[X = x] = \frac{-1}{\log(1-\alpha)}\frac{\alpha}{(1-\alpha)}$$

$$-\frac{(1-\alpha)\log(1-\alpha)}{\alpha}\sum_{x=1}^{\infty} x.P[X = x] = 1 \quad (1.3.18)$$

$$P[X = x] = \alpha^{x-1}(1-\alpha); x = 1,2...... \quad (1.2.19)$$

1.3.5 Generalized Negative Binomial Distribution

On account of wide-variety of available discrete distributions besides binomial, Poisson and negative binomial, to choose the most suitable was the problem for research workers in applied fields.With the aim of reducing this problem, the generalized negative binomial distribution (GNBD) was first introduced by Jain and Consul (1971).With probability function given by compounding the negative binomial distribution with another parameter which takes into account the variations in the mean and the variance. The parameter is such that both mean and variance are positively correlated with the value of the parameter, though the variance increase or decrease faster than the mean. Its value gives an indication of the nature of the data and its variation from binomial to negative binomial distribution. The probability function of Generalized Negative Binomial Distribution (GNBD) is given by:

$$P[X=x]=\frac{m\Gamma(m+\beta x)}{x!\Gamma(m+\beta x-x+1)}\alpha^{x}(1-\alpha)^{m+\beta x-x};\ x=0,1,2.... \qquad (1.3.20)$$

$$=0;\ \text{for } x>t \text{ when } \beta<0 \text{ or } 0<\beta<1$$

where $0 < \alpha < 1$, $m > 0$ and $\beta = 0$ or $0 < \beta < \alpha^{-1}$ and t is the largest positive integer for which $m+1+(\beta-1)t>0$

It can be easily seen that the distribution reduces to the classical negative binomial distribution (1.3.20) at β = 1, to the classical binomial distribution (1.2.1) at β = 0 and for β = ½ this distribution resembles with the Poisson distribution (1.3.1)

1.3.5.1 Size-biased Generalized Negative Binomial Distribution

A size biased generalized negative binomial distribution a particular case of the weighted generalized negative binomial, taking weights as the variate value can be obtained from (1.3.20) as:

$$\sum_{x=0}^{\infty} x\cdot P\ (X=x)=\frac{m\alpha}{(1-\alpha\beta)}$$

$$\frac{1-\alpha\beta}{m\alpha}\sum_{x=0}^{\infty} x\cdot P\ (X=x)=1$$

This gives the size-biased generalized negative binomial distribution (SBGNBD) as

$$P\ (X=x)=(1-\alpha\beta)\binom{m+\beta x-1}{x-1}\ \alpha^{x-1}\ (1-\alpha)^{m+\beta x-x}\ ;\ x=1,2.... \qquad (1.3.21)$$

$$where\ 0<\alpha<1,\ m>0,\ -\infty<\beta<\infty,$$

At $\beta=0$ and $\beta=1$, we get size biased binomial SBBD (1.3.21) and size biased negative binomial distributions SBNBD (1.3.12) respectively.

1.3.6 Generalized Geometric Series Distribution

As the geometric series distribution is a particular case of negative binomial distribution, it is supposed that a particular case of GNBD may give a generalized geometric series distribution. Mishra (1982) using the results of lattice path analysis obtained the following distribution.

$$P[X = x] = \frac{\Gamma(\beta x + 1)}{x!\,\Gamma(\beta x - x + 2)}\alpha^{x}(1-\alpha)^{1+\beta x - x} \quad ; x = 0, 1, 2, \cdots \qquad (1.3.22)$$

where P[X=x] = 0 for x ≥ m if 1+ β m < 0 and which is the same as obtained by taking m=1 in the GNBD (1.3.20).

Needless to say that, the GGSD is a member of Lagrangian distribution and can be obtained by taking $g(t) = (1-\alpha+\alpha t)^{\beta}$ and $f(t) = (1-\alpha+\alpha t)$. It can be seen that at β = 1, it reduces to the one parameter geometric series distribution and hence it is called as generalized geometric series distribution.

The moments of this distribution can be found simply by putting m = 1 in the expressions for moments of the GNBD. As it is a particular case of the GNBD, all the properties of the GNBD are supposed to be possessed by the GGSD also.

1.3.6.1 Size- Biased Generalized Geometric Series Distribution

The pmf of size-biased GGSD can be obtained by putting m=1 in size-biased GNBD (1.3.22)

$$P(X = x) = (1-\alpha\beta)\binom{\beta x}{x-1}\alpha^{x-1}(1-\alpha)^{1+\beta x - x} \; ; x = 1,2.... \qquad (1.3.23)$$

$$where\ 0 < \alpha < 1,\ -\infty < \beta < \infty$$

1.3.7 Generalized Poisson Distribution

The generalized Poisson distributions (GPDs) arise when the populations are Poissonian type having unequal mean and variance. Consul and Jain (1973a) are the early workers who derived a class of discrete distributions of the Poissonian type. The different aspects of these distributions have been studied by Consul and Jain (1973b), Jain (1975), Consul and Shoukri (1985, 1986), Consul (1986), Famoye and Lee (1992), Tuenter (2000). The detailed review works on the book authored by Consul (1989) have been done by Kemp (1992). Consul and Jain (1973a) defined the Generalized Poisson Distribution by taking m and β very large and α very small, such that $m\alpha = \lambda_1$ and $m\beta = \lambda$ where λ_1 is finite and positive and $|\lambda| < 1$, in GNBD, the pmf is given by

$$P[X = x] = \frac{\lambda_1(\lambda_1 + x\,\lambda_2)^{x-1}\,Exp\,[-(\lambda_1 + x\,\lambda_2)]}{x!} \; ; x = 0, 1, 2\ldots \qquad (1.3.24)$$

For $\lambda_2 = 0$, the distribution (1.3.24) reduces to Poisson distribution. The model (1.3.24) has been found to be a member of the Consul and Shenton's (1972) family of Lagrangian distributions.

1.3.7.1 Zero-Truncated Generalized Poisson Distribution

Shoukri and Consul (1989) redefined the distribution (1.3.24) by taking $\lambda_1 = \alpha$ and $\lambda_2 = \alpha\beta$ as

$$P_2(X = x) = \frac{(1+\beta x)^{x-1}\alpha^x \exp[-\alpha(1+\beta x)]}{x!}; \quad x = 0, 1, 2, \cdots, \alpha > 0, 0 < \beta < \frac{1}{\alpha}. \qquad (1.3.25)$$

The distribution (1.3.25) can be truncated at x = 0 and is defined with the probability function as:

$$P_3(X = x) = \frac{(1+\beta x)^{x-1}\alpha^x \exp[-\alpha(1+\beta x)](1-e^{-\alpha})^{-1}}{x!}; x=1,2.., \alpha > 0, 0 < \beta < \frac{1}{\alpha}. \qquad (1.3.26)$$

For β = 0, the distributions (1.3.25) and (1.3.26) reduce to Poisson distribution and David and Johnson's (1952) truncated Poisson distribution. The different aspects of the distribution have been studied by Consul and Famoye (1990), Jani and Shah (1981), Hassan and Mir (2007a), Hassan et al (2007b). A brief list of authors and their works can be seen in Consul (1989), Johnson, Kotz and Kemp (2005) and Consul and Famoye (2006).

1.3.7.2 Size-Biased Generalized Poisson distribution (SBGPD)

The pmf of size-biased generalized Poisson distribution can be obtained as

$$\sum_{x=0}^{\infty} x.P[X = x] = \frac{\lambda_1}{(1-\lambda_2)}$$

$$\frac{(1-\lambda_2)}{\lambda_1}\sum_{x=0}^{\infty} x.P[X = x] = 1$$

This gives the pmf of SBGPD as

$$P[X = x] = \frac{(1-\lambda_2)(\lambda_1 + x\lambda_2)^{x-1}\text{EXP}[-(\lambda_1 + x\lambda_2)]}{(x-1)!}; \quad x = 1,2,\ldots.. \qquad (1.3.27)$$

$$\lambda_1 > -1, |\lambda_1\lambda| < 1 + \lambda_1$$

Mir et al (2013) proposed the probability density function of Size-biased Generalized Poisson distribution.

At $\lambda_2 = 0$, we get size-biased Poisson distribution (1.3.8).

The mean and variance of the distribution is given as

$$\mu_1' = \text{Mean} = \frac{\lambda_1}{(1-\lambda)} + \frac{1}{(1-\lambda)^2} \qquad (1.3.28)$$

$$\mu_2 = \text{Variance} = \frac{2\lambda}{(1-\lambda)^4} + \frac{\lambda_1}{(1-\lambda)^3} \qquad (1.3.29)$$

The Graphical representation of Size biased generalized Poisson distribution in figures 1,2,3,4 and 5 considering various values of λ_1 and λ_2.

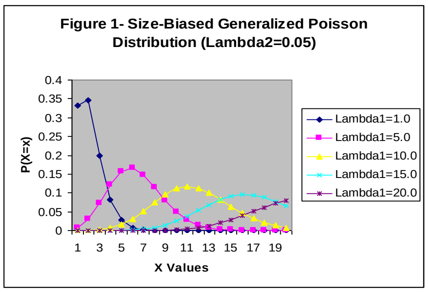


Figure 1- Size-Biased Generalized Poisson Distribution (Lambda2=0.05)

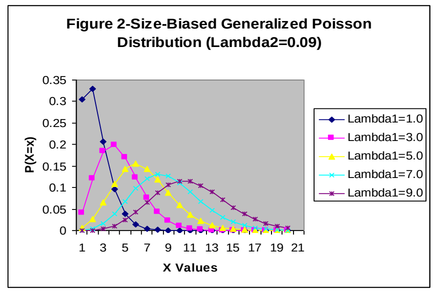


Figure 2-Size-Biased Generalized Poisson Distribution (Lambda2=0.09)

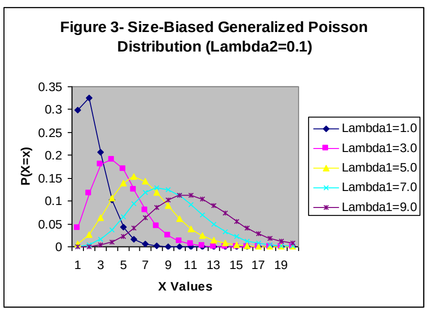


Figure 3- Size-Biased Generalized Poisson Distribution (Lambda2=0.1)

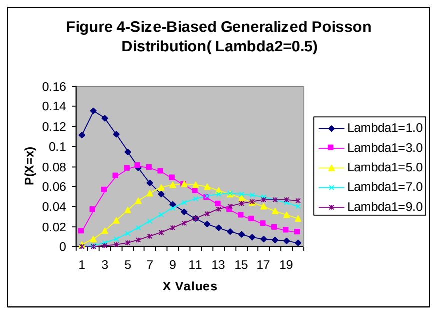


Figure 4-Size-Biased Generalized Poisson Distribution(Lambda2=0.5)

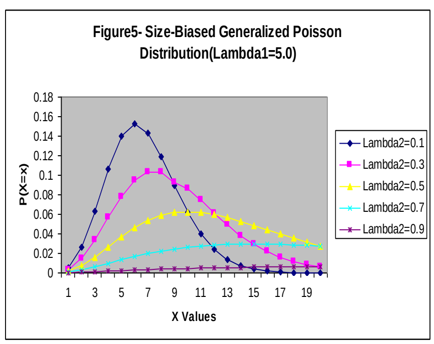


Figure5- Size-Biased Generalized Poisson Distribution(Lambda1=5.0)

In figure 1, for each small value of λ_1, the SBGPD curve changes from L-shaped to symmetric and with the considerable change in the value of λ_1, it becomes positively skewed. In figures 2, 3 and 4, we consider $\lambda_2 = 0.09, 0.1, 0.5$. For each small value of λ_1, the SBGPD curve is unimodal and extremely positively skewed. But it gradually changes to bell-shaped as the value of λ_1 and λ_2 increase. In figure 5, we take $\lambda_1 = 5.0$ and the different values of λ_2. It is observed that the variation in the values of λ_2 alters the shape of the distribution substantially. For larger values of λ_2 the bell-shaped form becomes more flattened.

1.3.8 Generalized Logarithmic Series Distribution

A Generalized Logarithmic Series Distribution (GLSD) was obtained by Jain and Gupta (1973) with probability function

$$P[X = x] = -\frac{1}{\log(1-\alpha)} \frac{\Gamma(\beta x)}{x!\,\Gamma(\beta x - x + 1)} \alpha^{x} (1-\alpha)^{\beta x - x} \text{ ; } x = 1, 2, 3, \cdots \qquad (1.3.30)$$

$$0 < \alpha < 1, \alpha\beta < 1$$

At β=1 in (1.3.30), the distribution is reduced to the logarithmic series distribution (1.3.15). Many others have also obtained the GLSD using different approaches. Mishra (1979) obtained it as a limiting case of zero-truncated GNBD; whereas Jani (1977) obtained it as a member of Modified Power Series Distribution (MPSD). Patel (1981) found it independently again as a limiting case of the zero-truncated GNBD. Mishra et al (1996) estimates the parameters of Generalized Logarithmic Series Distribution using Bayesian approach.

1.3.8.1 Size-Biased Generalized Logarithmic Series Distribution

The size-biased generalized logarithmic series distribution can be obtained from (1.3.30) as

$$\sum_{x=1}^{\infty} x.P[X = x] = \frac{\theta\alpha}{(1-\alpha\beta)}$$

$$\frac{(1-\alpha\beta)}{\theta\alpha} \sum_{x=1}^{\infty} xP.[X = x] = 1$$

This gives the pmf of SBGLSD as

$$P[X = x] = (1-\alpha\beta)\binom{\beta x - 1}{x - 1} \alpha^{x-1} (1-\alpha)^{\beta x - x} ; x = 1,2..... \qquad (1.3.31)$$

At $\beta = 1$, we get size-biased logarithmic distribution (1.3.17).

1.3.9 Geeta Distribution

Consul (1990a) defined the Geeta distribution over the set of all positive integers with the probability mass function as

$$P[X=x]=\frac{1}{\beta x-1}\binom{\beta x-1}{x}\alpha^{x-1}(1-\alpha)^{\beta x-x} \quad ;x=1,2\ldots \tag{1.3.32}$$

$$1<\beta<\alpha^{-1} \text{ and } 0<\alpha<1$$

The Geeta distribution has a maximum at x=1 and is L-shaped for all values of α and β. It may have a short tail or a long tail depending upon the values of α and β. Its mean and variance are given by

$$\mu=(1-\alpha)(1-\alpha\beta)^{-1} \tag{1.3.33}$$

$$\mu_2'=\frac{\alpha(1-\alpha)(\beta-1)+(1-\alpha)^2(1-\alpha\beta)}{(1-\alpha\beta)^3}$$

$$\sigma^2=(\beta-1)\alpha(1-\alpha)(1-\alpha\beta)^{-3}=\mu(\mu-1)(\beta\mu-1)(\beta-1)^{-1}. \tag{1.3.34}$$

The family of Geeta probability models belongs to the classes of the modified power series distributions (MPSD) and the Lagrangian series distributions. Consul (1990b) also expressed it as a location-parameter probability distribution given below:

$$P_1[X=x]=\frac{1}{\beta x-1}\binom{\beta x-1}{x}\left[\frac{\mu-1}{\mu(\beta-1)}\right]^{x-1}\left[\frac{\mu(\beta-1)}{\beta\mu-1}\right]^{\beta x-1} ;x=1,2,3\ldots. \tag{1.3.35}$$

1.3.9.1 Size-biased Geeta Distribution

A size-biased Geeta distribution (SBGET) is obtained by taking the weight of the Geeta distribution (1.3.32) as x.

We have from (1.3.32) and (1.3.33)

$$\sum_{x=1}^{\infty} x\cdot P(X=x)=(1-\alpha)(1-\alpha\beta)^{-1}$$

This gives the size-biased Geeta distribution (SBGET) as

$$P_2[X=x]=(1-\alpha\beta)\binom{\beta x-2}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x-1};x=1,2\ldots \tag{1.3.36}$$

$$1<\beta<\alpha^{-1} \quad \text{and } 0<\alpha<1$$

The mean and variance of SBGET are given by

$$\mu_1'(s) = \text{Mean} = \frac{\left(1-2\alpha+\alpha^2\beta\right)}{\left(1-\alpha\beta\right)^2}, \tag{1.3.37}$$

$$\mu_2(s) = \frac{2(\beta-1)\alpha(1-\alpha)}{(1-\alpha\beta)^4} \tag{1.3.38}$$

1.3.10 Consul Distribution

The probability function of Consul distribution is defined as

$$P[X=x] = \frac{1}{x}\binom{\beta x}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x+1}; \quad x=1, 2\ldots \tag{1.3.39}$$

$$1<\beta<\alpha^{-1} \quad \text{and} \quad 0<\alpha<1$$

It reduces to the geometric distribution when $\beta=1$. For this reason the distribution is also called as generalized geometric distribution. Famoye (1997) obtained the model (1.3.39) by using Lagrange expansion on the pgf of a geometric distribution and called it a generalized geometric distribution. He studied some of its properties and applications. Most of the interesting properties of the distribution can be seen in Consul and Famoye (2006). Consul (1990) showed that the model (1.3.39) belongs to the class of location-parameter discrete distributions. The moments of the model (1.3.39) are given as

$$\mu = (1-\alpha\beta)^{-1} \tag{1.3.40}$$

$$\mu_2' = \frac{\left(1-\alpha^2\beta\right)}{\left(1-\alpha\beta\right)^3} \tag{1.3.41}$$

$$\sigma^2 = \beta\alpha(1-\alpha)(1-\alpha\beta)^{-3} \tag{1.3.42}$$

The model (1.3.39) can be expressed as a location-parameter probability distribution in the form

$$P_1[X=x] = \frac{1}{x}\binom{\beta x}{x-1}\left[\frac{\mu-1}{\beta\mu}\right]^{x-1}\left[1-\frac{\mu-1}{\beta\mu}\right]^{\beta x-x+1}; x=1,2,3\ldots. \tag{1.3.43}$$

1.3.10.1 Size-biased Consul Distribution

The probability function of the size-biased Consul distribution (SBCOND) is given as

$$P_2[X=x]=(1-\alpha\beta)\binom{\beta x}{x-1}\alpha^{x-1}(1-\alpha)^{\beta x-x+1}\ ;\quad x=1,2\ldots \qquad (1.3.44)$$

$$1<\beta<\alpha^{-1} \quad \text{and } 0<\alpha<1$$

The mean and variance of SBCOND are given by

$$\text{Mean}=\frac{\left(1-\alpha^2\beta\right)}{\left(1-\alpha\beta\right)^2}, \qquad (1.3.45)$$

$$\mu_2(s)=\frac{2\alpha(1-\alpha)\beta\left(1-\alpha^2\beta\right)}{\left(1-\alpha\beta\right)^4}-\frac{2\alpha^2(1-\alpha)\beta}{\left(1-\alpha\beta\right)^3}. \qquad (1.3.46)$$

1.4 Akaike and Bayesian information criterion

In order to introducing of an approach for model selection, we remember Akaike and Bayesian information criterion based on entropy estimation. Akaike's information criterion, developed by Hirotsugu Akaike (1973) under the name of" an information criterion" (AIC) in 1971 and proposed in Akaike (1974), is a measure of the goodness of fit of an estimated statistical model. The concept of entropy, in effect offering a relative measure of the information lost when a given model is used to describe reality and can be said to describe the tradeoff between bias and variance in model construction, or loosely speaking that of precision and complexity of the model. The AIC is not a test of the model in the sense of hypothesis testing; rather it is a test between models - a tool for model selection. Given a data set, several competing models may be ranked according to their AIC, with the one having the lowest AIC being the best. From the AIC value one may infer that e.g. the top three models are in a tie and the rest are far worse, but it would be arbitrary to assign a value above which a given model is "rejected". In the general case, the AIC is

$$AIC=2K-2\log L\left(\hat{\theta}\right) \qquad (1.4.1)$$

Where k is the number of parameters in the statistical model and L is the maximized value of the likelihood function for the estimated model.

The Bayesian information criterion (BIC) or Schwarz Criterion is a criterion for model selection among a class of parametric models with different numbers of parameters. Choosing a model to optimize BIC is a form of regularization. It is very closely related to AIC. In BIC, the penalty for additional parameters is stronger than that of the AIC.
The formula for the BIC is

$$BIC = K \log n - 2 \log L(\hat{\theta}). \tag{1.4.2}$$

1.5 Shannon's entropy

In information theory, entropy is a measure of the uncertainty in a random variable. In this context, the term usually refers to the Shannon entropy, which quantifies the expected value of the information contained in a message. Entropy is typically measured in bits, nats or bans. Shannon entropy is the average unpredictability in a random variable, which is equivalent to its information context. Shannon entropy provides an absolute limit on the best possible lossless encoding or compression of any communication, assuming that the communication may be represented as a sequence of independent and identical distributed random variables. This definition of "entropy" was introduced by Claude. E. Shannon (1948) paper “A Mathematical theory of Communication". Shannon's definition of entropy, when applied to an information source, can determine the minimum channel capacity required to reliably transmit the source as encoded binary digits. It measures the information contained in a message as opposed to the portion of the message that is determined (or predictable). Entropy is a measure of unpredictability or information content. The concept of Shannon's entropy is the central role of information theory, sometimes referred as measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. Henceforth we assume that log is to the base 2 and entropy is expressed in bits. The concept of entropy was extensively used in literature as a quantitative measure of uncertainty associated with random phenomena. In the context of equilibrium thermodynamics, physicists originally developed the notion of entropy which was later extended through the development of statistical mechanics and information theory. Shannon (1948) was the one who formally introduced entropy, known as

Shannon's entropy or Shannon's information measure into information theory. For deriving the entropy of Probability distributions, we need the following two definitions that are more discussed in Cover *et al* (1991).

Definition 1.5.1: The entropy of the discrete alphabet random variable f defined on the probability space is given by:

$$H_P(f) = -\sum_{i=1}^{n} p(f = a)\log(p(f = a)) \tag{1.5.1}$$

It is obvious that

$$H_P(f) \geq 0$$

Definition 1.5.2: The oblivious generalizations of the definition of entropy for a probability density function f defined on the real line as:

$$H(f) = -\int_0^{\infty} f(x)\log f(x) = E(-\log(x)) \tag{1.5.2}$$

1.6 Generalized entropy

The generalized entropy index is a general formula for measuring redundancy in data. The redundancy can be viewed as inequality, lack of diversity, non-randomness, compressibility, or segregation in the data. The primary use is for income equality. It is equal to the definition of redundancy in information theory that is based on Shannon entropy when α = 1 which is also called the Theil Index (T_T) in income inequality research. Completely "diverse" data has no redundancy so that GE=0, so that it increases in the opposite direction of a Diversity index. It increases with order rather than disorder, so it is a negated measure of entropy. Generalized entropy is often used in econometrics. It is indexed by a single parameter α . The generalized entropy is defined to be

$$I_\alpha = \frac{v_\alpha u^{-\alpha} - 1}{\alpha(\alpha - 1)}; \alpha \neq 0,1 \tag{1.6.1}$$

$$\text{and } v_\alpha = \int_0^{\infty} x^\alpha f(x;\theta)dx \tag{1.6.2}$$

1.7 Fisher's information matrix

Fisher information (sometimes simply called information) is the variance of the score or the expected value of the observed information. In Bayesian statistics, the asymptotic distribution of the posterior mode depends on the Fisher information and not on the prior. The role of the Fisher information in the asymptotic theory of maximum likelihood estimator was emphasized by the statistician R. A Fisher. The Fisher information is also used in the calculation of the Jeffrey's prior, which is used in Bayesian statistics. The Fisher-information matrix is used to calculate the covariance matrices associated with maximum-likelihood estimates. It can also be used in the formulation of test statistics, such as the Wald's test. The Fisher information was discussed by several early statisticians, notably Edgeworth (1908). The Fisher information is a way of measuring the amount of information that an observable random variable *X* carries about an unknown parameter θ upon which the probability of *X* depends. The probability function for *X*, which is also the likelihood function for θ, is a function $f(X;\theta)$; it is the probability mass function (or probability density function of the random variable *X* conditional on the value of θ. The partial derivative with respect to θ of the natural logarithm of the likelihood function is called the score. The Fisher information is that a random variable 'X' contains about the parameter θ is given by;

$$I(\theta) = E\left[\frac{\partial}{\partial\theta}\log\left(f(x;\theta)\right)\right]^2 \qquad (1.7.1)$$

Now, if log f(x; θ) is twice differentiable with respect to θ under certain regularity conditions, Fisher's information is given by:

$$I(\theta) = -E_{\theta}\left[\frac{\partial^2}{\partial\theta^2}\log\left(f(x;\theta)\right)\right] \qquad (1.7.2)$$

1.8 Estimation Techniques

There are a few occasions when population is studied as a whole. As a matter of fact, generally a sample is drawn from the population and population constants are determined on the basis of sample values. Population parameters are usually those constant which occur in probability density or mass function or the moments or some other constants of the population like median.

We know that various sampling procedures do exist and also there are many techniques to determine the value of population constants through sample values. The constant determined through sample observations which stands for population parameter θ or a function t (θ) though f (θ) in many cases is equal to θ.

The choice of a technique depends on the type of the estimator vis-a-vis estimate and the purpose of study. The goodness of an estimator is governed by certain properties. An estimator possessing the maximum properties will be considered as a good estimator.

So in estimation theory we are concerned with the properties of estimators and methods of estimation. The merits of an estimator are judged by the properties of the distribution of estimates obtained through estimators i.e. by the properties of the sampling distribution. Further, it is emphasized that estimation is possible only if there is a random sample.

The theory of estimation was founded by Fisher (1930) in a series of fundamental papers and is divided into two groups (i) point estimation and (ii) Internal estimation. In point estimation, a sample statistic (numerical values) is used to provide an estimate of the population parameter whereas in Interval Estimation, probable range is specified within which the true value of the parameter might be expected to lie.

The word estimator stands for the function, and the word, estimate stands for a value of that function. In estimator, we take a random sample from the distribution to elicit some information about some unknown parameter θ. That is, we repeat the experiment n independent times, observe the sample $x_1, x_2, ..., x_n$. The function of $x_1, x_2, ..., x_n$ use to estimate θ; say the statistic $U(x_1, x_2, ..., x_n)$ called an estimator of θ. We want it to be such that the computed estimate $U(x_1, x_2, ..., x_n)$ is usually close to θ. Thus any statistic whose values are used to estimate $r(\theta)$ where r (.) is some function of the parameter θ, is defined to be an estimator $r(\theta)$. An estimator is always a statistic which is both a random variable and a function.

1.8.1 Methods of estimation

A number of methods to estimate the unknown parameters have been in use. The common used methods are:

a) Method of moment.

b) Method of maximum likelihood estimation.

c) Bayesian method of estimation.

1.8.2 Method of Moment

The method of moments is a method of estimation of population parameters such as mean, variance, median, etc. (which need not be moments), by equating sample moments with unobservable population moments and then solving those equations for the quantities to be estimated. One of the simplest and oldest methods of estimation is the substitution principle. The method of moments was discovered and studied in detail by Karl Pearson. The method of moments is special case when we need to estimate some known function of finite number of unknown moments.

Let $f(y;\theta_1,\theta_2,...,\theta_k)$ be density function of the parent population with k parameters $\theta_1,\theta_2,..,\theta_k$. If μ_r' denotes the rth moment about origin, then

$$\mu_r' = \int_{-\infty}^{\infty} y^r f(y;\theta_1,\theta_2,\ldots,\theta_k); r = 1,2,\ldots,k$$

In general $\mu_1',\mu_2',\ldots,\mu_k'$ will be functions of the parameters $\theta_1,\theta_2,\ldots,\theta_k$ .Let $y_i, i = 1,2,3,\ldots,n$ n be a random sample of size n from the given population. The method of moments consists in solving the k-equation (i) for $\theta_1,\theta_2,\ldots,\theta_k$ in terms of $\mu_1',\mu_2',\ldots,\mu_k'$ and then replacing these moments by the sample moments

e.g. $\hat{\theta}_i = \hat{\theta}(\hat{\mu}_1',\hat{\mu}_2',...,\hat{\mu}_k') = \theta_i(m_1',m_2',...,m_k'); i = 1,2,...,k$

Where m_i is the ith moment about origin in the sample.

Then by the method of moments $\hat{\theta}_1,\hat{\theta}_2,\ldots,\hat{\theta}_k$ are the estimators of respectively.

1.8.3 Method of maximum likelihood estimation (MLE)

The most general method of estimation known is the method of maximum likelihood estimators (MLE) which was initially formulated by C.F.Gauss but as a general method of estimation was first introduced by Fisher in the early (1920) and later on developed by him in a series of papers. He demonstrated the advantages of this method by showing that it yields sufficient estimators, which are asymptotically MVUES's. Thus, the essential feature of this method is that we look at the value of the random sample and then choose our estimate of the unknown population parameter, the value of which the probability of obtaining the observed data is maximum. If the observed data sample values are $x_1, x_2, \ldots\ldots, x_n$ we can write in the discrete case.

$$P(X_1 = x_1, X_2 = x_2, \ldots\ldots, X_n = x_n) = f(x_1, x_2, \ldots, x_n) \tag{1.8.1}$$

which is just the value of joint probability distribution of the random values $x_1, x_2, \ldots, x_n$ at the sample point $x_1, x_2, \ldots, x_n$ since the sample values has been observed and are therefore fixed numbers, we regard $f(x_1, x_{2,} \ldots, x_n; \theta)$ as the value of a function of the parameter θ, referred to as the likelihood function. A similar definition applies when the random sample comes from a continuous population but in that case $f(x_1, x_{2,} \ldots, x_n; \theta)$ is the value of joint pdf at the sample point $x_1, x_2, \ldots, x_n$ i.e.; the likelihood function at the sample value $x_1, x_2, \ldots, x_n$

$$L = \prod_{i=1}^{n} f(x_i, \theta) \tag{1.8.2}$$

Since the principle of maximum likelihood consists in finding an estimator of the parameter which maximizes L for variation in the parameter. Thus if there exists a function $\hat{\theta} = \hat{\theta}(x_1, x_2, \ldots., x_n)$ of the sample values which maximizes L for variation in θ, then $\hat{\theta}$ is to be taken as the estimator of θ. $\hat{\theta}$ is usually called ML estimators. Thus $\hat{\theta}$ is the solution if and only if

$$\frac{\partial L}{\partial \theta} = 0 and \frac{\partial^2 L}{\partial \theta^2} < 0 \tag{1.8.3}$$

Since L > 0, so Log L which shows that L and Log L attains their extreme values at the $\hat{\theta}$. Therefore, the equation becomes

$$\frac{1}{L}\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial \log L}{\partial \theta} = 0 \qquad (1.8.4)$$

a form which is more convenient from practical point of view.

1.8.4 Bayesian Method of Estimation

Bayesian approach to statistical inference exploits the idea that the only satisfactory description of uncertainty is by means of probability. Bayesian statistics is an approach in which estimates are based on a synthesis of a prior distribution and current sample data. Bayesian statistics requires the mathematics of probability and the interpretation of probability which most closely corresponds to the standard use of this word in everyday language: it is no accident that some of the more important seminal books on Bayesian statistics such as the works of De Laplace (1812), Jeffery's (1939) and de Finetti (1970) are actually entitled "probability theory". Indeed, Bayesian methods (i) reduce statistical inference to problems in probability theory, thereby minimizing the need for completely new concepts, and (ii) serve to discriminate among conventional statistical techniques either providing a logical justification to some (and making explicit the conditions which they are valid) or proving the logical in consistency of others.

Bayesian statistics have been used to deal with a wide variety of problems in many scientific and engineering areas. Whenever a quantity is to be inferred, or some conclusion is to be drawn, from observed data, Bayesian principles and tools can be used. The idea that forms the basis of the Bayesian approach is as:

i. Since we are uncertain about the true value of the parameters, we will consider them to be random variables.
ii. The rules of probability are used directly to make inferences about the parameters.
iii. Probability statements about parameters must be interpreted as "degree of belief". The prior distribution must be subjective.
iv. We revise our beliefs about parameters after getting the data by using Bayes' theorem. This gives our posterior distribution which gives the relative weights to each parameter value after analyzing the data.

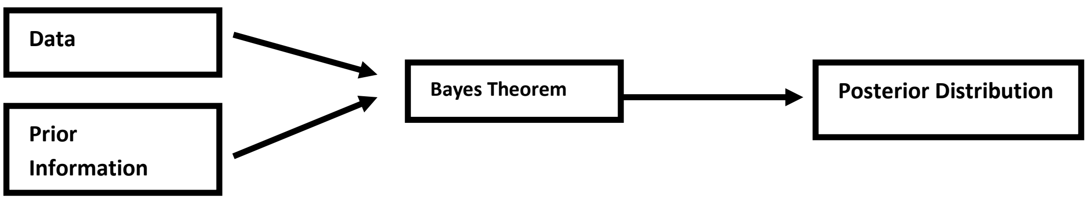
Bayesian statistics is predictive, unlike conventional frequentist statistics. This means we can easily find the conditional probability distribution of the next observation given the sample data. Bayesian approach to statistics is very different from the classical methodology, it formally seeks use of prior information and Bayes' theorem provides the basis for making use of this information. When significant prior is available, the Bayesian approach shows how to utilize it sensibly. This is not possible with the most non-Bayesian approaches. The business of statistics is to provide information or conclusions about uncertain quantities. The language of uncertainty is possible. Bayesian approach consistently uses this language to directly address uncertainty.

The classical or frequentists interpret probability as the limit of the success ratio as the number of trails 'n' conceptually tends to infinity. Under this interpretation the parameter θ in a statistical model is treated as an unknown constant and the sample of observations is regarded as the random sample from some underlying distribution. The classical school believes in Fishers Likelihood Principle which claims that all the information about the unknown parameter(s) is contained in the sample as summarized by the likelihood function. This principle leads to Fishers maximum likelihood estimator.

In Bayesian framework, the parameter is justifiably regarded as a random variable and the data once obtained is given or fixed for example, in the exponential model the mean life θ may be regarded as varying from batch to batch overtime and this variation is represented by a probability distribution over parameter space Ω. Thus the basic difference in the two approaches may be explained in the single sentence that to a frequentist, the parameter is constant and he is suspicious about the data, where as to a Bayesian data is given (or fixed) and he is suspicious about the parameter. Bayesian approach is an excellent alternative to use large sample procedures and is likely to be more reasonable for moderate and especially small sample sizes where non Bayesian procedures break down (e.g., Berger 1985).

1.8.4.1 Bayes Theorem

Bayesian analysis is based upon a theorem first developed by an 18th century English mathematician, logician, and clergy man Thomas Bayes (1701-1761). He developed the theorem in his study of the theory of logic and inductive reasoning. The theorem provides a mathematical basis for relating the degree to which an observation (or new information) confirms the various hypothesized causes or state of nature. His major mathematical works, including the theorem, were published in 1763. Later, in 1774 the theorem was proved independently by Laplace. Bayes’ theorem is an essential element of the Bayesian approach to statistical inference is the direct qualification of uncertainty in terms of probabilistic statements. Often, we begin our analysis with initial or prior probability estimates for specific events of interest then, from sources such as a sample, a special report, a product test and so on we obtain some additional information about the events. Given this new information we update the prior probability values by calculating revised probabilities, referred to as posterior probabilities. The steps in this probability revision process are shown in the following diagram



Suppose that $X' = x_1, x_2,...., x_n$ is a vector of n observations whose probability distribution $P(X \mid \boldsymbol{\theta})$ depends upon the values of k parameters $\boldsymbol{\theta}' = \theta_1, \theta_2,...., \theta_k$. Suppose also that $\boldsymbol{\theta}$ itself has a probability distribution $P(\boldsymbol{\theta})$. Then,

$$P(X \mid \theta)P(\theta) = P(X,\theta) = P(\theta \mid X)P(X). \qquad (1.8.5)$$

Given the observed data X, the conditional distribution of $\boldsymbol{\theta}$ is

$$P(\boldsymbol{\theta} \mid X) = \frac{P(X \mid \boldsymbol{\theta})P(\boldsymbol{\theta})}{P(X)} \qquad (1.8.6)$$

Also we can write

$$P(X) = E[P(X \mid \boldsymbol{\theta})] = k^{-1} = \int P(X \mid \boldsymbol{\theta})P(\boldsymbol{\theta})d\boldsymbol{\theta} = \sum P(X \mid \boldsymbol{\theta})P(\boldsymbol{\theta}) \qquad (1.8.7)$$

where the sum or the integral is taken over the admissible range of $\boldsymbol{\theta}$, and where E indicates averaging with respect to distribution of $\boldsymbol{\theta}$ (e.g., Box and Tiao, 1973; Gelman, Carlin, Stern and Rubin, 1995; Lee, 1997 and Carlin and Louis, 2001). Thus, we may write (1.8.6) alternatively as

$$P(\boldsymbol{\theta} \mid X) \propto P(X \mid \boldsymbol{\theta})P(\boldsymbol{\theta})$$

which is referred to as Bayes theorem. In this expression, $P(\theta)$ which tells us what is known about θ without knowledge of data, is called prior distribution of θ, or the distribution of θ a priori the density $P(X \mid \theta)$ is likelihood function of θ which represents the contribution of X(data) to knowledge about θ (e.g., Berger,1985 and Zellner, 1986). Correspondingly, $P(\theta \mid X)$, which tells us what is known about θ given knowledge of the data X, is called the posterior distribution of θ given X. The quantity 'k' is a normalizing constant.

The term 'Bayesian' however, came into use only around 1950 and in fact it is not clear that Bayes' would endorsed the very broad interpretation of probability now called "Bayesian". Laplace independently proved a more general version of Bayes' theorem and put it to good use in solving problems in celestial mechanics, medical statistics and, by some accounts, even jurisprudence.

1.8.4.2 Sequential Nature of Bayes' Theorem:

Now given the data X, $P(X \mid \theta)$ in (1.8.6) may be regarded as a function not of X but of θ. When so regarded, following Fisher (1922), it is called the likelihood function of θ forgiven X and can be written as $L(\theta \mid X)$ .We can thus write Bayes' formula as

$$P(\theta \mid X) = L(\theta \mid X)P(\theta) \qquad (1.8.8)$$

The theorem in (1.8.8) is appealing because it provides a mathematical formulation of how previous knowledge may be combined with new knowledge. Indeed the theorem allows us to continually update information about a set of parameters θ as more observations are taken. Thus, suppose we have an initial sample of observations X_1, then Bayes initial formula gives,

$$P(\theta \mid X_1) \propto P(\theta)L(\theta \mid X_1) \qquad (1.8.9)$$

Now suppose we have a second sample of observation X_2, distributed independently of first sample, then

$$P(\theta \mid X_1, X_2) \propto P(\theta)L(\theta \mid X_1)L(\theta \mid X_2)$$

$$P(\theta \mid X_1, X_2) \propto P(\theta \mid X_1)L(\theta \mid X_2) \qquad (1.8.10)$$

The expression (1.8.10) precisely of the same form as (1.8.9) except that $P(\theta \mid X_1)$, the posterior distribution for θ given X_1, plays the role of the prior distribution for the second sample. Obviously this process can be repeated any number of times. In particular, if we have n independent observations the posterior distribution can, if desired, be recalculated after each new observation, so that at the mth stage the likelihood associated with the mth observation is combined with the posterior distribution of θ after m-1 observations to give the new posterior distribution.

$$P(\theta \mid X_1, X_2,........, X_m) \propto P(\theta \mid X_1, X_2,......., X_{m-1}) L(\theta \mid X_m) : m = 1,2,......., n \qquad (1.8.11)$$

where $P(\theta \mid X_1) \propto P(\theta) L(\theta \mid X_1)$.

Thus, Bayes' theorem describes in a fundamental way, the process of learning from experience and shows how knowledge about the state of nature represented by θ is continually modified as new data becomes available (e.g., Box and Tiao, 1973).

1.8.4.3 Likelihood to Bayesian Analysis:

An informal summary of the likelihood principle may be that the inferences from data to hypothesis should depend on how likely the actual data are under competing hypothesis and not on how likely imaginary data would have been under a single "null" hypothesis or any other properties of merely possible data.

A more precise interpretation may be that inference procedures which make inferences about simple hypothesis should not be justified by appealing to probabilities assigned to observations that have not occurred. The usual interpretation is that any two probability models with the same likelihood function yield the same inference for θ. Some authors mistakenly claim that frequentist inference, such as the use of maximum likelihood estimation (MLE), obeys the likelihood, though it does not. Some argue that, although the subject of priors gets more attention, the true contention between frequentist and Bayesian inference is the likelihood principle, which Bayesian inference obeys, and frequentist inference does not. Some Bayesians have argued that Bayesian inference is incompatible with the likelihood principle on the grounds that there is no such thing as an isolated likelihood function Bayarri and DeGroot (1987). They argue that in a Bayesian analysis there is no principled distinction between the likelihood function and the prior probability function.

Although the likelihood principle is implicit in Bayesian statistics, it was developed as a separate principle by Barnard (1949), and became a focus of interest when Birnbaum (1962) showed that it followed from the widely accepted sufficiency and conditionality principles Bernardo and Smith (2000). Using Bayes' rule with a chosen probability model means that the data X affect posterior inference only through the function $L(X \mid \theta)$, which, when regarded as a function of θ, for fixed X, is called the `likelihood function'. In this way Bayesian inference obeys what is sometimes called the `likelihood principle', which states that for a given sample of data, any two probability models $L(X \mid \theta)$ that have the same likelihood function yield the same inference for θ Gelman et.al. (1995). The likelihood principle, by itself, is not sufficient to build a method of inference but should be regarded as a minimum requirement of any viable form of inference. This is a controversial point of view for anyone familiar with modern econometrics literature. Much of this literature is devoted to methods that do not obey the likelihood principle (Rossi, Allenby, and McCulloch, 2005).

Suppose $L(\theta \mid X)$ is the assumed likelihood function. Under MLE estimation, we would compute the mode (the maximal value of L, as a function of θ given the data X) of the likelihood function and use the local curvature to construct the confidence intervals. Hypothesis testing follows using likelihood ratio (LR) statistics. The strength of ML estimation rely on its large sample properties, namely that when the sample size is sufficiently large, we can assume both normality of the test statistic about its mean and that LR tests follows χ^2 distributions. These nice features don't necessarily hold for small samples Gianola & Fernando (1986).

An alternate way to proceed is to start with some initial knowledge/guess about the distribution of the unknown parameter(s), $P(\theta)$. From Bayes' theorem the data (likelihood) augments the prior distribution to produce a posterior distribution,

$$P(\theta \mid X) = \frac{1}{P(X)} P(X \mid \theta) P(\theta) \qquad (1.8.12)$$

$= \text{(normalizing constant)}\ P(X\mid\theta)P(\theta)$ (1.8.13)

$= \text{constant .likelihood .prior}$ (1.8.14)

As $P(X\mid\theta) = L(\theta\mid X)$ is just the likelihood function. 1/P(X) is constant (with respect to θ), because our concern is the distribution over θ. Because of this, the posterior is often written as

$$P(\theta\mid X) \propto L(\theta\mid X_1)P(\theta) \qquad (1.8.15)$$

where the symbol $\propto$ means "proportional to" (equal up to a constant). Note that the constant P(X) normalizes $P(X\mid\theta)P(\theta)$ to one, and hence can be obtained by integration

$$P(X) = \int_\theta P(X\mid\theta)P(\theta)d\theta \qquad (1.8.16)$$

The dependence of the posterior on the prior (which can easily be assessed by trying different prior) provides an indication of how much information on the unknown parameter values is contained in the data. If the posterior is highly dependent on the prior, then the data likely has little signal, while if the posterior is largely unaffected under different priors, the data are likely highly informative. To see this taking logs on equation (1.8.15) (and ignoring the normalizing constant) gives

Log (posterior) =log (likelihood) +log (prior) (1.8.17)

The Standard Likelihood

When the integral $L(\theta\mid X)d\theta$ taken over the admissible range of θ is finite, then occasionally it will be convenient to refer to the quantity

$$\frac{l(\theta\mid X)}{\int l(\theta\mid X)d\theta} \qquad (1.8.18)$$

We shall call this the standardized likelihood that is the likelihood scaled so that the area, volume or hyper volume under the curve, surface or hyper surface is one.

1.8.4.4 Prior Distribution and Some Important Types of Priors

A prior distribution of a parameter is the probability that represents uncertainty about the parameter before the current data are examined. A random variable can be thought of as a variable that takes on a set of values with specified probability. In frequentist statistics, parameters are not repeatable random things but are fixed quantities, which mean that they cannot be considered as random variables. In contrast, in Bayesian statistics anything about which we are uncertain, including the true value of the parameter, can be thought of as being a random variable to which we can assign a probability distribution, known specifically as prior information. A fundamental feature of the Bayesian approach to statistics is the use of prior information in addition to the (sample) data. A proper Bayesian analysis will always incorporate genuine prior information, which will help to strengthen inferences about the true value of the parameter and ensure that any relevant information about it is not wasted.

Obviously, a important feature of any Bayesian analysis is the use of prior. According to Diaconis and Ylvisaker (1985), there are three distinct Bayesian approaches for the selection of prior distributions. The classical Bayesian approach considers flat priors to represent objectivity in the analysis. The modern approach allows the priors to have characteristics like closure under sampling (conjugacy) suggested by G.Barnard (1954) and later developed by Raiffa & Schlaifer (1961)) and specification of hyper parameter values according to some specific criteria. The third approach is followed by subjective Bayesians, depends on elicitation of prior distributions based on pre-existing scientific knowledge in the area of investigation.

Some standard approaches of priors are discussed in brief as:

i) Non-informative Priors: A prior distribution is non-informative if the prior is “flat” relative to the likelihood function. Such a prior is also known as “vague” or “diffuse” prior. Thus, a prior $P(\theta)$ is non-informative if it has minimal impact on the posterior distribution of θ. We may prefer non-informative priors because they appear to be more objective. Non-informative priors provide a formal way of expressing ignorance of the value of the parameter over the permitted range Jeffery (1961).

ii) Informative prior: An informative prior is a prior that is not dominated by the likelihood and that has an impact on the posterior distribution. If a prior distribution dominates the likelihood, it is clearly an informative prior. On the other hand, the proper use of prior distributions illustrates the power of the Bayesian method: information gathered from the previous study, past experience, or expert opinion can be combined with current information in a natural way.

iii) Improper prior: A prior $P(\theta)$ is said to be improper if $\int P(\theta)\, d\theta = \infty$. For example, a uniform prior distribution on the real line, $P(\theta) \propto 1$, for $-\infty < \theta < \infty$, is an improper prior. Improper priors are often used in Bayesian inference since they usually yield non-informative priors and proper posterior distributions. Improper prior distributions can lead to posterior impropriety (improper posterior distribution). To determine whether a posterior distribution is proper, you need to make sure that the normalizing constant $\int L(X \mid \theta) P(\theta)\, d\theta$ is finite for all x. If an improper prior distribution leads to an improper posterior distribution, inference based on the improper posterior distribution is invalid.

iv) Conjugate Priors: A prior is said to be a conjugate prior for a family of distributions if the prior and posterior distributions are from the same family, which means that the form of the posterior has the same distributional form as the prior distribution. For example, if the likelihood is binomial, $X \sim B(n, \theta)$ a conjugate prior on θ is the beta distribution; it follows that the posterior distribution of θ is also a beta distribution. Other commonly used conjugate prior/likelihood combinations include the normal/normal, gamma/Poisson, gamma/gamma, and gamma/beta cases. The development of conjugate priors was partially driven by a desire for computational convenience-conjugacy provides a practical way to obtain the posterior distributions.

v) Jeffery's' Prior: A very useful prior is Jeffery's' prior (1961). It satisfies the local uniformity property: a prior that does not change much over the region in which the likelihood is significant and does not assume large values outside that range. It is based on the Fisher information matrix. Jeffrey's prior is defined as

$$P(\Theta) \propto \left| I(\Theta) \right|^{-1/2} \tag{1.8.19}$$

where $I(\theta)$ denotes the Fisher information matrix based on the likelihood function

$$I(\theta) = -\left[E\left\{ \frac{\partial^2 \log L(X \mid \theta)}{\partial \theta^2} \right\} \right] \tag{1.8.20}$$

Jeffrey's prior is locally uniform and hence non-informative. It provides an automated scheme for finding a non-informative prior for any parametric model $L(X \mid \Theta)$. Another appealing property of Jeffrey's prior is that it is invariant with respect to one-to-one transformations. The invariance property means that if you have a locally uniform prior on θ and $\phi(\theta)$ is a one-to-one function of θ, then $P(\phi(\Theta)) = P(\Theta).|\phi'(\Theta)|^{-1}$ is a locally uniform prior for $\phi(\theta)$. This invariance principle carries through to multidimensional parameters as well. While Jeffrey's prior provides a general recipe for obtaining non-informative priors, it has some shortcomings: the prior is improper for many models, and it can lead to improper posterior in some cases; and the prior can be cumbersome to use in high dimensions.

Bayesian analysis syntheses two sources of information about the unknown parameters of interest. The first of these is the sample data, expressed formally by the likelihood function. The second is the prior distribution, which represents additional information that is available to investigator. Suppose we have a random sample of size n say $x_1, x_2 ...., x_n$ which we regard as independent identically distributed random variables with distribution function $F(X \mid \theta)$ and pdf $f(x \mid \theta)$ and where θ a labeling parameter, real valued or a vector valued as the case may be. Also we assume that we do not know the exact value of parameter θ there are cases in which one can assume a little more about a parameter. Here Ω is the parameter space. We could assume that θ is itself a random variable with distribution function $F(\theta)$ or pdf $P(\theta)$.

Now suppose n items are put to test and it is assumed that their recorded life items from a random sample of size n from a population with pdf $f(x \mid \theta)$ to be specific we will assume θ to be real valued. We agree to regard θ itself as random variable with a pdf $P(\theta)$. The joint pdf of $P(\theta)$ is given by

$$P(\theta \mid x_1, x_{2,} ...., x_n) = \left\{ \prod_{i=1}^{n} f(x_i \mid \theta) \right\} = L(x_1, x_{2,} ...., x_n \mid \theta) \qquad (1.8.21)$$

The marginal pdf of ($x_1, x_{2,} ...., x_n$) is given by

$$p(x_1, x_2, \ldots, x_n) = \int_{\Omega} p(x_1, x_2, \ldots, x_n | \theta) d\theta \qquad (1.8.22)$$

And the conditional pdf of θ given data ($x_1, x_2, \ldots, x_n$) is given by

$$P(\theta | x_1, x_2, \ldots, x_n) = \frac{p(x_1, x_2, \ldots, x_n | \theta)}{p(x_1, x_2, \ldots, x_n)}$$

$$P(\theta | x_1, x_2, \ldots, x_n) = \frac{L(x_1, x_2, \ldots, x_n | \theta) p(\theta)}{\int_{\Omega} L(x_1, x_2, \ldots, x_n | \theta) p(\theta) d\theta} \qquad (1.8.23)$$

Thus, prior to obtaining ($x_1, x_2, \ldots, x_n$) the variations in θ where represented by $P(\theta)$, known as prior distribution on θ however, after the data ($x_1, x_2, \ldots, x_n$) has been obtained in the light of the new information, the variation in θ are represented by $P(\theta | x_1, x_2, \ldots, x_n)$ the posterior distribution of θ. The uncertainty about the parameter θ. Prior to experiment is represented by prior pdf $P(\theta)$ and the same after the experiment is represented by posterior pdf $P(\theta | x_1, x_2, \ldots, x_n)$ this process is the straight forward application pdf the Bayes theorem. Once the posterior distribution has been obtained it becomes the main object of study.

1.8.4.5 Marginal and Conditional inferences

Often only a subset of unknown parameter is really of concern to us, the rest being nuisance parameter that are of no concern to us. A very strong feature of Bayesian analysis is that we can remove the effect of nuisance parameters by simply integrating them out of the posterior distribution to generate a marginal posterior distribution for the parameters of interest. For example, if θ is partitioned as (θ_1, θ_2), with θ_1 a p dimensional vector and θ_2 as (k-p) dimensional vector, then the marginal posterior density for θ_1 is given by

$$P(\theta_1 | x) = \frac{\int_{R_2} P(x | \theta) P(\theta) d\theta_2}{\int_{R} P(x | \theta) P(\theta) d\theta} \qquad (1.8.24)$$

Similarly, the marginal posterior density for θ_2 is given by

$$P(\theta_2 \mid x) = \frac{\int_{R_1} P(x \mid \theta) P(\theta) d\theta_1}{\int_R P(x \mid \theta) P(\theta) d\theta} \tag{1.8.25}$$

The requirement of orthogonality between nuisance parameter and the parameter of interest is not required in this frame work Cox and Reid (1987). Moreover, marginal posterior densities are better substitutes of conditional profile likelihoods.

Conditional inferences for θ_1 given θ_2; and θ_2 given θ_1 can also be made using the posteriors

$$P(\theta_1 \mid x, \theta_2) = \frac{P(x \mid \theta_1, \theta_2) P(\theta_1 \mid \theta_2)}{\int_{R_1} P(x \mid \theta_1, \theta_2) P(\theta_1 \mid \theta_2) d\theta_1} \tag{1.8.26}$$

and $$P(\theta_2 \mid x, \theta_1) = \frac{P(x \mid \theta_1, \theta_2) P(\theta_2 \mid \theta_1)}{\int_{R_2} P(x \mid \theta_1, \theta_2) P(\theta_2 \mid \theta_1) d\theta_2} \tag{1.8.27}$$

Marginal and conditional inferences procedures are two entirely different things. In the former, we ignore one of the components of θ by integrating it out from the joint posterior $P(\Theta \mid x)$, while in the later we control (or adjust) one of the components of θ Khan (1997).

1.8.4.6 Predictive Distribution

It is the pdf (or pmf) of the as yet unobserved observation x given sample information X. let us write $f(x, \theta \mid y) = f(x, \mid \theta, y) P(\theta \mid y)$ as the joint pdf of x and the parameter θ, given the sample information Y. Here $f(x \mid \theta, Y)$ is the conditional pdf for x given θ and X, where $P(\theta \mid Y)$ is the conditional pdf for θ given Y the predictor pdf $f(x \mid y)$ is obtained as:

$$f(x \mid y) = \int f(x, \theta \mid y) d\theta = \int f(x \mid \theta, y) p(\theta \mid y) d\theta \tag{1.8.28}$$

In case, the unobserved observation of x is independent of sample information Y, that is x and y have independent conditional pdf's then

$$f(x \mid y) = \int f(x \mid y) p(\theta \mid y) d\theta \tag{1.8.29}$$

1.8.4.7 Methods of Posterior Modes

Asymptotic normality of the posterior is the basic tool of large sample Bayesian inference. Under certain regularity conditions, in particular, if the likelihood is a continuous function of θ and that the maximum likelihood estimate, $\hat{\theta}$ of θ is not the boundary of the parameter space, the unimodal and almost symmetric posterior distribution of θ approaches normality with mean $\hat{\theta}$ and precision $I(\hat{\theta})$, Fisher Information evaluated at $\hat{\theta}$,for large sample sizes. It may be noted that for large samples, the likelihood dominates the prior distribution and, therefore the knowledge of likelihood is enough to obtain the normal approximation. Gelman et al. (1995) give a number of counter examples to illustrate limitations of the large sample approximation to the posterior distribution. The Bayesian approach to parametric inference is conceptually simple and probabilistically elegant. However its numerical implication is not convenient since the posterior distributions are available as complicated functions. Although these approximations provide useful results in applications, neither gives any account for the cases when the mode is at boundary.

In the development of new simulation techniques, Laplace's method uses asymptotic arguments. Laplace's method is easier to implement and thus faster than the Monte Carlo methods, such as Gibbs sampling (Gelfand and Smith 1990), which requires a large number of simulations from the conditional densities. Laplace approximations to marginal densities and expectations can provide further insights to the problem at hand.

1.8.4.8 Normal approximation to posterior distribution

The numerical implementation of a Bayesian procedure is not always straight forward since the involved posterior distribution is complicate functions. One of the important steps in simplifying the computations is to investigate the large sample behavior of the posterior distribution and its characteristics. The basic result of the large sample Bayesian inference is that the posterior distribution of the parameter approaches a normal distribution. Relatively little has been written on the practical implications of asymptotic theory for Bayesian analysis. The overview by Edwards, Lindeman, and Savage (1963) remains one of the best and includes a detailed discussion of the principle of 'stable estimation' or when prior information can be satisfactorily approximated by a uniform density function. Some good sources on the topic from the Bayesian point of view include Lindley (1958), Pratt (1965), and Berger and Wolpert (1984). An example of the use of the normal approximation with small samples is provided by Rubin and Schenker (1987), who approximated the posterior distribution of the logit of the binomial parameter in real application and evaluate the frequentists operating characteristics of their procedure. Clogg et al. (1991) provide additional discussion of this approach in a more complicated setting. Sequential monitoring and analysis of clinical trials in medical research is an important area of practical application that has been dominated by frequentists thinking but has recently seen considerable discussion of the merits of a Bayesian approach; a recent review is provided by Freedman, Spiegel halter and Parmer (1994), Khan, A.A (1997) and Khan et al. (1996).

If the posterior distribution $P(\theta \mid y)$ is unimodal and roughly symmetric, it is convenient to approximate it by a normal distribution centered at the mode; that is logarithm of the posterior is approximated by a quadratic function, yielding the approximation

$$P(\theta \mid y) \sim N\left(\hat{\theta}, \left[I\left(\hat{\theta}\right)\right]^{-1}\right)$$

where $I\left(\hat{\theta}\right) = -\dfrac{\partial^2 \log P(\theta \mid x)}{\partial \theta^2}$ (1.8.30)

if the mode, $\hat{\theta}$ is in the interior parameter space, then $I(\theta)$ is positive; if $\hat{\theta}$ is a vector parameter, then $I(\theta)$ is a matrix.

1.8.4.9 Laplace's Approximation

Laplace's method is a family of asymptotic methods used to approximate integrals presented as a potential candidate for the tool box of techniques used for knowledge acquisition and probabilistic inference in belief networks with continuous variables. The method is promising for computing approximation for Bayes' factor for use in the context of model selection, model uncertainty and mixtures of pdf's. It is simple and remarkable method of asymptotic expansion of integrals generally attributed to Laplace (Laplace, 1986, 1774, Stigler, 1986) is widely used in applied mathematics. This method has been applied by many authors (Lindley, 1961, 1980; Mostller and Wallace, 1964; Johnson, 1970; DiCiccio, 1986; Hartigan, 1965; Khan et al., 1996; and Tierney and Kadane, 1986 to find approximations to the ratios of integrals of the interest, especially in Bayesian analysis. If we approximate the integrals involved in the posterior density using approximation

$$P(\theta \mid X) = (2\pi)^{\frac{-k}{2}} \left|I(\hat{\theta})\right|^{\frac{1}{2}} \exp\left[\log P(\hat{\theta} \mid x)\right]\left(1 + O(n^{-1})\right)$$

where $\left|I(\hat{\theta})\right|$ stands for determinant of $I(\hat{\theta})$ then posterior density can be approximated with error of order $O(n^{-1})$ i.e.

$$P(\theta \mid X) = (2\pi)^{\frac{-k}{2}} \left|I(\hat{\theta})\right|^{\frac{1}{2}} \exp\left[\log P(\theta \mid x) - \log P(\hat{\theta} \mid x)\right]\left(1 + O(n^{-1})\right) \qquad (1.8.31)$$

Approximation (1.8.31) is the well known Laplace's approximation of integrals Tierney and Kadane (1986). Laplace's approximation (1.9.2b) of posterior density can be compared with normal approximation which has error of order $O(n^{-\frac{1}{2}})$. Perhaps more importantly, Laplace's approximation is of order $O(n^{-1})$ uniformly on any neighborhood of the mode. This means that it should provide a good approximation in the tails of distribution also (e.g., Tierney and Kadane, 1986; Tierney, Kass and Kadane, 1989a; and Wong and Li, 1992).

CHAPTER – 2

SIZE-BIASED CLASSICAL CONTINUOUS DISTRIBUTIONS

2.1 Introduction

Gamma distribution is a two-parameter family of continuous probability distributions .This distribution is used as a lifetime model Gupta and Groll (1961), though not, nearly as much as the Weibull distribution. It is most widely used model for precipitation data. It also arises in some situations involving the exponential distribution; because of the well known results that the sum of independently and identically distributed exponential random variables has a Gamma distribution. Inference for Gamma model has been considered by Engelhard and Bain (1978),Chao and Glaser (1978), Jamali et al. (2006), Kalbfleisch and Prentice (2002), Lawless (2003), Zaman et al. (2005), Saal et al. (2008), Ahmad (2006) & Ahmad et al. (2011) has made significant contributions. It is frequently a probability model for waiting times; for instance, in life testing, the waiting time until death is a random variable that is frequently modeled with a Gamma distribution.

The probability density function of Gamma distribution is given by:

$$f(x;\alpha,\beta)=\frac{\alpha^{\beta}}{\Gamma\beta}e^{-\alpha x}x^{\beta-1};\alpha,\beta>0 \qquad (2.1.1)$$

where α and β are parameters; α is a scale parameter and β is sometimes called the index or shape parameter. $\Gamma(\beta)$ is the well-known Gamma function which for integral values of equals $(\beta-1)!$. For β =1, the distribution reduces to exponential distribution.

The moments of the Gamma Distribution are:

$$E(x) = \frac{\beta}{\alpha} \; and V(x) = \frac{\beta}{\alpha^2}$$

The hazard function of the model can be increasing, decreasing or constant depending on α > 1, α < 1 or α = 1 respectively. Whereas as in exponential distribution, the hazard rate is constant (1/α).

In this chapter, we have considered a new general class of Size biased Gamma, Beta and exponential distributions. Several structural properties of these new models have been discussed. The estimation of parameters of these new models is obtained by employing the methods of moments, maximum likelihood and Bayesian method of estimation. The Bayes' estimators are obtained by using Jeffrey's and extension of Jeffrey's prior under different loss functions. A comparison has been made of the Bayes' estimator with the corresponding maximum likelihood estimator. Also, a likelihood ratio test of size-biasedness is conducted. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator.

2.2 Size biased Gamma Distribution

A size- biased Gamma distribution (SBGMD) is obtained by applying the weights x^c, where c =1 to the Gamma distribution.

$$\mu_1' = \int_0^\infty x \, f(x;\theta) dx = \frac{\beta}{\alpha}$$

$$\int_0^\infty \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \; e^{-\alpha x} \; x^{\beta} \; dx = 1$$

This gives the size –biased Gamma distribution (SBGMD) as:

$$f(x;\alpha,\beta+1) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \; e^{-\alpha x} \; x^{\beta} \qquad (2.2.1)$$

= 0, otherwise; $0 < x < \infty$

where $\alpha > 0$ and $\beta \geq 0$ are parameters; α is a scale parameter and β is sometimes called the index or shape parameter.

Special Cases

Case 1: When $\beta = 0$, then Size-biased Gamma distribution (SBGMD) (2.2.1) reduces to exponential distribution (EPD) with probability density function as:

$$f(x;\alpha) = \alpha\, e^{-\alpha x}; 0 < x < \infty \tag{2.2.2}$$

Case 2: When $\beta = 1$, then Size-biased Gamma distribution (SBGMD) (2.2.1) reduces to size biased exponential distribution (SBEPD) with probability density function as:

$$f(x;\alpha) = \alpha^2\, x\, e^{-\alpha x}; 0 < x < \infty \tag{2.2.3}$$

Case 3: When $\alpha = 1$, then Size-biased Gamma distribution (SBGMD) (2.2.1) reduces to one parameter size biased Gamma distribution with probability density as:

$$f(x;\beta+1) = \frac{e^{-x}\, x^{\beta}}{\beta!}; \beta \geq 0 \tag{2.2.4}$$

2.3 Structural Properties of the Size-biased Gamma Distribution

In this section, we derive some important properties of Size-biased Gamma distribution.

2. 3.1 Moments of Size -biased Gamma distribution

The rth moment of Size-biased Gamma distribution (2.2.1) about origin is obtained as:

$$\mu_r' = \int_0^{\infty} x^r\; f(x;\alpha,\beta+1)\; dx$$

$$\mu_r' = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \int_0^{\infty} e^{-\alpha x}\; x^{\beta+r}\; dx$$

$$\mu_r' = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \frac{\Gamma(\beta+r+1)}{\alpha^{\beta+r+1}} \tag{2.3.1}$$

Using above relation (2.3.1), the mean and variance of the SBGMD is given as

$$\mu_1' = \frac{\beta+1}{\alpha} \tag{2.3.2}$$

$$\mu_2 = \frac{(\beta+1)}{\alpha^2} \tag{2.3.3}$$

$$C.V = \frac{1}{\sqrt{\beta+1}}$$

2.3.2 Moment generating function:

The moment generating of SBG distribution can be obtained as:

$$E(e^{tx}) = \int_0^\infty e^{tx} f(x;\alpha,\beta+1)dx$$

$$E(e^{tx}) = \int_0^\infty e^{tx} \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} \; x^\beta \; dx$$

$$E(e^{tx}) = \int_0^\infty e^{tx} \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} \; x^\beta \; dx$$

$$E(e^{tx}) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \int_0^\infty e^{-(\alpha-t)x} \; x^\beta \; dx$$

$$E(e^{tx}) = \frac{\alpha^{\beta+1}}{(\alpha-t)^{\beta+1}}$$

$$E(e^{tx}) = \left(\frac{\alpha}{\alpha-t}\right)^{\beta+1} \tag{2.3.4}$$

2.3.3 Characteristic function:

The characteristic function of SBG distribution is obtained as

$$\Phi_x(t) = \int_0^\infty e^{itx} f(x;\alpha,\beta+1)dx$$

$$\Phi_x(t) = \int_0^\infty e^{itx} \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} \; x^\beta \; dx$$

$$\Phi_x(t) = \int_0^\infty e^{itx} \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} \; x^\beta \; dx$$

$$\Phi_x(t) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \int_0^\infty e^{-(\alpha-it)x} \; x^\beta \; dx$$

$$\Phi_x(t) = \frac{\alpha^{\beta+1}}{(\alpha - it)^{\beta+1}}$$

$$\Phi_x(t) = \left(\frac{\alpha}{\alpha - it}\right)^{\beta+1} \tag{2.3.5}$$

For values of α and β, we get the following table.

Distribution name	α	β	Mean	variance	C.V	MGF	CF
Exponential	α	0	$\frac{1}{\alpha}$	$\frac{1}{\alpha^2}$	1	$\frac{\alpha}{(\alpha - t)}$	$\frac{\alpha}{(\alpha - it)}$
Size-biased Exponential	α	1	$\frac{2}{\alpha}$	$\frac{2}{\alpha^2}$	$\frac{1}{\sqrt{2}}$	$\frac{\alpha^2}{(\alpha - t)^2}$	$\frac{\alpha^2}{(\alpha - it)^2}$

2.3.4 Shannon's entropy of size-biased Gamma Distribution

Shannon's entropy is obtained as:

$$H[f(x;\alpha,\beta+1)] = E[-\log\{f(x;\alpha,\beta+1\}]$$

$$H[f(x;\alpha,\beta+1)] = E\left[-\log\left\{\frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} x^{\beta}\right\}\right]$$

$$H[f(x;\alpha,\beta+1)] = E\left[-\log\left(\frac{\alpha^{\beta+1}}{\Gamma(\beta+1)}\right)\right] + \alpha E(X) - \beta E(\log X)$$

$$H[f(x;\alpha,\beta+1)] = \log\frac{\Gamma(\beta+1)}{\alpha^{\beta+1}} + \alpha\left(\frac{\beta+1}{\alpha}\right) - \beta E(\log X) \tag{2.3.6}$$

Now, $E(\log(x)) = \int_0^\infty \log x f(x;\alpha,\beta+1)dx$

$$E(\log(x)) = \int_0^\infty \log x \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} x^{\beta} dx$$

$$E(\log(x)) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \int_0^\infty \log x\, e^{-\alpha x} x^{\beta} dx$$

Put, $\alpha x = t,\ x = \frac{t}{\alpha} \quad dx = \frac{dt}{\alpha}, x \to 0, t \to 0. x \to \infty, t \to \infty$

$$E(\log(x)) = \frac{1}{\Gamma(\beta+1)} \int_0^\infty \left(\log t - \log \alpha\right) e^{-t} t^{\beta} dt$$

$$E(\log x) = \frac{\Gamma'(\beta+1)}{\Gamma(\beta+1)} - \log \alpha$$

$$E(\log x) = \Psi(\beta+1) - \log \alpha \tag{2.3.7}$$

Using equation (2.3.7) in equation (2.3.6), we get the Shannon's entropy of Size-Biased Gamma Distribution

$$H\left[f(x;\alpha,\beta+1)\right] = \log \frac{\Gamma(\beta+1)}{\alpha^{\beta+1}} + \alpha\left(\frac{\beta+1}{\alpha}\right) - \beta\left(\Psi(\beta+1) - \log \alpha\right)$$

$$H\left[f(x;\alpha,\beta+1)\right] = \log \frac{\Gamma(\beta+1)}{\alpha^{\beta+1}} - \beta\left(\Psi(\beta+1) - \log \alpha\right) + \beta + 1 \tag{2.3.8}$$

2.3.5 Fisher's information matrix of size-biased Gamma Distribution

The Gamma distribution has a probability density function of the form

$$f(x;\alpha,\beta+1) \quad = \quad \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} \ e^{-\alpha x} x^{\beta} \tag{2.3.9}$$

Applying log on both sides in equation (2.3.9), we have

$$\log f(x;\alpha,\beta+1) = \log\left(\frac{\alpha^{\beta+1}}{\Gamma(\beta+1)}\right) - \alpha x + \beta \log x \tag{2.3.10}$$

Differentiating equation (2.3.10) partially with respect to α and β, we get

$$\frac{\partial}{\partial \alpha} \log f(x;\alpha,\beta+1) = \frac{\beta+1}{\alpha} - x$$

$$\frac{\partial^2}{\partial \alpha^2} \log f(x;\alpha,\beta+1) = -\left(\frac{\beta+1}{\alpha^2}\right) \tag{2.3.11}$$

$$\frac{\partial}{\partial \beta\, \partial \alpha} \log f(x;\alpha,\beta+1) = \frac{1}{\alpha} \tag{2.3.12}$$

$$\frac{\partial}{\partial \beta} \log f(x;\alpha,\beta+1) = \log \alpha - \Psi(\beta+1) + \log x$$

Where, $\Psi(\beta+1)=\dfrac{\Gamma'(\beta+1)}{\Gamma(\beta+1)}$

$$\frac{\partial}{\partial\alpha\,\partial\beta}\log f(x;\alpha,\beta+1)=\frac{1}{\alpha} \tag{2.3.13}$$

$$\frac{\partial^2}{\partial\beta^2}\log f(x;\alpha,\beta+1)=-\Psi'(\beta+1)+\left(\Psi(\beta+1)\right)^2 \tag{2.3.14}$$

Taking expectations on both sides of the equations, we get

$$I(1,1)=-E\left[\frac{\partial^2}{\partial\alpha^2}\log f(x;\alpha,\beta+1)\right]=\frac{\beta+1}{\alpha^2}$$

$$I(1,\,2)=-E\left[\frac{\partial^2}{\partial\beta\,\partial\alpha}\log f(x;\alpha,\beta+1)\right]=\frac{-1}{\alpha}$$

$$I(2,1)=-E\left[\frac{\partial^2}{\partial\alpha\,\partial\beta}\log f(x;\alpha,\beta+1)\right]=\frac{-1}{\alpha}$$

$$I(2,2)=-E\left[\frac{\partial^2}{\partial\beta^2}\log f(x;\alpha,\beta+1)\right]=\Psi'(\beta+1)-\left[\Psi(\beta+1)\right]^2$$

Now, the Fisher's information matrix of size-biased Gamma Distribution is given by:

$$I(\alpha,\beta+1)=\begin{pmatrix} -E\left(\dfrac{\partial^2}{\partial\alpha^2}\log f(x;\alpha,\beta+1)\right) & -E\left(\dfrac{\partial^2}{\partial\alpha\partial\beta}\log f(x;\alpha,\beta+1)\right) \\ -E\left(\dfrac{\partial^2}{\partial\beta\partial\alpha}\log f(x;\alpha,\beta+1)\right) & -E\left(\dfrac{\partial^2}{\partial\beta^2}\log f(x;\alpha,\beta+1)\right) \end{pmatrix}$$

$$I(\alpha,\beta+1)=\begin{pmatrix} \dfrac{\beta+1}{\alpha^2} & -\dfrac{1}{\alpha} \\ -\dfrac{1}{\alpha} & \Psi'(\beta+1)-\left[\Psi(\beta+1)\right]^2 \end{pmatrix} \tag{2.3.15}$$

The above relation (2.3.19) is the Fisher's information matrix of Size-biased Gamma Distribution.

2.3.6. Characterization of Size biased Gamma distribution.

Theorem 1: If X and Y are independent size biased Gamma varieties with parameters $\beta+1$ and $\alpha+1$ respectively .The distribution of U=X+Y, are independent and follows size biased Gamma distribution with parameter $\alpha+\beta+2$ and Z=$\frac{X}{Y}$ follows size biased beta distribution of second kind with parameters $(\alpha+1, \beta+1)$.

Proof: If X ~ $SBG(\beta+1)$, then

$$f(x;\beta+1) \quad = \quad \frac{1}{\Gamma(\beta+1)} \quad e^{-x}\, x^{\beta};0<x<\infty, \beta \geq 0$$

If y ~ $SBG(\alpha+1)$, then

$$f(y;\alpha+1) \quad = \quad \frac{1}{\Gamma(\alpha+1)} \quad e^{-y}\, y^{\alpha};0<y<\infty, \alpha \geq 0$$

Since X and Y are independently distributed, their joint probability differential is given by the compound probability theorem as shown below:

$$dF(x,y)=f(x)f(y)dxdy$$

$$dF(x,y)=\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}e^{-(x+y)}x^{\beta}y^{\alpha}dxdy \qquad (2.3.16)$$

Put U=X+Y, Z=$\frac{X}{Y}$

$$X=\frac{ZU}{1+Z}, Y=\frac{U}{1+Z}, J=\frac{-U}{(1+Z)^2}$$

As X and Y ranges from $0\ to\ \infty$, both U and Z ranges from $0 to \infty$.Hence the joint probability differential of random variables U and Z becomes:

$$f(U,Z)=f(X,Y)|J|dxdy$$

$$f(U,Z)=\int_0^{\infty}\frac{1}{\beta(\alpha+1,\beta+1)}\frac{Z^{(\beta+1)-1}}{(1+Z)^{\alpha+\beta+2}}dZ.\frac{1}{\Gamma(\alpha+\beta+2)}e^{-U}U^{(\alpha+\beta+2)-1}dU$$

$$f(U,Z)=SB\beta 2(\alpha+1,\beta+1).SBG(\alpha+\beta+2) \qquad (2.3.17)$$

Hence, U and Z are independently distributed; U is a size biased Gamma distribution with parameter $\alpha+\beta+2$ and Z as a size biased beta distribution of second kind with parameters $(\alpha+1, \beta+1)$.

Theorem 2: If X and Y are independent size biased Gamma varieties with parameters $\beta+1$ *and* $\alpha+1$ respectively. Then the distribution of Z= $X+Y$ are independent and follows size biased Gamma distribution with parameter $\alpha+\beta+2$ and U $=\frac{X}{X+Y}$ follows size biased beta distribution of first kind with parameters $(\alpha+1,\beta+1)$.

Proof: If X ~ $SBG(\beta+1)$, Then

$$f(x;\beta+1) \quad = \frac{1}{\Gamma(\beta+1)} \quad e^{-x} x^{\beta};0<x<\infty,\beta\geq 0$$

If Y ~ $SBG(\alpha+1)$, Then

$$f(y;\alpha+1) \quad = \frac{1}{\Gamma(\alpha+1)} \quad e^{-y} x^{\alpha};0<y<\infty,\alpha\geq 0$$

Since X and Y are independently distributed, their joint probability differential is given by the compound probability theorem as shown below:

$$dF(x,y)=f(x)f(y)dxdy$$

$$dF(x,y)=\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}e^{-(x+y)}x^{\beta}y^{\alpha}dxdy \qquad (2.3.18)$$

Put U= $\frac{X}{X+Y}$, Z=X+Y

$X=UZ, Y=Z(1-U)$, $J=Z$

As X and Y ranges from $0 \, to \, \infty$, but U ranges from 0 to 1 and Z ranges from $0 \, to \, \infty$ .Hence the joint probability differential of random variables U and Z becomes:

$$f(U,Z)=f(X,Y)|J|dxdy$$

$$f(U,Z)=\int_0^{\infty}\frac{1}{\beta(\alpha+1,\beta+1)}Z^{(\beta+1)-1}(1-Z)^{(\alpha+1)-1}dZ.\frac{1}{\Gamma(\alpha+\beta+2)}e^{-U}U^{(\alpha+\beta+2)-1}dU$$

$$f(U,Z)=SB\beta 1(\alpha+1,\beta+1).SBG(\alpha+\beta+2) \qquad (2.3.19)$$

Hence, U and Z are independently distributed; U follows as a size biased beta distribution of first kind with parameters $(\alpha+1,\beta+1)$ and Z as a size Gamma biased distribution with parameter $\alpha+\beta+2$.

Theorem 3: If $X \sim SB\beta 1(\mu, v) and Y \sim SBG(\lambda, \mu + v + 1)$ be independent random variables, then XY is distributed as a size biased Gamma variate with parameters $\lambda and \mu + 1, i, e, XY \sim SBG(\lambda, \mu + 1).$

Proof: Since X and Y are independently distributed, their joint probability differential is given by the compound probability theorem as shown below:

$$F(x, y) = f(x) f(y) dxdy$$

$$f(x, y) = \int_0^\infty \frac{1}{\beta(\mu + 1, v)} x^{(\mu+1)-1} (1 - x)^{v-1} . \frac{\lambda^{\mu+v+1}}{\Gamma(\alpha + \beta + 2)} e^{-\lambda y} y^{(\mu+v+1)-1} \qquad (2.3.20)$$

Let us transform to the new variables U and Z by the transformation:

Put U=XY, Z=X

$$Y = \frac{U}{Z}, J = \frac{-1}{Z}$$

Hence the joint probability of random variables U and Z becomes:

$$f(U, Z) = \int_0^\infty \frac{1}{\beta(\mu + 1, v)} Z^{-V-1} (1 - Z)^{v-1} . \frac{\lambda^{\mu+v+1}}{\Gamma(\alpha + \beta + 1)} e^{-\lambda\left(\frac{U}{Z}\right)} \left(\frac{U}{Z}\right)^{(\mu+v+1)-1} \qquad (2.3.21)$$

Integrating w.r.to z in the range 0 < Z < 1, the marginal p.d.f of U is given as:

$$f_1(U) = \frac{\lambda^{\mu+v+1} U^{\mu+v}}{\Gamma(\mu + 1)\Gamma v} \int_0^1 \frac{(1 - z)^{v-1}}{z^{v+1}} e^{-\lambda\left(\frac{U}{Z}\right)} dZ$$

$$f_1(U) = \frac{\lambda^{\mu+v+1} U^{\mu+v}}{\Gamma(\mu + 1)\Gamma v} \int_0^1 t^{v-1} e^{-\lambda U(1+t)} dt$$

$$f_1(U) = \frac{\lambda^{\mu+v+1} U^{\mu+v} e^{-\lambda U}}{\Gamma(\mu + 1)\Gamma v} \int_0^1 t^{v-1} e^{-\lambda U t} dt$$

$$f_1(U) = \frac{\lambda^{\mu+v+1} U^{\mu+v} e^{-\lambda U}}{\Gamma(\mu + 1)\Gamma v} \frac{\Gamma v}{(\lambda U)^v}$$

$$f_1(U) = \frac{\lambda^{\mu+1}}{\Gamma(\mu + 1)} e^{-\lambda U} u^{\mu}, 0 < U < \infty \qquad (2.3.22)$$

Hence U=XY, is distributed as a size biased Gamma variate with parameters λ and $\mu+1, i, e, XY \sim SBG\ (\lambda, \mu+1)$.

2.5.7 Test for Size-biasedness of Size biased Gamma Distribution

Let X_1, X_2, X_3… Xn be random samples can be drawn from Gamma distribution or size-biased Gamma distribution. We test the hypothesis

$H_o : f(x) = f(x;\alpha,\beta)$ againest $H_1 : f(x) = f_s^*(x;\alpha,\beta+1)$

For testing whether the random sample of size n comes from the Gamma distribution or Size-biased Gamma distribution, then the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left(\frac{f_s^*(x;\alpha,\beta+1)}{f(x;\alpha,\beta)}\right)$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left(\frac{\dfrac{\alpha^{\beta+1}e^{-\alpha x}x^{\beta}}{\Gamma(\beta+1)}}{\dfrac{\alpha^{\beta}e^{-\alpha x}x^{\beta-1}}{\Gamma(\beta)}}\right)$$

$$\Delta = \left[\frac{\alpha}{\beta}\right]^n \prod_{i=1}^{n} x_i \qquad (2.3.23)$$

We reject the null hypothesis.

$$\left[\frac{\alpha}{\beta}\right]^n \prod_{i=1}^{n} x_i > k$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, where\, k^* = k\left[\frac{\alpha}{\beta}\right]^n > 0 \qquad (2.3.24)$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution.

2.6 Estimation of parameters

In this section, we discuss the various estimation methods for size biased Gamma Distribution and verifying their efficiencies.

2.7 Method of moments

In the method of moments replacing the population mean and variance by the corresponding sample mean and variance, we get the following estimates

$$\hat{\alpha} = \frac{\bar{x}}{s^2} \tag{2.5.1}$$

$$\hat{\beta} = \frac{\bar{x}^2}{s^2} - 1 \tag{2.5.2}$$

2. 6 Method of maximum likelihood estimation (MLE)

Let $x_1, x_2, x_3 \ldots\ldots. x_n$ be a random sample can be drawn from the size biased Gamma distribution, and then the corresponding likelihood function is given as

$$L(x; \alpha, \beta + 1) = \frac{\alpha^{n\beta+n}}{[\Gamma(\beta+1)]^n} e^{-\alpha \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x^{\beta} \tag{2.6.1}$$

The $\log$ likelihood of (2.6.1) can be written as

$$\log\ L = (n\beta + n) \log \alpha - n \log \Gamma(\beta + 1) - \alpha \sum_{i=1}^{n} x_i + \beta \sum_{i=1}^{n} \log x_i$$

The corresponding likelihood equations are given as

$$\frac{\partial \log L}{\partial \alpha} = \frac{n\beta + n}{\alpha} - \sum_{i=1}^{n} x_i = 0$$

$$\hat{\alpha} = \frac{\beta + 1}{\bar{x}} \tag{2.6.2}$$

(2.6.2)

$$\frac{\partial \log L}{\partial \beta} = n \log \alpha - n \frac{\Gamma'(\beta+1)}{\Gamma(\beta+1)} - \sum_{i=1}^{n} \log x_i = 0 \tag{2.6.3}$$

Solving the above equation gives the MLE estimate of β.

2.7 Bayesian method of estimation

Bayesian analysis is an important approach to statistics, which formally seeks use of prior information and Bayes' Theorem provides the formal basis for using this information. In this approach, parameters are treated as random variables and data remains fixed.

2.7.1 Parameter estimation under squared error loss function

In this section, two different prior distributions are used for estimating the parameter of the size biased Gamma distribution namely; Jeffery's prior and extension of Jeffrey's prior information.

2.7.1.1 Bayes' estimation of parameter of size biased Gamma distribution under Jeffrey's prior

Consider there are n recorded values, $\underline{x} = (x_1, \dots x_n)$ from (2.2.1). We consider the extended Jeffrey's prior as:

$$g(\theta) \propto \sqrt{[I\alpha]}$$

Where $[I(\alpha)] = -nE\left[\frac{\partial^2 \log f(x; \alpha, \beta+1)}{\partial \alpha^2}\right]$ is the Fisher's information matrix. For the model (2.2.1),

$$g(\alpha) = k\,\frac{\sqrt{n(\beta+1)}}{\alpha} \tag{2.7.1}$$

Then the joint probability density function is given by

$$f(\underline{x}, \alpha) = L(x; \alpha) g(\alpha)$$

$$f(\underline{x}, \alpha) = \frac{k\sqrt{n(\beta+1)}}{[\Gamma(\beta+1)]^n} e^{-\alpha \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\beta} \alpha^{n\beta+n-1}$$

And the corresponding marginal PDF of $\underline{x} = (x_1, \dots x_n)$ is obtained as:

$$p(\underline{x}) = \int_0^\infty f(\underline{x}, \alpha) d\alpha$$

$$p(\underline{x}) = \int_0^\infty \frac{k\sqrt{n(\beta+1)}}{(\Gamma(\beta+1)^n)} \prod_{i=1}^{n} x_i^{\beta} e^{-\alpha \sum_{i=1}^{n} x_i} \alpha^{n\beta+n-1}\, d\alpha$$

$$p(\underline{x}) = \frac{k\sqrt{n(\beta+1)}}{[\Gamma(\beta+1)]^n} \prod_{i=1}^{n} x_i^{\beta} \frac{\Gamma(n\beta+n)}{\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n}} \tag{2.7.2}$$

The posterior PDF of α has the following form $\pi_1(\alpha/\underline{x}) = \frac{f(x, \alpha)}{p(\underline{x})}$

$$\pi_1(\alpha/\underline{x}) = \frac{\frac{k\sqrt{n(\beta+1)}}{[\Gamma(\beta+1)]^n} e^{-\alpha\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\beta} \alpha^{n\beta+n-1}}{\frac{k\sqrt{n(\beta+1)}}{\Gamma(\beta+1)^n} \prod_{i=1}^{n} x_i^{\beta} \frac{\Gamma(n\beta+n)}{\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n}}}$$

$$\pi_1(\alpha/\underline{x}) = \frac{e^{-\alpha\sum_{i=1}^{n} x_i} \alpha^{n\beta+n-1} \left(\sum_{i=1}^{n} x_i\right)^{n\beta+n}}{\Gamma(n\beta+n)} \qquad (2.7.3)$$

By using a squared error loss function $l_1(\hat{\alpha},\alpha) = c(\hat{\alpha}-\alpha)^2$ for some constant c, the risk function is:

$$R(\hat{\alpha}) = \int_0^{\infty} c(\hat{\alpha}-\alpha)^2 \pi_1(\alpha/\underline{x}) d\alpha$$

$$R(\hat{\alpha}) = \int_0^{\infty} c(\hat{\alpha}-\alpha)^2 \frac{e^{-\alpha\sum_{i=1}^{n} x_i} \alpha^{n\beta+n-1} \left(\sum_{i=1}^{n} y_i\right)^{n\beta+n}}{\Gamma(n\beta+n)} d\alpha$$

$$R(\hat{\alpha}) = c\hat{\alpha}^2 + \frac{c(n\beta+n)(n\beta+n+1)}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2c\hat{\alpha}(n\beta+n)}{\sum_{i=1}^{n} x_i} \qquad (2.7.4)$$

Now $\frac{\partial R(\hat{\alpha})}{\partial \hat{\alpha}} = 0$, Then the Bayes' estimator is

$$\hat{\alpha}_1 = \frac{n(\beta+1)}{\sum_{i=1}^{n} x_i}$$

$$\hat{\alpha}_1 = \frac{\beta+1}{\bar{x}} \qquad (2.7.5)$$

2.7.1.2 Estimation of Survival function:

By using posterior probability density function, we can find the Survival function, such that

$$\hat{S}_1(\underline{x}) = \int_0^{\infty} e^{-\alpha x} \pi_1(\alpha/\underline{x}) d\alpha$$

$$\hat{S}_2(\underline{x}) = \int_0^\infty e^{-\alpha\left(x+\sum_{i=1}^n x_i\right)} \alpha^{(n\beta+n--1)} \frac{\left(\sum_{i=1}^n x_i\right)^{n\beta+n}}{\Gamma(n\beta+n)} d\alpha$$

$$\hat{S}_1(\underline{x}) = \left(\frac{\sum_{i=1}^n x_i}{x+\sum_{i-1}^n x_i}\right)^{n\beta+n} \tag{2.7.6}$$

2.7.1.3 Bayes' estimation of parameter of size biased Gamma distribution using extension of Jeffrey's prior.

We consider the extended Jeffrey's prior are given as:

$$g(\alpha) \propto [I(\alpha)]^{c_1}; c_1 \varepsilon R^+$$

Where $[I(\alpha)] = -nE\left[\frac{\partial^2 \log f(x;\alpha,\beta+1)}{\partial\alpha^2}\right]$ is the Fisher's information matrix. For the model (2.2.1),

$$g(\alpha) = k\left[\frac{n(\beta+1)}{\alpha^2}\right]^{c_1} \tag{2.7.7}$$

Then the joint probability density function is given by:

$$f(\underline{x},\alpha) = L(x;\alpha)g(\alpha)$$

$$f(\underline{x},\alpha) = \frac{k(n(\beta+1))^{c_1}}{[\Gamma(\beta+1)]^n} e^{-\alpha\sum_{i=1}^n x_i} \prod_{i=1}^n x_i^{\beta} \alpha^{n\beta+n-2c_1} \tag{2.7.8}$$

And the corresponding marginal PDF of $\underline{x} = (x_1,\ldots x_n)$ is obtained as:

$$p(\underline{x}) = \int_0^\infty f(\underline{x},\alpha) d\alpha$$

$$p(\underline{x}) = \int_0^\infty \frac{k(n(\beta+1))^{c_1}}{(\Gamma(\beta+1)^n)} \prod_{i=1}^n x_i^{\beta} e^{-\alpha\sum_{i=1}^n x_i} \alpha^{n\beta+n-2c_1+1-1} d\alpha$$

$$p(\underline{x}) = \frac{k(n(\beta+1))^{c_1}}{[\Gamma(\beta+1)]^n} \prod_{i=1}^n x_i^{\beta} \frac{\Gamma(n\beta+n-2c_1+1)}{\left(\sum_{i=1}^n x_i\right)^{n\beta+n-2c_1+1}} \tag{2.7.9}$$

The posterior PDF of α has the following form

$$\pi_1(\alpha/\underline{x}) = \frac{f(\underline{x,\alpha})}{p(\underline{x})}$$

$$\pi_2(\alpha/\underline{x}) = \frac{e^{-\alpha\sum_{i=1}^{n} x_i}\alpha^{n\beta+n-2c_1}\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n-2c_1+1}}{\Gamma(n\beta+n-2c_1+1)} \tag{2.7.10}$$

By using a squared error loss function $l_1(\hat{\alpha},\alpha) = c(\hat{\alpha}-\alpha)^2$ for some constant c, the risk function is:

$$R(\hat{\alpha}) = \int_0^\infty c(\hat{\alpha}-\alpha)^2 \pi_1(\alpha/\underline{x})d\alpha$$

$$R(\hat{\alpha}) = \int_0^\infty c(\hat{\alpha}-\alpha)^2 \frac{e^{-\alpha\sum_{i=1}^{n} x_i}\alpha^{n\beta+n-2c_1}\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n-2c_1+1}}{\Gamma(n\beta+n-2c_1+1)}d\alpha$$

$$R(\hat{\alpha}) = c\hat{\alpha}^2 + \frac{c(n\beta+n-2c_1+2)(n\beta+n-2c_1+1)}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2c\hat{\alpha}(n\beta+n-2c_1+1)}{\sum_{i=1}^{n} x_i} \tag{2.7.11}$$

Now $\frac{\partial R(\hat{\alpha})}{\partial\hat{\alpha}} = 0$, Then the Bayes' estimator is

$$\hat{\alpha}_2 = \frac{n(\beta+1)-2c_1+1}{\sum_{i=1}^{n} x_i} \tag{2.7.12}$$

The Bayes' estimator under a precautionary loss function is denoted by $\hat{\alpha}$, and is given by the following equation:

$\hat{\alpha}_P = E[\alpha^2]^{\frac{1}{2}}$ and the corresponding Bayes' estimator comes out to be:

$$\hat{\alpha}_2 = \frac{n(\beta+1)-2c_1+1}{\sum_{i=1}^{n} x_i}$$

The risk function under precautionary loss function is given by:

$$R(\hat{\alpha}) = c\hat{\alpha} + \frac{c(n\beta+n-2c_1+2)(n\beta+n-2c_1+1)}{\hat{\alpha}\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2c(n\beta+n-2c_1+1)}{\sum_{i=1}^{n} x_i} \tag{2.7.13}$$

Remark: Replacing c_1= 1/2 in (2.7.12), the same Bayes' estimator is obtained as in (2.7.5) corresponding to the Jeffrey's prior. By Replacing c_1= 3/2 in (2.7.12), the Bayes' estimator becomes the estimator under Hartigan's prior (Hartigan (1964)). By Replacing c_1= 0 in (2.7.12), thus we get uniform prior.

2.7.1.3 Estimation of Survival function: By using posterior probability density function, we can found the Survival function, such that

$$\hat{S}_2(\underline{x}) = \int_0^\infty e^{-\alpha x}\pi_1(\alpha/\underline{x})d\alpha$$

$$\hat{S}_2(\underline{x}) = \int_0^\infty e^{-\alpha\left(x+\sum_{i=1}^{n}x_i\right)}\alpha^{(n\beta+n-2c_1+1)-1}\frac{\left(\sum_{i=1}^{n}x_i\right)^{n\beta+n-2c_1+1}}{\Gamma(n\beta+n-2c_1+1)}d\alpha$$

$$\hat{S}_1(\underline{x}) = \left(\frac{\sum_{i=1}^{n}x_i}{x+\sum_{i-1}^{n}x_i}\right)^{n\beta+n-2c_1+1} \qquad (2.7.14)$$

2.7.2 Parameter estimation under a new loss function.

This section uses a new loss function introduced by Al-Bayyati (2002). Employing this loss function, we obtain Bayes' estimators under Jeffrey's and extension of Jeffrey's prior information.

Al-Bayyati introduced a new loss function of the form:

$$l_A(\hat{\alpha},\alpha) = \alpha^{c_2}(\hat{\alpha}-\alpha)^2; c_2 \varepsilon R. \qquad (2.7.15)$$

Here, this loss function is used to obtain the estimator of the parameter of the size biased Gamma distribution.

2.7.2.1 Bayes' estimation of parameter of size biased Gamma distribution under Jeffrey's prior.

By using the loss function in the form given in (2.7.15), we obtained the following risk function:

$$R(\hat{\alpha}) = \int_0^\infty \alpha^{c_2} (\hat{\alpha} - \alpha)^2 \pi_1(\alpha/\underline{x}) d\alpha$$

$$R(\hat{\alpha}) = \int_0^\infty \alpha^{c_2} (\hat{\alpha} - \alpha)^2 \frac{e^{-\alpha \sum_{i=1}^n x_i} \alpha^{n\beta+n-1} \left(\sum_{i=1}^n x_i\right)^{n\beta+n}}{\Gamma(n\beta+n)} d\alpha$$

$$R(\hat{\alpha}) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\beta+n1}}{\Gamma(n\beta+n)} \frac{1}{\left(\sum_{i=1}^n x_i\right)^{n\beta+n+c_2}} \left[\hat{\alpha}^2 \Gamma(n\beta+n+c_2) + \frac{\Gamma(n\beta+n+c_2+2)}{\left(\sum_{i=1}^n x_i\right)^2} - \frac{2\hat{\alpha}\Gamma(n\beta+n+c_2+1)}{\sum_{i=1}^n x_i} \right]$$

Now $\frac{\partial R(\hat{\alpha})}{\partial \hat{\alpha}} = 0$, Then the Bayes' estimator is

$$\hat{\alpha}_3 = \frac{n(\beta+1)+c_2}{\sum_{i=1}^n x_i} \tag{2.7.16}$$

The Bayes' estimator under a precautionary loss function is denoted by $\hat{\alpha}$, and is given by the following equation:

$\hat{\alpha}_P = E[\alpha^2]^{\frac{1}{2}}$ and the corresponding Bayes' estimator comes out to be:

$$\hat{\alpha}_3 = \frac{n(\beta+1)+c_2}{\sum_{i=1}^n x_i} \tag{2.7.17}$$

The risk function under precautionary loss function is given by:

$$R_p(\hat{\alpha}_p) = \frac{\left(\sum_{i=1}^n x_i\right)^{n\beta+n}}{\Gamma(n\beta+n)} \frac{1}{\left(\sum_{i=1}^n x_i\right)^{n\beta+n+c_2}} \left[\hat{\alpha}\Gamma(n\beta+n+c_2) + \frac{\Gamma(n\beta+n+c_2+2)}{\hat{\alpha}\left(\sum_{i=1}^n x_i\right)^2} - \frac{2\Gamma(n\beta+n+c_2+1)}{\sum_{i=1}^n x_i} \right] \tag{2.7.18}$$

Remark: Replacing $c_2 = 0$ in (2.7.17), the same Bayes' estimator is obtained as in (2.7.5) corresponding to the Jeffrey's prior. By Replacing c_2 =-2 in (2.7.17), the Bayes' estimator becomes the estimator under Hartigan's prior (Hartigan (1964)). By Replacing c_2 =1 in (2.7.17), thus we get uniform prior.

2.7.2.2 Bayes' estimation of parameter of size biased Gamma distribution using extension of Jeffrey's prior.

By using the loss function in the form given in (2.7.15), we obtained the following risk function:

$$R(\hat{\alpha}) = \int_0^\infty \alpha^{c_2} (\hat{\alpha} - \alpha)^2 \pi_2(\alpha/\underline{x}) d\alpha$$

$$R(\hat{\alpha}) = \int_0^\infty \alpha^{c_2} (\hat{\alpha} - \alpha)^2 \frac{e^{-\alpha \sum_{i=1}^{n} x_i} \alpha^{n\beta+n-2c_1} \left(\sum_{i=1}^{n} x_i\right)^{n\beta+n-2c_1+1}}{\Gamma(n\beta + n - 2c_1 + 1)} d\alpha$$

$$R(\hat{\alpha}) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n-2c_1+1}}{\Gamma(n\beta + n - 2c_1 + 1)} \frac{1}{\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n+c_2-2c_1+1}} \left[\begin{array}{l} \hat{\alpha}^2 \Gamma(n\beta + n + c_2 - 2c_1 + 1) + \dfrac{\Gamma(n\beta + n + c_2 - 2c_1 + 3)}{\left(\sum_{i=1}^{n} x_i\right)^2} \\ - \dfrac{2\hat{\alpha}\Gamma(n\beta + n + c_2 - 2c_1 + 2)}{\sum_{i=1}^{n} x_i} \end{array} \right] \quad (2.7.19)$$

Now $\frac{\partial R(\hat{\alpha})}{\partial \hat{\alpha}} = 0$, Then the Bayes' estimator is

$$\hat{\alpha}_4 = \frac{n(\beta + 1) + c_2 - 2c_1 + 1}{\sum_{i=1}^{n} x_i} \quad (2.7.20)$$

The Bayes' estimator under a precautionary loss function is denoted by $\hat{\alpha}$, and is given by the following equation: $\hat{\alpha}_P = E[\alpha^2]^{\frac{1}{2}}$ and the corresponding Bayes' estimator comes out to be: $\hat{\alpha}_4 = \frac{n(\beta + 1) + c_2 - 2c_1 + 1}{\sum_{i=1}^{n} x_i}$

The risk function under precautionary loss function is given by:

$$R(\hat{\alpha}) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n-2c_1+1}}{\Gamma(n\beta + n - 2c_1 + 1)} \frac{1}{\left(\sum_{i=1}^{n} x_i\right)^{n\beta+n+c_2-2c_1+1}} \left[\begin{array}{l} \hat{\alpha}\Gamma(n\beta + n + c_2 - 2c_1 + 1) + \dfrac{\Gamma(n\beta + n + c_2 - 2c_1 + 3)}{\hat{\alpha}\left(\sum_{i=1}^{n} x_i\right)^2} \\ - \dfrac{2\Gamma(n\beta + n + c_2 - 2c_1 + 2)}{\sum_{i=1}^{n} x_i} \end{array} \right] \quad (2.7.21)$$

Remark: Replacing c_1= 1/2 and c_2 = 0 in (2.7.20), the same Bayes' estimator is obtained as in (2.7.5) corresponding to the Jeffrey's prior. By Replacing c_1= 3/2 and c_2 =0 in (2.7.20), the Bayes estimator becomes the estimator under Hartigan's prior (Hartigan (1964)). By Replacing c_1= 0 and c_2 =0 in (2.7.20), thus we get uniform prior.

2.7.3 Simulation Study of Size biased Gamma Distribution

In our simulation study, we choose a sample size of n=25, 50 and 100 to represent small, medium and large data set. The scale parameter is estimated for Size biased Gamma Distribution with Maximum Likelihood and Bayesian using Jeffrey's & extension of Jeffrey's prior methods. For the scale parameter we have considered α = 1.5, .2.0 and 2.5.The values of Jeffrey's extension were c_1 = 0.5, 1.0, 1.5 and 2.0. The value for the loss parameter c_2 = -1, 0 and +1.This was iterated 5000 times and the scale parameter for each method was calculated. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes with different values for the Extension of Jeffrey's' prior and the loss functions. The results are presented in tables (2.1), (2.2) for different selections of the parameters and c extension of Jeffrey's prior.

Table 2.1: Structural properties of Size biased classical Gamma distribution

n	α	$\beta+1$	Mean	variance	S.D	C.V	Shannon's Entropy
25	1.0	1.5	0.515819	0.219917	0.468954	1.099935	1.639386
	1.5	2.0	2.647555	1.128777	1.062439	2.491958	1.998861
	2.0	2.5	3.073902	1.310549	1.144792	2.685118	1.905132
50	1.0	1.5	0.942166	0.401689	0.633790	1.486559	1.694657
	1.5	2.0	2.221208	0.947005	0.973142	2.282511	2.026569
	2.0	2.5	3.500249	1.492321	1.221606	2.865285	1.938364
100	1.0	1.5	1.368513	0.583461	0.763846	1.791607	1.744428
	1.5	2.0	1.794861	0.765233	0.874776	2.051794	2.052823
	2.0	2.5	3.926596	1.674093	1.293867	3.034775	1.969526

Table 2.2: Mean Squared Error for under $\hat{\alpha}$ Jeffrey's prior

n	α	$\beta+1$	α_{ML}	α_{sl}	α_{Nl}		
					C2=-1.0	C2=0	C2=1.0
25	1.0	1.5	0.694831	0.694831	0.598016	0.694831	0.544739
	1.5	2.0	1.616179	1.616179	1.349812	1.585304	1.436083
	2.0	2.5	2.485953	2.485953	2.597593	2.456618	2.389094
50	1.0	1.5	0.610410	0.610409	0.5293161	0.530716	0.513424
	1.5	2.0	1.311115	1.311115	1.185384	1.311115	1.209764
	2.0	2.5	2.359689	2.379689	2.282167	2.359689	2.331872
100	1.0	1.5	0.466204	0.466204	0.3765582	0.466204	0.378032
	1.5	2.0	1.204456	1.204456	1.138956	1.127385	1.136965
	2.0	2.5	2.274085	2.274065	2.144278	2.274065	2.284561

Table 2.3: Mean Squared Error for ($\widehat{\hat{\alpha}}$) under extension of Jeffrey's prior

n	α	$\beta+1$	C1	α_{ML}	α_{sl}	α_{Nl}		
						C2=-1.0	C2=0	C2=1.0
25	1.0	1.5	0.5	1.751815	0.605711	0.491789	0.605711	0.675668
			1.0	1.751815	0.678151	0.785964	0.678151	0.715066
			1.5	1.751815	0.756965	0.796710	0.707501	0.761006
			2.0	1.751815	1.743262	1.551585	1.743262	1.873616
	2.0	2.5	0.5	1.465009	1.465009	1.530603	1.478005	1.525623
			1.0	1.465009	1.664499	1.393877	1.154286	1.600484
			1.5	1.465009	1.579438	1.178935	1.336005	0.743068
			2.0	1.465009	1.685124	1.469333	1.685124	1.753295
50	1.0	1.5	0.5	0.523688	0.523688	0.442794	0.523688	0.621982
			1.0	0.523688	0.581831	0.591908	0.581831	0.553100
			1.5	0.523688	0.580325	0.357746	0.441632	0.602326
			2.0	0.523688	0.718147	0.737926	0.471831	0.421187
	2.0	2.5	0.5	1.170480	1.170480	1.203897	1.132317	1.095543
			1.0	1.170480	1.275611	1.309292	1.181767	0.993167
			1.5	1.170480	1.421131	1.148665	1.221135	0.904535
			2.0	1.170480	1.656525	1.365212	1.656525	1.532779
100	1.0	1.5	0.5	0.494552	0.494552	0.357457	0.494552	0.560155
			1.0	0.494552	0.522866	0.532341	0.574154	0.410739
			1.5	0.494552	0.523018	0.332578	0.405039	0.480748
			2.0	0.494552	0.499482	0.459488	0.469988	0.344415
	2.0	2.5	0.5	1.069772	1.069772	1.192434	1.069772	0.981817
			1.0	1.069772	1.195533	1.222817	1.164875	0.879562
			1.5	1.069772	1.129912	1.146871	0.8618621	1.321517
			2.0	1.069772	1.346683	1.317127	1.346683	1.250574

ML= Maximum Likelihood, SL=Squared Error Loss Function, NL= New Loss Function,

In table 2.2, Bayes' estimation with New Loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is ±1. Similarly, in table

2.3, Bayes' estimation with New Loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is ± 1 whether the extension of Jeffrey's prior is 0.5, 1.0, 1.5 or 2.0.

2.8 Beta distribution of first kind

A continuous random variable X is said to be beta distribution of first kind with parameters a and b and its probability density function (pdf) is given by:

$$f(x;a,b) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \qquad (2.8.1)$$

$$for\, 0 < x < 1, a > 0\, and\ b > 0, where$$

$\beta(a,b) = \int_o^1 t^{a-1}(1-t)^{b-1}dt$ denotes the beta function.

Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions. Many generalizations of beta distributions involving algebraic and exponential functions have been proposed in the literature; see in Johnson et al. (2004) and Gupta and NadarSajah (2004) for detailed accounts.

2.9 Size-Biased Beta Distribution of first kind

A size biased beta distribution of first kind (SBBD1) is obtained by applying the weights x^c, where c =1 to the weighted beta distribution of first kind.

$$\mu_1' = \int_0^1 x\, f(x;a,b)dx = \frac{a}{a+b}$$

$$\int_0^1 x \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}dx = \frac{a}{a+b}$$

$$\int_0^1 \frac{1}{\beta(a,b)} \frac{a+b}{a} x^{a}(1-x)^{b-1}dx = 1$$

$$f(x;a+1,b) = \frac{1}{\beta(a+1,b)} x^{a}(1-x)^{b-1}$$

where $f(x;a,b+1)$ represents a probability density function. This gives the size –biased beta distribution of first kind (SBBD1) as:

$$f(x;a+1,b)=\frac{1}{\beta(a+1,b)}x^{a}(1-x)^{b-1};a\geq 0,b>0 \qquad (2.9.1)$$

$$=0;otherwise \qquad 0<x<1$$

Where

$$\beta(a,b)=\int_{o}^{1}t^{a-1}(1-t)^{b-1}dt$$

2.10 Structural properties of size biased beta distribution of first kind

In this section, we derive some important properties of Size-biased beta distribution of first kind.

2.10.1 Moments of Size biased beta distribution of first kind

The rth moment of Size biased beta distribution of first kind (2.9.1) about origin is obtained as:

$$\mu_r' =\int_0^1 x^r f(x;a+1,b)dx$$

$$\mu_r' =\int_0^1 x^r \frac{1}{\beta(a+1,b)}x^{a}(1-x)^{b-1}dx$$

On solving the above equation, we get

$$\mu_r' =\int_0^1 \frac{1}{\beta(a+1,b)}x^{a+r}(1-x)^{b-1}dx$$

$$\mu_r' =\frac{1}{\beta(a+1,b)}\beta(a+r+1,b) \qquad (2.10.1)$$

Using the equation (2.10.1), the mean of the SBBD1 is given by

$$\mu_1' =\frac{1}{\beta(a+1,b)}\beta(a+2,b)$$

$$\mu_1' =\frac{\Gamma a+b+1}{\Gamma a+1\ \Gamma b}\ \frac{\Gamma a+2\ \Gamma b}{\Gamma a+b+2}$$

$$\mu_1' =\frac{a+1}{a+b+1} \qquad (2.10.2)$$

Using the equation (2.10.1), the second moments of the SBBD1 is given by

$$\mu_2' = \frac{1}{\beta(a+1,b)}\beta(a+3,b)$$

$$\mu_2' = \frac{\Gamma a+b+1}{\Gamma a+1\,\Gamma b}\,\frac{\Gamma a+3\ \Gamma b}{\Gamma a+b+3} \qquad (2.10.3)$$

Using the equations (2.10.2) and (2.10.3) the variance of the SBBD1 is given by

$$\mu_2 = \frac{(a+1)b}{(a+b+1)^2(a+b+2)} \qquad (2.10.4)$$

Using the equation (2.10.1), the third and fourth moments of the SBBD1 are given by

$$\mu_3' = \frac{(a+1)(a+2)(a+3)}{(a+b+1)(a+b+2)(a+b+3)} \qquad (2.10.5)$$

$$\mu_4' = \frac{(a+1)(a+2)(a+3)(a+4)}{(a+b+1)(a+b+2)(a+b+3)(a+b+4)} \qquad (2.10.6)$$

The coefficient of variation of Size biased beta distribution is given as:

$$CV = \left[\frac{b}{(a+1)(a+b+2)}\right]^{\frac{1}{2}} \qquad (2.10.7)$$

2.10.2 Harmonic mean of Size biased Beta distribution of first kind

The harmonic mean (H) is given as:

$$\frac{1}{H} = \int_0^1 \frac{1}{x}\ f(x;a+1,b)dx$$

$$\frac{1}{H} = \int_0^1 \frac{1}{x}\ \frac{1}{\beta(a+1,b)}x^a(1-x)^{b-1}dx$$

$$\frac{1}{H} = \int_0^1 \frac{1}{\beta(a+1,b)}x^{a-1}(1-x)^{b-1}dx$$

$$\frac{1}{H} = \frac{1}{\beta(a+1,b)}\beta(a,b)$$

$$\frac{1}{H} = \frac{\Gamma a+b+1}{\Gamma a+1\,\Gamma b}\,\frac{\Gamma a\ \Gamma b}{\Gamma a+b}$$

$$H = \frac{a}{a+b} \qquad (2.10.8)$$

2.10.3 Mode of size biased beta distribution of first kind

The probability distribution of Size- biased Beta distribution of first kind is:

$$f(x;a+1,b)=\frac{1}{\beta(a+1,b)}x^{a}(1-x)^{b-1};a\geq 0,b>0$$
$$=0;otherwise \qquad 0<x<1$$

In order to discuss monotonicity of size biased beta distribution of first kind. We take the logarithm of its pdf:

$$\ln(f(x:a+1,b))=\ln C+a\ln x+(b-1)\ln(1-x)$$

Where C is a constant. Note that

$$\frac{\partial\ln f(x;a+1,b)}{\partial x}=\frac{a}{x}-\frac{b-1}{1-x}$$

Where $x>0,a\geq 0,b>0.$ It follows that

$$\frac{\partial\ln f(x;a+1,b)}{\partial x}>0\Leftrightarrow x<\frac{a}{a+b-1}$$

$$\frac{\partial\ln f(x;a+1,b)}{\partial x}=0\Leftrightarrow x=\frac{a}{a+b-1}$$

$$\frac{\partial\ln f(x;a+1,b)}{\partial x}<0\Leftrightarrow x>\frac{a}{a+b-1}$$

Therefore, the mode of size biased beta distribution of first kind is given as:

$$x_0=\frac{a}{a+b-1} \qquad (2.10.9)$$

2.10.4 Simulation Study of Size biased Beta distribution of first kind

In our simulation study, we choose a sample size of n=25, 50 and 100 to represent small, medium and large data set. For the scale parameter we have considered θ=0.5, 1.0 and 1.5. This was iterated 2000 times and the structural properties were calculated. A simulation study was conducted R-software to obtain the structural properties – mean, median, mode, variance, standard deviation and coefficient of variation. The results are presented in table (2.4) for different values of the parameters and having different sample sizes.

Table 2.4: Structural properties of Size biased Beta distribution of first kind

n	a+1	b	Mean	variance	Mode	H.M	S.D	C.V
25	1.5	2.0	0.4285714	0.05442177	0.3243333	0.2012413	0.2332847	0.5443310
	2.0	2.5	0.4323456	0.03756533	0.3276546	0.2434578	0.1938178	0.4482937
	2.5	3.0	0.4024565	0.04563651	0.3367899	0.2356788	0.2136270	0.5308077
50	1.5	2.0	0.4356746	0.06245671	0.3567888	0.2144566	0.2499134	0.5736240
	2.0	2.5	0.4245678	0.03934564	0.3123556	0.1956677	0.1983574	0.4671984
	2.5	3.0	0.3905663	0.03456367	0.3346778	0.2567890	0.1859131	0.4760091
100	1.5	2.0	0.4367899	0.06345889	0.3435678	0.1899909	0.2519105	0.5767315
	2.0	2.5	0.4599456	0.03245677	0.3134566	0.2456678	0.1801576	0.3916933
	2.5	3.0	0.4435632	0.04577788	0.3534677	0.2096778	0.2139577	0.4823613

2.11. Estimation of parameters of Size-biased Beta distribution of first kind

In this method of moments replacing the population mean and variance by the corresponding sample mean and variance, we have:

$$\mu_1' = \bar{x}$$

$$\frac{a+1}{a+b+1} = \bar{x}$$

$$\hat{b} = \frac{-(a+1)(\bar{x}-1)}{\bar{x}} \tag{2.11.1}$$

Also, $\mu_2 = S^2$

$$\frac{(a+1)b}{(a+b+1)^2(a+b+2)} = S^2$$

$$\hat{a} = \frac{\bar{x}^2(1-\bar{x}) - S^2(1+\bar{x})}{S^2} \tag{2.11.2}$$

Substitute the value of $\hat{a}$ in the above equation; we can get the estimated value of parameter b.

$$\hat{b} = \frac{(\bar{x}^2 + S^2\bar{x}^2)(\bar{x}-1)}{\bar{x}S^2} \tag{2.11.3}$$

2.12 Test for Size-biased beta distribution of first kind.

Let X_1, X_2, X_3… Xn be random samples can be drawn from beta distribution of first kind or size-biased beta distribution of first kind. We test the hypothesis $H_o : f(x) = f(x,a,b) \text{ vs } H_1 : f(x) = f_s^*(x;a+1,b)$

For test whether the random sample of size n comes from the beta distribution of first kind or Size-biased beta distribution of first kind the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \left(\frac{f_s^*(x;a+1,b)}{f(x;a,b)} \right)$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \left(\frac{\frac{\Gamma(a+b+1)x^a}{\Gamma(a+1)}}{\frac{\Gamma(a+b)x^{a-1}}{\Gamma(a)}} \right)$$

$$\Delta = \left[\frac{\Gamma(a)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(a+b)} \right]^n \prod_{i=1}^{n} x_i \qquad (2.12.1)$$

We reject the null hypothesis

$$\left[\frac{\Gamma(a)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(a+b)} \right]^n \prod_{i=1}^{n} x_i > k$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, \, where\, k^* = k \left[\frac{\Gamma(a+1)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b)} \right]^n > 0 \qquad (2.12.2)$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution.

2.13 Exponential distribution

The exponential distribution occupies an important position in the analysis of data. In probability theory and statistics, the exponential distribution is a family of continuous probability distribution. Historically, the exponential distribution was the first lifetime model for which statistical methods were extensively developed. It describes the time between events in the Poisson process i.e., a process in which events occur continuously and independently at a constant rate. The exponential distribution occurs naturally when describing the lengths of the inter arrival times in a homogeneous Poisson process. Exponential variables can also be used to model situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand, or between road kills on a given road. Reliability theory and reliability engineering also make extensive use of the exponential distribution. Because of the memory less property of this distribution, it is well-suited to model the constant hazard rate portion of the bathtub curve used in reliability theory. Failure rate is the frequency with which an engineered system or component fails, expressed for example in failures per hour. It is important in reliability engineering. By calculating the failure rate for smaller and smaller intervals of time, the interval becomes infinitely small. Work by Sukhatmi (1937), Epstein and Sobel (1955) and Epstein (1954), Bartholomew (1957), gave numerous results and popularized the exponential Distribution as a lifetime distribution, especially in the area of industrial life testing. Many authors have contributed to the statistical methodology of the distribution. The lengthy bibliographies of Mendenhall (1958), Govindara julu (1964), Johnson and Kotz (1970), Johnson, Kotz and Balakrishnan (1994) and Lawless (2003), Ahmad (2006), Ahmed et. al. (2007 & 2010), contains a large number of papers in this area.

A random variable X has an exponential distribution with parameter $\theta(\theta > 0)$ if its probability density function is of the form

$$f(x) = \theta \exp(-\theta x), x \geq 0; \theta > 0 \qquad (2.13.1)$$

with mean $\frac{1}{\theta}$ and variance $\frac{1}{\theta^2}$ respectively.

2.14 Size biased exponential distribution

A size biased exponential distribution (SBED) is obtained by applying the weights x^c, where c =1 to the weighted exponential distribution.

We have from relation (2.13.1), we have

$$\int_0^\infty x\, f(x;\theta)dx = \frac{1}{\theta}$$

$$\int_0^\infty x\theta e^{-\theta x}dx = \frac{1}{\theta}$$

Where $f(x;\theta)$ represents a probability density function. This gives the Size biased exponential distribution as:

$$f(x;\theta) = \theta^2 x e^{-\theta x}; \theta > 0 \qquad (2.14.1)$$

2.15 Structural Properties of the Size-biased exponential Distribution

In this section, we derive some properties of Size-biased exponential distribution.

2.15.1 Method of Moments

The rth moment of Size biased exponential distribution (2.14.1) about origin is obtained as

$$\mu_r' = E(x)^r$$

$$= \int_0^\infty x^r \; f(x;\theta) \; dx$$

$$= \int_0^\infty x^{r+1} \theta^2 e^{-\theta x} \; dx$$

$$\mu_r' = \theta^2 \frac{\Gamma r + 2}{\theta^{r+2}} \qquad (2.15.1)$$

Using above relation, the mean and variance of the SBEPD are given as

$$\mu_1' = \frac{2}{\theta} \qquad (2.15.2)$$

$$\mu_2 = \frac{2}{\theta^2} \qquad (2.15.3)$$

2.16 Estimation of parameters

In this section, we discuss the various estimation methods for size biased exponential distribution and verifying their efficiencies.

2.16.1 Methods of Moments

In the method of moments replacing the population mean and variance by the corresponding sample mean and variance, we have

$\mu_1' = \bar{x}$

Then $\hat{\theta} = \frac{2}{\bar{x}}$ (2.16.1)

2.16.2 Maximum likelihood estimation

Let $(x_1, x_2, \ldots, x_n)$ be a random sample of size n having the probability density function as

$$f(x;\theta) = \theta^2 \, x e^{-\theta x} \qquad for \;\; x \geq 0, \theta > 0$$

Then, the likelihood function is given by

$$L(x \mid \theta) = \theta^{2n} \prod_{i=1}^{n} x_i e^{-\theta \sum_{i=1}^{n} x_i}$$

$$\therefore Log \; L(x \mid \theta) = 2n \log \theta + \sum_{i=1}^{n} \log x_i - \theta \sum_{i=1}^{n} x_i$$

The ML estimator of θ is obtained by solving the

$$\frac{\partial}{\partial \theta} \log L(x \mid \theta) = 0$$

$$\Rightarrow \qquad \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\hat{\theta} = \frac{2}{\bar{x}} \tag{2.16.2}$$

2.16.3 Parameter estimation under squared error loss function

In this section, two different prior distributions are used for estimating the parameter of the size biased exponential distribution namely; Jeffery's prior and extension of Jeffrey's prior information.

2.16.3.1 Bayes' estimation of parameter of size biased exponential distribution under Jeffrey's prior

Consider there are n recorded values, $\underline{x} = (x_1, \ldots x_n)$ from (2.14.1). We consider the extended Jeffrey's prior as: $g(\theta) \propto \sqrt{[I(\theta)]}$

Where $I(\theta) = -nE\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right]$

$$\Rightarrow I(\theta) = -nE\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right] = \frac{2n}{\theta^2}$$

Therefore, $g(\theta) \propto \frac{\sqrt{2n}}{\theta}$ (2.16.3.1)

Then the joint probability density function is given by:

$$f(\underline{x},\theta) \propto L(x;\theta)g(\theta)$$

$$f(\underline{x},\theta) \propto \theta^{2n-1}\prod_{i=1}^{n} x_i\, e^{\left(-\theta\sum_{i=1}^{n} x_i\right)} \quad (2.16.3.2)$$

And the corresponding marginal PDF of $\underline{x} = (x_1,\dots x_n)$ is obtained as:

$$p(\underline{x}) = k\prod_{i=1}^{n} x_i \int_0^\infty \theta^{2n-1} e^{-\theta\sum_{i=1}^{n} x_i} d\theta$$

$$p(\underline{x}) = k\frac{(2n-1)!}{\left(\sum_{i=1}^{n} x_i\right)^{2n}}\prod_{i=1}^{n} x_i \quad (2.16.3.3)$$

The posterior PDF of θ has the following form

$$\pi_1(\theta/\underline{x}) = \frac{\theta^{2n-1}e^{\left(-\theta\sum_{i=1}^{n} x_i\right)}}{(2n-1)!}\left(\sum_{i=1}^{n} x_i\right)^{2n} \quad (2.16.3.4)$$

By using a squared error loss function $L(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2$ for some constant c, the risk function is:

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi_1(\theta/\underline{x})d\theta$$

$$R(\hat{\theta}) = \prod_{i=1}^{n} x_i\left[c\hat{\theta}^2 + \frac{2n(2n-1)c}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{4cn\hat{\theta}}{\sum_{i=1}^{n} x_i}\right] \quad (2.16.3.5)$$

Now $\frac{\partial R(\hat{\theta})}{\partial\hat{\theta}} = 0$, Then the Bayes' estimator is

$$\hat{\theta}_{B1} = \frac{2n}{\sum_{i=1}^{n} x_i}$$

$$\hat{\theta}_{B1} = \frac{2}{\bar{x}} \tag{2.16.3.6}$$

This coincides with maximum likelihood estimator.

2.16.3.2 Estimator of survival function

By using conditional density function, we can found the survival function such that

$$\hat{S}_{B1}(x) = \int_0^\infty \exp(-\theta x) p(\theta \mid x_1, \ldots, x_n) d\theta$$

$$\hat{S}_{B1}(x) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n}}{(2n-1)!} \int_0^\infty e^{(-\theta x)} \theta^{2n-1} e^{\left(-\theta \sum_{i=1}^{n} x_i\right)} d\theta$$

$$\hat{S}_{B1}(x) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n}}{(2n-1)!} \int_0^\infty e^{\left[-\theta\left(\sum_{i=1}^{n} x_i + x\right)\right]} \theta^{2n-1} d\theta$$

$$\hat{S}_{B1}(x) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n}}{(2n-1)!} \frac{\Gamma 2n}{\left(\sum_{i=1}^{n} x_i + x\right)^{2n}}$$

$$\hat{S}_{B1}(x) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n}}{\left(\sum_{i=1}^{n} x_i + x\right)^{2n}}$$

$$\hat{S}_{B1}(x) = \left(1 + \frac{\sum_{i=1}^{n} x_i}{x}\right)^{-2n}$$

$$\hat{S}_{B1}(x)=\left(1+\frac{n\bar{x}}{x}\right)^{-2n} \tag{2.16.3.7}$$

2.16.3.3 Bayes' estimation of parameter of size biased exponential distribution using extension of Jeffrey's prior

We consider the extended Jeffrey's prior are given as:

$$g(\theta)\propto\left[\frac{2n}{\theta^2}\right]^{c_1}$$

Then the joint probability density function is given by:

$$f(\underline{x},\theta)\propto L(x;\theta)g(\theta)$$

$$f(\underline{x},\theta)\propto\theta^{2n-2c_1}e^{\left(-\theta\sum_{i=1}^{n}x_i\right)}\prod_{i=1}^{n}x_i \tag{2.16.3.8}$$

And the corresponding marginal PDF of $\underline{x}=(x_1,\ldots x_n)$ is obtained as:

$$p(\underline{x})=\ k(2n)^{c_1}\prod_{i=1}^{n}x_i\int_0^{\infty}\ \theta^{2n-2c_1}e^{\left(-\theta\sum_{i=1}^{n}x_i\right)}d\theta$$

$$p(\underline{x})=k(2n)^{2c_1}\frac{\Gamma(2n-2c_1+1)}{\left(\sum_{i=1}^{n}x_i\right)^{2n-2c_1+1}}$$

$$p(\underline{x})=k(2n)^{2c_1}\frac{(2n-2c_1)!}{\left(\sum_{i=1}^{n}x_i\right)^{2n-2c_1+1}}\prod_{i=1}^{n}x_i \tag{2.16.3.9}$$

The posterior PDF of θ has the following form

$$\pi_2(\theta/\underline{x})=\frac{\theta^{2n-2c_1}e^{\left(-\theta\sum_{i=1}^{n}x_i\right)}}{(2n-2c_1)!}\left(\sum_{i=1}^{n}x_i\right)^{2n-2c_1+1} \tag{2.16.3.10}$$

By using a squared error loss function $L(\hat{\theta},\theta)=c(\hat{\theta}-\theta)^2$ for some constant c, the risk function is:

$$R(\hat{\theta})=\int_0^{\infty}c(\hat{\theta}-\theta)^2\pi_1(\theta/\underline{x})d\theta$$

$$R\left(\hat{\theta}-\theta\right)=\int_0^\infty c\left(\hat{\theta}-\theta\right)^2 \pi_1(\theta/\underline{x})d\theta$$

$$R\left(\hat{\theta}-\theta\right)=\int_0^\infty c\left(\hat{\theta}^2+\theta^2-2\theta\hat{\theta}\right)\pi_1(\theta/\underline{x})d\theta$$

$$R\left(\hat{\theta}-\theta\right)=\int_0^\infty c\left(\hat{\theta}^2+\theta^2-2\theta\hat{\theta}\right)\frac{\left(\sum_{i=1}^n x_i\right)^{2n-2c_1+1}\theta^{2n-2c_1}}{(2n-2c_1)!}e^{\left(-\theta\sum_{i=1}^n x_i\right)}d\theta$$

$$R\left(\hat{\theta}-\theta\right)=c\hat{\theta}^2+\frac{(2n-2c_1+2)(2n-2c_1+1)c}{\left(\sum_{i=1}^n x_i\right)^2}-\frac{2c\hat{\theta}(2n-2c_1+1)}{\sum_{i=1}^n x_i} \qquad (2.16.3.11)$$

Now $\frac{\partial R\left(\hat{\theta}\right)}{\partial\hat{\theta}}=0$, Then the Bayes' estimator is

$$2c\hat{\theta}-\frac{2c(2n-2c_1+1)}{\sum_{i=1}^n x_i}=0$$

$$\hat{\theta}_{B2}=\frac{(2n-2c_1+1)}{\sum_{i=1}^n x_i} \qquad (2.16.3.12)$$

The Bayes' estimator under a precautionary loss function is denoted by $\hat{\theta}$, and is given by the following equation:

$\hat{\theta}_P=E\left[\theta^2\right]^{\frac{1}{2}}$ and the corresponding Bayes' estimator comes out to be $\hat{\theta}_{B2}=\frac{(2n-2c_1+1)}{\sum_{i=1}^n x_i}$

The risk function under precautionary loss function is given by:

$$R_P\left(\hat{\theta}_P\right)=c\hat{\theta}+\frac{(2n-2c_1+2)(2n-2c_1+1)c}{\hat{\theta}\left(\sum_{i=1}^n x_i\right)^2}-\frac{2c(2n-2c_1+1)}{\sum_{i=1}^n x_i} \qquad (2.16.3.13)$$

Remarks1: If $c_1 = \frac{1}{2}$, we get, the Jeffrey's prior and the corresponding Bayes' estimator is $\hat{\theta}_{B2} = \frac{2n}{\sum_{i=1}^{n} x_i} = \frac{2}{\bar{x}}$, If $c_1 = \frac{3}{2}$, we get, the Hartigan prior [Hartigan [1964]] and the corresponding Bayes' estimator becomes: $\hat{\theta}_{B2} = \frac{(2n-2)}{\sum_{i=1}^{n} x_i}$. If $c_1 = 0$, we get, the uniform prior and the corresponding Bayes' estimator becomes: $\hat{\theta}_{B2} = \frac{(2n+1)}{\sum_{i=1}^{n} x_i}$

2.16.3.4 Estimator using new extension of survival function

$$\hat{S}_{B2}(x) = \int_0^\infty e^{(-\theta x)} p(\theta \mid x_1, \ldots, x_n) d\theta$$

$$\hat{S}_{B2}(x) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n-2c_1+1}}{(2n-2c_1)!} \int_0^\infty e^{(-\theta x)} \theta^{(2n-2c_1+1)-1} e^{\left(-\theta \sum_{i=1}^{n} x_i\right)} d\theta$$

$$\hat{S}_{B2}(x) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n-2c_1+1}}{(2n-2c_1)!} \int_0^\infty e^{\left[-\theta\left(\sum_{i=1}^{n} x_i + x\right)\right]} \theta^{(2n-2c_1+1)-1} d\theta$$

$$\hat{S}_{B2}(x) = \left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i + x}\right)^{2n-2c_1+1}$$

$$\hat{S}_{B2}(y) = \left(1 + \frac{x}{n\bar{x}}\right)^{-(2n-2c_1+1)} \qquad (2.16.3.14)$$

2.16.4 Parameter estimation under a new loss function.

We use a new loss function introduced by Al-Bayyati (2002). Employing this loss function, we obtain Bayes' estimators using Jeffrey's and extension of Jeffrey's prior information.

Al-Bayyati introduced a new loss function of the form:

$$l_A(\hat{\theta}, \theta) = \theta^{c_2} (\hat{\theta} - \theta)^2; c_2 \ \varepsilon \ R. \qquad (2.16.4.1)$$

Here, this loss function is used to obtain the estimator of the parameter of the size biased exponential distribution.

2.16.4.1 Bayes' estimation of parameter of size biased exponential distribution under Jeffrey's prior.

By using the loss function in the form given in (2.16.4.1), we obtained the following risk function:

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2} \left(\hat{\theta} - \theta\right)^2 \pi_1(\theta / \underline{x}) d\theta$$

$$R(\hat{\theta} - \theta) = \int_0^\infty \theta^{c_2} \left(\hat{\theta}^2 + \theta^2 - 2\theta\hat{\theta}\right) \frac{\left(\sum_{i=1}^n x_i\right)^{2n} \theta^{2n-1}}{(2n-1)!} \prod_{i=1}^n x_i e^{\left(-\theta \sum_{i=1}^n x_i\right)} d\theta$$

$$R(\hat{\theta} - \theta) = \frac{\left(\sum_{i=1}^n x_i\right)^{2n}}{(2n-1)!} \prod_{i=1}^n x_i \left\{ \hat{\theta}^2 \int_0^\infty e^{-\theta \sum_{i=1}^n x_i} \theta^{2n+c_2-1} d\theta + \int_0^\infty e^{-\theta \sum_{i=1}^n x_i} \theta^{2n+c_2+2-1} d\theta - 2\hat{\theta} \int_0^\infty e^{-\theta \sum_{i=1}^n x_i} \theta^{2n+c_2+1-1} d\theta \right\}$$

$$R(\hat{\theta} - \theta) = \frac{\left(\sum_{i=1}^n x_i\right)^{2n}}{(2n-1)!} \prod_{i=1}^n x_i \left\{ \hat{\theta}^2 \frac{\Gamma(2n+c_2)}{\left(\sum_{i=1}^n x_i\right)^{2n+c_2}} + \frac{\Gamma(2n+c_2+2)}{\left(\sum_{i=1}^n x_i\right)^{2n+c_2+2}} - 2\hat{\theta} \frac{\Gamma(2n+c_2+1)}{\left(\sum_{i=\1}^n x_i\right)^{2n+c_2+1}} \right\}$$

$$R(\hat{\theta} - \theta) = \frac{\left(\sum_{i=1}^n x_i\right)^{2n}}{(2n-1)!} \prod_{i=1}^n x_i \frac{1}{\left(\sum_{i=1}^n x_i\right)^{2n+c_2}} \left\{ \hat{\theta}^2 \left(2n + c_2 - 1\right)! + \frac{(2n+c_2+1)!}{\left(\sum_{i=1}^n x_i\right)^2} - 2\hat{\theta} \frac{(2n+c_2)!}{\left(\sum_{i=\1}^n x_i\right)} \right\} \quad (2.16.4.2)$$

Now $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, Then the Bayes' estimator is

$$2\hat{\theta}(2n+c_2-1)! - \frac{2(2n+c_2)!}{\sum_{i=1}^n x_i} = 0$$

$$\hat{\theta}_{B3} = \frac{(2n+c_2)}{\sum_{i=1}^n x_i} \qquad (2.16.4.3)$$

Remarks 2: If $c_2 = 0$, we get, the Jeffrey's prior and the corresponding Bayes' estimator is $\hat{\theta}_{B3} = \frac{2}{\bar{x}}$.If $c_2 = -2$, we get, the Hartigan prior [Hartigan [(1964)] and the corresponding Bayes' estimator becomes: $\hat{\theta}_{B3} = \frac{(2n-2)}{\sum_{i=1}^{n} x_i}$.If $c_2 = 1$, we get, the uniform prior and the corresponding Bayes' estimator becomes $\hat{\theta}_{B3} = \frac{(2n+1)}{\sum_{i=1}^{n} x_i}$

2.16.4.2 Bayes' estimation of parameter of size biased exponential distribution using extension of Jeffrey's prior.

By using the loss function in the form given in (2.16.4.1), we obtained the following risk function:

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2} \left(\hat{\theta} - \theta\right)^2 \pi_1(\theta/\underline{x}) d\theta$$

$$R(\hat{\theta} - \theta) = \int_0^\infty \theta^{c_2} \left(\hat{\theta}^2 + \theta^2 - 2\theta\hat{\theta}\right) \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n-2c_1+1} \theta^{2n-2c_1}}{(2n-2c_1)!} \prod_{i=1}^{n} x_i e^{\left(-\theta \sum_{i=1}^{n} x_i\right)} d\theta$$

$$R(\hat{\theta} - \theta) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n-2c_1+1}}{(2n-2c_1)!} \prod_{i=1}^{n} x_i \left\{ \hat{\theta}^2 \int_0^\infty e^{-\theta \sum_{i=1}^{n} x_i} \theta^{(2n+c_2-2c_1+1)-1} d\theta + \int_0^\infty e^{-\theta \sum_{i=1}^{n} x_i} \theta^{(2n+c_2-2c_1+3)-1} d\theta - 2\hat{\theta} \int_0^\infty e^{-\theta \sum_{i=1}^{n} x_i} \theta^{2n+c_2-2c_1-1} d\theta \right\}$$

$$R(\hat{\theta} - \theta) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n-2c_1+1}}{(2n-2c_1)!} \prod_{i=1}^{n} x_i \left\{ \hat{\theta}^2 \frac{\Gamma(2n+c_2-2c_1+1)}{\left(\sum_{i=1}^{n} x_i\right)^{2n+c_2-2c_1+1}} + \frac{\Gamma(2n+c_2-2c_1+3)}{\left(\sum_{i=1}^{n} x_i\right)^{2n+c_2-2c_1+3}} - 2\hat{\theta} \frac{\Gamma(2n+c_2-2c_1+2)}{\left(\sum_{i=\backslash 1}^{n} x_i\right)^{2n+c_2-2c_1+2}} \right\}$$

$$R(\hat{\theta} - \theta) = \frac{\left(\sum_{i=1}^{n} x_i\right)^{2n-2c_1+1}}{(2n-2c_1)!} \prod_{i=1}^{n} x_i \frac{1}{\left(\sum_{i=1}^{n} x_i\right)^{2n+c_2-2c_1+1}} \left\{ \begin{array}{l} \hat{\theta}^2 (2n+c_2-2c_1)! + \frac{(2n+c_2-2c_1+2)!}{\left(\sum_{i=1}^{n} x_i\right)^2} \\ -2\hat{\theta} \frac{(2n+c_2-2c_1+1)!}{\left(\sum_{i=\backslash 1}^{n} x_i\right)} \end{array} \right\} \quad (2.16.4.4)$$

Now $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, Then the Bayes' estimator is

$$2\hat{\theta}(2n+c_2-2c_1)! - \frac{2(2n+c_2-2c_1+1)!}{\sum_{i=1}^{n} x_i} = 0$$

$$\hat{\theta}_{B4} = \frac{(2n+c_2-2c_1+1)}{\sum_{i=1}^{n} x_i} \qquad (2.16.4.5)$$

Remarks 3: If $c_2 = 0, c_1 = \frac{1}{2}$, we get, the Jeffrey's prior and the corresponding Bayes' estimator is $\hat{\theta}_{B4} = \frac{2}{\bar{x}}$. If $c_2 = 0, c_1 = \frac{3}{2}$, we get, the Hartigan prior [Hartigan (1964)] and the corresponding Bayes' estimator becomes: $\hat{\theta}_{B4} = \frac{(2n-2)}{\sum_{i=1}^{n} x_i}$. If $c_2 = 2, c_1 = 1$, we get, the uniform prior and the corresponding Bayes' estimator becomes $\hat{\theta}_{B4} = \frac{(2n+1)}{\sum_{i=1}^{n} x_i}$

2.17 Simulation Study of size-biased exponential distribution

In our simulation study, we chose a sample size of n=25, 50 and 100 to represent small, medium and large data set. The scale parameter is estimated for size-biased exponential distribution by the methods of Maximum Likelihood and Bayesian using Jeffrey's & extension of Jeffrey's prior. For the scale parameter we have considered θ= 1.0 and 1.5. The values of Jeffrey's extension were c_1 = 0.5, 1.0, 1.5 and 2.0. The value for the loss parameter c_2 is 1, 0 and -1. This was iterated 5000 times and the scale parameter for each method was calculated. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes with different values for the Extension of Jeffrey's' prior and the loss functions. The results are presented in tables (2.6), (2.7) for different selections of the parameters and c extension of Jeffrey's prior.

Table 2.5: Structural properties of Size biased exponential distribution

n	θ	Mean	variance	S.D	C.V	Shannon's Entropy
25	1.5	0.8526943	0.03635438	0.1906682	0.2236067	0.5945349
	2.0	0.9867888	0.05156773	0.2270853	0.2301255	0.3068528
	2.5	0.8056772	0.03245661	0.1801572	0.2236097	0.0837092
50	1.5	0.9289910	0.03867889	0.1966695	0.2117023	0.6134556
	2.0	1.0234671	0.04899090	0.2213389	0.2162638	0.3356778
	2.5	0.7945678	0.03156767	0.1776729	0.2721779	0.0967888
100	1.5	0.9368991	0.03789921	0.1946772	0.2077889	0.6356778
	2.0	0.9856778	0.05356778	0.2314471	0.2348102	0.3556778
	2.5	0.8345677	0.03335664	0.1826380	0.2188414	0.1078867

Table 2.6: Mean Squared Error for ($\hat{\theta}$) under Jeffrey's prior

n	θ	θ_{ML}	θ_{SL}	θ_{NL}		
				C2=1.0	C2=0	C2=-1.0
25	0.5	0.220509	0.220509	0.239097	0.220509	0.202673
	1.0	0.046714	0.046714	0.049379	0.046714	0.044122
	1.5	0.281341	0.281341	0.261145	0.281341	0.302288
50	0.5	0.121840	0.121840	0.127840	0.121840	0.115985
	1.0	0.010920	0.010920	0.008897	0.010920	0.013150
	1.5	0.254824	0.254824	0.258140	0.254824	0.251529
100	0.5	0.092848	0.092848	0.101594	0.092848	0.084176
	1.0	0.000925	0.000925	0.000121	0.002480	0.002480
	1.5	0.180579	0.180579	0.177164	0.180579	0.1840687

Table2.7: Mean Squared Error for ($\hat{\theta}$) under extension of Jeffrey's prior

n	θ	C_1	θ_{ML}	θ_{SL}	θ_{NL}		
					C2=1.0	C2=0	C2=-1.0
25	1.0	0.5	0.046714	0.046714	0.049379	0.046714	0.044122
		1.0	0.046714	0.044122	0.046714	0.044122	0.041605
		1.5	0.046714	0.041605	0.044122	0.041605	0.039161
		2.0	0.046714	0.039161	0.041605	0.039161	0.036791

	1.5	0.5	0.281341	0.281341	0.261145	0.281341	0.302288
		1.0	0.281341	0.302288	0.281341	0.302288	0.323987
		1.5	0.281341	0.323987	0.302288	0.323987	0.346439
		2.0	0.281341	0.346439	0.323987	0.346439	0.369642
50	1.0	0.5	0.010920	0.010920	0.008897	0.010920	0.013150
		1.0	0.010920	0.013150	0.010920	0.013150	0.015588
		1.5	0.010920	0.015588	0.013150	0.015588	0.018233
		2.0	0.010920	0.018233	0.015588	0.018233	0.021084
	1.5	0.5	0.254824	0.254824	0.258140	0.254824	0.251529
		1.0	0.254824	0.251529	0.254824	0.251529	0.248256
		1.5	0.254824	0.248256	0.251529	0.248256	0.245004
		2.0	0.254824	0.245004	0.248256	0.245004	0.241774
100	1.0	0.5	0.000925	0.000925	0.000121	0.000925	0.002480
		1.0	0.000925	0.002480	0.000925	0.002480	0.004788
		1.5	0.000925	0.004788	0.002480	0.004788	0.007848
		2.0	0.000925	0.007848	0.004788	0.007848	0.011660
	1.5	0.5	0.180579	0.180579	0.177164	0.180579	0.184068
		1.0	0.180579	0.184068	0.180579	0.184068	0.187631
		1.5	0.180579	0.187631	0.184068	0.187631	0.191268
		2.0	0.180579	0.191268	0.187631	0.191268	0.194979

ML= Maximum Likelihood, SL=Squared Error Loss Function, NL= New Loss Function

In table 2.6, Bayes' estimation with New Loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is ±1. Similarly in table 2.7, Bayes' estimation with New Loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is ± 1 whether the extension of Jeffrey's prior is 0.5, 1.0, 1.5 or 2.0. Moreover, when the sample size increases from 25 to 100, the MSE decreases quite significantly.

CHAPTER – 3

SIZE-BIASED GENERALIZED GAMMA DISTRIBUTION

3.1 INTRODUCTION

The generalized Gamma (GG) distribution presents a flexible family in the varieties of shapes and hazard functions for modeling duration. The study of life testing models begins with the estimation of the unknown parameters involved in the models. Amorose Stacy (1962) proposed a generalized Gamma model and studied its characteristics. Shukla and Kumar (2006) used this model in a bit little transformed form to cover more real life situations. Distributions that are used in duration analysis in economics include exponential, lognormal, Gamma, and Weibull. Stacy and Mihram (1965) and Harter (1967) have derived maximum likelihood estimators of generalized Gamma model under different situations. Prantice (1974) has considered maximum likelihood estimators for generalized Gamma model by using the technique of reparametrization. The GG family, which encompasses exponential, Gamma, and Weibull as subfamilies, and lognormal as a limiting distribution, has been used in economics by Jaggia (1991). Some authors like Yamaguchi (1992) and Allenby et al (1999) have argued that the flexibility of GG makes it suitable for duration analysis, while others have advocated use of simpler models because of estimation difficulties caused by the complexity of GG parameter structure. Obviously, there would be no need to endure the costs associated with the application of a complex GG model if the data do not discriminate between the GG and members of its subfamilies, or if the fit of a simpler model to the data is as good as that for the complex GG. Hager and Bain (1970) inhibited applications of the GG model. However, despite its long history and growing use in various applications, the GG family and its properties has been remarkably presented in different papers. Maximum-likelihood estimation of the parameters and quasi maximum likelihood estimators for its subfamily (two-parameter Gamma distribution) can be found in Stacy (1973). Hwang T. et al (2006) introduced a new moment estimation of parameters of the generalized Gamma distribution using its characterization. In information theory, thus far a maximum entropy (ME) derivation of GG is found in Kapur (1989), where it is referred to as generalized Weibull distribution, and the entropy of GG has appeared in the context of flexible families of distributions. Some concepts of this family in information theory has introduced by Dadpay et al (2007).

The pdf of the generalized Gamma distribution is given by:

$$f(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1}e^{-(\lambda x)^{\beta}} \quad , for\, x>0\, and\, \lambda,\beta,k>0 \tag{3.1.1}$$

where $\Gamma(.)$ is the Gamma function, K and β are shape parameters, and α is the scale parameter. The generalized Gamma family is flexible in that it includes several well-known models as subfamilies. The subfamilies of generalized Gamma thus far considered in the literature are exponential, Gamma and Weibull. The lognormal distribution is also obtained as a limiting distribution when $n \to \infty$.

The CDF of the generalized Gamma distribution is given by:

$$F(x;\lambda,\beta,k)=\frac{\gamma\left(k,(kx)^{\beta}\right)}{\Gamma k}$$

The Structural properties of the generalized Gamma distribution are given as:

$$E(X)=\frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\lambda\Gamma k} \tag{3.1.2}$$

$$V(X)=\frac{\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^{2}\left(k+\frac{1}{\beta}\right)}{\lambda^{2}\Gamma^{2}(k)} \tag{3.1.3}$$

$$C.V(X)=\frac{\left[\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^{2}\left(k+\frac{1}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(k+\frac{1}{\beta}\right)} \qquad (3.1.4)$$

In this chapter, a new class of Size biased Generalized Gamma distribution have been considered. The several structural properties, reliability and information measures are introduced and derived. The estimation of parameters of this new model is obtained by employing the new methods of moments, maximum likelihood and Bayesian method of estimation. The Bayes' estimators are obtained by using Jeffrey's and extension of Jeffrey's prior under different loss functions. A comparison has been made of the Bayes' estimator with the corresponding maximum likelihood estimator. Also, a likelihood ratio test of size biased generalized Gamma distribution is to be conducted. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator. Also, survival functions of new model are derived using Jeffrey and extension of Jeffrey prior. It has been observed that Bayes' estimator provides better results and estimates as compared to classical estimators. In this chapter, the AIC, BIC and DIC values of exponential model are smaller as compared to size biased Gamma and size biased exponential models, so exponential model is more preferable than the size biased Gamma and size biased exponential models for the real data in hand.

3.2 Size biased Generalized Gamma Distribution

A size- biased Generalized Gamma distribution (SBGGMD) is obtained by applying the weights x^{c}, where c =1 to the Generalized Gamma distribution.

$$f_S(x;\lambda,\beta,k)=\int_0^{\infty}\frac{x\, f(x;\lambda,\beta,k)}{E(X)}\; dx$$

$$f_S(x;\lambda,\beta,k)=\int_0^{\infty}\frac{x\,\frac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1}\,e^{-(\lambda x)^{\beta}}}{\frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\lambda\Gamma(k)}}\; dx$$

$$f_S(x;\lambda,\beta,k)=\int_0^\infty \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^\beta}dx$$

where $f_s(x;\lambda,\beta,k)$ represents a probability density function. This gives the size -biased generalized Gamma distribution (SBGGMD) and its pdf is given by

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^\beta} \quad (3.2.1)$$

For $\lambda>0, k\geq 0, \beta>0$ *and* $k\beta>1$

where $\Gamma(.)$ is the Gamma function, K and β are shape parameters, and α is the scale parameter.

3.2.1 Special cases

1. When $\beta=k=1$, then Size biased Generalized Gamma distribution reduced to Size-biased exponential distribution and its probability distribution is given by

   $$f_S(x;\lambda)=\lambda^2 x\ e^{-\lambda x};\lambda>0 \quad (3.2.2)$$

2. When $\beta=1$ then Size biased Generalized Gamma distribution reduced to size-biased Gamma distribution and its probability distribution is given by

   $$f_S(x;\lambda,k+1)=\frac{\lambda^{k+1}}{\Gamma k+1}\ e^{-\lambda x}\ x^k;\lambda>0, k\geq 0\ ,0<x<\infty \quad (2.2.3)$$

3. When $k=0\, and\, \beta=1$ then Size biased Generalized Gamma distribution reduced to exponential distribution and its probability distribution is given by

   $$f_s(x;\lambda)=\lambda\ e^{-\lambda x};\lambda>0\ ,0<x<\infty \quad (3.2.4)$$

3.2.2 Hazard functions

The hazard function for the Size biased generalized Gamma distribution is given as:

$$h_s(x;\lambda,\beta,k)=\frac{f(x;\lambda,\beta,k)}{1-F(x;\lambda,\beta,k)}$$

$$h(x;\lambda,\beta,k)=\frac{\lambda\beta(\lambda x)^{k\beta}e^{-(\lambda x)^\beta}}{\Gamma\left(k+\frac{1}{\beta}\right)-\gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^\beta\right)} \quad (3.2.5)$$

The reverse hazard function for the Size biased generalized Gamma distribution is given as:

$$h_{rv}(x;\lambda,\beta,k)=\frac{f(x;\lambda,\beta,k)}{F(x;\lambda,\beta,k)}$$

$$h_{rv}(x;\lambda,\beta,k)=\frac{\lambda\beta(\lambda x)^{k\beta}e^{-(\lambda x)^{\beta}}}{\gamma\left(\left(k+\frac{1}{\beta}\right),(\lambda x)^{\beta}\right)} \qquad (3.2.6)$$

Theorem 3.2.2.1: Let $f(x;\lambda,\beta,k)$ be a twice differentiable probability density function of a continuous random variable X. Define $n(x;\lambda,\beta,k)=-\frac{f'(x;\lambda,\beta,k)}{f(x,\lambda,\beta,k)}$ where $f'(x;\lambda,\beta,k)$ is the first derivative of $f(x;\lambda,\beta,k)$with respect to x. Furthermore, suppose that the first derivative of $n(x;\lambda,\beta,k)$ exist.

a) If $n'(x;\lambda,\beta,k)<0,$for all x > 0, then the hazard function is monotonically decreasing.

b) If $n'(x;\lambda,\beta,k)>0,$for all x > 0, then the hazard function is monotonically increasing.

c) Suppose there exist x_0 such that $n'(x;\lambda,\beta,k)<0,\ \textit{for all}\ 0<x<x_0,\ n'(x_0;\lambda,\beta,k)=0$

And $n'(x;\lambda,\beta,k)>0,\ \textit{for all}\ 0>x_0$. In addition, $\lim_{x\to 0} f(x)=\infty$, then the hazard function is upside down bathtub shape.

Proof: Using equation (3.2.1), the derivative of the $f(x;\lambda,\beta,k)$is given by:

$$f'(x;\lambda,\beta,k)=f(x;\lambda,\beta,k)\left(\frac{k\beta-\beta(\lambda x)^{\beta}}{x}\right)$$

Therefore, $n(x;\lambda,\beta,k)=-\frac{f'(x;\lambda,\beta,k)}{f(x,\lambda,\beta,k)}$

$$n(x;\lambda,\beta,k)=\frac{\beta\left[(\lambda x)^{\beta}-k\right]}{x}$$

And $n'(x;\lambda,\beta,k)=\frac{\beta(\beta-1)(\lambda x)^{\beta}+k\beta}{x^{2}}$ (3.2.7)

a) If $\beta>1$, then $n'(x;\lambda,\beta,k)>0,$for all x > 0, then the hazard function is monotonically increasing.

b) If $\beta < 1$, then $n'(x;\lambda,\beta,k) < 0$, then the hazard function is monotonically decreasing.

c) If $0 < \beta < 1$, then the hazard function is upside down bathtub shape.

3.3 Structural properties of Size biased Generalized Gamma Distribution

In this section, we derive some structural properties of Size-biased generalized Gamma distribution.

3.3.1 Moments of Size biased Generalized Gamma distribution

Using equation (3.2.1), the mth moments are obtained as:

$$E(X^m) = \int_0^\infty x^m \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)} (\lambda x)^{k\beta} \quad e^{-(\lambda x)^\beta} dx$$

$$E(X^m) = \frac{\beta\,\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)} \int_0^\infty x^{(k\beta+m+1)-1} e^{-(\lambda x)^\beta} dx \tag{3.3.1}$$

On solving the above equation (3.3.1), we get

$$E(X^m) = \frac{\Gamma\left(k+\frac{m+1}{\beta}\right)}{\lambda^m\,\Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.3.2}$$

Using the equation (3.3.2), the mean and variance of the SBGGMD are given as:

$$E(X) = \frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\,\Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.3.3}$$

$$V(X) = \frac{\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right) - \Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\frac{1}{\beta}\right)} \tag{3.3.4}$$

The coefficient of variation of SBGGMD is given by

$$CV = \frac{\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)-\Gamma^2\left(k+\frac{2}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(k+\frac{2}{\beta}\right)} \qquad (3.3.5)$$

3.3.2 Mode of Size biased generalized Gamma distribution:

The probability distribution of size biased Generalized Gamma distribution can be obtained as:

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}$$

In order to discuss monotonicity of size biased Generalized Gamma distribution. We take the logarithm of its pdf:

$$\ln f_S(x;\lambda,\beta,k)=\ln\left(\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)+k\beta\ln(\lambda x)-(\lambda x)^{\beta} \qquad (3.3.6)$$

Where C is a constant. Note that

$$\frac{\partial\ln f(x;\lambda,\beta,k)}{\partial x}=0 \Leftrightarrow x=\frac{k^{\frac{1}{\beta}}}{\lambda^2}$$

The mode of size generalized Gamma distribution is given as:

$$x=\frac{k^{\frac{1}{\beta}}}{\lambda^2} \qquad (3.3.7)$$

3.3.3 Moment generating function of Size biased Generalized Gamma Distribution

The moment generating function of SBGG distribution is obtained as:

$$E\left(e^{tx^{\beta}}\right)=\int_0^{\infty}e^{tx^{\beta}}\ f\left(x;\lambda,\beta,k\right)dx$$

$$E\left(e^{tx^{\beta}}\right)=\int_0^{\infty}e^{tx^{\beta}}\ \frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}x^{k\beta}\ e^{-(\lambda x)^{\beta}}dx$$

$$E\left(e^{tx^\beta}\right)=\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\int_0^\infty x^{k\beta}\, e^{-x^\beta\left(\lambda^\beta-t\right)}dx \tag{3.3.8}$$

Put $x^\beta=t \Rightarrow x=t^{\frac{1}{\beta}}$ $\quad dx=\frac{dt}{\beta x^{\beta-1}}$

If $x\to 0, t\to 0$ *and* $x\to\infty, t\to\infty$

$$E\left(e^{tx^\beta}\right)=\frac{\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\int_0^\infty t^k\, e^{-t\left(\lambda^\beta-t\right)}\frac{dt}{t^{1-\frac{1}{\beta}}}$$

$$E\left(e^{tx^\beta}\right)=\frac{\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\int_0^\infty t^{k+\frac{1}{\beta}-1}\, e^{-t\left(\lambda^\beta-t\right)}dt \tag{3.3.9}$$

$$E\left(e^{tx^\beta}\right)=\frac{\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\left(\lambda^\beta-t\right)^{k+\frac{1}{\beta}}}$$

$$E\left(e^{tx^\beta}\right)=\frac{\lambda^{k\beta+1}}{\left(\lambda^\beta-t\right)^{k+\frac{1}{\beta}}} \tag{3.3.10}$$

Substitute $\beta=1$ in the above relation, we have

$$E\left(e^{tx}\right)=\left(\frac{\lambda}{\lambda-t}\right)^{k+1} \tag{3.3.11}$$

3.3.4 Characteristic function of Size biased Generalized Gamma Distribution

The Characteristic function of SBG distribution is obtained as:

$$\Phi_x(t)=\int_0^\infty e^{itx^\beta} f\left(x;\lambda,\beta,k\right)dx$$

$$\Phi_x(t)=\int_0^\infty e^{itx^\beta}\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}x^{k\beta}\, e^{-(\lambda x)^\beta}dx$$

$$\Phi_x(t) = \frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\int_0^\infty x^{k\beta}\, e^{-x^{\beta}\left(\lambda^{\beta}-it\right)}dx \tag{3.3.12}$$

Put $x^{\beta} = t \Rightarrow x = t^{\frac{1}{\beta}};\ dx = \frac{dt}{\beta x^{\beta-1}}$

If $x \to 0, t \to 0$ *and* $x \to \infty, t \to \infty$

$$\Phi_x(t) = \frac{\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\int_0^\infty t^{k}\, e^{-t\left(\lambda^{\beta}-it\right)}\frac{dt}{t^{1-\frac{1}{\beta}}}$$

$$\Phi_x(t) = \frac{\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\int_0^\infty t^{k+\frac{1}{\beta}-1}\, e^{-t\ \left(\lambda^{\beta}-it\right)}dt \tag{3.3.13}$$

$$\Phi_x(t) = \frac{\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\left(\lambda^{\beta}-it\right)^{k+\frac{1}{\beta}}}$$

$$\Phi_x(t) = \frac{\lambda^{k\beta+1}}{\left(\lambda^{\beta}-it\right)^{k+\frac{1}{\beta}}} \tag{3.3.14}$$

Substitute $\beta = 1$ in the above relation, we have

$$\Phi_x(t) = \left(\frac{\lambda}{\lambda-it}\right)^{k+1} \tag{3.3.15}$$

3.3.5 Shannon's entropy of size-biased Generalized Gamma Distribution

For deriving the entropy of the size-biased Generalized Gamma distribution, we need the two definitions that are more details of them can be found in Cover (1991).

Theorem.3.5.5.1 Let $x_1, x_2, x_3 ... x_n$ be an n positive identical independently distributed random samples drawn from a population having a size-biased generalized Gamma density

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}\ ,\ x>0,\lambda>0,\beta>0,k\geq 0$$

Then Shannon's entropy of Size-biased Generalized Gamma Distribution is:

$$H\big(f(x;\lambda,k,\beta)\big)=-\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)-k\left[\Psi\left(k+\frac{1}{\beta}\right)-\log\lambda^{\beta}\right]+\lambda^{\beta}\beta\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)}$$

Proof: Shannon's entropy of Size biased Generalized Gamma Distribution is obtained as:

$$H[f(x;\alpha,\beta,k)]=E[-\log\{f(x;\alpha,\beta,k\}]$$

$$H[f(x;\alpha,\beta,k)]=E\left[-\log\left\{\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}x^{k\beta}e^{-(\lambda x)^{\beta}}\right\}\right]$$

$$H[f(x;\alpha,\beta,k)]=\left[-\log\left\{\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right\}-k\beta E(\log x)+\lambda^{\beta}\beta\ E(x)\right]$$

$$H\big(f(x;\lambda,k,\beta)\big)=-\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right)-k\beta E(\log x)+\lambda^{\beta}\beta\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)} \qquad (3.3.16)$$

Now, $E(\log(x))=\int_0^{\infty}\log xf(x;\alpha,\beta,k)dx$

$$E(\log(x))=\int_0^{\infty}\log x\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}x^{k\beta}\ e^{-(\lambda x)^{\beta}}dx \qquad (3.3.17)$$

Let $(\lambda x)^{\beta}=t\Rightarrow x^{\beta}=\frac{t}{\lambda^{\beta}}\Rightarrow\ x=\left(\frac{t}{\lambda^{\beta}}\right)^{\frac{1}{\beta}} \qquad dx=\frac{dt}{\lambda\beta t^{1-\frac{1}{\beta}}}$

$$E(\log(x)) = \frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)} \int_0^\infty \log\left(\frac{t}{\lambda^\beta}\right)^{\frac{1}{\beta}} \left(\frac{t}{\lambda^\beta}\right)^{k\beta} e^{-t} \frac{dt}{\lambda\beta t^{1-\frac{1}{\beta}}}$$

$$E(\log(x)) = \frac{1}{\Gamma\left(k+\frac{1}{\beta}\right)} \int_0^\infty \log\left(\frac{t}{\lambda^\beta}\right)^{\frac{1}{\beta}} t^{k+\frac{1}{\beta}-1} e^{-t} dt$$

$$E(\log(x)) = \frac{1}{\beta\Gamma\left(k+\frac{1}{\beta}\right)} \int_0^\infty \left(\log t\, e^{-t}\, t^{k+\frac{1}{\beta}-1} dt\right) - \frac{\log\lambda^\beta}{\Gamma\left(k+\frac{1}{\beta}\right)} \int_0^\infty e^{-t} t^{k+\frac{1}{\beta}-1} dt$$

$$E(\log(x)) = \frac{\Gamma'\left(k+\frac{1}{\beta}\right)}{\beta\Gamma\left(k+\frac{1}{\beta}\right)} - \frac{\log\lambda^\beta\Gamma\left(k+\frac{1}{\beta}\right)}{\beta\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$E(\log(x)) = \frac{\Psi\left(k+\frac{1}{\beta}\right) - \log\lambda^\beta}{\beta} \qquad (3.3.18)$$

Substitute the value of equation (3.3.18) in equation (3.3.16), we have

$$H\left(f\left(x;\lambda,k,\beta\right)\right) = -\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k+\frac{1}{\beta}\right)}\right) - k\left[\Psi\left(k+\frac{1}{\beta}\right) - \log\lambda^\beta\right] + \lambda^\beta\beta\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)} \qquad (3.3.19)$$

3.3.6 The generalized entropy of size-biased Generalized Gamma Distribution

Generalized entropy is often used in econometrics. It is indexed by a single parameter α .The generalized entropy is defined to be

$$I_\alpha = \frac{v_\alpha u^{-\alpha} - 1}{\alpha(\alpha-1)}; \alpha \neq 0,1 \text{ and } v_\alpha = \int_0^\infty x^\alpha f_s\left(x;\alpha,\beta,k\right)dx$$

We know that

$$v_\alpha = \frac{\Gamma\left(k+\frac{\alpha+1}{\beta}\right)}{\lambda^\alpha \, \Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.3.20}$$

Substituting above values, we get

$$I_\alpha = \frac{\frac{\Gamma\left(k+\frac{\alpha+1}{\beta}\right)}{\lambda^\alpha \, \Gamma\left(k+\frac{1}{\beta}\right)}\left[\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)}\right]^{-\alpha} -1}{\alpha(\alpha-1)} \tag{3.3.21}$$

$$I_\alpha = \frac{\Gamma\left(k+\frac{\alpha+1}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)^{\alpha-1} - \Gamma^\alpha\left(k+\frac{2}{\beta}\right)}{\alpha(\alpha-1)\left[\Gamma^\alpha\left(k+\frac{2}{\beta}\right)\right]} \tag{3.3.22}$$

3.3.7 Fisher's information matrix of size-biased Generalized Gamma Distribution

The Size biased generalized Gamma distribution has a probability density function of the form:

$$f_s(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta} \, e^{-(\lambda x)^\beta} \tag{3.3.23}$$

Applying log on both sides in equation, we have

$$\log f_s(x;\lambda,\beta,k) = \log\beta + (k\beta+1)\log\lambda - \log\Gamma\left(k+\frac{1}{\beta}\right) - \lambda^\beta x^\beta \tag{3.3.24}$$

Differentiating equation (3.3.19) partially with respect to λ, β and k we get

$$\frac{\partial \log f_s(x;\lambda,\beta,k)}{\partial\lambda} = \frac{k\beta - \beta(\lambda x)^\beta}{\lambda} \tag{3.3.25}$$

$$\frac{\partial \log f_s(x;\lambda,\beta,k)}{\partial\beta} = \frac{\psi\left(k+\frac{1}{\beta}\right)}{\beta^2} + \frac{1}{\beta} + k\log\lambda - (\lambda x)^\beta \log(\lambda x) \tag{3.3.26}$$

$$\frac{\partial \log f_s(x;\lambda,\beta,k)}{\partial k} = \beta\log\lambda - \psi\left(k+\frac{1}{\beta}\right) \tag{3.3.27}$$

Differentiating again the above equation partially with respect to λ, β and k we have

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \lambda^2} = \frac{-(k\beta+1)-\beta(\beta-1)(\lambda)^{\beta} x^{\beta}}{\lambda^2} \qquad (3.3.28)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \beta \partial \lambda} = \frac{k-\lambda^{\beta}\left[(\beta \log \lambda + 1)x^{\beta} + \beta x^{\beta} \log x\right]}{\lambda} \qquad (3.3.29)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k \partial \lambda} = \frac{\beta}{\lambda} \qquad (3.3.30)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \beta^2} = -\begin{pmatrix} \frac{1}{\beta^2} + \psi'\left(k+\frac{1}{\beta}\right) + 2\beta\psi\left(k+\frac{1}{\beta}\right) + \lambda^{\beta} \ x^{\beta}(\log x)^2 \\ + \lambda^{\beta}(\log \lambda)^2 x^{\beta} + 2\lambda^{\beta} \log \lambda x^{\beta} \log x \end{pmatrix} \qquad (3.3.31)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \lambda \partial \beta} = \frac{k-\lambda^{\beta}\left[(\beta \log \lambda + 1)x^{\beta} - \beta\lambda^{\beta} x^{\beta} \log x\right]}{\lambda} \qquad (3.3.32)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k \partial \beta} = \log \lambda + \frac{\psi'\left(k+\frac{1}{\beta}\right) - \psi^2\left(k+\frac{1}{\beta}\right)}{\beta^2} \qquad (3.3.33)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k^2} = -\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} + \psi^2\left(k+\frac{1}{\beta}\right) \qquad (3.3.34)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \lambda \partial k} = \frac{\beta}{\lambda^2} \qquad (3.3.35)$$

$$\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \beta \partial k} = -\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)} + \psi^2\left(k+\frac{1}{\beta}\right) \qquad (3.3.36)$$

Taking expectations on both sides of the above equations, we get

$$\mathrm{I}(1,1) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \lambda^2}\right) = \frac{(k\beta+1)+\beta(\beta-1)(\lambda)^{\beta} E(x)^{\beta}}{\lambda^2}$$

$$\mathrm{I}(1,2) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \beta \partial \lambda}\right) = \frac{-k+\lambda^{\beta}\left[(\beta \log \lambda + 1)x^{\beta} + \beta E\left(x^{\beta} \log x\right)\right]}{\lambda}$$

$$I(1,3) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k \partial \lambda}\right) = -\frac{\beta}{\lambda}$$

$$I(2,1) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \lambda \partial \beta}\right) = \frac{-k + \lambda^\beta \left[(\beta \log \lambda + 1) E(x)^\beta - \beta \lambda^\beta E\left(x^\beta \log x\right)\right]}{\lambda}$$

$$I(2,2) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \beta^2}\right) = \begin{pmatrix} \frac{1}{\beta^2} + \psi'\left(k + \frac{1}{\beta}\right) + 2\beta\psi\left(k + \frac{1}{\beta}\right) + \lambda^\beta \, E\left[x^\beta (\log x)^2\right] \\ + \lambda^\beta (\log \lambda)^2 \, E(x)^\beta + 2\lambda^\beta \log \lambda E\left(x^\beta \log x\right) \end{pmatrix}$$

$$I(2,3) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k \partial \beta}\right) = \log \lambda + \frac{\psi'\left(k + \frac{1}{\beta}\right) - \psi^2\left(k + \frac{1}{\beta}\right)}{\beta^2}$$

$$I(3,1) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \lambda \partial k}\right) = \frac{\beta}{\lambda^2}$$

$$I(3,2) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial \beta \partial k}\right) = -\frac{\Gamma''\left(k + \frac{1}{\beta}\right)}{\Gamma\left(k + \frac{1}{\beta}\right)} + \psi^2\left(k + \frac{1}{\beta}\right)$$

$$I(3,3) = -E\left(\frac{\partial^2 \log f_s(x;\lambda,\beta,k)}{\partial k^2}\right) = -\frac{\Gamma''\left(k + \frac{1}{\beta}\right)}{\Gamma\left(k + \frac{1}{\beta}\right)} + \psi^2\left(k + \frac{1}{\beta}\right)$$

We know that, $E\left(x^\beta \log x\right) = \int_0^\infty x^\beta \log x f_s(x;\lambda,\beta,k) dx$

$$E\left(x^\beta \log x\right) = \int_0^\infty x^\beta \log x \frac{\lambda\beta}{\Gamma\left(k + \frac{1}{\beta}\right)} (\lambda x)^{k\beta} \, e^{-(\lambda x)^\beta} \, dx$$

$$E\left(x^\beta \log x\right) = \frac{\Gamma'\left(k + \frac{1}{\beta} + 1\right) - \beta \log \lambda \Gamma\left(k + \frac{1}{\beta} + 1\right)}{\beta \lambda^\beta \Gamma\left(k + \frac{1}{\beta}\right)} \tag{3.3.37}$$

Also, $E\left(x^{\beta}(\log x)^2\right)=\int_0^{\infty} x^{\beta}(\log x)^2 f_s\ (x;\lambda,\beta,k)dx$

$$E\left(x^{\beta}(\log x)^2\right)=\int_0^{\infty} x^{\beta}(\log x)^2 \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}\ dx$$

$$E\left(x^{\beta}(\log x)^2\right)=\frac{\Gamma''\left(k+\frac{1}{\beta}+1\right)-2\beta\log\lambda\Gamma'\left(k+\frac{1}{\beta}+1\right)+\beta^2(\log\lambda)^2\Gamma\left(k+\frac{1}{\beta}+1\right)}{\beta^2\lambda^{\beta}\Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.3.38}$$

Also, $E(X^{\beta})=\dfrac{\Gamma\left(k+\frac{\beta+1}{\beta}\right)}{\lambda^{\beta}\Gamma\left(k+\frac{1}{\beta}\right)}$ (3.3.39)

Substitute the values of equations (3.3.37), (3.3.38) and (3.3.39) in the above entries of a Fisher information matrix, we get

$$I(1,1)=\frac{(k\beta+1)\Gamma\left(k+\frac{1}{\beta}\right)+\beta(\beta-1)\Gamma\left(k+\frac{1}{\beta}+1\right)}{\lambda^2\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(1,2)=\frac{-k\Gamma\left(k+\frac{1}{\beta}\right)+\left[(\beta\log\lambda-\beta^2\log\lambda+1)\Gamma\left(k+\frac{1}{\beta}+1\right)+\beta\Gamma'\left(k+\frac{1}{\beta}+1\right)\right]}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(1,3)=-\frac{\beta}{\lambda}$$

$$I(2,1)=\frac{-k\Gamma\left(k+\frac{1}{\beta}\right)+\left[\Gamma\left(k+\frac{1}{\beta}+1\right)+\left(\Gamma'\left(k+\frac{1}{\beta}+1\right)\right)\right]}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(2,2)=\frac{\left(\Gamma\left(k+\frac{1}{\beta}\right)+\beta^{2}\Gamma''\left(k+\frac{1}{\beta}\right)+2\beta^{3}\Gamma'\left(k+\frac{1}{\beta}\right)+\Gamma''\left(k+\frac{1}{\beta}+1\right)\right)}{\beta^{2}\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$I(2,3)=-\log\lambda-\left[\frac{\psi'\left(k+\frac{1}{\beta}\right)-\psi^{2}\left(k+\frac{1}{\beta}\right)}{\beta^{2}}\right]$$

$$I(3,1)=-\frac{\beta}{\lambda^{2}}$$

$$I(3,2)=-\log\lambda-\frac{1}{\beta^{2}}\left[\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)}+\psi^{2}\left(k+\frac{1}{\beta}\right)\right]$$

$$I(3,3)=\frac{\Gamma''\left(k+\frac{1}{\beta}\right)}{\Gamma\left(k+\frac{1}{\beta}\right)}-\psi^{2}\left(k+\frac{1}{\beta}\right)$$

3.3.8 Entropy estimation:

Consider the pdf of size biased generalized Gamma distribution (3.2.1)

$$f_S(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}} \tag{3.3.40}$$

$$\log L^{*}(X;\lambda,\beta,k)\ =\ n(1+k\beta)\log\lambda+n\log\beta-n\log\Gamma\left(k+\frac{1}{\beta}\right)+k\beta\sum_{i=1}^{n}\log x_i-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta} \tag{3.3.41}$$

$$l(X;\lambda,\beta,k)\ =\ n\left((1+k\beta)\log\lambda+\log\beta-\log\Gamma\left(k+\frac{1}{\beta}\right)\right)+k\beta\sum_{i=1}^{n}\log x_i-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta} \tag{3.3.42}$$

$$\frac{l(x;\lambda,\beta,k)}{n}\ =\ (1+k\beta)\log\lambda+\log\beta-\log\Gamma\left(k+\frac{1}{\beta}\right)+k\beta\overline{\log x}-\lambda^{\beta}\overline{x^{\beta}} \tag{3.3.43}$$

The Shannon’s entropy of Size-biased Generalized Gamma Distribution is given as:

$$\hat{H}(SBGG) = -\left[(1+\hat{k}\hat{\beta})\log\hat{\lambda} + \log\hat{\beta} - \log\Gamma\left(\hat{k}+\frac{1}{\hat{\beta}}\right) + \hat{k}\hat{\beta}\overline{\log x} - \hat{\lambda}^{\hat{\beta}}\overline{x^{\hat{\beta}}}\right] \quad (3.3.44)$$

From equation (3.3.43) and (3.3.44) , we can write

$$\hat{H}(SBGG) = -\frac{l(x;\hat{\lambda},\hat{\beta},\hat{k})}{n} \quad (3.3.45)$$

3.3.9 Akaike and Bayesian information criterion

The AIC and BIC methodology attempts to find the model that best explains the data with a minimum of their values, from (3.3.45) we have

$$l(x;\hat{\lambda},\hat{\beta},\hat{k}) = -n\hat{H}(SBGG)$$

Then for SBGG family we have

$$AIC = 2K + 2n\hat{H}(SBGG) \quad (3.3.46)$$

and $BIC = K\log n + 2n\hat{H}(SBGG)$ (3.3.47)

3.4 Test for Size-biasedness of Size biased generalized Gamma distribution

Let X_1, X_2, X_3… Xn be random samples can be drawn from generalized Gamma distribution or Size biased generalized Gamma distribution. We test the hypothesis $H_0 : f(x) = f(x;\lambda,\beta,k)$ againest $H_1 : f(x) = f_S(x;\lambda,\beta,k)$.

For testing whether the random sample of size n comes from the generalized Gamma distribution or Size biased generalized Gamma distribution, and the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{f_S(x;\lambda,\beta,k))}{f(x;\lambda,\beta,k))}\right]$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{\frac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1}e^{-(\lambda x)^{\beta}}}{\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}e^{-(\lambda x)^{\beta}}}\right]$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \left[\frac{\lambda\Gamma(k)}{\Gamma\left(k + \frac{1}{\beta}\right)} x \right].$$

$$\Delta = \left[\frac{\lambda\Gamma(k)}{\Gamma\left(k + \frac{1}{\beta}\right)} \right]^n \prod_{i=1}^{n} x_i \qquad (3.4.1)$$

We reject the null hypothesis.

$$\left[\frac{\lambda\Gamma(k)}{\Gamma\left(k + \frac{1}{\beta}\right)} \right]^n \prod_{i=1}^{n} x_i \ > k \qquad (3.4.2)$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, where\, k^* = k \left[\frac{\lambda\Gamma(k)}{\Gamma\left(k + \frac{1}{\beta}\right)} \right]^n > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution. Also, we can reject the reject the null hypothesis, when probability value s given by:

$P(\Delta^* > \lambda^*), Where\ \lambda^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the test statistic Δ^*.

3.5 Estimation of parameters in the size-biased Generalized Gamma Distribution.

In this section, we obtain estimates of the parameters for the Size-biased Generalized Gamma distribution by employing the method of moment (MOM) and maximum likelihood (ML) estimators.

3.5.1 Method of Moment Estimators

Let X_1, X_2, X_3… Xn be an independent random samples from the SBGGMD with weight c=1. The method of moment estimators are obtained by setting the raw moments equal to the sample moments, that is $E(X^r)=M_r$ where is the sample moment M_r corresponding to the $E(X^r)$ .The following equations are obtained using the first and second sample moments.

$$\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)} = \frac{1}{n}\sum_{i=1}^{n} X_j \qquad (3.5.1)$$

$$\frac{\Gamma\left(k+\frac{3}{\beta}\right)}{\lambda^2\Gamma\left(k+\frac{1}{\beta}\right)} = \frac{1}{n}\sum_{i=1}^{n} X_j^{\,2} \qquad (3.5.2)$$

1(*a*).*When* β and k are fixed and from equation (3.5.1), we obtain an estimate $\hat{\lambda}$ for λ, that is

$$\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\Gamma\left(k+\frac{1}{\beta}\right)} = \bar{X}$$

$$\hat{\lambda} = \frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\bar{X}\,\Gamma\left(k+\frac{1}{\beta}\right)} \qquad (3.5.3)$$

1(b). When $\beta = 1 \, and \; \lambda$ are fixed and dividing (3.5.1) by equations (3.5.2), we get

$$\frac{\bar{X}}{M_2} = \frac{\Gamma(k+2)}{\lambda\,\Gamma(k+1)}\,\frac{\lambda^2\,\Gamma(k+1)}{\Gamma(k+3)}$$

$$\Rightarrow \quad \frac{\bar{X}}{M_2} = \frac{\lambda}{k+2}$$

$$\Rightarrow \quad \hat{k} = \frac{M_2}{\bar{X}}.\lambda - 2 \qquad (3.5.4)$$

1.(c): When λ *and* k are fixed, the estimate for β can be obtained by numerical methods.

3.5.2 Maximum likelihood Estimators

Let $X_1, X_2, ..., X_n$ be a random sample from a Size-biased Generalized Gamma Distribution. Then the likelihood function of SBGGMD is given by

$$L(X;\lambda,\beta,k) = \prod_{i=1}^{n} f_s(x;\lambda,\beta,k)$$

$$L(X;\lambda,\beta,k) = \prod_{i=1}^{n}\left[\frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta} e^{-(\lambda x)^{\beta}}\right]$$

$$L(X;\lambda,\beta,k) = \frac{\lambda^{n(1+k\beta)}\beta^{n}}{\Gamma^{n}\left(k+\frac{1}{\beta}\right)}\prod_{i=1}^{n} x_i^{k\beta} e^{-\lambda^{\beta}\sum_{i=1}^{n} x_i^{\beta}} \qquad (3.5.5)$$

Using equation (3.5.5), the log likelihood function is given by

$$\log L^{*}(X;\lambda,\beta,k) = n(1+k\beta)\log\lambda + n\log\beta - n\log\Gamma\left(k+\frac{1}{\beta}\right) + k\beta\sum_{i=1}^{n}\log x_i - \lambda^{\beta}\sum_{i=1}^{n} x_i^{\beta} \qquad (3.5.6)$$

Now differentiate $\log L^{*}(X;\lambda,\beta,k)$ with respect to λ, β *and* k, we get

$$\frac{\partial L^{*}}{\partial\lambda} = \frac{n(1+k\beta)}{\lambda} - \sum_{i=1}^{n} x_i^{\beta}\beta\lambda^{\beta-1}$$

$$\frac{\partial L^{*}}{\partial\beta} = nk\log\lambda + \frac{n}{\beta} - \frac{n\log(\lambda^{\beta})}{\beta^{2}} + \frac{n\Psi\left(k+\frac{1}{\beta}\right)}{\beta^{2}} + k\sum_{i=1}^{n}\log x_i - \lambda^{\beta}\sum_{i=1}^{n} x_i^{\beta}\log x_i - \lambda^{\beta}\log\lambda\sum_{i=1}^{n}\log x_i$$

$$\frac{\partial L^{*}}{\partial k} = n\beta\log\lambda - n\Psi\left(k+\frac{1}{\beta}\right) + \beta\sum_{i=1}^{n}\log x_i$$

Equating these equations to zero, leads to the normal equations:

$$\frac{n(1+k\beta)}{\lambda} - \sum_{i=1}^{n} x_i^{\beta}\beta\lambda^{\beta-1} = 0 \qquad (3.5.7)$$

$$nk\log\lambda + \frac{n}{\beta} - \frac{n\log(\lambda^{\beta})}{\beta^{2}} + \frac{n\Psi\left(k+\frac{1}{\beta}\right)}{\beta^{2}} + k\sum_{i=1}^{n}\log x_i - \lambda^{\beta}\sum_{i=1}^{n} x_i^{\beta}\log x_i - \lambda^{\beta}\log\lambda\sum_{i=1}^{n}\log x_i = 0 \qquad (3.5.8)$$

$$n\beta \log \lambda - n\Psi\left(k + \frac{1}{\beta}\right) + \beta\sum_{i=1}^{n} \log x_i \quad = 0 \qquad (3.5.9)$$

1. When β *and* λ are fixed, It follows from equation (3.5.7), that

$$\hat{k} = \frac{\lambda^{\beta}\sum_{i=1}^{n} x_i^{\beta}}{n} - \frac{1}{\beta} \qquad (3.5.10)$$

2. When β *and* k are fixed, It follows from equation (3.5.9), that

$$\hat{\lambda} = \left[\exp\left(\Psi\left(k + \frac{1}{\beta}\right) - \frac{\beta\sum_{i=1}^{n} \log x_i}{n}\right)\right]^{\frac{1}{\beta}} \qquad (3.5.11)$$

3. When λ *and* k are fixed, the estimate for β can be obtained by numerical methods.

3.6. New method of estimation of Size-Biased Generalized Gamma Distribution

Although Prentice (1974) have presented a procedure to obtain the three parameters of the generalized Gamma distribution, his procedure still quit complicated. In this research, we propose a simple procedure to obtain the three estimators by using its characterization and moment estimation approach. Note that Hwang .T and Huang. P (2006) have obtained more general characterizations with the independence of sample coefficient of variation V_n with sample mean $\overline{X}_n$ as one of its special cases when random samples are drawn from the generalized Gamma distribution. Their characterization is used to derive the expectation and the variance of V_n^2 and then the new estimators for the three parameters of size-biased generalized Gamma distribution are proposed. For deriving new moment estimators of three parameters of the size-biased generalized Gamma distribution, we need the following theorem obtained by using the similar approach of Hwang .T and Huang .P (Theorems of 2006).

Theorem 3.6.1: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Gamma density

$$f_S(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}\ ,\ x>0,\lambda>0,\beta\geq 0,k>0$$

Then $E(S_n{}^2) = \dfrac{\left[\Gamma\left(k+\frac{1}{\beta}\right)\Gamma\left(k+\frac{3}{\beta}\right)-\Gamma\left(k+\frac{2}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\frac{1}{\beta}\right)}$

Proof: We know that

$$E(X^m) = \frac{\Gamma\left(k+\frac{m+1}{\beta}\right)}{\lambda^m\,\Gamma\left(k+\frac{1}{\beta}\right)} \quad ,m=1,2,3,\ldots$$

$$E(\overline{X}_n) = \frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\,\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$E\left(\overline{X}_n{}^2\right) = \frac{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(K+\frac{1}{\beta}\right)+(n-1)\Gamma^2\left(k+\frac{2}{\beta}\right)}{nk\,\Gamma^2\left(k+\frac{1}{\beta}\right)}$$

And $E(S_n{}^2) = n.V(\overline{X}_n)$

$$E(S_n{}^2) = \frac{\left[\Gamma\left(k+\frac{1}{\beta}\right)\Gamma\left(k+\frac{3}{\beta}\right)-\Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\frac{1}{\beta}\right)} \qquad (3.6.1)$$

where $\overline{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Theorem 3.6.2: Let $n\geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Gamma density

$$f_S(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^{\beta}}\ ,\ x>0, \lambda>0, \beta\geq 0, k>0$$

Then

$$E\left(\frac{S_n^{\ 2}}{\overline{X}_n^{\ 2}}\right) = \frac{n\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)-\Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)+(n-1)\Gamma^2\left(k+\frac{2}{\beta}\right)}$$

where $\overline{X}_n$ and $S_n^{\ 2}$ are respectively their sample mean and sample variance.

Proof: By theorem 3.6.1, we have

$$E(S_n^{\ 2}) = E\left(\frac{S_n^{\ 2}}{\overline{X}_n^{\ 2}}.\overline{X}_n^{\ 2}\right) = E\left(\frac{S_n^{\ 2}}{\overline{X}_n^{\ 2}}\right)..E\left(\overline{X}_n^{\ 2}\right)$$

And hence

$$E\left(\frac{S_n^{\ 2}}{\overline{X}_n^{\ 2}}\right) = \frac{E\left(S_n^{\ 2}\right)}{E\left(\overline{X}_n^{\ 2}\right)}$$

Applying theorem 3.6.1 to the above identity yields that

$$E\left(\frac{S_n^{\ 2}}{\overline{X}_n^{\ 2}}\right) = \frac{n\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)-\Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)+(n-1)\Gamma^2\left(k+\frac{2}{\beta}\right)} \qquad (3.6.2)$$

Thus 3.6.2 is established.

Theorem 3.6.4: Let $n\geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Gamma density, then $E\left(\frac{S_n^{\ 2}}{\overline{X}_n}\right) \to \frac{\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)\right]}{\Gamma^2\left(k+\frac{2}{\beta}\right)} - 1$ as $n\to\infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n^{\ 2}}{\overline{X}_n^{\ 2}}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation

Proof: Furthermore, if SBGG distribution, we have

$$E(S_n{}^2) = \frac{\left[\Gamma\left(k+\frac{1}{\beta}\right)\Gamma\left(k+\frac{3}{\beta}\right) - \Gamma\left(k+\frac{2}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\frac{1}{\beta}\right)}$$

$$E\left(\frac{S_n{}^2}{\overline{X}_n}\right) = \frac{n\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right) - \Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right) + (n-1)\Gamma^2\left(k+\frac{2}{\beta}\right)} \tag{3.6.3}$$

$$\frac{\sigma^2}{\mu^2} = \frac{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)}{\Gamma^2\left(k+\frac{2}{\beta}\right)} - 1 \tag{3.6.4}$$

And it can be show that

$$E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right) \to \frac{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right)}{\Gamma^2\left(k+\frac{2}{\beta}\right)} - 1 \tag{3.6.5}$$

After comparing the above equations, we have

$E\left(\frac{S_n{}^2}{\overline{X}_n}\right) \to \frac{\sigma^2}{\mu^2}$ as $n \to \infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n{}^2}{\overline{X}_n{}^2}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation. Hence, we conclude that the independence of the sample mean $\overline{X}_n$ and the sample coefficient of variation $V_n = \frac{S_n}{\overline{X}_n}$ is equivalent to that $f(x)$ is a size-biased generalized Gamma density where S_n is the sample standard deviation.

From (3.3.2) and we know that

$$1. E(\overline{X}_n) = \frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda \Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.6.6}$$

$$2. E(\overline{X}_n{}^T) = \frac{\Gamma\left(k+\frac{T+1}{\beta}\right)}{\lambda^T \Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.6.7}$$

$$3. E\left(\frac{S_n{}^2}{n\ \overline{X}_n{}^2}\right) = \frac{\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right) - \Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right) + (n-1)\Gamma^2\left(k+\frac{2}{\beta}\right)} \tag{3.6.8}$$

Then we can solve numerically via moment method the below equations for estimating of SBGG parameters

$$\frac{\sum_{i=1}^{n} x_i}{n} = \frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda \Gamma\left(k+\frac{1}{\beta}\right)} \tag{3.6.9}$$

$$\frac{S_n{}^2}{n\,\overline{X}_n{}^2} = \frac{\left[\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right) - \Gamma^2\left(k+\frac{2}{\beta}\right)\right]}{\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(k+\frac{1}{\beta}\right) + (n-1)\Gamma^2\left(k+\frac{2}{\beta}\right)} \tag{3.6.10}$$

3.7 Bayesian Method of Estimation

Bayesian analysis is an important approach to statistics, which formally seeks use of prior information and Bayes' Theorem provides the formal basis for using this information. In this approach, parameters are treated as random variables and data is treated fixed.

3.7.1 Parameter estimation under squared error loss function.

In this section, two different prior distributions are used for estimating the parameter of the size biased Generalized Gamma distribution namely; Jeffery's prior and extension of Jeffrey's prior information.

3.7.1.1Bayes estimation of parameter of size biased Generalized Gamma distribution under Jeffrey's prior.

Consider there are n recorded values, $\underline{x} = (x_1,...x_n)$ from (3.2.1). We consider the extended Jeffrey's prior as:

$$g(\lambda) \propto \sqrt{I(\lambda)}$$

Where $[I(\lambda)] = -nE\left[\frac{\partial^2 \log f(x;\lambda,\beta,k)}{\partial\lambda^2}\right]$ is the Fisher's information matrix. For the model (3.2.1),

$$g(\lambda) = \frac{k}{\lambda} \tag{3.7.1}$$

Then the joint probability density function is given by:

$$f(\underline{x};\lambda) \propto L(x;\lambda,\beta,k)g(\lambda)$$

$$f(\underline{x};\lambda) \propto \frac{\lambda^{nk\beta+n-1}\beta^n}{\left[\Gamma^n\left(k+\frac{1}{\beta}\right)\right]} e^{-\lambda^\beta \sum_{i=1}^{n} x_i^\beta} \prod_{i=1}^{n}(x_i)^{k\beta}$$

And the corresponding marginal PDF of $\underline{x} = (x_1,...x_n)$ is obtained as:

$$p(\underline{x}) = \int_0^\infty f(\underline{x},\lambda)d\lambda$$

$$p(\underline{x}) = \int_0^\infty k\left(\frac{\lambda^{nk\beta+n-1}\beta^n}{\left[\Gamma^n\left(k+\frac{1}{\beta}\right)\right]} e^{-\lambda^\beta \sum_{i=1}^{n} x_i^\beta} \prod_{i=1}^{n}(x_i)^{k\beta} d\alpha\right)$$

$$p(\underline{x}) = \frac{k\beta^n}{\left[\Gamma\left(k+\frac{1}{\beta}\right)\right]^n} \prod_{i=1}^{n} x_i^{k\beta} \frac{\Gamma\left(\frac{nk\beta+n-1}{\beta}+1\right)}{\left(\sum_{i=1}^{n} x_i^\beta\right)^{\frac{nk\beta+n-1}{\beta}+1}}$$

$$p(\underline{x}) = \frac{k\beta^n}{\left[\Gamma\left(k+\frac{1}{\beta}\right)\right]^n} \prod_{i=1}^{n} x_i^{k\beta} \frac{\Gamma^n\left(\frac{(k\beta+1)-1}{\beta}+1\right)}{\left(\sum_{i=1}^{n} x_i^\beta\right)^{\frac{nk\beta+n-1}{\beta}+1}} \tag{3.7.2}$$

The posterior PDF of λ has the following form

$$\pi_1(\lambda/\underline{x}) = \frac{f(\underline{x,\lambda})}{p(\underline{x})}$$

$$\pi_1(\lambda/\underline{x}) = \frac{e^{-\lambda^{\beta}\sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta+n-1} \left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{nk\beta+n-1}{\beta}+1}}{\Gamma^n\left[\frac{(k\beta+1)-1}{\beta}+1\right]} \quad (3.7.3)$$

By using a squared error loss function $l_1(\hat{\lambda},\lambda) = c(\hat{\lambda}-\lambda)^2$ for some constant c, the risk function is:

$$R(\hat{\lambda}) = \int_0^{\infty} c(\hat{\lambda}-\lambda)^2 \pi_1(\lambda/\underline{x})\, d\lambda$$

$$R(\hat{\lambda}) = \int_0^{\infty} c(\hat{\lambda}-\lambda)^2 \frac{e^{-\lambda^{\beta}\sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta+n-1} \left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{nk\beta+n-1}{\beta}+1}}{\Gamma^n\left[\frac{(k\beta+1)-1}{\beta}+1\right]} d\lambda$$

$$R(\hat{\lambda}) = c\hat{\lambda}^2 + \frac{c(nk\beta+n)(nk\beta+n+1)}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2c\hat{\lambda}(n\beta+n)}{\sum_{i=1}^{n} x_i} \quad (3.7.4)$$

Now $\frac{\partial R(\hat{\lambda})}{\partial \hat{\lambda}} = 0$, Then the Bayes estimator is

$$\hat{\lambda}_1 = \frac{n\left(\frac{k\beta+1}{\beta}\right)}{\sum_{i=1}^{n} x_i}$$

$$\hat{\lambda}_1 = \frac{n\left(k+\frac{1}{\beta}\right)}{\sum_{i=1}^{n} x_i} \quad (3.7.5)$$

3.7.1.2 Estimation of Survival function

By using posterior probability density function, we can found the Survival function, such that

$$\hat{S}_1(\underline{x}) = \int_0^\infty e^{-(\lambda x)^\beta} \pi_1(\lambda/\underline{x}) d\lambda$$

$$\hat{S}_1(\underline{x}) = \int_0^\infty e^{-\lambda^\beta\left(x_i^\beta + \sum_{i=1}^n x_i^\beta\right)} \lambda^{\beta\left(\frac{nk\beta+n--1}{\beta}\right)} \frac{\left(\sum_{i=1}^n x_i^\beta\right)^{\frac{n\beta+n-1}{\beta}+1}}{\Gamma^n\left(\frac{(k\beta+1)-1}{\beta}+1\right)} d\lambda$$

$$\hat{S}_1(\underline{x}) = \left(\frac{\sum_{i=1}^n x_i^\beta}{x_i^\beta + \sum_{i-1}^n x_i^\beta}\right)^{\frac{n(k\beta+1)-1}{\beta}+1} \tag{3.7.6}$$

3.7.1.3 Bayes' estimation of parameter of size biased Generalized Gamma distribution under the extension Jeffrey's prior

Consider there are n recorded values, $\underline{x} = (x_1,\ldots x_n)$ from (3.2.1). We consider the extended Jeffrey's prior as:

$$g(\lambda) \propto [I(\lambda)]^{c_1}$$

Where $[I(\lambda)] = -nE\left[\frac{\partial^2 \log f(x;\lambda,\beta,k)}{\partial\lambda^2}\right]$ is the Fisher's information matrix. For the model (3.2.1),

$$g(\lambda) = k\left(\frac{1}{\lambda}\right)^{2c_1} \tag{3.7.7}$$

Then the joint probability density function is given by:

$$f(\underline{x};\lambda) \propto L(x;\lambda,\beta,k)g(\lambda)$$

$$f(\underline{x};\lambda) \propto \frac{\lambda^{nk\beta+n-2c_1}\beta^n}{\left[\Gamma^n\left(k+\frac{1}{\beta}\right)\right]} e^{-\lambda^\beta \sum_{i=1}^n x_i^\beta} \prod_{i=1}^n (x_i)^{k\beta}$$

And the corresponding marginal PDF of $\underline{x} = (x_1,\ldots x_n)$ is obtained as:

$$p(\underline{x}) = \int_0^\infty f(\underline{x},\lambda) d\lambda$$

$$p(\underline{x}) = \int_0^{\infty} k\left(\frac{\lambda^{nk\beta+n-2c_1}\beta^n}{\left[\Gamma^n\left(k+\frac{1}{\beta}\right)\right]} e^{-\lambda^\beta \sum_{i=1}^{n} x_i^{\beta}} \prod_{i=1}^{n} (x_i)^{k\beta} d\alpha\right)$$

$$p(\underline{x}) = \frac{k\beta^n}{\left[\Gamma\left(k+\frac{1}{\beta}\right)\right]^n} \prod_{i=1}^{n} x_i^{k\beta} \frac{\Gamma\left(\frac{nk\beta+n-2c_1}{\beta}+1\right)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{nk\beta+n-2c_1}{\beta}+1}}$$

$$p(\underline{x}) = \frac{k\beta^n}{\left[\Gamma\left(k+\frac{1}{\beta}\right)\right]^n} \prod_{i=1}^{n} x_i^{k\beta} \frac{\Gamma^n\left(\frac{(k\beta+1)-2c_1}{\beta}+1\right)}{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{nk\beta+n-2c_1}{\beta}+1}} \qquad (3.7.8)$$

The posterior PDF of λ has the following form

$$\pi_2(\lambda/\underline{x}) = \frac{f(\underline{x},\lambda)}{p(\underline{x})}$$

$$\pi_2(\lambda/\underline{x}) = \frac{e^{-\lambda^\beta \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta+n-2c_1} \left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{nk\beta+n-2c_1}{\beta}+1}}{\Gamma^n\left[\frac{(k\beta+1)-2c_1}{\beta}+1\right]} \qquad (3.7.9)$$

By using a squared error loss function $l_1(\hat{\lambda},\lambda) = c(\hat{\lambda}-\lambda)^2$ for some constant c, the risk function is:

$$R(\hat{\lambda}) = \int_0^{\infty} c(\hat{\lambda}-\lambda)^2 \pi_2(\lambda/\underline{x}) d\lambda$$

$$R(\hat{\lambda}) = \int_0^{\infty} c(\hat{\lambda}-\lambda)^2 \frac{e^{-\lambda^\beta \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta+n-2c_1} \left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{nk\beta+n-2c_1}{\beta}+1}}{\Gamma^n\left[\frac{(k\beta+1)-2c_1}{\beta}+1\right]} d\lambda$$

$$R(\hat{\lambda}) = c\hat{\lambda}^2 + \frac{c\left(\frac{nk\beta+n-2c_1+2}{\beta}\right)\left(\frac{nk\beta+n-2c_1+1}{\beta}\right)}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2c\hat{\lambda}\,\frac{(nk\beta+n)-2c_1+1}{\beta}}{\sum_{i=1}^{n} x_i} \qquad (3.7.10)$$

Now $\frac{\partial R(\hat{\lambda})}{\partial \hat{\lambda}} = 0$, Then the Bayes' estimator is

$$\hat{\lambda}_2 = \frac{n\left(\frac{k\beta+1}{\beta}\right) - 2c_1 + 1}{\sum_{i=1}^{n} x_i}$$

$$\hat{\lambda}_2 = \frac{n\left(k + \frac{1}{\beta}\right) - 2c_1 + 1}{\sum_{i=1}^{n} x_i} \qquad (3.7.11)$$

3.7.1.4 Estimation of Survival function:

By using posterior probability density function, we can found the Survival function, such that

$$\hat{S}_1(\underline{x}) = \int_0^\infty e^{-(\lambda x)^\beta} \pi_2(\lambda/\underline{x}) d\lambda$$

$$\hat{S}_1(\underline{x}) = \int_0^\infty e^{-\lambda^\beta\left(x_i^{\beta} + \sum_{i=1}^{n} x_i^{\beta}\right)} \lambda^{\beta\left(\frac{nk\beta+n-2c_1}{\beta}\right)} \frac{\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{\frac{n\beta+n-2c_1}{\beta}+1}}{\Gamma^n\left(\frac{(k\beta+1)-2c_1}{\beta}+1\right)} d\lambda$$

$$\hat{S}_1(\underline{x}) = \left(\frac{\sum_{i=1}^{n} x_i^{\beta}}{x_i^{\beta} + \sum_{i-1}^{n} x_i^{\beta}}\right)^{\frac{n(k\beta+1)-2c_1}{\beta}+1} \qquad (3.7.12)$$

3.7.2 Parameter estimation under a new loss function.

This section uses a new loss function introduced by Al-Bayyati (2002). Employing this loss function, we obtain Bayes' estimators using Jeffrey's and extension of Jeffrey's prior information.

Al-Bayyati introduced a new loss function of the form:

$$l_A(\hat{\alpha}, \alpha) = \alpha^{c_2} (\hat{\alpha} - \alpha)^2; c_2 \varepsilon R. \qquad (3.7.13)$$

Here, this loss function is used to obtain the estimator of the parameter of the size biased Generalized Gamma distribution.

3.7.2.1 Bayes' estimation of parameter of size biased Generalized Gamma distribution under Jeffrey's prior.

By using the loss function in the form given in (2.7.13), we obtained the following risk function:

$$R(\hat{\lambda}) = \int_0^\infty \lambda^{c_2} (\hat{\lambda} - \lambda)^2 \pi_1(\lambda/\underline{x}) d\lambda$$

$$R(\hat{\lambda}) = \int_0^\infty \lambda^{c_2} (\hat{\lambda} - \lambda)^2 \frac{e^{-\lambda^\beta \sum_{i=1}^n x_i^\beta} \lambda^{nk\beta+n-1} \left(\sum_{i=1}^n x_i^\beta\right)^{\frac{nk\beta+n-1}{\beta}+1}}{\Gamma^n\left[\frac{(k\beta+1)-1}{\beta}+1\right]}$$

$$R(\hat{\lambda}) = \frac{\left(\sum_{i=1}^n x_i^\beta\right)^{\frac{n(k\beta+n)}{\beta}}}{\Gamma^n\left[\frac{(k\beta+1)}{\beta}\right]} \frac{1}{\left(\sum_{i=1}^n x_i^\beta\right)^{\frac{nk\beta+n+c_2}{\beta}}} \left[\hat{\lambda}^2 \Gamma^n\left(\frac{(k\beta+1)+c_2}{\beta}\right) + \frac{\Gamma^n\left(\frac{(k\beta+1)+c_2-1}{\beta}+2\right)}{\left(\sum_{i=1}^n x_i^\beta\right)^{\frac{2}{\beta}}} - 2\hat{\lambda}\frac{\Gamma^n\left(\frac{(k\beta+1)+c_2-1}{\beta}+1\right)}{\left(\sum_{i=1}^n x_i^\beta\right)^{\frac{1}{\beta}}}\right]$$

Now $\frac{\partial R\hat{\lambda}}{\partial \lambda} = 0$, Then the Bayes' estimator is

$$\hat{\lambda}_3 = \frac{\frac{n(k\beta+1)+c_2}{\beta}}{\sum_{i=1}^n x_i} \qquad (3.7.14)$$

Remark 3.1: Replacing $c_2 = 0$ in (3.7.14), the same Bayes' estimator is obtained as in (3.7.5) corresponding to the Jeffrey's prior. By replacing and $c_2 = -2$ in (3.7.14), the Bayes' estimator becomes the estimator under Hartigan's prior (Hartigan (1964)). By replacing $c_2 = 1$ in (3.7.14), thus we get uniform prior.

3.7.2.2 Bayes' estimation of parameter of size biased Generalized Gamma distribution using extension of Jeffrey's prior.

By using the loss function in the form given in (3.7.13), we obtained the following risk function:

$$R(\hat{\lambda}) = \int_0^\infty \lambda^{c_2} (\hat{\lambda} - \lambda)^2 \pi_2(\lambda/\underline{x}) d\lambda$$

$$R(\hat{\lambda}) = \int_0^{\infty} \lambda^{c_2} (\hat{\lambda} - \lambda)^2 \frac{e^{-\lambda^{\beta} \sum_{i=1}^{n} x_i^{\beta}} \lambda^{nk\beta+n-2c_1} \left(\sum_{i=1}^{n} x_i^{\beta} \right)^{\frac{nk\beta+n-2c_1}{\beta}+1}}{\Gamma^n \left[\frac{(k\beta+1)-2c_1}{\beta} + 1 \right]} d\lambda$$

$$R(\hat{\lambda}) = c \frac{\left(\sum_{i=1}^{n} x_i^{\beta} \right)^{\frac{n(k\beta+n)-2c_1}{\beta}+1}}{\Gamma^n \left[\frac{(k\beta+1)-2c_1}{\beta} + 1 \right]} \frac{1}{\left(\sum_{i=1}^{n} x_i^{\beta} \right)^{\frac{n\beta+n+c_2-2c_1}{\beta}}} \left[\hat{\lambda}^2 \frac{\Gamma^n \left(\frac{(k\beta+1)+c_2-2c_1}{\beta} + 1 \right)}{\left(\sum_{i=1}^{n} x_i^{\beta} \right)^{\frac{1}{\beta}}} + \frac{\Gamma^n \left(\frac{(k\beta+1)+c_2-2c_1}{\beta} + 3 \right)}{\left(\sum_{i=1}^{n} x_i^{\beta} \right)^{\frac{3}{\beta}}} - 2\hat{\lambda} \frac{\Gamma^n \left(\frac{(k\beta+1)+c_2-2c_1}{\beta} + 2 \right)}{\left(\sum_{i=1}^{n} x_i^{\beta} \right)^{\frac{2}{\beta}}} \right]$$

Now $\frac{\partial R(\hat{\lambda})}{\partial \hat{\lambda}} = 0$, Then the Bayes' estimator is

$$\hat{\lambda}_4 = \frac{\frac{n(k\beta+1)-2c_1+c_2+1}{\beta}}{\sum_{i=1}^{n} x_i} \qquad (3.7.15)$$

Remark 3.2: Replacing $c_1 = 1/2$ and $c_2 = 0$ in (3.7.15), the same Bayes' estimator is obtained as in (3.7.5) corresponding to the Jeffrey's prior. By replacing $c_1 = 3/2$ and $c_2 = 0$ in (3.7.15), the Bayes' estimator becomes the estimator under Hartigan's prior (Hartigan (1964)). By replacing $c_1 = 0$ and $c_2 = 0$ in (3.7.15), thus we get uniform prior.

3.8 Simulation Study of Size biased Generalized Gamma Distribution:

For description of this manner, we generate different random samples of size 25,50 and 100 from the Size biased Generalized Gamma distribution, a simulation study is carried out 10,000 times for each pairs of (λ, β, k) where $(\lambda = 1.0,1.5,2.0), (\beta = 0.5,1.0,1.5)$ and $(k = 0.5,1.0,1.0)$.

Table 3.1: AIC and BIC criteria of Size-biased Generalized Gamma Distribution.

n	λ	β	k	Shannon's entropy	AIC	BIC
25	1.0	0.5	0.5	26.66715	1339.358	1343.014
	1.5	1.0	1.0	26.34335	1323.168	1326.824
	2.0	1.5	1.0	26.25954	1318.977	1322.634
50	1.0	0.5	0.5	13.53889	1359.889	1365.625
	1.5	1.0	1.0	13.16118	1322.118	1327.854
	2.0	1.5	1.0	13.12484	1318.484	1324.220

100	1.0	0.5	0.5	6.968103	1399.621	1407.436
	1.5	1.0	1.0	6.617217	1329.443	1337.259
	2.0	1.5	1.0	6.539594	1313.918	1321.734

From the above table 3.1, we can conclude that the Size-biased Generalized Gamma Distribution have the smallest AIC and BIC values when sample size is 100 and scale parameter is 2.0 and shape parameters are $\beta = 1.5$, k=1.0.

Table 3.2: AIC and BIC criteria of different subfamilies of Size-biased Generalized Gamma Distribution

n	Distribution	Shannon's entropy	AIC	BIC
25	Size biased Gamma	1.639386	85.9693	88.40705
	Size biased exponential	0.5945349	31.72674	32.94562
	Exponential	0.4812101	26.0605	27.27938
50	Size biased Gamma	1.694657	173.4657	177.2897
	Size biased exponential	0.6134556	63.34556	65.25758
	Exponential	0.378411	39.8411	41.75312
100	Size biased Gamma	2.052823	414.5646	419.7749
	Size biased exponential	0.6356778	129.1356	131.7407
	Exponential	0.221281	46.2562	48.86137

From the above table 3.2, it has been observed that the exponential distribution have the smallest AIC and BIC values as compared to other family of Size-biased Generalized Gamma Distribution, when sample sizes of distributions are 25, 50 and 100.Hence we can concluded that the exponential distribution gives better results and estimates as compared to Size biased exponential and Size biased Gamma distributions.

3.8.1 Estimation of Parameters

In our simulation study, we choose a sample size of n=25, 50 and 100 to represent small, medium and large data set. The scale parameter is estimated for Size biased Generalized Gamma Distribution with Maximum Likelihood and Bayesian using Jeffrey's & extension of Jeffrey's prior methods. For the scale parameter we have considered λ = 1.0, 1.5 and .2.0 .The values of Jeffrey's extension were c_1 = 0.5, 1.0, 1.5 and 2.0. The value for the loss parameter a = -1, 0 and +1.This was iterated 5000 times and the scale parameter for each method was calculated. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes with different values for the Extension of Jeffrey's' prior and the loss functions. The results are presented in tables for different selections of the parameters and c extension of Jeffrey's prior.

Table 3.3: Mean Squared Error for $\hat{\lambda}$ under Jeffrey's prior

n	λ	β	k	λ_{ML}	λ_{sl}	λ_{Nl}		
						C2=-1.0	C2=0	**C2=1.0**
25	1.0	0.5	0.5	0.5811284	0.5811284	0.5715454	0.5811284	0.5907911
	1.5	1.0	1.0	0.5678211	0.5678211	0.553246	0.5678211	0.5834567
	2.0	1.5	1.0	0.4122289	0.4122289	0.3834204	0.4122289	0.4420807
50	1.0	0.5	0.5	0.2566281	0.2566281	0.2366366	0.2566281	0.2774302
	1.5	1.0	1.0	0.3692554	0.3692554	0.3299744	0.3692554	0.3828399
	2.0	1.5	1.0	0.3274210	0.3274210	0.3152655	0.3274210	0.3398064
100	1.0	0.5	0.5	0.1638196	0.1638196	0.1565776	0.1638196	0.1712253
	1.5	1.0	1.0	0.2922827	0.2922827	0.2866838	0.2922827	0.2979358
	2.0	1.5	1.0	0.2910748	0.2910748	0.2854935	0.2910748	0.2967102

Table 3.4: Mean Squared Error for $(\hat{\theta})$ under extension of Jeffrey's prior.

n	λ	β	k	C_1	λ_{ML}	λ_{sl}	λ_{Nl}		
							C2=-1.0	C2=0	C2=1.0
25	1.0	0.5	0.5	0.5	0.5811284	0.5811284	0.5715454	0.5811284	0.5907911
				1.0	0.5811284	0.5715454	0.562042	0.5715454	0.5811284
				1.5	0.5811284	0.562042	0.5526183	0.562042	0.5715454
				2.0	0.5811284	0.5526183	0.5432743	0.5526183	0.562042
	2.0	1.5	1.0	0.5	0.4122289	0.4122289	0.3834204	0.4122289	0.4420807
				1.0	0.4122289	0.3834204	0.4122289	0.3834204	**0.3556554**
				1.5	0.4122289	0.3556554	0.3834204	0.3556554	**0.3289338**
				2.0	0.4122289	0.3289338	0.3032556	0.3289338	0.3556554
50	1.0	0.5	0.5	0.5	0.2566281	0.2566281	0.2366366	0.2566281	0.2774302
				1.0	0.2566281	0.2366366	0.2174556	0.2366366	0.2566281
				1.5	0.2566281	0.2174556	0.1990852	0.2174556	0.2366366
				2.0	0.2566281	0.1990852	0.1815254	0.1990852	0.2174556
	2.0	1.5	1.0	0.5	0.327421	0.327421	0.3152655	0.327421	0.3398064
				1.0	0.327421	0.3152655	0.3033399	0.3152655	0.327421
				1.5	0.327421	0.3033399	0.2916442	0.3033399	0.3152655
				2.0	0.327421	0.2916442	0.2801784	0.2916442	0.3033399
100	1.0	0.5	0.5	0.5	0.1638196	0.1638196	0.1565776	0.1638196	0.1712253
				1.0	0.1638196	0.1565776	0.1494993	0.1565776	0.1638196
				1.5	0.1638196	0.1494993	0.1425847	0.1494993	0.1712253
				2.0	0.1638196	0.1425847	0.1358339	0.1425847	0.1494993
	2.0	1.5	1.0	0.5	0.2910748	0.2910748	0.2854935	0.2910748	0.2967102
				1.0	0.2910748	0.2854935	0.2799663	0.2854935	0.2910748
				1.5	0.2910748	0.2799663	0.274493	0.2799663	0.2854935
				2.0	0.2910748	0.274493	0.2690738	0.274493	0.2799663

ML= Maximum Likelihood, SL=Squared Error Loss Function, NL= New Loss Function,

In table 3.3, Bayes' estimation with New Loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is ± 1. Similarly, in table 3.4, Bayes' estimation with New Loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is ± 1 whether the extension of Jeffrey's prior is 0.5, 1.0, 1.5 or 2.0.

CHAPTER – 4

WEIGHTED AND SIZE – BIASED GENERALIZED BETA DISTRIBUTIONS

4.1. Introduction

Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions, see for example Barlow and Proschan (1975). Many generalizations of beta distributions involving algebraic and exponential functions have been proposed in the literature; see in Johnson et al. (2004) and Gupta and NadarSajah (2004) for detailed accounts. J.B.McDonald (1984) introduced the generalized beta distribution of first kind. It captures the characteristics of income distribution including skewness, peakedness in low-middle range, and long right hand tail. The Generalized Beta distribution of first kind includes several other distributions as special or limiting cases, such as generalized gamma (GGD), Dagum, beta of the second kind (BD2), Sing-Maddala (SM), gamma, Weibull and exponential distributions.

The probability density function (pdf) of the generalized beta distribution of first kind (GBD1) is given by:

$$f(x;a,b,p,q)=\frac{a}{b^{ap}\beta(p,q)}x^{ap-1}\left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1} \quad for\ x>0 \qquad (4.1.1)$$
$$= 0, otherwise$$

Where a,p,q are shape parameters and b is a scale parameter, $\beta(p,q)=\frac{\Gamma p\Gamma q}{\Gamma p+q}$ is a beta function, a,b,p,q and are positive real values.

The cth moment of generalized beta distribution of first kind is given by McDonald (1995):

$$E(X^{c})=\frac{b^{c}\beta\left(p+\frac{c}{a},q\right)}{\beta(p,q)} \qquad (4.1.2)$$

Put c =1 in relation (4.1.2), we have

$$E(X)=\frac{b\beta\left(p+\frac{1}{a},q\right)}{\beta(p,q)} \qquad (4.1.3)$$

In this chapter, we have introduce a new class of weighted Generalized Beta distribution of first kind, Size biased Generalized Beta distribution of first and second kind. The several structural properties of these probability models includes mean, variance, coefficient of variation, mode and harmonic mean has been studied and derived. The estimation of parameters of this new model is obtained by employing the new methods of moments Also, a likelihood ratio test of Weighted and size biased probability distributions are to be conducted. Some important theorems have been derived to estimates the parameters of four parametric weighted and size-biased beta distributions. It was found that the square of the sample coefficient of variation is asymptotically unbiased estimator of square of the population coefficient of variation.

4.4 Derivation of Weighted Generalized Beta Distribution of first kind

A Weighted generalized Beta distribution of first kind (WGBD1) is obtained by applying the weights x^{c}, to the weighted Generalized Beta distribution of first kind.

We have from relation (4.1.1) and (4.1.2), we have

$$f^*(x;\theta) = \frac{x^c f(x;\theta)}{\mu'_c}$$

$$f_w^*(x;a,b,p,q) = \int_0^\infty x^c \frac{a}{b^{ap}} \frac{x^{ap-1}}{\beta(p,q)} \left(1-\left(\frac{x}{b}\right)^a\right)^{q-1} \cdot \frac{\beta(p,q)}{b^c\ \beta\left(p+\frac{c}{a},q\right)}\ dx$$

$$f_w^*(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{ap+c-1} \left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

Where $f_w^*(x;a,b,p,q)$ represents a probability density function. This gives the weighted generalized beta distribution of first kind (WGBD1) as:

$$f_w^*(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{ap+c-1} \left(1-\left(\frac{x}{b}\right)^a\right)^{q-1} \tag{4.2.1}$$

where a,p,q are shape parameters and b is a scale parameter, $\beta(p+\frac{c}{a},q) = \frac{\Gamma\left(p+\frac{c}{a}\right)\Gamma q}{\Gamma\left(p+q+\frac{c}{a}\right)}$

is a beta function, a,b,p,q are positive real values.

4.2.1 Special cases

1. The distribution like the weighted beta distributions of first kind as special case when $a=b=1$, then the probability density function is given as:

$$f_w^*(x;p,q) = \frac{1}{\beta(p+c,q)} x^{p+c-1} (1-x)^{q-1},\ p>0, q>0 \tag{4.2.2}$$

2. The distribution like the Size-biased beta distribution of first kind as particular case when $a=b=c=1$, then the probability density function is given as:

$$f_s^*(x;p,q) = \frac{1}{\beta(p+1,q)} x^p (1-x)^{q-1},\ p>0, q>0 \tag{4.2.3}$$

3. The distribution like the area-biased beta distribution of first kind as particular case when $a=b=1, c=2$, then the probability density function is given as:

$$f_A^{\ *}(x;p,q) = \frac{1}{\beta(p+2,q)} x^{p-1}(1-x)^{q-1}, p>0, q>0 \qquad (4.2.4)$$

$\beta(p+2,q) = \frac{\Gamma p + 2\Gamma q}{\Gamma p + q + 2}$ is a beta function, a,b,p,q are positive real values.

4.3 Structural properties of weighted Generalized beta distribution of first kind

In this section, we derive some structural properties of weighted generalized beta distribution of first kind.

4.3.1 Moments of Weighted Generalized beta distribution of first kind

The rth moment of weighted generalized beta distribution of first kind (4.2.1) about origin is obtained as:

$$\mu_r' = \int_0^\infty x^r f_w(x;a,b,p,q)dx$$

$$\mu_r' = \int_0^\infty x^r \frac{a\, x^{ap+c-1}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \int_0^\infty \frac{a\, x^{ap+r+c-1}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty \left(\frac{x}{b}\right)^{ap+r+c-1}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty \left[\left(\frac{x}{b}\right)^a\right]^{p+\frac{r+c-1}{a}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

Put $\left(\frac{x}{b}\right)^a = t$, *then* $x = bt^{\frac{1}{a}}, dx = \frac{b}{a}t^{\frac{1}{a}-1}dt$

$$\mu_r' = \frac{b^r}{\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty [t]^{p+\frac{r+c}{a}-1}[1-t]^{q-1}dt$$

$$\mu_r' = \frac{b^r}{\beta\left(p+\frac{c}{a},q\right)}\beta\left(p+\frac{r+c}{a},q\right) \tag{4.3.1}$$

Using the equation (4.3.1), the mean and second moment of the WGBD1 is given by

$$\mu_1' = \frac{b\,\beta\left(p+\frac{c+1}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} \tag{4.3.2}$$

$$\mu_2' = \frac{b^2\,\beta\left(p+\frac{c+2}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} \tag{4.3.3}$$

Using the equation (4.3.2) and (4.3.3), the variance of the WGBD1 is given by

$$\mu_2 = b^2\left[\frac{\beta\left(p+\frac{c+2}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)} - \left[\frac{\beta\left(p+\frac{c+1}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)}\right]^2\right] \tag{4.3.4}$$

The Coefficient of variation of Weighted Generalized Beta Distribution of first kind is:

$$CV = \sqrt{\frac{\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)}{\beta^2\left(p+\frac{c+1}{a},q\right)} - 1} \tag{4.3.5}$$

4.3.2 Mode of weighted generalized beta distribution of first kind

The probability distribution of weighted Generalized Beta distribution of first kind is given as:

$$f_w^*(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)}x^{ap+c-1}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

In order to discuss monotonicity of weighted generalized beta distribution of first kind. We take the logarithm of its pdf:

$$\ln(f_w(x:a,b,p,q)) = \ln\left(\frac{a}{b^{ap+c}\beta\left(p+\frac{c}{a},q\right)}\right) + \ln x^{ap+c-1} + \ln\left\{\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}\right\} \qquad (4.3.6)$$

Where C is a constant. Note that

$$\frac{\partial \ln f_w^*(x;a,b,p,q)}{\partial x} = \frac{(ap+c-1)\left(b^a - x^a\right) - (q-1)ax^a}{x\left(b^a - x^a\right)}$$

Where a,p,q are shape parameters and b is a scale parameter, It follows that

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} > 0 \Leftrightarrow x < b\left[\frac{ap+c-1}{a(p+q-1)+c-1}\right]^{\frac{1}{a}}$$

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} < 0 \Leftrightarrow x > b\left[\frac{ap+c-1}{a(p+q-1)+c-1}\right]^{\frac{1}{a}}$$

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} = 0 \Leftrightarrow x = b\left[\frac{ap+c-1}{a(p+q-1)+c-1}\right]^{\frac{1}{a}}$$

The mode of weighted generalized beta distribution of first kind is given as:

$$x = b\left[\frac{ap+c-1}{a(p+q-1)+c-1}\right]^{\frac{1}{a}} \qquad (4.3.7)$$

4.3.3 Harmonic mean of weighted generalized beta distribution of first kind

The probability distribution of weighted Generalized Beta distribution of first kind is given by:

$$f_w^*(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{a+c-1}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

The harmonic mean (H) is obtained as:

$$\frac{1}{H} = \int_0^\infty \frac{1}{x}\ f_w(x;a,b,p,q)dx$$

$$\frac{1}{H}=\int_0^\infty \frac{1}{x}\frac{a\,x^{ap+c-1}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}dx$$

$$\frac{1}{H}=\int_0^\infty \frac{a\,x^{ap+c-2}}{\beta\left(p+\frac{c}{a},q\right)b^{ap+c}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}dx$$

$$\frac{1}{H}=\frac{a}{b^2\beta\left(p+\frac{c}{a},q\right)}\int_0^\infty\left[\left(\frac{x}{b}\right)^a\right]^{p+\frac{c-2}{a}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}dx \qquad (4.3.8)$$

Put $\left(\frac{x}{b}\right)^a=t,\ then\quad x=bt^{\frac{1}{a}},dx=\frac{b}{a}t^{\frac{1}{a}-1}dt$

$$\frac{1}{H}=\frac{a}{b\ \beta\left(p+\frac{c}{a},q\right)}\int_0^\infty[t]^{p+\frac{c-1}{a}-1}[1-t]^{q-1}dt$$

$$\frac{1}{H}=\frac{\beta\left(p+\frac{c-1}{a},q\right)}{b\beta\left(p+\frac{c}{a},q\right)}$$

$$H=\frac{b\beta\left(p+\frac{c}{a},q\right)}{\beta(p+\frac{c-1}{a},q)} \qquad (4.3.9)$$

4. 4 Estimation of parameters of the weighted Generalized Beta Distribution of first kind

In this section, we obtain estimates of the parameters for the weighted Generalized Beta distribution of first kind by employing the new method of moment (MOM) estimator.

New Method of Moment Estimators

Let X_1, X_2, X_3… Xn be an independent sample from the WGBD1. The method of moment estimators are obtained by setting the row moments equal to the sample moments, that is E (X^r) = M_r where is the sample moment M_r corresponding to the E (X^r). The following equations are obtained using the first and second sample moments.

$$\frac{1}{n}\sum_{j=1}^{n} X_j = \frac{b\,\beta\left(p+\frac{c+1}{a}, q\right)}{\beta\left(p+\frac{c}{a}, q\right)} \qquad (4.4.1)$$

$$\frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} = \frac{b^2\,\beta\left(p+\frac{c+2}{a}, q\right)}{\beta\left(p+\frac{c}{a}, q\right)} \qquad (4.4.2)$$

Case 1: When p and q are fixed and a=1, then

$$\frac{\overline{X}}{M_2} = \frac{\Gamma(p+q+c+1)}{b\,\Gamma(p+c+1)}$$

$$\hat{b} = \frac{M_2}{\overline{X}}\left[1+\frac{q}{p+c+1}\right] \qquad (4.4.3)$$

Case 2: When p and b are fixed and a=1, then dividing equation (4.4.1) by (4.4.2), we have:

$$\frac{\overline{X}}{M_2} = \frac{\Gamma(p+q+c+1)}{b\,\Gamma(p+c+1)}$$

$$\hat{q} = (p+c+1)\left[\frac{b\overline{X}}{M_2}-1\right] \qquad (4.4.4)$$

Case 3: When b and q are fixed and a=1, then dividing equation (4.4.1) by (4.4.2), we have:

$$\frac{\overline{X}}{M_2} = \frac{\Gamma(p+q+c+1)}{b\,\Gamma(p+c+1)}$$

$$\hat{p} = \frac{qM_2}{b\,\overline{X}-M_2}-(c+1) \qquad (4.4.5)$$

Case 4: When p and q are fixed, b=1 then we can calculate the value of $\hat{a}$ estimator by numerical methods.

4.5 New moment estimation method of Weighted Generalized Beta distribution of first kind

Although Prentice (1974) have presented a procedure to obtain the three parameters of the generalized gamma distribution, his procedure still quit complicated. In this research, we propose a simple procedure to obtained three estimators by using its characterization and moment estimation approach. Note that Hwang .T and Huang. P (2006) have obtained more general characterizations with the independence of sample coefficient of variation V_n with sample mean $\overline{X}_n$ as one of its special cases when random samples are drawn from the generalized gamma distribution. Their characterization is used to derive the expectation and the variance of $V_n^{\ 2}$ and then the new estimators for the three parameters of size-biased generalized gamma distribution are proposed. For deriving new moment estimators of three parameters of the weighted generalized Beta distribution of first kind, we need the following theorem obtained by using the similar approach of Ahmed *et al* (Theorems of 2013).

Theorem 4.5.1: Let $n \geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be a n positive identical independently distributed random samples drawn from a population having a weighted generalized Beta distribution of first kind

$$f_w^{\ *}(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{a+c-1}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

Then $E(S_n^{\ 2}) = \dfrac{b^2\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)-\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{\beta^2\left(p+\frac{c}{a},q\right)}$

Proof: Here, $E(X)^r = \dfrac{b^r}{\beta\left(p+\frac{c}{a},q\right)}\beta\left(p+\frac{r+c}{a},q\right)$

$$E(\overline{X}_n) = \frac{b\,\beta\left(p+\frac{c+1}{a},q\right)}{\beta\left(p+\frac{c}{a},q\right)}$$

$$E(X_n{}^2)=\frac{b^2\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)+(n-1)\,\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{n\beta^2\left(p+\frac{c}{a},q\right)}$$

And

$$E(S_n{}^2)=\frac{b^2\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)-\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{\beta^2\left(p+\frac{c}{a},q\right)} \qquad (4.5.1)$$

Where $\overline{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Theorem 4.5.2: Let $n\geq 3$ and let $X_1,X_2,X_3...X_n$ be a n positive identical independently distributed random samples drawn from a population having a weighted generalized Beta distribution of first kind

$$f_w{}^*(x;a,b,p,q)=\frac{a}{b^{ap+c}\;\beta\left(p+\frac{c}{a},q\right)}\,x^{a+c-1}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

Then $E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right)=\frac{n\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)-\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)+(n-1)\,\beta^2\left(p+\frac{c+1}{a},q\right)\right]}$

Where $\overline{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Proof: By theorem 4.5.1, we have

$$E(S_n{}^2)=E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}.\overline{X}_n{}^2\right)=E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right)..E\left(\overline{X}_n{}^2\right)$$

And hence $E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right)=\frac{E\left(S_n{}^2\right)}{E\left(\overline{X}_n{}^2\right)}$

Applying theorem 4.5.1 to the above identity yields that

$$E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right)=\frac{n\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)-\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)+(n-1)\,\beta^2\left(p+\frac{c+1}{a},q\right)\right]} \qquad (4.5.2)$$

Thus 4.5.2 is established.

Theorem 4.5.3: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a weighted generalized Beta distribution of first kind.

$$f_w^{\,*}(x;a,b,p,q) = \frac{a}{b^{ap+c}\ \beta\left(p+\frac{c}{a},q\right)} x^{a+c-1} \left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1}$$

$$E(S_n^{\,2}) = \frac{b^2\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)-\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{\beta^2\left(p+\frac{c}{a},q\right)}$$

$$E\left(\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}\right) = \frac{n\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)-\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)+(n-1)\beta^2\left(p+\frac{c+1}{a},q\right)\right]}$$

Furthermore, if WGBD1 distribution, we have

$$\frac{\sigma^2}{\mu^2} = \frac{\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)}{\beta^2\left(p+\frac{c+1}{a},q\right)} - 1 \qquad (4.5.3)$$

And it can be show that

$$E\left(\frac{S_n^{\,2}}{\overline{X}_n}\right) \to \frac{\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)}{\beta^2\left(p+\frac{c+1}{a},q\right)} - 1 \qquad (4.5.4)$$

Comparing above two equations, we have

Note that $E\left(\frac{S_n^{\,2}}{\overline{X}_n}\right) \to \frac{\sigma^2}{\mu^2}$ as $n \to \infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation.

4.6. Test for weighted generalized beta distribution of first kind

Let X_1, X_2, X_3… Xn be random samples can be drawn from generalized beta distribution of first kind or weighted generalized beta distribution of first kind. We test the hypothesis

$H_o : f(x) = f(x;a,b,p,q)$ vs $H_1 : f(x) = f_w^{\ *}(x;a,b,p,q)$.

To test whether the random sample of size n comes from the generalized beta distribution of first kind or weighted generalized beta distribution of first kind the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left(\frac{f_w^{\ *}(x;a,b,p,q)}{f(a,b,p,q)}\right)$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left(\frac{\frac{ax^{ap+c-1}}{b^{ap+c}\beta\left(p+\frac{c}{a},q\right)}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}}{\frac{ax^{ap-1}}{b^{ap}\beta(p,q)}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1}}\right)$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\frac{\beta(p,q)}{b^c\beta\left(p+\frac{c}{a},q\right)}.x^c$$

$$\Delta = \left[\frac{\beta(p,q)}{b^c\beta\left(p+\frac{c}{a},q\right)}\right]^n \prod_{i=1}^{n} x_i^{\ c} \qquad (4.6.1)$$

We reject the null hypothesis.

$$\left[\frac{\beta(p,q)}{b^c\Gamma\left(p+\frac{c}{a},q\right)}\right]^n \prod_{i=1}^{n} x_i^{\ c} > k$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i^{\ c} > k^*, where\, k^* = k\left[\frac{b^c\beta\left(p+\frac{c}{a},q\right)}{\beta(p,q)}\right]^n > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution. Also, we can reject the reject the null hypothesis, when probability value s given by:

$P(\Delta^* > \lambda^*), Where\ \lambda^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the test statistic Δ^*.

4.7 Size-Biased Generalized Beta Distribution of first kind:

A size biased generalized beta distribution of first kind (SBGBD1) is obtained by applying the weights x^c, where c =1 to the weighted Generalized beta distribution of first kind.

We have from relation (4.1.1) and (4.1.2),

$$f^*(x;a,b,p,q) = \frac{x\, f(x;a,b,p,q)}{\mu}$$

$$f^*(x;a,b,p,q) = \int_0^\infty x \frac{a \quad x^{ap-1}}{b^{ap}\ \beta(p,q)} \left(1-\left(\frac{x}{b}\right)^a\right)^{q-1} . \frac{\beta(p,q)}{b\beta\left(p+\frac{1}{a},q\right)}\ dx$$

$$f^*(x;a,b,p,q) = \int_0^\infty \frac{a \quad x^{ap}}{b^{ap+1}\ \beta\left(p+\frac{1}{a},q\right)} \left(1-\left(\frac{x}{b}\right)^a\right)^{q-1} dx$$

$$f^*(x;a,b,p,q) = \frac{a}{b^{ap+1}\ \beta\left(p+\frac{1}{a},q\right)} x^{ap} \left(1-\left(\frac{x}{b}\right)^a\right)^{q-1}$$

Where $f^*(x;a,b,p,q)$ represents a probability density function. This gives the size – biased generalized beta distribution of first kind (SBGBD1) as:

$$f^*(x;a,b,p,q) = \frac{a}{b^{ap+1}\,\beta\left(p+\frac{1}{a},q\right)} x^{ap}\left(1-\left(\frac{x}{b}\right)^a\right)^{q-1} \qquad (4.7.1)$$

Where a, p, q are shape parameters and b is a scale parameter, $\beta(p+1,q) = \frac{\Gamma p+1\Gamma q}{\Gamma p+q+1}$ is a beta function, a,b,p,q are positive real values.

Special case: The distribution like the Size-biased beta distribution of first kind as special case ($a = b = 1$), then the probability density function is given as:

$$f^*(x;p,q) = \frac{1}{\beta(p+1,q)} x^p (1-x)^{q-1}, p > 0, q > 0$$

$\beta(p+1,q) = \frac{\Gamma p+1\Gamma q}{\Gamma p+q+1}$ is a beta function, a,b,p,q are positive real values.

4.9 Structural properties of Size- biased Generalized beta distribution of first kind:

In this section, we derive some structural properties of Size-biased generalized beta distribution of first kind.

4.8.1 Moments of Size- biased Generalized beta distribution of first kind:

The rth moment of Size biased generalized beta distribution of first kind (4.7.1) about origin is obtained as:

$$\mu_r' = \int_0^\infty x^r f(x;a,b,p,q)dx$$

$$\mu_r' = \int_0^\infty x^r \frac{a\,x^{ap}}{\beta\left(p+\frac{1}{a},q\right)b^{ap+1}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \int_0^\infty \frac{a\,x^{ap+r}}{\beta\left(p+\frac{1}{a},q\right)b^{ap+1}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{1}{a},q\right)}\int_0^\infty \left(\frac{x}{b}\right)^{ap+r}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

$$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{1}{a},q\right)}\int_0^\infty \left[\left(\frac{x}{b}\right)^{a}\right]^{p+\frac{r}{a}}\left[1-\left(\frac{x}{b}\right)^a\right]^{q-1} dx$$

Put $\left(\frac{x}{b}\right)^a = t,\ then\ \ x = bt^{\frac{1}{a}}, dx = \frac{b}{a}t^{\frac{1}{a}-1}dt$

$$\Rightarrow \mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{1}{a},q\right)}\int_0^\infty [t]^{p+\frac{r}{a}}[1-t]^{q-1}\frac{b}{a}t^{\frac{1}{a}-1}dt$$

$$\mu_r' = \frac{b^{r}}{\beta\left(p+\frac{1}{a},q\right)}\int_0^\infty [t]^{p+\frac{r}{a}+\frac{t}{a}-1}[1-t]^{q-1}dt$$

$$\mu_r' = \frac{b^{r}}{\beta\left(p+\frac{1}{a},q\right)}\beta\left(p+\frac{r}{a}+\frac{1}{a},q\right) \tag{4.8.1}$$

Using the equation (4.8.1), the mean and variance of the SBGBD1 is given by

$$\mu_1' = \frac{b\,\beta\left(p+\frac{2}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)} \tag{4.8.2}$$

$$\mu_2 = b^2\left[\frac{\beta\left(p+\frac{3}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)} - \left[\frac{\beta\left(p+\frac{2}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)}\right]^2\right] \tag{4.8.3}$$

The Coefficient of variation of Size- biased Generalized Beta Distribution of first kind is given as:

$$CV = \frac{\sqrt{V(X)}}{E(X)} = \sqrt{\frac{\beta\left(p+\frac{3}{a},.q\right)\beta\left(p+\frac{1}{a},q\right)}{\beta^2\left(p+\frac{2}{a},q\right)} - 1} \tag{4.8.4}$$

Where, the first four moments about origin are given as:

$$\mu_1' = \frac{b\,\beta\left(p+\frac{2}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)}$$

$$\mu_2' = \frac{b^2\,\beta\left(p+\frac{3}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)}$$

$$\mu_2 = b^2\left[\frac{\beta\left(p+\frac{3}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)} - \left[\frac{\beta\left(p+\frac{2}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)}\right]^2\right]$$

$$\mu_3' = \frac{b^3\,\beta\left(p+\frac{4}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)}$$

$$\mu_4' = \frac{b^4\,\beta\left(p+\frac{5}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)}$$

4.8.2 Mode of Size-biased generalized beta distribution of first kind

The probability distribution of Size-biased Generalized Beta distribution of first kind is given as:

$$f^{*}(x;a,b,p,q)=\frac{a}{b^{ap+1}\ \beta\left(p+\frac{1}{a},q\right)}x^{ap}\left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1}$$

In order to discuss monotonicity of size-biased generalized beta distribution of first kind. We take the logarithm of its pdf:

$$\ln(f(x:a,b,p,q))=\ln\left(\frac{a}{b^{ap+1}\beta\left(p+\frac{1}{a},q\right)}\right)+\ln x^{ap}+\ln\left\{\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}\right\} \qquad (4.8.5)$$

Where C is a constant. Note that

$$\frac{\partial\ln f^{*}(x;a,b,p,q)}{\partial x}=\frac{ap\left(b^{a}-x^{a}\right)-(q-1)ab^{a}x^{a}}{x\left(b^{a}-x^{a}\right)}$$

Where a,p,q are shape parameters and b is a scale parameter, It follows that

$$\frac{\partial\ln f^{*}(x;a,b,p,q)}{\partial x}>0\Leftrightarrow x<b\left[\frac{ap}{a(p+q-1)}\right]^{\frac{1}{a}}$$

$$\frac{\partial\ln f^{*}(x;a,b,p,q)}{\partial x}<0\Leftrightarrow x>b\left[\frac{ap}{a(p+q-1)}\right]^{\frac{1}{a}}$$

$$\frac{\partial\ln f^{*}(x;a,b,p,q)}{\partial x}=0\Leftrightarrow x=b\left[\frac{ap}{a(p+q-1)}\right]^{\frac{1}{a}}$$

The mode of size-biased generalized beta distribution of first kind is:

$$x=b\left[\frac{ap}{a(p+q-1)}\right]^{\frac{1}{a}} \qquad (4.8.6)$$

4.8.3 Harmonic mean of Size-biased generalized beta distribution of first kind

The probability distribution of Size-biased Generalized Beta distribution of first kind is given as:

$$f^{*}(x;a,b,p,q)=\frac{a}{b^{ap+1}\ \beta\left(p+\frac{1}{a},q\right)}x^{ap}\left(1-\left(\frac{x}{b}\right)^{a}\right)^{q-1}$$

The harmonic mean (H) is obtained as:

$$\frac{1}{H}=\int_{0}^{\infty}\frac{1}{x}\ f(x;a,b,p,q)dx$$

$$\frac{1}{H}=\int_{0}^{\infty}\frac{1}{x}\ \frac{a\,x^{ap}}{b^{ap+1}\beta\left(p+\frac{1}{a},q\right)}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}dx$$

$$\frac{1}{H}=\frac{a}{b^{2}\ \beta\left(p+\frac{1}{a},q\right)}\int_{0}^{\infty}\left(\frac{x}{b}\right)^{ap-1}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}dx$$

$$\frac{1}{H}=\frac{a}{b^{2}\ \beta\left(p+\frac{1}{a},q\right)}\int_{0}^{\infty}\left[\left(\frac{x}{b}\right)^{a}\right]^{p-\frac{1}{a}}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}dx$$

$$\text{Put}\left(\frac{x}{b}\right)^{a}=t,\ \ then\ \ \ x=bt^{\frac{1}{a}},dx=\frac{b}{a}t^{\frac{1}{a}-1}dt$$

$$\frac{1}{H}=\frac{a}{b^{2}\ \beta\left(p+\frac{1}{a},q\right)}\int_{0}^{\infty}[t]^{p-\frac{1}{a}}\ [1+t]^{q-1}\frac{b}{a}t^{\frac{1}{a}-1}dt$$

$$\frac{1}{H}=\frac{1}{b\,\beta\left(p+\frac{1}{a},q\right)}\int_{0}^{\infty}[t]^{p-1}\ [1+t]^{q-1}dt$$

$$\frac{1}{H}=\frac{\beta(p,q)}{b\,\beta\left(p+\frac{1}{a},q\right)}$$

$$H=\frac{b\beta\left(p+\frac{1}{a}\right)\beta(q)}{\beta(p,q)} \qquad (4.8.7)$$

4.9. Estimation of parameters in the size-biased Generalized Beta Distribution of first kind

In this section, we obtain estimates of the parameters for the Size-biased Generalized Beta distribution of first kind by employing the new method of moment (MOM) estimator.

4.9.1 New Method of Moment Estimators

Let X_1, X_2, X_3… Xn be an independent sample from the SBGBD1with weight c=1. The method of moment estimators are obtained by setting the row moments equal to the sample moments, that is $E(X^r) = M_r$ where is the sample moment M_r corresponding to the $E(X^r)$. The following equations are obtained using the first and second sample moments.

$$\frac{1}{n}\sum_{j=1}^{n} X_j = \frac{b\,\beta\left(p+\frac{2}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)} \tag{4.9.1}$$

$$\frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} = \frac{b^2\,\beta\left(p+\frac{3}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)} \tag{4.9.2}$$

Case 1: When p and q are fixed and a=1, then

$$\overline{X} = \frac{b\,\beta\left(p+\frac{2}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)}$$

$$\overline{X} = \frac{b\Gamma(p+2)\Gamma(p+q+1)}{\Gamma(p+q+2)\Gamma(p+1)}$$

$$\hat{b} = \overline{X}\,\frac{(p+q+1)}{p+1} \tag{4.9.3}$$

Case 2: When p and b are fixed and a=1, then dividing equation (4.9.1) by (4.9.2), we have:

$$\frac{\overline{X}}{M_2}=\frac{\Gamma(p+2)\Gamma(p+q+3)}{b\Gamma(p+q+2)\Gamma(p+3)}$$

$$\frac{\overline{X}}{M_2}=\frac{p+q+2}{b(p+2)}$$

$$\hat{q}=(p+2)\left[\frac{b\overline{X}}{M_2}-1\right] \qquad (4.9.4)$$

Case 3: When b and q are fixed and a=1, then dividing equation (4.9.1) by (4.9.2), we have:

$$\frac{\overline{X}}{M_2}=\frac{p+q+2}{b(p+2)}$$

$$\hat{p}=\frac{qM_2}{b\,\overline{X}-M_2}-2 \qquad (4.9.5)$$

Case 4: When p and q are fixed, b=1 then we can calculate the value of $\hat{a}$ estimator by numerical methods.

4.10 Test for size-biased generalized beta distribution of second kind

Let X_1, X_2, $X_3 \ldots X_n$ be a random samples drawn from generalized beta distribution of first kind or size-biased generalized beta distribution of first kind. We test the hypothesis

$$H_o : f(x) = f(x,a,b,p,q) \text{ vs } H_1 : f(x) = f_s^{\,*}(a,b,p,q).$$

To test whether the random sample of size n comes from the generalized beta distribution of first kind or size-biased generalized beta distribution of first kind the following test statistic is used.

$$\Delta=\frac{L_1}{L_0}=\prod_{i=1}^{n}\frac{f^{*}{}_{s}(x;a,b,p,q)}{f(a,b,p,q)}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\dfrac{ax^{ap}}{b^{ap+1}\beta\left(p+\dfrac{1}{a},q\right)}\left[1-\left(\dfrac{x}{b}\right)^{a}\right]^{q-1}}{\dfrac{ax^{ap-1}}{b^{ap}\beta(p,q)}\left[1-\left(\dfrac{x}{b}\right)^{a}\right]^{q-1}}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\beta(p,q)}{b\beta\left(p+\dfrac{1}{a},q\right)}.x$$

$$\Delta = \left[\frac{\beta(p,q)}{b\beta\left(p+\dfrac{1}{a},q\right)}\right]^{n} \prod_{i=1}^{n} x_i \qquad (4.10.1)$$

We reject the null hypothesis

$$\left[\frac{\beta(p,q)}{b\Gamma\left(p+\dfrac{1}{a},q\right)}\right]^{n} \prod_{i=1}^{n} x_i > k$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, where\, k^* = k\left[\frac{b\beta\left(p+\dfrac{1}{a},q\right)}{\beta(p,q)}\right]^{n} > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution.

4.11 Generalized Beta Distribution of Second kind.

The probability density function (pdf) of the generalized beta distribution of second kind (GBD2) is given by:

$$f(x;a,b,p,q) = \frac{ax^{ap-1}}{b^{ap}\beta(p.q)\left[1+\left(\dfrac{x}{b}\right)^{a}\right]^{p+q}} \qquad for\ x > 0 \qquad (4.11.1)$$

$$= \ 0, otherwise$$

Where a,p,q are shape parameters and b is a scale parameter, $\beta(p,q)=\frac{\Gamma p\Gamma q}{\Gamma p+q}$ is a beta function, a,b,p,q and are positive real values.

The rth moment of generalized beta distribution of second kind is given as:

$$E(X^r)=\frac{b^r\Gamma\left(p+\frac{r}{a}\right)\Gamma\left(q-\frac{r}{a}\right)}{\Gamma p\ \Gamma q} \tag{4.11.2}$$

Put r =1 in relation (4.11.2), we have

$$E(X)=\frac{b\ \Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)}{\Gamma p\ \Gamma q} \tag{4.11.3}$$

4.12 Size Biased Generalized Beta Distribution of first kind

A size biased generalized beta distribution of second kind (SBGBD2) is obtained by applying the weights x^c, where c =1 to the weighted Generalized beta distribution of second kind.

We have from relation (4.11.1) and (4.11.3)

$$f^*(x;a,b,p,q)=\frac{x\,f(x;a,b,p,q)}{\mu}$$

$$f^*(x;a,b,p,q)=\int_0^\infty x\frac{a\quad x^{ap-1}}{b^{ap}\ \beta(p,q)\left[\left(1+\frac{x}{b}\right)^a\right]^{p+q}}\cdot\frac{\Gamma p\,\Gamma q}{b\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)}\,dx$$

$$f^*(x;a,b,p,q)=\frac{a\quad x^{ap}}{b^{ap+1}\ \beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}}$$

Where $f^*(x;a,b,p,q)$ represents a probability density function. This gives the size –biased generalized beta distribution of second kind (SBGBD2) as:

$$f^*(x;a,b,p,q)=\frac{a\quad x^{ap}}{b^{ap+1}\ \beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}} \tag{4.12.1}$$

where a,p,q are shape parameters and b is a scale parameter.

$$\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)=\frac{\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)}{\Gamma(p+q)}$$ is a beta function, a,b,p,q are positive real values.

Special case: The distribution like the Size-biased beta distribution of second kind as special case ($a=b=1$), then the probability density function is given as:

$$f^*(x;p,q)=\frac{x^p}{\beta(p+1,q-1)[1+x]^{p+q}} \quad ;p,\ q>0$$

4.13 Structural properties of Size- biased Generalized beta distribution of second kind

In this section, we derive some structural properties of Size-biased generalized beta distribution of kind.

4.13.1 Moments of Size- biased generalized beta distribution of second kind:

The rth moment of Size biased generalized beta distribution of second kind (4.12.1) about origin is obtained as:

$$\mu_r' = \int_0^\infty x^r f(x;a,b,p,q)dx$$

$$\mu_r' = \int_0^\infty x^r \frac{a\,x^{ap}}{\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)b^{ap+1}\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}}dx$$

$$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\int_0^\infty\left[\left(\frac{x}{b}\right)^a\right]^{p+\frac{r}{a}}\left[1+\left(\frac{x}{b}\right)^a\right]^{-(p+q)}dx$$

Put $\left(\frac{x}{b}\right)^a=t,\ then\ \ x=bt^{\frac{1}{a}},dx=\frac{b}{a}t^{\frac{1}{a}-1}dt$

and $$\mu_r' = \frac{ab^{r-1}}{\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\int_0^\infty[t]^{p+\frac{r}{a}}[1+t]^{-(p+q)}\frac{b}{a}t^{\frac{1}{a}-1}dt$$

$$\mu_r' = \frac{b^r}{\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}\beta\left(p+\frac{r}{a}+\frac{1}{a}, q-\frac{r}{a}-\frac{1}{a}\right) \quad (4.13.1)$$

Using the equation (4.13.1), the mean and variance of the SBGBD2 is given by

$$\mu_1' = \frac{b\beta\left(p+\frac{2}{a}, q-\frac{2}{a}\right)}{\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)} \quad (4.13.2)$$

$$\mu_2 = b^2\left[\frac{\beta\left(p+\frac{3}{a}, q-\frac{3}{a}\right)}{\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)} - \left[\frac{\beta\left(p+\frac{2}{a}, q-\frac{2}{a}\right)}{\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}\right]^2\right] \quad (4.13.3)$$

The Coefficient of variation of Size- biased Generalized Beta Distribution of second kind is:

$$CV = \frac{\sqrt{V(X)}}{E(X)} = \sqrt{\frac{\beta\left(p+\frac{3}{a}.q-\frac{3}{a}\right)\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}{\beta^2\left(p+\frac{2}{a}, q-\frac{2}{a}\right)} - 1} \quad (4.13.4)$$

4.13.2 Mode of Size-biased generalized beta distribution of first second

The probability distribution of Size-biased Generalized Beta distribution of second kind is given by:

$$f^*(x;a,b,p,q) = \frac{a \quad x^{ap}}{b^{ap+1}\ \beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}}$$

In order to discuss monotonicity of size-biased generalized beta distribution of second kind. We take the logarithm of its pdf:

$$\ln(f(x:a,b,p,q)) = \ln\left(\frac{a}{b^{ap+1}\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}\right) + \ln x^{ap} + \ln\left\{\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}\right\} \quad (4.13.5)$$

Where C is a constant. Note that

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} = \frac{a\,p\,b^a - a\,q\,x^a}{x(b^a + x^a)}$$

Where a, p, q are shape parameters and b is a scale parameter, It follows that

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} > 0 \Leftrightarrow x < b\left(\frac{p}{q}\right)^{\frac{1}{a}}$$

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} < 0 \Leftrightarrow x > b\left(\frac{p}{q}\right)^{\frac{1}{a}}$$

$$\frac{\partial \ln f^*(x;a,b,p,q)}{\partial x} = 0 \Leftrightarrow x = b\left(\frac{p}{q}\right)^{\frac{1}{a}}$$

Therefore, the mode of size-biased generalized beta distribution of second kind is given as:

$$x_0 = b\left(\frac{p}{q}\right)^{\frac{1}{a}} \tag{4.13.6}$$

4.13.3 Harmonic mean of Size-biased generalized beta distribution of second kind

The probability distribution of Size-biased Generalized Beta distribution of second kind is given as:

$$f^*(x;a,b,p,q) = \frac{a \quad x^{ap}}{b^{ap+1}\ \beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}}$$

The harmonic mean (H) is obtained as:

$$\frac{1}{H} = \int_0^\infty \frac{1}{x}\ f(x;a,b,p,q)dx$$

$$\frac{1}{H} = \int_0^\infty \frac{1}{x}\ \frac{a\,x^{ap}}{b^{ap+1}\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}\left[1+\left(\frac{x}{b}\right)^a\right]^{-(p+q)} dx$$

$$\frac{1}{H} = \frac{a}{b^2\ \beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}\int_0^\infty \left(\frac{x}{b}\right)^{ap-1}\left[1+\left(\frac{x}{b}\right)^a\right]^{-(p+q)} dx$$

$$\frac{1}{H}=\frac{a}{b^2\,\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\int_0^\infty\left[\left(\frac{x}{b}\right)^a\right]^{p-\frac{1}{a}}\left[1+\left(\frac{x}{b}\right)^a\right]^{-(p+q)}dx$$

Put $\left(\frac{x}{b}\right)^a=t,\ then\quad x=bt^{\frac{1}{a}},dx=\frac{b}{a}t^{\frac{1}{a}-1}dt$

$$\frac{1}{H}=\frac{a}{b^2\,\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\int_0^\infty[t]^{p-\frac{1}{a}}\,[1+t]^{-(p+q)}\,\frac{b}{a}t^{\frac{1}{a}-1}dt$$

$$\frac{1}{H}=\frac{1}{b\,\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\int_0^\infty[t]^{p-1}\,[1+t]^{-(p+q)}dt$$

$$\frac{1}{H}=\frac{\beta(p,q)}{b\,\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}$$

$$H=\frac{b\beta\left(p+\frac{1}{a}\right)\beta\left(q-\frac{1}{a}\right)}{\beta(p,q)}\tag{4.13.7}$$

4.14 Estimation of parameters in the size-biased Generalized Beta Distribution of second kind

In this section, we obtain estimates of the parameters for the Size-biased Generalized Beta distribution of second kind by employing the new method of moment (MOM) estimator.

4.14.1 New Method of Moment Estimators

Let X_1, X_2, X_3… Xn be an independent random samples from the SBGBD2 with weight c=1. The method of moment estimators are obtained by setting the raw moments equal to the sample moments, that is $E(X^r)=M_r$ where is the sample moment M_r corresponding to the .$E(X^r)$ The following equations are obtained using the first and second sample moments.

$$\frac{1}{n}\sum_{j=1}^{n}X_j=\frac{b\,\beta\left(p+\frac{2}{a},q-\frac{2}{a}\right)}{\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\tag{4.14.1}$$

$$\frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} = \frac{b\,\beta\left(p+\frac{3}{a}, q-\frac{3}{a}\right)}{\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)} \qquad (4.14.2)$$

Case 1: When p and q are fixed and a=1, then

$$\bar{X} = \frac{b\,\beta\left(p+\frac{2}{a}, q-\frac{2}{a}\right)}{\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)}$$

$$\hat{b} = \bar{X}\,\frac{(q-2)}{p+1} \qquad (4.14.3)$$

Case 2: When p and b are fixed and a=1, then dividing equation (4.14.2) by (4.14.1), we have:

$$\frac{\bar{X}}{M_2} = \frac{\Gamma(p+2)\Gamma(q-2)}{b\,\Gamma(p+3)\Gamma(q-3)}$$

$$\hat{q} = (p+2)b\,\frac{\bar{X}}{M_2} + 3 \qquad (4.14.4)$$

Case 3: When b and q are fixed and a=1, then dividing equation (4.14.1) by (4.14.2), we have:

$$\frac{\bar{X}}{M_2} = \frac{\Gamma(p+2)\Gamma(q-2)}{b\,\Gamma(p+3)\Gamma(q-3)}$$

$$\frac{\bar{X}}{M_2} = \frac{q-3}{b(p+2)}$$

$$\hat{p} = \frac{M_2(q-3)}{b\bar{X}} - 2 \qquad (4.14.5)$$

Case 4: When p and q are fixed, b =1 then we can calculate the value of $\hat{a}$ estimator by numerical methods.

4.14.2: New method of estimation of Size- biased generalized Beta distribution of second kind

This section is based on a new moment estimation method of parameters of SBG family using its characterization. The characterization is used to derive the expectation and the variance of $V_n^{\ 2}$ and then the new estimators for the four parameters of size-biased generalized Beta distribution of second kind are proposed. For deriving new moment estimators of three parameters of the size-biased generalized Beta distribution of second kind, we need the following theorem obtained by using the similar approach of Ahmed *et al* (Theorems of 2013).

Theorem 4.14.2.1: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be an n positive identical independently distributed random samples drawn from a population having a Size- biased generalized Beta distribution of second kind.

$$f^{\,*}(x; a, b, p, q) = \frac{a \quad x^{ap}}{b^{ap+1} \ \beta\left(p + \frac{1}{a}, q - \frac{1}{a}\right)\left[1 + \left(\frac{x}{b}\right)^{a}\right]^{p+q}}$$

Then $E(S_n^{\ 2}) = \dfrac{b^2\left[\beta\left(p + \frac{3}{a}, q - \frac{3}{a}\right)\beta\left(p + \frac{1}{a}, q - \frac{1}{a}\right) - \beta^2\left(p + \frac{2}{a}, q - \frac{2}{a}\right)\right]}{\beta^2\left(p + \frac{1}{a}, q - \frac{1}{a}\right)}$

Proof: Here, $E(X)^r = \dfrac{b^r}{\beta\left(p + \frac{1}{a}, q - \frac{1}{a}\right)}\beta\left(p + \frac{r}{a} + \frac{1}{a}, q - \frac{r}{a} - \frac{1}{a}\right)$

$$E(\overline{X}_n) = \frac{b\,\beta\left(p + \frac{2}{a}, q - \frac{2}{a}\right)}{\beta\left(p + \frac{1}{a}, q - \frac{1}{a}\right)}$$

$$E(X_n^{\ 2}) = \frac{b^2\left[\beta\left(p + \frac{3}{a}, q - \frac{3}{a}\right)\beta\left(p + \frac{1}{a}, q - \frac{1}{a}\right) - (n-1)\,\beta^2\left(p + \frac{2}{a}, q - \frac{2}{a}\right)\right]}{n\beta^2\left(p + \frac{1}{a}, q - \frac{1}{a}\right)}$$

And $E(S_n^{\ 2}) = \dfrac{b^2\left[\beta\left(p + \frac{3}{a}, q - \frac{3}{a}\right)\beta\left(p + \frac{1}{a}, q - \frac{1}{a}\right) - \beta^2\left(p + \frac{2}{a}, q - \frac{2}{a}\right)\right]}{\beta^2\left(p + \frac{1}{a}, q - \frac{1}{a}\right)}$ (4.14.6)

Where $\overline{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Theorem 4.14.2.2: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a Size- biased generalized Beta distribution of second kind

$$f^*(x;a,b,p,q) = \frac{a \quad x^{ap}}{b^{ap+1} \ \beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right)\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}}$$

Then $$E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right) = \frac{n\left[\beta\left(p+\frac{3}{a}, q-\frac{3}{a}\right)\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right) - \beta^2\left(p+\frac{2}{a}, q-\frac{2}{a}\right)\right]}{\left[\beta\left(p+\frac{3}{a}, q-\frac{3}{a}\right)\beta\left(p+\frac{1}{a}, q-\frac{1}{a}\right) - (n-1)\beta^2\left(p+\frac{2}{a}, q-\frac{2}{a}\right)\right]}$$

Where $\overline{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Proof: By theorem 4.14.2.1, we have

$$E(S_n{}^2) = E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}.\overline{X}_n{}^2\right) = E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right)..E\left(\overline{X}_n{}^2\right)$$

And hence $$E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right) = \frac{E\left(S_n{}^2\right)}{E\left(\overline{X}_n{}^2\right)}$$

Applying theorem 4.14.2.2 to the above identity yields that

$$E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right) = \frac{n\left[\beta\left(p+\frac{c+2}{a}, q\right)\beta\left(p+\frac{c}{a}, q\right) - \beta^2\left(p+\frac{c+1}{a}, q\right)\right]}{\left[\beta\left(p+\frac{c+2}{a}, q\right)\beta\left(p+\frac{c}{a}, q\right) + (n-1)\beta^2\left(p+\frac{c+1}{a}, q\right)\right]} \qquad (4.14.7)$$

Thus 4.14.2.2 is established.

Theorem 4.14.2.3: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a Size- biased generalized Beta distribution of second kind

$$f^*(x;a,b,p,q)=\frac{a\ \ x^{ap}}{b^{ap+1}\ \beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)\left[1+\left(\frac{x}{b}\right)^{a}\right]^{p+q}}$$

$$E(S_n{}^2)=\frac{b^2\left[\beta\left(p+\frac{3}{a},q-\frac{3}{a}\right)\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)-\beta^2\left(p+\frac{2}{a},q-\frac{2}{a}\right)\right]}{\beta^2\left(p+\frac{1}{a},q-\frac{1}{a}\right)}$$

$$E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right)=\frac{n\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)-\beta^2\left(p+\frac{c+1}{a},q\right)\right]}{\left[\beta\left(p+\frac{c+2}{a},q\right)\beta\left(p+\frac{c}{a},q\right)+(n-1)\beta^2\left(p+\frac{c+1}{a},q\right)\right]}$$

Furthermore, if SBGBD2 distribution, we have

$$\frac{\sigma^2}{\mu^2}=\frac{\beta\left(p+\frac{3}{a}.q-\frac{3}{a}\right)\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}{\beta^2\left(p+\frac{2}{a},q-\frac{2}{a}\right)}-1 \qquad (4.14.8)$$

And it can be show that

$$E\left(\frac{S_n{}^2}{\overline{X}_n}\right)\rightarrow\frac{\beta\left(p+\frac{3}{a}.q-\frac{3}{a}\right)\beta\left(p+\frac{1}{a},q-\frac{1}{a}\right)}{\beta^2\left(p+\frac{2}{a},q-\frac{2}{a}\right)}-1 \qquad (4.14.9)$$

Comparing above two equations, we have

Note that $E\left(\frac{S_n{}^2}{\overline{X}_n}\right)\rightarrow\frac{\sigma^2}{\mu^2}$ as $n\rightarrow\infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n{}^2}{\overline{X}_n{}^2}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation.

4.15 Test for size-biased generalized beta distribution of second kind.

Let X_1, X_2, X_3… Xn be random samples can be drawn from generalized beta distribution of second kind or size-biased generalized beta distribution of second kind.

We test the hypothesis

$$H_o : f(x) = f(x;a,b,p,q) \, vs \, H_1 : f(x) = f_s^*(x;a,b,p,q).$$

To test whether the random sample of size n comes from the generalized beta distribution of second kind or size-biased generalized beta distribution of second kind the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \left(\frac{f^*{}_s(x;a,b,p,q)}{f(a,b,p,q)} \right)$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \left(\frac{\dfrac{ax^{ap}\Gamma(p+q)}{b^{ap+1}\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}}}{\dfrac{ax^{ap-1}\Gamma(p+q)}{b^{ap}\Gamma p \Gamma q\left[1+\left(\frac{x}{b}\right)^a\right]^{p+q}}} \right)$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \left(\frac{x^{ap}}{b\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)} \frac{\Gamma p \Gamma q}{x^{ap-1}} \right)$$

$$\Delta = \left[\frac{\Gamma p \Gamma q}{b\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)} \right]^n \prod_{i=1}^{n} x_i \qquad (4.15.1)$$

We reject the null hypothesis, if

$$\left[\frac{\Gamma p \Gamma q}{b\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)} \right]^n \prod_{i=1}^{n} x_i > k$$

Equalivalently, we rejected the null hypothesis when

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, where \, k^* = k \left[\frac{kb\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)}{\Gamma p \Gamma q} \right]^n > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution.

CHAPTER – 5

SIZE-BIASED GENERALIZED RAYLEIGH DISTRIBUTION

5.1 Introduction

Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. In statistic literature, the Rayleigh distribution is a continuous probability distribution. The problem of estimating the unknown parameters in statistical distributions used to study a certain phenomenon is one of the important problems facing constantly those who are interested in applied statistics. This distribution was introduced by Lord Rayleigh (1980). Surles and Padgett (2005) introduced two-parameter Burr Type X distribution and correctly named as the generalized Rayleigh distribution. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution, originally proposed by Mudholkar and Srivastava (1993).Several aspects of the one-parameter (scale parameter equals one) generalized Rayleigh distribution were studied by Sartawi and Abu-Salih (1991), and Surles and Padgett (1998). It presents a flexible family in the varieties of shapes and is suitable for modeling data with different types of hazard rate function: increasing, decreasing and upside down bathtub shape (UBT).The Generalized Rayleigh distribution includes several other distributions as special or limiting cases, such as gamma, Weibull and exponential distributions.

The probability distribution of Generalized Rayleigh distribution is given as:

$$f(x;\theta,k) = \frac{k}{\theta^{\frac{1}{k}}\Gamma\left(\frac{1}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right) \qquad for\ x \geq 0, k\ and\ \theta > 0 \tag{5.1.1}$$

$$= 0, otherwise$$

Its mean and variance are given by:

$$\mu = \frac{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)} \tag{5.1.2}$$

$$\mu_2 = \frac{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)} - \left[\frac{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)}\right]^2 \tag{5.1.3}$$

In this chapter, we have introduced a new class of Size biased Generalized Rayleigh distribution. The several structural properties, reliability and information measures are introduced and derived. The estimation of parameters of this new model is obtained by employing the new methods of moments, maximum likelihood and Bayesian method of estimation. The Bayes' estimators are obtained by using Jeffrey's and extension of Jeffrey's prior under different loss functions. A comparison has been made of the Bayes' estimator with the corresponding maximum likelihood estimator. Also, a likelihood ratio test of size biased generalized Rayleigh distribution is to be conducted. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator. Also, survival functions of new model are derived using Jeffrey and extension of Jeffrey prior. It has been observed that Bayes' estimator provides better results and estimates as compared to classical estimators.

5.2 Size Biased Generalized Rayleigh Distribution

A size biased generalized Rayleigh distribution (SBGRD) is obtained by applying the weights x^c, where c =1 to the weighted Generalized Rayleigh distribution.

We have from relation (5.1.1) and (5.1.2)

$$\int_0^\infty x\, f(x;\theta,k)dx = \frac{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)}$$

This gives the size-biased generalized Rayleigh distribution (SBGRD) as:

$$f_S(x;\theta,k) = \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right) \qquad for\ x \geq 0, k\ and\ \theta > 0 \qquad (5.2.1)$$

$$= 0, otherwise$$

The CDF of the Size biased generalized Rayleigh distribution is given by:

$$F(x;\theta,k) = \frac{\gamma\left(\frac{2}{k}, x\right)}{\Gamma\left(\frac{2}{k}\right)} \qquad (5.2.2)$$

5.2.1 Special Cases

The distribution like the Size-biased exponential distributions as a special case when k = 1, then the probability density function is given as:

$$f_S(x;\theta) = \frac{x}{\theta^2}\exp\left(-\frac{x}{\theta}\right) \qquad for\ x \geq 0, k > 0$$

$$= 0, otherwise$$

The distribution like the Size-biased Rayleigh distribution as a special case, when k=2 then the probability density function is given as:

$$f_S(x;\theta) = \frac{2x}{\theta}\exp\left(-\frac{x^2}{\theta}\right) \qquad for\ x \geq 0, \theta > 0$$

$$= 0, otherwise$$

5.2.2 Hazard functions

The hazard function for the Size biased generalized Rayleigh distribution is given as:

$$h_s(x;\theta,k) = \frac{f(x;\theta,,k)}{1 - F(x;\theta,,k)}$$

$$h(x;\theta,k) = \frac{kxe^{-\frac{x^k}{\theta}}}{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{2}{k}\right) - \gamma\left(\frac{2}{k},x\right)\right]} \tag{5.2.3}$$

The reverse hazard function for the Size biased generalized Rayleigh distribution is given as:

$$h_{rv}(x;\theta,k) = \frac{f(x;\theta,k)}{F(x;\theta,k)}$$

$$h(x;\theta,k) = \frac{kxe^{-\frac{x^k}{\theta}}}{\theta^{\frac{2}{k}}\left[\gamma\left(\frac{2}{k},x\right)\right]} \tag{5.2.4}$$

Theorem 5.2.3: Let $f(x;\theta,k)$ be a twice differentiable probability density function of a continuous random variable X. Define $n(x;\theta,k) = -\frac{f'(x;\theta,k)}{f(x,\theta,k)}$,where $f'(x;\theta,k)$ is the first derivative of $f(x;\theta,k)$with respect to x. Furthermore, suppose that the first derivative of $n(x;\theta,k)$ exist.

a) If $n'(x;\theta,k) < 0$, for all x > 0, then the hazard function is monotonically decreasing.
b) If $n'(x;\theta,k) > 0$, for all x > 0, then the hazard function is monotonically increasing.
c) Suppose there exist x_0 such that $n'(x;\theta,k) < 0,$ *for all* $0 < x < x_0$, $n'(x_0;\theta,k) = 0$

And $n'(x;\theta,k) > 0,$ for all $0 > x_0$. In addition, $\lim_{x\to 0} f(x) = \infty$, then the hazard function is upside down bathtub shape.

Proof: Using equation (5.2.1), the derivative of the $f(x;\theta,k)$ is given by:

$$f'(x;\theta,k)=\frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}e^{\frac{-x^k}{\theta}}$$

Therefore, $n(x;\theta,k)=-\frac{f'(x;\theta,k)}{f(x,\theta,k)}$

$$n(x;\theta,k)=-\frac{kx^k-\theta}{\theta x}$$

The derivative of $n(x;\theta,k)$ is given as:

And

$$n'(x;\theta,k)=\frac{k(k-1)x^{k-2}}{\theta}+\frac{1}{x^2} \qquad (5.2.5)$$

Collory:

a) If $k\geq 1$, then $n'(x;\theta,k)>0$, for all x > 0, then the hazard function is monotonically increasing.

b) If $k<1$, then $n'(x;\theta,k)<0$, then the hazard function is monotonically decreasing.

c) If $0<k<1$, then the hazard function is upside down bathtub shape.

5.2 Structural properties of Size-biased generalized Rayleigh distribution

In this section, we derive some structural properties of Size-biased generalized Rayleigh distribution.

5.3.1 Moments of Size-biased generalized Rayleigh distribution (SBGRD)

The rth moment of SBGRD (5.2.1) about origin is obtained as:

$$\mu_r' = \int_0^\infty x^r f(x;\theta,k)dx$$

$$\mu_r' = \int_0^\infty x^{r+1}\frac{k}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right)dx$$

Let $x^k = z \Rightarrow x = z^{\frac{1}{k}}, dx=\frac{1}{k}z^{\frac{1}{k}-1}dz$, $x\to 0, z\to 0; x\to\infty, z\to\infty$

$$\mu_r' = \frac{1}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\int_0^\infty z^{\frac{r+2}{k}-1}\exp\left(-\frac{z}{\theta}\right)dz$$

$$\mu_r' = \frac{1}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\Gamma\left(\frac{r+2}{k}\right).\theta^{\frac{r+2}{k}} \qquad (5.3.1)$$

By putting r = 1, 2 in equation (5.3.1), the mean, variance and coefficient of variation are given as:

$$\mu = Mean = \frac{\Gamma\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}.\theta^{\frac{1}{k}} \qquad (5.3.2)$$

$$\mu_2 = \frac{\theta^{\frac{2}{k}}}{\Gamma\left(\frac{2}{k}\right)}\left[\Gamma\left(\frac{4}{k}\right) - \frac{\Gamma^2\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}\right] \qquad (5.3.3)$$

$$CV = \frac{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) - \Gamma^2\left(\frac{3}{k}\right)}{\Gamma\left(\frac{3}{k}\right)} \qquad (5.3.4)$$

5.3.2 Moment generating function of Size-biased generalized Rayleigh distribution

The moment generating function of Size-biased generalized Rayleigh distribution is obtained as:

$$E(e^{tx^k}) = \int_0^\infty e^{tx^k} f_s(x;\theta,k)dx$$

$$E(e^{tx^k}) = \int_0^\infty e^{tx^k} \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} e^{-\frac{x^k}{\theta}} dx$$

$$E(e^{tx^k}) = \frac{k}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\int_0^\infty x e^{-x^{k}\left(\frac{1}{\theta}-t\right)} dx$$

Put $x^k = \frac{z}{\frac{1}{\theta}-t} \Rightarrow x = \frac{z^{\frac{1}{k}}}{\left(\frac{1}{\theta}-t\right)^{\frac{1}{k}}}, dx = \frac{z^{\frac{1}{k}-1}}{k\left(\frac{1}{\theta}-t\right)^{\frac{1}{k}}} dz$

$$E(e^{tx^k}) = \frac{k}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} \cdot \frac{1}{\left(\frac{1}{\theta}-t\right)^{\frac{2}{k}}} \int_0^\infty e^{-z} z^{\frac{2}{k}-1} dz$$

$$E(e^{tx^k}) = \frac{1}{\theta^{\frac{2}{k}}\left(\frac{1}{\theta}-t\right)^{\frac{2}{k}}}$$

$$E(e^{tx^k}) = \frac{1}{(1-\theta t)^{\frac{2}{k}}} \qquad (5.3.5)$$

5.3.3 Characteristic function of Size-biased generalized Rayleigh distribution

The Characteristic function of Size-biased generalized Rayleigh distribution is obtained as:

$$E(e^{itx^k}) = \int_0^\infty e^{itx^k} f_s(x;\theta,k)dx$$

$$E(e^{itx^k}) = \int_0^\infty e^{itx^k} \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} e^{-\frac{x^k}{\theta}} dx$$

$$E(e^{itx^k}) = \frac{k}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} \int_0^\infty x e^{-x^k\left(\frac{1}{\theta}-it\right)} dx$$

Put $x^k = \frac{z}{\frac{1}{\theta}-t} \Rightarrow x = \frac{z^{\frac{1}{k}}}{\left(\frac{1}{\theta}-t\right)^{\frac{1}{k}}}, dx = \frac{z^{\frac{1}{k}-1}}{k\left(\frac{1}{\theta}-t\right)^{\frac{1}{k}}} dz$

$$E(e^{itx^k}) = \frac{k}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} \cdot \frac{1}{\left(\frac{1}{\theta}-it\right)^{\frac{2}{k}}} \int_0^\infty e^{-z} z^{\frac{2}{k}-1} dz$$

$$E(e^{itx^k}) = \frac{1}{\theta^{\frac{2}{k}}\left(\frac{1}{\theta} - it\right)^{\frac{2}{k}}}$$

$$E(e^{itx^k}) = \frac{1}{(1-\theta it)^{\frac{2}{k}}} \qquad (5.3.6)$$

5.3.4 Shannon's entropy of size-biased Generalized Rayleigh Distribution

The Shannon entropy of a random variable X is a measure of the uncertainty and is given by $E[-\log(f(x)]$, where $f(x)$ is the probability function of the random variable X.Shannon entropy of Size biased Generalized Rayleigh Distribution are obtained as:

$$H[f_s(x;\theta,k)] = E[-\log f_s(x;\theta,k)]$$

$$H[f_s(x;\theta,k)] = -\int_0^{\infty} \log f_s(x;\theta,k)[f_s(x;\theta,k)]$$

Note that

$$f_S(x;\theta,k) = \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} \exp\left(-\frac{x^k}{\theta}\right) \qquad for\ x \geq 0, k\ and\ \theta > 0$$

$$= 0, otherwise$$

$$\log f_S(x;\theta,k) = \log c + \log x - \frac{x^k}{\theta}$$

Where $c = \frac{k}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}$

So that, $H[f_s(x;\theta,k)] = -\int_0^{\infty}\left[\log c + \log x - \frac{x^k}{\theta}\right] cx \exp\left[-\frac{x^k}{\theta}\right] dx$

$$H\left[f_s(x;\theta,k)\right] = --c\log c\int_0^\infty x\exp\left[-\frac{x^k}{\theta}\right]dx - c\int_0^\infty x\log x\exp\left[-\frac{x^k}{\theta}\right]dx + \frac{c}{\theta}\int_0^\infty x^{k+1}\exp\left[-\frac{x^k}{\theta}\right]dx$$

Put, $\frac{x^k}{\theta} = z, x = \theta^{\frac{1}{k}} z^{\frac{1}{k}}, dx = \frac{1}{k}\theta^{\frac{1}{k}} z^{\frac{1}{k}-1} dz, \log x = \frac{1}{k}\log \theta z$

So that

$$H\left[f_s(x;\theta,k)\right] = -\frac{c\log c}{k}\int_0^\infty \theta^{\frac{2}{k}}\exp(-z)\, z^{\frac{2}{k}-1}dz - \frac{c\theta^{\frac{2}{k}}}{k^2}\int_0^\infty \log(\theta z)\exp(-z) z^{\frac{2}{k}-1}dz + \frac{c\theta^{\frac{2}{k}}}{k}\int_0^\infty \exp(-z) z^{\frac{k+2}{k}}dz \qquad (5.3.7)$$

But, $\frac{c\log c}{k}\int_0^\infty \theta^{\frac{2}{k}} \exp(-z) z^{\frac{2}{k}-1} dz = \log k - \frac{z\log\theta}{k} - \log\Gamma\left(\frac{2}{k}\right)$ (5.3.8)

$$\frac{c\theta^{\frac{2}{k}}}{k^2}\int_0^\infty \log(\theta z)\exp(-z) z^{\frac{2}{k}-1} dz = \frac{\log\theta}{k} + \frac{\Gamma\left(\frac{2}{k}+1\right)}{k\Gamma\left(\frac{2}{k}\right)} \quad (5.3.9)$$

$$\frac{c\theta^{1+\frac{2}{k}}}{\theta k}\int_0^{\infty} \exp(-z) z^{\frac{k+2}{k}-1} dz = \frac{\Gamma\left(\frac{k+2}{k}\right)}{\Gamma\left(\frac{2}{k}\right)} \tag{5.3.10}$$

Substitute the values of equations (5.3.7), (5.3.8), (5.3.9) in equation (5.3.10), we get the Shannon's entropy of Size-biased generalized Rayleigh distribution which is given as:

$$H[f_s(x;\theta,k)] = -\log k + \frac{2\log\theta}{k} + \log\Gamma\left(\frac{2}{k}\right) - \frac{\log\theta}{k} - \frac{\Gamma\left(\frac{2}{k}+1\right)}{k\Gamma\left(\frac{2}{k}\right)} + \frac{\Gamma\left(\frac{2}{k}+1\right)}{\Gamma\left(\frac{2}{k}\right)}$$

$$H[f_s(x;\theta,k)] = \frac{\log\theta}{k} - \log k + \log\Gamma\left(\frac{2}{k}\right) + [k-1]\frac{\Gamma\left(\frac{2}{k}+1\right)}{k\Gamma\left(\frac{2}{k}\right)} \tag{5.3.11}$$

The above relation (5.3.11) represents the Shannon's entropy of Size-biased generalized Rayleigh distribution.

5.3.5 Fisher's information matrix of size-biased Generalized Rayleigh Distribution

The Size biased generalized Rayleigh distribution has a probability density function of the form:

$$f_S(x;\theta,k) = \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right)$$

Applying log on both sides in equation, we have

$$\log f_s(x;\theta,k) = \log k + \log x - \log(\theta)^{\frac{2}{k}} - \log\Gamma\left(\frac{2}{k}\right) - \frac{x^k}{\theta}$$

Differentiating equation (3.3.19) partially with respect to θ and k , we get

$$\frac{\partial \log f_s(x;\theta,k)}{\partial\theta} = \frac{-2}{k\theta} + \frac{x^k}{\theta^2} \tag{5.3.12}$$

$$\frac{\partial \log f_s(x;\theta,k)}{\partial k} = \frac{1}{k} - \Psi\left(\frac{2}{k}\right) + \frac{2\log\theta}{k^2} - \frac{x^k \log k}{\theta} \tag{5.3.13}$$

Differentiating again the above equation partially with respect to θ and k we have

$$\frac{\partial^2 \log f_s(x;\theta,k)}{\partial\theta^2} = \frac{2}{k\theta^2} - \frac{2x^k}{\theta^3} \tag{5.3.14}$$

$$\frac{\partial^2 \log f_s(x;\theta,k)}{\partial k^2} = \frac{-x^k}{\theta k} - \frac{x^k(\log k)^2}{\theta} \tag{5.3.15}$$

$$\frac{\partial^2 \log f_s(x;\theta,k)}{\partial\theta\partial k} = \frac{2}{\theta k^2} + \frac{x^k \log k}{\theta^2} \tag{5.3.16}$$

$$\frac{\partial^2 \log f_s(x;\theta,k)}{\partial k\partial\theta} = \frac{2}{\theta k^2} + \frac{x^k \log k}{\theta^2} \tag{5.3.17}$$

Taking expectations on both sides of the above equations, we get

$$\mathrm{I}(1,1) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial\theta^2}\right) = \frac{2E(x)^\beta}{\theta^3} - \frac{2}{k\theta^2}$$

$$\mathrm{I}(1,2) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial\theta\partial k}\right) = -\left(\frac{2}{\theta k^2} + \frac{E(x^k)\log k}{\theta^2}\right)$$

$$\mathrm{I}(2,1) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial k\partial\theta}\right) = -\left(\frac{2}{\theta k^2} + \frac{E(x^k)\log k}{\theta^2}\right)$$

$$\mathrm{I}(2,2) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial k^2}\right) = \left(\frac{1}{\theta k} - \frac{(\log k)^2}{\theta}\right)E(x)^k$$

We know that

$$E(x^k) = \frac{\Gamma\left(\frac{k+2}{k}\right)\theta^{\frac{k+2}{k}}}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} \tag{5.3.18}$$

Substitute the values of equation (5.3.18) in the above entries of a Fisher information matrix, we get

$$\mathrm{I}(1,1) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial\theta^2}\right) = \frac{2}{k\theta^2}$$

$$\mathrm{I}(1,2) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial\theta\partial k}\right) = -2\left(\frac{1+\theta}{\theta k^2}\right)$$

$$\mathrm{I}(2,1) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial k\partial\theta}\right) = -2\left(\frac{1+k\log k}{\theta k^2}\right)$$

$$\mathrm{I}\,(2,2) = -E\left(\frac{\partial^2 \log f_s(x;\theta,k)}{\partial k^2}\right) = \frac{2\left(1-k(\log k)^2\right)}{k^2}$$

5.3.6 Test for Size-biasedness of Size biased generalized Rayleigh distribution.

Let X_1, X_2, X_3… Xn be random samples can be drawn from generalized Rayleigh distribution or Size biased generalized Rayleigh distribution. We test the hypothesis

$H_0 : f(x) = f(x;\lambda,\beta,k)$ vs $H_1 : f(x) = f_S(x;\lambda,\beta,k)$

To test whether the random sample of size n comes from the generalized Rayleigh distribution or Size biased generalized Rayleigh distribution, then the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{f_S(x;\theta,k))}{f(x;\theta,k))}\right]$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{\dfrac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\dfrac{x^k}{\theta}\right)}{\dfrac{k}{\theta^{\frac{1}{k}}\Gamma\left(\frac{1}{k}\right)}\exp\left(-\dfrac{x^k}{\theta}\right)}\right]$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{\Gamma\left(\frac{1}{k}\right)}{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}x\right].$$

$$\Delta = \left[\frac{\Gamma\left(\frac{1}{k}\right)}{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}\right]^n \prod_{i=1}^{n} x_i \qquad (5.3.19)$$

We reject the null hypothesis.

$$\left[\frac{\Gamma\left(\frac{1}{k}\right)}{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}\right]^n \prod_{i=1}^{n} x_i > k \qquad (5.3.20)$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, where\, k^* = k\left[\frac{\Gamma\left(\frac{1}{k}\right)}{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}\right]^n > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution. Also, we can reject the reject the null hypothesis, when probability value s given by:

$P(\Delta^* > \lambda^*), Where\ \lambda^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the test statistic.

5.3 Estimation of parameters

In this section, we discuss the various estimation methods for size biased Generalized Rayleigh distribution and verifying their efficiencies.

5.4.1 Methods of Moments

Replacing sample moments with population moments, we get

$$\bar{X} = \frac{\Gamma\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}.\theta^{\frac{1}{k}} \tag{5.4.1}$$

$$S^2 = \frac{\theta^{\frac{2}{k}}}{\Gamma\left(\frac{2}{k}\right)}\left[\Gamma\left(\frac{4}{k}\right) - \frac{\Gamma^2\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}\right] \tag{5.4.2}$$

From above two equations, we get

$$\frac{S^2}{\bar{X}^4} = \frac{\Gamma^3\left(\frac{2}{k}\right)\Gamma\left(\frac{4}{k}\right) - \Gamma^2\left(\frac{2}{k}\right)\Gamma^2\left(\frac{3}{k}\right)}{\Gamma^2\left(\frac{3}{k}\right)} \tag{5.4.3}$$

Solving above equation for k, we get the estimate for k and substituting that value in equation (5.4.1), we get the estimate of θ.

5.5 Method of Maximum Likelihood estimator

Maximum likelihood estimation has been the most widely used method for estimating the parameters of the Size biased generalized Rayleigh distribution. Let $x_1, x_2, x_3 .... x_n$ be a random sample from the size biased generalized Rayleigh distribution, and then the corresponding likelihood function is given as

$$L(X;\theta,k)=\frac{k^n}{\theta^{\frac{2n}{k}}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n}x_i \exp\left(\frac{-\sum_{i=1}^{n}x_i^{\ k}}{\theta}\right) \tag{5.5.1}$$

The log-likelihood function is:

$$\log L(x;\theta,k)=n\log k-\frac{2n\log\theta}{k}-n\log\Gamma\left(\frac{2}{k}\right)+\sum_{i=1}^{n}\log x_i-\frac{\sum_{i=1}^{n}x_i^{\ k}}{\theta} \tag{5.5.2}$$

Now, we obtain the normal equations, we get

$$\frac{2n}{\theta k}+\frac{\sum_{i=1}^{n}x_i^{\ k}}{\theta^2}=0 \tag{5.5.3}$$

$$\frac{n}{k}+\frac{2n\log\theta}{k^2}-2n\frac{\Gamma'\left(\frac{2}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}-\frac{\sum_{i=1}^{n}x_i^{\ k}\log\sum_{i=1}^{n}x_i}{\theta}=0 \tag{5.5.4}$$

After solving equation (5.5.3), we have

$$\hat{\theta}=\frac{k\sum_{i=1}^{n}x_i^{\ k}}{2n} \tag{5.5.5}$$

Substitute the value of $\hat{\theta}$ in equation (5.5.4), we get the estimate of k.

5.6 Bayesian analysis of Size Biased Generalized Rayleigh Distribution

Bayesian analysis is an important approach to statistics, which formally seeks use of prior information and Bayes' Theorem provides the formal basis for using this information. In this approach, parameters are treated as random variables and data is treated fixed. Ghafoor et al. (2005) and Rahul et al. (2009) have discussed the application of Bayesian methods**.** An important requisite in Bayesian estimation is the appropriate choice of prior(s) for the parameters. Very often, priors are chosen according to ones subjective knowledge and beliefs. However, if one has adequate information about the parameter(s) one should use informative prior(s), otherwise it is preferable to use non informative prior(s).

5.6.1 Parameter estimation under squared error loss function

In this section, two different prior distributions are used for estimating the parameter of the size biased generalized Rayleigh distribution namely; Jeffery's prior and extension of Jeffrey's prior information.

5.6.1.1 Bayes' estimation of parameter of size biased generalized Rayleigh distribution under Jeffrey's prior

Consider there are n recorded values, $\underline{x} = \left(x_1,\ldots x_n\right)$ from (5.2.1). We consider the extended Jeffrey's prior as:

$$g(\theta) \propto \sqrt{[I(\theta)]}$$

Where $[I(\theta)] = -nE\left[\frac{\partial^2 \log f\left(x;\theta,k\right)}{\partial\theta^2}\right]$ is the Fisher's information matrix. For the model (5.2.1),

$$g(\theta) = k\sqrt{\frac{1}{\theta}}$$

Then the joint probability density function is given by:

$$f(\underline{x},\theta) = L(x;\theta)g(\theta)$$

$$f(\underline{x},\theta) = \frac{k^n}{\theta^{\frac{2n}{k}+\frac{1}{2}}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right) \tag{5.6.1}$$

And the corresponding marginal PDF of $\underline{x} = \left(x_1,\ldots x_n\right)$ is obtained as:

$$p(\underline{x}) = \int_0^\infty \frac{k^n}{\theta^{\frac{2n}{k}+\frac{1}{2}}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right) d\theta$$

$$p(\underline{x}) = \frac{k^n}{\Gamma^n\left(\frac{2}{k}\right)} \prod_{i=1}^{n} x_i \int_0^\infty \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\ k}}{\theta}\right)\left(\frac{1}{\theta}\right)^{\frac{2n}{k}+\frac{3}{2}-1} d\theta$$

$$p(\underline{x}) = \frac{k^n}{\Gamma^n\left(\frac{2}{k}\right)} \prod_{i=1}^{n} x_i \frac{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}{\left[\left(\sum_{i=1}^{n} x_i^{\ k}\right)\right]^{\frac{2n}{k}+\frac{3}{2}}} \qquad (5.6.2)$$

The posterior PDF of θ has the following form

$$\pi_1(\theta/\underline{x}) = \frac{\left[\sum_{i=1}^{n} x_i^{\ k}\right]^{\frac{2n}{k}+\frac{3}{2}}}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)} \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\ k}}{\theta}\right)\left(\frac{1}{\theta}\right)^{\frac{2n}{k}+\frac{1}{2}} \qquad (5.6.3)$$

By using a squared error loss function $L(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2$ for some constant c, the risk function is:

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi_1(\theta/\underline{x}) d\theta$$

$$R(\hat{\theta}) = c\hat{\theta}^2 + \frac{c\Gamma\left(\frac{2n}{k}-\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\left(\sum_{i=1}^{n} x_i^{\ k}\right)^2 - \frac{2c\hat{\theta}\Gamma\left(\frac{2n}{k}+\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\sum_{i=1}^{k} x_i^{\ k}$$

Now $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, Then the Bayes' estimator is

$$\hat{\theta}_1 = \frac{\Gamma\left(\frac{2n}{k}+\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\sum_{i=1}^{k} x_i^{\ k} \qquad (5.6.4)$$

5.6.1.2 Bayes' estimation of parameter of size biased generalized Rayleigh distribution using extension of Jeffrey's prior

We consider the extended Jeffrey's prior are given as:

$$g(\theta) \propto [I(\theta)]^{c_1}; c_1 \varepsilon R^+$$

Where $[I(\theta)] = -nE\left[\frac{\partial^2 \log f(x;\theta,k)}{\partial\theta^2}\right]$ is the Fisher's information matrix. For the model (5.2.1), $g(\theta) = k\left[\frac{1}{\theta}\right]^{c_1}$

Then the joint probability density function is given by:

$$f(\underline{x},\theta) = L(x;\theta)g(\theta)$$

$$f(\underline{x},\theta) = \frac{k^n}{\theta^{\frac{2n}{k}+c_1}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right)$$

And the corresponding marginal PDF of $\underline{x} = (x_1, \ldots x_n)$ is obtained as:

$$p(\underline{x}) = \int_0^\infty \frac{k^n}{\theta^{\frac{2n}{k}+c_1}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right)d\theta$$

$$p(\underline{x}) = \frac{k^n}{\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \int_0^\infty \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right)\left(\frac{1}{\theta}\right)^{\frac{2n}{k}+c_1+1-1} d\theta$$

$$p(\underline{x}) = \frac{k^n}{\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \frac{\Gamma\left(\frac{2n}{k}+c_1+1\right)}{\left[\left(\sum_{i=1}^{n} x_i^{\,k}\right)\right]^{\frac{2n}{k}+c_1+1}} \tag{5.6.5}$$

The Posterior PDF of θ has the following form

$$\pi_2(\theta/\underline{x}) = \frac{\left[\sum_{i=1}^{n} x_i^{\,k}\right]^{\frac{2n}{k}+c_1+1}}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right)\left(\frac{1}{\theta}\right)^{\frac{2n}{k}+c_1} \tag{5.6.6}$$

By using a squared error loss function $L(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2$ for some constant c, the risk function is:

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi_1(\theta/\underline{x})d\theta$$

$$R(\hat{\theta}) = c\hat{\theta}^2 + \frac{c\Gamma\left(\frac{2n}{k}+c_1-1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\left(\sum_{i=1}^{n} x_i^{\,k}\right)^2 - \frac{2c\hat{\theta}\Gamma\left(\frac{2n}{k}+c_1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\sum_{i=1}^{k} x_i^{\,k}$$

Now $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, Then the Bayes' estimator is

$$\hat{\theta}_2 = \frac{\Gamma\left(\frac{2n}{k}+c_1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\sum_{i=1}^{k} x_i^{\,k} \tag{5.6.7}$$

The Bayes' estimator under a precautionary loss function is denoted by $\hat{\theta}$, and is given by the following equation:

$\hat{\theta}_P = E[\theta^2]^{\frac{1}{2}}$ and the corresponding Bayes' estimator comes out to be:

$$\hat{\theta}_2 = \frac{\Gamma\left(\frac{2n}{k}+c_1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\sum_{i=1}^{k} x_i^{\,k}$$

The risk function under precautionary loss function is given by:

$$R_P(\hat{\theta}_P) = c\hat{\theta} + \frac{1}{\hat{\theta}}\frac{c\Gamma\left(\frac{2n}{k}+c_1-1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\left(\sum_{i=1}^{n} x_i^{\,k}\right)^2 - \frac{2c\Gamma\left(\frac{2n}{k}+c_1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\sum_{i=1}^{k} x_i^{\,k} \tag{5.6.8}$$

5.6.2 Parameter estimation under a new loss function.

This section uses a new loss function introduced by Al-Bayyati (2005). Employing this loss function, we obtain Bayes' estimators using Jeffrey's and extension of Jeffrey's prior information.

Al-Bayyati introduced a new loss function of the form:

$$l_A(\hat{\theta},\theta) = \theta^{c_2}(\hat{\theta}-\theta)^2; c_2 \varepsilon R. \tag{5.6.9}$$

Here, this loss function is used to obtain the estimator of the parameter of the size biased generalized Rayleigh distribution.

5.6.2.1 Bayes' estimation of parameter of size biased generalized Rayleigh distribution under Jeffrey's prior.

By using the loss function in the form given in (5.6.9), we obtained the following risk function:

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2} \left(\hat{\theta} - \theta\right)^2 \pi_1(\theta/\underline{x}) d\theta$$

$$R(\hat{\theta}) = \hat{\theta}^2 \frac{\Gamma\left(\frac{2n}{k} + c_2 + \frac{3}{2}\right)}{\Gamma\left(\frac{2n}{k} + \frac{3}{2}\right)} \left[\frac{1}{\sum_{i=1}^{n} x_i^k}\right]^{c_2} + \frac{\Gamma\left(\frac{2n}{k} + c_2 - \frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k} + \frac{3}{2}\right)} \left(\frac{1}{\sum_{i=1}^{k} x_i^k}\right)^{c_2+2} - \frac{2\hat{\theta}\Gamma\left(\frac{2n}{k} + c_2 + \frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k} + \frac{3}{2}\right)} \left[\frac{1}{\sum_{i=1}^{n} x_i^k}\right]^{c_2-1}$$

Now $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, Then the Bayes' estimator is

$$\hat{\theta}_3 = \frac{\Gamma\left(\frac{2n}{k} + \frac{1}{2} + c_2\right)}{\Gamma\left(\frac{2n}{k} + \frac{3}{2} + c_2\right)} \left(\sum_{i=1}^{k} x_i^k\right) \tag{5.6.10}$$

Remarks:

If $c_2 = 0$, we get, the Jeffrey's prior and the corresponding Bayes' estimator is:

$$\hat{\theta}_3 = \frac{\Gamma\left(\frac{2n}{k} + \frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k} + \frac{3}{2}\right)} \sum_{i=1}^{k} x_i^k$$

If $c_2 = 1$, we get, the Hartigan prior [Hartigan (1964)] and the corresponding Bayes' estimator becomes:

$$\hat{\theta}_3 = \frac{\Gamma\left(\frac{2n}{k} + \frac{3}{2}\right)}{\Gamma\left(\frac{2n}{k} + \frac{5}{2}\right)} \sum_{i=1}^{k} x_i^k$$

If $c_2 = 0$, we get, the uniform prior and the corresponding Bayes' estimator becomes:

$$\hat{\theta}_3 = \frac{\Gamma\left(\frac{2n}{k}\right)}{\Gamma\left(\frac{2n}{k}+1\right)}\sum_{i=1}^{k} x_i^{\,k}$$

5.6.2.2 Bayes' estimation of parameter of size biased generalized Rayleigh distribution using extension of Jeffrey's prior.

By using the loss function in the form given in (5.6.9), we obtained the following risk function:

$$R(\hat{\theta}) = \int_0^{\infty} \theta^{c_2}\left(\hat{\theta}-\theta\right)^2 \pi_1(\theta/\underline{x})\,d\theta$$

$$R(\hat{\theta}) = \hat{\theta}^2 \frac{\Gamma\left(\frac{2n}{k}+c_2+c_1+1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\left[\frac{1}{\sum_{i=1}^{n} x_i^{\,k}}\right]^{c_2} + \frac{\Gamma\left(\frac{2n}{k}+c_2+c_1-1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\left(\frac{1}{\sum_{i=1}^{k} x_i^{\,k}}\right)^{c_2+2} - \frac{2\hat{\theta}\Gamma\left(\frac{2n}{k}+c_2+c_1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\left[\frac{1}{\sum_{i=1}^{n} x_i^{\,k}}\right]^{c_2-1}$$

Now $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, Then the Bayes' estimator is

$$\hat{\theta}_4 = \frac{\Gamma\left(\frac{2n}{k}+c_1+c_2\right)}{\Gamma\left(\frac{2n}{k}+c_1+c_2+1\right)}\left(\sum_{i=1}^{k} x_i^{\,k}\right) \qquad (5.6.11)$$

If $c_1 = \frac{1}{2}$ and $c_2 = 0$, we get, the Jeffrey's prior and the corresponding Bayes' estimator is:

$$\hat{\theta}_4 = \frac{\Gamma\left(\frac{2n}{k}+\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\sum_{i=1}^{k} x_i^{\,k}$$

If $c_1 = \frac{3}{2}$ and $c_2 = 0$, we get, the Hartigan prior [Hartigan [(1964)] and the corresponding Bayes' estimator becomes:

$$\hat{\theta}_4 = \frac{\Gamma\left(\frac{2n}{k} + \frac{3}{2}\right)}{\Gamma\left(\frac{2n}{k} + \frac{5}{2}\right)} \sum_{i=1}^{k} x_i^{\,k}$$

If $c_1 = 0$ and $c_2 = 0$, we get the uniform prior and the corresponding Bayes' estimator becomes:

$$\hat{\theta}_4 = \frac{\Gamma\left(\frac{2n}{k}\right)}{\Gamma\left(\frac{2n}{k} + 1\right)} \sum_{i=1}^{k} x_i^{\,k}$$

The Bayes' estimator under a precautionary loss function is denoted by $\hat{\theta}$, and is given by the following equation:

$\hat{\theta}_P = E\left[\theta^2\right]^{\frac{1}{2}}$ and the corresponding Bayes' estimator comes out to be:

$$\hat{\theta}_4 = \frac{\Gamma\left(\frac{2n}{k} + c_1 + c_2\right)}{\Gamma\left(\frac{2n}{k} + c_1 + c_2 + 1\right)} \left(\sum_{i=1}^{k} x_i^{\,k}\right)^{c_{2-1}}$$

The risk function under precautionary loss function is given by:

$$R_p\left(\hat{\theta}_p\right) = \hat{\theta}\,\frac{\Gamma\left(\frac{2n}{k} + c_2 + c_1 + 1\right)}{\Gamma\left(\frac{2n}{k} + c_1 + 1\right)} \left[\frac{1}{\sum_{i=1}^{n} x_i^{\,k}}\right]^{c_2} + \frac{1}{\hat{\theta}}\,\frac{\Gamma\left(\frac{2n}{k} + c_2 + c_1 - 1\right)}{\Gamma\left(\frac{2n}{k} + c_1 + 1\right)} \left(\frac{1}{\sum_{i=1}^{k} x_i^{\,k}}\right)^{c_2+2} - \frac{2\Gamma\left(\frac{2n}{k} + c_2 + c_1\right)}{\Gamma\left(\frac{2n}{k} + c_1 + 1\right)} \left[\frac{1}{\sum_{i=1}^{n} x_i^{\,k}}\right]^{c_2-1} \quad (5.6.12)$$

5.7 New method of estimation of Size biased Generalized Rayleigh distribution.

Note that Hwang T. and Huang P. (2006) have obtained more general characterizations with the independence of sample coefficient of variation V_n with sample mean $\overline{X}_n$ as one of its special cases when random samples are drawn from the generalized gamma distribution. Their characterization is used to derive the expectation and the variance of $V_n^{\,2}$ and then the new estimators for the three parameters of size-biased generalized Rayleigh distribution are proposed. For deriving new moment estimators of three parameters of the size-biased generalized Rayleigh distribution, we need the following theorem obtained by using the similar approach of Hwang .T and Huang .P (Theorems of 2006).

Theorem 5.7.1: Let $n \geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be a n positive identical independently distributed random variables having a probability density function $f(x)$.Then the independence of the sample mean $\overline{X}_n$ and the sample coefficient of variation $V_n = \frac{S_n}{\overline{X}_n}$ is equivalent to that $f(x)$ is a size-biased generalized Rayleigh distribution where S_n is the sample standard deviation.

The next theorem is easy to prove and need to derive the expectation and the variance of $V_n^{\ 2} = \left(\frac{S_n}{\overline{X}_n}\right)^2$, where $\overline{X}_n$ and S_n are respectively the sample mean and the sample standard deviation.

Theorem 5.7.2: Let $n \geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Rayleigh distribution

$$f_S(x;\theta,k) = \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} \exp\left(-\frac{x^k}{\theta}\right) \qquad for\ x \geq 0, k\ and\ \theta > 0$$

$$= 0, otherwise$$

Then $E(S_n^{\ 2}) = \dfrac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) - \Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma^2\left(\frac{2}{k}\right)}$

Proof: Here, $E\left(X^m\right) = \dfrac{1}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)} \Gamma\left(\frac{m+2}{k}\right).\theta^{\frac{m+2}{k}}$

$$E\left(\overline{X}_n\right) = \frac{\Gamma\left(\frac{3}{k}\right)\theta^{\frac{1}{k}}}{\Gamma\left(\frac{2}{k}\right)}$$

$$E(\overline{X}_n^{\,2}) = \frac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) + (n-1)\theta^{\frac{2}{k}}\,\Gamma^2\left(\frac{3}{k}\right)\right]}{n\Gamma^2\left(\frac{2}{k}\right)}$$

And $E(S_n^{\,2}) = \dfrac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) - \Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma^2\left(\frac{2}{k}\right)}$ (5.7.1)

Where $\overline{X}_n$ and $S_n^{\,2}$ are respectively their sample mean and sample variance.

Theorem 5.7.3: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Rayleigh distribution

$$f_S(x;\theta,k) = \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right) \qquad for\ x \geq 0, k\ and\ \theta > 0$$

$$= 0, otherwise$$

Then $E\left(\dfrac{S_n^{\,2}}{\overline{X}_n}\right) = \dfrac{n\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) - \Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) + (n-1)\Gamma^2\left(\frac{3}{k}\right)}$

Where $\overline{X}_n$ and $S_n^{\,2}$ are respectively their sample mean and sample variance.

Proof: By theorem 5.7.1, we have

$$E(S_n^{\,2}) = E\left(\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}.\overline{X}_n^{\,2}\right) = E\left(\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}\right)..E\left(\overline{X}_n^{\,2}\right)$$

And hence $E\left(\dfrac{S_n^{\,2}}{\overline{X}_n^{\,2}}\right) = \dfrac{E\left(S_n^{\,2}\right)}{E\left(\overline{X}_n^{\,2}\right)}$

Applying theorem 5.7.2 to the above identity yields that

$$E\left(\frac{S_n^{\,2}}{\bar{X}_n}\right)=\frac{n\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)+(n-1)\Gamma^2\left(\frac{3}{k}\right)} \tag{5.7.2}$$

Thus 5.7.3 is established.

Theorem 5.7.4: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Rayleigh distribution

$$E(S_n^{\,2})=\frac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma^2\left(\frac{2}{k}\right)}$$

$$E\left(\frac{S_n^{\,2}}{\bar{X}_n}\right)=\frac{n\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)+(n-1)\Gamma^2\left(\frac{3}{k}\right)}$$

Furthermore, if SBGR distribution, we have

$$\frac{\sigma^2}{\mu^2}=\frac{\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)\right]}{\Gamma^2\left(\frac{3}{k}\right)}-1 \tag{5.7.3}$$

And it can be show that

$$E\left(\frac{S_n^{\,2}}{\bar{X}_n}\right)\rightarrow\frac{\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)\right]}{\Gamma^2\left(\frac{3}{k}\right)}-1 \tag{5.7.4}$$

Comparing above two equations, we have

Note that $E\left(\frac{S_n^{\ 2}}{\bar{X}_n}\right) \to \frac{\sigma^2}{\mu^2}$ as $n \to \infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n^{\ 2}}{\bar{X}_n^{\ 2}}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation.

5.8 Simulation Study of Size biased Generalized Rayleigh distribution

In our simulation study, we chose a sample size of n=25, 50 and 75 to represent small, medium and large data set. The scale parameter is estimated for Size biased Generalized Rayleigh distribution by the methods of Maximum Likelihood and Bayesian using Jeffrey's & extension of Jeffrey's prior methods. For the scale parameter we have considered θ= 0.5 and 1.0. The values of Jeffrey's extension were c_1 = 0.5, 1.0, 1.5 and 2.0. The value for the loss parameter c_2 = 0 and ± 1.0. This was iterated 5000 times and the scale parameter for each method was calculated. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes with different values for the Extension of Jeffrey's' prior and the loss functions. The results are presented in tables for different selections of the parameters and c extension of Jeffrey's prior.

Table 5.1Structural properties of Size biased Generalized Rayleigh distribution

n	θ	k	Mean	variance	S.D	C.V	Shannon's Entropy
25	0.5	1.0	1.3057878	0.0013881	0.0372575	0.028532	3.414851
	1.0	1.5	1.310549	0.0182512	0.1350972	0.103084	4.724048
50	0.5	1.0	0.3054362	0.0011856	0.0344329	0.112733	1.879161
	1.0	1.5	0.3199525	0.0407701	0.2019163	0.631082	2.362021
75	0.5	1.0	1.3256778	1.425e-05	0.0037708	0.002844	1.365678
	1.0	1.5	1.310549	0.0319192	0.1786595	0.136324	1.181019

Table 5.2 Mean Squared Error for ($\hat{\theta}$) under Jeffrey's prior

n	θ	k	θ_{ML}	θ_{SL}	θ_{NL}		
					C2=-1.0	C2=-0	C2=1.0
25	0.5	1.0	0.4184437	0.02261071	0.02053641	0.02261071	0.02469849
	1.0	1.5	0.35385413	0.35744087	0.35031408	0.35744087	0.36473563
50	0.5	1.0	0.3912145	0.01621413	0.01527433	0.01621413	0.01716265
	1.0	1.0	0.3218592	0.3243453	0.3193622	0.3243453	0.3292847
75	0.5	1.0	0.3897367	0.01572284	0.01517348	0.01572284	0.16547846
	1.0	1.0	0.2638086	0.2654711	0.2621403	0.2654711	0.2687786

Table 5.3: Mean Squared Error for ($\hat{\theta}$) under extension of Jeffrey's prior

n	**θ**	k	C_1	**θ_{ML}**	**θ_{SL}**	θ_{NL}		
						C2=-1.0	C2=0	C2=1.0
25	0.5	1.0	0.5	0.41844371	0.02261071	0.02053641	0.02261071	0.02469849
			1.0	0.41844371	0.02365331	0.02157146	0.02365331	0.02574551
			1.5	0.41844372	0.02469849	0.02261071	0.02469849	0.02679371
			2.0	0.41844371	0.02574551	0.02365331	0.02574551	0.02784246
	1.0	1.0	0.5	0.35385413	0.35744087	0.35031408	0.35744087	0.36473563
			1.0	0.35385413	0.36106952	0.35385413	0.36106952	0.36843503
			1.5	0.35385413	0.36473563	0.35744087	0.36473563	0.37216383
			2.0	0.35385413	0.36843503	0.36106952	0.36843503	0.37591841
50	0.5	1.0	0.5	0.39121450	0.01621413	0.01527433	0.01621413	0.01716265
			1.0	0.39121450	0.01668737	0.01574307	0.01668737	0.01763985
			1.5	0.39121450	0.01716265	0.01621413	0.01716265	0.01811884
			2.0	0.39121450	0.01763985	0.01668737	0.01763985	0.0185995
	1.0	1.0	0.5	0.33218592	0.3243453	0.3193622	0.3243453	0.3292847
			1.0	0.33218592	0.3268205	0.3218592	0.3268205	0.3317378
			1.5	0.33218592	0.3292847	0.3243453	0.3292847	0.3341798
			2.0	0.33218592	0.3317378	0.3268205	0.3317378	0.3366108
75	0.5	1.0	0.5	0.38973671	0.01572284	0.01517348	0.01572284	0.01654786
			1.0	0.38973671	0.0163256	0.01544774	0.0153256	0.01653488
			1.5	0.38973671	0.01507789	0.01468892	0.01507789	0.01569146
			2.0	0.38973671	0.01604798	0.01543174	0.01604798	0.01666804
	1.0	1.0	0.5	0.26547112	0.2654711	0.2621403	0.26547112	0.2687786
			1.0	0.26547112	0.2671278	0.2638086	0.2671278	0.2704236

			1.5	0.26547112	0.2687786	0.2654711	0.2687786	0.2720628
			2.0	0.26547112	0.2704236	0.2671278	0.2704236	0.2736962

ML= Maximum Likelihood, SL=Squared Error Loss Function, NL= New Loss Function,

In table 5.2, Bayes'estimation with New Loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is -1. Similarly, in table 5.3, Bayes' estimation with New Loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is - 1 whether the extension of Jeffrey's prior is 0.5, 1.0, 1.5 or 2.0. Moreover, when the sample size increases from 25 to 75, the MSE decreases quite significantly.

CHAPTER – 6

SUMMARY AND CONCLUSIONS

The main focus in the present work has been made on the size biased probability distributions, particularly of some basic and most widely used member of it. Various contributions have been made about these distributions by various Statisticians and mathematicians in the past. All these which are still scattered in various journals of Statistics and Mathematics have been reviewed and critically examined. Size-biased probability distributions are a special case of a general form known as weighted distributions. The Size biased Distributions are obtained by taking the weights as the variate values has been defined. We have proposed a new general class of Size biased Gamma, Beta and exponential distributions. The size biased Gamma (SBG) Distribution that is a flexible distribution in statistical literature, and has size biased exponential and exponential distribution as a subfamilies are introduced and also consists of presentation of general review of some important properties of SBG family. The power and logarithmic moments of this family is defined. Some important theorems of SBG family has been derived and studied, also identify the relation of SBG family with other related distributions. The estimation of parameters of these new models is obtained by employing the methods of moments, maximum likelihood and Bayesian method of estimation. We have also present Bayes' estimator of the parameter of Size biased classical Distribution that stems from an extension of Jeffery's prior (Al-Kutubi (2005)) with a new loss function (Al-Bayyati (2002)). We are proposing four different types of estimators. Under squared error loss function, there are two estimators formed by using Jaffrey prior and an extension of Jaffrey's prior. The two remaining estimators are derived using the same Jeffrey's prior and extension of Jeffrey's prior under a new loss function. We are also derive the survival function of the size biased Gamma and exponential distributions. A comparison has been made of the Bayes' estimator with the corresponding maximum likelihood estimator. Also, a likelihood ratio test of size-biasedness is conducted. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator. We have considered a new class of Size biased Generalized Gamma Distribution. The several structural properties, reliability and information measures of Size biased Generalized Gamma model are introduced and derived. The estimation of parameters of this new model is obtained by employing the new methods of moments, maximum likelihood and Bayesian method of estimation. The Bayes' estimators are obtained by using Jeffrey's and extension of Jeffrey's prior under different loss functions. A comparison has been made of the Bayes' estimator with the corresponding maximum likelihood estimator. Also, a likelihood ratio test of size biased generalized gamma distribution is to be conducted. We have derived the survival function of the size biased Generalized Gamma distribution. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator. We have derived the survival functions of the size biased Generalized Gamma distribution under Jaffrey and extension of Jaffrey's prior. It has been observed that Bayes' estimator provides better results and estimates as compared to classical estimators. In this chapter third , the AIC, and BIC values of exponential model are smaller as compared to size biased Gamma and size biased exponential models, so exponential model is more preferable than the size biased Gamma and size biased exponential models for the real data in hand. In this chapter fourth, a new class of weighted Generalized Beta Distribution of first kind, Size biased Generalized Beta Distribution of first and second kind has been considered. The several structural properties, of these probability models includes mean, variance, coefficient of variation, mode and harmonic mean has been studied and derived. The estimation of parameters of this new model is obtained by employing the new methods of moments Also, a likelihood ratio test of Weighted and size biased probability distributions are to be conducted. Some important theorems have been derived to estimates the parameters of four parametric weighted and size-biased beta distributions. It was found that the square of the sample coefficient of variation is asymptotically unbiased estimator of square of the population coefficient of variation. In this chapter fifth, we have made an attempt to introduce a new class of Size biased Generalized Rayleigh distribution. The several structural properties, reliability and information measures are introduced and derived. The estimation of parameters of this new model is obtained by employing the new methods of moments, maximum likelihood and Bayesian method of estimation. The Bayes' estimators are obtained by using Jeffrey's and extension of Jeffrey's prior under different loss functions. A comparison has been made of the Bayes' estimator with the corresponding maximum likelihood estimator. Also, a likelihood ratio test of size biased generalized Rayleigh distribution is to be conducted. A simulation study has been performed for the comparison of Bayes' estimators with the MLE estimator. Also, survival functions of new model are derived using Jeffrey and extension of Jeffrey prior. It has been observed that Bayes' estimator provides better results and estimates as compared to classical estimators. The main objective of the research work is to introduce new class of Size biased probability distributions and obtain its structural and characterizing properties.Also,estimates the parameters of these new models by using different estimation techniques includes method of moment, method of maximum likelihood estimator and Bayesian method of estimation etc.Simulate the data in different Statistical packages and estimate the parameters and compared the estimators with different estimation techniques and interpret the whole data, draw your valid conclusion.

The work is expected to be useful to those who are interested in applied estimation theory in general and estimations of probability distributions in particular .This work suggests about some new classes, different forms and new estimation procedures for these distributions.

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