SOME CONTRIBUTIONS TO OPTIMALITY CRITERIA AND DUALITY IN MULTIOBJECTIVE MATHEMATICAL PROGRAMMING

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Certificate

This is to certify that the work embodied in this thesis entitled **"SOME CONTRIBUTIONS TO OPTIMALITY CRITERIA AND DUALITY IN MULTIOBJECTIVE MATHEMATICAL PROGRAMMING"** is the original work carried out by **Ms. Rumana Gulzar Mattoo** under our supervision and is suitable for the award of the degree of **Doctor of Philosophy** in Statistics.

The thesis has reached the standard fulfilling the requirements of regulations relating to the degree. The results contained in the thesis have not been submitted earlier to this or any other university or institute for the award of degree or diploma.

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Dedicated To My Parents

Since the fabric of the universe is most perfect and the work of the most Wise Creator, nothing at all takes place in the Universe in which some rule of the maximum or minimum does not appear.

(Leonhard Euler)

ABSTRACT

This thesis entitled, **"SOME CONTRIBUTIONS TO OPTIMALITY CRITERIA AND DUALITY IN MULTIOBJECTIVE MATHEMATICAL PROGRAMMING"**, offers an extensive study on optimality, duality and mixed duality in a variety of multiobjective mathematical programming that includes nondifferentiable nonlinear programming, variational problems containing square roots of a certain quadratic forms and support functions which are prominent nondifferentiable convex functions. This thesis also deals with optimality, duality and mixed duality for differentiable and nondifferentiable variational problems involving higher order derivatives, and presents a close relationship between the results of continuous programming problems through the problems with natural boundary conditions between results of their counter parts in nonlinear programming. Finally it formulates a pair of mixed symmetric and self dual differentiable variational problems and gives the validation of various duality results under appropriate invexity and generalized invexity hypotheses. These results are further extended to a nondifferentiable case that involves support functions.

This thesis comprises seven chapters. The chapters are divided into various subsections to give specific descriptions of the results quoted in the thesis.

Chapter 1: is devoted to surveying the relevant literature which is required in the development of existing results in multiobjective mathematical programming.

Chapter 2: The purpose of the chapter 2nd is to study multiobjective duality in nonlinear programming involving support functions. This chapter contains two sections, 2.1 and 2.2. In section 2.1, Wolfe type duality and Mond-Weir type duality are investigated under invexity and generalized invexity, and special cases are derived. Section 2.2, unifies the nondifferentiable Wolfe type

dual and Mond-Weir type dual problems considered in section 2.1 and also incorporates particular cases.

Chapter 3 studies multiobjective continuous programming in which the components of the objective and constraint functions contain support functions. In this chapter Wolfe and Mond-Weir type duality are studied under invexity / generalized invexity requirements. The results of this chapter are regarded as dynamic versions of the duality results of nondifferentiable nonlinear problems derived in chapter second.

Chapter 4 is focused on the study of optimality criteria and duality in multiobjective variational problems involving higher order derivatives. The models of the variational problems presented in this chapter are obviously more general than those of the preceding chapters. It consists of three sections, 4.1, 4.2 and 4.3. In section 4.1, optimality conditions, both Fritz-John and Karush-Kuhn-Tucker type optimality conditions are derived for the variational problem and the extended notion of invexity / generalized invexity. As an application of Karush-Kuhn-Tucker optimality conditions, Wolfe type dual is formulated and various duality results are established under invexity / generalized invexity defined in this section. In this section, it is also shown that our results can be considered as continuous time extension of nonlinear problem existing in the literature. Section 4.2 formulates Mond-Weir dual for multiobjective variational problem considered in section 4.1 to relax the invexity requirements for various duality results to hold and gives relationship between the results of this section and those of nonlinear programming. Section 4.3 is meant to unify the dual formulations of the variational problems in section 4.1 and 4.2 and prove various duality results under invexity and generalized invexity.

Chapter 5 consists of two sections, 5.1 and 5.2. In section 5.1, optimality conditions are derived and Wolfe type duality and Mond-Weir type duality problems for a class of nondifferentiable variational problems involving higher order derivatives with nondifferentiable terms of square root of certain quadratic form are treated. The second section 5.2 is meant to present mixed type duality for the class of nondifferentiable multiobjective variational programming considered in section 5.1. The subsection 5.2.2 considers the variational problem with natural boundary conditions instead of fixed point conditions. These formulations are indicated to possess close relationship of the duality results of section 5.2 with those of nondifferentiable nonlinear programming.

Chapter 6 is devoted to the study of mixed type symmetric and self duality for multiobjective variational programming. In essence the formulation of the problem of this chapter combines the Wolfe and Mond-Weir type symmetric dual multiobjective variational problems already studied in the literature. Further, it is also shown that our results are closely related to those of their static counter parts.

Chapter 7 is the last chapter of the thesis. It has two sections, 7.1 and 7.2. In section 7.1, Wolfe and Mond-Weir type symmetric dual models for multiobjective variational problems with support functions are treated. For these pairs of problems, weak, strong and converse duality theorems are validated under convexity-concavity and pseudoconvexity-pseudoconcavity assumptions on certain combination of functionals. Self duality theorems for both the pairs are established. The problems with natural boundary values are formulated. In section 7.2, we present the mixed formulation that unifies two existing pairs, Wolfe and Mond-Weir type symmetric dual multiobjective variational problems containing support functions and various duality theorems are established under convexity-concavity and pseudoconvexity-pseudoconcavity of certain combination of functionals appearing in the formulation. A self-duality theorem under additional assumptions on the kernel functions that occur in the problems is validated. A pair of mixed type nondifferentiable multiobjective variational problem with natural boundary values is also formulated to indicate the validity of various duality theorems and to find linkage between the duality results of this section and those of the results of nonlinear programming.

The subject matter of the present research thesis is almost fully published / under publication in the form of the following research papers written by the author:

1) Optimality Criteria and Duality for Multiobjective Variational Problems Involving Higher Order Derivatives, *Journal of Applied Mathematics And Informatics*, Vol. 27(2009), No. 1 - 2, pp. 123 -137.

2) Optimality and Duality for Nondifferentiable Multiobjective Variational Problems with Higher Order Derivatives, *European Journal of Pure and Applied Mathematics,* Vol. 2, No 3 (2009), pp. 372-400.

3) Mixed type Multiobjective Variational Problems with Higher Order Derivatives, *Journal of Applied Mathematics and Informatics,* Vol. 27(2009), No. 1 - 2, pp. 245 – 257.

4) Multiobjective Continuous Programming Containing Support Functions, *Journal of Applied Mathematics And Informatics*, Vol. 27(2009), No. 3 - 4, pp. 603 – 619.

5) On Multiobjective Nonlinear Programming with Support Functions, to appear in *Journal of Applied Analysis in* 2010.

6) Mixed Type Symmetric and Self-Duality for Multiobjective Variational Problems, *European Journal of Pure and Applied Mathematics,* Vol. 2, No 4, 2009, pp. 578-603.

7) On Mixed Type Duality for Nondifferentiable Multiobjective Variational Problems, *European Journal of Pure and Applied Mathematics,* Vol.3, No 1, 2010, pp. 81-97.

8) Multiobjective Duality in Variational Problems with Higher Order Derivatives, C*ommunications and Network* Vol.2, No 1, 2010, pp. 138-144.

9) On Mixed Type Duality for Multiobjective Programming Containing Support Functions, to appear in *Journal of Informatics and Mathematics* 2010.

10) Symmetric Duality for Multiobjective Variational Problems Containing Support Functions", *Istanbul University Journal of School of Business Administration* Vol.39, No 1, 2010, pp. 1-20.

11) Mixed Type Symmetric Duality for Multiobjective Variational Problems Containing Support Functions", Submitted For Publication.

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"Our Lord! bestow on us Mercy from Thyself, and dispose of our affair for us in the right way!"

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Rumana Gulzar Mattoo

CONTENTS

Abstract

Chapter-1

INTRODUCTION

1.1 GENERAL INTRODUCTION

In this chapter we present a brief survey of related work done in the fascinating area of multiobjective mathematical programming followed by a precise summary of our own findings in the subsequent chapters of this thesis.

Many problems of practical importance can be transformed into different forms of minimization or maximization problems no matter whether such problems are from the field of engineering, science, business or finance. These problems share the characteristics of requirements of finding the most advantageous solution that offers certain optimal criteria under several limitations. Many of these problems concentrate primarily on optimizing the gain or the quality of performance: for instance the problem of optimal control (discrete or continuous), structural design, mechanical design, electrical network, water resource management, stochastic resource allocation, location facilities, etc., can be cast into optimization problems.

Euler has truly said,

> "***Since the fabric of the universe is most perfect and the work of the most Wise Creator, nothing at all takes place in the Universe in which some rule of the maximum or minimum does not appear".***

Almost all practical optimization problems are concerned with more than a single objective function**.** Real life problems require the optimization of multiple objectives at the same time. For example in planning of transportation or distribution schedule, the total transit time is usually not the only criterion to consider. A planner may also consider the cost and the reliability of the planner schedule before it is actually executed. When the evaluation process deals with more than one objective, the problems of this nature are described as multiple objective optimization. These objectives are often inter-conflicting. When objectives are conflicting, this implies that an objective cannot be improved without affecting the optimality of the other objectives. A possible solution to multiple criteria optimization should provide balance in objectives. These solutions may be suboptimal with respect to single objective programming problem. In fact, they are called trade-off solutions that are regarded as the best solution. Multiple criteria optimization is most often applied to deterministic problem in which the number of feasible alternatives is large. It is more useful with less controversial in business and government such as in oil refinery, scheduling, production planning, capital budgeting, forest management, determining reservoir release policy, allocation of audit staff in a firm, transportation and many others.

Optimality criteria play a very significant role in determining the solution of the problem as the classical calculus suggests. Fritz-John [81] was the first to derive necessary optimality conditions for constrained single objective optimization problem using Lagrange multiplier rule. Later Kuhn and Tucker [88] established necessary optimality conditions for the existence of optimal solution under certain constraint qualification in 1951. It was revealed after wards that W.Karush [83] had presented way back in 1939 without imposing any constraint qualification; thus the Kuhn-tucker conditions are known as Karush-Kuhn-Tucker optimality conditions. Abadie [1] established a regularity condition that enabled him to derive Karush-

Kuhn-Tucker conditions from Fritz John optimality conditions. Subsequently, Mangasarian and Fromovitz [93] generalized Fritz-John optimality conditions to treat equality and inequality constraints. Sufficiency of these conditions under convexity and generalized convexity were extensively treated by many authors notably, Mangasarian [92] and Martos [95].

The first notion of optimality in the setting of multiobjective programming goes back to Edgeworth in 1881 and Pareto in 1896 as well known historical references and is still the most extensively used. In (Edgeworth-) Pareto-optimality every feasible alternative that is not dominated by any other in terms of the componentwise partial order is considered to be optimal. Hence each solution is considered optimal that is not definitely worse than another. Thus, multiobjective optimization does not yield a single or a set of equally good answers, but rather suggests a range of potentially very different answers.

Optimality criteria are extensively studied as a vital organic part in the theory of single or multiple optimization theory because these criteria lay the foundation of duality which is a natural and significant concept. The term duality used in our daily life means the sort of harmony of two opposite or complementary parts by which they integrate into whole. Inner beauty in natural phenomena is bound up with duality, which has always been a rich source of inspiration in human knowledge through the centuries. The theory of duality is a vast subject, significant in art and natural science. Mathematics lies in it roots. The concept of duality has proved to be valuable notion in analysis of linear and nonlinear programming. According to Dantzig [47] the notion of duality was first introduced by Von-Neumann [120] and was subsequently formulated in the precise form by Gale, Kuhn and Tucker [57]. The concept of duality is to associate with each mathematical programming, called primal, (a Latin word, which means original) another mathematical programming called dual program. This idea is useful in economics where the dual problem can be stated in terms of price, in mechanics where primal and dual problems are two well-known forms of conservation principles.

Duality in nonlinear programming problems originated with duality results of quadratic programming, initially studied by Dannis [46]. Dual of convex primal program was given by Dorn [53], Mangasarian [92] and Wolfe [156]. Schechter [132] extended the duality results of Wolfe to nondifferentiable case by replacing gradients by subgradients using Slater's constraint qualification.

Mond and Weir [116] modified the Wolfe dual moving a part of objective function of Wolfe dual to the constraints and thus introducing Mond-Weir dual programming problem. The resulting pair of dual programming was nonconvex program and was found that there was no involution between primal and dual that is, the dual of the dual was not primal in general. In the literature of mathematical programming, a primal-dual pair of problem is called symmetric if the dual of the dual is primal problem. In the sense, a linear problem and its dual is symmetric. However, the majority of the formulation, in nonlinear programming does not possess this property. The first symmetric dual formulation in nonlinear programming was proposed by Dantzig, Eisenberg and Cottle [48] which subsumed the duality formulation of linear programming and certain duality formations in quadratic programming. Making use of the Fritz John optimality conditions, they proved weak and strong duality theorems for their pair of symmetric dual programming problems under differentiability conditions. These ideas were further extended to single and multiple objective variational problems.

Kuhn and Tucker [88] were the first to incorporate some interesting results concerning multiobjective optimization in 1951 .Since then, research in this area has made remarkable progress both theoretically and practically.

Some of the earliest attempts to obtain conditions for efficiency were carried out by Kuhn and Tucker [88], Arrow et al [2]. Their research has been inherited by Da Cunha and Polak [45], Neustadt [122], Ritter [126-128], Smale [139], Aubin [3], Husain et al. [62-68] and others.

Duality, which plays an important role in traditional mathematical programming, has been extended to multiobjective optimization since the late 1970's. Isermann [75-78] developed multiobjective duality in linear case while results for nonlinear cases have been given by Schonfeld [134], Tanino and Sawaragi [144], Mazzoleni [96], Corley [44], Nakayama [119] and others.

Concept of mixed type multiobjective duality seems to be quite interesting and useful from practical as well as from algorithmic point of view. The computational advantage of mixed type dual formulations involves the flexibility of the choice of constraints to be put in the Lagrange function can be exploited to develop certain efficient solution procedures for solving mathematical programming problems.

The main contribution of this thesis is to derive optimality criteria for differentiable as well as nondifferentiable multiobjective mathematical programming problems which contain both nonlinear programming problems and variational problems and study duality and symmetric duality for these problems. The variational problems are often described as continuous programming problems. In these problems nondifferentiability occurs due to the terms of square root of a certain quadratic forms and support functions. As the mixed type duality in mathematical programming is interesting from theoretical as well as computational point of view, mixed type multiobjective duality for the problems of this research are elaborately discussed and linkage between variational problems and the corresponding nonlinear programming problems is incorporated in most of the cases.

1.2 PRE-REQUISITES

1.2.1 Notations

In this section, we shall incorporate major symbols which are used throughout the research work reported in this thesis.

R^n = n-dimensional Euclidean space,

R_+^n = The non-negative orthant in R^n,

A^T = Transpose of the matrix A,

Let f be a numerical function defined on an open set Γ in R^n, then $\nabla f(\bar{x})$ denotes the gradient of f at $\bar{x}$, that is '

$$\nabla f(\bar{x}) = \left[\frac{\partial f(\bar{x})}{\partial x^1}, \ldots, \frac{\partial f(\bar{x})}{\partial x^n}\right]^T$$

Let ϕ be a real valued twice continuously differentiable function defined on an open set contained in $R^n \times R^m$. Then $\nabla_x \phi(x, y)$ and $\nabla_y \phi(x, y)$ denote the gradient (column) vector of ϕ with respect to x and y respectively i.e.,

$$\nabla_x \phi(\bar{x}, \bar{y}) = \left(\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \ldots, \frac{\partial \phi}{\partial x^n}\right)^T_{(\bar{x}, \bar{y})}$$

$$\nabla_y \phi(\bar{x}, \bar{y}) = \left(\frac{\partial \phi}{\partial y^1}, \frac{\partial \phi}{\partial y^2}, \ldots, \frac{\partial \phi}{\partial y^m}\right)^T_{(\bar{x}, \bar{y})}$$

Further $\nabla_x^2 \phi(\bar{x}, \bar{y})$ and $\nabla_{xy}^2 \phi(\bar{x}, \bar{y})$ denote respectively the $(n \times n)$ and $(n \times m)$ matrices of second order partial derivative i.e.,

$$\nabla_x^2 \phi(\bar{x}, \bar{y}) = \left(\frac{\partial^2 \phi}{\partial x^i \partial x^j}\right)_{(\bar{x}, \bar{y})}$$

$$\nabla_{xy}^2 \phi(\bar{x}, \bar{y}) = \left(\frac{\partial^2 \phi}{\partial x^i \partial y^j}\right)_{(\bar{x}, \bar{y})}$$

The symbols $\nabla_y^2\phi(\bar{x},\bar{y})$ and $\nabla_{yx}^2\phi(\bar{x},\bar{y})$ are similarly defined.

1.2.2 Definitions

Definition 1.1: Let $X \subseteq R^n$ be an open and convex set and $f : X \to R$ be differentiable. Then we define f to be

1. **Convex**, if for all $x_1, x_2 \in X$,

$$f(x_1) - f(x_2) \geq (x_1 - x_2)\nabla f(x_2)$$

2. **Strictly convex**, if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$

$$f(x_1) - f(x_2) > (x_1 - x_2)\nabla f(x_2)$$

3. **Quasi convex**, if for all $x_1, x_2 \in X$,

$$f(x_1) \leq f(x_2) \Rightarrow (x_1 - x_2)\nabla f(x_2) \leq 0$$

4. **Pseudo convex**, if for all $x_1, x_2 \in X$,

$$(x_1 - x_2)\nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2)$$

5. **Strictly pseudoconvex**, if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$

$$(x_1 - x_2)\nabla f(x_2) \geq 0 \Rightarrow f(x_1) > f(x_2)$$

6. **Invex,** if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$f(x_1) - f(x_2) \geq \eta(x_1, x_2)^T \nabla f(x_2)$$

7. **Pseudoinvex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$\eta^T(x_1, x_2)\nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2)$$

8. **Quasi-invex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$f(x_1) \leq f(x_2) \Rightarrow \eta^T(x_1, x_2)\nabla f(x_2) \leq 0.$$

$$(x_1 - x_2)^T \nabla f(x_2) + (x_1 - x_2)^T \nabla^2 f(x_2) p \geq 0 \Rightarrow f(x_1) \geq f(x_2) - \frac{1}{2} p^T \nabla^2 f(x_2) p$$

Clearly, a differentiable convex, pseudoconvex, quasiconvex function is invex, pseudoinvex or quasi invex respectively with $\eta^T(x_1, x_2) = (x_1 - x_2)$. Further we define f to be concave, strictly concave pseudoconcave, quasiconcave, strictly pseudo convex on X according as $-f$ is convex, strictly convex, quasi convex, pseudoconvex, strictly pseudoconvex.

In the following definitions we shall use D and D^2 for customary symbols $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$.

Definition 1.2:

1. **Invexity**, If there exists vector function $\eta(t,x,u) \in R^n$ with $\eta = 0$ and $x(t) = u(t)$, $t \in I = [a,b]$, a real interval, such that for a scalar function $\phi(t,x,\dot{x})$, the functional $\Phi(x) = \int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\Phi(u)\text{-}\Phi(x) \geqq \int_I \left\{ \eta\phi_x(t,x,\dot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x}) \right\} dt,$$

Φ is said to be invex in x and $\dot{x}$ on I with respect to η.

2. **Pseudoinvexity,** Φ is said to be pseudoinvex in x and $\dot{x}$ with respect to η if

$$\int_I \left\{ \eta^T\phi_x(t,x,\dot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x}) \right\} dt \geqq 0$$

implies $\Phi(x,\dot{u}) \geqq \Phi(x,\dot{x})$.

3. **Quasi-invexity,** The functional Φ is said to quasi-invex in x and $\dot{x}$ with respect to η if

$$\Phi(x,\dot{u}) \leqq \Phi(x,\dot{x}) \text{ implies}$$

$$\int_I \left\{ \eta^T\phi_x(t,x,\dot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x}) \right\} dt \leqq 0.$$

Consider the multiobjective variational problem (VP).

(VP): Minimize $\int_I \left(f^1(t,x,\dot{x}),\ldots, f^p(t,x,\dot{x}) \right) dt$

Subject to

$$g(t,x,\dot{x}) \leq 0 \ , \ t \in I$$

***Definition 1.3* (*Efficient Solution*):** A feasible solution $\bar{x}$ is efficient for (VP) if there exist no other feasible x for (VP) such that for some $i \in P = \{1,2,\ldots,p\}$,

$$\int_I f^i(t,x,\dot{x})\,dt < \int_I f^i(t,\bar{x},\dot{\bar{x}})\,dt$$

and

$$\int_I f^j(t,x,\dot{x})\,dt \leqq \int_I f^j(t,\bar{x},\dot{\bar{x}})\,dt \ \text{ for all } \ j \in P, \ j \neq i.$$

***Definition 1.4* (*Support function*):** Let K be a compact set in R^n, then the support function of K is defined by

$$s(x(t)|K) = \max\left\{ x(t)^T v(t) : v(t) \in K, t \in I \right\}$$

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis i.e., there exist $z(t) \in R^n$, $t \in I$, such that

$$s(y(t)|C) - s(x(t)|C) \geq (y(t) - x(t))^T z(t)$$

From [114], subdifferential of $s(x(t)|K)$ is given by

$$\partial s(x(t)|K) = \left\{ z(t) \in K,\ t \in I \text{ such that} \left| x(t)^T z(t) = s(x(t)|K) \right. \right\}.$$

For any set $\Gamma \subset R^n$, the normal cone to Γ at a point $x(t) \in \Gamma$ is defined by

$$N_\Gamma(x(t)) = \left\{ y(t) \in R^n \left| y(t)^T (z(t) - x(t)) \leq 0, \forall z(t) \in \Gamma \right. \right\}$$

It can be verified that for a compact convex set K, $y(t) \in N_K(x(t))$ if and only if

$$s(y(t)|K) = x(t)^T y(t) \ , \ t \in I$$

Definition 1.5 (Skew Symmetry): The function $f : I \times R^n \times R^n \times R^n \times R^n \to R$ is said to be skew symmetric if for all x and y in the domain of f if

$$f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) = -f(t, y(t), \dot{y}(t), x(t), \dot{x}(t)),\ t \in I$$

where x and y are piecewise smooth on I .

Definition 1.6: Let $f: R^n \to R$ be a convex function, then a subgradient of f at a point $x \in R^n$ is a vector $\xi \in R^n$ satisfying

$$f(y) \ge f(x) + \xi^T (y - x), \text{ for all } y \in R^n$$

There are number of constraint qualifications [92], which are required to be satisfied by the constraints in establishing the necessary optimality criteria to ensure that certain Lagrange multipliers are non zero. Here we describe only four of them for completeness.

i) **Slater's Constraint Qualification:** Let X^0 be a convex set in R^n. The m-dimensional convex vector function g on X^0 which defines the convex feasible region $X = \{x : x \in X^o, g(x) \le 0\}$ is said to satisfy Slater's constraint qualification on X^0 if there exists an $\bar{x} \in X^o$ such that $g(\bar{x}) \le 0$.

ii) **The Kuhn-Tucker Constraint Qualification:** Let X^0 be an open set in R^n.Let g be m-dimensional vector function on X^0 and let $X = \{x : x \in X^o, g(x) \le 0\}$.Then the constraints are said to satisfy Kuhn-Tucker constraint qualification at $\bar{x} \in X$, if g is differentiable at $\bar{x}$ and if

$$\left.\begin{array}{l} y \in R^n \\ \\ \\ \nabla g_i(\bar{x})\, y \le 0 \end{array}\right\rangle \Rightarrow \left\langle\begin{array}{l} \text{There exists an n-dimentional vector function e} \\ \text{in the interval } [0,1] \text{ such that} \\ (a)\quad e(0) = \bar{x} \\ (b)\quad e(\tau) \in X \ \ for\ \ 0 \le \tau \le 1 \\ (c)\quad e \text{ is differentiable } at\ \tau = 0 \text{ and} \\ \qquad \dfrac{d\, e(0)}{d\tau} = \lambda y \text{ for some } \lambda > 0. \end{array}\right.$$

where $I = \{i \mid g_i(\bar{x}) = 0\}$.

iii) **The reverse convex constraint qualification:** Let X^0 be an open set in R^n .let g be m-dimensional vector function defined on X^0 and let $X = \{x : x \in X^o, g(x) \le 0\}$, g is said to satisfy the reverse constraint qualification at $\bar{x} \in X$,if g is differentiable at $\bar{x}$ and if for each $i \in I$ either g_i is concave at $\bar{x}$ or g_i is linear on R^n ,where $I = \{i | g_i(\bar{x}) = 0\}$.

iv) **Linear independence constraint qualification:** The condition that the vectors $\nabla g_1(x_0),.....,\nabla g_m(x_0)$ are linearly independent is often referred to as linearly independence constraint qualification.

1.3 REVIEW OF THE RELATED WORK

1.3.1 Duality in Mathematical Programming

Nonlinear Programming

Consider the following nonlinear programming problem (P):

(P): Minimize $f(x)$

Subject to

$$h_j(x) \le 0, \quad j = 1, 2, \ldots, m$$

where $f : R^n \to R$ and $h_j : R^n \to R$, $j = 1, 2, \ldots, m$ are continuously differentiable.

The following problem (WD) is the Wolfe type dual to the problem (P):

(WD): Maximize $f(x) + y^T h(x)$

Subject to

$$\nabla\left(f(x) + y^T h(x)\right) = 0,$$

$$y \ge 0, \ y \in R^m$$

Mangasarian [92] explained by means of an example that certain duality theorems may not be valid if the objective or the constraint function is a generalized convex function. This motivated Mond and Weir [116] to introduce a different dual for (P) which is given below:

(MWD): Maximize $f(x)$

Subject to

$$\nabla f(x) + \nabla y^T h(x) = 0.$$

$$y^T h(x) \geq 0$$

$$y \in R_+^m$$

and they proved various duality theorems under pseudoconvexity of f and quasiconvexity of $y^T h(x)$ for all feasible solution of (P) and (MWD).

Later Weir and Mond [153] derived sufficiency of Fritz John optimality criteria under pseudoconvexity of the objective and quasiconvexity or semi-strict convexity of constraint functions. They formulated the following dual using Fritz John optimality conditions instead of Karush-Kuhn-Tucker optimality conditions and proved various duality theorems-thus the requirement of constraint qualification is eliminated.

(FrD): Maxmize $f(x)$

Subject to

$$\lambda_\circ \nabla f(x) + \nabla \lambda^T h(x) = 0.$$

$$\lambda^T h(x) \geq 0$$

$$(\lambda_\circ, \lambda) \geq 0, \ (\lambda_\circ, \lambda) \neq 0$$

Duality in Nondifferentiable Mathematical Programming

Mond [100] considered the following class of nondifferentiable mathematical programming problems:

(NP): Minimize $f(x)+\left(x^T Bx\right)^{\frac{1}{2}}$

Subject to

$$h_j(x)\leq 0, \quad j=1,2,\ldots,m$$

Here f and h_j , $j=1,2,\ldots,m$ are twice differentiable function from R^n to R and B is an n x n positive semidefinite (symmetric) matrix. It is assumed that the functions f and h_j , $j=1,2,\ldots,m$ are convex functions. They established a duality theorem between (NP) and the following problem

(ND): Maximize $f(u)+y^T h(u)-u^T\nabla\left[f(u)+y^T h(u)\right]$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+Bw=0,$$

$$w^T Bw\leq 1$$

$$y\geq 0.$$

Further on the lines of Mond and Weir [116], Chandra, Craven and Mond [33] introduced another dual program:

(NWD): Maximize $f(u)-u^T\nabla\left[f(u)+y^T h(u)\right]$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+Bw=0,$$

$$y^T h(u)\geq 0,$$

$$w^T Bw\leq 1,$$

$$y\geq 0.$$

and established duality theorems by assuming the function $f(\cdot)+(\cdot)^T Bw$ to be pseudoconvex and $y^T h(\cdot)$ to be quasiconvex for all feasible solutions of (NP) and (NWD).

Later Mond and Schechter [113] replaced the square root term by the norm term and considered the nondifferentiable nonlinear programming problems as:

(NP)$_1$: Minimize $f(x)+\|Sx\|_p$

Subject to

$$h_j(x)\le 0, \quad j=1,2,\ldots,m$$

Here f and h_j , $(j=1,2,\ldots,m)$ are twice differentiable function from R^n to R. The dual for (NP)$_1$ is the problem:

(ND)$_1$: Maximize $f(u)+y^T h(u)-u^T S^T v$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+S^T v=0,.$$

$$\|v\|_q\le 1,$$

$$y\ge 0.$$

where p and q are conjugate exponents.

Later Schechter [132] replaced the norm term or the square root term by a more general function as a support function of a compact set. The problem considered by Schechter [132] is:

(NP)$_2$: Minimize $f(x)+S(x|C)$

Subject to

$$h_j(x)\le 0, \quad j=1,2,\ldots,m,$$

where f and h_j , $(j=1,2,\ldots,m)$ are twice differentiable function from R^n to R and $S(x|C)$ is a support function of a compact convex set $C\subseteq R^n$. Using the subdifferential of the support function of $S(x|C)$, the dual of (NP)$_2$ is the problem:

(ND)$_2$: Maximize $f(u)+w^T u+y^T h(u)$

Subject to

$$\nabla f(u)+\nabla y^T h(u)+w=0,$$

$$y \geq 0,\ w \in C.$$

Duality in Multiobjective Mathematical Programming

Whenever we shall study multiobjective programming problem we shall follow the following conventions for vectors in R^n

$$x<y, \quad \Leftrightarrow \quad x_i<y_i, \quad i=1,2,\ldots,n.$$

$$x \leqq y, \quad \Leftrightarrow \quad x_i \leqq y_i, \quad i=1,2,\ldots,n.$$

$$x \leq y, \quad \Leftrightarrow \quad x_i \leq y_i, \quad i=1,2,\ldots,n, \text{ but } x \neq y$$

$x \not\leq y$, is the negation of $x \leq y$.

Consider the multiobjective programming problem:

(VP): V- Min $F(x)=\left(f_1(x), f_2(x),..., f_p(x)\right)$

Subject to

$$h_j(\bar{x}) \leqq 0,\ \left(j=1,2,...,m\right)$$

where $X \subseteq R^n$ is an open and convex set and f_i and h_j are differentiable functions where, $f_i : X \to R,\ i=1,2,\ldots,p$ and $h_j : X \to R,\ j=1,2,\ldots,m$. Here the symbol "V-Min" stands for vector minimization and minimality is taken in terms of either "*efficient points*" or "*properly efficient points*" given by Koopman [87] and Geoffrin [58] respectively.

Definition 1.7 **[58]:** A feasible point $\bar{x}$ for is said to be efficient solution of (VP), if there does not exist any feasible x for (VP) such that

$$f_r(x)<f_r(\bar{x}) \text{ for some } r,$$

$$f_i(x) \leqq f_i(\bar{x}) \text{ for all } i=1,2,...,k,\ i \neq r.$$

***Definition* 1.7 [58]:** A feasible point $\bar{x}$ is said to be properly efficient solution of (VP), if it is an efficient solution of (VP) and if there exists a scalar $M > 0$ such that for each i and $x \in X_0$ satisfying $f_i(x) < f_i(\bar{x})$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq M,$$

for some *j*, satisfying $f_j(x) < f_j(\bar{x})$.

Geoffrion [58] considered the following single objective minimization problems for fixed $\lambda \in R^p$:

(VP)$_\lambda$: Minimize $\sum_{i=1}^{p} \lambda_i f_i(x)$

Subject to

$$h_j(\bar{x}) \leqq 0, \ (j = 1, 2, ..., m)$$

and proved the following lemma connecting (VP) and (VP)$_\lambda$.

Lemma 1.1:

(i) Let $\lambda_i > 0$, $(i = 1, 2, ..., p)$, $\sum_{i=1}^{p} \lambda_i = 1$ be fixed. If $\bar{x}$ is optimal for $(VP)_\lambda$, then $\bar{x}$ is properly efficient for (VP).

(ii) Let f_i and h_j be convex functions Then $\bar{x}$ is properly efficient for (VP) iff $\bar{x}$ is optimal for are differentiable functions $(VP)_\lambda$ for some $\lambda > 0$, $\sum_{i=1}^{p} \lambda_i = 1$.

If f_i and h_j are differentiable convex functions then $(VP)_\lambda$ is a convex programming problem. Therefore in relation to $(VP)_\lambda$ consider the scalar maximization problem:

(VD)$_\lambda$: Maximize $\lambda^T f(x) + y^T h(x) e = \lambda^T \left(f(x) + y^T h(x) \right)$

Subject to

$$\nabla\left(\lambda^T f(x) + y^T h(x)\right) = 0$$

$$\lambda \in \Lambda^+ \ , \ y \geqq 0,$$

where $e = (1,1,\ldots,1) \in R^p$ and $\Lambda^+ = \left\{\lambda \in R^P : \lambda > 0 \ , \lambda^T e = 1\right\}$

Now as $(VD)_\lambda$ is a dual program of $(VP)_\lambda$, Weir [148] considered the following vector optimization problem in relation to (VP) as

(DV): Maximize $\lambda^T f(x) + y^T h(x) e$

Subject to

$$\nabla\left(w^T f(x) + y^T h(x)\right) = 0$$

$$w \in \Lambda^+ \ , \ y \geqq 0,$$

They termed (DV) as the dual of (VP) and proved various duality theorems between (VP) and (DV) under the assumption that f and h are convex functions.

Further, for the purpose of weakening the convexity requirements on objective and constraint functions, Weir [148] introduced another dual program (DV1).

(DV1): Maximize $f(x)$

Subject to

$$\nabla\left(\lambda^T f(x) + y^T h(x)\right) = 0$$

$$y^T h(x) \geq 0$$

$$\lambda \in \Lambda^+ \ , \ y \geqq 0,$$

For these problems, various duality theorems are proved by assuming the function f to be pseudo convex and $y^T h$ to be quasiconvex for their feasible solutions.

1.3.2 Symmetric Duality in Mathematical Programming

Symmetric Duality in Differentiable Mathematical Programming

Consider a function $f(x,y)$ which is differentiable in $x \in R^m$ and $y \in R^m$. Dantzig et al [48] introduced the following pair of problems:

(SP): Minimize $f(x,y) - y^T \nabla_y f(x,y)$

Subject to

$$\nabla_y f(x,y) \leq 0$$

$$(x,y) \geq 0.$$

(SD): Maximize $f(x,y) - x^T \nabla_x f(x,y)$

Subject to

$$\nabla_x f(x,y) \geq 0$$

$$(x,y) \geq 0.$$

and proved the existence of a common optimal solution to the primal (SP) and (SD), when **(i)** an optimal solution of $(x_\circ, y_\circ)$ to the primal (SP) exists **(ii)** f is convex in x for each y, concave in y for each x and **(iii)** f, twice differentiable, has the property that at $(x_\circ, y_\circ)$ its matrix of second partials with respect to y is negative definite.

Mond [99] further gave the following formulation of symmetric dual programming problems:

(MSP): Maximize $f(x,y) - y^T \nabla_y f(x,y)$

Subject to

$$\nabla_y f(x,y) \leq 0$$

$$x \geq 0.$$

(MSD): Maximize $f(x,y) - x^T \nabla_x f(x,y)$

Subject to

$$\nabla_x f(x,y) \geq 0$$

$$y \geq 0.$$

It may be remarked here that in [48], the constraints of both (SP) and (SD) include $x \geq 0$, $y \geq 0$**,** but only $x \geq 0$ is required in the primal and only $y \geq 0$ in the dual.

Later Mond and Weir [118] gave the following pair of symmetric dual nonlinear programming problems which allows the weakening of the convexity-concavity assumptions to pseudoconvexity-pseudoconcavity.

(M-WSP): Minimize $f(x,y)$

Subject to

$$\nabla_y f(x,y) \leq 0$$

$$y^T \nabla_y f(x,y) \geq 0,$$

$$x \geq 0.$$

(M-WSD): Maximize $f(x,y)$

Subject to

$$\nabla_x f(x,y) \geq 0$$

$$x^T \nabla_y f(x,y) \leq 0,$$

$$y \geq 0.$$

Symmetric Duality in Nondifferentiable Mathematical Programming

Let $f(x,y)$ which is differentiable in $x \in R^m$ and $y \in R^m$. Chandra and Husain [28] introduced pair of symmetric dual nondifferentiable programs and proved duality results assuming convexity-concavity conditions on the kernel function $f(x,y)$:

(NP): Minimize $f(x,y) - y^T \nabla_y f(x,y) + (x^T Bx)^{\frac{1}{2}}$

Subject to

$$-\nabla_y f(x,y) + Cw \geq 0 \; ,$$

$$w^T Cw \leq 1,$$

$$(x,y) \geq 0.$$

(ND): Maximize $f(x,y) - x^T \nabla_x f(x,y) - (y^T Cy)^{\frac{1}{2}}$

Subject to

$$-\nabla_x f(x,y) - Bz \leq 0$$

$$z^T Cz \leq 1 \; ,$$

$$(x,y) \geq 0.$$

where B and C are $n \times m$ and $m \times m$ positive semidefinite matrices.

Further on the lines of Mond and Weir [116], Chandra, Craven and Mond [33] presented another pair of symmetric dual nondifferentiable programs by weakening the convexity conditions on the kernel function $f(x,y)$ to the pseudoconvexity and pseudoconcavity. The problems considered in [33] are:

(PS): Minimize $f(x,y) + (x^T Bx)^{\frac{1}{2}} - y^T Cz$

Subject to

$$\nabla_y f(x,y) - Cz \leq 0 \; ,$$

$$y^T \left[\nabla_y f(x,y) - Cz \right] \geq 0$$

$$z^T Cz \leq 1,$$

$$x \geq 0.$$

(DS): Maximize $f(x,y) + (y^T Cy)^{\frac{1}{2}} - x^T Bw$

Subject to

$$\nabla_x f(x,y) + Bw \leq 0 \; ,$$

$$x^T\left[\nabla_x f(x,y)+Bw\right]\leq 0,$$

$$w^T Bw \leq 1,$$

$$y \geq 0.$$

Subsequently Mond and Schechter [111] introduced the following two pairs of symmetric dual programs with support functions ─ one of which is Wolfe [156] type and another is Mond and Weir [116] type.

(P): Minimize $f(x,y)-y^T\nabla_y f(x,y)+S(x|C_1)$

Subject to

$$\nabla_y f(x,y)-z\leq 0\ ,$$

$$z\in C_2,\ x\geq 0.$$

(D): Maximize $f(u,v)-u^T\nabla_x f(u,v)+S(v|C_2)$

Subject to

$$\nabla_x f(u,v)+w\geq 0\ ,$$

$w\in C_1,\ v\geq 0.$ and

(P1): Minimize $f(x,y)-y^T z+S(x|C_1)$

Subject to

$$\nabla_y f(x,y)-z\leq 0\ ,$$

$$y^T\left(\nabla_y f(x,y)-z\right)\geq 0,$$

$$z\in C_2,\ x\geq 0.$$

(D1): Minimize $f(u,v)+u^T w+S(v|C_2)$

Subject to

$$\nabla_x f(u,v)+w\geq 0\ ,$$

$$u^T\left(\nabla_x f(u,v)+w\right)\leq 0,$$

$$w\in C_1,\ v\geq 0.$$

Symmetric Duality in Multiobjective Programming

Mond and Weir [118] discussed symmetric duality in multiobjective programming by considering the following pair of programs:

(PS): Minimize $f(x,y)-\left(y^T\nabla_y\lambda^T f(x,y)\right)e$

Subject to

$$\nabla_y\lambda^T f(x,y)\leqq 0\ ,$$

$$x\geqq 0\ ,\ \lambda\in\Lambda^+$$

(DS): Maximize $f(x,y)-\left(x^T\nabla_x\lambda^T f(x,y)\right)e$

Subject to

$$\nabla_x\lambda^T f(x,y)\geqq 0\ ,$$

$$y\geqq 0\ ,\ \lambda\in\Lambda^+$$

where $f:R^n\times R^m\to R^p$, and proved the symmetric duality theorem under the convexity – concavity assumptions on $f(x,y)$. Here the minimization is taken in the sense of proper efficiency as given by Geoffrion [58].

Further on the lines of scalar case (Mond and Weir [116]) also considered another pair of symmetric dual programs and proved symmetric duality results under pseudoconvexity-pseudoconcavity:

(PS1): Minimize $f(x,y)$

Subject to

$$\nabla_2\lambda^T f(x,y)\leqq 0\ ,$$

$$y^T\nabla_2\lambda^T f(x,y)\geqq 0$$

$$x\geqq 0\ ,\ \lambda\in\Lambda^+$$

(DS1): Maximize $f(x,y)-\left(x^T\nabla_1\lambda^T f(x,y)\right)e$

Subject to

$$\nabla_1\lambda^T f(x,y)\geqq 0\ ,$$

$$x^T\nabla_1\lambda^T f(x,y)\leqq 0,$$

$$y\geqq 0\ ,\ \lambda\in\Lambda^+.$$

Later Chandra and Durga Prasad [34] introduced the following pair of multiobjective programs by associating a vector valued infinite game:

(PS*): Minimize $f(x,y)-\left(y^T\nabla_y\mu^T f(x,y)\right)e$

Subject to

$$\nabla_y\mu^T f(x,y)\leqq 0\ ,$$

$$x\geqq 0\ ,\ \mu\in\Lambda^+.$$

(DS*): Maximize $f(x,y)-\left(x^T\nabla_x\lambda^T f(x,y)\right)e$

Subject to

$$\nabla_x\lambda^T f(x,y)\geqq 0\ ,$$

$$y\geqq 0\ ,\ \lambda\in\Lambda^+$$

Here it may be noted that not the same λ is appearing in (PS*) and (DS*) and this creates certain difficulties which are also discussed in [34].

1.3.3 Variational Problems

Differentiable Variational Problems

A variational problem can be considered as a particular case of an optimal control problem in which the control function is the derivative of a state function.

In [43] Courant and Hilbert, quoting an earlier work of Friedrichs [56], gave a dual relationship for a simple type of unconstrained variational problem. Subsequently, Hanson [60] pointed out that some of the duality results of mathematical programming have analogues in variational calculus. Exploring this relationship between mathematical programming and the classical calculus of variations, Mond and Hanson [105] formulated a constrained variational problem as a mathematical programming problem and using Valentine's [145] optimality conditions for the same, presented its

Wolfe type dual variational problem for validating various duality results under convexity.

Mathematically, a variational problem is of the form:

(VP): Minimize $\int_I f(t,x,\dot{x})\,dt$

Subject to

$$x(a)=\alpha\ ,\ x(b)=\beta$$

$$g(t,x,\dot{x})\le 0,\qquad t\in I,$$

$$x\in C(I,R^n).$$

where $I=[a,b]$ is a real time interval, $\dot{x}$ denotes derivative of x with respect to t, $f: I\times R^n\times R^n\to R$ and $g: I\times R^n\times R^n\to R$ are continuously differentiable functions with respect to each of their arguments; $C(I,R^n)$ is the space of continuously differentiable functions $x: I\to R^n$, and is equipped with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differentiation operator D is given by $u=Dx \Leftrightarrow x(t)=\alpha+\int_a^t u(s)\,ds$ except at discontinuities.

The following necessary conditions for the existence for (VP) are derived by Valentine [145].

Theorem 1.1: For every minimizing arc $x=x^\circ(t)$ of the problem (VP), there exists a function of the form

$$H=\lambda_\circ f(t,x,\dot{x})-\lambda(t)^T g(t,x,\dot{x})$$

Such that

$$H_{\dot{x}}=\frac{d}{dt}H_x$$

$$\lambda(t)^T g(t,x,\dot{x})=0$$

$$(\lambda_\circ,\lambda(t))\ge 0\ ,\ (\lambda_\circ,\lambda(t))\ne 0\ ,\ t\in I$$

hold throughout I (except at corners of x° where $H_{\dot{x}} = \frac{d}{dt} H_x$, holds for unique right and left limits). Here $\lambda_\circ$ is constant and $\lambda(\cdot)$ is continuous except possibly for values of t corresponding to corners of x°.

Following is the Wolfe type dual variational problem [105] for validating various duality results under convexity:

(WD): Maximize $\int_i \left(f(t,u,\dot{u}) + y(t)^T g(t,u,\dot{u}) \right) dt$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$\left(f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) \right) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) = 0$$

$$y(t) \geqq 0 \ , \ t \in I$$

Later Bector, Chandra and Husain [15] studied Mond-Weir type duality for the problem of [105] for weakening its convexity requirement.

(MWD): Maximize $\int_i f(t,u,\dot{u}) dt$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$\left(f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) \right) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) = 0$$

$$\int_I y(t)^T g_{\dot{u}}(t,u,\dot{u}) dt > 0$$

$$y(t) \geqq 0 \ , \ t \in I$$

Nondifferentiable Variational Problems

Chandra, Craven and Husain [30] obtained necessary optimality conditions for a constrained continuous programming problem having term with a square root of a quadratic form in the objective function, and using these optimality conditions formulated Wolfe type dual which is given below:

(NVP): Minimize $\int_I \left\{ f(t,x,\dot{x})dt + \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \right\} dt$

Subject to

$$x(a) = \alpha \ , \ x(b) = \beta$$

$$g(t,x,\dot{x}) \leq 0, \qquad t \in I,$$

(WNVP): Maximize $\int_I \left\{ f(t,u,\dot{u}) + u(t)^T B(t) z(t) + y(t)^T g(t,u,\dot{u}) \right\} dt$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$f_u(t,u,\dot{u})dt + B(t)z(t) + y(t)^T g_u(t,u,\dot{u}) = D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) \ , \ t \in I$$

$$\bar{z}(t)^T B(t) \bar{z}(t) \leqq 1 \ , \ t \in I$$

$$y(t) \geq 0 \ , \ t \in I$$

Subsequently, Bector, Chandra and Husain [16] constructed a Mond-Weir dual which allows weakening of convexity hypothesis of [30] and derived various duality results under generalized convexity of functionals.

The Mond-Weir dual model to the problem (NVP) is given as:

(MWNVP): Maximize $\int_I \left(f(t,u,\dot{u}) + u(t)^T B(t) z(t) \right) dt$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$f_u(t,u,\dot{u})dt + B(t)z(t) + y(t)^T g_u(t,u,\dot{u}) = D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) \ , \ t \in I$$

$$\int_I y^j(t) g^j(t,u,\dot{u}) dt \geqq 0 \ , \ t \in I$$

$$\bar{z}(t)^T B(t) \bar{z}(t) \leqq 1 \ , \ t \in I$$

$$y(t) \geq 0 \ , \ t \in I$$

Husain and Jabeen [70] replaced the square root of quadratic form by the support function of a compact convex set that is somewhat more general and for which the subdifferential may be simply expressed. Fritz John and Karush-Kuhn-Tucker type necessary optimality conditions for this nondifferentiable continuous programming are derived in which nondifferentiability enters due to appearance of support functions in the integrand of the objective functional as well as in each constraint function.

The model for this nondifferentiable variational problem is:-

(CP): Minimize $\int_I \left(f\left(t, x(t), \dot{x}(t)\right) + s\left(x(t)\middle| K\right)\right) dt$

Subject to

$$x(a) = \alpha \ , \ x(b) = \beta$$

$$g^j\left(t, x(t), \dot{x}(t)\right) + s\left(x(t)\middle| C^j\right) \le 0, \quad j = 1, 2, ..., m$$

Where f and g are continuously differentiable and each C^j , $j = 1, 2, ..., m$ is a compact convex set in R^n .

The following is the Wolfe type dual problem to the problem (CP):

(WCD): $\underset{u\in X, \lambda, z, \omega}{\text{Maximize}}\ \psi\left(u, \lambda, z, \omega^1, ..., \omega^m\right) = \int_I \Big[f\left(t, u, \dot{u}\right) + u(t)^T z(t)$

$$+ \sum_{j=1}^{m} \lambda^j(t)\Big(g^j\left(t, u, \dot{u}\right) + u(t)^T \omega^j(t)\Big)\Big] dt$$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$f_u\left(t, u, \dot{u}\right) - z(t) + \sum_{j=1}^{m} \lambda^j(t)\Big(g_u^j\left(t, u, \dot{u}\right) + \omega^j(t)\Big)$$

$$= D\Big(f_{\dot{u}}\left(t, u, \dot{u}\right) + \lambda(t)^T g_{\dot{u}}^j\left(t, u, \dot{u}\right)\Big) \ , \ t \in I$$

$$\bar{z}(t) \in K_i \ , \ \omega^j(t) \in C^j \quad , t \in I, \quad j = 1, 2, \ldots, m,$$

$$\lambda(t) \ge 0 \ , \ t \in I$$

The Mond-Weir type dual to the problem (CP) is given as,

(M-WCD): $\underset{u\in X,\lambda,z,\omega}{\text{Maximize}}\ \psi\left(u,\lambda,z,\omega^1,\ldots,\omega^m\right)=\int_I\left(f\left(t,u,\dot{u}\right)+u\left(t\right)^T z\left(t\right)\right)dt$

Subject to

$$u(a)=\alpha\ ,\ u(b)=\beta$$

$$f_u\left(t,u,\dot{u}\right)+z\left(t\right)+\sum_{j=1}^{m}\lambda^j\left(t\right)\left(g_u^j\left(t,u,\dot{u}\right)+\omega^j\left(t\right)\right)$$

$$=D\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\lambda\left(t\right)^T g_{\dot{u}}^j\left(t,u,\dot{u}\right)\right)\ ,\ t\in I$$

$$\sum_{j=1}^{m}\int_I\lambda^j\left(t\right)\left(g^j\left(t,u,\dot{u}\right)+u\left(t\right)^T\omega^j\left(t\right)\right)dt\geq 0\ ,\ t\in I\ ,$$

$$\bar{z}\left(t\right)\in K_i\ ,\ \omega^j\left(t\right)\in C^j\quad ,t\in I,\quad j=1,2,\ldots,m,$$

$$\lambda\left(t\right)\geq 0\ ,\ t\in I\ .$$

Following the scheme of formulation in Bector *et.al.* [12] and Xu [157], Husain and Jabeen [71] formulated the following mixed type dual (Mix CD) to (CP).

(Mix CD): $\underset{u\in X,\lambda,z,\omega^1,\ldots,\omega^m}{\text{Maximize}}\int_I\Big[f\left(t,u,\dot{u}\right)+u\left(t\right)^T z\left(t\right)$

$$+\sum_{j\in J_\circ}^{m}\lambda^j\left(t\right)\left(g^j\left(t,u,\dot{u}\right)+u\left(t\right)^T\omega^j\left(t\right)\right)\Big]dt$$

Subject to

$$u(a)=\alpha\ ,\ u(b)=\beta$$

$$f_u\left(t,u,\dot{u}\right)+z\left(t\right)+\sum_{j=1}^{m}\lambda^j\left(t\right)\left(g_u^j\left(t,u,\dot{u}\right)+\omega^j\left(t\right)\right)$$

$$=D\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\lambda\left(t\right)^T g_{\dot{u}}^j\left(t,u,\dot{u}\right)\right)\ ,\ t\in I$$

$$\sum_{j=J_\alpha}\int_I\lambda^j\left(t\right)\left(g^j\left(t,u,\dot{u}\right)+u\left(t\right)^T\omega^j\left(t\right)\right)dt\geq 0\ ,\ t\in I\ ,\ \alpha=1,\ldots,r\ ,$$

$$\bar{z}\left(t\right)\in K_i\ ,\ \omega^j\left(t\right)\in C^j\quad ,t\in I,\quad j=1,2,\ldots,m,$$

$$\lambda\left(t\right)\geq 0\ ,\ t\in I\ .$$

where $J_\alpha\subseteq M=\{1,2,\ldots,m\}$, $\alpha=1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r}J_\alpha=M$ and $J_\alpha\bigcap J_\beta=\phi$, if $\alpha\neq\beta$.

If $J_\circ = M$ and $J_\alpha = \phi$ for $\alpha \in \{1,2,\ldots,r\}$, then the (Mix CD) becomes the problem (WCD). In case $J_\circ = \phi$ and $J_\alpha = M$ for some $\alpha \in \{1,2,\ldots,r\}$, then (Mix CD) becomes (M-WCD).

Multiobjective Variational Problems

Differentiable Multiobjective Variational Problems

Many authors have studied optimality and duality for multiobjective variational problems. Bector and Husain [19] were probably the first to introduce multiobjective programming in calculus of variation. They considered the following multiobjective variational problem (VP):

$$\textbf{(VP):}\ \text{Minimize}\left(\int_I \left(f^1\left(t,x(t),\dot{x}(t)\right)\right)dt,\ldots,\int_I \left(f^p\left(t,x(t),\dot{x}(t)\right)\right)dt\right)$$

Subject to

$$x(a)=\alpha\ ,\ x(b)=\beta$$

$$g\left(t,x(t),\dot{x}(t)\right) \leqq 0\ ,\ t\in I,$$

$$x \in C\left(I,R^n\right)$$

where, $f^i : I\times R^n \times R^n \to R$, $i \in P = 1,2,\ldots,p$, $g: I\times R^n\times R^n \to R^m$, are assumed to be continuously differentiable functions, for each $t\in I$, $i\in P$, $B^i(t)$ is an $n\times n$ positive semidefinite symmetric matrix with $B^i(\cdot)$ continuous on I.

Bector and Husain [19] constructed Wolfe type dual and Mond-Weir type dual and proved various duality theorems under convexity and generalized convexity of functionals.

$$\textbf{(WD):}\ \text{Maximize}\ \left(\int_i \left(f^1(t,u,\dot{u})+y(t)^T g(t,u,\dot{u})\right)dt\right.$$

$$\left.,\ldots,\int_i \left(f^p(t,u,\dot{u})+y(t)^T g(t,u,\dot{u})\right)dt\right)$$

Subject to

$$u(a)=\alpha\ ,\ u(b)=\beta$$

$$\left(\lambda^T f_u + y(t)^T g_u\right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}}\right) = 0 \;,\; t \in I$$

$$y(t) \geqq 0 \;,\; t \in I$$

$$\lambda > 0 \;,\; \lambda^T e = 1$$

where $e = (1,1,...,1)^T$ and $\lambda \in R^k$.

The following Mond-Weir type dual to the problem (VP):

(M-WD): Maximize $\left(\int_I f^1(t,u,\dot{u})dt, \ldots, \int_I f^p(t,u,\dot{u})dt \right)$

Subject to

$$u(a) = \alpha \;,\; u(b) = \beta \;,$$

$$\left(\lambda^T f_u + y(t)^T g_u\right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}}\right) = 0,\; t \in I$$

$$\int_1 y(t)^T g(t,u,\dot{u})dt \geqq 0 \;,$$

$$y(t) \geqq 0 \;,\; t \in I,$$

$$\lambda > 0.$$

Nondifferentiable Multiobjective Variational Problems

In [97], Mishra and Mukerjee discussed duality for multiobjective variational problems involving generalized (F,ρ)-convex functions. In [91], Liu proved only some weak duality theorem for nondifferentiable multiobjective variational problems involving generalized (F,ρ)-convex functions.

Kim *et al.* [85] considered the following nondifferentiable variational problem:

(MP): Minimize $\left(\int_I \left(f^1(t,x(t),\dot{x}(t)) + \left(x(t)^T B^1(t) x(t)\right)^{\frac{1}{2}} \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,x(t),\dot{x}(t)) + \left(x(t)^T B^p(t) x(t)\right)^{\frac{1}{2}} \right) dt \right)$$

Subject to

$$x(a)=\alpha \ , \ x(b)=\beta$$

$$g(t,x(t),\dot{x}(t)) \leqq 0 \ , \ t \in I,$$

$$x \in C(I,R^n)$$

where, $f^i : I \times R^n \times R^n \times R^n \to R, i \in P = 1,2,\ldots,p$, $g : I \times R^n \times R^n \times R^n \to R^m$, are assumed to be continuously differentiable functions, for each $t \in I, i \in P$, $B^i(t)$ is an $n \times n$ positive semidefinite symmetric matrix with $B^i(\cdot)$ continuous on I.

The following are the Wolfe type and Mond-Weir type dual model for the problem (MP) considered in Kim et al [85]:

(MDP1): Maximize

$$\Bigg(\int_I \Big(f^1(t,y(t),\dot{y}(t))dt + \big(y(t)^T B^1(t)\omega^1(t)\big) + \lambda(t)^T g(t,y(t),\dot{y}(t))\Big)dt$$
$$,\ldots, \int_I \Big(f^p(t,y(t),\dot{y}(t))dt + \big(y(t)^T B^p(t)\omega^p(t)\big) + \lambda(t)^T g(t,y(t),\dot{y}(t))\Big)dt \Bigg)$$

Subject to

$$y(a)=\alpha \ , \ y(b)=\beta$$

$$\sum_{i=1}^{p} \tau^i \Big(f_x^i(t,y(t),\dot{y}(t)) + B^i(t)\omega(t) + \lambda(t)^T g_x(t,y(t),\dot{y}(t))\Big)$$
$$= D\left\{ \sum_{i=1}^{p} \tau^i f_{\dot{x}}^i(t,y(t),\dot{y}(t)) + \lambda(t)^T g_{\dot{x}}(t,y(t),\dot{y}(t)) \right\}$$

$$\omega^i(t)^T B^i(t)\omega^i(t) \le 1 \ , \ i \in P$$

$$\lambda(t) \geq 0 \ , \ \tau^i \geq 0 \ , \ \sum_{i=1}^{p} \tau^i = 1$$

$$y \in C(I,R^n) \ , \ \omega \in C(I,R^n) \ , \ \lambda \in C(I,R^m).$$

(MDP2): Maximize $\left(\int_I \left(f^1\left(t, y(t), \dot{y}(t)\right) dt + y(t)^T B^1(t)\,\omega^1(t)\right) dt\right.$

$$\left. ,\ldots, \int_I \left(f^p\left(t, y(t), \dot{y}(t)\right) dt + \left(y(t)^T B^p(t)\,\omega^p(t)\right)\right) dt \right)$$

Subject to

$$y(a) = \alpha \ , \ y(b) = \beta$$

$$\sum_{i=1}^{p} \tau^i \left(f_x^i\left(t, y(t), \dot{y}(t)\right) + B^i(t)\,\omega(t) + \lambda(t)^T g_x\left(t, y(t), \dot{y}(t)\right)\right)$$

$$= D\left\{ \sum_{i=1}^{p} \tau^i f_{\dot{x}}^i\left(t, y(t), \dot{y}(t)\right) + \lambda(t)^T g_{\dot{x}}\left(t, y(t), \dot{y}(t)\right)\right\}$$

$$\int_I \lambda(t)^T g\left(t, y(t), \dot{y}(t)\right) dt \geq 0$$

$$\omega^i(t)^T B^i(t)\,\omega^i(t) \leq 1 \, , \ i \in P \ , \ t \in I$$

$$\lambda(t) \geq 0 \ , \ \tau_i \geq 0 \ , \ \sum_{i=1}^{p} \tau_i = 1$$

$$y \in C\left(I, R^n\right) \ , \ \omega \in C\left(I, R^n\right), \ \lambda \in C\left(I, R^m\right).$$

1.3.4 Symmetric Duality for Variational Problems

Mond and Hanson [108] and Bector, Chandra and Husain [15] extended symmetric duality to Variational problems. In [108] they investigated Wolfe type duality symmetric duality for the variational problems (VP). Later [15] Bector, Chandra and Husain studied Mond-Weir type symmetric dual variational problems in order to weaken the convexity-concavity assumptions. Smart and Mond [141] applied invexity for Variational problems introduced by Mond, Chandra and Husain [103] to symmetric dual Variational problems without nonnegetivity constraints of Mond and Hanson [105], but subjecting invexity to an additional condition.

Mond and Hanson [108] studied symmetric duality for the following variational problem under convexity / concavity assumptions:

(Primal): Minimize $\int_a^b \Big\{ f\left(t,x,\dot{x},y,\dot{y}\right) - y(t)^T f_y\left(t,x,\dot{x},y,\dot{y}\right)$

$$+y(t)\frac{d}{dt} f_{\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right)\Big\} dt$$

Subject to

$$x(a)=\alpha \quad , \quad x(b)=\beta$$

$$y(a)=\gamma \quad , \quad y(b)=\delta$$

$$\frac{d}{dt} f_{\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right) \geq f_y\left(t,x,\dot{x},y,\dot{y}\right)$$

$$x(t)\geq 0$$

(Dual): Maximize $\int_a^b \Big\{ f\left(t,u,\dot{u},v,\dot{v}\right) - u(t)^T f_x\left(t,u,\dot{u},v,\dot{v}\right)$

$$+u(t)\frac{d}{dt} f_{\dot{x}}\left(t,u,\dot{u},v,\dot{v}\right)\Big\} dt$$

Subject to

$$u(a)=\alpha \quad , \quad u(b)=\beta$$

$$v(a)=\gamma \quad , \quad v(b)=\delta$$

$$\frac{d}{dt} f_{\dot{x}}\left(t,u,\dot{u},v,\dot{v}\right) \leq f_x\left(t,u,\dot{u},v,\dot{v}\right)$$

$$v(t)\geq 0$$

Let $I=[a,b]$ be the real interval, $x: I \to R^n$ and $y: I \to R^m$, x and $\dot{x}$ denote derivatives of x and y respectively with respect to t and $f\left(t,x,\dot{x},y,\dot{y}\right)$ is a continuously differentiable scalar function. They needed f to be convex in x and $\dot{x}$ for each y and $\dot{y}$ and concave in y and $\dot{y}$ for each x and $\dot{x}$.

If the constraints $x(t)\geq 0$ and $y(t)\geq 0$ are removed from the above problem primal and dual problems respectively, we get the pair considered by Smart and Mond [141], wherein weak duality theorem is proved assuming the functional $\int_a^b f\,dt$ to be invex in x and $\dot{x}$ and $-\int_a^b f\,dt$ to be invex in y and $\dot{y}$.

Subsequently, Bector, Chandra and Husain [15] presented a pair of Mond-Weir type symmetric dual variational problems in order to relax convexity-concavity to pseudoconvexity-pseudoconcavity.

The following are the primal and dual problems formulated in [15]:

Problem I (Primal) = P

Minimize $\int_a^b f(t, x, \dot{x}, y, \dot{y})\,dt$

Subject to

$$x(a) = \alpha \quad , \quad x(b) = \beta$$
$$y(a) = \gamma \quad , \quad y(b) = \delta$$
$$f_y(t, x, \dot{x}, y, \dot{y}) - Df_{\dot{y}}(t, x, \dot{x}, y, \dot{y}) \leq 0$$
$$\int_a^b y(t)^T \left(f_y(t, x, \dot{x}, y, \dot{y}) - Df_{\dot{y}}(t, x, \dot{x}, y, \dot{y}) \right) dt \geq 0$$
$$x(t) \geq 0$$

Problem II (Dual) = D

Maximize $\int_a^b f(t, x, \dot{x}, y, \dot{y})\,dt$

Subject to

$$x(a) = \alpha \quad , \quad x(b) = \beta$$
$$y(a) = \gamma \quad , \quad y(b) = \delta$$
$$f_x(t, x, \dot{x}, y, \dot{y}) - Df_{\dot{x}}(t, x, \dot{x}, y, \dot{y}) \geq 0$$
$$\int_a^b x(t)^T \left(f_x(t, x, \dot{x}, y, \dot{y}) - Df_{\dot{x}}(t, x, \dot{x}, y, \dot{y}) \right) dt \leq 0$$
$$y(t) \geq 0.$$

The usual duality results are derived for above pair of Mond-Weir problems under pseudoconvexity and pseudoconcavity. The close relationship between the duality results for the pair in [15] and those of its counterpart is pointed out.

Symmetric Duality for Multiobjective Variational Problems

Gulalti, Husain and Ahmed [59] studied symmetric duality for multiobjective variational problems under appropriate invexity assumptions.

Following is the pair of the Wolfe type multiobjective symmetric dual variational problems constructed in [59]:

(VP): Minimize $\int_a^b \Big[f\left(t,x,\dot{x},y,\dot{y}\right) - \Big\{ y\left(t\right)^T \left(\lambda^T f_y\left(t,x,\dot{x},y,\dot{y}\right) \right.$

$$\left. - D\lambda^T f_{\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right) \right) \Big\} e \Big] dt$$

Subject to

$$x\left(a\right)=0=x\left(b\right) \quad , \quad y\left(a\right)=0=y\left(b\right)$$

$$\dot{x}\left(a\right)=0=\dot{x}\left(b\right) \quad , \quad \dot{y}\left(a\right)=0=\dot{y}\left(b\right)$$

$$\lambda^T f_y\left(t,x,\dot{x},y,\dot{y}\right) \leqq D\lambda^T f_{\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right) \, , \, t \in I$$

$$\lambda > 0$$

$$\lambda^T e = 1$$

The dual to this problem is:

(VD): Maximize $\int_a^b \Big[f\left(t,u,\dot{u},v,\dot{v}\right) - \Big\{ u\left(t\right)^T \left(\lambda^T f_x\left(t,u,\dot{u},v,\dot{v}\right) \right.$

$$\left. - D\lambda^T f_{\dot{x}}\left(t,x,\dot{x},y,\dot{y}\right) \right) \Big\} e \Big] dt$$

Subject to

$$u\left(a\right)=0=u\left(b\right) \quad , \quad v\left(a\right)=0=v\left(b\right)$$

$$\dot{u}\left(a\right)=0=\dot{u}\left(b\right) \quad , \quad \dot{v}\left(a\right)=0=\dot{u}\left(b\right)$$

$$\lambda^T f_x\left(t,u,\dot{u},v,\dot{v}\right) \leqq D\lambda^T f_{\dot{x}}\left(t,u,\dot{u},v,\dot{v}\right) \, , \, t \in I$$

$$\lambda > 0$$

$$\lambda^T e = 1.$$

where $f : I \times R^n \times R^n \times R^m \times R^m \to R^p$ is twice continuously differentiable and $e = \left(1,1,...,1\right)^T \in R^p$. If $p = 1$, the problems (VP) and (VD) reduce to single objective symmetric dual variational problems considered by Smart and Mond [141].

In [59], the following pair of Mond-Weir type multiobjective symmetric dual variational problems are considered:

(VP): Minimize $\left(\int_a^b f^1\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) dt \right.$

$$\left. , \ldots, \int_a^b f^p\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) dt \right)$$

Subject to

$$x(a) = 0 = x(b) \quad , \quad y(a) = 0 = y(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b) \quad , \quad \dot{y}(a) = 0 = \dot{y}(b)$$

$$\lambda^T f_y\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) \leqq D\lambda^T f_{\dot{y}}\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right), \; t \in I$$

$$\int_I y(t)^T \left(\lambda^T f_y\right)\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) dt$$

$$\geqq \int_a^b y(t)^T D\left(\lambda^T g_{\dot{y}}\right)\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) dt$$

$$x(t) \geq 0 \quad , \quad t \in I$$

$$\lambda > 0 \; .$$

(VD): Maximize $\left(\int_a^b f^1\left(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\right) dt \right.$

$$\left. , \ldots, \int_a^b f^p\left(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\right) dt \right)$$

Subject to

$$u(a) = 0 = u(b) \quad , \quad v(a) = 0 = v(b)$$

$$\dot{u}(a) = 0 = \dot{u}(b) \quad , \quad \dot{v}(a) = 0 = \dot{u}(b)$$

$$\lambda^T f_y\left(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\right) \leqq D\lambda^T f_{\dot{y}}\left(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\right), \; t \in I$$

$$\int_I y(t)^T \left(\lambda^T f_y\right)\left(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\right) dt$$

$$\geqq \int_a^b y(t)^T D\left(\lambda^T g_{\dot{y}}\right)\left(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\right) dt$$

$$v(t) \geq 0 \quad , \quad t \in I$$

$$\lambda > 0 .$$

1.3.5 Variational Problems with Higher Order Derivative

Husain and Jabeen [73] presented the following variational problem (P) with higher order derivatives as to study optimality and duality.

(P): $\underset{x\in X}{\text{Minimize}} \int_I f\left(t,x,\dot{x},\ddot{x}\right)dt$

Subject to

$$x(a)=\alpha \quad , \quad x(b)=\beta$$
$$\dot{x}(a)=\gamma \quad , \quad \dot{x}(b)=\delta$$
$$g\left(t,x,\dot{x},\ddot{x}\right)\geq 0 \quad , \quad t\in I$$
$$h\left(t,x,\dot{x},\ddot{x}\right)=0 \quad , \quad t\in I$$

where

1) $f^i : I\times R^n\times R^n\times R^n \to R$, $i=1,2,\ldots,p$, $g : I\times R^n\times R^n\times R^n \to R^m$ and $h : I\times R^n\times R^n\times R^n \to R^k$ are continuously differentiable function

2) X designates the space of piecewise functions $x : I\to R^n$ possessing derivatives $\dot{x}$ and $\ddot{x}$ with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty+\|D^2x\|_\infty$, where the differentiation operator D is given by

$$u=Dx \Leftrightarrow x(t)=\alpha+\int_a^t u(s)\,ds,$$

where α is given boundary value; thus $D\equiv\frac{d}{dt}$ except at discontinuities.

Theorem 1.2 [73] (*Fritz John Optimality Conditions*): If $\bar{x}$ is an optimal solution of (P) and $h_x\left(x(\cdot),\dot{x}(\cdot),\ddot{x}(\cdot)\right)$ maps X into the subspace of $C\left(I,R^k\right)$, then there exists Lagrange multiplier $\bar{\tau}\in R$, the piecewise smooth $\bar{y} : I\to R^m$ and $\bar{z} : I\to R^k$ satisfying

$$\left(\bar{\tau}\, f_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\lambda(t)^T g_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\mu(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)_x\right)$$
$$=D\left(\bar{\tau}\, f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\lambda(t)^T g_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\mu(t)^T h_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)$$
$$-D^2\left(\bar{\tau}\, f_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\lambda(t)^T g_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\mu(t)^T h_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right),\ t\in I,$$

$$\lambda(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0\ ,\ t\in I\ ,$$

$$\left(\bar{\tau},\lambda(t)\right)\geqq 0\ ,\ t\in I\ ,$$

$$\left(\tau,\lambda(t),\mu(t)\right)\neq 0\ ,\ t\in I\ .$$

Husain *et al.* [73] formulated Wolfe type dual to variational problem (P).

Dual (WD): $\underset{x\in X,\lambda,\mu}{\text{Maximize}}\int_I\left(f\left(t,x,\dot{x},\ddot{x}\right)-\lambda(t)^T g\left(t,x,\dot{x},\ddot{x}\right)-\mu(t)^T h\left(t,x,\dot{x},\ddot{x}\right)\right)dt$

Subject to

$$x(a)=\alpha\quad,\quad x(b)=\beta$$
$$\dot{x}(a)=\gamma\quad,\quad \dot{x}(b)=\delta$$

$$\left(f_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\lambda(t)^T g_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\mu(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)_x\right)$$
$$=D\left(f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\lambda(t)^T g_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\mu(t)^T h_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)$$
$$-D^2\left(f_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\lambda(t)^T g_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\mu(t)^T h_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right),\ t\in I,$$

$$\lambda(t)\geqq 0\ ,\ t\in I$$

Following Mond-Weir dual model was formulated by Husain et al [73] in order to further weaken pseudoinvexity requirements.

(M-WD) $\underset{x\in X,\lambda,\mu}{\text{Maximize}}\int_I f\left(t,x,\dot{x},\ddot{x}\right)dt$

Subject to

$$x(a)=\alpha\quad,\quad x(b)=\beta$$
$$\dot{x}(a)=\gamma\quad,\quad \dot{x}(b)=\delta$$

$$\left(f_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\lambda(t)^T g_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\mu(t)^T h_x\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)$$
$$=D\left(f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\lambda(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\mu(t)^T h_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)$$
$$-D^2\left(f_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\lambda(t)^T g_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)-\mu(t)^T h_{\ddot{x}}\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right),\ t\in I,$$

$$\int_I \left(\lambda(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\mu(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt \le 0$$

$$\lambda(t) \geqq 0\ ,\quad t\in I$$

Summary of the Thesis

The research work reported in this thesis is presented in chapters 2-7. The results in these chapters are briefly summarized as follows.

Chapter 2

Chapter 2 has two sections, 2.1 and 2.2. In the section 2.1, we consider the following nondifferentiable nonlinear programming problem with support functions:

(NP): Minimize $\left(f^1(x)+S\left(x\middle|C^1\right),\ldots, f^p(x)+S\left(x\middle|C^p\right)\right)$

Subject to

$$g^j(x)+S\left(x\middle|D^j\right)\leqq 0\ ,\ j=1,2,\ldots,m$$

where,

i. $f^i : R^n \to R$ and $g^j : R^n \to R$, $j=1,2,\ldots,m$ are differentiable functions and

ii. $S\left(\cdot\middle|C^i\right)$, $i=1,2,\ldots,p$ and $S\left(\cdot\middle|D^j\right)$, $j=1,2,\ldots,m$ are support functions of a compact convex set C^i, $i=1,2,\ldots,p$ and D^j, $j=1,2,\ldots,m$ in R^n.

For this problem, we formulate the following Wolfe and Mond-Weir type dual problems and establish various duality results under invexity / generalized invexity assumptions.

The following Wolfe type dual to the problem (NP) is presented:

(WND): Maximize $\left(f^1(u)+u^T z^1+\sum_{j=1}^{m} y^j \left(g^j(u)+u^T w^j \right) \right.$

$$\left. ,\ldots, f^p(u)+u^T z^p+\sum_{j=1}^{m} y^j \left(g^j(u)+u^T w^j \right) \right)$$

Subject to

$$\sum_{i=1}^{p} \lambda^i \nabla \left(f^i(u)+u^T z^i \right)+\sum_{j=1}^{m} y^j \nabla \left(g^j(u)+u^T w^j \right)=0$$

$$z^i \in C^i \quad , \quad i=1,2,\ldots,p\,,$$

$$w^j \in D^j \quad , \quad j=1,2,\ldots,m\ ,$$

$$y \geqq 0\,,$$

$$\lambda > 0\ ,\sum_{i=1}^{p} \lambda^i =1 \ .$$

The Mond-Weir type dual to the problem (NP) is given as:

(M-WND): Maximize $\left(f^1(u)+u^T z^1,\ldots, f^p(u)+u^T z^p \right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \nabla \left(f^i(u)+u^T z^i \right)+\sum_{j=1}^{m} y^j \left(g^j(u)+u^T w^j \right)=0\,,$$

$$z^i \in C^i \quad , \quad i=1,2,\ldots,p,$$

$$\sum_{j=1}^{m} y^j \left(g^j(u)+u^T w^j \right) \geqq 0\,,$$

$$\lambda > 0 \quad , \qquad y \geqq 0$$

Several known results are deduced as special cases.

In section 2.2, mixed type duality is studied under suitable invexity / generalized invexity requirements to unify Wolfe and Mond-Weir type dual problem considered in section 2.1.

The following is the mixed type dual (Mix D) to (NP):

(Mix D): Maximize $\left(f^1(u)+u^T z^1+\sum_{j\in J_\circ} y^j\left(g^j(u)+u^T w^j\right)\right.$

$$\left., \ldots, f^p(u)+u^T z^p+\sum_{j\in J_\circ} y^j\left(g^j(u)+u^T w^j\right)\right)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^T z^i\right)+\sum_{j=1}^{m} y^j\nabla\left(g^j(u)+u^T w^j\right)=0,$$

$$\sum_{j\in J_\alpha} y^j\left(g^j(u)+u^T w^j\right)\geqq 0,\ \alpha=1,2,\ldots,r,$$

$$z^i\in C^i \quad , \quad i=1,2,\ldots,p,$$

$$w^j\in D^j \quad , \quad j=1,2,\ldots,m \ ,$$

$$y\geqq 0,$$

$$\lambda\in\Lambda,$$

where $\quad \Lambda=\left\{\lambda\in R^p \,\middle|\, \lambda>0\ ,\ \sum_{i=1}^{p}\lambda^i=1\right\}.$

where $J_\alpha\subseteq M=\{1,2,\ldots,m\}$, $\alpha=0,1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r} J_\alpha=M$ and $J_\alpha\bigcap J_\beta=\phi$, if $\alpha\neq\beta$.

For the above pair of problems (VP) and (Mix D), various duality results have been established and special cases are also deduced.

Chapter 3

The main purpose of this chapter is to present continuous-time version of the results of section 2.1 of chapter 2. This chapter deals with Wolfe type duality and Mond-Weir type duality for multiobjective variational problems containing support functions in objective as well as in constraint functions.

In this chapter we present the following nondifferentiable variational problem:

(CP): Minimize $\left(\int_I \left(f^1(t,x,\dot{x}) + S\left(x \middle| C^1\right)\right)dt, \ldots, \int_I \left(f^p(t,x,\dot{x}) + S\left(x \middle| C^p\right)\right)dt\right)$

Subject to

$$x(a) = \alpha \quad , \quad x(b) = \beta$$

$$g^j(t,x,\dot{x}) + S\left(x \middle| D^j\right) \leqq 0 \ , \ j = 1,2,\ldots,m \ , \ t \in I \ ,$$

where,

$f^i : I \times R^n \times R^n \to R, (i = 1,2,\ldots,p)$, $\quad g^j : I \times R^n \times R^n \to R$,$(j = 1,2,\ldots,m)$ are continuously differentiable function, and for each C^i , $i = 1,\ldots,p$ and D^j , $j = 1,\ldots,m$ are compact convex set in R^n .

For this problem (CP), the Wolfe and Mond-Weir type dual variational problems are constructed and various duality theorems are proven under invexity / generalized invexity requirements.

The following is Wolfe type dual (WCD) for the problem (CP).

(WCD): Maximize

$$\left(\int_I \left(f^1(t,u,\dot{u}) + u(t)^T z^1(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t)\right)\right)dt\right.$$

$$\left.,\ldots, \int_I \left(f^p(t,u,\dot{u}) + u(t)^T z^p(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t)\right)\right)dt\right)$$

Subject to

$$u(a) = \alpha \quad , \quad u(b) = \beta$$

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t)\right) + \sum_{j=1}^{m} y^j(t)\left(g_x^j + w^j(t)\right) = D\left(\lambda^T f_{\dot{x}}^i + y(t) g_{\dot{x}}\right) , \ t \in I$$

$$z^i(t) \in C^i \quad , \quad i = 1, 2, \ldots, p$$

$$w^j(t) \in D^j \quad , \quad j = 1, 2, \ldots, m$$

$$\lambda > 0 \ , \sum_{i=1}^{p} \lambda^i = 1$$

We further weaken the invexity requirements by formulating Mond-Weir type dual to the problem (CP).

(M-WCD): Maximize

$$\left(\int_I \left(f^1(t,u,\dot{u}) + u(t)^T z^1(t) \right) dt, \ldots, \int_I \left(f^p(t,u,\dot{u}) + u(t)^T z^p(t) \right) dt \right)$$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) \right) + \sum_{j=1}^{m} y^j(t) \left(g_x^j + w^j(t) \right) = D\left(\lambda^T f_{\dot{x}}^i + y(t)^T g_{\dot{x}} \right), \ t \in I$$

$$\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) dt \geqq 0,$$

$$z^i(t) \in C^i \quad , \quad i = 1, 2, \ldots, p,$$

$$w^j(t) \in D^j \quad , \quad j = 1, 2, \ldots, m,$$

$$y(t) \geqq 0 \ , \ t \in I$$

$$\lambda > 0 \ .$$

The problems related to the above variational problems are also discussed.

Chapter 4

This chapter is devoted to the study a more general class of variational problems than the existing variational problem, treated by Mond and Hanson [105]. The basic purpose of this chapter is to study optimality criteria and duality for multiobjective variational problems having higher order derivatives. This chapter consists of three sections, 4.1, 4.2 and 4.3. In section 4.1, we present the following variational problem:

(VPE) Minimize $\left(\int_I f^1(t,x,\dot{x},\ddot{x})dt, \ldots, \int_I f^p(t,x,\dot{x},\ddot{x})dt\right)$

Subject to

$$x(a)=0=x(b)$$

$$\dot{x}(a)=0=\dot{x}(b)$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \quad , \quad t \in I$$

$$h(t,x,\dot{x},\ddot{x})=0 \quad , \quad t \in I$$

where

1. $f^i : I \times R^n \times R^n \times R^n \to R,\ i=1,2,\ldots,p$, $g: I \times R^n \times R^n \times R^n \to R^m$ and $h: I \times R^n \times R^n \times R^n \to R^k$ are continuously differentiable function, and

2. X designates the space of piecewise functions $x: I \to R^n$ possessing derivatives $\dot{x}$ and $\ddot{x}$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\,ds,$$

where α is given boundary value; thus $D \equiv \frac{d}{dt}$ except at discontinuities.

In the results to follow, we use $C(I,R^m)$ to denote the space of continuous functions $\phi: I \to R^k$ with the uniform norm $\|\phi\| = Sup|\phi|_{t\in I}$; the partial derivatives of g and h are $m \times n$ and $k \times n$ matrices respectively; superscript T denotes matrix transpose.

For this variational problem, Fritz John and Karush-Kuhn-Tuker type optimality conditions are derived.

Theorem 4.1 (*Fritz John Optimality Conditions*): If $\bar{x}$ is an optimal solution of (P_0) and $h_x(x(\cdot),\dot{x}(\cdot),\ddot{x}(\cdot))$ maps X into the subspace of $C(I,R^k)$, then there exists Lagrange multiplier $\bar{\tau} \in R$, the piecewise smooth $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that

$$\left(\bar{\tau}\,\phi_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x\right) - D\left(\bar{\tau}\,\phi_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}}\right)$$
$$+ D^2\left(\bar{\tau}\,\phi_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}\right) = 0 \ , \ t \in I,$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) = 0 \ , \ t \in I \ ,$$

$$\left(\bar{\tau}, \bar{y}(t)\right) \geqq 0 \ , \ t \in I \ ,$$

$$\left(\bar{\tau}, \bar{y}(t), \bar{z}(t)\right) \neq 0 \ , \ t \in I \ .$$

If $\bar{\tau} = 1$, then the above optimality conditions will reduce to the Karush-Kuhn-Tucker type optimality conditions and the solution $\bar{x}$ is commonly referred to as a normal solution.

Theorem 4.2 (*Karush-Kuhn-Tucker Conditions*): Let $\bar{x}$ be an efficient solution for (VPE) which is assumed to be normal for $\left(\bar{P}_r\right)$ for each $r = 1, 2, \ldots, p$. Let the constraints of (P_r) satisfy Slater's Constraint Qualification [5] for each $r = 1, 2, \ldots, p$. Then there exist $\bar{\lambda}^T \in R_+^k$, $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that the following relation hold for all $t \in I$,

$$\left(\bar{\lambda}^T f_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x\right) - D\left(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}}\right)$$
$$+ D^2\left(\bar{\lambda}^T f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}\right) = 0 \ , \ t \in I$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) = 0 \ , \ t \in I$$

$$\bar{\lambda} > 0 \quad , \quad y(t) \geqq 0 \ , \ t \in I$$

As an application of the above Karush-Kuhn-Tucker type optimality conditions, the following Wolfe dual problem is formulated.

(WD): Maximize $\left(\int_{i} \left(f^{1}(t,u,\dot{u},\ddot{u}) + y(t)^{T} g(t,u,\dot{u},\ddot{u}) \right) dt \right.$

$$\left. ,\ldots, \int_{i} \left(f^{p}(t,u,\dot{u},\ddot{u}) + y(t)^{T} g(t,u,\dot{u},\ddot{u}) \right) dt \right)$$

Subject to

$$u(a) = 0 = u(b)$$

$$\dot{u}(a) = 0 = \dot{u}(b)$$

$$\left(\lambda^{T} f_{x} + y(t)^{T} g_{x} \right) - D\left(\lambda^{T} f_{\dot{x}} + y(t)^{T} g_{\dot{x}} \right)$$

$$+ D^{2} \left(\lambda^{T} f_{\ddot{x}} + y(t)^{T} g_{\ddot{x}} \right) = 0 \ , \ t \in I$$

$$y(t) \geqq 0 \ , \ t \in I$$

$$\lambda > 0 \ , \ \lambda^{T} e = 1$$

For the pair of dual problems (VP) and (WD), usual duality theorems are validated under the extended invexity conditions. In this section, it is also shown that the results of this section can be related to those of the nonlinear programs studied earlier.

In the section 4.2, using the optimality conditions derived in the section 4.1, the following Mond-Weir type dual variational problem is formulated and various duality results are proved under generalized invexity defined in the section 4.1.

(M-WD): Maximize $\left(\int_{I} f^{1}(t,u,\dot{u},\ddot{u}) dt, \ldots, \int_{I} f^{p}(t,u,\dot{u},\ddot{u}) dt \right)$

Subject to

$$x(a) = 0 = x(b) \ ,$$

$$\dot{x}(a) = 0 = \dot{x}(b),$$

$$\left(\lambda^{T} f_{x} + y(t)^{T} g_{x} \right) - D\left(\lambda^{T} f_{\dot{x}} + y(t)^{T} g_{\dot{x}} \right)$$

$$+ D^{2} \left(\lambda^{T} f_{\ddot{x}} + y(t)^{T} g_{\ddot{x}} \right) = 0 \ , \ t \in I$$

$$\int_1 y(t)^T g(t,u,\dot{u},\ddot{u})\,dt \geqq 0 \ ,$$

$$y(t) \geqq 0 \ , \quad t \in I \ ,$$

$$\lambda > 0 \ .$$

As in the section 4.1, in this section also, the relationship between our results and those of static cases is outlined. In the section 4.3, we present the following mixed type dual problem for unifying the two dual models incorporated in the section 4.1 and 4.2 and present similar results to those in these sections.

(Mix VD): Maximize $\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u})\,dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u}) \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u})\,dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u}) \right) dt \right)$$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$\left(\lambda f_u + y(t)^T g_u \right) - D\left(\lambda f_{\dot{u}} + y(t)^T g_{\dot{u}} \right)$$
$$+ D^2 \left(\lambda f_{\ddot{u}} + y(t)^T g_{\ddot{u}} \right) = 0 \ , \ t \in I$$

$$\sum_{j \in I_\alpha} \int_I y^j(t)^T g^j(t,u,\dot{u},\ddot{u})\,dt \geqq 0, \ \ \alpha = 1,2,\ldots,r$$

$$y(t) \geqq 0 \ , \ t \in I,$$

$$\lambda \in \Lambda^+$$

where $I_\alpha \subseteq M = \{1,2,\ldots,m\}$, $\alpha = 1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$, if $\alpha \neq \beta$.

Related problems are also deduced.

Chapter 5

In chapter 5, we study optimality and duality for a class of nondifferentiable variational problem involving higher order derivatives containing nondifferentiable terms for the following problem (VP):

(VP): Minimize $\left(\int_I \left(f^1(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^1(t)\, x(t) \right)^{\frac{1}{2}} \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^p(t)\, x(t) \right)^{\frac{1}{2}} \right) dt \right)$$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \text{ , } t \in I,$$

where, $f^i : I \times R^n \times R^n \times R^n \to R, (i = 1,2,\ldots,p)$, $g : I \times R^n \times R^n \times R^n \to R^m$, are assumed to be continuously differentiable functions, for each $i \in P = \{ i = 1,2,\ldots,p \}$, $B^i(t)$ is an $n \times n$ positive semidefinite symmetric matrix with $B^i(\cdot)$ continuous on I .

The purpose of this chapter is to extend the duality results of the chapter 4 to nondifferentiable case. This chapter contains two sections 5.1 and 5.2. In subsection 5.1.1 optimality conditions for the variational problems are obtained and these conditions are used to formulate duals to study Wolfe type vector duality and Mond-Weir type vector duality under invexity / generalized invexity defined in chapter 4. The related problems are also incorporated.

The following is the Wolfe type dual to the problem (VP):

(MWD): Maximize $\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t) + y(t)^T g(t,u,\dot{u},\ddot{u}) \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + u(t)^T B^p(t) z^p(t) + y(t)^T (t,u,\dot{u},\ddot{u}) \right) dt \right)$$

Subject to

$$u(a) = 0 = u(b)$$

$$\dot{u}(a) = 0 = \dot{u}(b)$$

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},\ddot{u})dt + B^i(t)z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right)$$

$$-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u})\right)$$

$$+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) = 0 \ , \ t \in I$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1 \ , \ t \in I \ , \ i \in P$$

$$y(t) \geqq 0 \ , \ t \in I$$

$$\lambda > 0 \ , \ \lambda^T e = 1$$

Following is the Mond-Weir type vector dual to the problem (VP):

(M-WVD): Maximize $\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t) \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + u(t)^T B^p(t) z^p(t) \right) dt \right)$$

Subject to

$$u(a) = 0 = u(b)$$

$$\dot{u}(a) = 0 = \dot{u}(b)$$

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,u,\dot{u},\ddot{u})dt + B^i(t)z^i(t) + y(t)^T g_x(t,u,\dot{u},\ddot{u}) \right)$$

$$-D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right) = 0 \ , \ t \in I$$

$$\sum_{j=1}^{m} \int_I y^j(t) g^j(t,u,\dot{u},\ddot{u}) dt \geqq 0 \ , \ t \in I$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1 \ , \ t \in I \ , \ i \in P$$

$$\lambda > 0 \ , \quad y(t) \geqq 0 \ , \ t \in I$$

In the section 5.2, under invexity and generalized invexity, we study mixed type duality for (VP) that combines Wolfe and Mond-Weir type duals presented section in 5.1. As in section 5.1, here also the results of variational problems are shown to be connected with those of nonlinear programming.

Following is the mixed type dual to the problem (VP):

(Mix D): Maximize
$$\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t) + \sum_{j \in J_\circ} y^j(t) g^j(t,u,\dot{u},\ddot{u}) \right) dt \right.$$
$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + \left(u(t)^T B^p(t) z^p(t) \right) + \sum_{j \in J_\circ} y^j(t) g^j(t,u,\dot{u},\ddot{u}) \right) dt \right)$$

Subject to

$$u(a) = 0 = u(b),$$

$$\dot{u}(a) = 0 = \dot{u}(b),$$

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},\ddot{u}) dt + B^i(t) z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right)$$

$$-D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}} \right) + D^2 \left(\lambda^T f_{\ddot{u}} + y(t)^T g_{\ddot{u}} \right) = 0 \ , \ t \in I \ ,$$

$$\sum_{j \in J_\alpha} \int_I y^j(t) g^j(t,u,\dot{u},\ddot{u}) dt \geqq 0 \ , \ \alpha = 1, 2, \ldots, r \, ,$$

$$\bar{z}^i(t)^T B^i(t) \bar{z}^i(t) \leqq 1 \, , \ t \in I \ , \ i \in P \ .$$

$$y(t) \geqq 0 \ , \ t \in I \, ,$$

$$\lambda \in \Lambda^+ .$$

where

(i) $\Lambda^+ = \left\{ \lambda \in R^p \,|\, \lambda > 0, \ \lambda^T e = 1, e = (1,1,\ldots,1)^T \in R^p \right\}$

(ii) $J_\alpha \subseteq M = \{1, 2, \ldots, m\}, \ \alpha = 0, 1, 2, \ldots, r$ with $\bigcup_{\alpha=0}^{r} J_\alpha = M$ and $J_\alpha \bigcap J_\beta = \phi$, if $\alpha \neq \beta$.

Chapter 6

This chapter is aimed to unify the existing pairs of formulations of Wolfe and Mond-Weir type symmetric dual multiobjective variational problems

Now, we state the following pair of mixed type multiobjective symmetric dual variational problems involving vector functions f and g.

(Mix SP): Minimize $F\left(x^1,x^2,y^1,y^2\right)=\int_I\left\{f\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+g\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right.$

$$\left.-y^1(t)^T\left(\lambda^T f_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-D\lambda^T f_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right)e\right\}dt$$

Subject to

$$x^1(a)=0=x^1(b)\ ,\quad y^1(a)=0=y^1(b)\ ,$$

$$x^2(a)=0=x^2(b)\ ,\quad y^2(a)=0=y^2(b)\ ,$$

$$\lambda^T f_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-D\lambda^T f_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\leqq 0\ ,\ t\in I\ ,$$

$$\lambda^T g_{y^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-D\lambda^T g_{\dot{y}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\leqq 0\ ,\ t\in I\ ,$$

$$\int_I y^2(t)^T\left(\lambda^T g_{y^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right.$$

$$\left.-D\lambda^T g_{\dot{y}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)\geqq 0\ ,$$

$$\lambda\in\Lambda^+ .$$

The dual formulation of the above problem is:

(Mix SD): Maximize $G\left(u^1,u^2,v^1,v^2\right)=\int_I\left\{f\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+g\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right.$

$$-u^1(t)^T\left(\lambda^T f_{y^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right.$$

$$\left.\left.-D\lambda^T f_{\dot{y}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right)e\right\}dt$$

Subject to

$$u^1(a)=0=u^1(b)\ ,\quad v^1(a)=0=v^1(b),$$

$$u^2(a)=0=u^2(b)\ ,\quad v^2(a)=0=v^2(b),$$

$$\lambda^T f_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-D\lambda^T f_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\geqq 0\ ,\ t\in I\ ,$$

$$\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-D\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\geqq 0\ ,\ t\in I\ ,$$

$$\int_I u^2(t)^T\left(\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-D\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right)\geqq 0,$$

$$\lambda\in\Lambda^+ .$$

where $\quad \Lambda^+=\left\{\lambda\in R^p \,\middle|\, \lambda>0,\ \lambda^T e=1, e=(1,1,\ldots,1)^T\in R^p\right\}$

For the above pair of problems, weak and strong and self duality theorem under invexity / generalized invexity, are proved. Special cases are deduced and it is also pointed out that our results can be considered as dynamic generalizations of corresponding (static) symmetric duality results in multiobjective nonlinear programming.

Chapter 7

Chapter 7 is divided into two sections, 7.1 and 7.2. In the section 7.1 we consider the pairs of Wolfe type symmetric dual (SWP) and (SWD) and Mond-Weir type symmetric dual (SM-WP) and (SM-WD) multiobjective variational problems containing support functions. This chapter is essentially an extension of the results of chapter 7 of nondifferentiable case.

In the section 7.1, we consider the following is the pair of Wolfe type symmetric multiobjective dual variational problems, (SWP) and (SWD):

(SWP): Minimize: $\int_I \left(H^1, H^2, ..., H^p\right) dt$

Subject to:

$$x(a) = 0 = x(b)$$

$$y(a) = 0 = y(b)$$

$$\sum_{i=1}^{p} \lambda^i \left(f_y^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right) \leqq 0 \ , \ t \in I$$

$$z^i(t) \in C^i \ , \ i = 1,...,p, \quad t \in I$$

$$x(t) \geqq 0 \ , \ t \in I$$

$$\lambda \in \Lambda^+ = \left\{ \lambda \in R^p \,|\, \lambda > 0, \ \lambda^T e = 1, e = (1,1,\ldots,1)^T \in R^p \right\}$$

(SWD): Maximize: $\int_I \left(G^1, G^2, ..., G^p\right) dt$

Subject to:

$$u(a) = 0 = u(b)$$

$$v(a) = 0 = v(b)$$

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},v,\dot{v}) + \omega^i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right) \geqq 0 \ , \ t \in I$$

$$\omega^i(t) \in K^i \ , \ i = 1,...,p \ , \ t \in I$$

$$v(t) \geqq 0 \ , \ t \in I$$

$$\lambda \in \Lambda^{+}$$

where,

1. $H^{i} = f^{i}(t, x, \dot{x}, y, \dot{y}) + s(x(t)|C^{i})$

$$- y(t)^{T} \sum_{i=1}^{p} \lambda^{i} \left(f_{y}^{i}(t, x, \dot{x}, y, \dot{y}) + z^{i}(t) - Df_{\dot{y}}^{i}(t, x, \dot{x}, y, \dot{y}) \right) - y(t)^{T} z(t)$$

2. $f^{i} : I \times R^{n} \times R^{n} \rightarrow R, (i = 1, 2, \ldots, p)$, is continuously differentiable function.

3. $G^{i} = f^{i}(t, u, \dot{u}, v, \dot{v}) + s(v(t)|K^{i})$

$$-u(t)^{T} \sum_{i=1}^{p} \lambda^{i} \left(f_{u}^{i}(t, u, \dot{u}, v, \dot{v}) + \omega^{i}(t) - Df_{\dot{u}}^{i}(t, u, \dot{u}, v, \dot{v}) \right) - x(t)^{T} \omega(t)$$

The following is the pair of Mond-Weir type symmetric multiobjective dual variational problems (SM-WP) and (SM-WD):

(SM-WP): Maximize: $\int_{I} \left(\Phi^{1}, \Phi^{2}, \ldots, \Phi^{p} \right) dt$

Subject to:

$$x(a) = 0 = x(b)$$

$$y(a) = 0 = y(b)$$

$$\sum_{i=1}^{p} \lambda^{i} \left(f_{y}^{i}(t, x, \dot{x}, y, \dot{y}) - z^{i}(t) - Df_{\dot{x}}^{i}(t, x, \dot{x}, y, \dot{y}) \right) \leqq 0 \ , \ t \in I$$

$$\int_{I} y^{T}(t) \sum_{i=1}^{p} \lambda^{i} \left(f_{y}^{i}(t, x, \dot{x}, y, \dot{y}) - z^{i}(t) - Df_{\dot{x}}^{i}(t, x, \dot{x}, y, \dot{y}) \right) dt \geqq 0$$

$$z^{i}(t) \in C^{i} \ , \ i = 1, \ldots, p, \quad t \in I$$

$$x(t) \geqq 0 \ , \ t \in I$$

$$\lambda > 0$$

(SM-WD): Minimize: $\int_{I} \left(\psi^{1}, \psi^{2}, \ldots, \psi^{p} \right) dt$

Subject to:

$$u(a) = 0 = u(b)$$

$$v(a) = 0 = v(b)$$

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},v,\dot{v}\right)+\omega^i\left(t\right)-Df_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right)\geqq 0\ ,\ t\in I$$

$$\int_I y^T\left(t\right)\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},v,\dot{v}\right)+\omega^i\left(t\right)-Df_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right)dt\leqq 0$$

$$\omega^i\left(t\right)\in K^i\ \ ,i=1,\ldots,p$$

$$v\left(t\right)\geqq 0\ ,\ \ t\in I$$

$$\lambda>0$$

where

1. $\Phi^i=f^i\left(t,x,\dot{x},y,\dot{y}\right)+s\left(x(t)\middle|C^i\right)-y\left(t\right)^T z\left(t\right)\quad ,i=1,\ldots,p$

2. $\psi^i=f^i\left(t,u,\dot{u},v,\dot{v}\right)-s\left(v\left(t\right)\middle|K^i\right)+u\left(t\right)^T\omega\left(t\right)\quad ,i=1,\ldots,p$

For above pairs of problems, weak, strong and self duality theorems under convexity-concavity and pseudoconvexity-pseudoconcavity, are proved. The problems with natural boundary values are also formulated.

In section 7.2, we consider the following pair of mixed type symmetric dual multiobjective variational problems containing support functions:

(Mix SP): Minimize: $\int_I\Big(H^1\left(t,x^1,x^2,y^1,y^2,\dot{x}^1,\dot{x}^2,\dot{y}^1,\dot{y}^2,z^1,z^2,\lambda\right),\ldots,$
$$H^p\left(t,x^1,x^2,y^1,y^2,\dot{x}^1,\dot{x}^2,\dot{y}^1,\dot{y}^2,z^1,z^2,\lambda\right)\Big)dt$$

Subject to:

$$x^1\left(a\right)=0=x^1\left(b\right)\ \ ,\quad y^1\left(a\right)=0=y^1\left(b\right)\ ,$$

$$x^2\left(a\right)=0=x^2\left(b\right)\ \ ,\quad y^2\left(a\right)=0=y^2\left(b\right).$$

$$\sum_{i=1}^{p}\lambda^i\left[f_{y^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-z_i^1\left(t\right)-Df_{\dot{y}^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right]\leqq 0\ ,\ t\in I\ ,$$

$$\sum_{i=1}^{p}\lambda^i\left[g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-z_i^2\left(t\right)-Dg_{\dot{y}^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right]\leqq 0\ ,\ t\in I\ ,$$

$$\int_I y^2\left(t\right)^T\left[\sum_{i=1}^{p}\lambda^i\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-z_i^2\left(t\right)-Dg_{\dot{y}^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)\right]\geqq 0,$$

$$\left(x^1(t), x^2(t)\right) \geqq 0\ ,\ t \in I\ ,$$

$$z_i^1(t) \in K_i^1 \quad \text{and} \quad z_i^2(t) \in K_i^2\ ,$$

$$\lambda > 0,\ \ \lambda^T e = 1\ ,\ e^T = (1,.....,1).$$

where,

$$\begin{aligned} H^i = {} & f^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g^i\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) + s\left(x^1(t)\middle| C_i^1\right) + s\left(x^2(t)\middle| C_i^2\right) \\ & - y^1(t)\sum_{i=1}^{p}\lambda^i\left[f_{y^1}^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - z_i^1(t) - Df_{\dot{y}^1}^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right] \\ & - z_i^1(t)\, y^1(t) - z_i^2(t)\, y^2(t) \end{aligned}$$

(Mix SD): Maximize: $\int_I \Big(G^1\left(t,u^1,u^2,v^1,v^2,\dot{u}^1,\dot{u}^2,\dot{v}^1,\dot{v}^2,w^1,w^2,\lambda\right),\ldots,$

$$G^p\left(t,u^1,u^2,v^1,v^2,\dot{u}^1,\dot{u}^2,\dot{v}^1,\dot{v}^2,w^1,w^2,\lambda\right)\Big)dt$$

Subject to:

$$u^1(a) = 0 = u^1(b)\ \ ,\quad v^1(a) = 0 = v^1(b)\ ,$$

$$u^2(a) = 0 = u^2(b)\ \ ,\quad v^2(a) = 0 = v^2(b),$$

$$\sum_{i=1}^{p}\lambda^i\left[f_{u^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right) + \omega_i^1(t) - Df_{\dot{u}^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right] \geqq 0\ ,\ t \in I\ ,$$

$$\sum_{i=1}^{p}\lambda^i\left[g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right) + \omega_i^2(t) - Dg_{\dot{u}^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right] \geqq 0\ ,\ t \in I\ ,$$

$$\int_I u^2(t)^T\left[g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right) + \omega_i^2(t) - Dg_{\dot{u}^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right] \leqq 0\ ,\ t \in I\ ,$$

$$\left(v^1(t), v^2(t)\right) \geqq 0\ ,\ t \in I\ ,$$

$$\omega_i^1(t) \in C_i^1 \quad \text{and} \quad \omega_i^2(t) \in C_i^2 \quad ,\ i = 1, 2, ..., p$$

$$\lambda > 0,\ \ \lambda^T e = 1\ ,\ e^T = (1,.....,1)\ ,$$

where,

$$\begin{aligned} G^i = {} & f^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g^i\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \\ & + s\left(v^1(t)\middle| K_i^1\right) + s\left(v^2(t)\middle| K_i^2\right) + u^1(t)\,\omega_i^1(t) + u_i^2(t)\,\omega^2(t) \\ & - u^1(t)\sum_{i=1}^{p}\lambda^i\left[f_{u^1}^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + \omega_i^1(t) - Df_{\dot{u}^1}^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right]. \end{aligned}$$

For the above pair of mixed type nondifferentiable multiobjective variational problems, weak, strong and self duality theorems are established under convexity-concavity and pseudoconvexity-pseudoconcavity of certain combination of functionals appearing in the formulations. Special cases are also derived. A pair of mixed type nondifferentiable multiobjective variational problem with natural boundary values is also formulated to investigate various duality theorem and to have linkage between the duality results of this chapter and results for static cases surveyed in this thesis.

Chapter-2

ON MULTIOBJECTIVE NONLINEAR PROGRAMMING WITH SUPPORT FUNCTIONS

2.1 NONDIFFERENTIABLE MULTIOBJECTIVE NONLINEAR PROGRAMMING

2.1.1 Introductory Remarks

The usefulness of study of duality in mathematical programming lies in the fact that the duality helps to develop numerical algorithms as it provides stopping rules for primal and dual problems. Motivated with these observations, in this chapter we consider vector version of the problem by Husain et al [74], considered in section 2.1.3 and study duality byformulating Wolfe and Mond-Weir type dual problems to multiobjective problem with support functions under invexity and generalized invexity requirements. It is remarked in this chapter that the primal problem is a nondifferentiable multiobjective problem but its Wolfe and Mond-Weir problems are differentiable multiobjective programming problems. Obviously it is easier to solve differentiable programming problems than to solve a nondifferentiable programming problem. In essence, this observation explains the advantage of these dual problems over the primal problems. The problems considered in this chapter are hard to solve. So to expect an immediate application of these problems would be far from reality. Unfortunately, there has not always been sufficient flow between the researchers in the multiple criteria decision making and the researchers applying it to their problems. Of course, we can find many problems of facility location and portfolio selection modeled as multiobjective programming problems which reflect the utility of our problems.

This chapter is divided into two sections, 2.1 and 2.2. In section 2.1, Wolfe type duality and Mond-Weir type duality are investigated under invexity and generalized invexity. Special cases are derived. The section 2.2 unifies the nondifferentiable Wolfe type dual and Mond-Weir type dual problems considered in section 2.1 and also incorporates particular cases.

2.1.2 Statement of the Primal Problem

In [74] Husain et al considered the following problem:

(P): Minimize $f(x)+S(x|C)$

Subject to

$$g^j(x)+S\left(x\middle|D^j\right)\leqq 0 \ , \ j=1,2,\ldots,m$$

where

i. $f:R^n \to R$ and $g_j:R^n \to R$, $j=1,2,\ldots,m$ are continuously differentiable and

ii. $S(\cdot|C)$ and $S\left(\cdot\middle|D^j\right)$, $j=1,2,\ldots,m$ are support functions of a compact convex sets C and D^j , $j=1,2,\ldots,m$ in R^n .

The authors in [74], constructed the following Wolfe type dual (WD) and Mond-Weir type dual (M-WD) to the problem (NP) and established various duality results under convexity and generalized convexity assumptions.

(WD): Maximize $f(u)+u^T z+\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T u\right)$

Subject to

$$\nabla\left(f(u)+u^T z\right)+\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T u\right)=0,$$

$$z\in C,$$

$$w^j\in D^j \ , \ j=1,2,\ldots,m.$$

(M-WD): Maximize $f(u)+u^T z$

Subject to

$$\nabla\left(f(u)+u^T z\right)+\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T u\right)=0,$$

$$\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T u\right)\geq 0,$$

$$z\in C,\ w_j\in D_j\ ,\quad j=1,2,\ldots,m\ ,$$

$$y\geq 0$$

We present the following nondifferentiable multiobjective programming problem containing support function as the primal problem.

(NP): Minimize $\left(f^1(x)+S\left(x\middle|C^1\right),\ldots,f^p(x)+S\left(x\middle|C^p\right)\right)$

Subject to

$$g^j(x)+S\left(x\middle|D^j\right)\leqq 0\ ,\ j=1,2,\ldots,m$$

where

i. $f^i:R^n\to R$ and $g^j:R^n\to R$, $j=1,2,\ldots,m$ are differentiable functions and

ii. $S\left(\cdot\middle|C^i\right)$, $i=1,2,\ldots,p$ and $S\left(\cdot\middle|D^j\right)$, $j=1,2,\ldots,m$ are support functions of a compact convex set C^i, $i=1,2,\ldots,p$ and D^j, $j=1,2,\ldots,m$ in R^n.

2.1.3 Wolfe Type Duality

We formulate the following Wolfe type dual to the problem (NP) and establish various duality results under invexity defined in the preceding section.

(WND): Maximize $\Bigg(f^1(u)+u^T z^1+\sum_{j=1}^{m} y^j\left(g^j(u)+u^T w^j\right)$

$$,\ldots,f^p(u)+u^T z^p+\sum_{j=1}^{m} y^j\left(g^j(u)+u^T w^j\right)\Bigg)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^Tz^i\right)+\sum_{j=1}^{m}y^j\nabla\left(g^j(u)+u^Tw^j\right)=0 \qquad (2.1)$$

$$z^i\in C^i \quad , \quad i=1,2,\ldots,p\,, \qquad (2.2)$$

$$w^j\in D^j \quad , \quad j=1,2,\ldots,m\;, \qquad (2.3)$$

$$y\geqq 0\,, \qquad (2.4)$$

$$\lambda>0\;\;,\sum_{i=1}^{p}\lambda^i=1 \qquad (2.5)$$

Theorem 2.1: (*Weak Duality*)**:** Let x be feasible for (NP) and $\left(u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ be feasible for (WND). If for all feasible $\left(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ $,\sum_{i=1}^{p}\lambda^i\left(f^i(\cdot)+(\cdot)^Tz^i\right)+\sum_{j=1}^{m}y^j\left(g^j(\cdot)+(\cdot)^Tw^j\right)$ is pseudoinvex with respect to η, then the following cannot hold.

$$f^i(x)+S\left(x\middle|C^i\right)\leq f^i(u)+u^Tz^i+\sum_{j=1}^{m}y^j\left(g^j(u)+u^Tw^j\right)$$

for all $i\in\{1,\ldots,p\}$, and

$$f^r(x)+S\left(x\middle|C^r\right)<f^r(u)+u^Tz^r+\sum_{j=1}^{m}y^j\left(g^j(u)+u^Tw^j\right)$$

for some $r\in\{1,2,\ldots,p\}$, $r\neq i.$

Proof: Suppose this the conclusion of the theorem hold. Then from the feasibility of (NP) together with $x^Tz^i\leqq S\left(x\middle|C^i\right)$, $i=1,2,\ldots,p$ and $x^Tw^j\leqq S\left(x\middle|D^j\right)$, $j=1,2,\ldots,m$.

$$f^i(x)+x^Tz^i+\sum_{j=1}^{m}y^j\left(g^j(x)+x^Tw^j\right)$$

$$\leq f^i(u)+u^Tz^i+\sum_{j=1}^{m}y^j\left(g^j(u)+u^Tw^j\right)$$

for all $i\in\{1,\ldots,p\}$

$$f^r(x)+x^Tz^r+\sum_{j=1}^{m}y^j\left(g^j(x)+x^Tw^j\right)$$

$$<f^r(u)+u^Tz^r+\sum_{j=1}^{m}y^j\left(g^j(u)+u^Tw^j\right)$$

for some $r \in \{1,2,\ldots,p\}$.

In view of $\lambda > 0$ and $\sum_{i=1}^{p} \lambda^i = 1$, these inequalities yield

$$\sum_{i=1}^{p} \lambda^i \left(f^i(x) + x^T z^i \right) + \sum_{j=1}^{m} y^j \left(g^j(x) + x^T w^j \right)$$
$$< \sum_{i=1}^{p} \lambda^i \left(f^i(u) + u^T z^i \right) + \sum_{j=1}^{m} y^j \left(g^j(u) + u^T w^j \right)$$

This in view of the pseudoinvexity of $\sum_{i=1}^{p} \lambda^i \left(f^i(\cdot) + (\cdot)^T z^i \right) + \sum_{j=1}^{m} y^j \left(g^j(\cdot) + (\cdot)^T w^j \right)$ gives,

$$\eta^T \left(\sum_{i=1}^{p} \lambda^i \nabla \left(f^i(x) + u^T z^i \right) + \sum_{j=1}^{m} y^j \nabla \left(g^j(u) + u^T w^j \right) \right) < 0 \tag{2.6}$$

From the equality constraint of the (WND), we have

$$\eta^T \left(\sum_{i=1}^{p} \lambda^i \nabla \left(f^i(x) + u^T z^i \right) + \sum_{j=1}^{m} y^j \nabla \left(g^j(u) + u^T w^j \right) \right) = 0$$

This is a contradiction to (2.6). Hence the proof of the theorem follows.

In order to prove the strong duality theorem, we require the following lemma due to Chankong and Haimes [35].

Lemma 2.1 [35]: A point $\bar{x} \in X$ is an efficient for (NP), if and only if $\bar{x} \in X$ solves.

$\left(\mathrm{P}_k(\bar{x}) \right)$: Minimize $\quad f^k(x) + S\left(x \middle| C^k \right)$

Subject to

$$f^i(x) + S\left(x \middle| C^i \right) \leqq f^i(\bar{x}) + S\left(\bar{x} \middle| C^i \right)$$

for all $i = \{1,2,\ldots,p\}$, $i \neq k$

$$g^j(x) + S\left(x \middle| D^j \right) \leqq 0, \; j = 1,2,\ldots,m$$

where X is the set of feasible solutions of (NP).

Theorem 2.2 (*Strong Duality*): Let $\bar{x} \in X$ be an efficient solution of (NP) and for at least one i , $i \in \{1, 2, \ldots, p\}$, $\bar{x}$ satisfies the constraint qualification [30] for the problem $\left(\mathrm{P}_k\left(\bar{x}\right)\right)$. Then there exist $\lambda \in R^p$ with $\lambda^T = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^i, \ldots, \bar{\lambda}^p\right)$, $\bar{y} \in R^m$ with $\bar{y}^T = \left(\bar{y}^1, \ldots, \bar{y}^i, \ldots, \bar{y}^m\right)$, $z^i \in R^n$, $i = \{1, 2, \ldots, p\}$ and $w^j \in R^n$, $j = 1, 2, \ldots, m$ such that $\left(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda\right)$ is feasible for (WND) and $\sum_{j=1}^{m} \bar{y}^j \nabla\left(g^j\left(\bar{x}\right) + \bar{x}^T w^j\right) = 0$.

Further, if the hypotheses of Theorem 2.1 are met, then $\left(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda\right)$ is an efficient solution of (WND).

Proof: Since $\bar{x} \in X$ is an efficient solution of (NP), by Lemma 2.1, $\bar{x}$ is an optimal solution of $\left(\mathrm{P}_k\left(\bar{x}\right)\right)$. Consequently, there exists $\tau \in R^p$ with $\tau^T = \left(\tau^1, \ldots, \tau^i, \ldots, \tau^p\right)$, $\bar{\upsilon}^T = \left(\bar{\upsilon}^1, \ldots, \bar{\upsilon}^i, \ldots, \bar{\upsilon}^m\right)$, $z^i \in R^n, i = \{1, 2, \ldots, p\}$ and $w^j \in R^n$, $j = 1, 2, \ldots, m$ such that the following optimality conditions [30] hold:

$$\tau^k\left[\nabla\left(f^k\left(x\right) + \bar{x}^T \bar{z}^k\right)\right] + \sum_{\substack{i=1\\ i\neq k}}^{p} \tau^i \nabla\left(f^i\left(x\right) + \bar{x}^T \bar{z}^i\right)$$
$$+\sum_{j=1}^{m} \upsilon^j \nabla\left(g^j\left(x\right) + x^T w^j\right) = 0, \tag{2.7}$$

$$\sum_{j=1}^{m} \upsilon^j \nabla\left(g^j\left(x\right) + x^T w^j\right) = 0, \tag{2.8}$$

$$z^i \in C^i \quad , \quad i = 1, 2, \ldots, p, \tag{2.9}$$

$$w^j \in D^j \quad , \quad j = 1, 2, \ldots, m \ , \tag{2.10}$$

$$\bar{x}^T \bar{z}^i = S\left(\bar{x} \middle| C^i\right) \quad , \quad i = 1, 2, \ldots, p, \tag{2.11}$$

$$\bar{x}^T \bar{w}^j = S\left(\bar{x} \middle| D^j\right) \quad , \quad j = 1, 2, \ldots, m, \tag{2.12}$$

$$\tau > 0 \quad , \quad \upsilon \geqq 0 \tag{2.13}$$

The relation (2.7) can reduce to

$$\sum_{i=1}^{p}\tau^{i}\nabla\left(f^{i}\left(\bar{x}\right)+\bar{x}^{T}\bar{z}^{i}\right)+\sum_{j=1}^{m}\bar{\upsilon}^{j}\nabla\left(g^{j}\left(x\right)+\bar{x}^{T}\bar{w}^{j}\right)=0 \tag{2.14}$$

Dividing (2.8), (2.13) and (2.14) by $\sum_{i=1}^{p}\tau^{i}\neq 0$, and putting $\lambda^{i}=\dfrac{\tau^{i}}{\sum_{i=1}^{p}\tau^{i}}$ and $y^{i}=\dfrac{\upsilon^{i}}{\sum_{i=1}^{p}\tau^{i}}$, we have,

$$\sum_{j=1}^{m}\bar{y}^{j}\nabla\left(g^{j}\left(x\right)+\bar{x}^{T}\bar{w}^{j}\right)=0 \tag{2.15}$$

$$\lambda>0\ ,\sum_{i=1}^{p}\lambda^{i}=1 \tag{2.16}$$

$$\sum_{i=1}^{p}\lambda^{i}\nabla\left(f^{i}\left(\bar{x}\right)+\bar{x}^{T}\bar{z}^{i}\right)+\sum_{j=1}^{m}\bar{y}^{j}\nabla\left(g^{j}\left(x\right)+\bar{x}^{T}\bar{w}^{j}\right)=0 \tag{2.17}$$

Consequently (2.9), (2.10), (2.16) and (2.17) implies that $\left(x,u,y,z^{1},\ldots,z^{p},w^{1},\ldots,w^{m},\lambda\right)$ is feasible for (WD). In view of (2.11) and (2.15)

$$\begin{aligned} & f^{i}\left(\bar{x}\right)+\bar{x}^{T}\bar{z}^{i}+\sum_{j=1}^{m}\bar{y}^{j}\left(g^{j}\left(x\right)+\bar{x}^{T}\bar{w}^{j}\right) \\ &\qquad = f^{i}\left(\bar{x}\right)+S\left(\bar{x}\middle|C^{i}\right),\quad i=1,2,\ldots,p. \end{aligned}$$

In view of Theorem 2.1, this implies the efficiency of $\left(x,u,y,z^{1},\ldots,z^{p},\right.$ $\left.w^{1},\ldots,w^{m},\lambda\right)$ for (WND).

Theorem 2.3 (*Converse Duality*): Let $\left(\bar{x},\bar{u},\bar{y},\bar{z}^{1},\ldots,\bar{z}^{p},\bar{w}^{1},\ldots,\bar{w}^{m},\bar{\lambda}\right)$ be an efficient solution: Assume that

(A_1) f and g are twice differentiable, and

(A_2) $\nabla^{2}\left(\lambda^{T}f^{i}\left(\bar{x}\right)+\bar{y}^{T}g\left(x\right)\right)$ is positive or negative definite.

Further, if the assumptions of Theorem 2.1 are satisfied, then $\bar{x}$ is an efficient solution of (NP).

Proof: Since $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda}\right)$be an efficient solution of (WND), then there exist $\alpha\in R^p$ with $\alpha^T=\left(\alpha^1,\ldots,\alpha^i,\ldots,\alpha^p\right)$, $\beta\in R^n$, $\eta\in R^p$ with $\eta^T=\left(\eta^1,\ldots,\eta^i,\ldots,\eta^p\right)$, and $\kappa\in R$ such that the following Fritz-John conditions [37] holds,

$$-\sum_{i=1}^{p}\alpha^i\left(\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)+\sum_{j=1}^{m}\bar{y}^j\nabla\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)\right) + \beta^T\nabla^2\left(\lambda^T f\left(\bar{x}\right)+\bar{y}^T g\left(x\right)\right)=0 \tag{2.18}$$

$$-\left(\alpha^T e\right)\left(g^j+\bar{x}^T\bar{w}^j\right)-\mu^j=0\ ,\quad j=1,2,\ldots,m \tag{2.19}$$

$$\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)\beta-\eta^i+\kappa=0 \tag{2.20}$$

$$\left(\beta\lambda^i-\alpha^i\bar{x}\right)\in N_{C^i}\left(\bar{z}^i\right),\ i=1,\ldots,p \tag{2.21}$$

$$\left(\beta-\left(\alpha^T e\right)\bar{x}\right)y^j\in N_{D^j}\left(\bar{w}^j\right),\ j=1,\ldots,m \tag{2.22}$$

$$\eta^T\lambda=0 \tag{2.23}$$

$$\kappa\left(\sum_{i=1}^{p}\lambda^i-1\right)=0 \tag{2.24}$$

$$\mu^T\bar{y}=0 \tag{2.25}$$

$$\left(\alpha,\mu,\eta,\kappa\right)\geqq 0 \tag{2.26}$$

$$\left(\alpha,\beta,\mu,\eta,\kappa\right)\neq 0 \tag{2.27}$$

Since $\lambda>0$, (2.23) implies $\eta=0$. Consequently (2.20) implies

$$\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)\beta=-\kappa \tag{2.28}$$

From the equality constraint of (WND), we have

$$\sum_{j=1}^{m}\bar{y}^j\nabla\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)\beta=-\sum_{i=1}^{p}\lambda^i\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)\beta\ =\kappa \tag{2.29}$$

Using (2.28) and (2.29) in (2.18), we have

$$\beta^T \nabla^2 \left(\lambda^T f(\bar{x}) + \bar{y}^T g(x) \right) \beta = 0$$

In view of the hypothesis (A_2), this yields

$$\beta = 0 \tag{2.30}$$

Suppose $\alpha = 0$, then from (2.19) we have $\mu = 0$, and from (2.20) we get $\kappa = 0$.

Consequently $(\alpha, \beta, \mu, \eta, \kappa) = 0$. This contradicts (2.27).

Hence $\alpha > 0$. From (2.21) and (2.22) in view of (2.30) implies,

$$\bar{x}^T \bar{z}^i = S\left(\bar{x} \middle| C^i\right) \quad , \quad i = 1, 2, \ldots, p \tag{2.31}$$

$$\bar{x}^T \bar{w}^j = S\left(\bar{x} \middle| D^j\right) \quad , \quad j = 1, 2, \ldots, m \tag{2.32}$$

From (2.19) along with (2.32), (2.28) and $\alpha > 0$, we have

$$g^j(x) + S\left(\bar{x} \middle| D^j\right) \leqq 0 \quad , \quad j = 1, 2, \ldots, m$$

This implies $\bar{x}$ is feasible for (NP).

Again from (2.19) in view of (2.25), we have

$$\sum_{j=1}^{m} \bar{y}^j \nabla \left(g^j(x) + \bar{x}^T \bar{w}^j \right) = 0 \tag{2.33}$$

In view of (2.31) and (2.33), we have

$$f^i(\bar{x}) + \bar{x}^T \bar{z}^i + \sum_{j=1}^{m} \bar{y}^j \nabla \left(g^j(x) + \bar{x}^T \bar{w}^j \right)$$

$$= f^i(\bar{x}) + S\left(\bar{x} \middle| C^i\right) \quad , \quad i = 1, 2, \ldots, p$$

Since the requirements of the Theorem 2.1 are fulfilled, this implies $\bar{x}$ is an efficient solution of (NP).

2.1.4 Mond-Weir Type Duality

In this section, we construct the following problem as the Mond-Weir type dual to the problem (NP).

(M-WND): Maximize $\left(f^1(u)+u^Tz^1,\ldots,f^i(u)+u^Tz^p\right)$

Subject to

$$\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^Tz^i\right)+\sum_{j=1}^{m}y^j\left(g^j(u)+u^Tw^j\right)=0, \quad (2.34)$$

$$z^i\in C^i \quad , \quad i=1,2,\ldots,p, \quad (2.35)$$

$$\sum_{j=1}^{m}y^j\left(g^j(u)+u^Tw^j\right)\geqq 0, \quad (2.36)$$

$$\lambda>0 \quad , \quad y\geqq 0 \quad (2.37)$$

Theorem 2.4 (*Weak Duality*): Let $\bar{x}\in X$ be feasible for (NP) and $\left(\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda}\right)$ be feasible for (M-WND). If for all feasible $\left(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ $,\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(\cdot)+(\cdot)^Tz^i\right)$ is pseudoinvex and $\sum_{j=1}^{m}y^j\left(g^j(\cdot)+(\cdot)^Tw^j\right)$ is quasi-invex with respect to η, then the following cannot hold.

$$f^i(x)+S\left(x|C^i\right)\leq f^i(u)+u^Tz^i \quad (2.38)$$

for all $i\in\{1,\ldots,p\}$. and

$$f^r(x)+S\left(x|C^r\right)< f^r(u)+u^Tz^r \quad (2.39)$$

for some $r\in\{1,2,\ldots,p\}$.

Proof: Suppose (2.38) and (2.39) of the theorem hold. Then by feasibility of (NP) and (M-WND) along with $\lambda>0$ and $x^Tz^i\leqq S\left(x|C^i\right)$, $i=1,2,\ldots,p$, we have

$$\sum_{i=1}^{p}\lambda^i\left(f^i(x)+x^Tz^i\right)<\sum_{i=1}^{p}\lambda^i\left(f^i(u)+u^Tz^i\right)$$

which because of the pseudo-invexity of $\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(\cdot)+(\cdot)^Tz^i\right)$ implies,

$$\eta^T\left(\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+(u)^Tz^i\right)\right)<0. \quad (2.40)$$

Now from feasibility of (NP) and (M-WND) with $x^T w^j \leqq S\left(x \middle| D^j\right)$, $j = 1, 2, \ldots, m$

We have

$$\sum_{j=1}^{m} y^j \left(g^j(x) + (x)^T w^j\right) \leqq \sum_{j=1}^{m} y^j \left(g^j(u) + (u)^T w^j\right)$$

this, due to quasi-invexity of $\sum_{j=1}^{m} y^j \left(g^j(\cdot) + (\cdot)^T w^j\right)$ with respect to η gives

$$\eta^T \sum_{j=1}^{m} y^j \nabla\left(g^j(u) + (u)^T w^j\right) \leqq 0. \tag{2.41}$$

Combining (2.40) and (2.41), we have

$$\eta^T \left(\sum_{i=1}^{p} \lambda^i \nabla\left(f^i(u) + u^T z^i\right) + \sum_{j=1}^{m} y^j \left(g^j(u) + u^T w^j\right) \right) < 0.$$

This contradicts the equality constraint. Hence the validation of the theorem follows.

The following duality theorem can be proved on the line of Theorem 2.2.

Theorem 2.5 (*Strong Duality*): Let $\bar{x} \in X$ be an efficient solution of (MNP) and for at least one i, $i \in \{1, 2, \ldots, p\}$, $\bar{x}$ satisfies the constraint qualification [30] for the problem $\left(\mathrm{P}_k(\bar{x})\right)$. Then there exist $\lambda \in R^p$ with $\lambda^T = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^i, \ldots, \bar{\lambda}^p\right)$, $\bar{y} \in R^m$ with $\bar{y}^T = \left(\bar{y}^1, \ldots, \bar{y}^i, \ldots, \bar{y}^m\right)$, $z^i \in R^n$, $i = \{1, 2, \ldots, p\}$ and $w^j \in R^n$, $j = 1, 2, \ldots, m$ such that $\left(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda\right)$ is feasible for (M-WND).

Further, if the hypotheses of Theorem 2.4 are satisfied, then $\left(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda\right)$ is an efficient solution of (M-WND).

Theorem 2.6 (*Converse Duality*): Let $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda}\right)$ be an efficient solution for (M-WND). Assume that

(A_1) f and g are twice continuously differentiable,

(A₂) $\nabla\left(f^i(\bar{x})+\bar{x}^T\bar{z}^i\right)$, $i\in\{1,2,\ldots,p\}$ are linearly independent.

(A₃) $\nabla^2\left(\lambda^T f^i(\bar{x})+\bar{y}^T g(x)\right)$ is positive or negative definite.

Further, if the assumptions of Theorem 2.4 are satisfied, then $\bar{x}$ is an efficient solution.

Proof: Since $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda}\right)$ be an efficient solution of (M-WND), then there exist $\alpha\in R^p$ with $\alpha^T=\left(\alpha^1,\ldots,\alpha^i,\ldots,\alpha^p\right)$, $\beta\in R^n$, $\eta\in R^p$ with $\eta^T=\left(\eta^1,\ldots,\eta^i,\ldots,\eta^p\right)$, $\gamma\in R$ and $\mu\in R^m$ such that the following Fritz-John conditions [37] are satisfied

$$-\sum_{i=1}^{p}\alpha^i\nabla\left(f^i(\bar{x})+\bar{x}^T\bar{z}^i\right)+\beta^T\nabla^2\left(\lambda^T f(\bar{x})+\bar{y}^T g(\bar{x})\right)$$
$$-\gamma\sum_{j=1}^{m}\bar{y}^j\nabla\left(g^j(x)+\bar{x}^T\bar{w}^j\right)=0 \qquad (2.42)$$

$$-\gamma\left(g^j(\bar{x})+\bar{x}^T\bar{w}^j\right)-\mu^j=0\ ,\quad j=1,2,\ldots,m, \qquad (2.43)$$

$$\nabla\left(f^i(\bar{x})+\bar{x}^T\bar{z}^i\right)\beta-\eta^i=0, \qquad (2.44)$$

$$\left(\beta\lambda^i-\alpha^i\bar{x}\right)\in N_{C^i}\left(\bar{z}^i\right),\ i=1,\ldots,p, \qquad (2.45)$$

$$\left(\beta-\gamma\bar{x}\right)y^j\in N_{D^j}\left(\bar{w}^j\right),\ j=1,\ldots,m, \qquad (2.46)$$

$$\gamma\sum_{j=1}^{m}\bar{y}^j\nabla\left(g^j(\bar{x})+\bar{x}^T\bar{w}^j\right)=0, \qquad (2.47)$$

$$\eta^T\lambda=0, \qquad (2.48)$$

$$\mu^T\bar{y}=0, \qquad (2.49)$$

$$\left(\alpha,\mu,\eta,\gamma\right)\geqq 0, \qquad (2.50)$$

$$\left(\alpha,\beta,\mu,\eta,\gamma\right)\neq 0. \qquad (2.51)$$

Since $\lambda>0$, (2.48) implies $\eta=0$. Consequently (2.44) implies

$$\nabla\left(f^i(\bar{x})+\bar{x}^T\bar{z}^i\right)\beta=0 \qquad (2.52)$$

Using the equality constraint of (M-WND) in (2.42), we have

$$-\sum_{i=1}^{p}\left(\alpha^i-\gamma\lambda^i\right)\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)+\beta^T\nabla^2\left(\lambda^T f\left(\bar{x}\right)+\bar{y}^j g^j\left(\bar{x}\right)\right)=0 \qquad (2.53)$$

Postmultiplying (2.53) by β and then using (2.52), we have

$$\beta^T\nabla^2\left(\lambda^T f\left(\bar{x}\right)+\bar{y}^T g\left(x\right)\right)\beta=0$$

This because of positive or negative definiteness of $\nabla^2\left(\lambda^T f^i\left(\bar{x}\right)+\bar{y}^T g\left(x\right)\right)$, yields

$$\beta=0 \qquad (2.54)$$

Using (2.54) in (2.53), we have

$$\sum_{i=1}^{p}\left(\alpha^i-\gamma\lambda^i\right)\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)=0$$

This, due to the hypotheses (A_2) gives,

$$\alpha^i-\gamma\lambda^i=0\ ,\ i=1,2,\dots,p \qquad (2.55)$$

Suppose $\gamma=0$, then from (2.55) we have $\alpha=0$. From (2.43), we have $\mu=0$.

Consequently $\left(\alpha,\beta,\mu,\eta,\gamma\right)=0$. This contradicts (2.51).

Hence $\gamma>0$ and from (2.55), $\alpha>0$.

For $\beta=0$, (2.45) and (2.46), we have,

$$\bar{x}^T\bar{z}^i=S\left(\bar{x}\middle|C^i\right)\quad,\quad i=1,2,\dots,p \qquad (2.56)$$

$$\bar{x}^T\bar{w}^j=S\left(\bar{x}\middle|D^j\right)\quad,\quad j=1,2,\dots,m \qquad (2.57)$$

The relation (2.43) along with (2.57) and (2.50) gives

$$g^j\left(x\right)+S\left(\bar{x}\middle|D^j\right)\leqq 0\quad,\quad j=1,2,\dots,m$$

This implies the feasibility of $\bar{x}$ for (MNP).

In view of (2.56), we have

$$f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i=f^i\left(\bar{x}\right)+S\left(\bar{x}\middle|C^i\right)\quad,\quad i=1,2,\dots,p.$$

This in view of the hypotheses of Theorem 2.4, gives the efficiency of $\bar{x}$ for (NP).

2.1.5 Related Problems

In this section, we specialize our problem (NP) and its dual problems (WND) and (M-WND). As discussed in [111] we may write $S\left(x\middle|C^i\right)=\left(x^T B^i x\right)^{\frac{1}{2}}$, $i=1,\ldots,p$ and $S\left(x\middle|D^j\right)=\left(x^T E^j x\right)^{\frac{1}{2}}$, $j=1,\ldots,m$ and the matrices B^i, $i=1,\ldots,p$ and E^j, $j=1,\ldots,m$ are positive semidefinite. Putting these in our problems, we have

(MNP)$_1$: Minimize $\left(f^1(x)+\left(x^T B^1 x\right)^{\frac{1}{2}},\ldots,f^p(x)+\left(x^T B^p x\right)^{\frac{1}{2}}\right)$

Subject to

$$g^j(x)+\left(x^T E^j x\right)^{\frac{1}{2}}\leqq 0,\ j=1,2,\ldots,m.$$

For the dual problem, we get

(WND)$_1$: Maximize

$$\left(f^1(u)+u^T B^1 z^1+\sum_{j=1}^{m} y^j\left(g^j(u)+u^T E^j w^j\right)\right.$$
$$\left.,\ldots,f^p(u)+u^T B^p z^p+\sum_{j=1}^{m} y^j\left(g^j(u)+u^T E^j w^j\right)\right)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^T B^i z^i\right)+\sum_{j=1}^{m} y^j\left(g^j(u)+u^T E^j w^j\right)=0,$$

$$z^T B^i z\le 1,\quad i=1,2,\ldots,p,$$

$$\left(w^j\right)^T E^j w^j\le 1,\quad j=1,2,\ldots,m\ ,$$

$$y\geqq 0\ ,\quad \lambda>0\ ,\sum_{i=1}^{p}\lambda^i=1$$

(M-WND)$_1$: Maximize $\left(f^1(u)+u^T B^1 z^1,\ldots,f^p(u)+u^T B^p z^p\right)$

Subject to

$$\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^T B^i z^i\right)+\sum_{j=1}^{m} y^j\left(g^j(u)+u^T D^j w^j\right)=0$$

$$\sum_{j=1}^{m} y^j\left(g^j(u)+u^T E^j w^j\right)\geqq 0$$

$$z^T B^i z\le 1,\quad i=1,2,\ldots,p$$

$$\left(w^j\right)^T E^j w^j\le 1,\quad j=1,2,\ldots,m$$

$$\lambda>0\ ,\quad y\geqq 0$$

2.2 MIXED TYPE DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING

2.2.1 Mixed Type Duality

In this section we present the following mixed type dual formulation of nondifferentiable nonlinear programming (Mix D) which combines Wolfe and Mond-Weir dual models studied in the preceding section:

(Mix D): Maximize $\Big(f^1(u)+u^T z^1+\sum_{j\in J_\circ} y^j\left(g^j(u)+u^T w^j\right)$

$$,\ldots, f^p(u)+u^T z^p+\sum_{j\in J_\circ} y^j\left(g^j(u)+u^T w^j\right)\Big)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^T z^i\right)+\sum_{j=1}^{m} y^j\nabla\left(g^j(u)+u^T w^j\right)=0, \quad (2.58)$$

$$\sum_{j\in J_\alpha} y^j\left(g^j(u)+u^T w^j\right)\geqq 0\ ,\ \alpha=1,2,\ldots,r, \quad (2.59)$$

$$z^i\in C^i \quad , \quad i=1,2,\ldots,p, \quad (2.60)$$

$$w^j\in D^j \quad , \quad j=1,2,\ldots,m\ , \quad (2.61)$$

$$y\geqq 0, \quad (2.62)$$

$$\lambda\in\Lambda^+, \quad (2.63)$$

where $J_\alpha\subseteq M=\{1,2,\ldots,m\}$, $\alpha=0,1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r} J_\alpha=M$ and $J_\alpha\bigcap J_\beta=\phi$, if $\alpha\neq\beta$.

If $J_\circ=M$, then (Mix D) becomes Wolfe type dual considered in the subsection 2.1.3. If $J_\circ=\phi$ and $J_\alpha=M$ for some $\alpha\in\{1,2,\ldots,r\}$, then (Mix D) becomes Mond-Weir type dual considered in the subsection 2.1.4.

Theorem 2.7 (*Weak Duality*): Let $\bar{x}$ be feasible for (NP) and $\left(u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ be feasible for (Mix D).If for all feasible $\left(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$, $\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(\cdot)+(\cdot)^T z^i\right)+\sum_{j\in J_\circ} y^j\left(g^j(\cdot)+(\cdot)^T w^j\right)$ is pseudoinvex and $\sum_{j\in J_\alpha} y^j\left(g^j(\cdot)+(\cdot)^T w^j\right)$, $\alpha=1,2,\dots,r$ is quasi-invex with respect to η, then the following cannot hold.

$$f^i(x)+s\left(x\middle|C^i\right)\leq f^i(u)+u^Tz^i+\sum_{j\in J_\circ} y^j\left(g^j(u)+u^Tw^j\right) \tag{2.64}$$

for all $i\in\{1,\dots,p\}$, and

$$f^k(x)+s\left(x\middle|C^k\right)< f^k(u)+u^Tz^k+\sum_{j\in J_\circ} y^j\left(g^j(u)+u^Tw^j\right) \tag{2.65}$$

for some k.

Proof: Suppose that (2.64) and (2.65) hold. Then in view of $\lambda>0$ and $\sum_{i=1}^p \lambda^i=1$, (2.64) and (2.65) together with $x^Tz^i\leqq s\left(x\middle|C^i\right)$, $i=1,2,\dots,p$ and $x^Tw^j\leqq s\left(x\middle|D^j\right)$, $j=1,2,\dots,m$ and the feasibility for (NP) and (Mix D) imply

$$\sum_{i=1}^p \lambda^i\left(f^i(x)+x^Tz^i\right)+\sum_{j\in J_\circ} y^j\left(g^j(x)+x^Tw^j\right)$$
$$<\sum_{i=1}^p \lambda^i\left(f^i(u)+u^Tz^i\right)+\sum_{j\in J_\circ} y^j\left(g^j(u)+u^Tw^j\right)$$

This, in view of the pseudoinvexity of $\sum_{i=1}^p \lambda^i\left(f^i(\cdot)+(\cdot)^T z^i\right)+\sum_{j\in J_\circ} y^j\left(g^j(\cdot)+(\cdot)^T w^j\right)$ with respect to η, implies,

$$\eta^T(x,u)\left(\sum_{i=1}^p \lambda^i\nabla\left(f^i(u)+u^Tz^i\right)+\sum_{j\in J_\circ} y^j\nabla\left(g^j(u)+u^Tw^j\right)\right)<0 \tag{2.66}$$

Since $\bar{x}$ is feasible for (VP), $\left(u,y,z^1,\dots,z^p,w^1,\dots,w^m,\lambda\right)$ is feasible for (Mix D), and $x^Tw^j\leqq s\left(x\middle|D^j\right)$, $j=1,2,\dots,m$, we have

$$\sum_{j\in J_\alpha} y^j\left(g^j(x)+x^Tw^j\right)\leqq\sum_{j\in J_\alpha} y^j\left(g^j(u)+u^Tw^j\right),\ \alpha=1,2,\dots,r$$

This in view of quasi-invexity of $\sum_{j\in J_\alpha} y^j\left(g^j(\cdot)+(\cdot)^T w^j\right)$, $\alpha=1,2,\dots,r$ with respect to η, gives

$$\eta^T(x,u)\left(\sum_{j\in J_\alpha} y^j\nabla\left(g^j(x)+x^Tw^j\right)\right)\leqq 0,\ \alpha=1,2,\dots,r$$

Hence,

$$\eta^T(x,u)\nabla\left(\sum_{j\in M-J_\circ} y^j\left(g^j(x)+x^Tw^j\right)\right)\leqq 0\ ,\ \ \alpha=1,2,\ldots,r \tag{2.67}$$

Combining (2.66) and (2.67), we have

$$\eta^T(x,u)\left(\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^Tz^i\right)+\sum_{j=1}^{m}y^j\nabla\left(g^j(u)+u^Tw^j\right)\right)<0 \tag{2.68}$$

From the equality constraint of (Mix D), we have

$$\eta^T(x,u)\left(\sum_{i=1}^{p}\lambda^i\nabla\left(f^i(u)+u^Tz^i\right)+\sum_{j=1}^{m}y^j\nabla\left(g^j(u)+u^Tw^j\right)\right)=0 \tag{2.69}$$

The relation (2.69) contradicts (2.68). Hence the conclusion of the theorem is true.

Theorem 2.8 (*Strong Duality*): Let $\bar{x}$ be an efficient solution of (NP) and for at least one i , $i\in\{1,2,\ldots,p\}$, $\bar{x}$ satisfies the regularity condition [74] for the problem $\left(\mathrm{P}_k(\bar{x})\right)$. Then there exist $\lambda\in R^p$ with $\lambda^T=\left(\bar{\lambda}^1,\ldots,\bar{\lambda}^i,\ldots,\bar{\lambda}^p\right)$, $\bar{y}\in R^m$ with $\bar{y}^T=\left(\bar{y}^1,\ldots,\bar{y}^i,\ldots,\bar{y}^m\right)$, $z^i\in R^n$, $i=\{1,2,\ldots,p\}$ and $w^j\in R^n$, $j=1,2,\ldots,m$ such that $\left(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ is feasible for (Mix D) and the objectives of (NP) and (Mix D) are equal.

Further, if the hypotheses of Theorem 2.7 are satisfied, then $\left(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ is an efficient solution of (Mix D).

Proof: Since $\bar{x}$ is an efficient solution for $\left(\mathrm{P}_k(\bar{x})\right)$, this implies that there exists $\xi\in R^p$ $\upsilon\in R^m$ with $\bar{\upsilon}^T=\left(\bar{\upsilon}^1,\ldots,\bar{\upsilon}^i,\ldots,\bar{\upsilon}^m\right)$ and $z^i\in R^n$, $i=\{1,2,\ldots,p\}$ such that

$$\bar{\xi}^k\nabla\left(f^k(x)+\bar{x}^T\bar{z}^k\right)+\sum_{\substack{i=1\\ i\neq k}}^{p}\bar{\xi}^i\nabla\left(f^i(x)+\bar{x}^T\bar{z}^i\right)+\sum_{j=1}^{m}y^j\nabla\left(g^j(x)+x^Tw^j\right)=0\ , \tag{2.70}$$

$$\sum_{j=1}^{m}\bar{\upsilon}^j\nabla\left(g^j(x)+x^Tw^j\right)=0\ , \tag{2.71}$$

$$\bar{x}^T\bar{z}^i = S\left(\bar{x}\middle|C^i\right), \quad i=1,2,\ldots,p \ , \tag{2.72}$$

$$\bar{x}^T\bar{w}^j = S\left(\bar{x}\middle|D^j\right), \quad j=1,2,\ldots,m, \tag{2.73}$$

$$z^i \in C^i \ , \quad i=1,2,\ldots,p, \tag{2.74}$$

$$w^j \in D^j, \quad j=1,2,\ldots,m \ , \tag{2.75}$$

$$\xi > 0 \ , \quad \bar{\upsilon} \geqq 0 \tag{2.76}$$

Dividing (2.70), (2.71) and (2.76) by $\sum_{i=1}^{p}\xi^i \neq 0$, and putting $\bar{\lambda}^i = \dfrac{\bar{\xi}^i}{\sum_{i=1}^{p}\xi^i}$ and

$\bar{y}^i = \dfrac{\bar{\upsilon}^i}{\sum_{i=1}^{p}\xi^i}$, we have

$$\sum_{i=1}^{p}\bar{\lambda}^i\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)+\sum_{j=1}^{m}\bar{y}^j\nabla\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)=0 \tag{2.77}$$

$$\sum_{j=1}^{m}\bar{y}^j\nabla\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)=0 \tag{2.78}$$

$$\lambda > 0 \ , \sum_{i=1}^{p}\lambda^i = 1 \ , \ \bar{y} \geqq 0 \tag{2.79}$$

The equation (2.78) implies

$$\sum_{j\in J_\circ}\bar{y}^j\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)=0 \tag{2.80}$$

and $$\sum_{j\in J_\alpha}\bar{y}^j\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)=0 \ , \ \alpha=1,2,\ldots,r \tag{2.81}$$

The relation (2.77), (2.79) and (2.81) imply that $\left(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ is feasible for (Mix D).

$$f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i+\sum_{j\in J_\circ}\bar{y}^j\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)=f^i\left(\bar{x}\right)+S\left(\bar{x}\middle|C^i\right), \quad i=1,2,\ldots,p.$$

This implies the objective of the primal and dual problems are equal.

Further, in view of the assumptions Theorem 2.7, the efficiency of $\bar{x}$ for (NP) is immediate.

Theorem 2.9 (*Converse Duality*): Let $\left(\bar{x},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda}\right)$ be an efficient solution for (Mix D). Assume that

(A$_1$) f and g are twice continuously differentiable,

(A$_2$) $\nabla f^i\left(\bar{x}\right)+\bar{z}^i+\sum_{j\in J_\circ}\bar{y}^j\left(\nabla g^j\left(\bar{x}\right)+\bar{w}^j\right),\quad i=1,2,\ldots,p$ are linearly independent,

(A$_3$) $\nabla^2\left(\lambda^T f^i\left(\bar{x}\right)+\bar{y}^T g\left(\bar{x}\right)\right)$ is positive or negative definite.

Further, if the assumptions of Theorem 2.7 are met, then $\bar{x}$ is an efficient solution.

Proof: Since $\left(\bar{x},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda}\right)$ be an efficient solution of (Mix D), then there exist $\tau\in R^p$, $\beta\in R^n$, $\gamma\in R$ for each γ constraints, $\eta\in R^p$ with $\eta^T=\left(\eta^1,\ldots,\eta^i,\ldots,\eta^p\right)$ and $\mu\in R^m$ such that the following Fritz-John optimality conditions [37] are satisfied,

$$-\sum_{i=1}^{p}\tau^i\left(\nabla\left(f^i\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)+\sum_{j\in J_\circ}\bar{y}^j\nabla\left(g\left(\bar{x}\right)+\bar{x}^T\bar{w}^j\right)\right)$$

$$+\beta^T\nabla^2\left(\lambda^T f\left(\bar{x}\right)+\bar{y}^T g\left(\bar{x}\right)\right)-\gamma\sum_{\alpha=1}^{r}\sum_{j\in J_\circ}\bar{y}^j\nabla\left(g^j\left(x\right)+\bar{x}^T\bar{w}^j\right)=0 \tag{2.82}$$

$$-\left(\tau^T e\right)\left(g^j\left(\bar{x}\right)+\bar{x}^T\bar{w}^j+\beta^T\nabla\left(g^j\left(\bar{x}\right)+\bar{x}^T\bar{w}^j\right)\right)-\mu^j=0\ ,\quad j\in J_\circ \tag{2.83}$$

$$-\gamma\left(g^j\left(\bar{x}\right)+\bar{x}^T\bar{w}^j+\beta^T\nabla\left(g^j\left(\bar{x}\right)+\bar{x}^T\bar{w}^j\right)\right)-\mu^j=0\ ,\ j\in J_\alpha\ ,\ \alpha=1,\ldots,r \tag{2.84}$$

$$\beta^T\nabla\left(f\left(\bar{x}\right)+\bar{x}^T\bar{z}^i\right)+\sum_{j\in J_\circ}\bar{y}^j\left(\nabla g^j\left(\bar{x}\right)+\bar{w}^j\right)-\eta^i=0 \tag{2.85}$$

$$\left(\lambda^i\beta-\tau^i\bar{x}\right)\in N_{C^i}\left(\bar{z}^i\right),\ i=1,\ldots,p \tag{2.86}$$

$$\left(\beta-\left(\tau^T e\right)\bar{x}\right)\bar{y}^j\in N_{D^j}\left(\bar{w}^j\right),\ j\in J_\circ \tag{2.87}$$

$$\left(\beta-\gamma\bar{x}\right)\bar{y}^j\in N_{D^j}\left(\bar{w}^j\right),\ j\in J_\alpha\ ,\ \alpha=i,\ldots,r \tag{2.88}$$

$$\mu^T\bar{y}=0 \tag{2.89}$$

$$\eta^T \lambda = 0 \tag{2.90}$$

$$\gamma \sum_{j \in J_\alpha} \bar{y}^j \nabla \left(g^j (\bar{x}) + \bar{x}^T \bar{w}^j \right) = 0 \quad , \quad \alpha = 1, \ldots, r \tag{2.91}$$

$$(\tau, \mu, \eta, \gamma) \geqq 0 \tag{2.92}$$

$$(\tau, \beta, \mu, \eta, \gamma) \neq 0 \tag{2.93}$$

Since $\lambda > 0$, (2.90) implies $\eta = 0$. Consequently (2.85) implies

$$\left(\nabla \left(f^i (\bar{x}) + \bar{x}^T \bar{z}^i \right) + \sum_{j \in J_\circ} \bar{y}^j \left(\nabla g^j (\bar{x}) + \bar{w}^j \right) \right) \beta = 0 \tag{2.94}$$

Using the equality constraint of (Mix D) in (2.82), we have

$$-\sum_{i=1}^{p} \left(\tau^i - \gamma \lambda^i \right) \left(\nabla f^i (\bar{x}) + \bar{z}^i + \sum_{j \in J_\circ} \bar{y}^T \left(\nabla g^j (\bar{x}) + \bar{w}^j \right) \right)$$
$$+ \beta^T \nabla^2 \left(\lambda^T f (\bar{x}) + \bar{y} g (\bar{x}) \right) = 0 \tag{2.95}$$

Postmultiplying (2.95) by β and then using (2.94), we have

$$\beta^T \nabla^2 \left(\lambda^T f (\bar{x}) + \bar{y}^T g (x) \right) \beta = 0$$

This because of (A_3), yields

$$\beta = 0 \tag{2.96}$$

Using (2.96) along with (A_2), we have

$$\tau^i - \gamma \lambda^i = 0 \, , \qquad i = 1, 2, \ldots, p \tag{2.97}$$

Suppose $\gamma = 0$, then from (2.97) we have $\tau = 0$. Consequently we have from (2.83) and (2.84), $\mu = 0$.

Thus $(\tau, \beta, \mu, \eta, \gamma) = 0$, contradicting (2.93).

Hence $\gamma > 0$ and $\tau > 0$.

In view of (2.96), (2.86), (2.87) and (2.88)we have,

$$\bar{x}^T \bar{z}^i = S \left(\bar{x} \middle| C^i \right) \quad , \quad i = 1, 2, \ldots, p \tag{2.98}$$

$$\bar{x}^T \bar{w}^j = S \left(\bar{x} \middle| D^j \right) \quad , \quad j = 1, 2, \ldots, m \tag{2.99}$$

From (2.83) and (2.84) along with (2.99) and (2.92), we have

$$g^j(x) + s(\bar{x}|D^j) \leqq 0 \quad , \quad j = 1, 2, \ldots, m$$

This implies the feasibility of $\bar{x}$ for (VP).

From (2.83) and (2.89), we have

$$\sum_{j \in J_\circ} \bar{y}^j \nabla\left(g^j(\bar{x}) + \bar{x}^T \bar{w}^j\right) = 0$$

In view of this together with (2.98), we have

$$f^i(\bar{x}) + \bar{x}^T \bar{z}^i + \sum_{j \in J_\circ} \bar{y}^j \left(g^j(\bar{x}) + \bar{x}^T \bar{w}^j\right)$$

$$= f^i(\bar{x}) + S(\bar{x}|C^i) \quad , \quad i = 1, 2, \ldots, p$$

This establishes the equality of objective values of (NP)

This in view of the hypothesis of Theorem 2.7 gives the efficiency of $\bar{x}$ for (NP).

2.2.2 Special Cases

In this section, we specialize our problem (NP) and its mixed dual problem (Mix D). As discussed in [111] we may write $S(x|C^i) = \left(x^T B^i x\right)^{\frac{1}{2}}$, $i = 1, \ldots, p$ and $S(x|D^j) = \left(x^T E^j x\right)^{\frac{1}{2}}$, $j = 1, \ldots, m$ and the matrices B^i, $i = 1, \ldots, p$ and E^j, $j = 1, \ldots, m$ are positive semidefinite. Putting these in our problems, we have

(NP)$_1$: Minimize $\left(f^1(x) + \left(x^T B^1 x\right)^{\frac{1}{2}}, \ldots, f^p(x) + \left(x^T B^p x\right)^{\frac{1}{2}}\right)$

Subject to

$$g^j(x) + \left(x^T E^j x\right)^{\frac{1}{2}} \leqq 0 \ , \ j = 1, 2, \ldots, m$$

For the dual (Mix D) problem, we get

(Mix D)$_1$: Maximize $\Bigg(f^1(u) + u^T B^i z^1 + \sum_{j \in J_\circ} y^j \left(g^j(u) + u^T E^j w^j\right)$

$$f^p(u) + u^T B^p z^p + \sum_{j \in J_\circ} y^j \left(g^j(u) + u^T E^j w^j\right)\Bigg)$$

Subject to

$$\sum_{i=1}^{p} \lambda^i \nabla \left(f^i(u) + u^T B^i z^i \right) + \sum_{j=1}^{m} y^j \left(g^j(u) + u^T D^j w^j \right) = 0 ,$$

$$\sum_{j \in J_\alpha} y^j \left(g^j(u) + u^T D^j w^j \right) \geqq 0 \ , \ \alpha = 1, 2, \ldots, r \ ,$$

$$z^T B^i z \leq 1 \ , \quad i = 1, 2, \ldots, p ,$$

$$\left(w^j \right)^T E^j w^j \leq 1, \quad j = 1, 2, \ldots, m \ ,$$

$$\lambda > 0 \quad , \quad y \geqq 0 \ .$$

Chapter-3

MULTIOBJECTIVE CONTINUOUS PROGRAMMING CONTAINING SUPPORT FUNCTIONS

3.1 MULTIOBJECTIVE CONTINUOUS PROGRAMMING

3.1.1 Introductory Remarks

Recently, Husain and Jabeen [70] derived optimality conditions for a nondifferentiable continuous programming problem in which nondifferentiability enters due to appearance of support functions in the integrand of the objective functional as well as in each constraint function. As an application of these optimality conditions, the authors in [70] formulated both Wolfe and Mond-weir type duals to the nondifferentiable continuous programming problem and established various duality results under invexity and generalized invexity.

There exist an extensive literature relating to optimality and duality in multiobjective nonlinear programming. But the status of continuous programming for optimality and duality is not very accomplished. Duality and optimality for multiobjective variational problems which can be referred to as continuous programming problems have been studied by a number of authors notably Bector and Husain [19], Chen [36] and many others cited in these references.

Any real world problem can be identified as multiple objective problem. Motivated with this cursory observation, in this chapter we formulate multiobjective nondifferentiable variational programming problem where nondifferentiability occurs due to support functions and study its duality under suitable invexity hypotheses. The close relationship of our duality results with those of nonlinear programming is also outlined.

3.1.2 Multiobjective Variational Problems and Related Results

We present the following nondifferentiable continuous programming problem containing support function.

(CP): Minimize $\left(\int_I \left(f^1(t,x,\dot{x}) + S\left(x \middle| C^1\right)\right) dt, \ldots, \int_I \left(f^p(t,x,\dot{x}) + S\left(x \middle| C^p\right)\right) dt \right)$

Subject to

$$x(a) = \alpha \quad , \quad x(b) = \beta$$

$$g^j(t,x,\dot{x}) + S\left(x \middle| D^j\right) \leqq 0 \ , \ j = 1,2,\ldots,m \ , \ t \in I$$

where $f^i : I \times R^n \times R^n \to R, (i = 1,2,\ldots,p)$, $g : I \times R^n \times R^n \to R^m$, $j = 1,2,\ldots,m$ are continuously differentiable function, and for each C^i , $i = 1,\ldots,p$ and D^j , $j = 1,\ldots,m$ are compact convex set in R^n.

In order to validate the strong duality theorem, we will require the following lemma of Chankong and Haimes [35].

Lemma 3.1: A point $\bar{x}$ is an efficient for (CP) if and only if $\bar{x}$ is an optimal solution for all

$\left(\mathrm{P_k}(\bar{x})\right)$: Minimize $\left(\int_I \left(f^k(t,x,\dot{x}) + S\left(x \middle| C^k\right)\right) dt \right)$

Subject to

$$x(a) = \alpha \quad , \quad x(b) = \beta$$

$$\int_I \left(f^i(t,x,\dot{x}) + S\left(x \middle| C^i\right)\right) dt \leq \int_I \left(f^i(t,\bar{x},\dot{\bar{x}}) + S\left(\bar{x} \middle| C^i\right)\right) dt \ ,$$

for all $i \neq k$

$$g^j(t,x,\dot{x}) + S\left(x \middle| D^j\right) \leqq 0 \ , \ j = 1,2,\ldots,m \ , \ t \in I$$

3.1.3 Wolfe Type Duality

The following problem is formulated as Wolfe type dual for the problem (CP).

(WCD): Maximize

$$\left(\int_I \left(f^1(t,u,\dot{u}) + u(t)^T z^1(t) + \sum_{j=1}^m y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt\right.$$

$$\left. ,\ldots,\int_I \left(f^p(t,u,\dot{u}) + u(t)^T z^p(t) + \sum_{j=1}^m y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt \right)$$

Subject to

$$u(a) = \alpha \quad , \quad u(b) = \beta \tag{3.1}$$

$$\sum_{i=1}^p \lambda^i \left(f_x^i + z^i(t) \right) + \sum_{j=1}^m y^j(t) \left(g_x^j + w^j(t) \right) = D\left(\lambda^T f_{\dot{x}}^i + y(t) g_{\dot{x}} \right), \; t \in I \tag{3.2}$$

$$z^i(t) \in C^i \quad , \quad i = 1,2,\ldots,p \tag{3.3}$$

$$w^j(t) \in D^j \quad , \quad j = 1,2,\ldots,m \tag{3.4}$$

$$\lambda > 0 \; , \sum_{i=1}^p \lambda^i = 1 \tag{3.5}$$

Theorem 3.1 (*Weak Duality*): Let $\bar{x}$ be feasible for (CP) and $(u, y, z^1,\ldots,z^p, w^1,\ldots,w^m,\lambda)$ be feasible for (WCD). If for all feasible $(x, u, y, z^1,\ldots,z^p, w^1,\ldots,w^m,\lambda)$ and with respect to $\eta \equiv \eta(t,x,u)$, $\int_I \left(\sum_{i=1}^p \lambda^i \left(f^i(t,.,.) + (\cdot)^T z^i(t) \right) dt + \sum_{j=1}^m y^j(t) \left(g^j(t,.,.) + (\cdot)^T w^j(t) \right) \right) dt$ is pseudoinvex then the following cannot hold.

$$\int_I \left(f^i(t,x,\dot{x}) + S\left(x(t) \middle| C^i\right) \right) dt \le \int_I \left(f^i(t,u,\dot{u}) + u(t)^T z^i(t) dt \right.$$

$$\left. + \sum_{j=1}^m y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt$$

for all $i \in \{1,\ldots,p\}$, and

$$\int_I \left(f^r(t,x,\dot{x}) + S\left(x(t)\middle|C^r\right)\right) dt < \int_I \left(f^r(t,u,\dot{u}) + u(t)^T z^r(t) dt \right.$$
$$\left. + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt$$

for some $r \in \{1,2,\ldots,p\}$.

Proof: Suppose that the conclusion of the theorem hold. With the feasibility of the problems (CP) and (WCD), together with $x^T(t) z^i(t) \leqq S\left(x(t)\middle|C^i\right), i = 1,2,\ldots,p$, we have

$$\int_I \left(f^i(t,x,\dot{x}) + x(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,x,\dot{x}) + x(t)^T w^j(t) \right) \right) dt$$
$$\leq \int_I \left(f^i(t,u,\dot{u}) + u(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt$$

for all $i \in \{1,\ldots,p\}$

$$\int_I \left(f^r(t,x,\dot{x}) + x(t)^T z^r(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,x,\dot{x}) + x(t)^T w^j(t) \right) \right) dt$$
$$< \int_I \left(f^r(t,u,\dot{u}) + u(t)^T z^r(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt$$

for some $r \in \{1,2,\ldots,p\}$.

Now in view of $\lambda > 0$ and $\sum_{i=1}^{p} \lambda^i = 1$, these inequalities yield

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x}) + x(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,x,\dot{x}) + x(t)^T w^j(t) \right) \right) dt$$
$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u}) + u(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt$$

This in view of the pseudoinvexity of

$\int_I \left(\sum_{i=1}^{p} \lambda^i \left(f^i(t,.,.) + (\cdot)^T z^i(t) \right) dt + \sum_{j=1}^{m} y^j(t) \left(g^j(t,.,.) + (\cdot)^T w^j(t) \right) \right) dt$ gives,

$$0 > \int_I \left\{ \eta^T \sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)\left(g_x^j + w^j(t)\right) \right) \right.$$
$$\left. + (D\eta)^T \left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} \right) \right\} dt$$

$$= \int_I \eta^T \left\{ \sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)\left(g_x^j + w^j(t)\right) \right) \right.$$
$$\left. - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} \right) \right\} dt + \eta^T \left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} \right)\Big|_{t=a}^{t=b}$$

(by integration by parts)

Using the boundary conditions which at $t = a$, $t = b$ give $\eta = 0$, we have

$$= \int_I \eta^T \left\{ \sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)\left(g_x^j + w^j(t)\right) \right) \right.$$
$$\left. - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} \right) \right\} dt < 0 \tag{3.6}$$

From the equality constraint of the (WCD), we have

$$\int_I \eta^T \left\{ \sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t)\left(g_x^j + w^j(t)\right) \right) \right.$$
$$\left. - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} \right) \right\} dt = 0$$

This relation (3.6) contradicts the equality constraint. Hence the conclusion of the theorem is true.

Theorem 3.2 (*Strong Duality*): Let $\bar{x}$ be a feasible solution of (CP) and for at least one i , $i \in \{1, 2, \ldots, p\}$, $\bar{x}$ satisfies the regularity condition [30] for $\left(\mathrm{P}_i(x)\right)$. Then there exist $\bar{\lambda} \in R^p$ with $\bar{\lambda}^T = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^i, \ldots, \bar{\lambda}^p\right)$, $z^i(t) \in C^i$, $i = 1, 2, \ldots, p$, $w^j(t) \in D^j$, $j = 1, 2, \ldots, m$ piecewise smooth $\bar{\upsilon} : I \to R^m$ with $\bar{\upsilon}^T = \left(\bar{\upsilon}^1, \ldots, \bar{\upsilon}^i, \ldots, \bar{\upsilon}^m\right)$, such that $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda}\right)$ is feasible for (WCD) and the objective values of (CP) and (WCD) are equal, and $\sum_{j=1}^{m} \int_I \bar{y}^j(t)\left(g^j\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{x}(t)^T \bar{w}(t)^j\right) dt = 0$.

Further, if the hypothesis of Theorem 3.1 is met, then $\left(x,u,y,z^1,\dots,z^p,w^1,\dots,w^m,\lambda\right)$ is an efficient solution of (WCD).

Proof: Since $\bar{x}$ is an efficient solution of (CP), by Lemma 3.1, $\bar{x}$ is an optimal solution of $\left(\mathrm{P}_k\left(\bar{x}\right)\right)$. Consequently, by Theorem 1 [70] there exists $\tau \in R^p$ with $\tau^T = \left(\tau^1,\dots,\tau^i,\dots,\tau^p\right)$, $z^i\left(t\right) \in C^i$, $i = 1,2,\dots,p$ and $w^j\left(t\right) \in D^j$, $j = 1,2,\dots,m$ and piecewise smooth $\bar{\upsilon} : I \to R^m$ with $\bar{\upsilon}^T = \left(\bar{\upsilon}^1,\dots,\bar{\upsilon}^i,\dots,\bar{\upsilon}^m\right)$ such that the following optimality conditions [30] hold:

$$\tau^k\left(f_x^k + \bar{z}^k\left(t\right) - Df_{\dot{x}}^k\right) + \sum_{j=1}^{m} \bar{\upsilon}^j\left(t\right)\left(g_x^j + \bar{w}^j\left(t\right)\right) - D\upsilon\left(t\right)^T g_{\dot{x}} + \sum_{\substack{i=1 \\ i \neq k}}^{p} \tau^i\left(f_x^i + \bar{z}^i\left(t\right) - Df_{\dot{x}}^i\right) = 0 \tag{3.7}$$

$$\sum_{j=1}^{m} y^j\left(t\right)\left(g^j\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{x}\left(t\right)^T \bar{w}^j\left(t\right)\right)dt = 0 \tag{3.8}$$

$$\bar{x}\left(t\right)^T \bar{z}^i\left(t\right) = S\left(\bar{x}\left(t\right)\middle| C^i\right) \quad , \quad i = 1,2,\dots,p \ , \ t \in I \tag{3.9}$$

$$\bar{x}\left(t\right)^T \bar{w}^j\left(t\right) = S\left(\bar{x}\left(t\right)\middle| D^j\right) \quad , \quad j = 1,2,\dots,m \ , \ t \in I \tag{3.10}$$

$$z^i\left(t\right) \in C^i \quad , \quad i = 1,2,\dots,p \tag{3.11}$$

$$w^j\left(t\right) \in D^j \quad , \quad j = 1,2,\dots,m \tag{3.12}$$

$$\tau > 0 \quad , \quad \upsilon\left(t\right) \geqq 0 \ , \ t \in I \tag{3.13}$$

Dividing (3.7), (3.8) and (3.13) by $\sum_{i=1}^{p} \tau^i \neq 0$, and setting $\lambda^i = \dfrac{\tau^i}{\sum_{i=1}^{p} \tau^i}$ and $y^i\left(t\right) = \dfrac{\upsilon^i\left(t\right)}{\sum_{i=1}^{p} \tau^i}$, we have

$$\sum_{j=1}^{m} \lambda^i\left(f_x^i + \bar{z}^i\left(t\right)\right) + \sum_{j=1}^{m} y^j\left(t\right)\left(g_x^j + \bar{x}\left(t\right)\bar{w}^j\left(t\right)\right) = D\left(\lambda^T f_{\dot{x}} + y\left(t\right)^T g_{\dot{x}}\right) \tag{3.14}$$

$$\sum_{j=1}^{m}\bar{y}^{j}(t)\left(g^{j}(t,x,\dot{x})+\bar{x}(t)^{T}\bar{w}^{j}(t)\right)=0\ ,\ t\in I \tag{3.15}$$

$$\lambda>0\ ,\quad y(t)\geqq 0,t\in I\ ,\quad \sum_{i=1}^{p}\lambda^{i}=1 \tag{3.16}$$

Consequently from (3.11), (3.12), (3.14), (3.15) and (3.16), the feasibility of $\left(\bar{x},\bar{u},\bar{y},\bar{z}^{1},\ldots,\bar{z}^{p},\bar{w}^{1},\ldots,\bar{w}^{m},\bar{\lambda}\right)$ for (WCD) follows.

In view of (3.9), (3.10) and (3.15), we have for each $i=1,\ldots,p$.

$$\int_{I}\left(f^{i}\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{x}(t)^{T}\bar{z}^{i}(t)+\sum_{j=1}^{m}\bar{y}^{j}(t)\left(g^{j}\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{x}(t)^{T}\bar{w}^{j}(t)\right)\right)dt=0$$

$$=\int_{I}f^{i}\left(t,\bar{x},\dot{\bar{x}}\right)+S\left(\bar{x}(t)\middle|C^{i}\right)dt,\quad i=1,2,\ldots,p.$$

This in view of Theorem 3.1, yields the efficiency of $\left(x,u,y,z^{1},\ldots,z^{p},w^{1},\ldots,w^{m},\lambda\right)$ for (WCD)

For the converse duality, we make the assumption that X denote the space of the piecewise differentiable function $x:I\to R^{n}$ for which $x(a)=0=x(b)$ equipped with the norm $\|x\|_{\infty}+\|Dx\|_{\infty}+\|D^{2}x\|_{\infty}$.

The problem (WCD) may be rewritten in the following form:

Minimize $\left(\int_{I}-\left(f^{1}(t,u,\dot{u})+u(t)^{T}z^{1}(t)+\sum_{j=1}^{m}y^{j}(t)^{T}\left(g^{j}(t,u,\dot{u})+u(t)^{T}w^{j}(t)\right)\right)dt\right.$

$$\left.,\ldots,\int_{I}-\left(f^{p}(t,u,\dot{u})+u(t)^{T}z^{p}(t)+\sum_{j=1}^{m}y^{j}(t)^{T}\left(g^{j}(t,u,\dot{u})+u(t)^{T}w^{j}(t)\right)\right)dt\right)$$

Subject to

$$u(a)=\alpha\ ,\quad u(b)=\beta$$

$$\theta(t,x,\dot{x},y,\lambda)=0$$

$$z^{i}(t)\in C^{i}\quad ,\quad i=1,2,\ldots,p$$

$$w^{j}(t)\in D^{j}\quad ,\quad j=1,2,\ldots,m$$

$$\lambda>0\ ,\sum_{i=1}^{p}\lambda^{i}=1$$

where

$$\theta = \theta\left(t, x(t), \dot{x}(t), y(t), \lambda \right)$$
$$= \sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) \right) + \sum_{j=1}^{m} y^j(t) \left(g_x^j + w^j(t) \right) - D\left(\lambda^T f_{\dot{x}}^i + y(t) g_{\dot{x}} \right),\ t \in I$$

with $\ddot{x} = D^2 x(t)$ and $\dot{y} = Dy(t)$

Consider $\theta\left(t, x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot), y(\cdot), \dot{y}(\cdot), \lambda \right) = 0$ as defining a mapping $\psi : I \times X \times Y \times R^p \to B$ where Y is a space of piecewise differentiable function and B is the Banach Space. In order to apply results of Craven [37] to the problem (WCD), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $\left[\psi_x(x, y, \lambda), \psi_y(x, y, \lambda), \psi_\lambda(x, y, \lambda) \right]$.

Theorem 3.3 (*Converse Duality*): Let $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda} \right)$ be an efficient solution for (WCD) Assume that

(H$_1$) The Frèchèt derivative ψ' has a (weak*) closed range,

(H$_2$) f and g are twice continuously differentiable, and

(H$_3$) $\left(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}} \right) \beta(t) = 0, \Rightarrow \beta(t) = 0$, $t \in I$

Further, if the assumptions of Theorem 3.1 are satisfied, then $\bar{x}$ is an efficient solution of (CP).

Proof: Since $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda} \right)$ with ψ' having a (weak*) closed range, is an efficient solution of (WD), then there exist $\alpha \in R^p$ with $\alpha^T = \left(\alpha^1, \ldots, \alpha^i, \ldots, \alpha^p \right)$, piecewise smooth $\beta : I \to R^n$ and $\mu : I \to R^m$ with $\mu(t)^T = \left(\mu^1(t), \ldots, \mu^m(t) \right)$, $\eta \in R^p$ and $\kappa \in R$ such that the following Fritz-John conditions [37] holds,

$$-\sum_{i=1}^{p}\alpha^i\left(\left(f_x^i+z^i(t)\right)+\sum_{j=1}^{m}y^j(t)\left(g_x^j+w^j(t)\right)-D\left(\alpha^T f_{\dot{x}}+y(t)^T g_{\dot{x}}\right)\right)$$
$$+\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}=0\ ,\ t\in I \qquad (3.17)$$

$$-\left(\alpha^T e\right)\left(g^j+\bar{x}(t)^T\bar{w}^j(t)\right)+\beta(t)^T\theta_{y^j}-D\beta(t)^T\theta_{\dot{y}^j}-\mu^j(t)=0\ ,\ t\in I$$
$$j=1,2,\ldots,m \qquad (3.18)$$

$$\left(f_x^i+z^i(t)-Df_{\dot{x}}^i\right)\beta(t)-\eta^i+\kappa=0\ \ i=1,\ldots,p \qquad (3.19)$$

$$\left(\beta(t)\lambda^i-\alpha^i\bar{x}(t)\right)\in N_{C^i}\left(\bar{z}^i(t)\right),\ i=1,\ldots,p\ ,\ t\in I \qquad (3.20)$$

$$\left(\beta(t)-\left(\alpha^T e\right)\bar{x}(t)\right)\bar{y}^j(t)\in N_{D^j}\left(\bar{w}^j(t)\right),\ j=1,\ldots,m\ ,\ t\in I \qquad (3.21)$$

$$\eta^T\bar{\lambda}=0 \qquad (3.22)$$

$$\kappa\left(\sum_{i=1}^{p}\lambda^i-1\right)=0 \qquad (3.23)$$

$$\mu(t)^T\bar{y}(t)=0\ ,\ t\in I \qquad (3.24)$$

$$\left(\alpha,\eta,\kappa,\mu(t),\right)\geqq 0 \qquad (3.25)$$

$$\left(\alpha,\beta(t),\eta,\kappa,\mu(t)\right)\neq 0 \qquad (3.26)$$

Since $\lambda>0$, (3.22) implies $\eta=0$. Consequently (3.19) implies

$$\left(f_x^i+z^i(t)-Df_{\dot{x}}^i\right)\beta(t)=-\kappa \qquad (3.27)$$

From the equality constraint of (WCD), we have

$$\left(\sum_{j=1}^{m}y^j(t)\left(g_x^j+\bar{w}^j(t)-Dy(t)^T g_{\dot{x}}\right)\right)\beta(t)=-\sum_{i=1}^{p}\lambda^i\left(f_x^i+\bar{z}^i(t)+Df_{\dot{x}}^i\right)\beta(t)$$
$$=-\sum_{i=1}^{p}\lambda^i(-\kappa)=\kappa \qquad (3.28)$$

From (3.17) have

$$-\sum_{i=1}^{p}\alpha^i\left(f_x^i+z^i(t)+Df_{\dot{x}}^i\right)\beta(t)-\left(\alpha^T e\right)\left(\sum_{j=1}^{m}\bar{y}^j(t)\left(g_x^j+\bar{w}^j(t)-D\bar{y}(t)^T g_{\dot{x}}\right)\right)\beta(t)$$
$$+\left(\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}\right)\beta(t)=0\ ,\ t\in I$$

Using (3.27) and (3.28) in this relation, we have

$$\left(\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}=0\right)\beta(t)\ ,\ t\in I$$

This because of the hypothesis (H_3), gives

$$\beta(t)=0 \ , \ t \in I \tag{3.29}$$

Suppose $\alpha = 0$, then from (3.18) we have $\mu^j(t)=0$, $j=1,2,\ldots,m$, and from (3.19) we get $\kappa=0$.

Consequently $(\alpha, \beta(t), \eta, \kappa, \mu(t))=0$. This contradicts (3.26).

Hence $\alpha > 0$.

From (3.20) and (3.21) in view of (3.29) implies,

$$\bar{x}(t)^T \bar{z}^i(t) = S(\bar{x}(t) | C^i) \quad , \quad i=1,2,\ldots,p \ , \ t \in I \tag{3.30}$$

$$\bar{x}(t)^T \bar{w}^j(t) = S(\bar{x}(t) | D^j) \quad , \quad j=1,2,\ldots,m \ , \ t \in I \tag{3.31}$$

From (3.18) along with (3.25), (3.29) and (3.31), we have

$$g^j(t,x,\dot{x}) + S(\bar{x}(t) | D^j) \leqq 0 \quad , \quad j=1,2,\ldots,m \ , \ t \in I$$

This implies $\bar{x}$ is feasible for (CP).

The relation (3.18) along with (3.29) and (3.24) gives

$$\sum_{j=1}^{m} \bar{y}^j(t)\left(g^j(t,x,\dot{x}) + \bar{x}(t)^T \bar{w}^j(t)\right) = 0 \ , \ t \in I$$

(3.32)

Now for each $i \in \{1,\ldots,p\}$, in view of (3.30) and (3.32), we have

$$\int_I \left(f^i(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T z^i(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T w^j(t) \right) \right) dt = 0$$

$$= \int_I \left(f^i(t,\bar{x},\dot{\bar{x}}) + S(\bar{x} | C^i) \right) dt \quad , \quad i=1,2,\ldots,p$$

This along with the requirements of Theorem yields the efficiency of $\bar{x}$ for (CP).

3.1.4 Mond-Weir Type Duality

We further weaken the invexity requirements in the preceeding section by formulating Mond-Weir type dual to the problem (CP).

(M-WCD): Maximize $\left(\int_I \left(f^1(t,u,\dot{u})+u(t)^T z^1(t)\right)dt,\ldots,\int_I \left(f^p(t,u,\dot{u})+u(t)^T z^p(t)\right)dt\right)$

Subject to

$$u(a)=\alpha \quad , \quad u(b)=\beta \tag{3.33}$$

$$\sum_{i=1}^{p}\lambda^i\left(f_x^i+z^i(t)\right)+\sum_{j=1}^{m}y^j(t)\left(g_x^j+w^j(t)\right)$$
$$=D\left(\lambda^T f_{\dot{x}}^i+y(t)^T g_{\dot{x}}\right),\ t\in I \tag{3.34}$$

$$z^i(t)\in C^i \quad , \quad i=1,2,\ldots,p, \tag{3.35}$$

$$w^j(t)\in D^j \quad , \quad j=1,2,\ldots,m, \tag{3.36}$$

$$y(t)\geqq 0 \ , \ t\in I \tag{3.37}$$

$$\sum_{j=1}^{m}\int_I y^j(t)^T\left(g^j(t,u,\dot{u})+u(t)^T w^j(t)\right)dt\geqq 0 \ , \ t\in I \tag{3.38}$$

$$\lambda>0 \ . \tag{3.39}$$

Theorem 3.4 (*Weak Duality*): Let $\bar{x}$ be feasible for (CP) and $\left(u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ be feasible for (M-WCD). If for all feasible $\left(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda\right)$ with respect to $\eta\equiv\eta(t,x,u)$,

i. $\sum_{i=1}^{p}\lambda^i\left(f^i(t,.,.)+(\cdot)^T z^i(t)\right)dt$ is pseudoinvex and

ii. $\sum_{j=1}^{m}\int_I y^j(t)\left(g^j(t,.,.)+(\cdot)^T w^j(t)\right)dt$ is quasi-invex with respect to same $\eta\equiv\eta(t,x,u)$ following cannot hold.

$$\int_I\left(f^i(t,x,\dot{x})+S\left(x(t)\middle|C^i\right)\right)dt\leq\int_I\left(f^i(t,u,\dot{u})+u(t)^T z^i(t)dt \tag{3.40}$$

for all $i\in\{1,\ldots,p\}$, and

$$\int_I\left(f^r(t,x,\dot{x})+S\left(x(t)\middle|C^r\right)\right)dt<\int_I\left(f^r(t,u,\dot{u})+u(t)^T z^r(t)dt \tag{3.41}$$

for some $r\in\{1,2,\ldots,p\}$.

Proof: Suppose that (3.40) and (3.41) hold, then in view of $\lambda > 0$ and $x(t)^T z^i(t) \leqq S\left(x(t)\middle|C^i\right)$, $i = 1, 2, \ldots, p$ we have

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t, x, \dot{x}) + x(t)^T z^i(t) \right) dt < \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t, u, \dot{u}) + u(t)^T z^i(t) \right) dt$$

This in view of the pseudoinvexity of $\sum_{i=1}^{p} \lambda^i \left(f^i(t, ., .) + (\cdot)^T z^i(t) \right) dt$ yields,

$$\int_I \left\{ \eta^T \left(\sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) \right) \right) + (D\eta)^T \left(\lambda^T f_{\dot{x}} \right) \right\} dt < 0$$

This on integration by parts gives

$$= \int_I \eta^T \left(\sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) \right) - D\left(\lambda^T f_{\dot{x}} \right) \right) dt + \eta^T \left(\lambda^T f_{\dot{x}} \right) \Big|_{t=a}^{t=b}$$

Using the boundary conditions which at $t = a$, $t = b$ gives $\eta = 0$, we have

$$\int_I \eta^T \left(\sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) \right) - D\left(\lambda^T f_{\dot{x}} \right) \right) dt < 0$$

From the feasibility requirements of (CP) and (M-WCD) together with $x(t)^T z^i(t) \leqq S\left(x(t)\middle|C^i\right)$, we have

$$\sum_{j=1}^{m} \int_I y^j(t) \left(g^j(t, x, \dot{x}) + x(t)^T w^j(t) \right) dt \leqq \sum_{j=1}^{m} \int_I y^j(t) \left(g^j(t, u, \dot{u}) + u(t)^T w^j(t) \right) dt$$

By quasi-invexity of $\sum_{j=1}^{m} \int_I y^j(t) \left(g^j(t, ., .) + (\cdot)^T w^j(t) \right) dt$, implies

$$\int_I \left[\eta^T \sum_{j=1}^{m} y^j(t) \left(g_x^j + w^j(t) \right) + (D\eta)^T y^T(t) g_{\dot{x}}^j \right] dt \leqq 0 \tag{3.42}$$

This, by integration by parts, as earlier gives

$$\int_I \eta^T \left[\sum_{j=1}^{m} y^j(t) \left(g_x^j + w^j(t) \right) - D y^T(t) g_{\dot{x}}^j \right] dt \leqq 0 \tag{3.43}$$

Combining (3.42) and (3.43), we have

$$\int_I \eta^T \left\{ \sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) + \sum_{j=1}^{m} y^j(t) \left(g_x^j + w^j(t) \right) \right) \right.$$
$$\left. - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} \right) \right\} dt < 0$$

which contradicts (3.34), and the conclusion of the theorem is established.

The following strong duality can be proved on the lines of Theorem 3.2 with slight modifications.

Theorem 3.5 (*Strong Duality*): Let $\bar{x}$ be a efficient solution of (CP) and for at least one $i \quad , i \in \{1, 2, \ldots, p\}$, $\bar{x}$ satisfies the regularity condition [30] for $\left(\mathrm{P}_i\left(\bar{x}\right)\right)$. Then there exist $\bar{\lambda} \in R^p$ with $\bar{\lambda}^T = \left(\bar{\lambda}^1, \ldots, \bar{\lambda}^i, \ldots, \bar{\lambda}^p\right)$ and piecewise smooth $\bar{y} : I \to R^m$ with $\bar{y}^T = \left(\bar{y}^1, \ldots, \bar{y}^i, \ldots, \bar{y}^m\right)$, $z^i\left(t\right) \in C^i$, $i = 1, 2, \ldots, p$ and $w^j\left(t\right) \in D^j$, $j = 1, 2, \ldots, m$ such that $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda}\right)$ is feasible for (M-WCD) and the objective values of (CP) and (M-WCD) are equal.

Further, if the hypotheses of Theorem 3.4 are met, then $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda}\right)$ is an efficient solution of (M-WCD).

(M-WCD) can be rewritten in the following form:

$$\text{Minimize } -\left(\int_I \left(f^1\left(t, u, \dot{u}\right) + u\left(t\right)^T z^1\left(t\right)\right) dt, \ldots, \int_I \left(f^p\left(t, u, \dot{u}\right) + u\left(t\right)^T z^p\left(t\right)\right) dt\right)$$

Subject to

$$u\left(a\right) = \alpha \quad , \quad u\left(b\right) = \beta$$

$$\theta\left(t, x\left(t\right), \dot{x}\left(t\right), y\left(t\right), \lambda\right) = 0$$

$$z^i\left(t\right) \in C^i \quad , \quad i = 1, 2, \ldots, p,$$

$$w^j\left(t\right) \in D^j \quad , \quad j = 1, 2, \ldots, m,$$

$$y\left(t\right) \geqq 0 \ , \ t \in I$$

$$\sum_{j=1}^{m} \int_I y^j\left(t\right)^T \left(g^j\left(t, u, \dot{u}\right) + u\left(t\right)^T w^j\left(t\right)\right) dt \geqq 0 \ , \ t \in I$$

$$\lambda > 0 \ .$$

where

$$\theta = \theta\left(t, x\left(t\right), \dot{x}\left(t\right), y\left(t\right), \lambda\right)$$
$$= \sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i\left(t\right)\right) + \sum_{j=1}^{m} y^j\left(t\right)\left(g_x^j + w^j\left(t\right)\right) - D\left(\lambda^T f_{\dot{x}}^i + y\left(t\right) g_{\dot{x}}\right), \ t \in I$$

Theorem 3.6 (*Converse Duality*): Let $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda}\right)$ be an efficient solution for (M-WCD). Assume that

(A$_1$) The Frèchèt derivative ψ' has a (weak*) closed range,

(A$_2$) f and g are twice continuously differentiable,

(A$_3$) $f_x^i+\bar{z}^i\left(t\right)-Df_{\dot{x}}^i$, $i\in\{1,2,\ldots,p\}$ are linearly independent and

(A$_4$) $\left(\beta\left(t\right)^T\theta_x-D\beta\left(t\right)^T\theta_{\dot{x}}+D^2\beta\left(t\right)^T\theta_{\ddot{x}}\right)\beta\left(t\right)=0, \Rightarrow \beta\left(t\right)=0$, $t\in I$

Further, if the hypotheses of Theorem 3.4 are met then $\bar{x}$ is an efficient solution of (CP)

Proof: Since $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda}\right)$ with ψ' having a (weak*) closed range, is an efficient solution of (MWD), then there exist $\alpha\in R^p$ with $\alpha^T=\left(\alpha^1,\ldots,\alpha^i,\ldots,\alpha^p\right)$, piecewise smooth $\beta: I\rightarrow R^n$ and $\mu: I\rightarrow R^m$, $\eta\in R^p$ with $\eta^T=\left(\eta^1,\ldots,\eta^p\right)$ satisfying the following Fritz-John conditions [37],

$$-\sum_{i=1}^{p}\alpha^i\left(f_x^i+z^i\left(t\right)-Df_{\dot{x}}^i\right)-\gamma\left(\sum_{j=1}^{m}y^j\left(t\right)\left(g_x^j+w^j\left(t\right)\right)-Dy\left(t\right)^T g_{\dot{x}}\right)$$
$$+\beta\left(t\right)^T\theta_x-D\beta\left(t\right)^T\theta_{\dot{x}}+D^2\beta\left(t\right)^T\theta_{\ddot{x}}=0\ ,\ t\in I \tag{3.44}$$

$$-\gamma\left(g^j+\bar{x}\left(t\right)\bar{w}^j\left(t\right)\right)+\beta\left(t\right)^T\theta_{y^j}-D\beta\left(t\right)^T\theta_{\dot{y}^j}-\mu^j\left(t\right)=0\ ,\ t\in I$$
$$j=1,2,\ldots,m \tag{3.45}$$

$$\left(f_x^i+z^i\left(t\right)-Df_{\dot{x}}^i\right)\beta\left(t\right)-\eta^i=0\ \ ,\ i=1,\ldots,p \tag{3.46}$$

$$\left(\beta\left(t\right)\lambda^i-\alpha^i\bar{x}\left(t\right)\right)\in N_{C^i}\left(\bar{z}^i\left(t\right)\right),\ i=1,\ldots,p\ ,\ t\in I \tag{3.47}$$

$$\left(\beta\left(t\right)-\gamma\bar{x}\left(t\right)\right)\bar{y}^j\left(t\right)\in N_{D^j}\left(\bar{w}^j\left(t\right)\right),\ j=1,\ldots,m\ ,\ t\in I \tag{3.48}$$

$$\gamma\sum_{j=1}^{m}\int_I y^j\left(t\right)\left(g^j+\bar{x}\left(t\right)^T\bar{w}^j\left(t\right)\right)dt=0 \tag{3.49}$$

$$\eta^T\lambda=0 \tag{3.50}$$

$$\bar{\mu}^T\left(t\right)\bar{y}\left(t\right)=0\ ,\ t\in I \tag{3.51}$$

$$\left(\alpha,\mu(t),\eta,\gamma\right) \geqq 0 \tag{3.52}$$

$$\left(\alpha,\beta(t),\mu(t),\eta,\gamma\right) \neq 0 \tag{3.53}$$

Since $\lambda > 0$, (3.50) implies $\eta = 0$. Consequently (3.46) implies

$$\left(f_x^i + z^i(t) - Df_{\dot{x}}^i\right)\beta(t) = 0 \quad , \quad i = 1,\ldots,p \tag{3.54}$$

Using the equality constraint of (M-WCD) in (3.44), we have

$$-\sum_{i=1}^{p}\left(\alpha^i - \gamma\lambda^i\right)\left(f_x^i + z^i(t) - Df_{\dot{x}}^i\right) + \beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} = 0 \ , \ t \in I \tag{3.55}$$

Using (3.54) and (3.55) in (3.44), we have

$$\left(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}}\right)\beta(t) = 0 \ , \ t \in I$$

which because of the hypothesis (A_4) implies,

$$\beta(t) = 0 \quad , \quad t \in I \tag{3.56}$$

The relation (3.55) along with (3.56), gives,

$$\sum_{i=1}^{p}\left(\alpha^i - \gamma\lambda^i\right)\left(f_x^i + z^i(t) - Df_{\dot{x}}^i\right) = 0$$

This, due to the hypothesis (A_3) gives,

$$\alpha^i = \gamma\lambda^i = 0 \ , \ i = 1,2,\ldots,p \tag{3.57}$$

Suppose $\gamma = 0$; Then from (3.57) we have $\alpha = 0$ .From (3.45), we have,

$$\mu(t) = 0 \quad , \quad t \in I \ .$$

Consequently, $\left(\alpha,\beta(t),\mu(t),\eta,\gamma\right) = 0$ contradicting the Fritz-John condition (3.53).

Hence $\gamma > 0$ and from (3.57), $\alpha > 0$.

In view of (3.56) together with $\gamma > 0$ and $\alpha > 0$, (3.47) and (3.48), respectively imply

$$\bar{x}(t)^T\bar{z}^i(t) = S\left(\bar{x}(t)\middle|C^i\right) \quad , \quad i = 1,2,\ldots,p \ , \ t \in I \tag{3.58}$$

$$\bar{x}(t)^T\bar{w}^j(t) = S\left(\bar{x}(t)\middle|D^j\right) \quad , \quad j = 1,2,\ldots,m \ , \ t \in I \tag{3.59}$$

The relation (3.45) along with $\gamma > 0$ and $y(t) \geqq 0$, $t \in I$ and (3.59) imply

$$g^j\left(t,\bar{x},\dot{\bar{x}}\right) + S\left(\bar{x}(t)\middle|D^j\right) \leqq 0 \quad , \quad j = 1,2,\dots,m$$

This implies the feasibility of $\bar{x}$ for (CP).

In view of (3.58), we have

$$f^i(\bar{x}) + \bar{x}^T \bar{z}^i$$

$$= f^i(\bar{x}) + S\left(\bar{x}\middle|C^i\right) \quad , \quad i = 1,2,\dots,p$$

This, in view of the hypothesis of Theorem 3.4 gives the efficiency of $\bar{x}$ for (CP).

3.1.5 Related Problems

It is possible to extend duality theorems established in the previous sections to the corresponding multiobjective variational problem containing support functions with natural boundary values rather than fixed end point.

PRIMAL $(CP)_0$: Minimize

$$\left(\int_I \left(f^1(t,x,\dot{x}) + S\left(x(t)\middle|C^1\right)\right)dt, \dots, \int_I \left(f^p(t,x,\dot{x}) + S\left(x(t)\middle|C^p\right)\right)dt \right)$$

Subject to

$$g^j(t,x,\dot{x}) + S\left(x(t)\middle|D^j\right) \leqq 0 \ , \ j = 1,2,\dots,m \ , \ t \in I$$

DUAL $(WDP)_0$: Maximize

$$\left(\int_I \left(f^1(t,u,\dot{u}) + u(t)^T z^1(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt \right.$$

$$\left., \dots, \int_I \left(f^p(t,u,\dot{u}) + u(t)^T z^p(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) \right) dt \right)$$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) \right) + \sum_{j=1}^{m} y^j(t) \left(g_x^j + w^j(t) \right) = D\left(\lambda^T f_{\dot{x}}^i + y(t) g_{\dot{x}} \right), \ t \in I$$

$$\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} = 0, \quad \text{at } t = a \text{ and } t = b$$

$$z^i(t) \in C^i \quad , \quad i = 1, 2, \ldots, p$$

$$w^j(t) \in D^j \quad , \quad j = 1, 2, \ldots, m$$

$$\lambda > 0 \ , \sum_{i=1}^{p} \lambda^i = 1$$

DUAL (M-WD)$_0$: Maximize

$$\left(\int_I \left(f^1(t,u,\dot{u}) + u(t)^T z^1(t) \right) dt, \ldots, \int_I \left(f^p(t,u,\dot{u}) + u(t)^T z^p(t) \right) dt \right)$$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i + z^i(t) \right) + \sum_{j=1}^{m} y^j(t) \left(g_x^j + w^j(t) \right)$$

$$= D\left(\lambda^T f_{\dot{x}}^i + y(t)^T g_{\dot{x}} \right), \ t \in I$$

$\lambda^T f_{\dot{x}} = 0 = \lambda^T y(t) g_{\dot{x}}$, at $t = a$ and $t = b$

$$z^i(t) \in C^i \quad , \quad i = 1, 2, \ldots, p,$$

$$w^j(t) \in D^j \quad , \quad j = 1, 2, \ldots, m,$$

$$y(t) \geqq 0 \ , \ t \in I$$

$$\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) dt \geqq 0 \ , \ t \in I$$

$$\lambda > 0 \ .$$

If the functions in the problem mentioned in Section 3.5 are independent of t, they will reduce to the following.

PRIMAL (NP): Minimize$\left(f^1(x) + S\left(x \middle| C^1\right), \ldots, f^p(x) + S\left(x \middle| C^p\right) \right)$

Subject to

$$g^j(x) + S\left(x \middle| D^j\right) \leqq 0 \ , \ j = 1, 2, \ldots, m$$

DUAL (WNP): Maximixe $\left(f^1(u) + u^T z^1 + \sum_{j=1}^{m} y^j \left(g^j(u) + u^T w^j \right) \right.$

$$\left. \ldots, f^p(u) + u^T z^p + \sum_{j=1}^{m} y^j \left(g^j(u) + u^T w^j \right) \right)$$

Subject to

$$\sum_{i=1}^{p}\lambda^{i}\left(f_{x}^{i}+z^{i}\right)+\sum_{j=1}^{m}y^{j}\left(g_{x}^{j}+w^{j}\right)=0$$

$$z^{i}\in C^{i}\quad,\quad i=1,2,\ldots,p$$

$$w^{j}\in D^{j}\quad,\quad j=1,2,\ldots,m$$

$$\lambda>0\ ,\sum_{i=1}^{p}\lambda^{i}=1$$

DUAL (M-WNP): Maximize $\left(f^{1}(u)+u^{T}z^{1},\ldots,f^{p}(u)+u^{T}z^{p}\right)$

Subject to

$$\sum_{i=1}^{p}\lambda^{i}\left(f_{x}^{i}+z^{i}\right)+\sum_{j=1}^{m}y^{j}\left(g_{x}^{j}+w^{j}\right)=0$$

$$z^{i}\in C^{i}\quad,\quad i=1,2,\ldots,p$$

$$w^{j}\in D^{j}\quad,\quad j=1,2,\ldots,m$$

$$y\geqq 0\ ,\ t\in I$$

$$\sum_{j=1}^{m}y^{j}\left(g^{j}(u)+u^{T}w^{j}\right)\geqq 0$$

$$\lambda>0$$

Chapter-4

OPTIMALITY CONDITIONS AND MULTIOBJECTIVE DUALITY FOR VARIATIONAL PROBLEMS INVOLVING HIGHER ORDER DERIVATIVES

4.1 OPTIMALITY CRITERIA AND DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS INVOLVING HIGHER ORDER DERIVATIVES

4.1.1 Introductory Remarks

In [73], Husain and Jabeen studied a wider class of variational problem in which the arc function is twice differentiable by extending the notion of invexity given in [103]. They obtained Fritz John as well as Karush- Kuhn-Tucker necessary optimality conditions as an application of Karush-Kuhn-Tucker optimality conditions studied various duality results for Wolfe and Mond-Weir type models.

Since mathematical programming and classical calculus of variations have undergone independent development, it is felt that mutual adaptation of ideas and techniques may prove useful. Motivated with this idea in this exposition, we propose to study optimality criteria and duality for a wider class of multiobjective variational problems involving higher order derivative. These results not only generalize the results of Husain and Jabeen [73] and Bector and Husain [19] but also present a dynamic generalization of some of the results in multiobjective nonlinear programming already existing.

This chapter is divided into three sections, 4.1, 4.2 and 4.3. In the section 4.1, optimality conditions, both Fritz-John and Karush-Kuhn-Tucker type optimality conditions are derived for the variational problem and the notion of invexity / generalized invexity are extended. As an application of Karush-Kuhn-Tucker optimality conditions, Wolfe type dual is formulated and various duality results are established under invexity defined in this section. In this section, it is also shown that our results can be considered as continuous time extension of nonlinear problem existing in the literature. The section 4.2 formulates Mond-Weir dual for multiobjective variational problem considered in the subsection 4.2.1 to relax the invexity requirements of section 4.1 for various duality results to hold and gives relationship between the results of this section to those of nonlinear programming. The section 4.3 is meant to unify the dual formulations of the variational problems in the section 4.1 and 4.2 and prove various duality results under invexity and generalized invexity results.

4.1.2 Invexity and Generalized Invexity

For ready reference, we reproduce the following definition extended by Husain and Jabeen [73].

Definition 4.1 (*Invexity*): If there exists vector function $\eta(t,\dot{u},\ddot{u},x,\dot{x},\ddot{x}) \in R^n$ with $\eta = 0$ and $x(t) = u(t), t \in I$ and $D\eta = 0$ for $\dot{x}(t) = \dot{u}(t), t \in I$ such that for a scalar function $\phi(t,x,\dot{x},\ddot{x})$, the functional $\Phi(x,\dot{x},\ddot{x}) = \int_I \phi(t,x,\dot{x},\ddot{x})dt$ satisfies,

$$\Phi(u)\text{-}\Phi(x) \geqq \int_I \left\{ \eta^T \phi_x(t,x,\dot{x},\ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x},\ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t,x,\dot{x},\ddot{x}) \right\} dt,$$ Φ is

said to be invex in $x, \dot{x}$ *and* $\ddot{x}$ on I with respect to η.

Definition 4.2 (*Pseudoinvexity*): Φ is said to be pseudoinvex in $x, \dot{x}$ and $\ddot{x}$ with respect to η if $\int_I \left\{ \eta^T \phi_x(t,x,\dot{x},\ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x},\ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t,x,\dot{x},\ddot{x}) \right\} dt \geqq 0$ implies $\Phi(u) \geqq \Phi(x)$.

Definition 4.3 (*Quasi-Invex*): The functional Φ is said to quasi-invex in x, $\dot{x}$ and $\ddot{x}$ with respect to η if

$$\Phi(u) \leqq \Phi(x) \quad \text{implies}$$

$$\int_I \left\{ \eta^T \phi_x(t,x,\dot{x},\ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t,x,\dot{x},\ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t,x,\dot{x},\ddot{x}) \right\} dt \leqq 0 .$$

4.1.3 Variational Problem and Optimality Conditions

We present the following multiobjective variational problem with higher order derivatives:

(VPE): Minimize $\left(\int_I f^1(t,x,\dot{x},\ddot{x})\,dt, \ldots, \int_I f^p(t,x,\dot{x},\ddot{x})\,dt \right)$

Subject to

$$x(a) = 0 = x(b) \tag{4.1}$$

$$\dot{x}(a) = 0 = \dot{x}(b) \tag{4.2}$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \;\;,\;\; t \in I \tag{4.3}$$

$$h(t,x,\dot{x},\ddot{x}) = 0 \;\;,\;\; t \in I \tag{4.4}$$

where

1) $f^i : I \times R^n \times R^n \times R^n \to R$, $i = 1,2,\ldots,p$, $g : I \times R^n \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \times R^n \to R^k$ are continuously differentiable function, and

2) $\hat{X}$ designates the space of piecewise functions $x : I \to R^n$ possessing derivatives $\dot{x}$ and $\ddot{x}$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$.

We require the following definition of efficient solution for our further analysis.

Definition 4.4 (*Efficient Solution*): A feasible solution $\bar{x}$ is efficient for (VPE) if there exist no other feasible x for (VPE) such that for some $i \in P = \{1,2,\ldots,p\}$,

$$\int_I f^i(t,x,\dot{x},\ddot{x})dt < \int_I f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt$$

and

$$\int_I f^j(t,x,\dot{x},\ddot{x})dt \leqq \int_I f^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt \text{ for all } j \in P,\ j \neq i.$$

In relation to **(**VPE**)**, we introduce the following set of problems $\bar{P}_r$ for each $r = 1, 2, \ldots, p$ in the spirit of [35], with a single objective,

$(\bar{\mathbf{P}}_\mathbf{r})$: Minimize $\int_I f^r(t,x,\dot{x},\ddot{x})dt$

Subject to

$$x(a) = 0 = x(b),$$

$$\dot{x}(a) = 0 = \dot{x}(b),$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \ , \ t \in I,$$

$$h(t,x,\dot{x},\ddot{x}) = 0 \ , \ t \in I,$$

$$\int_I f^i(t,x,\dot{x},\ddot{x})dt \leqq \int_I f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt \ , \ i = 1, 2, \ldots, p \ , \ i \neq r.$$

The following lemma can be proved on the lines of Chankong and Haimes [35].

Lemma 4.1: x^* is an efficient solution of (VPE) if and only if $\bar{x}$ is an optimal solution of $(\bar{\mathrm{P}}_\mathrm{r})$ for each $r = 1, 2, \ldots, p$.

Consider the following single objective variational problem considered in [73].

(P$_0$): Minimize $\int_I \phi(t,x,\dot{x},\ddot{x})dt$

Subject to

$$x(a) = 0 = x(b),$$

$$\dot{x}(a) = 0 = \dot{x}(b),$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \ , \ t \in I,$$

$$h(t,x,\dot{x},\ddot{x}) = 0 \ , \ t \in I,$$

where $\phi : I \times R^n \times R^n \times R^n \to R$.

The following proposition gives the Fritz-John type necessary optimality conditions obtained by Husain and Jabeen [73]. In this proposition, we have written the functions without arguments for brevity.

Proposition 4.1 [73**]:** (*Fritz John Optimality Conditions*) If $\bar{x}$ is an optimal solution of (P_0) and $h_x\left(x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot)\right)$ maps X into the subspace of $C\left(I, R^k\right)$, then there exists Lagrange multiplier $\bar{\tau} \in R$, the piecewise smooth $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that

$$\left(\bar{\tau}\,\phi_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x\right) - D\left(\bar{\tau}\,\phi_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}}\right)$$
$$+ D^2\left(\bar{\tau}\,\phi_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}\right) = 0\ ,\ t \in I,$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) = 0\ ,\ t \in I\ ,$$

$$\left(\bar{\tau}, \bar{y}(t)\right) \geqq 0\ ,\ t \in I\ ,$$

$$\left(\bar{\tau}, \bar{y}(t), \bar{z}(t)\right) \neq 0\ ,\ t \in I\ .$$

If $\bar{\tau} = 1$, then the above optimality conditions will reduce to the Karush-Kuhn-Tucker type optimality conditions and the solution $\bar{x}$ is referred to as a normal solution.

We now establish the following theorem that gives the necessary optimality conditions for (VPE).

Theorem 4.1 (*Fritz-John Conditions*): Let $\bar{x}$ be an efficient solution of **(**VPE**)** and $h_x\left(x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot)\right)$ maps X into the subspace of $C\left(I, R^k\right)$, then there exist $\bar{\lambda} \in R^k$ and the piecewise smooth $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that

$$\left(\bar{\lambda}^T f_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x\right) - D\left(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}}\right)$$
$$+ D^2\left(\bar{\lambda}^T f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}\right) = 0\ ,\ \ t \in I\ , \tag{4.5}$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0\ ,\ t \in I\ , \tag{4.6}$$

$$(\bar{\lambda},\bar{y}(t)) \geqq 0\ ,\ t \in I\ , \tag{4.7}$$

$$(\bar{\lambda},\bar{y}(t),\bar{z}(t)) \neq 0\ ,\ t \in I\ . \tag{4.8}$$

Proof: Since $\bar{x}$ is an efficient solution of **(**VPE**)** by Lemma 4.1, $\bar{x}$ is an optimal solution of $(\bar{P}_r)$, for each $r = 1, 2, \ldots, p$. From Proposition1, it follows that, there exist scalars $\bar{\lambda}^{1r},\bar{\lambda}^{2r},\ldots,\bar{\lambda}^{pr}$ and piecewise smooth function $\bar{y}: I \to R^m$ and $\bar{z}: I \to R^k$, such that

$$\bar{\lambda}^{rr} f_x^r + \sum_{\substack{i=1\\ i\neq r}}^{p} \bar{\lambda}^{ir} f_x^i + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_x^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_x^l$$
$$- D\left(\bar{\lambda}^{rr} f_{\dot{x}}^r + \sum_{\substack{i=1\\ i\neq r}}^{p} \bar{\lambda}^{ir} f_{\dot{x}}^i + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_{\dot{x}}^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_{\dot{x}}^l \right)$$
$$+ D^2\left(\bar{\lambda}^{rr} f_{\ddot{x}}^r + \sum_{\substack{i=1\\ i\neq r}}^{p} \bar{\lambda}^{ir} f_{\ddot{x}}^i + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_{\ddot{x}}^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_{\ddot{x}}^l \right) = 0\ ,\ t \in I,$$

$$\bar{y}^T(t) g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0\ ,\ t \in I\ ,$$

$$\left(\bar{\lambda}^{1r},\bar{\lambda}^{2r},\ldots,\bar{\lambda}^{pr},\ \bar{y}^{1r}(t),\bar{y}^{2r}(t),\ldots,\bar{y}^{mr}(t)\right) \geqq 0\ ,\ t \in I\ ,$$

$$\left(\bar{\lambda}^{1r},\bar{\lambda}^{2r},\ldots,\bar{\lambda}^{pr},\ \bar{y}^{1r}(t),\bar{y}^{2r}(t),\ldots,\bar{y}^{mr}(t),\bar{z}^{1r}(t),\bar{z}^{2r}(t),\ldots,\bar{z}^{lr}(t)\right) \neq 0\ ,\ t \in I\ .$$

Summing over r, we have

$$\sum_{r=1}^{p}\left(\sum_{i=1}^{p}\bar{\lambda}^{ir}\right) f_x^i + \sum_{r=1}^{p}\left(\sum_{j=1}^{m}\bar{y}^{jr}(t)\right) g_x^j + \sum_{r=1}^{p}\left(\sum_{l=1}^{k}\bar{z}^{lr}(t)\right) h_x^l$$
$$- D\left(\sum_{r=1}^{p}\left(\sum_{i=1}^{p}\bar{\lambda}^{ir}\right) f_{\dot{x}}^i + \sum_{r=1}^{p}\left(\sum_{j=1}^{m}\bar{y}^{jr}(t)\right) g_{\dot{x}}^j + \sum_{r=1}^{p}\left(\sum_{l=1}^{k}\bar{z}^{lr}(t)\right) h_{\dot{x}}^l\right)$$
$$+ D^2\left(\sum_{r=1}^{p}\left(\sum_{i=1}^{p}\bar{\lambda}^{ir}\right) f_{\ddot{x}}^i + \sum_{r=1}^{p}\left(\sum_{j=1}^{m}\bar{y}^{jr}\right) g_{\ddot{x}}^j + \sum_{r=1}^{p}\left(\sum_{l=1}^{k}\bar{z}^{lr}(t)\right) h_{\ddot{x}}^l\right) = 0\ ,\ t \in I\ ,$$

$$\bar{y}^T(t) g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0\ ,\ t \in I\ ,$$

$$\left(\sum_{r=1}^{p}\bar{\lambda}^{1r},\ldots,\sum_{i=1}^{p}\bar{\lambda}^{pr};\sum_{i=1}^{p}\bar{y}^{1r}(t),\ldots,\sum_{i=1}^{p}\bar{y}^{mr}(t)\right)\geqq 0\ ,\ t\in I\ ,$$

$$\left(\sum_{r=1}^{p}\bar{\lambda}^{1r},\ldots,\sum_{i=1}^{p}\bar{\lambda}^{pr};\sum_{i=1}^{p}\bar{y}^{1r}(t),\ldots,\sum_{i=1}^{p}\bar{y}^{mr}(t);\sum_{r=1}^{p}\bar{z}^{1r}(t),\ldots,\sum_{r=1}^{p}\bar{z}^{lr}(t)\right)\neq 0\ ,\ t\in I$$

Setting $\bar{\lambda}^{i}=\sum_{r=1}^{p}\bar{\lambda}^{ir}$, $\bar{y}^{j}(t)=\sum_{r=1}^{p}\bar{y}^{jr}(t)$, $t\in I$ and $\bar{z}^{l}(t)=\sum_{r=1}^{l}\bar{z}^{lr}(t)$, $t\in I$, we have,

$$\left(\bar{\lambda}^{T}f_{x}+\bar{y}(t)^{T}g_{x}+\bar{z}(t)^{T}h_{x}\right)-D\left(\bar{\lambda}^{T}f_{\dot{x}}+\bar{y}(t)^{T}g_{\dot{x}}+\bar{z}(t)^{T}h_{\dot{x}}\right)$$
$$+D^{2}\left(\bar{\lambda}^{T}f_{\ddot{x}}+\bar{y}(t)^{T}g_{\ddot{x}}+\bar{z}(t)^{T}h_{\ddot{x}}\right)=0\ ,\ \ t\in I,$$

$$\bar{y}(t)^{T}g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0\ \ ,\ t\in I\ ,$$

$$\left(\bar{\lambda},\bar{y}(t)\right)\geqq 0\ ,\ t\in I\ ,$$

$$\left(\bar{\lambda},\bar{y}(t),\bar{z}(t)\right)\neq 0\ ,\ t\in I\ .$$

Theorem 4.2 (*Karush-Kuhn-Tucker Conditions*): Let $\bar{x}$ be an efficient solution for (VPE) which is assumed to be normal for $\left(\bar{P}_{r}\right)$ for each $r=1,2,\ldots,p$. Let the constraints of (P_r) satisfy Slater's Constraint Qualification [30] for each $r=1,2,\ldots,p$. Then there exist $\bar{\lambda}^{T}\in R_{+}^{k}$, $\bar{y}:I\rightarrow R^{m}$ and $\bar{z}:I\rightarrow R^{k}$, such that the following relation hold for all $t\in I$,

$$\left(\bar{\lambda}^{T}f_{x}+\bar{y}(t)^{T}g_{x}+\bar{z}(t)^{T}h_{x}\right)-D\left(\bar{\lambda}^{T}f_{\dot{x}}+\bar{y}(t)^{T}g_{\dot{x}}+\bar{z}(t)^{T}h_{\dot{x}}\right)$$
$$+D^{2}\left(\bar{\lambda}^{T}f_{\ddot{x}}+\bar{y}(t)^{T}g_{\ddot{x}}+\bar{z}(t)^{T}h_{\ddot{x}}\right)=0\ ,\ t\in I \tag{4.9}$$

$$\bar{y}(t)^{T}g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0\ ,\ \ t\in I \tag{4.10}$$

$$\bar{\lambda}>0\quad ,\quad y(t)\geqq 0\ ,\ t\in I \tag{4.11}$$

Proof: Since $\bar{x}$ is an efficient solution of (VPE) by lemma 4.1, $\bar{x}$ is an optimal solution of $\left(\bar{P}_{r}\right)$, for each $r=1,2,\ldots,p$ then there exist scalars $\bar{\lambda}_{1r},\bar{\lambda}_{2r},\ldots,\bar{\lambda}_{pr}$ with $\bar{\lambda}_{rr}=1$, $\bar{y}:I\rightarrow R^{m}$ and $\bar{z}:I\rightarrow R^{k}$, such that the following conditions are satisfied for all

$$f_x^r + \sum_{\substack{i=1 \\ i \neq r}}^{k} \bar{\lambda}^{ir} f_x^r + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_x^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_x^l$$

$$- D\left(f_x^r + \sum_{\substack{i=1 \\ i \neq r}}^{k} \bar{\lambda}^{ir} f_{\dot{x}}^r + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_{\dot{x}}^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_{\dot{x}}^l \right)$$

$$+ D^2\left(f_{\ddot{x}}^r + \sum_{\substack{i=1 \\ i \neq r}}^{k} \bar{\lambda}^{ir} f_{\ddot{x}}^r + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_{\ddot{x}}^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_{\ddot{x}}^l \right) = 0\ ,\ t \in I,$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) = 0\ ,\ t \in I\ ,$$

$$\bar{y}(t) \geqq 0,$$

$\bar{\lambda}_{ir} \geq 0\ ,\ i = 1, 2, \ldots, p\ ,\ i \neq r$. Summing over r and setting $\bar{\lambda}_i = \sum_{r=1}^{p} \bar{\lambda}^{ir}$ with $\bar{\lambda}^{ii} = 1$,

$\bar{y}^j(t) = \sum_{r=1}^{p} \bar{y}^{jr}(t), \bar{z}^l(t) = \sum_{r=1}^{p} \bar{z}^{lr}(t)$, we have,

$$\left(\bar{\lambda}^T f_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x\right) - D\left(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}}\right)$$
$$+ D^2\left(\bar{\lambda}^T f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}\right) = 0\ ,\ t \in I,$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) = 0\ ,\ t \in I\ ,$$

$$\bar{\lambda} > 0 \quad , \quad \bar{y}(t) \geqq 0\ ,\ t \in I\ .$$

Remark: If $\lambda > 0$, then Theorem 4.1 reduces to Theorem 4.2 and then an efficient solution is called a normal solution as an analogy to the normality condition given in [30].

4.1.4 Wolfe Type Duality

In this section, we consider the following variational problem (VP) involving higher order derivatives, by suppressing the equality constraint in (VPE).

(VP): Minimize $\left(\int_I f^1(t,x,\dot{x},\ddot{x})dt, \ldots, \int_I f^p(t,x,\dot{x},\ddot{x})dt\right)$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \quad , \quad t \in I$$

We formulate the following Wolfe type dual to the problem (VP) and establish various duality results under invexity defined in the preceding section.

(WD): Maximize $\left(\int_i \left(f^1(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u})\right)dt\right.$

$$\left., \ldots, \int_i \left(f^p(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u})\right)dt\right)$$

Subject to

$$u(a) = 0 = u(b)$$

$$\dot{u}(a) = 0 = \dot{u}(b)$$

$$\left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right) = 0 \quad , \quad t \in I \tag{4.12}$$

$$y(t) \geqq 0 \quad , \quad t \in I \tag{4.13}$$

$$\lambda > 0 \ , \quad \lambda^T e = 1 \tag{4.14}$$

where $e = (1,1,\ldots,1)^T$ and $\lambda \in R^k$.

Theorem 4.3 (*Weak Duality*): Let $x \in X$ be feasible for (VP) and (u,λ,y) be feasible for (WD), if $\int_I \lambda^T f\, dt$ is invex and $\int_I y(t)^T g\, dt$ is invex with respect to the same η. Then

$$\int_1 f(t,x,\dot{x},\ddot{x})dt \not\leq \int_I \left\{f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u})e\right\}dt$$

Proof: $\lambda^T\left(\int_I f^i(t,x,\dot{x},\ddot{x})dt-\int_I\left\{f(t,u,\dot{u},\ddot{u})+y(t)^T g(t,u,\dot{u},\ddot{u})e\right\}dt\right)$

$$=\int\left(\lambda^T f(t,x,\dot{x},\ddot{x})-\lambda^T f(t,u,\dot{u},\ddot{u})-\left(\lambda^T e\right)y(t)^T g(t,u,\dot{u},\ddot{u})\right)dt$$

$$=\int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt-\int_I \lambda^T f(t,u,\dot{u},\ddot{u})dt-\int_I y(t)^T g(t,u,\dot{u},\ddot{u})dt,$$

(By using $\lambda^T e=1$)

$$\geqq\int_I\left\{\eta^T\left(\lambda^T f_x\right)+(D\eta)^T\left(\lambda^T f_{\dot{x}}\right)+\left(D^2\eta\right)^T\left(\lambda^T f_{\ddot{x}}\right)\right\}dt$$
$$-\int_I y(t)^T g(t,u,\dot{u},\ddot{u})dt \qquad (4.15)$$

This is possible by invexity of $\int_I \lambda^T f\,dt$

Also from the feasibility of (VP) and (WD), we have

$$\int_I y(t)^T(t,x,\dot{x},\ddot{x})dt-\int_I y(t)^T g(t,u,\dot{u},\ddot{u})dt$$
$$\geqq\int_I\left\{\eta^T\left(y^T g\right)_x+(D\eta)^T\left(y^T g\right)_{\dot{x}}+\left(D^2\eta\right)^T\left(y^T g\right)_{\ddot{x}}\right\}dt$$

(By definition of Invexity)

This implies

$$-\int_I y(t)^T g(t,u,\dot{u},\ddot{u})dt\geqq\int_I\left\{\eta^T\left(y^T g\right)_x+(D\eta)^T\left(y^T g\right)_{\dot{x}}+\left(D^2\eta\right)^T\left(y^T g\right)_{\ddot{x}}\right\}dt$$
$$-\int_I y(t)^T g(t,x,\dot{x},\ddot{x})dt$$

Using this in (4.15), we have,

$$\lambda^T\left(\int_I f(t,x,\dot{x},\ddot{x})dt-\int_I\left\{f(t,u,\dot{u},\ddot{u})+y(t)^T g(t,u,\dot{u},\ddot{u})e\right\}dt\right)$$
$$\geqq\int_I\eta^T\left\{\lambda^T f_x+(D\eta)^T\left(\lambda^T f_{\dot{x}}\right)+\left(D^2\eta\right)^T\left(\lambda^T f_{\ddot{x}}\right)\right\}dt$$
$$+\int_I\left\{\eta^T\left(y^T g\right)_x+(D\eta)^T\left(y^T g\right)_{\dot{x}}+\left(D^2\eta\right)^T\left(y^T g\right)_{\ddot{x}}\right\}dt$$
$$-\int_I y(t)^T g(t,x,\dot{x},\ddot{x})dt$$

$$= \int_I \eta^T \left\{ \left(\lambda^T f_x + y^T g_x \right) + (D\eta)^T \left(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}} \right) \right.$$

$$\left. + \left(D\eta^2 \right)^T \left(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}} \right) \right\} dt - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})\, dt$$

$$= \int_I \eta^T \left(\lambda^T f_x + y^T g_x \right) dt + \eta^T \left(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}} \right) \Big|_{t=a}^{t=b}$$

$$- \int_I \eta^T D\left(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}} \right) dt + (D\eta)^T \left(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}} \right) \Big|_{t=a}^{t=b}$$

$$- \int_I (D\eta)^T D\left(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}} \right) dt - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})\, dt$$

(By integration by parts)

Using the boundary conditions which give $D\eta = 0 = \eta \ at \ t = a, t = b$

$$= \int_I \eta^T \left(\lambda^T f_x + y^T g_x \right) dt - \int_I \eta^T D\left(\lambda^T f_x + y^T g_x \right) dt + \eta^T D\left(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}} \right) \Big|_{t=a}^{t=b}$$

$$+ \int_I \eta^T D^2 \left(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}} \right) dt - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})\, dt$$

(By integration by parts)

Using the boundary conditions which give $D\eta = 0 = \eta \ at \ t = a, t = b$

$$= \int_I \eta^T \left\{ \left(\lambda^T f_x + y^T g_x \right) - D\left(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}} \right) + D^2 \left(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}} \right) \right\} dt$$

$$- \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})\, dt$$

$\geqq - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})\, dt,$ (By equation (4.12))

$\geqq 0$ (by (4.3) and (4.13))

That is,

$$\int_I \lambda^T f(t, x, \dot{x}, \ddot{x})\, dt \geqq \int_I \left\{ \lambda^T f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u}) \right\} dt$$

or

$$\lambda^T \left(\int_I f(t, x, \dot{x}, \ddot{x}) \right) dt \geqq \lambda^T \left(\int_I \left\{ f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u}) e \right\} \right) dt$$

This yields,

$$\int_I f(t, x, \dot{x}, \ddot{x})\, dt \not\leq \int_I \left\{ f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u}) e \right\} dt.$$

Theorem 4.4 (*Strong Duality*): If $\bar{x}$ is efficient and normal solution of (VP), then there exist piecewise smooth $\bar{y}: I \to R^m$ such that $(\bar{x}, \bar{y})$ is feasible for (WD) and the corresponding objective values of the problems (VP) and (WD) are equal. If the hypotheses of Theorem 4.2 are satisfied, then $(\bar{x}, \bar{y})$ is an efficient solution of (WD).

Proof: Since $\bar{x}$ is efficient and normal for (VP), by Theorem 4.2, it implies that there exist $\mu \in R^p$ and piecewise smooth $u: I \to R^m$ such that,

$$\left(\mu^T f_x + u(t)^T g_x\right) - D\left(\mu^T f_{\dot{x}} + u(t)^T g_{\dot{x}}\right) + D^2\left(\mu^T f_{\ddot{x}} + u(t)^T g_{\ddot{x}}\right) = 0,\ t \in I,$$

$$\bar{u}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) = 0,\ t \in I,$$

$$\mu > 0 \quad \bar{u}(t) \geqq 0,\ t \in I.$$

Since $\mu > 0$, $\mu^T e \neq 0$.

$$\left(\left(\frac{\mu}{\mu^T e}\right)^T f_x + \frac{y(t)^T}{\mu^T e} g_x\right) - D\left(\left(\frac{\mu}{\mu^T e}\right)^T f_{\dot{x}} + \frac{y(t)^T}{\mu^T e} g_{\dot{x}}\right) + D^2\left(\left(\frac{\mu}{\mu^T e}\right)^T f_{\ddot{x}} + \frac{y(t)^T}{\mu^T e} g_{\ddot{x}}\right) = 0,\ t \in I$$

$$\left(\frac{y(t)}{\mu^T e}\right)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) = 0,\ t \in I$$

$$\frac{\mu}{\mu^T e} > 0,\ \frac{y(t)}{\mu^T e} \geqq \ ,\ t \in I$$

Setting $\frac{\mu}{\mu^T e} = \bar{\lambda}_i$ and $\frac{y(t)}{\mu^T e} = \bar{y}(t)$ in the above relations, we have,

$$\left(\bar{\lambda}^T f_x + \bar{y}(t)^T g_x\right) - D\left(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}}\right) + D^2\left(\bar{\lambda}^T f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}}\right) = 0,\ t \in I \tag{4.16}$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})=0 \ , \ t \in I \tag{4.17}$$

$$\left.\begin{array}{l}\bar{\lambda}>0 \quad \bar{y}(t) \geqq 0 \\ \quad \lambda^T e=1\end{array}\right\} ,t \in I \tag{4.18}$$

From (4.5) and (4.7), it follows that $(\bar{x},\bar{\lambda},\bar{y})$ is feasible for (WD). The equality of the objective of (VP) and (WD) is obvious in view of (4.17). The efficiency of $(\bar{x},\bar{\lambda},\bar{y})$ for (WD) follows from Theorem 4.3.

As in [105], by employing chain rule in calculus, it can be easily seen that the expression $\left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right)$, may be regarded as a function θ of variables $t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}$ and λ, where $\dddot{x} = D^3 x$ and $\ddot{y} = D^2 y$. That is, we can write

$$\theta(t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},\lambda) = \left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right)$$

In order to prove converse duality between (VP) and (WD), the space X is now replaced by a smaller space X_2 of piecewise smooth thrice differentiable function $x : I \to R^n$ with the norm $\|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty + \|D^3 x\|_\infty$. The problem (WD) may now be briefly written as,

$$\text{Minimize } -\left(\int_i \left(f^1(t,x,\dot{x},\ddot{x}) + y(t)^T g(t,x,\dot{x},\ddot{x})\right)dt\right.$$
$$\left. ,\dots,\int_i \left(f^p(t,x,\dot{x},\ddot{x}) + y(t)^T g(t,x,\dot{x},\ddot{x})\right)dt\right)$$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$\theta(t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},\lambda) = 0 \ , \quad t \in I$$

$$y(t) \geqq 0 \ , \quad t \in I$$

$$\lambda > 0 \ , \quad \lambda^T e = 1$$

where $e = (1,1,\dots,1)^T \in R^p$ and $\lambda \in R^p$.

Consider $\theta\left(t, x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot), \dddot{x}(\cdot), y(\cdot), \dot{y}(\cdot), \ddot{y}(\cdot), \lambda\right)=0$ as defining a mapping $\psi: X_2 \times Y \times R^p \rightarrow B$ where Y is a space of piecewise twice differentiable function and B is the Banach Space. In order to apply Theorem 4.1 to the problem (WD), the infinite dimensional inequality must bne restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $\left[\psi_x(x, y, \lambda), \psi_y(x, y, \lambda), \psi_\lambda(x, y, \lambda)\right]$.

Theorem 4.5 (*Converse Duality*): Let $\left(\bar{x}, \bar{\lambda}, \bar{y}\right)$ be an efficient solution of (WD) and ψ' has a (weak*) closed range. Assume that

(H₁) f and g are twice differentiable,

(H₂) the hypotheses of Theorem 4.3 hold, and

(H₃) $\sigma(t)^T\left(\sigma(t)^T \theta_x - D\sigma(t)^T \theta_{\dot{x}} + D^2\sigma(t)^T \theta_{\ddot{x}} - D^3\sigma(t)^T \theta_{\dddot{x}}\right)=0, \quad t \in I$

$\Rightarrow \sigma(t)=0$, $t \in I$

Then $\bar{x}$ is an efficient solution of (VP).

Proof: Since $\left(\bar{x}, \bar{\lambda}, \bar{y}\right)$ is an efficient solution of (WD) and ψ' has a closed range, then by Theorem 4.1, there exist $\alpha \in R^k$ and piecewise smooth $\beta: I \rightarrow R^n$, $\xi: I \rightarrow R^m$ and $\mu^T \in R^p$ such that

$$\left[-\left(\alpha f_x+\left(\alpha^T e\right) y(t)^T g_x\right)+\beta(t)^T \theta_x\right]-D\left[-\left(\alpha f_{\dot{x}}+\left(\alpha^T e\right) y(t)^T g_{\dot{x}}\right)+\beta(t)^T \theta_{\dot{x}}\right]$$
$$+\left[-\left(\alpha f_{\ddot{x}}+\left(\alpha^T e\right) y(t)^T g_{\ddot{x}}\right)+\beta(t)^T \theta_{\ddot{x}}\right]-D^3 \beta(t)^T \theta_{\dddot{x}}=0 \tag{4.19}$$

$$-\left(\alpha^T e\right) g+\beta(t)^T \theta_y-D\left(\beta(t)^T \theta_{\dot{y}}\right)+D^2\left(\beta(t)^T \theta_{\ddot{y}}\right)-\xi(t)=0 \tag{4.20}$$

$$\beta(t)\left(f_x-D f_{\dot{x}}+D^2 f_{\ddot{x}}\right)+\mu^i+\gamma=0 \tag{4.21}$$

$$\mu^T \lambda=0 \tag{4.22}$$

$$\bar{\xi}(t)^T \bar{y}(t)=0 \tag{4.23}$$

$$\gamma\left(\sum_{i=1}^{p}\lambda^i - 1\right) = 0 \tag{4.24}$$

$$\left(\alpha,\mu,\gamma,\xi(t)\right) \geqq 0 \tag{4.25}$$

$$\left(\alpha,\beta(t),\mu,\gamma,\xi(t)\right) \neq 0 \tag{4.26}$$

Since $\lambda > 0$, (4.22) implies $\mu = 0$. Consequently (4.21) implies

$$\beta(t)^T\left(f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i\right) = -\gamma \tag{4.27}$$

From the equality constraint of the dual problem (WD) together with (4.27), it follows

$$y(t)^T g_x - Dy(t)^T g_{\dot{x}} + D^2 y(t)^T g_{\ddot{x}} = -\sum_{i=1}^{p}\lambda^i\left(f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i\right)$$

$$\left(y(t)^T g_x - Dy(t)^T g_{\dot{x}} + D^2 y(t)^T g_{\ddot{x}}\right)\beta(t) = -\sum_{i=1}^{p}\lambda^i\left(f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i\right)\beta(t)$$

$$= -\sum_{i=1}^{p}\lambda^i(-\gamma) = \gamma \tag{4.28}$$

Postmultiplying (4.19) by $\beta(t)$ and then using (4.27) and (4.28), we have

$$\left(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}}\right)\beta(t) = 0$$

This because of the hypothesis of (H_3) yields

$$\beta(t) = 0,\ t \in I \tag{4.29}$$

Therefore from (4.27), we have $\gamma = 0$.

Suppose $\alpha = 0$, then from (4.20), $\xi(t) = 0$, $t \in I$. Consequently we have $\left(\alpha,\beta(t),\mu,\gamma,\xi(t)\right) = 0$, $t \in I$. This is in contradiction to (4.26).

Hence $\alpha > 0$. The relation (4.20) in conjunction with (4.29) yields,

$$g(t,x,\dot{x},\ddot{x}) = -\frac{\xi(t)}{\alpha^T e} \leqq 0 \tag{4.30}$$

This implies the feasibility of $\bar{x}$ for (VP). The relation (4.30) with (4.23) yields

$$y(t)^T g(t,x,\dot{x},\ddot{x}) = 0,\ t \in I \tag{4.31}$$

This implies,

$$\int_i \left(f^i\left(t,u,\dot{u},\ddot{u}\right) + y\left(t\right)^T g\left(t,u,\dot{u},\ddot{u}\right)\right) dt = \int_i f^i\left(t,x,\dot{x},\ddot{x}\right) dt$$

This along with an application of Theorem 4.3 accomplishes the efficiency of $\bar{x}$ for (VP).

4.1.5 Natural Boundary Values

The duality results obtained in the preceding sections can easily be extended to the multiobjective variational problems with natural boundary values rather than fixed end points.

Primal (P_1): Minimize $\left(\int_I f^1\left(t,x,\dot{x},\ddot{x}\right) dt, \ldots, \int_I f^p\left(t,x,\dot{x},\ddot{x}\right) dt \right)$

Subject to

$$g\left(t,x,\dot{x},\ddot{x}\right) \leqq 0 \quad , \quad t \in I$$

Dual (D_1): Maximize $\int_I \left(f\left(t,u,\dot{u},\ddot{u}\right) + y\left(t\right)^T g\left(t,u,\dot{u},\ddot{u}\right) e \right) dt$

Subject to

$$\left(\lambda^T f_x + y\left(t\right)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y\left(t\right)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y\left(t\right)^T g_{\ddot{x}}\right) = 0 \quad t \in I$$

$$\left(\lambda^T f_{\dot{x}} + y\left(t\right)^T g_{\dot{x}}\right) = 0 \text{ , at } t = a \text{ and } t = b$$

$$\left(\lambda^T f_{\ddot{x}} + y\left(t\right)^T g_{\ddot{x}}\right) = 0 \text{ , at } t = a \text{ and } t = b$$

$$y \geqq 0 \quad , \quad \lambda \in \Lambda^+$$

4.1.6 Nonlinear Programming

If the problems (P_1) and (D_1) are independent of t, then they will reduce the following multiobjective nonlinear programming problems: studied in [55]

(NP): Minimize $f(x)$

Subject to

$$g(x) \leq 0.$$

(ND): Maximize $f(x) + y^T g(x) e$

Subject to

$$\lambda^T f_x + y^T g_x = 0$$

$$y \geqq 0.$$

4.2 MULTIOBJECTIVE DUALITY IN VARIATIONAL PROBLEMS WITH HIGHER ORDER DERIVATIVES

4.2.1 Mond-Weir type Duality

We formulate the following Mond-Weir type dual to the problem (VP) and establish various duality results under invexity defined in the preceding section.

(M-WD): Maximize $\left(\int_I f^1(t,u,\dot{u},\ddot{u})\,dt, \ldots, \int_I f^p(t,u,\dot{u},\ddot{u})\,dt \right)$

Subject to

$$x(a) = 0 = x(b)\ , \tag{4.32}$$

$$\dot{x}(a) = 0 = \dot{x}(b), \tag{4.33}$$

$$\left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right) = 0\ ,\ t \in I \tag{4.34}$$

$$\int_1 y(t)^T g(t,u,\dot{u},\ddot{u})\,dt \geqq 0\ , \tag{4.35}$$

$$y(t) \geqq 0 \ , \ \ t \in I \ , \tag{4.36}$$

$$\lambda > 0 \, . \tag{4.37}$$

Theorem 4.6 (*Weak Duality*): Let $x \in X$ be feasible for (VP) and (u, λ, y) be feasible for (M-WD) if for all feasible (x, u, λ, y) $\int_I \lambda^T f(t, u, \dot{u}, \ddot{u}) dt$ is pseudoinvex and $\int_I y(t)^T g(t, u, \dot{u}, \ddot{u}) dt$ is quasi-invex with respect to the same η.

Then,

$$\int_I f(t, x, \dot{x}, \ddot{x}) dt \not\leq \int_I f(t, u, \dot{u}, \ddot{u}) dt \, .$$

Proof: The relations $g(t, x, \dot{x}, \ddot{x}) \leqq 0$, $y(t) \geqq 0$, $t \in I$ imply

$$\int_I y(t)^T g(t, x, \dot{x}, \ddot{x}) dt \leqq \int_I y(t)^T g(t, u, \dot{u}, \ddot{u}) dt$$

This because of the quasi-invexity of $\int_I y(t)^T g(t, u, \dot{u}, \ddot{u}) dt$**,** implies that

$$0 \geqq \int_I \left\{ \eta^T y^T(t) g_u + (D\eta)^T y^T g_{\dot{u}} + (D^2\eta)^T y^T g_{\ddot{u}} \right\} dt$$

$$= \int_I \eta^T y(t)^T g_u dt + \eta^T y(t)^T g_{\dot{u}} \Big|_{t=a}^{t=b}$$

$$- \int_I \eta^T D y(t)^T g_{\dot{u}} dt + (D\eta)^T y(t)^T g_{\ddot{u}} \Big|_{t=a}^{t=b} - \int_I (D\eta)^T D y(t)^T g_{\ddot{u}} dt$$

(By integration by parts)

Using the boundary conditions which gives $D\eta = 0 = \eta$ at $t = a, t = b$

$$= \int_I \eta^T y(t)^T g_u dt - \int_I \eta^T D y(t)^T g_{\dot{u}} dt - \eta^T D y(t)^T g_{\ddot{u}} \Big|_{t=a}^{t=b} + \int_I \eta^T D^2 y(t)^T g_{\ddot{u}} dt$$

(By integration by parts)

Using the boundary conditions which give $D\eta = 0 = \eta$ at $t = a, t = b$

$$\int_I \eta^T y(t)^T g_u dt - \int_I \eta^T D y(t)^T g_{\dot{u}} dt + \int_I \eta^T D^2 y(t)^T g_{\ddot{u}} dt \leqq 0$$

$$\int_I \eta^T \left(y(t)^T g_u - Dy(t)^T g_{\dot{u}} + D^2 y(t)^T g_{\ddot{u}} \right) dt \leqq 0$$

From equation (4.34) this yields,

$$\int_I \eta^T \left\{ \lambda^T f_x - D\lambda^T f_{\dot{x}} + D^2 \lambda^T f_{\ddot{x}} \right\} dt \geqq 0$$

This by integration by parts and then using boundary conditions gives,

$$\int_I \left\{ \eta^T \left(\lambda^T f_x \right) + (D\eta)^T \left(\lambda^T f_{\dot{x}} \right) + \left(D^2 \eta \right)^T \left(\lambda^T f_{\ddot{x}} \right) \right\} dt \geqq 0,$$

This, in view of pseudoinvexity of $\int_I \lambda^T f(t, x, \dot{x}, \ddot{x}) dt$ implies that

$$\lambda^T \int_I f(t, x, \dot{x}, \ddot{x}) dt \geqq \lambda^T \int_I f(t, u, \dot{u}, \ddot{u}) dt .$$

For this , it follows

$$\int_I f(t, x, \dot{x}, \ddot{x}) dt \not\leq \int_I f(t, u, \dot{u}, \ddot{u}) dt .$$

Theorem 4.7 (*Strong Duality*): If $\bar{x}$ be a feasible solution for (VP) and assume that

- **i.** $\bar{x}$ is an efficient solution.
- **ii.** for at least one i, $i \in P$, $\bar{x}$ satisfies a regularity condition in [30] for $\mathrm{P_k}(\bar{x})$.

Then there exists one $\bar{\lambda} \in R^p$, $\bar{y} \in R^m$ such that $(\bar{x}, \bar{y}, \bar{\lambda})$ is efficient for (VD). Further if the assumptions of Theorem 4.6 are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda})$ is an efficient solution of (VD).

Proof: Since $\bar{x}$ is efficient solution by Lemma 4.2, it is an optimal solution of $\mathrm{P_k}(\bar{x})$. By Proposition 4.2, this implies that there exists $\lambda^T = (\lambda^1, \ldots, \lambda^p)$ and piecewise smooth $y : I \to R^m$ such that,

$$\bar{\lambda}_k \left(f_x^k - Df_{\dot{x}}^k + D^2 f_{\ddot{x}}^k \right) + \sum_{i \neq k} \bar{\lambda}_i \left(f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i \right)$$

$$+ \left(y(t)^T g_x - Dy(t)^T g_{\dot{x}} + D^2 y(t)^T g_{\ddot{x}} \right) = 0 \ , \qquad (4.38)$$

$$\left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right) = 0 \ , \ t \in I, \tag{4.39}$$

$$\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt = 0 \quad , \ t \in I, \tag{4.40}$$

$$\left(\bar{\lambda}, \bar{y}(t)\right) \geqq 0 \quad , \ t \in I, \tag{4.41}$$

$$\left(\bar{\lambda}, \bar{y}(t)\right) \neq 0 \quad , \ t \in I \tag{4.42}$$

From (4.40), we have

$$\int_1 y(t)^T g(t, x, \dot{x}, \ddot{x}) dt = 0 \tag{4.43}$$

Equations (4.39), (4.40) and (4.41) imply that $\left(\bar{x}, \bar{\lambda}, \bar{y}\right)$ is feasible for (M-WD). The equality of objective functional of the primal and dual problems is obvious from their formulations. Efficiency of $\left(\bar{x}, \bar{\lambda}, \bar{y}\right)$ is immediate from the application of Theorem 4.6.

As in [105], by employing chain rule in calculus, it can be easily seen that the expression $\left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right)$, may be regarded as a function θ of variables $t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}$ and λ, where $\dddot{x} = D^3 x$ and $\ddot{y} = D^2 y$. That is, we can write

$$\theta(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda) = \left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right)$$

In order to prove converse duality between (VP) and (M-WD), the space X is now replaced by a smaller space X_2 of piecewise smooth thrice differentiable function $x : I \to R^n$ with the norm $\|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty + \|D^3 x\|_\infty$. The problem (M-WD) may now be briefly written as,

Minimize $\left(-\int_i f^1(t, x, \dot{x}, \ddot{x}) dt, \ldots, -\int_i f^p(t, x, \dot{x}, \ddot{x}) dt\right)$

Subject to

$$x(a) = 0 = x(b),$$

$$\dot{x}(a) = 0 = \dot{x}(b),$$

$$\theta(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda) = 0$$

$$\int_1 y(t)^T g(t, x, \dot{x}, \ddot{x})\, dt \geqq 0$$

$$y(t) \geqq 0 \ , \ \ t \in I$$

Consider $\theta(t, x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot), \dddot{x}(\cdot), y(\cdot), \dot{y}(\cdot), \ddot{y}(\cdot), \lambda) = 0$ as defining a mapping $\psi : X_2 \times Y \times R^p \to B$ where Y is a space of piecewise twice differentiable function and B is the Banach Space. In order to apply Theorem 4.1 to the problem (M-WD), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $\left[\psi_x(x, y, \lambda), \psi_y(x, y, \lambda), \psi_\lambda(x, y, \lambda)\right]$.

Theorem 4.8 (*Converse Duality*): Let $(\bar{x}, \bar{\lambda}, \bar{y})$ be an efficient solution with $x \in X_2$, $y \in Y_2$ and $\lambda^T \in R^p$ and ψ' have a (weak*)closed range hypothesis. Let f and g be twice continuously differentiable. Assume that

(H1) $\int_I \lambda^T f\, dt$ be pseudoinvex and $\int_I y(t)^T g\, dt$ be quasi-invex with respect to same η.

(H2) $\sigma(t)^T \left(\sigma(t)\theta_x - D\sigma(t)\theta_{\dot{x}} + D^2\sigma(t)\theta_{\ddot{x}} - D^3\sigma(t)\theta_{\dddot{x}}\right) = 0$

$\Rightarrow \sigma(t) = 0,\ t \in I$.

(H3) $f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i$, $i = 1, 2, \ldots, p$ are linearly independent.

Then $\bar{x}$ is an efficient solution of (VP).

Proof: Since $(\bar{x}, \bar{\lambda}, \bar{y})$ where $\bar{x} \in X$ and ψ' having a closed range, is an efficient solution of (M-WD), by Theorem 4.6, it implies that there exist $\alpha \in R,\ \gamma \in R, \eta \in R^p$ and piecewise smooth $\beta : R \to R^n$ and $\mu : R \to R^m$ satisfying the following conditions.

$$-\alpha\left(f_x - Df_{\dot{x}} + D^2 f_{\ddot{x}}\right) - \gamma\left(y(t)^T g_x - D\left(y(t)^T g_{\dot{x}}\right) + D^2\left(y(t)^T g_{\ddot{x}}\right)\right)$$
$$+\beta(t)^T \theta_x - D\left(\beta(t)^T \theta_{\dot{x}}\right) + D^2\left(\beta(t)^T \theta_{\ddot{x}}\right) - D^3\left(\beta(t)^T \theta_{\dddot{x}}\right) = 0 \tag{4.44}$$

$$\beta(t)^T \theta_y - D\left(\beta(t)^T \theta_{\dot{y}}\right) + D^2\left(\beta(t)^T \theta_{\ddot{y}}\right) - \gamma g - \mu(t) = 0 \tag{4.45}$$

$$\beta(t)^T\left(f_x - Df_{\dot{x}} + D^2 f_{\ddot{x}}\right) - \eta = 0 \tag{4.46}$$

$$\gamma\int_1 y(t)^T g(t,x,\dot{x},\ddot{x})dt = 0 \tag{4.47}$$

$$\eta^T\lambda = 0, \quad \mu(t)^T y(t) = 0, \quad t\in I \tag{4.48}$$

$$\left(\alpha,\gamma,\eta,\mu(t)\right) \geqq 0, \; t\in I \text{ and } \left(\alpha,\gamma,\eta,\mu(t),\beta(t)\right) \neq 0, \quad t\in I \tag{4.49}$$

Since $\lambda > 0$, $\eta^T\lambda = 0,$ which implies $\eta = 0$

This yields from (4.46)

$$\beta(t)^T\left(f_x - Df_{\dot{x}} + D^2 f_{\ddot{x}}\right) = 0 \tag{4.50}$$

Using the equality constraint (4.34) in (4.44), we have

$$-(\alpha - \gamma\lambda)^T\left(f_x - Df_{\dot{x}} + D^2 f_{\ddot{x}}\right) + \beta(t)^T \theta_x$$
$$-D\left(\beta(t)^T \theta_{\dot{x}}\right) + D^2\left(\beta(t)^T \theta_{\ddot{x}}\right) - D^3\left(\beta(t)^T \theta_{\dddot{x}}\right) = 0 \tag{4.51}$$

Postmultiplying equation (4.44) by $\beta(t)$ and using (4.50) in (4.51) we get,

$$\beta(t)^T\left(\beta(t)^T \theta_{\dot{x}}\right) + D^2\left(\beta(t)^T \theta_{\ddot{x}}\right) - D^3\left(\beta(t)^T \theta_{\dddot{x}}\right) = 0 \, , \, t\in I$$

This by hypothesis (H_2) implies $\beta(t) = 0, \; t\in I$

Also from (4.51) we have

$$(\alpha - \gamma\lambda)^T\left(f_x - Df_{\dot{x}} + D^2 f_{\ddot{x}}\right) = 0$$

This, because of linear independence of $f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i$, $i = 1, 2, \ldots, p$, gives

$$\alpha - \gamma\lambda = 0 \tag{4.52}$$

Now suppose $\gamma = 0$, then, from (4.45) and (4.52) we have

$\mu(t) = 0$, $t\in I$ and $\alpha = 0$ respectively.

This implies $(\alpha, \beta(t), \gamma, \eta, \mu(t)) = 0$, which is the contradiction to

$$(\alpha, \beta(t), \gamma, \eta, \mu(t)) \neq 0, \ t \in I .$$

Hence $\gamma > 0$ and by (4.52) we have, $\alpha > 0$.

The relation (4.45) in conjunction with $\beta(t) = 0$, and $\mu(t) \geq 0$, $t \in I$ gives

$$g(t, x, \dot{x}, \ddot{x}) \leqq 0 \quad , t \in I$$

This implies the feasibility of $\bar{x}$ for (VP) and its efficiency is evident from and application of Theorem 4.6.

4.2.2 Natural Boundary Values

The duality results obtained in the preceding sections can easily be extended to the following multiobjective variational problems with natural boundary values rather than fixed end points:

Primal (P$_1$): Minimize $\left(\int_I f^1(t, x, \dot{x}, \ddot{x}) dt, \ldots, \int_I f^p(t, x, \dot{x}, \ddot{x}) dt \right)$

Subject to

$$g(t, x, \dot{x}, \ddot{x}) \leqq 0 \quad , \quad t \in I$$

Dual (D$_1$): Maximize $\left(\int_I f^1(t, x, \dot{x}, \ddot{x}) dt, \ldots, \int_I f^p(t, x, \dot{x}, \ddot{x}) dt \right)$

Subject to

$$\left(\lambda^T f_x + y(t)^T g_x \right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}} \right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}} \right) = 0 \quad t \in I,$$

$$y(t)^T g_{\dot{x}} = 0 \text{ , at } t = a \text{ and } t = b,$$

$$y(t)^T g_{\ddot{x}} = 0 \text{ , at } t = a \text{ and } t = b,$$

$$y(t) \geqq 0 \text{ , } t \in I .$$

4.2.3 Nonlinear Programming

If the problems (P_1) and (D_1) are independent of t, then they will reduce to the following multiobjective nonlinear programming problems studied in [55]

(NP): Minimize $f(x)$

Subject to

$$g(x) \leqq 0..$$

(ND): Maximize $f(x)$

Subject to

$$\lambda^T f_x + y^T g_x = 0$$

$$\lambda > 0 \ \ , \ \ y \geqq 0 .$$

4.3 MIXED TYPE MULTIOBJECTIVE VARIATIONAL PROBLEMS WITH HIGHER ORDER DERIVATIVES

4.3.1 Mixed Type Multiobjective Duality

In the spirit of Xu [157], we formulate a mixed type dual for a wider class of variational problems involving higher order derivatives to unify the duality results of section 4.1 and 4.2, under invexity and generalized invexity conditions.

The following is the mixed type dual for multiobjective variational problem (VP):

(Mix VD): Maximize $$\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u})dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u}) \right) dt \right.$$
$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u})dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u}) \right) dt \right)$$

Subject to

$$x(a) = 0 = x(b) \tag{4.53}$$

$$\dot{x}(a) = 0 = \dot{x}(b) \tag{4.54}$$

$$\left(\lambda f_u + y(t)^T g_u\right) - D\left(\lambda f_{\dot{u}} + y(t)^T g_{\dot{u}}\right) + D^2\left(\lambda f_{\ddot{u}} + y(t)^T g_{\ddot{u}}\right) = 0 \text{ , } t \in I \tag{4.55}$$

$$\sum_{j \in I_\alpha} \int_I y^j(t)^T g^j(t,u,\dot{u},\ddot{u})\,dt \geqq 0, \ \alpha = 1,2,\ldots,r \tag{4.56}$$

$$y(t) \geqq 0 \text{ , } t \in I, \tag{4.57}$$

$$\lambda \in \Lambda^+ \tag{4.58}$$

where $I_\alpha \subseteq M = \{1,2,\ldots,m\}$, $\alpha = 1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$, if $\alpha \neq \beta$.

If $I_\circ = M$, then (Mix VD) becomes (WD) considered in the section 4.1. If $I_\circ = \phi$ for $I_\alpha = M$ (for some $\alpha \in \{1,2,\ldots,r\}$, then the (Mix VD) becomes the problem (M-WD) considered in the section 4.2.

Theorem 4.9(*Weak Duality*): Let $x \in X$ for feasible (VP) and (u,y,λ) be feasible for (Mix VD). If, for all feasible (x,u,y,λ) $\int_I \left(\lambda^T f(t,u,\dot{u},\ddot{u})\,dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u})\right) dt$ is pseudoinvex and $\sum_{j \in I_\circ} \int_I y^j(t) g^j(t,u,\dot{u},\ddot{u})\,dt$, $\alpha = 1,2,\ldots,r$ is quasi-invex with respect to same η, then,

$$\int_I f(t,x,\dot{x},\ddot{x})\,dt \not\leq \int_I \left(f(t,u,\dot{u},\ddot{u})\,dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u}) e \right) dt$$

Proof: The relations $g(t,x,\dot{x},\dot{x}) \leqq 0$ and $y(t) \geqq 0$, $t \in I$ imply

$$\sum_{j \in I_\circ} \int_I y^j(t) g^j(t,x,\dot{x},\ddot{x})\,dt \leqq \sum_{j \in I_\circ} \int_I y^j(t) g^j(t,u,\dot{u},\ddot{u})\,dt$$

This because of the quasi-invexity of $\sum_{j\in I_\circ}\int_I y^j(t)g^j(t,u,\dot{u},\ddot{u})dt$, $\alpha=1,2,\ldots,r$, implies,

$$0\geqq\sum_{j\in I_\circ}\int_I\Big\{\eta^T y^j(t)g^j(t,u,\dot{u},\ddot{u})dt+(D\eta)^T y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})$$
$$+\left(D^2\eta\right)^T y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\Big\}dt,\ \alpha=1,2,\ldots,r$$

This by integrating by parts gives,

$$0\geqq\sum_{j\in I_\alpha}\Bigg[\int_I\eta^T y^j(t)g^j(t,u,\dot{u},\ddot{u})dt-\eta^T y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})\Big|_{t=a}^{t=b}$$
$$-\int_I\eta^T D\left(y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})\right)dt+(D\eta)^T y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\Big|_{t=a}^{t=b}$$
$$-\int_I\eta^T D\left(y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)dt\Bigg]\qquad,\qquad\alpha=1,2,\ldots,r$$

Integrating by parts and using the boundary conditions which at $t=a$, $t=b$ give $D\eta=0=\eta$ and from this we have,

$$0\geqq\sum_{j\in I_\alpha}\Bigg[\int_I\eta^T y^j(t)g^j(t,u,\dot{u},\ddot{u})dt-\int_I\eta^T D\left(y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})\right)dt$$
$$-\eta^T D\left(y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)\Big|_{t=a}^{t=b}+\int_I\eta^T D^2\left(y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)dt\Bigg]$$

again using the boundary conditions which at $t=a$, $t=b$ give $D\eta=0=\eta$

$$0\geqq\sum_{j\in M\setminus I_\circ}\Bigg[\int_I\eta^T\Big\{y^j(t)g^j(t,u,\dot{u},\ddot{u})dt-D\left(y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})\right)$$
$$+D^2\left(y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)\Big\}dt\Bigg]\tag{4.59}$$

Using equations (4.55) and (4.59) we have

$$0\leqq\int_I\eta^T\Bigg\{\left(\lambda^T f_u(t,u,\dot{u},\ddot{u})\right)+\sum_{j\in I_\circ}y^j(t)g^j_u(t,u,\dot{u},\ddot{u})$$
$$-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u})+\sum_{j\in I_\circ}y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})\right)dt$$
$$+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u})+\sum_{j\in I_\circ}y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right)\Bigg\}dt$$

This, on integrating by parts, implies

$$0 \leqq \int_I \Bigg[\eta^T \left(\left(\lambda^T f_u \left(t,u,\dot{u},\ddot{u}\right)\right) + \sum_{j\in I_\circ} y^j\left(t\right) g_u^j\left(t,u,\dot{u},\ddot{u}\right) \right)$$

$$+ \left(D\eta\right)^T \left(\lambda^T f_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right) + \sum_{j\in I_\circ} y^j\left(t\right) g_{\dot{u}}^j\left(t,u,\dot{u},\ddot{u}\right)\right)$$

$$+ \left(D^2\eta\right)^T \left(\lambda^T f_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right) + \sum_{j\in I_\circ} y^j\left(t\right) g_{\ddot{u}}^j\left(t,u,\dot{u},\ddot{u}\right)\right)\Bigg] dt \geqq 0$$

This, because of pseudoinvexity of $\int_I \left\{ \lambda^T f\left(t,.,.,.\right) + \sum_{j\in I_\circ} y^j\left(t\right) g\left(t,.,.,.\right)\right\} dt$ with respect to η implies

$$\int_I \left\{ \lambda^T f\left(t,x,\dot{x},\ddot{x}\right) + \sum_{j\in I_\circ} y^j\left(t\right) g^j\left(t,x,\dot{x},\ddot{x}\right)\right\} dt$$

$$\geqq \int_I \left\{ \lambda^T f\left(t,u,\dot{u},\ddot{u}\right) + \sum_{j\in I_\circ} y^j\left(t\right) g^j\left(t,u,\dot{u},\ddot{u}\right)\right\} dt$$

Because $y\left(t\right) \geqq 0$, $t \in I$ and $g\left(t,x,\dot{x},\dot{x}\right) \leqq 0$, $t \in I$, it follows

$$\int_I \lambda^T f\left(t,x,\dot{x},\ddot{x}\right) dt \geqq \int_I \left\{ \lambda^T f\left(t,u,\dot{u},\ddot{u}\right) + \sum_{j\in I_\circ} y^j\left(t\right) g^j\left(t,u,\dot{u},\ddot{u}\right)\right\} dt$$

Since $\lambda > 0$ and $\lambda^T e = 1$, this yields,

$$\lambda^T \int_I f\left(t,x,\dot{x},\ddot{x}\right) dt \geqq \lambda^T \int_I \left\{ f\left(t,u,\dot{u},\ddot{u}\right) + \sum_{j\in I_\circ} y^j\left(t\right) g^j\left(t,u,\dot{u},\ddot{u}\right) e\right\} dt$$

This implies,

$$\int_I f\left(t,x,\dot{x},\ddot{x}\right) dt \not\leq \int_I \left\{ f\left(t,u,\dot{u},\ddot{u}\right) + \sum_{j\in I_\circ} y^j\left(t\right) g^j\left(t,u,\dot{u},\ddot{u}\right) e\right\} dt$$

Theorem 4.10 (*Strong Duality*): Let $\bar{x}$ be a feasible solution for (VP) and assume that

i. $\bar{x}$ is an efficient solution of (VP), and

ii. for at least one $i, i \in P$, $\bar{x}$ satisfies a regularity condition for [30] for $\mathrm{P}_i\left(\bar{x}\right)$, Then there exist $\bar{\lambda} \in R^p$, $\bar{y} \in R^m$ such that $(\bar{x}, \bar{y}, \bar{\lambda})$ is efficient for (Mix VD). Further if the assumptions of Theorem 4.10 are satisfied, then $(\bar{x}, \bar{y}, \bar{\lambda})$ is efficient for (Mix VD).

Proof: Since $\bar{x}$ is an optimal solution for (VP) and is normal, then by Proposition 4.1, there exists piecewise smooth $\bar{y}: I \to R^m$ such that

$$\begin{aligned}&\bar{\lambda} f_x(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) + \bar{y}(t)^T g_x(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) \\ &\quad - D\left(\bar{\lambda} f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})\right) \\ &\quad\quad + D^2\left(\bar{\lambda} f_{\ddot{x}}(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) + \bar{y}(t)^T g_{\ddot{x}}(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})\right) = 0 \, , \; t \in I \end{aligned} \tag{4.60}$$

$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0 \, , \; t \in I \tag{4.61}$$

$$\bar{y}(t) \geqq 0 \, , \; t \in I \tag{4.62}$$

$$\bar{\lambda} > 0 \tag{4.63}$$

The relation $\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0$, $t \in I$ implies

$$\sum_{j \in I_\circ} \bar{y}^j(t)^T g^j(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0 \, , \; t \in I \tag{4.64}$$

and

$$\sum_{j \in I_\alpha} \bar{y}^j(t)^T g^j(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0 \, , \; t \in I \; , \; \alpha = 1, 2, \ldots, r \, , \tag{4.65}$$

giving

$$\sum_{j \in I_\alpha} \int_I \bar{y}^j(t)^T g^j(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0 \, , \; t \in I \; , \; \alpha = 1, 2, \ldots, r$$

From (4.60), (4.62) and (4.63), it follows that $(\bar{x}, \bar{y}, \bar{\lambda})$ is feasible for (Mix VD). Also

$$\int_I \left\{ f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) + \sum_{j \in I_\circ} y^j(t) g^j(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) \right\} dt = \int_I f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) dt \; , \; i = 1, \ldots, p$$

That is, the objective values of (VP) and (Mix VD) are equal. The efficiency of $(\bar{x}, \bar{y}, \bar{\lambda})$ follows from Theorem 4.9.

As in [105], by employing the chain rule in calculus, it can be easily seen that the expression $\left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right)$, may be regarded as a function θ of variables $t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}$ and λ, where $\dddot{x} = D^3 x$ and $\ddot{y} = D^2 y$. That is, we can write $\theta(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda) = \left(\lambda^T f_x + y(t)^T g_x\right) - D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right)$

In order to prove converse duality between (VP) and (Mix VD), the space X is now replaced by a smaller space X_2 of piecewise smooth thrice differentiable function $x : I \to R^n$ with the norm $\|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty + \|D^3 x\|_\infty$. The problem (Mix VD) may now be briefly written as,

$$\text{Minimize} \quad -\left(\int_I \left(f^1(t,x,\dot{x},\ddot{x})dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,x,\dot{x},\ddot{x}) \right)dt ,\ldots, \int_I \left(f^p(t,x,\dot{x},\ddot{x})dt + \sum_{j \in I_\circ} y^j(t)^T g^j(t,x,\dot{x},\ddot{x}) \right)dt \right)$$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$\theta(t, x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y}) = 0 \ , \ t \in I$$

$$\sum_{j \in I_\alpha} \int_I y^j(t)^T g^j(t, u, \dot{u}, \ddot{u})\, dt \geqq 0, \ \ \alpha = 1, 2, \ldots, r$$

$$y(t) \geqq 0 \ , \ t \in I,$$

$$\lambda > 0$$

where $\theta(\cdot) = \lambda f_u + y(t)^T g_u - D\left(\lambda f_{\dot{u}} + y(t)^T g_{\dot{u}}\right) + D^2\left(\lambda f_{\ddot{u}} + y(t)^T g_{\ddot{u}}\right), t \in I$

Consider $\theta\left(t, x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot), \dddot{x}(\cdot), y(\cdot), \dot{y}(\cdot), \ddot{y}(\cdot), \lambda\right) = 0$ as defining a mapping $\psi : X_2 \times Y \times R^p \to B$ where Y is a space of piecewise twice differentiable functions and B is the Banach space. In order to apply Theorem 4.1 to the problem (Mix D), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $\left[\psi_x(x,y,\lambda),\psi_y(x,y,\lambda),\psi_\lambda(x,y,\lambda)\right]$.

Theorem 4.11 (*Converse Duality*): Let $(\bar{x},\bar{y},\bar{\lambda})$ be an efficient solution of (Mix VD).

Assume that

H₁: The Frechet derivative ψ' has a (weak *) closed range,

H₂: f and g are twice continuously differentiable,

H₃: $$f_x^i+\sum_{j\in I_\circ} y_j(t)^T g_x^j - D\left(f_{\dot{x}}^i+\sum_{j\in I_\circ} y_j(t)^T g_{\dot{x}}^j\right)+D^2\left(f_{\ddot{x}}^i+\sum_{j\in I_\circ} y_j(t)^T g_{\ddot{x}}^j\right),$$

$t\in I,\ i=1,\dots,p$ are linearly independent, and

H₄: $$\left(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}}\right)\beta(t)=0$$

$\Rightarrow \beta(t)=0,\ t\in I$.

Further, if the hypotheses of Theorem 4.10 are satisfied, then $\bar{x}$ is an efficient solution of (Mix VD).

Proof: Since $(\bar{x},\bar{y},\bar{\lambda})$ with ψ' has closed (weak*) range is an efficient solution, by (Mix VD), there exist Lagrange multipliers $\tau\in R^p$, piecewise smooth $\beta: I\to R^n, \gamma\in R$ for each of r constraints, $\eta\in R^p$ and piecewise smooth $\mu(t): I\to R^m$, satisfying the Fritz-John conditions

$$\left(\tau^T f_x+(\alpha^T e)\sum_{j\in I_\circ} y^j(t)^T g_x^j\right)-D\left(\tau^T f_{\dot{x}}+(\alpha^T e)\sum_{j\in I_\circ} y^j(t)^T g_{\dot{x}}^j\right)$$
$$+D^2\left(\tau^T f_{\ddot{x}}+(\alpha^T e)\sum_{j\in I_\circ} y^j(t)^T g_{\ddot{x}}^j\right)+\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}}$$
$$-\gamma\sum_{\alpha=1}^{r}\sum_{j\in I_\alpha}\left[y^j(t)^T g_x^j - Dy^j(t)^T g_{\dot{x}}^j + D^2 y^j(t)^T g_{\ddot{x}}^j\right]=0,\ t\in I \qquad (4.66)$$

$$-\left(\tau^T e\right)g^i+\beta(t)^T\theta_{y^j}-D\beta(t)^T\theta_{\dot{y}^j}+D^2\beta(t)^T\theta_{\ddot{y}^j}-\mu^j(t)=0\ ,\ j\in I_{\circ}\,,\ t\in I \tag{4.67}$$

$$\gamma g^i+\beta(t)^T\theta_{y^j}-D\beta(t)^T\theta_{\dot{y}^j}+D^2\beta(t)^T\theta_{\ddot{y}^j}-\mu^j(t)=0\ ,\ j\in I_{\alpha} \quad t\in I,\alpha=1,\ldots,r \tag{4.68}$$

$$\left[\left(f_x^i+\sum_{j\in I_\circ}y^j(t)^T g_x^j\right)-D\left(f_{\dot{x}}^i+\sum_{j\in I_\circ}y^j(t)^T g_{\dot{x}}^j\right)+D^2\left(f_{\ddot{x}}^i+\sum_{j\in I_\circ}y^j(t)^T g_{\ddot{x}}^j\right)\right]\beta(t)-\eta^i=0 \tag{4.69}$$

$$\mu(t)^T y(t)=0\ ,\ t\in I, \tag{4.70}$$

$$\eta^T\lambda=0 \tag{4.71}$$

$$\sum_{j\in I_\alpha}\int_I y^j(t)^T g^j(t,u,\dot{u},\ddot{u})\,dt=0,\ \ \alpha=1,2,\ldots,r \tag{4.72}$$

$$\left(\tau,\gamma,\mu(t),\eta\right)\geqq 0,\ \ t\in I\,, \tag{4.73}$$

$$\left(\tau,\beta(t),\gamma,\mu(t),\eta\right)\neq 0,\ t\in I\,. \tag{4.74}$$

Since $\lambda>0$, (4.71) implies $\eta=0$. Consequently (4.69) yields,

$$\left[\left(f_x^i+\sum_{j\in I_\circ}y^j(t)^T g_x^j\right)-D\left(f_{\dot{x}}^i+\sum_{j\in I_\circ}y^j(t)^T g_{\dot{x}}^j\right)\right.$$
$$\left.+D^2\left(\tau^T f_{\ddot{x}}+\sum_{j\in I_\circ}y^j(t)^T g_{\ddot{x}}^j\right)\right]\beta(t)=0,\ t\in I \tag{4.75}$$

Using the equality constraint of (Mix VD) in (4.66) we have,

$$-\sum_{i=1}^{p}\left(\tau^i-\gamma\lambda^i\right)\left[\left(f_x+\sum_{j\in I_\circ}y^j(t)^T g_x^j\right)-D\left(f_{\dot{x}}+\sum_{j\in I_\circ}y^j(t)^T g_{\dot{x}}^j\right)\right.$$
$$\left.+D^2\left(f_{\ddot{x}}+\sum_{j\in I_\circ}y^j(t)^T g_{\ddot{x}}^j\right)\right]\beta(t)$$
$$+\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}=0,\ t\in I \tag{4.76}$$

Postmultiplying (4.76) by $\beta(t)$, we have

$$\left(\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}\right)\beta(t)=0\,,\ t\in I$$

This, because of the hypothesis (H_4) gives

$$\beta(t)=0,\ t\in I \tag{4.77}$$

Using (4.77) in (4.76), we have

$$-\sum_{i=1}^{p}\left(\tau^i-\gamma\lambda^i\right)\left[\left(f_x+\sum_{j\in I_\circ}y^j(t)^T g_x^j\right)-D\left(f_{\dot{x}}+\sum_{j\in I_\circ}y^j(t)^T g_{\dot{x}}^j\right)\right.$$
$$\left.+D^2\left(f_{\ddot{x}}+\sum_{j\in I_\circ}y^j(t)^T g_{\ddot{x}}^j\right)\right]=0,\quad t\in I$$

This because of the linear independence stated in (H_3) gives

$$\left(\tau^i-\gamma\lambda^i\right)=0\ ,i=1,2,\ldots,p \tag{4.78}$$

If possible, let $\gamma=0$.Then (4.78) implies $\tau=0$. The equations (4.67) and (4.68) imply that $\mu(t)=0$, $t\in I$. Then $\left(\tau,\beta(t),\gamma,\mu(t),\eta\right)=0$, contradicting $\tau>0$.

From (4.67) and (4.68) it follows,

$$g^j=-\frac{\mu^j(t)}{\left(\tau^T e\right)},\ j\in I_\circ,\ \ t\in I$$

$$g^j=-\frac{\mu^j(t)}{\gamma},\ j\in I_\alpha\ ,\ \alpha=1,2,\ldots,r,\ \ t\in I$$

This implies $g\leqq 0$, also in view of (4.70) $y^T g=0$. From $y^T g=0$, it implies $\sum_{j\in J_\circ}y^j(t)g^j=0$, $t\in I$

$$\int_I\left(f^i(t,x,\dot{x},\ddot{x})+\sum_{j\in J_\circ}y^j(t)g^j(t,x,\dot{x},\ddot{x})\right)dt=\int_I f^i(t,x,\dot{x},\ddot{x})dt$$

This along with the application of Theorem 4.9 establishes the efficiency of $\bar{x}$ for (VP).

4.3.2 Related Nonlinear Problems

If f and g do not explicitly depend on t, the variational problem considered in the preceeding section reduces to the following static problems similar to those by Xu [157] and Zhang and Mond [160].

(NP): Minimize $f(x)$

Subject to

$$g(x) \leqq 0$$

(Mix ND): Maximize $\int_I \left(f^1(u) + \sum_{j \in J_\circ} y^j g^j(u), \ldots, f^p(u) + \sum_{j \in J_\circ} y^j g^j(u) \right)$

Subject to

$$\nabla\left(\lambda^T f(u) + y^T g(u)\right) = 0$$

$$\sum_{j \in J_\alpha} y^j g^j(u) \geqq 0 \ , \alpha = 1, \ldots, r$$

$$y \geqq 0$$

$$\lambda \in \Lambda^+$$

Chapter-5

OPTIMALITY AND DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE VARIATIONAL PROBLEMS WITH HIGHER ORDER DERIVATIVES

5.1 OPTIMALITY CONDITIONS AND DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE VARIATIONAL PROBLEMS INVOLVING HIGHER ORDER DERIVATIVES

5.1.1 Introductory Remarks

Chandra, Craven and Husain [30] obtained necessary optimality conditions for a constrained continuous programming having term with a square root of a quadratic form in the objective function, and using these optimality conditions formulated Wolfe type dual and established weak, Strong and Huard [92] type converse duality theorems under convexity of functions. Subsequently, for the problems of [30], Bector, Chandra and Husain [16] constructed a Mond-Weir type dual which allows weakening of convexity hypotheses of [30] and derived various duality results under generalized convexity of functionals.

This chapter is divided into two sections, 5.1 and 5.2. In the section 5.1, we study optimality and duality for a class of nondifferentiable variational problem containing higher order derivatives. The popularity of this type of problems seem to originate from the fact that, even though the objective function and or/ constraint functions are non-smooth, a simple representation of the dual problem may be found. The theory of non-smooth mathematical programming deals with much more general types of functions by means of generalized subdifferentials [41] and quasi differentials [49]. However, the square root of a positive semi-definite quadratic form is one of the few cases of a nondifferentiable function for which one can write down the sub or quasi differentials explicitly. We formulate Wolfe and Mond-Weir type dual problems for this class of variational problems and prove various duality results under invexity and generalized invexity. The result of section 5.1 also serves as correction to some of the results obtained by Kim and Kim [85].In the section 5.2 we study mixed type duality for the class of nondifferentiable multiobjective variational programming considered in the section 5.1. The subsection 5.2.2 considers the variational problem with natural boundary conditions instead of fixed point conditions. These formulations provide close relationship of the duality results of the section 5.2 to those of nondifferentiable programming.

5.1.2 Problem Formulation

We present the following nondifferentiable multiobjective variational problem with higher order derivatives as:

(VP): Minimize $\left(\int_I \left(f^1(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^1(t) x(t) \right)^{\frac{1}{2}} \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^p(t) x(t) \right)^{\frac{1}{2}} \right) dt \right)$$

Subject to

$$x(a) = 0 = x(b) \tag{5.1}$$

$$\dot{x}(a) = 0 = \dot{x}(b) \tag{5.2}$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \ ,\ t \in I, \tag{5.3}$$

where, $f^i : I \times R^n \times R^n \times R^n \to R, (i = 1,2,\ldots,p)$, $g : I \times R^n \times R^n \times R^n \to R^m$, are assumed to be continuously differentiable functions, for each $i \in P$, $\{i = 1,2,\ldots,p\}$, $B^i(t)$ is an $n \times n$ positive semidefinite symmetric matrix with $B^i(\cdot)$ continuous on I.

The following generalized Schwartz inequality [125] is required in the sequel.

$$\left(x(t)^T B^i(t) z(t)\right) \le \left(x(t)^T B^i(t) x(t)\right)^{\frac{1}{2}} \left(z(t)^T B^i(t) z(t)\right)^{\frac{1}{2}}$$
$$\forall\, x(t) \in R^n, z(t) \in R^n, t \in I$$

In order to prove the strong duality theorem, we will invoke the following lemma due to Changkong and Haimes [35].

Lemma 5.1 [39**]:** A function $\bar{x} \in X$ be an efficient solution of (VP) if and only if $\bar{x} \in X$ is an optimal solution of the following problem $\left(\mathrm{P}_k(\bar{x})\right)$ for all k.

$\left(\mathrm{P}_k(\bar{x})\right)$**:** Minimize $\left(\int_I \left(f^k(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^k(t) x(t)\right)^{\frac{1}{2}} \right) dt\right.$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$g(t,x,\dot{x},\dot{x}) \leqq 0,\ t \in I,$$

$$\int_I \left(f^i\left(t,x(t),\dot{x}(t),\ddot{x}(t)\right) dt + \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \right) dt$$
$$\le \int_I \left(f^i\left(t,\bar{x}(t),\dot{\bar{x}}(t),\ddot{\bar{x}}(t)\right) dt + \left(\bar{x}(t)^T B(t) \bar{x}(t)\right)^{\frac{1}{2}} \right) dt,\ i \ne k$$

5.1.3 Optimality

In this section, we give necessary optimality conditions for the problem $\left(\mathrm{P}_k(\bar{x})\right)$ which are required to establish strong duality theorem for Wolfe and Mond-weir type vector dual.

In order to derive optimality conditions for $\left(\mathrm{P}_k(\bar{x})\right)$**,** we require the following Lemma 5.2.

Lemma 5.2 [85]: Define a function $h: R^n \to R$ by $h(x(t)) = \left(\bar{x}(t)^T B(t)\bar{x}(t)\right)^{\frac{1}{2}}$, where B is a symmetric positive semidefinite $n \times n$ matrix and continuous on I, then h is convex, and

$$\partial h(x(t)) = \left\{B(t)z(t) : z(t)^T B(t)z(t) \leq 1\right\},$$

where $\partial h(x(t))$ is subgradient of h at $x(t)$.

Using the analysis in [91] and [49], the Fritz-John optimality conditions for $(\mathrm{P}_k(\bar{x}))$ can be given by the following theorem.

Theorem 5.1: (*Fritz-John Optimality Conditions*): If $\bar{x}$ is optimal solution of $(\mathrm{P}_k(\bar{x}))$ there exist scalars $\tau^1, \tau^2, \ldots, \tau^p$, piecewise smooth $z^i : I \to R$, $i \in P$, such that

$$\sum_{i=1}^{p} \bar{\tau}^i \left(f_x^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + D^2 f_{\ddot{x}}^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + B^i(t)\bar{z}^i(t)\right)$$

$$t + \sum_{j=1}^{m} \bar{y}^j(t)\left(g_x^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) - Dg_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + D^2 g_{\ddot{x}}^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\right) = 0 \ , \ t \in I$$

$$y(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0 \ , \ t \in I$$

$$\bar{x}(t)^T B^i(t)\bar{z}^i(t) = \left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}} \ , \quad i \in P \ , t \in I$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leq 1,$$

$$(\bar{\tau}, \bar{y}(t)) \geqq 0 \ , \ (\bar{\tau}, \bar{y}(t)) \neq 0 \ , t \in I$$

Proof: The proof of the theorem easily follows on the line of analysis in [73] and [30]. Hence it is omitted for brevity.

5.1.4 Wolfe Type Vector Duality

In this section, we present Wolfe type vector dual to (VP) and establish various duality results.

(MWD): Maximize $\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t) + y(t)^T g(t,u,\dot{u},\ddot{u}) \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + u(t)^T B^p(t) z^p(t) + y(t)^T (t,u,\dot{u},\ddot{u}) \right) dt \right)$$

Subject to

$$u(a) = 0 = u(b) \tag{5.4}$$

$$\dot{u}(a) = 0 = \dot{u}(b) \tag{5.5}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},\ddot{u}) dt + B^i(t) z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right)$$
$$- D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+ D^2 \left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) = 0 \ , \ t \in I \tag{5.6}$$

$$\bar{z}^i(t)^T B^i(t) \bar{z}^i(t) \leqq 1 \ , \ t \in I \ , \ i \in P \tag{5.7}$$

$$y(t) \geqq 0 \ , \ t \in I \tag{5.8}$$

$$\lambda > 0 \ , \ \lambda^T e = 1 \tag{5.9}$$

Theorem 5.2: (*Weak Duality*): Let $\bar{x}$ be feasible for (VP) and $\left(u, \lambda, z^1, \ldots, z^p, y\right)$ be feasible for (MWD). If for all feasible $\left(x, u, \lambda, z^1, \ldots, z^p, y\right)$, $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + (\cdot)^T B^i(t) z^i(t) + y(t)^T g(t,.,.,.) \right) dt$ is pseudoinvex with respect to η,

Then the following cannot hold:

$$\int_I \left(f^i(t,x,\dot{x},\ddot{x}) + x(t)^T B^i(t) z^i(t) \right) dt$$
$$\leqq \int_I \left(f^i(t,u,\dot{u},\ddot{u}) dt + u(t)^T B^i(t) z^i(t) + y(t)^T g(t,u,\dot{u},\ddot{u}) \right) dt, \tag{5.10}$$

For all $i \in P$, and

$$\int_I \left(f^j(t,x,\dot{x},\ddot{x}) dt + x(t)^T B^j(t) z^j(t) \right) dt$$
$$< \int_I \left(f^j(t,u,\dot{u},\ddot{u}) dt + \left(u(t)^T B^j(t) z^j(t) \right) + y(t)^T g^j(t,u,\dot{u},\ddot{u}) \right) dt \tag{5.11}$$

for some $j \in P$.

Proof: Suppose that (5.10) and (5.11) hold. Then, from (5.3) and (5.8), we have,

$$\int_I \left(f^i(t,x,\dot{x},\ddot{x}) + x(t)^T B^i(t) z^i(t) + y(t)^T g(t,x,\dot{x},\ddot{x}) \right) dt$$
$$\leqq \int_I \left(f^i(t,u,\dot{u},\ddot{u}) dt + u(t)^T B^i(t) z^i(t) + y(t)^T g^j(t,u,\dot{u},\ddot{u}) \right) dt,$$

For all $i \in P$, and

$$\int_I \left(f^j(t,x,\dot{x},\ddot{x}) dt + x(t)^T B^j(t) z^j(t) + y(t)^T g(t,x,\dot{x},\ddot{x}) \right) dt$$
$$< \int_I \left(f^j(t,u,\dot{u},\ddot{u}) dt + \left(u(t)^T B^j(t) z^j(t) \right) + y(t)^T g^j(t,u,\dot{u},\ddot{u}) \right) dt$$

for some $j \in P$.

Now using $\lambda > 0$ and $\sum_{i=1}^{p} \lambda^i = 1$, these inequalities yield,

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x},\ddot{x}) dt + \left(x(t)^T B^i(t) z^i(t) \right) + y(t)^T g(t,x,\dot{x},\ddot{x}) \right) dt$$
$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u},\ddot{u}) dt + \left(u(t)^T B^i(t) z^i(t) \right) + y(t)^T g(t,u,\dot{u},\ddot{u}) \right) dt$$

This, because of the pseudoinvexity of

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + (\cdot)^T B^i(t) z^i(t) + y(t)^T g(t,.,.,.) \right) dt \quad \text{implies}$$
$$\sum_{i=1}^{p} \lambda^i \int_I \eta^T \Big[\left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right)$$
$$(D\eta)^T \left(f_{\dot{u}}^i(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+ \left(D^2 \eta \right)^T \left(f_{\ddot{u}}^i(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt < 0$$

Integrating by parts, we get

$$0 > \sum_{i=1}^{p} \lambda^i \int_I \eta^T \Big[\left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right)$$
$$- D\left(f_{\dot{u}}^i(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt$$

$$-\sum_{i=1}^{p}\lambda^i\int_I (D\eta)^T\left(f_{\dot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\dot u}(t,u,\dot u,\ddot u)\right)dt$$

$$+\sum_{i=1}^{p}\lambda^i\eta^T\left(f_{\ddot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)\Big|_{t=a}^{t=b}$$

$$+\sum_{i=1}^{p}\lambda^i(D\eta)^T\left(f_{\ddot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)\Big|_{t=a}^{t=b}$$

Using the boundary conditions which at $t=a$, $t=b$ gives $D\eta=0=\eta$, we have

$$=\sum_{i=1}^{p}\lambda^i\int_I \eta^T\Big[\left(f_u^i(t,u,\dot u,\ddot u)+B^i(t)z^i(t)+y(t)^T g_u(t,u,\dot u,\ddot u)\right)$$
$$-D\left(f_{\dot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\dot u}(t,u,\dot u,\ddot u)\right)\Big]dt$$
$$-\sum_{i=1}^{p}\lambda^i\int_I (D\eta)^T\left(f_{\ddot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)dt$$

Again, integrating by parts we obtain

$$=\sum_{i=1}^{p}\lambda^i\int_I \eta^T\Big[\left(f_u^i(t,u,\dot u,\ddot u)+B^i(t)z^i(t)+y(t)^T g_u(t,u,\dot u,\ddot u)\right)$$
$$-D\left(f_{\dot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\dot u}(t,u,\dot u,\ddot u)\right)+D^2\left(f_{\ddot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)\Big]dt$$
$$+\sum_{i=1}^{p}\lambda^i\eta^T\left(f_{\ddot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)\Big|_{t=a}^{t=b}$$

Again using boundary conditions which at $t=a$, $t=b$ gives $D\eta=0=\eta$,

$$\int_I \eta^T\sum_{i=1}^{p}\lambda^i\Big[\left(f_u^i(t,u,\dot u,\ddot u)+B^i(t)z^i(t)+y(t)^T g_u(t,u,\dot u,\ddot u)\right)$$
$$-D\left(f_{\dot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\dot u}(t,u,\dot u,\ddot u)\right)$$
$$+D^2\left(f_{\ddot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)\Big]dt<0 \qquad (5.12)$$

From the equality constraint (5.6), we have

$$\int_I \eta^T\sum_{i=1}^{p}\lambda^i\Big[\left(f_u^i(t,u,\dot u,\ddot u)+B^i(t)z^i(t)+y(t)^T g_u(t,u,\dot u,\ddot u)\right)$$
$$-D\left(f_{\dot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\dot u}(t,u,\dot u,\ddot u)\right)$$
$$+D^2\left(f_{\ddot u}^i(t,u,\dot u,\ddot u)+y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u)\right)\Big]dt=0 \qquad (5.13)$$

The inequality (5.12) contradicts (5.13). Hence our assumption is invalid and the theorem follows.

Theorem 5.3 (*Strong Duality*): Let $\bar{x} \in X$ be an efficient solution of (VP) and for at least one $k \in P$, $\bar{x}$ satisfies the regularity condition [30] for the problem $\left(\mathrm{P}_k\left(\bar{x}\right)\right)$. Then there exist multipliers $\bar{\lambda} \in R^p$, piecewise smooth $\bar{y} \in R^m, z^i\left(t\right) \in R^n$, $i = \{1, 2, \ldots, p\}$ such that $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \lambda\right)$ is feasible for (MWD) and the objectives of (VP) and (MWD) are equal.

Further, if the hypothesis of Theorem 5.2 is met, then $\left(x, u, y, z^1, \ldots, z^p, \lambda\right)$ is an efficient solution of (MWD).

Proof: By Lemma 5.1 $\bar{x}$ is an optimal solution of $\left(\mathrm{P}_k\left(\bar{x}\right)\right)$.This implies that there exist $\bar{\xi} \in R^p$ with $\bar{\xi}^1, \ldots, \bar{\xi}^p, \bar{z}^i\left(t\right) \in R^n$, $i = \{1, 2, \ldots, p\}$ and piecewise smooth $\bar{v} \in R^m$ such that, the following optimality conditions [73] hold:

$$\begin{aligned}
&\bar{\xi}^k\left(f_x^k\left(t, x, \dot{x}, \ddot{x}\right) + B^k\left(t\right)\bar{z}^k\left(t\right) - Df_{\dot{x}}^k\left(t, x, \dot{x}, \ddot{x}\right) + D^2 f_{\ddot{x}}^k\left(t, x, \dot{x}, \ddot{x}\right)\right) \\
&\quad + \sum_{\substack{i=1 \\ i \neq k}}^{p} \bar{\xi}^i\left(f_x^i\left(t, x, \dot{x}, \ddot{x}\right) + B^i\left(t\right)\bar{z}^i\left(t\right) - Df_{\dot{x}}^i\left(t, x, \dot{x}, \ddot{x}\right) + D^2 f_{\ddot{x}}^i\left(t, x, \dot{x}, \ddot{x}\right)\right) \\
&\quad\quad + \bar{v}\left(t\right)^T g_x\left(t, x, \dot{x}, \ddot{x}\right) - D\left(\bar{v}\left(t\right)^T g_{\dot{x}}\left(t, x, \dot{x}, \ddot{x}\right)\right) + D^2\left(\bar{v}\left(t\right)^T g_{\ddot{x}}\left(t, x, \dot{x}, \ddot{x}\right)\right) = 0
\end{aligned} \tag{5.14}$$

$$\left(\bar{x}\left(t\right)^T B^i\left(t\right)\bar{x}\left(t\right)\right)^{\frac{1}{2}} = \left(\bar{x}\left(t\right)^T B^i\left(t\right)\bar{z}^i\left(t\right)\right), \; i = 1, \ldots, p \tag{5.15}$$

$$\bar{v}\left(t\right)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt = 0 \tag{5.16}$$

$$\left(\bar{z}\left(t\right)^T B^i\left(t\right)\bar{z}^i\left(t\right)\right) \leqq 1, \; t \in I, \; i = 1, 2, \ldots, p \tag{5.17}$$

$$\bar{\xi} > 0 \;, \quad \bar{v}\left(t\right) \geqq 0, \; t \in I \tag{5.18}$$

From (5.14) we obtain

$$\begin{aligned}
&\sum_{i=1}^{p} \bar{\xi}^i\left(f_x^i\left(t, x, \dot{x}, \ddot{x}\right) + B^i\left(t\right) z^i\left(t\right) - Df_{\dot{x}}^i\left(t, x, \dot{x}, \ddot{x}\right) + D^2 f_{\ddot{x}}^i\left(t, x, \dot{x}, \ddot{x}\right)\right) \\
&\quad + \bar{v}\left(t\right)^T g_x\left(t, x, \dot{x}, \ddot{x}\right) - D\left(\bar{v}\left(t\right)^T g_{\dot{x}}\left(t, x, \dot{x}, \ddot{x}\right)\right) + D^2\left(\bar{v}\left(t\right)^T g_{\ddot{x}}\left(t, x, \dot{x}, \ddot{x}\right)\right) = 0
\end{aligned} \tag{5.19}$$

Dividing (5.16), (5.18) and (5.19) by $\bar{\xi}^T e(\neq 0)$, and setting

$\lambda^i = \left(\frac{\bar{\xi}}{\bar{\xi}^T e}\right)$, $i = 1,\dots,p$ and $\bar{y}(t) = \left(\frac{\bar{v}(t)}{\bar{\xi}^T e}\right)$, we have,

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f_x^i(t,x,\dot{x},\ddot{x}) + B^i(t)\bar{z}^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x}) + D^2 f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x}) \right)$$
$$+\bar{y}(t)^T g_x(t,x,\dot{x},\ddot{x}) - D\left(\bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\right) + D^2\left(\bar{y}(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right) = 0 \tag{5.20}$$

$$\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt = 0 \tag{5.21}$$

$$\bar{\lambda} > 0 \ , \ \ \lambda^T e = 1 \tag{5.22}$$

$$\bar{y}(t) \geqq 0 \ , \ t \in I \tag{5.23}$$

Consequently (5.17), (5.20), (5.22) and (5.23) implies that $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\dots,\bar{z}^p,\bar{\lambda}\right)$is feasible for (WD). Because of (5.21), the two objectives of the problem (VP)and (MWD) are equal. Hence by Theorem 5.2 $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\dots,\bar{z}^p,\bar{\lambda}\right)$is efficient solution for (MWD). This completes the proof.

For validating converse duality theorem, we regard (MWD) in term of function x for convenience instead of the function u.

As in [105], by employing chain rule in calculus, it can be easily seen that the expression

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,x,\dot{x},\ddot{x})dt + B^i(t)z^i(t) + y(t)^T g_x(t,x,\dot{x},\ddot{x}) \right)$$
$$-D\left(\lambda^T f_{\dot{x}}(t,x,\dot{x},\ddot{x}) + y(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\right)$$
$$+D^2\left(\lambda^T f_{\ddot{x}}(t,x,\dot{x},\ddot{x}) + y(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right) = 0 \ , \ t \in I$$

may be regarded as a function θ of variables $t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}$ and λ, where $\dddot{x} = \frac{d^3}{dt^3}x = D^3 x$ and $\ddot{y} = D^2 y$. That is, we can write

$$\theta\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda\right)=\sum_{i=1}^{p} \lambda^{i}\left(f_{x}^{i}(t, x, \dot{x}, \ddot{x}) dt+B^{i}(t) z^{i}(t)+y(t)^{T} g_{x}(t, x, \dot{x}, \ddot{x})\right)$$
$$-D\left(\lambda^{T} f_{\dot{x}}(t, x, \dot{x}, \ddot{x})+y(t)^{T} g_{\dot{x}}(t, x, \dot{x}, \ddot{x})\right)$$
$$+D^{2}\left(\lambda^{T} f_{\ddot{x}}(t, x, \dot{x}, \ddot{x})+y(t)^{T} g_{\ddot{x}}(t, x, \dot{x}, \ddot{x})\right)=0,\ t \in I$$

The problem (MWD) may now be briefly written as,

Minimize $\left(\int_{I}-\left(f^{1}(t, x, \dot{x}, \ddot{x})+\left(u(t)^{T} B^{1}(t) z^{1}(t)\right)+y^{T}(t) g^{T}(t, x, \dot{x}, \ddot{x})\right) dt\right.$

Subject to

$$x(a)=0=x(b)$$

$$\dot{x}(a)=0=\dot{x}(b)$$

$$\theta\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda\right)=0$$

$$\bar{z}^{i}(t)^{T} B^{i}(t) \bar{z}^{i}(t) \leqq 1,\ t \in I,\ i \in P$$

$$y(t) \geqq 0,\ t \in I$$

$$\lambda>0,\ \lambda^{T} e=1$$

Consider $\theta\left(t, x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot), \dddot{x}(\cdot), y(\cdot), \dot{y}(\cdot), \ddot{y}(\cdot), \lambda\right)=0$ as defining a mapping $\psi: X \times Y \times R^{p} \rightarrow Q$ where Y is a space of piecewise twice differentiable function and Q is the Banach Space. In order to apply Theorem 4.1 to the problem (MWD), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $\left[\psi_{x}(x, y, \lambda), \psi_{y}(x, y, \lambda), \psi_{\lambda}(x, y, \lambda)\right]$.

Theorem 5.4 (*Converse Duality*): Let $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^{1}, \ldots, \bar{z}^{p}, \bar{\lambda}\right)$ be an efficient solution for (MWD) Assume that

(H_1) The Frèchèt derivative ψ' has a (weak*) closed range,

(H_2) f and g be twice continuously differentiable, and

(H_3) $\left(\beta(t)^{T} \theta_{x}-D \beta(t)^{T} \theta_{\dot{x}}+D^{2} \beta(t)^{T} \theta_{\ddot{x}}\right) \beta(t)=0, \Rightarrow \beta(t)=0,\ t \in I$

Further, if the assumptions of Theorem 5.2 are satisfied, then $\bar{x}$ is an efficient solution of (VP).

Proof: Since $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{\lambda}\right)$ with ψ' having a (weak*) closed range, is an efficient solution of (MWD), then there exist $\alpha\in R^p$, $\eta\in R^p$, $\gamma\in R,\delta\in R,\xi\in R^m$ and piecewise smooth $\beta(t):I\to R^n$ and $\mu(t):I\to R^m$ such that the following Fritz-John optimality conditions hold

$$\begin{aligned}-\sum_{i=1}^{p}\alpha^i\Big(f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)z^i(t)+y(t)^T g_x(t,x,\dot{x},\ddot{x})\Big)\\ +D\Big(\alpha^T f_{\dot{x}}(t,x,\dot{x},\ddot{x})+\left(\alpha^T e\right)y(t)^T g_{\dot{x}}(t,x,\dot{x},\ddot{x})\Big)\\ -D^2\Big(\alpha^T f_{\ddot{x}}(t,x,\dot{x},\ddot{x})+\left(\alpha^T e\right)y(t)^T g_{\ddot{x}}(t,x,\dot{x},\ddot{x})\Big)\\ +\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}=0,\ t\in I\end{aligned} \tag{5.24}$$

$$-\left(\alpha^T e\right)g^j(t,x,\dot{x},\ddot{x})+\beta(t)^T\theta_{y^j}-D\beta(t)^T\theta_{\dot{y}^j}+D^2\beta(t)^T\theta_{\ddot{y}^j}-\mu^j(t)=0,\ t\in I \tag{5.25}$$

$$j=1,2,\ldots,m$$

$$\begin{aligned}\Big[f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)z^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})\\ +D^2f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\Big]\beta(t)+\eta^i+\gamma=0,\qquad i=1,\ldots,p\end{aligned} \tag{5.26}$$

$$-\alpha^i x(t)^T B^i(t)+\beta(t)\lambda^i B^i(t)+\delta^i 2B^i(t)z^i(t)=0 \tag{5.27}$$

$$\eta^T\bar{\lambda}=0 \tag{5.28}$$

$$\mu(t)^T\bar{y}(t)=0,\ t\in I \tag{5.29}$$

$$\gamma\left(\sum_{i=1}^{p}\lambda^i-1\right)=0 \tag{5.30}$$

$$\delta^i\Big(z^i(t)^T B^i(t)z^i(t)-1\Big)=0,\ t\in I \tag{5.31}$$

$$\left(\alpha,\lambda,\mu(t),\eta,\gamma,\delta,\right)\geqq 0,\ t\in I \tag{5.32}$$

$$\left(\alpha,\beta(t),\lambda,\mu(t),\eta,\gamma,\delta,\right)\neq 0,\ t\in I \tag{5.33}$$

Since $\lambda>0$, (5.28) implies $\eta=0$. Consequently (5.26) implies

$$\left(f_x^i\left(t,x,\dot{x},\ddot{x}\right)+B^i\left(t\right)z^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)+D^2 f_{\ddot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)\right)\beta\left(t\right)$$

$$=-\gamma=0 \tag{5.34}$$

From the equality constraint of (MWD), we have

$$\left(\bar{y}\left(t\right)^T g_x\left(t,x,\dot{x},\ddot{x}\right)-D\bar{y}\left(t\right)^T g_{\dot{x}}\left(t,x,\dot{x},\ddot{x}\right)+D^2\bar{y}\left(t\right)^T g_{\ddot{x}}\left(t,x,\dot{x},\ddot{x}\right)\right)$$

$$=-\sum_{i=1}^{p}\lambda^i\left(f_x^i\left(t,x,\dot{x},\ddot{x}\right)+B^i\left(t\right)\bar{z}^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)+D^2 f_{\ddot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)\right)$$

This, in view of (5.34), implies

$$\beta\left(t\right)^T\left(\bar{y}\left(t\right)^T g_x\left(t,x,\dot{x},\ddot{x}\right)-D\bar{y}\left(t\right)^T g_{\dot{x}}\left(t,x,\dot{x},\ddot{x}\right)+D^2\bar{y}\left(t\right)^T g_{\ddot{x}}\left(t,x,\dot{x},\ddot{x}\right)\right)$$

$$=-\sum_{i=1}^{p}\lambda^i\beta\left(t\right)^T\left(f_x^i\left(t,x,\dot{x},\ddot{x}\right)+B^i\left(t\right)\bar{z}^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)+D^2 f_{\ddot{x}}^i\right)\left(t,x,\dot{x},\ddot{x}\right)$$

$$=-\sum_{i=1}^{p}\lambda^i\left(-\gamma\right)=\gamma \tag{5.35}$$

Postmultiplying (5.24) by $\beta\left(t\right)$ and then using (5.34) and (5.35), we obtain

$$\left(\beta\left(t\right)^T\theta_x-D\beta\left(t\right)^T\theta_{\dot{x}}+D^2\beta\left(t\right)^T\theta_{\ddot{x}}=0\right)\beta\left(t\right)=0\ ,\ t\in I$$

This, because of the hypothesis (H_3), gives

$$\beta\left(t\right)=0\ \ ,\ \ t\in I$$

Suppose $\alpha=0$, then from (5.25) we have $\mu^j\left(t\right)=0$, $j=1,2,\ldots,m$, and from (5.26) it follows that $\gamma=0$.

Also from (5.27) we have $\delta^i B^i\left(t\right)z^i\left(t\right)=0$ which together with (5.31) implies $\delta=0$. Thus, $\left(\alpha,\beta\left(t\right),\lambda,\mu\left(t\right),\eta,\gamma,\delta,\right)=0$, which is a contradiction to (5.33). Hence $\alpha>0$.

From the equation (5.25), we have

$$g^j\left(t,x,\dot{x},\ddot{x}\right)=-\frac{\mu^j\left(t\right)}{\left(\alpha^T e\right)}\leqq 0\ \ ,\ \ t\in I$$

which implies $g^j\left(t,x,\dot{x},\ddot{x}\right)\leqq 0\ ,\ t\in I$.

Therefore, $\bar{x}$ is feasible for (VP). Multiplying (5.26) by $y^j(t)$, and using (5.29), we have

$$y^j(t)g^j(t,x,\dot{x},\ddot{x})=0 \ , \ t\in I$$

By generalized Schwarz inequality [125]

$$\left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right)\leqq\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\left(\bar{z}^i(t)B^i(t)\bar{z}^i(t)\right)^{\frac{1}{2}} \tag{5.36}$$

Now let $\dfrac{2\delta^i}{\alpha^i}=\xi^i$. Then $\xi^i \geqq 0$ and from (5.27), we have

$$B^i(t)x(t)=\xi^i 2B^i(t)z^i(t),\ i=1,2,\ldots,p$$

This is the condition for the equality in (5.36). Therefore, we have

$$\left(\bar{x}(t)^T B^i(t)z^i(t)\right)=\left(x(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\left(z^i(t)B^i(t)z^i(t)\right)^{\frac{1}{2}}$$

From (5.31), either $\delta^i=0$ or $z^i(t)^T B^i(t)z^i(t)=1$ and hence $B^i(t)\bar{x}(t)=0$..
Therefore, in either case $\left(x(t)^T B^i(t)z^i(t)\right)=\left(x(t)^T B^i(t)z^i(t)\right)^{\frac{1}{2}}$, $i=1,2,\ldots,p$.
Hence,

$$\int_I\left(f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt+\left(x(t)^T B^i(t)z^i(t)\right)^{\frac{1}{2}}+y^j(t)g^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\right)dt$$
$$=\int_I\left(f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt+\left(x(t)^T B^i(t)z^i(t)\right)^{\frac{1}{2}}\right)dt\ ,\ i=1,2,\ldots,p$$

The efficiency of $\bar{x}$ for (VP) is an immediate consequence of the application of Theorem 5.2.

Remarks: Theorem 5.4 serves as a correction to Theorem 5 of Kim and Kim [85] as its hypothesis (iii) is not required to establish it.

5.1.5 Mond-Weir Type Vector Duality

In this section, we establish various duality theorems for the Mond-Weir type nondifferentiable vector dual variational problems.

(M-WVD): Maximize $\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t) \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + u(t)^T B^p(t) z^p(t) \right) dt \right)$$

Subject to

$$u(a) = 0 = u(b) \tag{5.37}$$

$$\dot{u}(a) = 0 = \dot{u}(b) \tag{5.38}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_x^i(t,u,\dot{u},\ddot{u}) dt + B^i(t) z^i(t) + y(t)^T g_x(t,u,\dot{u},\ddot{u}) \right)$$
$$-D\left(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + D^2\left(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}\right) = 0 \ , \ t \in I \tag{5.39}$$

$$\sum_{j=1}^{m} \int_I y^j(t) g^j(t,u,\dot{u},\ddot{u}) dt \geqq 0 \ , \ t \in I \tag{5.40}$$

$$\bar{z}^i(t)^T B^i(t) \bar{z}^i(t) \leqq 1 \ , \ t \in I \ , \ i \in P \tag{5.41}$$

$$\lambda > 0 \ , \quad y(t) \geqq 0 \ , \ t \in I \tag{5.42}$$

Theorem 5.5 (*Weak Duality*): Let $\bar{x}$ be feasible for (VP) and $(u,\lambda,z^1,\ldots,z^p,y)$ be feasible for (M-WVD). If for feasible $(x,u,\lambda,z^1,\ldots,z^p,y)$, $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + (\cdot)^T B^i(t) z^i(t) \right) dt$ is pseudoinvex and $\int_I y(t)^T g(t,.,.,.) dt$ is quasi-invex with respect to same η, the following cannot hold:

$$\int_I \left(f^i(t,x,\dot{x},\ddot{x}) dt + \left(x(t)^T B^i(t) x(t) \right)^{\frac{1}{2}} \right) dt$$
$$\leq \int_I \left(f^i(t,u,\dot{u},\ddot{u}) dt + \left(u(t)^T B^i(t) z^i(t) \right)^{\frac{1}{2}} \right) dt, \ \text{for all } i \in P, \tag{5.43}$$

and

$$\int_I \left(f^j(t,x,\dot{x},\ddot{x}) dt + \left(x(t)^T B^j(t) x(t) \right)^{\frac{1}{2}} \right) dt$$
$$< \int_I \left(f^j(t,u,\dot{u},\ddot{u}) dt + \left(u(t)^T B^j(t) z^j(t) \right)^{\frac{1}{2}} \right) dt, \quad \text{for some } j \in P \tag{5.44}$$

Proof: Suppose that (5.43) and (5.44) hold. Using $\lambda > 0$ and $\sum_{i=1}^{p} \lambda^i = 1$, then in view of Schwartz inequality [125], this gives

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x},\ddot{x})\,dt + \left(x(t)^T B^i(t) z^i(t) \right)^{\frac{1}{2}} \right) dt$$
$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u},\ddot{u})\,dt + \left(u(t)^T B^i(t) z^i(t) \right)^{\frac{1}{2}} \right) dt$$

In view of $\left(x(t)^T B^i(t) z(t) \right) \leq \left(x(t)^T B^i(t) x(t) \right)^{\frac{1}{2}} \left(z(t)^T B^i(t) z(t) \right)^{\frac{1}{2}}$ and $\left(z(t)^T B^i(t) z(t) \right) \leq 1$, from this inequality we obtain,

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,x,\dot{x},\ddot{x})\,dt + x(t)^T B^i(t) z^i(t) \right) dt$$
$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,u,\dot{u},\ddot{u})\,dt + u(t)^T B^i(t) z^i(t) \right) dt$$

By pseudo invexity of $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + (\cdot)^T B^i(t) z^i(t) \right) dt$ with respect to η, this implies,

$$0 > \sum_{i=1}^{p} \lambda^i \int_I \Big[\eta^T \left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t) \right)$$
$$+ (D\eta)^T \left(f_{\dot{u}}^i(t,u,\dot{u},\ddot{u}) \right) + \left(D^2\eta \right)^T \left(f_{\ddot{u}}^i(t,u,\dot{u},\ddot{u}) \right) \Big] dt$$

This, by integration by parts and using boundary conditions as earlier, yields

$$\sum_{i=1}^{p} \lambda^i \int_I \eta^T \Big[\left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t) \right)$$
$$- D f_{\dot{u}}^i(t,u,\dot{u},\ddot{u}) + D^2 f_{\ddot{u}}^i(t,u,\dot{u},\ddot{u}) \Big] dt < 0 \qquad (5.45)$$

Now, from the feasibility of (VP) and (M-WVD), we have

$$\int_I y(t)^T g(t,x,\dot{x},\ddot{x})\,dt \leqq \int_I y(t)^T g(t,u,\dot{u},\ddot{u})\,dt$$

which, because of the quasi-invexity of $\int_I y(t)^T g(t,.,.,.)\,dt$ with respect to η implies

$$\int_I \eta^T y(t)^T g_u(t,u,\dot{u},\ddot{u}) + (D\eta)^T y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u})$$
$$+\left(D^2\eta\right)^T y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u})\,dt \leqq 0$$

This, as earlier, implies

$$\int_I \eta^T \left[y(t)^T g_u(t,u,\dot{u},\ddot{u}) - Dy(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + D^2 y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right] dt \leqq 0 \tag{5.46}$$

Combining (5.45) and (5.46), we have the inequality as,

$$\int_I \eta^T \left[\sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},\ddot{u}) + B^i(t) z^i(t) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right) \right.$$
$$- D\left(f_{\dot{u}}^i(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$\left. + D^2\left(f_{\ddot{u}}^i(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \right] dt < 0$$

which contradicts the dual equality constraint. Hence the theorem is validated.

Theorem 5.6 (*Strong Duality*): Let $\bar{x}$ be an efficient solution of (VP) and for at least one $k \in P$, $\bar{x}$ satisfies the regularity condition [30] for the problem $\left(\mathrm{P}_k(\bar{x})\right)$. Then there exist multipliers $\bar{\lambda} \in R^p$, piecewise smooth $\bar{y} \in R^m$ and $\bar{z}^i(t) \in R^n$, $i = \{1,2,\ldots,p\}$ such that $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\lambda\right)$ is feasible for (M-WVD) and the objectives of (VP) and (M-WVD) are equal.

Further, if the generalized invexity of hypothesis of Theorem 5.5 is met, then $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{\lambda}\right)$ is an efficient solution of (M-WVD).

Proof: Since $\bar{x}$ is an solution of the problem $\left(\mathrm{P}_k(\bar{x})\right)$, by analysis of Theorem 5.1, it implies that there exists $\bar{\lambda} \in R^p$, piecewise smooth $\bar{y} \in R^m$ and $\bar{z}^i(t) \in R^n$, $i = \{1,2,\ldots,p\}$ such that, (5.20), (5.21), (5.22), (5.23) and (5.17) holds:

From (5.21), it implies,

$$\int_I \bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right) dt = 0 \tag{5.47}$$

Now from (5.20), (5.47), (5.17) and (5.23) together with $\bar{\lambda} > 0$, it follows that $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda}\right)$ is feasible. From the equality of the objectives of (VP) and (M-WVD), along with the hypotheses of Theorem 5.5, the efficiency of $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda}\right)$ follows. This completes the proof.

(M-WVD) may be rewritten in the following form:

$$\text{Minimize} \left(-\int_I \left(f^1(t, x, \dot{x}, \ddot{x}) + u(t)^T B^1(t) z^1(t) \right) dt \right.$$
$$\left. , \ldots, \int_I -\left(f^p(t, x, \dot{x}, \ddot{x}) + u(t)^T B^p(t) z^p(t) \right) dt \right)$$

Subject to

$$x(a) = 0 = x(b)$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$\theta\left(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda \right) = 0$$

$$\sum_{j=1}^{m} \int_I y^j(t) g^j(t, x, \dot{x}, \ddot{x}) dt \geqq 0 \ , \ t \in I$$

$$\bar{z}^i(t)^T B^i(t) \bar{z}^i(t) \leqq 1 \ , \ t \in I \ , \ i \in P$$

$$\lambda > 0 \ , \quad y(t) \geqq 0 \ , \ t \in I$$

Theorem 5.7 (*Converse Duality*): Let $\left(\bar{x}, \bar{u}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{\lambda}\right)$ be an efficient solution for (M-WDP). Assume that

(A_1) The Frèchèt derivative ψ' has a (weak*) closed range,

(A_2) f and g are twice continuously differentiable,

(A_3) $f_x^i(t, x, \dot{x}, \ddot{x}) + B^i(t) z^i(t) - D f_{\dot{x}}^i(t, x, \dot{x}, \ddot{x}) + D^2 f_{\ddot{x}}^i(t, x, \dot{x}, \ddot{x})$, $i \in \{1, 2, \ldots, p\}$ are linearly independent and

(A_4) $\left(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2 \beta(t)^T \theta_{\ddot{x}} \right) \beta(t) = 0, \Rightarrow \beta(t) = 0 \ , \ t \in I$

Further, if the hypotheses of Theorem 5.5 are met, then $\bar{x}$ is an efficient solution of (VP)

Proof: Since $\left(\bar{x},\bar{u},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{\lambda}\right)$ with ψ' having a (weak*) closed range, is an efficient solution of (M-WDP), then there exist $\alpha\in R^p,\eta\in R^p$, $\gamma\in R,\delta\in R,\xi\in R^m$ and piecewise smooth $\beta(t):I\to R^n$ and $\mu(t):I\to R^m$ such that the following Fritz-John optimality conditions holds

$$-\sum_{i=1}^{p}\alpha^i\left(f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)z^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})+D^2f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)$$
$$-\gamma\left(y(t)^Tg_x(t,x,\dot{x},\ddot{x})-Dy(t)^Tg_{\dot{x}}(t,x,\dot{x},\ddot{x})+D^2y(t)^Tg_{\ddot{x}}(t,x,\dot{x},\ddot{x})\right)$$
$$+\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}=0\ ,\ t\in I \tag{5.48}$$

$$-\gamma g^j(t,x,\dot{x},\ddot{x})+\beta(t)^T\theta_{y^j}-D\beta(t)^T\theta_{\dot{y}^j}+D^2\beta(t)^T\theta_{\ddot{y}^j}-\mu^j(t)=0\ ,\ t\in I \tag{5.49}$$

$$j=1,2,\ldots,m$$

$$-\alpha^ix(t)^TB^i(t)+\lambda^i\beta(t)^TB^i(t)+2\delta^iB^i(t)z^i(t)=0 \tag{5.50}$$

$$\left(f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)z^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})\right.$$
$$\left.+D^2f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\beta(t)\right]-\eta^i=0\ ,\quad i=1,\ldots,p \tag{5.51}$$

$$\gamma\int_I y(t)^Tg\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt=0 \tag{5.52}$$

$$\eta^T\lambda=0 \tag{5.53}$$

$$\mu^T(t)\bar{y}(t)=0\ ,\ t\in I \tag{5.54}$$

$$\delta^i\left(z^i(t)^TB^i(t)z^i(t)-1\right)=0\ ,\ t\in I \tag{5.55}$$

$$\left(\alpha,\mu(t),\delta,\eta,\gamma\right)\geqq 0 \tag{5.56}$$

$$\left(\alpha,\beta(t),\mu(t),\delta,\eta,\gamma\right)\geqq 0 \tag{5.57}$$

Since $\lambda>0$, (5.53) implies $\eta=0$. Consequently (5.51) implies

$$\left(f_x^i(t,x,\dot{x},\ddot{x})+B^i(t)z^i(t)-Df_{\dot{x}}^i(t,x,\dot{x},\ddot{x})\right.$$
$$\left.+D^2f_{\ddot{x}}^i(t,x,\dot{x},\ddot{x})\right)\beta(t)=0\ \ i=1,\ldots,p \tag{5.58}$$

Using the duality constraint of (M-WVD) in (5.48), we have

$$-\sum_{i=1}^{p}\left(\alpha^i-\gamma\lambda^i\right)\Big(f_x^i\left(t,x,\dot{x},\ddot{x}\right)+B^i\left(t\right)z^i\left(t\right)$$
$$-Df_{\dot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)+D^2 f_{\ddot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)\Big)$$
$$+\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}=0\ ,\ t\in I \tag{5.59}$$

$$=-\sum_{i=1}^{p}\left(\alpha^i-\gamma\lambda^i\right)\Big(f_x^i\left(t,x,\dot{x},\ddot{x}\right)+B^i\left(t\right)z^i\left(t\right)$$
$$-Df_{\dot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)+D^2 f_{\ddot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)\Big)\beta(t)$$
$$+\left(\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}\right)\beta(t)=0\ ,\ t\in I$$

This in conjunction with (5.58) yields

$$\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}=0\ ,\ t\in I$$

which because of the hypothesis (A_4) implies

$$\beta(t)=0\ ,\ t\in I \tag{5.60}$$

Using (5.60) in (5.59), we have

$$-\sum_{i=1}^{p}\left(\alpha^i-\gamma\lambda^i\right)\Big(f_x^i\left(t,x,\dot{x},\ddot{x}\right)+B^i\left(t\right)z^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)+D^2 f_{\ddot{x}}^i\left(t,x,\dot{x},\ddot{x}\right)\Big)=0$$

This, due to the hypothesis (A_3) gives,

$$\alpha^i-\gamma\lambda^i=0\ ,\ i=1,2,\ldots,p \tag{5.61}$$

Suppose $\gamma=0$, then from (5.61) we have $\alpha=0$. The relation (5.49) gives, $\mu(t)=0\ , t\in I$.

As earlier, (5.4) implies $\delta=0$. Hence we get $\left(\alpha,\beta(t),\mu(t),\eta,\gamma,\delta\right)=0$, which contradicts (5.56). Hence $\gamma>0$. Consequently , (5.61) implies $\alpha>0$.

From (5.50) we have,

$$g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\leqq 0$$

This implies the feasibility of $\bar{x}$ for (VP).

In view of the explanations given in the proof of Theorem 5.3, (5.50) together with (5.55) readily yields,

$$\left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right)=\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}} \ , \ i=1,2,\ldots,p$$

Hence,

$$\int_I\left(f^i\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt+\left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right)\right)dt$$

$$=\int_I\left(f^i\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt+\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\right)dt \ , \ i=1,2,\ldots,p$$

This, in view of the hypothesis of Theorem 5.5, implies that $\bar{x}$ is efficient solution of (VP).

5.1.6 Related Problems

It is possible to extend the duality theorems established in the previous two sections to the corresponding variational problems with natural boundary values rather than fixed end points.

(VP)$_0$: Minimize $$\left(\int_I\left(f^1\left(t,x,\dot{x},\ddot{x}\right)dt+\left(x(t)^T B^1(t)x(t)\right)^{\frac{1}{2}}\right)dt\right.$$
$$\left.,\ldots,\int_I\left(f^p\left(t,x,\dot{x},\ddot{x}\right)dt+\left(x(t)^T B^p(t)x(t)\right)^{\frac{1}{2}}\right)dt\right)$$

Subject to

$$g^j\left(t,x,\dot{x},\dot{x}\right)\leqq 0 \ , \ t\in I, \quad j=1,\ldots,m$$

(MWD)$_0$: Maximize $$\left(\int_I\left(f^1\left(t,u,\dot{u},\ddot{u}\right)+u(t)^T B^1(t)z^1(t)+y^j(t)g^j\left(t,u,\dot{u},\ddot{u}\right)\right)dt\right.$$
$$\left.,\ldots,\int_I\left(f^p\left(t,u,\dot{u},\ddot{u}\right)+u(t)^T B^p(t)z^p(t)+y^j(t)g^j\left(t,u,\dot{u},\ddot{u}\right)\right)dt\right)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},\ddot{u}\right)+B^i(t)z^i(t)+y(t)^T g_u\left(t,u,\dot{u},\ddot{u}\right)\right)$$
$$-D\left(\lambda^T f_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y(t)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)$$
$$+D^2\left(\lambda^T f_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y(t)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)=0 \ , \ t\in I$$

$$\lambda^T f_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)=0\text{, at } t=a\ ,\ t=b,$$

$$\lambda^T f_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)=0\text{, at } t=a\ ,\ t=b,$$

$$\bar{z}^i\left(t\right)^T B^i\left(t\right)\bar{z}^i\left(t\right)\leqq 1\ ,\ t\in I\ ,\ i\in P$$

$$y\left(t\right)\geqq 0\ ,\ t\in I$$

$$\lambda>0\ ,\ \lambda^T e=1$$

(M-WVD)$_0$: Maximize $\left(\int_I\left(f^1\left(t,u,\dot{u},\ddot{u}\right)+u\left(t\right)^T B^1\left(t\right)z^1\left(t\right)\right)dt\right.$

$$\left.,\ldots,\int_I\left(f^p\left(t,u,\dot{u},\ddot{u}\right)+u\left(t\right)^T B^p\left(t\right)z^p\left(t\right)\right)dt\right)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},\ddot{u}\right)+B^i\left(t\right)z^i\left(t\right)+y\left(t\right)^T g_u\left(t,u,\dot{u},\ddot{u}\right)\right)$$
$$-D\left(\lambda^T f_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)$$
$$+D^2\left(\lambda^T f_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)=0\ ,\ t\in I$$

$$\lambda^T f_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)=0=y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\text{, at } t=a\ ,\ t=b,$$

$$\lambda^T f_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)=0=y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\text{, at } t=a\ ,\ t=b,$$

$$\sum_{j=1}^{m}\int_I y^j\left(t\right)g^j\left(t,u,\dot{u},\ddot{u}\right)dt\geqq 0\ ,\ t\in I$$

$$\bar{z}^i\left(t\right)^T B^i\left(t\right)\bar{z}^i\left(t\right)\leqq 1\ ,\ t\in I\ ,\ i\in P$$

$$\lambda>0\ ,\quad y\left(t\right)\geqq 0\ ,\ t\in I$$

If the function in the problem (WD) and (M-WD) are independent of t, then these problems reduce to those treated by Mond, Husain and Prasad [109].

(VP)$_1$: Minimize $\left(f^1\left(x\right)+\left(x^T B^1 x\right)^{\frac{1}{2}},\ldots,f^p\left(x\right)+\left(x^T B^p x\right)^{\frac{1}{2}}\right)$

Subject to

$$g\left(x\right)\leqq 0$$

(MWD)$_1$: Maximize $\left(f^1(u)+u^T B^1 z^1 + y^T g(u),\ldots, f^p(u)+u^T B^p z^p + y^T g(u)\right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_x(u)+B^i z^i \right)+ y^T g_x(u) = 0$$

$$\bar{z}^i B^i \bar{z}^i \leqq 1, \quad i \in P$$

$$y \geqq 0 \ , \ \lambda \in \Lambda^+$$

where $\Lambda^+ = \left\{ \lambda \in R^p \,\middle|\, \lambda > 0,\ \lambda^T e = 1, e = (1,1,\ldots,1)^T \in R^p \right\}$

(M-WVD)$_1$: Maximize $\left(f^1(u)+u^T B^1 z^1,\ldots, f^p(u)+u^T B^p z^p \right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_x(u)+B^i z^i \right)+ y^T g_x(u) = 0$$

$$y^T g(u) \geqq 0 \ , \ t \in I$$

$$\bar{z}^i B^i \bar{z}^i \leqq 1, \quad i \in P,$$

$$\lambda > 0 \ , \quad y(t) \geqq 0 \ , t \in I \quad .$$

5.2 MIXED TYPE DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE VARIATIONAL PROBLEMS

5.2.1 Mixed Type Multiobjective Duality

In this section, we present the following mixed type dual formulation of nondifferentiable multiobjective nonlinear programming (Mix D) which combines Wolfe and Mond-Weir dual models studied in the preceding section in the spirit of Husain and Jabeen [71] and Xu [157]

(Mix D): Maximize $\left(\int_I \left(f^1(t,u,\dot{u},\ddot{u}) + u(t)^T B^1(t) z^1(t) + \sum_{j \in J_\circ} y^j(t) g^j(t,u,\dot{u},\ddot{u}) \right) dt \right.$

$$\left. ,\ldots, \int_I \left(f^p(t,u,\dot{u},\ddot{u}) + \left(u(t)^T B^p(t) z^p(t) \right) + \sum_{j \in J_\circ} y^j(t) g^j(t,u,\dot{u},\ddot{u}) \right) dt \right)$$

Subject to

$$u(a)=0=u(b), \tag{5.62}$$

$$\dot{u}(a)=0=\dot{u}(b), \tag{5.63}$$

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i(t,u,\dot{u},\ddot{u})dt+B^i(t)z^i(t)+y(t)^T g_u(t,u,\dot{u},\ddot{u})\right)$$
$$-D\left(\lambda^T f_{\dot{u}}+y(t)^T g_{\dot{u}}\right)+D^2\left(\lambda^T f_{\ddot{u}}+y(t)^T g_{\ddot{u}}\right)=0\ ,\ t\in I\ , \tag{5.64}$$

$$\sum_{j\in J_\alpha}\int_I y^j(t)g^j(t,u,\dot{u},\ddot{u})dt \geqq 0\ ,\ \alpha=1,2,\ldots,r, \tag{5.65}$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t)\leqq 1\ ,\ t\in I\ ,\ i\in P\ , \tag{5.66}$$

$$y(t)\geqq 0\ ,\ t\in I\ , \tag{5.67}$$

$$\lambda\in\Lambda^+. \tag{5.68}$$

where

(i) $\Lambda^+=\left\{\lambda\in R^p \,\middle|\, \lambda>0,\ \lambda^T e=1, e=(1,1,\ldots,1)^T\in R^p\right\}$

(ii) $J_\alpha\subseteq M=\{1,2,\ldots,m\},\ \alpha=0,1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r}J_\alpha=M$ and $J_\alpha\bigcap J_\beta=\phi$, if $\alpha\neq\beta$.

If $J_\circ=M$, then (Mix D) becomes Wolfe type dual considered in the subsection 5.1.4, if $J_\circ=\phi$ and $J_\alpha=M$ for some $\alpha\in\{1,2,\ldots,r\}$, then (Mix D) becomes Mond-Weir type dual considered in subsection 5.1.5.

Theorem 5.8 (*Weak Duality*): Let $\bar{x}$ be feasible for (VP) and $(u,y,z^1,\ldots,z^p,\lambda)$ be feasible for (Mix D). If for feasible $(x,u,z^1,\ldots,z^p,\lambda)$, $\sum_{i=1}^{p}\lambda^i\int_I\left(f^i(t,.,.,.)+\sum_{j\in J_\circ}y^j(t)g^j(t,.,.,.)+(\cdot)^T B^i(t)z^i(t)\right)dt$ is pseudoinvex and $\sum_{j\in J_\alpha}\int_I y^j(t)g^j(t,.,.,.)dt\ ,\ \alpha=1,2,\ldots,r$ is quasi-invex with respect to same η, then the following cannot hold:

$$\int_I\left(f^i(t,x,\dot{x},\ddot{x})dt+\left(x(t)^T B^i(t)x(t)\right)^{\frac{1}{2}}\right)dt$$
$$\leqq\int_I\left(f^i(t,u,\dot{u},\ddot{u})dt+\left(u(t)^T B^i(t)z^i(t)\right)+\sum_{j\in J_\circ}y^j(t)g^j(t,u,\dot{u},\ddot{u})\right)dt \tag{5.69}$$

For all $i \in P$, and

$$\int_I \left(f^k\left(t,x,\dot{x},\ddot{x}\right)dt + \left(x(t)^T B^k(t)\,x(t)\right)^{\frac{1}{2}} \right)dt$$
$$\leqq \int_I \left(f^k\left(t,u,\dot{u},\ddot{u}\right)dt + \left(u(t)^T B^k(t)\,z^k(t)\right) + \sum_{j\in J_\circ} y^j(t) g^j\left(t,u,\dot{u},\ddot{u}\right) \right)dt \tag{5.70}$$

for some k.

Proof: Suppose contrary to the result, that (5.69) and (5.70) hold. In view of $y(t) \geqq 0$, $t \in I$, $\bar{z}^i(t)^T B^i(t)\bar{z}^i(t) \leqq 1$, $t \in I$, $i \in P$ and $g\left(t,x,\dot{x},\dot{x}\right) \leqq 0$, $t \in I$,, these inequalities yield,

$$\int_I \left(f^i\left(t,x,\dot{x},\ddot{x}\right)dt + \left(x(t)^T B^i(t)\,z^i(t)\right) + \sum_{j\in J_\circ} y^j(t) g^j\left(t,u,\dot{u},\ddot{u}\right) \right)dt$$
$$\leqq \int_I \left(f^i\left(t,u,\dot{u},\ddot{u}\right)dt + \left(u(t)^T B^i(t)\,z^i(t)\right) + \sum_{j\in J_\circ} y^j(t) g^j\left(t,u,\dot{u},\ddot{u}\right) \right)dt$$

for all $i \in P$, and

$$\int_I \left(f^k\left(t,x,\dot{x},\ddot{x}\right)dt + \left(x(t)^T B^k(t)\,z^k(t)\right) + \sum_{j\in J_\circ} y^j(t) g^j\left(t,u,\dot{u},\ddot{u}\right) \right)dt$$
$$\leqq \int_I \left(f^k\left(t,u,\dot{u},\ddot{u}\right)dt + \left(u(t)^T B^k(t)\,z^k(t)\right) + \sum_{j\in J_\circ} y^j(t) g^j\left(t,u,\dot{u},\ddot{u}\right) \right)dt$$

for some k.

Now using $\lambda > 0$ and $\lambda^T e = 1$, these inequalities yield

$$\sum_{i=1}^{p} \lambda^i \int_I \left(f^i\left(t,x,\dot{x},\ddot{x}\right)dt + \left(x(t)^T B^i(t)\,z^i(t)\right) + \sum_{j\in J_\circ} y^j(t) g^j\left(t,x,\dot{x},\ddot{x}\right) \right)dt$$
$$< \sum_{i=1}^{p} \lambda^i \int_I \left(f^i\left(t,u,\dot{u},\ddot{u}\right)dt + \left(u(t)^T B^i(t)\,z^i(t)\right) + \sum_{j\in J_\circ} y^j(t) g^j\left(t,x,\dot{x},\ddot{x}\right) \right)dt$$

By pseudo invexity of $\sum_{i=1}^{p} \lambda^i \int_I \left(f^i(t,.,.,.) + \sum_{j\in J_\circ} y^j(t) g^j(t,.,.,.) + (\cdot)^T B^i(t)\,z^i(t) \right)dt$ with respect to η, we have,

$$0 > \sum_{i=1}^{p} \lambda^i \int_I \eta^T \left[\left(f_x^i + B^i(t) z^i(t) + \sum_{j \in J_\circ} y^j(t) g_x^j \right) \right.$$
$$\left. + (D\eta)^T \left(f_{\dot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\dot{x}}^j \right) + (D^2\eta)^T \left(f_{\ddot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\ddot{x}}^j \right) \right] dt$$

This, by integration by parts, gives,

$$= \sum_{i=1}^{p} \lambda^i \int_I \eta^T \left[\left\{ \left(f_x^i + B^i(t) z^i(t) + \sum_{j \in J_\circ} y^j(t) g_x^j \right) - D\left(f_{\dot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\dot{x}}^j \right) \right\} dt \right.$$
$$+ \eta^T \left(f_{\dot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\dot{x}}^j \right) \Bigg|_{t=a}^{t=b} + (D\eta)^T \left(f_{\ddot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\ddot{x}}^j \right) \Bigg|_{t=a}^{t=b}$$
$$\left. + \int_I (D\eta)^T D\left(f_{\ddot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\ddot{x}}^j \right) dt \right]$$

Using the boundary conditions which at $t = a, t = b$ gives $D\eta = 0 = \eta$

$$= \sum_{i=1}^{p} \lambda^i \int_I \left[\eta^T \left\{ \left(f_x^i + B^i(t) z^i(t) + \sum_{j \in J_\circ} y^j(t) g_x^j \right) - D\left(f_{\dot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\dot{x}}^j \right) \right\} dt \right.$$
$$\left. + \int_I (D\eta)^T D\left(f_{\ddot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\ddot{x}}^j \right) \right] dt$$

$$= \sum_{i=1}^{p} \lambda^i \int_I \left[\eta^T \left\{ \left(f_x^i + B^i(t) z^i(t) + \sum_{j \in J_\circ} y^j(t) g_x^j \right) - D\left(f_{\dot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\dot{x}}^j \right) \right. \right.$$
$$\left. \left. + D^2\left(f_{\ddot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\ddot{x}}^j \right) \right\} \right] dt - \eta^T \left(f_{\ddot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\ddot{x}}^j \right) \Bigg|_{t=a}^{t=b}$$

$$0 > \sum_{i=1}^{p} \lambda^i \int_I \left[\eta^T \left\{ \left(f_x^i + B^i(t) z^i(t) + \sum_{j \in J_\circ} y^j(t) g_x^j \right) - D\left(f_{\dot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\dot{x}}^j \right) \right. \right.$$
$$\left. \left. + D^2\left(f_{\ddot{x}}^i + \sum_{j \in J_\circ} y^j(t) g_{\ddot{x}}^j \right) \right\} \right] dt \qquad (5.71)$$

Now, from the feasibility of (VP) and (Mix-D), we have

$$\sum_{j \in J_\alpha} \int_I y^j(t) g^j(t, x, \dot{x}, \ddot{x}) dt \leqq \sum_{j \in J_\alpha} \int_I y^j(t) g^j(t, u, \dot{u}, \ddot{u}) dt$$

This, because of quasi-invexity of $\sum_{j \in J_\alpha} \int_I y^j(t) g^j(t, ., ., .) dt$ with η implies,

$$\sum_{j\in J_\alpha}\int_I \left\{\eta^T\left(y^j(t)g_x^j\right)+(D\eta)^T y^j(t)g_{\dot{x}}^j+\left(D^2\eta\right)^T y^j(t)g_{\ddot{x}}^j\right\}dt \leqq 0$$

As earlier, integrating by parts and using the boundary conditions, we have,

$$\sum_{j\in J_\alpha}\int_I \eta^T\left\{\left(y^j(t)g_x^j\right)+D^T y^j(t)g_{\dot{x}}^j+D^2 y^j(t)g_{\ddot{x}}^j\right\}dt \leqq 0 \tag{5.72}$$

Combining (5.71) and 5.72), we have,

$$\int_I \eta^T\left(\sum_{i=1}^p \lambda^i\left(f_x^i+B^i(t)z^i(t)+y(t)^T g_x\right)\right.$$
$$\left.-D\left(\lambda^T f_{\dot{x}}+y(t)^T g_{\dot{x}}\right)+D^2\left(\lambda^T f_{\ddot{x}}+y(t)^T g_{\ddot{x}}\right)\right)dt<0$$

From the equality constraint of the dual, we have,

$$\int_I \eta^T\left(\sum_{i=1}^p \lambda^i\left(f_x^i+B^i(t)z^i(t)+y(t)^T g_x\right)\right.$$
$$\left.-D\left(\lambda^T f_{\dot{x}}+y(t)^T g_{\dot{x}}\right)+D^2\left(\lambda^T f_{\ddot{x}}+y(t)^T g_{\ddot{x}}\right)\right)dt=0$$

which is a contradiction. Hence the conclusion of the theorem is true.

Theorem 5.9 (*Strong Duality*): Let $\bar{x}\in X$ be an efficient solution of (VP) and for at least one $i\in P$, $\bar{x}$ satisfies the regularity condition [30] for the problem $\left(\mathrm{P}_k(\bar{x})\right)$. Then there exist multipliers $\bar{\lambda}\in R^p$, piecewise smooth $\bar{y}\in R^m$ and $z^i(t)\in R^n$, $i=\{1,2,\ldots,p\}$ such that $\left(\bar{x},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\lambda\right)$ is feasible for (Mix D) and the objectives of (VP) and (Mix D) are equal.

Further, if the hypotheses of Theorem 5.8 are met, then $\left(\bar{x},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\lambda\right)$ is an efficient solution of (Mix D).

Proof: Since $\bar{x}\in X$ is an optimal solution of $\left(\mathrm{P}_k(\bar{x})\right)$.This implies that there exist $\bar{\xi}\in R^p$ with $\bar{\xi}^1,\ldots,\bar{\xi}^p$, $z^i(t)\in R^n$, $i=\{1,2,\ldots,p\}$ and piecewise smooth $\bar{\upsilon}\in R^m$ such that, the following optimality conditions [30] hold:

$$\bar{\xi}^k\left(f_x^k+B^k(t)z^k(t)-Df_{\dot{x}}^k+D^2 f_{\ddot{x}}^k\right)+\sum_{\substack{i=1\\ i\neq k}}^p \bar{\xi}^i\left(f_x^i+B^i(t)z^i(t)-Df_{\dot{x}}^i+D^2 f_{\ddot{x}}^i\right)$$
$$+\bar{\upsilon}(t)^T g_x-D\left(\bar{\upsilon}(t)^T g_{\dot{x}}\right)+D^2\left(\bar{\upsilon}(t)^T g_{\ddot{x}}\right)=0 \tag{5.73}$$

$$\bar{\upsilon}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt = 0 \tag{5.74}$$

$$\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}} = \left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right), \; i = 1,\dots,p \tag{5.75}$$

$$\left(\bar{z}(t)^T B^i(t)\bar{z}^i(t)\right) \leqq 1, \; t \in I, \; i = 1,2,\dots,p \tag{5.76}$$

$$\bar{\xi} > 0 \;, \; \bar{\upsilon}(t) \geqq 0, \; t \in I \tag{5.77}$$

Dividing (5.73), (5.74) and (5.77) by $\sum_{i=1}^{p} \bar{\xi}^i$, and setting

$\bar{\lambda}^i = \dfrac{\bar{\xi}}{\sum_{i=1}^{p} \bar{\xi}^i}$, $i = 1,\dots,p$ and $\bar{y}(t) = \dfrac{\bar{\upsilon}(t)}{\sum_{i=1}^{p} \bar{\xi}^i}$, we have,

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f_x^i + B^i(t)\bar{z}^i(t) - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i \right) + \bar{y}(t)^T g_x - D\left(\bar{y}(t)^T g_{\dot{x}}\right) + D^2\left(\bar{y}(t)^T g_{\ddot{x}}\right) = 0 \tag{5.78}$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt = 0 \tag{5.79}$$

$$\bar{\lambda} > 0 \;, \; \bar{y}(t) \geqq 0, \; t \in I \tag{5.80}$$

The equation (5.79) implies

$$\sum_{j \in J_\circ} y^j(t) g^j(t,x,\dot{x},\ddot{x}) = 0 \;\text{ and }\; \sum_{j \in J_\alpha} y^j(t) g^j(t,x,\dot{x},\ddot{x}) = 0, \; \alpha = 1,\dots,r \tag{5.81}$$

This implies,

$$\sum_{j \in J_\alpha} \int_I y^j(t) g^j(t,x,\dot{x},\ddot{x}) = 0, \; \alpha = 1,\dots,r \tag{5.82}$$

Consequently (5.76), (5.78), (5.80) and (5.82) implies that $\left(\bar{x},\bar{y},\bar{z}^1,\dots,\bar{z}^p,\bar{\lambda}\right)$ is feasible for (Mix D).

$$\int_I \left(f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + \left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}} + \sum_{j \in J_\circ} y^j(t) g^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) \right) dt = \int_I \left(f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + \left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right)\right) dt \;, \; i = 1,2,\dots,p$$

This in view of Theorem 5.8, the efficiency of $\left(\bar{x},\bar{y},\bar{z}^1,\dots,\bar{z}^p,\bar{\lambda}\right)$ follows.

As in [105], by employing chain rule in calculus, it can be easily seen that the expression

$$\sum_{i=1}^{p}\lambda^i\left(f_x^i(t,u,\dot{u},\ddot{u})dt+B^i(t)z^i(t)+y(t)^T g_x(t,u,\dot{u},\ddot{u})\right)$$
$$-D\left(\lambda^T f_{\dot{x}}+y(t)^T g_{\dot{x}}\right)+D^2\left(\lambda^T f_{\ddot{x}}+y(t)^T g_{\ddot{x}}\right)=0,\ t\in I$$

may be regarded as a function θ of variables $t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y}$ and λ, where $\dddot{x}=D^3x$ and $\ddot{y}=D^2y$. That is, we can write

$$\theta(t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},\lambda)=\sum_{i=1}^{p}\lambda^i\left(f_x^i(t,u,\dot{u},\ddot{u})dt+B^i(t)z^i(t)+y(t)^T g_x(t,u,\dot{u},\ddot{u})\right)$$
$$-D\left(\lambda^T f_{\dot{x}}+y(t)^T g_{\dot{x}}\right)+D^2\left(\lambda^T f_{\ddot{x}}+y(t)^T g_{\ddot{x}}\right)=0$$

In order to prove converse duality between (VP) and (Mix D), the space X is now replaced by a smaller space X_2 of piecewise smooth thrice differentiable function $x:I\to R^n$ with the norm $\|x\|_\infty+\|Dx\|_\infty+\|D^2x\|_\infty+\|D^3x\|_\infty$. The problem (Mix D) may now be briefly written as,

$$\text{Minimize}\left(-\int_I\left(f^1(t,u,\dot{u},\ddot{u})+u(t)^T B^1(t)z^1(t)+\sum_{j\in J_\circ}y^j(t)g^j(t,u,\dot{u},\ddot{u})\right)dt\right.$$
$$\left.,\ldots,\int_I-\left(f^p(t,u,\dot{u},\ddot{u})+\left(u(t)^T B^p(t)z^p(t)\right)+\sum_{j\in J_\circ}y^j(t)g^j(t,u,\dot{u},\ddot{u})\right)dt\right)$$

Subject to

$$u(a)=0=u(b),$$

$$\dot{u}(a)=0=\dot{u}(b),$$

$$\theta(t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},\lambda)=0,\ t\in I,$$

$$\sum_{j\in J_\alpha}\int_I y^j(t)g^j(t,u,\dot{u},\ddot{u})dt\geqq 0,\ \alpha=1,2,\ldots,r,$$

$$\bar{z}^i(t)^T B^i(t)\bar{z}^i(t)\leqq 1,\ t\in I,\ i\in P,$$

$$y(t)\geqq 0,\ t\in I,$$

$$\lambda>0,\ \lambda^T e=1.$$

where

$$\theta\left(t,x,\dot{x},\ddot{x},\dddot{x},y,\dot{y},\ddot{y},\lambda\right)=\sum_{i=1}^{p}\lambda^{i}\left(f_{x}^{i}\left(t,u,\dot{u},\ddot{u}\right)dt+B^{i}\left(t\right)z^{i}\left(t\right)+y\left(t\right)^{T}g_{x}\left(t,u,\dot{u},\ddot{u}\right)\right)$$
$$-D\left(\lambda^{T}f_{\dot{x}}+y\left(t\right)^{T}g_{\dot{x}}\right)+D^{2}\left(\lambda^{T}f_{\ddot{x}}+y\left(t\right)^{T}g_{\ddot{x}}\right)=0$$

Consider $\theta\left(t,x(\cdot),\dot{x}(\cdot),\ddot{x}(\cdot),\dddot{x}(\cdot),y(\cdot),\dot{y}(\cdot),\ddot{y}(\cdot),\lambda\right)=0$ as defining a mapping $\psi: X\times Y\times R^{p}\to B$ where Y is a space of piecewise twice differentiable function and B is the Banach Space. In order to apply Theorem 4.1 to the problem (Mix D), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $\left[\psi_{x}\left(x,y,\lambda\right),\psi_{y}\left(x,y,\lambda\right),\psi_{\lambda}\left(x,y,\lambda\right)\right]$.

Theorem 5.10 (*Converse Duality*): Let $\left(\bar{x},\bar{y},\bar{z}^{1},\ldots,\bar{z}^{p},\bar{\lambda}\right)$ be an efficient solution for (Mix D). Assume that

(A₁) The Frèchèt derivative ψ' has a (weak*) closed range,

(A₂) f and g are twice continuously differentiable,

(A₃) $\left(f_{x}^{i}+B^{i}\left(t\right)z^{i}\left(t\right)+\sum_{j\in J_{\circ}}y^{j}\left(t\right)g_{x}^{i}\right)-D\left(f_{\dot{x}}^{i}+\sum_{j\in J_{\circ}}y^{j}\left(t\right)g_{\dot{x}}^{i}\right)$
$+D^{2}\left(f_{\ddot{x}}^{i}+\sum_{j\in J_{\circ}}y^{j}\left(t\right)g_{\ddot{x}}^{i}\right)$, $i\in\{1,2,\ldots,p\}$ are linearly independent, and

(A₄) $\left(\beta\left(t\right)^{T}\theta_{x}-D\beta\left(t\right)^{T}\theta_{\dot{x}}+D^{2}\beta\left(t\right)^{T}\theta_{\ddot{x}}-D^{3}\beta\left(t\right)^{T}\theta_{\dddot{x}}\right)\beta\left(t\right)=0,$
$\Rightarrow\beta\left(t\right)=0$, $t\in I$

Further, if the hypotheses of Theorem 5.8 are met then $\bar{x}$ is an efficient solution of (VP).

Proof: Since $\left(\bar{x},\bar{y},\bar{z}^{1},\ldots,\bar{z}^{p},\bar{\lambda}\right)$ with ψ' having a (weak*) closed range, is an efficient solution of (Mix D), then by Theorem 4.1, there exist multipliers $\tau\in R^{p}$, piecewise smooth $\beta: I\to R^{n}$, $z^{i}\left(t\right)\in R^{n}$, $i=1,\ldots,p$, $\gamma\in R$ for each of r constraints, $\eta\in R^{p}$, $\delta\in R$ and piecewise smooth $\mu: I\to R^{m}$ satisfying the following Fritz-John optimality conditions.

$$-\sum_{i=1}^{p}\tau^i\left[\left(f_x^i+B^i(t)z^i(t)+\sum_{j\in J_\circ}y^j(t)g_x^j\right)-D\left(f_{\dot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\dot{x}}^j\right)\right.$$

$$\left.+D^2\left(f_{\ddot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\ddot{x}}^j\right)\right]-\gamma\sum_{\alpha=1}^{r}\sum_{j\in J_\alpha}\left(y^j(t)g_x^j-Dy^j(t)g_{\dot{x}}^j+D^2y^j(t)g_{\ddot{x}}^j\right)$$

$$+\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}=0\ ,\ t\in I\,, \tag{5.83}$$

$$-\left(\tau^Te\right)g^j+\beta(t)^T\theta_{y^j}-D\beta(t)^T\theta_{\dot{y}^j}+D^2\beta(t)^T\theta_{\ddot{y}^j}-\mu^j(t)=0\ ,\ j\in J_\circ, \tag{5.84}$$

$$-\gamma g^j+\beta(t)^T\theta_{y^j}-D\beta(t)^T\theta_{\dot{y}^j}+D^2\beta(t)^T\theta_{\ddot{y}^j}-\mu^j(t)=0\ ,\ j\in J_\alpha,\alpha=1,\dots,r \tag{5.85}$$

$$-\tau^ix(t)^TB^i(t)+\lambda^i\beta(t)^TB^i(t)+2\delta^iB^i(t)z^i(t)=0\ ,\ i=1,\dots,p\,, \tag{5.86}$$

$$\left[f_x^i+B^i(t)z^i(t)+\sum_{j\in J_\circ}y^j(t)g_x^j-D\left(f_{\dot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\dot{x}}^j\right)\right.$$

$$\left.+D^2\left(f_{\ddot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\ddot{x}}^j\right)\right]-\eta^i=0\,, \tag{5.87}$$

$$\mu^T(t)\bar{y}(t)=0\ ,\ t\in I\,, \tag{5.88}$$

$$\eta^T\lambda=0\,, \tag{5.89}$$

$$\gamma\sum_{j\in J_\alpha}\int_I y(t)^Tg\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt=0\ ,\ \alpha=1,2,\dots,p\,, \tag{5.90}$$

$$\delta^i\left(z^i(t)^TB^i(t)z^i(t)-1\right)=0\ ,\ t\in I\,, \tag{5.91}$$

$$\left(\tau,\gamma,\mu(t),\delta,\eta\right)\geqq 0\,, \tag{5.92}$$

$$\left(\tau,\beta(t),\gamma,\mu(t),\delta,\eta\right)\neq 0\,. \tag{5.93}$$

Since $\lambda>0$, (5.89) implies $\eta=0$. Consequently (5.87) implies

$$\left[f_x^i+B^i(t)z^i(t)+\sum_{j\in J_\circ}y^j(t)g_x^j-D\left(f_{\dot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\dot{x}}^j\right)+D^2\left(f_{\ddot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\ddot{x}}^j\right)\right]=0 \tag{5.94}$$

Using the duality constraint of (Mix D) equation (5.83) reduces to

$$-\sum_{i=1}^{p}\left(\tau^i-\gamma\lambda^i\right)\left\{\left(f_x^i+B^i(t)z^i(t)+\sum_{j\in J_\circ}y^j(t)g_x^j\right)-D\left(f_{\dot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\dot{x}}^j\right)\right.$$

$$\left.+D^2\left(f_{\ddot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\ddot{x}}^j\right)\right\}+\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}=0\ ,\ t\in I \tag{5.95}$$

Post multiplying (5.95) by $\beta(t)$ and then using (5.94), we obtain,

$$\left(\beta(t)^T\theta_x-D\beta(t)^T\theta_{\dot{x}}+D^2\beta(t)^T\theta_{\ddot{x}}-D^3\beta(t)^T\theta_{\dddot{x}}\right)\beta(t)=0\ ,\ t\in I$$

This because of the hypothesis (A_4) yields,

$$\beta(t)=0\ ,\ t\in I \tag{5.96}$$

Using (5.96) in (5.95) in (5.83), we have,

$$-\sum_{i=1}^{p}\left(\tau^i-\gamma\lambda^i\right)\left\{\left(f_x^i+B^i(t)z^i(t)+\sum_{j\in J_\circ}y^j(t)g_x^j\right)-D\left(f_{\dot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\dot{x}}^j\right)\right.$$

$$\left.+D^2\left(f_{\ddot{x}}^i+\sum_{j\in J_\circ}y^j(t)g_{\ddot{x}}^j\right)\right\}=0$$

This, due to the hypothesis (A_3) gives,

$$\tau^i-\gamma\lambda^i=0\ ,\ i=1,2,\ldots,p \tag{5.97}$$

Suppose $\gamma=0$, then from (5.97) we have $\tau=0$. Consequently from (5.84) and (5.85) implies $\mu(t)=0\ ,\ t\in I$.

Also from (5.86) we have $\delta^i B^i(t)z^i(t)=0$, which together with (5.91) implies $\delta^i=0$, i.e., $\delta=0$. Thus, $(\tau,\beta(t),\mu(t),\eta,\gamma,\delta,)=0$, which is a contradiction to (5.93). Hence $\gamma>0$. From (5.95) it implies that $\tau>0$.

By the generalized Schwartz inequality [125],

$$\left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right)\leq\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\left(\bar{z}^i(t)B^i(t)\bar{z}^i(t)\right)^{\frac{1}{2}}\ ,\ i=1,2,\ldots,p \tag{5.98}$$

Now let $\dfrac{\delta^i}{\tau^i}=\alpha^i$. Then $\alpha^i\geqq 0$ and from (5.86), we have,

$$B^i(t)\bar{x}(t)=2\alpha^i B^i(t)\bar{z}^i(t),\ i=1,2,\ldots,p$$

which, is the condition for the equality in (5.98). Therefore, we have,

$$\left(\bar{x}(t)^T B^i(t)\bar{z}^i(t)\right)=\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\left(z^i(t)B^i(t)z^i(t)\right)^{\frac{1}{2}}$$

From (5.91), either $\delta^i=0$ or $\bar{z}^i(t)^T B^i(t)\bar{z}^i(t)=1$ or $\delta^i=0$, $i=1,\ldots,p$ and hence $B^i(t)\bar{x}(t)=0$. Therefore, in either case,

$$\left(x(t)^T B^i(t)z^i(t)\right)=\left(x(t)^T B^i(t)x(t)\right)^{\frac{1}{2}}\;,\; i=1,2,\ldots,p \tag{5.99}$$

From (5.84) and (5.85) we readily obtain,

$$g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\leqq 0\;,\; t\in I$$

which gives the feasibility of $\bar{x}$ for (P). Using (5.88) in (5.84) and (5.85) we have

$$\bar{y}(t)^T g=0,\;\text{ i.e., }\bar{y}^j(t)g^j=0\;,\; j=1,2,\ldots,m,\;\; t\in I$$

This obviously gives

$$\sum_{j\in J_\circ}\bar{y}^j(t)g^j=0\;,\; j=1,2,\ldots,m \tag{5.100}$$

Hence

$$\int_I\left(f^i\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt+\bar{x}(t)^T B^i(t)\bar{z}^i(t)+\sum_{j\in J_\circ}\bar{y}^j(t)g^j=0\right)dt$$

$$=\int_I\left(f^i\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt+\left(\bar{x}(t)^T B^i(t)\bar{x}(t)\right)^{\frac{1}{2}}\right)dt\;,\; i=1,2,\ldots,p$$

(by using (5.99) and (5.100))

The efficiency of $\bar{x}$ for (VP) follows by an application of Theorem 5.8.

5.2.2 Variational Problems with Natural Boundary Values

Here we shall consider the following variational problems with natural boundary values rather that fixed points.

(VP)$_0$: Minimize $$\left(\int_I\left(f^1(t,x,\dot{x},\ddot{x})dt+\left(x(t)^T B^1(t)x(t)\right)^{\frac{1}{2}}\right)dt\right.$$

$$\left.,\ldots,\int_I\left(f^p(t,x,\dot{x},\ddot{x})dt+\left(x(t)^T B^p(t)x(t)\right)^{\frac{1}{2}}\right)dt\right)$$

Subject to

$$g^j\left(t,x,\dot{x},\dot{x}\right)\leqq 0\ ,\ t\in I,\quad j=1,\ldots,m$$

(MIX D)$_0$: Maximize$\left(\int_I\left(f^1\left(t,u,\dot{u},\ddot{u}\right)+u\left(t\right)^T B^1\left(t\right)z^1\left(t\right)+\sum_{j\in J_\circ} y^j\left(t\right)g^j\left(t,u,\dot{u},\ddot{u}\right)\right)dt\right.$

$$\left.,\ldots,\int_I\left(f^p\left(t,u,\dot{u},\ddot{u}\right)+\left(u\left(t\right)^T B^p\left(t\right)z^p\left(t\right)\right)+\sum_{j\in J_\circ} y^j\left(t\right)g^j\left(t,u,\dot{u},\ddot{u}\right)\right)dt\right)$$

Subject to

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},\ddot{u}\right)+B^i\left(t\right)z^i\left(t\right)+y\left(t\right)^T g_u\left(t,u,\dot{u},\ddot{u}\right)\right)$$
$$-D\left(\lambda^T f_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)$$
$$+D^2\left(\lambda^T f_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)+y\left(t\right)^T g_{\ddot{u}}\left(t,u,\dot{u},\ddot{u}\right)\right)=0\ ,\ t\in I,$$

$$\left.\left(f_{\dot{x}}^i+\sum_{j\in J_\circ} y^j\left(t\right)g_{\dot{x}}^j\right)\right|_{t=a}=0,$$

$$\left.\left(f_{\dot{x}}^i+\sum_{j\in J_\circ} y^j\left(t\right)g_{\ddot{x}}^j\right)\right|_{t=b}=0,$$

$$\left.\left(f_{\ddot{x}}^i+\sum_{j\in J_\circ} y^j\left(t\right)g_{\dot{x}}^j\right)\right|_{t=a}=0,$$

$$\left.\left(f_{\ddot{x}}^i+\sum_{j\in J_\circ} y^j\left(t\right)g_{\ddot{x}}^j\right)\right|_{t=b}=0,$$

$$\forall\ i\in\left\{1,2,\ldots,p\right\}$$

$$\bar{z}^i\left(t\right)^T B^i\left(t\right)\bar{z}^i\left(t\right)\leqq 1\ ,\ t\in I\ ,\ i\in P\ ,$$

$$\sum_{j\in J_\alpha}\int_I y^j\left(t\right)g^j\left(t,u,\dot{u},\ddot{u}\right)dt\geqq 0\ \ ,\ \ \alpha=1,2,\ldots,r\ ,$$

$$\lambda>0\ \ ,\quad y\left(t\right)\geqq 0\ ,\ t\in I\ \ .$$

5.2.3 Nondifferentiable Nonlinear Programming

If all the functions are independent of t then the problems (VP)$_0$ and (MIX D)$_0$ become the following problems.

(VP)$_1$: Minimize$\left(f^1(x)+\left(x^T B^1 x\right)^{\frac{1}{2}},\ldots, f^p(x)+\left(x^T B^p x\right)^{\frac{1}{2}} \right)$

Subject to

$$g(x) \leqq 0$$

(Mix D)$_1$: Maximize$\left(f^1(u)+u^T B^1 z^1+\sum_{j\in J_\circ} y^j g^j(u),\ldots, f^P(u)+u^T B^P z^P+\sum_{j\in J_\circ} y^j g^j(u) \right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(u)dt + B^i z^i + y^T g_u(u) \right) = 0$$

$$\bar{z}^{iT} B^i \bar{z}^i \leqq 1 \ , \ i \in P$$

$$\sum_{j\in J_\alpha} y^j g^j(u) \geqq 0 \ , \ \alpha = 1,2,\ldots,r$$

$$y(t) \geqq 0 \ , \ t \in I$$

$$\lambda \in \Lambda^+$$

The duality results for this pair of problems are not explicitly reported in the literature but can be derived easily on the lines of the analysis of this research.

Chapter-6

MIXED TYPE SYMMETRIC AND SELF DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS

6.1 MIXED TYPE SYMMETRIC AND SELF DUALITY FOR MULTIOBJECTIVEVARIATIONAL PROBLEMS

6.1.1 Introductory Remarks

Motivated with the results of Dorn [51], symmetric duality results in mathematical programming have been derived by a number of authors, notably, Dantzig et al [48], Mond [99], Bazaraa and Goode [6]. In these researches, the authors have studied symmetric duality under the hypothesis of convexity-concavity of the kernel function involved. Mond and Cottle [104] presented self duality for the problems of [48] by assuming skew symmetric of the kernel function. Later Mond-Weir [14] formulated a different pair of symmetric dual nonlinear program with a view to generalize convexity-concavity of the kernel function to pseudoconvexity-pseudoconcavity.

Symmetric duality for variational problems was first introduced by Mond and Hanson [108] under the convexity-concavity conditions of a scalar functions like $\psi\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right)$ with $x(t) \in R^n$ and $y(t) \in R^m$. Bector, Chandra and Husain [15] presented a different pair of symmetric dual variational problems in order to relax the requirement of convexity-concavity to that of pseudoconvexity-pseudoconcavity while in [27] Chandra and Husain gave a fractional analogue.

Bector and Husain [19] probably were the first to study duality for multiobjective variational problems under appropriate convexity assumptions. Subsequently, Gulati, Husain and Ahmed [59] presented two distinct pairs of symmetric dual multiobjective variational problems and established various duality results under appropriate invexity requirements. In this reference, self duality theorem is also given under skew symmetric of the integrand of the objective functional. Husain and Jabeen [72] formulated a pair of mixed type symmetric dual variational problem in order to unify the Wolfe and Mond-Weir symmetric dual pairs of variational problems studied in [59].

The purpose of this chapter is to unify formulations of Wolfe and Mond-Weir type symmetric dual pairs of multiobjective variational problems incorporated by Gulati, Husain and Ahmed [59] and also present multiobjective version of the formulation of a pair of mixed type symmetric dual of Husain and Jabeen [72] and hence study symmetric and self duality for a pair of mixed multiobjective variational problem. This research is motivated by the work of Xu [157]. Problems with natural boundary values are formulated in the subsection 6.1.6, as in the previous chapters. In subsection 6.1.7, it is pointed out that our results can be considered as dynamic generalizations of corresponding (static) symmetric duality results of multiobjective nonlinear nonlinear treated by Bector. et.al.[15].

6.1.2 Notations and Preliminaries

Let $I=[a,b]$ be the real interval, and $\phi^i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)$ be a scalar function and twice differentiable function for $i=1,2,\ldots,p$ where $x:I\rightarrow R^n$ and $y:I\rightarrow R^n$ with derivatives $\dot{x}$ and $\dot{y}$. In order to consider $\phi^i\left(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\right)$ denote the first partial derivatives of ϕ^i with respect to $t,x(t),\dot{x}(t),y(t),\dot{y}(t)$ respectively, by $\phi^i_t,\phi^i_x,\phi^i_{\dot{x}},\phi^i_y,\phi^i_{\dot{y}}$, that is,

$$\phi^i_t=\frac{\partial\phi^i}{\partial t}$$

$$\phi^i_x=\left[\frac{\partial\phi^i}{\partial x_1},\frac{\partial\phi^i}{\partial x_2},\ldots,\frac{\partial\phi^i}{\partial x_n}\right],\qquad \phi^i_{\dot{x}}=\left[\frac{\partial\phi^i}{\partial\dot{x}_1},\frac{\partial\phi^i}{\partial\dot{x}_2},\ldots,\frac{\partial\phi^i}{\partial\dot{x}_n}\right]$$

$$\phi^i_y=\left[\frac{\partial\phi^i}{\partial y_1},\frac{\partial\phi^i}{\partial y_2},\ldots,\frac{\partial\phi^i}{\partial y_n}\right],\qquad \phi^i_{\dot{y}}=\left[\frac{\partial{\phi^i}_i}{\partial\dot{y}_1},\frac{\partial\phi^i}{\partial\dot{y}_2},\ldots,\frac{\partial\phi^i}{\partial\dot{y}_n}\right].$$

The twice partial derivatives of ϕ^i with respect to $t, x(t), \dot{x}(t),\ y(t)$ and $\dot{y}(t)$, respectively are the matrices

$$\phi^i_{xx} = \left(\frac{\partial^2 \phi^i}{\partial x_k x_s} \right)_{n\times n}, \qquad \phi^i_{x\dot{x}} = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{x}_s} \right)_{n\times n}, \qquad \phi^i_{xy} = \left(\frac{\partial^2 \phi^i}{\partial x_k y_s} \right)_{n\times n}$$

$$\phi^i_{x\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{y}_s} \right)_{n\times n}, \qquad \phi^i_{\dot{x}y} = \left(\frac{\partial^2 \phi^i}{\partial \dot{x}_k y_s} \right)_{n\times n}, \qquad \phi^i_{x\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{y}_s} \right)_{n\times n}$$

$$\phi^i_{yy} = \left(\frac{\partial^2 \phi^i}{\partial y_k y_s} \right)_{n\times n}, \qquad \phi^i_{y\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial y_k \dot{y}_s} \right)_{n\times n}, \qquad \phi^i_{\dot{y}\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial \dot{y}_k \dot{y}_s} \right)_{n\times n}$$

for $i = 1, 2, \ldots, p$.

Noting that

$$\frac{d}{dt}\phi^i_{\dot{y}} = \phi^i_{\dot{y}t} + \phi^i_{\dot{y}y}\dot{y} + \phi^i_{\dot{y}\dot{y}}\ddot{y} + \phi^i_{\dot{y}x}\dot{x} + \phi^i_{\dot{y}\dot{x}}\ddot{x}$$

and hence

$$\frac{\partial}{\partial y}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{y\dot{y}}, \qquad \frac{\partial}{\partial \dot{y}}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{\dot{y}\dot{y}} + \phi_{\dot{y}y}, \qquad \frac{d}{d\ddot{y}}\frac{d}{dt}\phi^i_{\dot{y}} = \phi^i_{\dot{y}\dot{y}}$$

$$\frac{\partial}{\partial x}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{\dot{y}x}, \qquad \frac{\partial}{\partial \dot{x}}\frac{d}{dt}\phi_{\dot{y}} = \frac{d}{dt}\phi_{\dot{y}\dot{x}} + \phi_{\dot{y}x}, \qquad \frac{\partial}{\partial \ddot{x}}\frac{d}{dt}\phi_{\dot{y}} = \phi_{\dot{y}\dot{x}}$$

In order to establish our main results, the following are needed.

Definition 6.1 (*Partially Invex*): If there exists a vector function $\eta(t, x(t), y(t), u(t), v(t)) \in \mathbf{R}^n_+$ with $\eta = 0$ at $x(t) = u(t)$ or $y(t) = v(t)$, such that for the scalar function $h(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ the functional

$$H(x, y) = \int_I h(t, x(t), \dot{x}(t), y(t), \dot{y}(t))\,dt$$

satisfies

$$H(x, y) - H(u, v) \geq \int_I \Big[\eta^T h_x\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) + (D\eta)^T h_{\dot{x}}\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right)\Big]dt$$

then $H(x, \dot{x}, y, \dot{y})$ is said to be partially invex in x and $\dot{x}$ on I with respect to η, for fixed y. If H satisfies

$$H(x,y)-H(x,v)\geq\int_I\left[\eta^T h_y\left(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\right)\right.$$
$$\left.+(D\eta)^T h_{\dot{y}}\left(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\right)\right]dt,$$

then $H(x,\dot{x},y,\dot{y})$ is said to be partially invex in y and $\dot{y}$ on I with respect to η, for fixed x. If $-H$ is partially invex in x and $\dot{x}$ (or in y and $\dot{y}$) on I with respect to η, for fixed y (or for fixed x), then H is said to be partially incave in x and $\dot{x}$ (or in y and $\dot{y}$) on I with respect to η, for fixed y (or for fixed x).

Definition 6.2 (*Partially Pseudoinvex*): The functional H is said to be partially pseudoinvex 2in x and $\dot{x}$ with respect to η, for fixed y if H satisfies

$$\int_I\left[\eta^T h_x(t,x,\dot{x},y,\dot{y})+(D\eta)^T h_{\dot{x}}(t,x,\dot{x},y,\dot{y})\right]dt \geqq 0$$

implies

$$H(x,u)\geqq H(x,y).$$

and

H is said to be partially pseudoinvex in y and $\dot{y}$ with respect to η, for fixed x If H satisfies

$$\int_I\left[\eta^T h_y(t,x,\dot{x},y,\dot{y})+(D\eta)^T h_{\dot{y}}(t,x,\dot{x},y,\dot{y})\right]dt \geqq 0$$

implies

$$H(x,v)\geqq H(x,y).$$

Definition 6.3 (*Partially Quasi-Invex*): The functional H is said to be partially quasi-invex in x and $\dot{x}$ with respect to η, for fixed y if H satisfies

$$H(x,u)\leqq H(x,y)$$

implies

$$\int_I\left[\eta^T h_x(t,x,\dot{x},y,\dot{y})+(D\eta)^T h_{\dot{x}}(t,x,\dot{x},y,\dot{y})\right]dt \leqq 0$$

and

H is said to be partially quasi-invex in y and $\dot{y}$ with respect to η, for fixed x if H satisfies

$$H(x,v) \leqq H(x,y)$$

implies

$$\int_I \left[\eta^T h_y(t,x,\dot{x},y,\dot{y}) + (D\eta)^T h_{\dot{y}}(t,x,\dot{x},y,\dot{y})\right] dt \leqq 0 .$$

If h is independent of t, then the above definitions become the usual definitions of invexity and generalized invexity, discussed by several authors, notably Ben-Israel and Mond [20], Martin [94], and Rueda and Hanson [130].

Now consider the following multiobjective variational problem considered in [19]:

(VP$_0$): Minimize $\left(\int_I \phi^1(t,x,\dot{x})\,dt, \int_I \phi^2(t,x,\dot{x})\,dt, \ldots, \int_I \phi^p(t,x,\dot{x})\,dt, \right)$

Subject to

$$x(a) = \alpha, \qquad x(b) = \beta$$

$$h(t,x,\dot{x}) \leqq 0 \ , \ t \in I,$$

where $\phi^i : I \times R^n \times R^n \times R^n \to R, (i = 1,2,\ldots,p)$ and $h : I \times R^n \times R^n \times R^n \to R^m$.Let the set of feasible solution of (VP$_0$) be represented by K.

Definition 6.4 (*Efficiency*): A point $\bar{x} \in K$ is an efficient (Pareto optimal) solution of (VP$_0$) if for all $x \in K$,

$$\int_I \phi^i(t,x,\dot{x})\,dt \not\leq \int_I \phi^i(t,\bar{x},\dot{\bar{x}})\,dt \quad , \quad (i = 1,2,\ldots,p)$$

6.1.3 Statement of the Problems

For $N = \{1,2,\ldots,n\}$ and $M = \{1,2,\ldots,m\}$, let $J_1 \subset N, K_1 \subset M, J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in the subset J_1. The other symbol $|J_2|, |K_1|$ and $|K_2|$ are similarly defined. Let $x^1 : I \to R^{|J_1|}$ and $x^2 : I \to R^{|J_2|}$, then any $x : I \to R^n$ can be written as $x = (x^1, x^2)$. Similarly for $y^1: I \to R^{|K_1|}$ and $y^2: I \to R^{|K_2|}$ can be written as $y=\left(y^1, y^2\right)$. Let $f: I \times R^{|J_1|} \times R^{|K_1|} \to R^p$ and $g: I \times R^{|J_2|} \times R^{|K_2|} \to R^p$ be twice continuously differentiable functions.

We state the following pair of mixed type multiobjective symmetric dual variational problems involving vector functions f and g.

(Mix SP): Minimize $F\left(x^1, x^2, y^1, y^2\right)=\int_I \Big\{ f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)+g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)$

$$-y^1(t)^T\left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right.$$

$$\left.\left.-D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right)e\right\} dt$$

Subject to

$$x^1(a)=0=x^1(b) \ , \quad y^1(a)=0=y^1(b) \ , \tag{6.1}$$

$$x^2(a)=0=x^2(b) \ , \quad y^2(a)=0=y^2(b) \ , \tag{6.2}$$

$$\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)-D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \leqq 0 \ , \ t \in I \ , \tag{6.3}$$

$$\lambda^T g_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)-D\lambda^T g_{\dot{y}^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \leqq 0 \ , \ t \in I \ , \tag{6.4}$$

$$\int_I y^2(t)^T\left(\lambda^T g_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\right.$$

$$\left.-D\lambda^T g_{\dot{y}^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\right) \geqq 0 \ , \tag{6.5}$$

$$\lambda \in \Lambda^+ . \tag{6.6}$$

(Mix SD): Maximize $G\left(u^1, u^2, v^1, v^2\right)=\int_I \Big\{ f\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)+g\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right)$

$$-u^1(t)^T\left(\lambda^T f_{y^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right.$$

$$\left.\left.-D\lambda^T f_{\dot{y}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right)e\right\} dt$$

Subject to

$$u^1(a)=0=u^1(b) \ , \quad v^1(a)=0=v^1(b), \tag{6.7}$$

$$u^2(a)=0=u^2(b) \ , \quad v^2(a)=0=v^2(b), \tag{6.8}$$

$$\lambda^T f_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)-D\lambda^T f_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \geqq 0 \ , \ t \in I \ , \tag{6.9}$$

$$\lambda^T g_{u^2}\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right)-D\lambda^T g_{\dot{u}^2}\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right) \geqq 0 \ , \ t \in I \ ,$$

$$\int_I u^2(t)^T \left(\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right. \tag{6.10}$$

$$\left. -D\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right) \geqq 0,, \tag{6.11}$$

$$\lambda \in \Lambda^+ . \tag{6.12}$$

where $\Lambda^+ = \left\{\lambda \in R^p \,\middle|\, \lambda > 0,\ \lambda^T e = 1, e = (1,1,\dots,1)^T \in R^p\right\}$

6.1.4 Mixed Type Multiobjective Symmetric Duality

In this section, we present various duality results and the appropriate invexity and generalized invexity assumptions.

Theorem 6.1. (*Weak Duality***):** Let $\left(x^1,x^2,y^1,y^2,\lambda\right)$ be feasible for (Mix SP) and $\left(u^1,u^2,v^1,v^2,\lambda\right)$ be feasible for (Mix SD).

Let

H

H_1 $\int_I \left\{ f\left(t,.,.,y^1(t),\dot{y}^1(t)\right) dt\right.$ be partially invex in $x^1,\dot{x}^1$ on I for fixed $y^1,\dot{y}^1$ with respect to $\eta_1\left(t,x^1,u^1\right) \in R^{|J_1|}$.

$\int_I \left\{ f\left(t,x^1(t),\dot{x}^1(t),.,.\right) dt\right.$ be partially incave in $y^1,\dot{y}^1$ on I for fixed $x^1,\dot{x}^1$ with respect to $\eta_2\left(t,y^1,v^1\right) \in R^{|K_1|}$.

H_2 $\int_I \lambda^T g\left(t,.,.,y^2(t),\dot{y}^2(t)\right) dt$ be partially pseudoinvex in $x^2,\dot{x}^2$ on I for fixed $y^2,\dot{y}^2$ with respect $\eta_3\left(t,x^2,u^2\right) \in R^{|J_2|}$ and

$\int_I \lambda^T g\left(t,x^2,\dot{x}^2,.,.\right) dt$ be partially pseudoincave in $y^2,\dot{y}^2$ on I for fixed $x^2,\dot{x}^2$ with respect to $\eta_4\left(t,y^2,v^2\right) \in R^{|K_2|}$.

H_3 $$\eta_1\left(t,x^1,u^1\right) + u^1(t) \geqq 0,\ t \in I , \tag{6.13}$$

$$\eta_2\left(t,v^1,y^1\right) + y^1(t) \geqq 0,\ t \in I , \tag{6.14}$$

$$\eta_3\left(t,x^2,u^2\right) + u^2(t) \geqq 0,\ t \in I , \tag{6.15}$$

$$\eta_4\left(t,v^2,y^2\right) + y^2(t) \geqq 0,\ t \in I , \tag{6.16}$$

then

$$F\left(x^1,x^2,y^1,y^2\right) \not\leq G\left(u^1,u^2,v^1,v^2\right).$$

Proof: Because of the partial invexity-incavity of the function f, we have for each $i=\{1,2,\dots,p\}$.

$$\int_I f^i\left(t,x^1,\dot{x}^1,v^1,\dot{v}^1\right)dt-\int_I f^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt$$
$$\geqq \int_I \left\{\eta_1^T f^i_{x^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+\left(D\eta_1\right)^T f^i_{\dot{x}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}dt \tag{6.17}$$

$$\int_I f^i\left(t,x^1,\dot{x}^1,v^1,\dot{v}^1\right)dt-\int_I f^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)dt$$
$$\leqq \int_I \left\{\eta_2^T f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+\left(D\eta_2\right)^T f^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}dt \tag{6.18}$$

Multiplying (6.17) by $\lambda^i>0$ and summing over i, we get,

$$\int_I \lambda^T f\left(t,x^1,\dot{x}^1,v^1,\dot{v}^1\right)dt-\int_I \lambda^T f\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt$$
$$\geqq \int_I \left\{\eta_1^T\left(\lambda^T f_{x^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+\left(D\eta_1\right)^T \lambda^T f_{\dot{x}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right)\right\}dt$$

Integrating by parts, the above inequality becomes,

$$\int_I \lambda^T f\left(t,x^1,\dot{x}^1,v^1,\dot{v}^1\right)-\int_I \lambda^T f\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt$$
$$\geqq \int_I \eta_1^T\lambda^T f_{x^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt+\eta_1^T\lambda^T f_{\dot{x}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\Big|_{t=a}^{t=b}$$
$$-\int_I \eta_1{}^T D\lambda^T f_{\dot{x}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt$$

Using the boundary conditions which at $t=a,\ t=b$ gives $\eta_1=0$, we have,

$$\int_I \lambda^T f\left(t,x^1,\dot{x}^1,v^1,\dot{v}^1\right)-\int_I \lambda^T f\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt$$
$$\geqq \int_I \eta_1^T\left[\lambda^T f_{x^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt-D\left(\lambda^T f_{\dot{x}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right)\right]dt \tag{6.19}$$

Multiplying (6.18) by $\lambda^i,\ i\in\{1,2,\dots,p\}$ and summing over i, we get,

$$\int_I \lambda^T f\left(t,x^1,\dot{x}^1,v^1,\dot{v}^1\right)-\int_I \lambda^T f\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)dt$$
$$\leqq \int_I \left\{\eta_2^T\left(\lambda^T f_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right)+\left(D\eta_2\right)^T \lambda^T f_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}dt$$

On integrating by parts the R.H.S of the above inequality and using the boundary conditions which at $t=a,\ t=b$ gives $\eta_2=0$, we have,

$$\int_I \lambda^T f\left(t,x^1,\dot{x}^1,v^1,\dot{v}^1\right)-\int_I \lambda^T f\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)dt$$

$$\leqq \int_I \eta_2^T \left[\left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right) - D\left(\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right) \right] dt \quad (6.20)$$

Multiplying (6.20) by (-1) and adding to (6.19), we have

$$\int_I \lambda^T f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) dt - \int_I \lambda^T f\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) dt$$
$$\geqq \int_I \eta_1^T \left[\left(\lambda^T f_{x^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) - D\left(\lambda^T f_{\dot{x}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) \right] dt$$
$$-\int_I \eta_2^T \left[\left(\lambda^T f_{y^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) - D\left(\lambda^T f_{\dot{y}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) \right] dt$$
$$(6.21)$$

Now from the inequality (6.9) along with (6.13), it follows

$$\int_I \eta_1^T \left(\lambda^T f_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) dt$$
$$\geqq -\int_I u^1(t)^T \left[\lambda^T f_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right] dt \quad (6.22)$$

Also from the inequality (6.3) together with (6.14) implies

$$-\int_I \eta_2^T \left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right) dt$$
$$\geqq \int_I y^1(t)^T \left[\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right] dt \quad (6.23)$$

Using (6.22) and (6.23), in (6.21), we have

$$\int_I \lambda^T f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) dt - y^1(t)^T \int_I \lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) dt$$
$$\geqq -\int_I u^1(t)^T \left[\left(\lambda^T f_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) - D\left(\lambda^T f_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) \right] dt$$
$$+\int_I y^1(t)^T \left[\left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right) - D\left(\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right) \right] dt,$$
$$(6.24)$$

which implies,

$$\int_I \left\{ \lambda^T f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - y^1(t)^T \left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right) \right\} dt$$
$$\geqq \int_I \left\{ \lambda^T f_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - u^1(t)^T \left(\lambda^T f_{x^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{x}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right) \right\} dt$$
$$(6.25)$$

Now from the inequality (6.10) along with (6.15), we have,

$$\int_I \left(\eta_3^T\left(t,x^2,\dot{x}^2,u^2,\dot{u}^2\right)+u^2(t)\right)\left(\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-D\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right)\geqq 0.$$

This implies,

$$\int_I \eta_3^T\left(\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-D\left(\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right)\right)dt$$
$$\geqq -\int_I u^2(t)^T\left[\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-D\left(\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right)\right]dt$$

Integrating by parts and using the boundary conditions which at $t=a,\ t=b$ gives $\eta_3=0$, we have,

$$\int_I \left\{\eta_3^T\left(\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)+\left(D\eta_3\right)^T\left(\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right)\right)\right\}dt\geqq 0$$

Because of the partial pseudo-invexity of $\int_I \lambda^T g_{u^2}\,dt$**,** this gives,

$$\int_I \lambda^T g\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)dt\geqq\int_I \lambda^T g\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)dt \tag{6.26}$$

Also from (6.4) together with (6.16), we have,

$$\int_I \left(\eta_4^T\left(t,v^2,\dot{v}^2,y^2,\dot{y}^2\right)+y^2(t)\right)\left(\lambda^T g_{y^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-D\lambda^T g_{\dot{y}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)dt\geqq 0$$

This implies,

$$\int_I \eta_4^T\left(\lambda^T g_{y^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)-D\left(\lambda^T g_{\dot{y}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)dt$$
$$\leqq -\int_I y^2(t)^T\left[\lambda^T g_{y^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-D\left(\lambda^T g_{\dot{y}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)\right]dt$$

This in view of (6.5) yields,

$$\int_I \eta_4^T\left\{\lambda^T g_{y^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-D\left(\lambda^T g_{\dot{y}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)\right\}dt\leqq 0$$

On integrating by parts and using the boundary conditions which at $t=a,\ t=b$ gives $\eta_4=0$, we have,

$$\int_I \eta_4^T\left\{\lambda^T g_{y^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)+\left(D\eta_4\right)^T\left(\lambda^T g_{\dot{y}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)\right\}dt\leqq 0$$

Because of partial pseudoincavity of $\int_I \lambda^T g_{y^2}\,dt$**,** we have,

$$\int_I \left(\lambda^T g\left(t,x^2,\dot{x}^2,v^2,\dot{v}^2\right)\right)dt\leqq\int_I \left(\lambda^T g\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)dt \tag{6.27}$$

From (6.26) and (6.27), we get,

$$\int_I \left(\lambda^T g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\right) dt \geqq \int_I \left(\lambda^T g\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right)\right) dt \tag{6.28}$$

Combining (6.25) and (6.28), we get,

$$\begin{aligned}
&\int_I \Big\{ \lambda^T f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - y^1(t)^T \left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right) \\
&-D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + \lambda^T g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \Big\} dt \\
&\geqq \int_I \Big\{ \lambda^T f\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - u^1(t)^T \left(\lambda^T f_{x^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right) \\
&-D\lambda^T f_{\dot{x}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) + \lambda^T g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \Big\} dt
\end{aligned}$$

This implies,

$$\begin{aligned}
&\lambda^T \int_I \Big\{ f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \\
&-y^1(t)^T \left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right) e \Big\} dt \\
&\quad \geqq \lambda^T \int_I \Big\{ f\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) + g\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right) \\
&-u^1(t)^T \left(\lambda^T f_{x^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{x}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right) e \Big\} dt
\end{aligned}$$

This implies,

$$\begin{aligned}
&\int_I \Big\{ f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \\
&-y^1(t)^T \left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right) e \Big\} dt \\
&\quad \not\leq \int_I \Big\{ f\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) + g\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right) \\
&-u^1(t)^T \left(\lambda^T f_{x^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{x}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right) e \Big\} dt
\end{aligned}$$

This was to be proved.

Theorem 6.2 (*Strong Duality*): Let $\left(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}\right)$ be an efficient solution of (Mix SP). Let $\lambda = \bar{\lambda}$ be fixed in (Mix SD) and

(C₁)
$$\begin{aligned}
&\int_I \Big[\Big\{ \left(\phi^1(t)\right)^T \left(\lambda^T f_{y^1 y^1} - D\lambda^T f_{y^1 \dot{y}^1}\right) - D\phi^1(t)^T \left(-D\lambda^T f_{\dot{y}^1 \dot{y}^1}\right) \\
&\qquad + D^2 \phi^1(t)^T \left(-\lambda^T f_{\dot{y}^1 \dot{y}^1}\right) \Big\} \phi^1(t) \Big] dt > 0,
\end{aligned}$$

and

$$\int_I \Big[\Big\{ \left(\phi^2(t)\right)^T \left(\lambda^T g_{y^2 y^2} - D\lambda^T g_{y^2 \dot{y}^2}\right) - D\phi^2(t)^T \left(-D\lambda^T g_{\dot{y}^2 \dot{y}^2}\right)$$
$$+ D^2 \phi^2(t)^T \left(-\lambda^T g_{\dot{y}^2 \dot{y}^2}\right) \Big\} \phi^2(t) \Big] dt > 0,$$

(C₂)

$$\int_I \Big[\Big\{ \left(\phi^1(t)\right)^T \left(\lambda^T f_{y^1 y^1} - D\lambda^T f_{y^1 \dot{y}^1}\right) - D\phi^1(t)^T \left(-D\lambda^T f_{\dot{y}^1 \dot{y}^1}\right)$$
$$+ D^2 \phi^1(t)^T \left(-\lambda^T f_{\dot{y}^1 \dot{y}^1}\right) \Big\} \phi^1(t) \Big] dt = 0, t \in I \Rightarrow \phi^1(t) = 0,\ t \in I$$

and

$$\int_I \Big[\Big\{ \left(\phi^2(t)\right)^T \left(\lambda^T g_{y^2 y^2} - D\lambda^T g_{y^2 \dot{y}^2}\right) - D\phi^2(t)^T \left(-D\lambda^T g_{\dot{y}^2 \dot{y}^2}\right)$$
$$+ D^2 \phi^2(t)^T \left(-\lambda^T g_{\dot{y}^2 \dot{y}^2}\right) \Big\} \phi^2(t) \Big] dt = 0, t \in I \Rightarrow \phi^1(t) = 0,\ t \in I.$$

(C₃) $g^i_{y^2} - Dg^i_{\dot{y}^2} = 0$, $i = 1, 2, \ldots, p$ are linearly independent.

Let $\int_I f\, dt$ and $\int_I \lambda^T g\, dt$ satisfy the invexity and generalized invexity as stated in Theorem 6.1, then $\left(\bar{x}^1, \bar{x}^2, \bar{y}, \bar{y}^2, \bar{\lambda}\right)$ and $\left(\bar{u}^1, \bar{u}^2, \bar{v}, \bar{v}^2, \bar{\lambda}\right)$ are efficient solution of (Mix SP) and (Mix SD) respectively.

Proof: Since $\left(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}\right)$ is efficient, it is weak minimum. Hence there exists $\tau \in R^p$, $\eta \in R^p, \gamma \in R$ and piecewise smooth functions $\theta^1(t) : I \to R^{|k_1|}$, $\theta^2(t) : I \to R^{|k_2|}$ and $\mu : I \to R^m$ such that the following Fritz-John optimality conditions, in view of the analysis on [59, 104, 116], are satisfied

$$H = \tau(f + g) + \left(\theta^1(t) - \left(\tau^T e\right) y^1(t)^T\right)\left(\lambda^T f_{y^1} - D\lambda^T f_{\dot{y}^1}\right)$$
$$+ \left(\theta^2(t) - \gamma y^2(t)^T\right)\left(\lambda^T g_{y^2} - D\lambda^T g_{\dot{y}^2}\right) + \eta^T \lambda$$

Satisfying

$$H_{x^1} - DH_{\dot{x}^1} + D^2 H_{\ddot{x}^1} = 0\ ,\ t \in I\ , \tag{6.29}$$

$$H_{x^2} - DH_{\dot{x}^2} + D^2 H_{\ddot{x}^2} = 0\ ,\ t \in I\ , \tag{6.30}$$

$$H_{y^1} - DH_{\dot{y}^1} + D^2 H_{\ddot{y}^1} = 0\ ,\ t \in I\ , \tag{6.31}$$

$$H_{y^2} - DH_{\dot{y}^2} + D^2 H_{\ddot{y}^2} = 0\ ,\ t \in I\ , \tag{6.32}$$

$$\left(\theta^1(t) - \left(\tau^T e\right) y^1(t)\right)^T \left(f_{y^1} - Df_{\dot{y}^1}\right)$$

$$+\left(\theta^2(t)-\gamma y^2(t)\right)^T\left(g_{y^2}-Dg_{\dot{y}^2}\right)-\eta=0, t\in I \tag{6.33}$$

$$\theta^1(t)\left(\lambda^T f_{y^1}-D\lambda^T f_{\dot{y}^1}\right)=0,\ t\in I, \tag{6.34}$$

$$\theta^2(t)\left(\lambda^T g_{y^2}-D\lambda^T g_{\dot{y}^2}\right)=0,\ t\in I, \tag{6.35}$$

$$\gamma\int_I y^2(t)^T\left(\lambda^T g_{y^2}-D\lambda^T g_{\dot{y}^2}\right)=0, \tag{6.36}$$

$$\eta^T\bar{\lambda}=0, \tag{6.37}$$

$$\left(\tau,\theta^1(t),\theta^2(t),\eta,\gamma\right)\geqq 0,\ t\in I\ , \tag{6.38}$$

$$\left(\tau,\theta^1(t),\theta^2(t),\eta,\gamma\right)\neq 0,\ t\in I\ , \tag{6.39}$$

hold throughout I (except at the corners of $\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t)\right)$ where (6.29)-(6.32) are valid for unique right and left hand limits). Here θ^1 and θ^2 are continuous except possibly at corner of $\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t)\right)$.

The relations (6.29)-(6.32) are all deducible from the classical Euler-Lagrange and Clebsch necessary optimality conditions. Particularly, the equations (6.29)-(6.32) are the famous Euler-Lagrange differential equation when second order derivatives appear in H.Using the analogies of the observation of $D\phi_{\ddot{y}}$ from the notational section, the equations (6.29)-(6.32) become,

$$\begin{aligned}\tau\left(f_{x^1}-Df_{\dot{x}^1}\right)&-\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1x^1}-D\lambda^T f_{\dot{y}^1x^1}\right)\\&-D\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1\dot{x}^1}-D\lambda^T f_{\dot{y}^1\dot{x}^1}-\lambda^T f_{\dot{y}^1x^1}\right)\\&+D^2\left(\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-\lambda^T f_{\dot{y}^1\dot{x}^1}\right)\right)=0,\end{aligned} \tag{6.40}$$

$$\begin{aligned}\tau\left(g_{x^2}-Dg_{\dot{x}^2}\right)&+\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(\lambda^T g_{y^2x^2}-D\lambda^T g_{\dot{y}^2x^2}\right)\\&-D\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(\lambda^T g_{y^2\dot{x}^2}-D\lambda^T g_{\dot{y}^2\dot{x}^2}-\lambda^T g_{\dot{y}^2x^2}\right)\\&+D^2\left(\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-\lambda^T g_{\dot{y}^2\dot{x}^2}\right)\right)=0,\end{aligned} \tag{6.41}$$

$$\left(\tau-\left(\tau^T e\right)\lambda\right)^T\left(f_{y^1}-Df_{\dot{y}^1}\right)+\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1y^1}-D\lambda^T f_{\dot{y}^1y^1}\right)$$

$$-D\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-D\lambda^T f_{\dot{y}^1\dot{y}^1}\right)+D^2\left(\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-\lambda^T f_{\dot{y}^1\dot{y}^1}\right)\right)=0, \tag{6.42}$$

$$\left(\tau-\gamma\lambda\right)^T\left(g_{y^2}-Dg_{\dot{y}^2}\right)+\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(\lambda^T g_{y^2y^2}-D\lambda^T g_{\dot{y}^2y^2}\right)$$
$$-D\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-D\lambda^T g_{\dot{y}^2\dot{y}^2}\right)+D^2\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-\lambda^T g_{\dot{y}^2\dot{y}^2}\right)=0 \tag{6.43}$$

Since $\lambda>0$, (6.27) implies $\eta=0$. Consequently, (6.33) reduces to

$$\left(\theta^1(t)-\left(\tau^T e\right)y^1(t)\right)^T\left(f_{y^1}-Df_{\dot{y}^1}\right)+\left(\theta^2(t)-\gamma y^2(t)\right)^T\left(g_{y^2}-Dg_{\dot{y}^2}\right)=0, t\in I \tag{6.44}$$

Postmultiplying (6.42) by $\left(\theta^1(t)-\left(\tau^T e\right)y^1(t)\right)$, (6.43) by $\left(\theta^2(t)-\gamma y^2(t)\right)$ and then adding, we have,

$$\Big\{\left(\tau-\left(\tau^T e\right)\lambda\right)^T\left(f_{y^1}-Df_{\dot{y}^1}\right)+\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1y^1}-D\lambda^T f_{\dot{y}^1y^1}\right)$$
$$-D\left[\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-D\lambda^T f_{\dot{y}^1\dot{y}^1}\right)\right]$$
$$+D^2\left[\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-\lambda^T f_{\dot{y}^1\dot{y}^1}\right)\right]\Big\}\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)$$
$$+\Big\{\left(\tau-\gamma\lambda\right)^T\left(g_{y^2}-Dg_{\dot{y}^2}\right)+\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(\lambda^T g_{y^2y^2}-D\lambda^T g_{\dot{y}^2y^2}\right)$$
$$-D\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-D\lambda^T g_{\dot{y}^2\dot{y}^2}\right)\right]$$
$$+D^2\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-\lambda^T g_{\dot{y}^2\dot{y}^2}\right)\right]\Big\}\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)=0 \tag{6.45}$$

Now multiplying (6.44) by $\bar{\lambda}$ and then using (6.35) and (6.36) we have,

$$\int_I\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1}-D\lambda^T f_{\dot{y}^1}\right)dt=0$$

that is,

$$\int_I\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1}-D\lambda^T f_{\dot{y}^1}\right)\left(\tau^T e\right)dt=0 \tag{6.46}$$

Multiplying (6.44) by τ, we have,

$$\int_I\left[\left(\theta^1(t)-\left(\tau^T e\right)y^1(t)\right)^T\left(\tau f_{y^1}-D\tau f_{\dot{y}^1}\right)+\left(\theta^2(t)-\gamma y^2(t)\right)^T\left(\tau g_{y^2}-D\tau g_{\dot{y}^2}\right)\right]dt=0 \tag{6.47}$$

Subtracting (6.45) and (6.47) and using (6.35) and (6.36), we have

$$\int_I \Big[\left(\theta^1(t)-\left(\tau^T e\right)y^1(t)\right)^T\left(f_{y^1}-Df_{\dot{y}^1}\right)\left(\tau-\left(\tau^T e\right)\bar{\lambda}\right)$$
$$+\left(\theta^2(t)-\gamma y^2(t)\right)^T\left(\tau g_{y^2}-D\tau g_{\dot{y}^2}\right)\left(\tau-\gamma\bar{\lambda}\right)\Big]dt=0 \qquad (6.48)$$

From (6.45) and (6.48), we obtain,

$$\int_I \Big[\Big\{\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1y^1}-D\lambda^T f_{y^1\dot{y}^1}\right)$$
$$-D\left[\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-D\lambda^T f_{\dot{y}^1\dot{y}^1}\right)\right]$$
$$+D^2\left[\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-\lambda^T f_{\dot{y}^1\dot{y}^1}\right)\right]\Big\}\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)\Big]dt$$
$$+\int_I \Big[\Big\{\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(\lambda^T g_{y^2y^2}-D\lambda^T g_{y^2\dot{y}^2}\right)$$
$$-D\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-D\lambda^T g_{\dot{y}^2\dot{y}^2}\right)\right]$$
$$+D^2\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-\lambda^T g_{\dot{y}^2\dot{y}^2}\right)\right]\Big\}\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)\Big]dt=0$$

In view of the hypothesis (C_1), we have

$$\int_I \Big[\Big\{\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(\lambda^T f_{y^1y^1}-D\lambda^T f_{y^1\dot{y}^1}\right)$$
$$-D\left[\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-D\lambda^T f_{\dot{y}^1\dot{y}^1}\right)\right]$$
$$+D^2\left[\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)^T\left(-\lambda^T f_{\dot{y}^1\dot{y}^1}\right)\right]\Big\}\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)\Big]dt=0$$

and

$$\int_I \Big[\Big\{\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(\lambda^T g_{y^2y^2}-D\lambda^T g_{y^2\dot{y}^2}\right)$$
$$-D\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-D\lambda^T g_{\dot{y}^2\dot{y}^2}\right)\right]$$
$$+D^2\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\left(-\lambda^T g_{\dot{y}^2\dot{y}^2}\right)\right]\Big\}\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)\Big]dt=0$$

This in view of the hypothesis (C_2) yields,

$$\phi^1(t)=\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)=0,\ t\in I \qquad (6.49)$$

$$\phi^2(t)=\theta^2(t)-\gamma\bar{y}^2(t)=0,\ t\in I \qquad (6.50)$$

From (6.43) and (6.50), we have,

$$\left(\tau-\gamma\lambda\right)^T\left(g_{y^2}-Dg_{\dot{y}^2}\right)=0$$

that is,

$$\sum_{i=1}^{p}\left(\tau^{i}-\gamma\lambda^{i}\right)^{T}\left(g_{y^{2}}-Dg_{\dot{y}^{2}}\right)=0$$

This in view of the (C_3) yields,

$$\tau^{i}=\gamma\lambda^{i}\ ,\ i=1,2,\ldots,p \tag{6.51}$$

Let if possible, $\gamma=0$. Then from (6.51), we have $\tau=0$ and therefore, from (6.49) and (6.50), we have,

$$\phi^{1}(t)=0,\theta^{2}(t)=0,\ t\in I$$

Hence $\left(\tau,\theta^{1}(t),\theta^{2}(t),\eta,\gamma\right)=0$, contradicting Fritz-John conditions (6.39). Hence $\gamma>0$ and consequently $\tau>0$.

From (6.40) and (6.41) along with (6.51), we obtain

$$\left(\bar{\lambda}^{T}f_{x^{1}}-D\bar{\lambda}^{T}f_{\dot{x}^{1}}\right)=0\ ,\ t\in I \tag{6.52}$$

$$\left(\bar{\lambda}^{T}g_{x^{2}}-D\bar{\lambda}^{T}g_{\dot{x}^{2}}\right)=0\ ,\ t\in I \tag{6.53}$$

which implies

$$\int_{I}x^{2}(t)^{T}\left(\bar{\lambda}^{T}g_{x^{2}}-D\bar{\lambda}^{T}g_{\dot{x}^{2}}\right)dt=0 \tag{6.54}$$

From (6.52)-(6.54) together with (6.49), we have

$$y^{1}(t)^{T}\left(\bar{\lambda}^{T}f_{y^{1}}-D\bar{\lambda}^{T}f_{\dot{y}^{1}}\right)=0\ ,\ t\in I \tag{6.55}$$

From the primal objective with (6.55)

$$\begin{aligned}&\int_{I}\left\{f\left(t,x^{1},\dot{x}^{1},y^{1},\dot{y}^{1}\right)+g\left(t,x^{2},\dot{x}^{2},y^{2},\dot{y}^{2}\right)-y^{1}(t)^{T}\left(\lambda^{T}f_{y^{1}}-D\lambda^{T}f_{\dot{y}^{1}}\right)\right\}dt\\&\qquad=\int_{I}\left\{f\left(t,x^{1},\dot{x}^{1},y^{1},\dot{y}^{1}\right)+g\left(t,x^{2},\dot{x}^{2},y^{2},\dot{y}^{2}\right)\right\}dt\end{aligned} \tag{6.56}$$

From the dual objective in view of (6.52), we have

$$\begin{aligned}&\int_{I}\left\{f\left(t,x^{1},\dot{x}^{1},y^{1},\dot{y}^{1}\right)+g\left(t,x^{2},\dot{x}^{2},y^{2},\dot{y}^{2}\right)-x^{1}(t)^{T}\left(\lambda^{T}f_{x^{1}}-D\lambda^{T}f_{\dot{x}^{1}}\right)\right\}dt\\&\qquad=\int_{I}\left\{f\left(t,x^{1},\dot{x}^{1},y^{1},\dot{y}^{1}\right)+g\left(t,x^{2},\dot{x}^{2},y^{2},\dot{y}^{2}\right)\right\}dt\end{aligned} \tag{6.57}$$

From (6.65) and (6.57), the equality of objective values is evident. Consequently, in view of the hypothesis of Theorem 6.1, the efficiency of $\left(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda}\right)$ follows.

We now state converse duality whose proof follows by symmetry.

Theorem 6.3 (*Converse Duality*): Let $\left(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda}\right)$ be an efficient solution of (Mix SP). Let $\lambda=\bar{\lambda}$ be fixed in (Mix SD) and

(A$_1$)
$$\int_I \Big[\Big\{\psi^1(t)^T\left(\lambda^T f_{x^1x^1}-D\lambda^T f_{x^1\dot{x}^1}\right)-D\psi^1(t)^T\left(-D\lambda^T f_{\dot{x}^1\dot{x}^1}\right) +D^2\psi^1(t)^T\left(-\lambda^T f_{\dot{x}^1\dot{x}^1}\right)\Big\}\psi^1(t)\Big]dt>0,$$

and

$$\int_I \Big[\Big\{\psi^2(t)^T\left(\lambda^T g_{x^2x^2}-D\lambda^T g_{x^2\dot{x}^2}\right)-D\psi^2(t)^T\left(-D\lambda^T g_{\dot{x}^2\dot{x}^2}\right) +D^2\psi^2(t)^T\left(-\lambda^T g_{\dot{x}^2\dot{x}^2}\right)\Big\}\psi^2(t)\Big]dt>0,$$

(A$_2$)
$$\int_I \Big[\Big\{\psi^1(t)^T\left(\lambda^T f_{x^1x^1}-D\lambda^T f_{x^1\dot{x}^1}\right)-D\psi^1(t)^T\left(-D\lambda^T f_{\dot{x}^1\dot{x}^1}\right) +D^2\psi^1(t)^T\left(-\lambda^T f_{\dot{x}^1\dot{x}^1}\right)\Big\}\psi^1(t)\Big]dt=0, t\in I\Rightarrow\psi^1(t)=0,\ t\in I,$$

and

$$\int_I \Big[\Big\{\psi^2(t)^T\left(\lambda^T g_{x^2x^2}-D\lambda^T g_{x^2\dot{x}^2}\right)-D\psi^2(t)^T\left(-D\lambda^T g_{\dot{x}^2\dot{x}^2}\right) +D^2\psi^2(t)^T\left(-\lambda^T g_{\dot{x}^2\dot{x}^2}\right)\Big\}\psi^2(t)\Big]dt=0, t\in I\Rightarrow\psi^2(t)=0,\ t\in I$$

and

(A$_3$) $g^i_{x^2}-Dg^i_{\dot{x}^2}=0$, $i=1,2,\ldots,p$ are linearly independent.

Let $\int_I f\,dt$ and $\int_I \lambda^T g\,dt$ satisfy the invexity and generalized invexity as stated in Theorem 6.1, then $\left(\bar{x}^1,\bar{x}^2,\bar{y},\bar{y}^2,\bar{\lambda}\right)$ and $\left(\bar{u}^1,\bar{u}^2,\bar{v},\bar{v}^2,\bar{\lambda}\right)$ are efficient solution of (Mix SP) and (Mix SD) respectively.

6.1.5 Self Duality

A problem is said to be self-dual if it is formally identical with its dual, in general, the problems (Mix SP) and (Mix SD) are not formally in the absence of an additional restrictions of the function f and g. Hence skew symmetry of f and g is assumed in order to validate the following self-duality theorem.

Theorem 6.4. (*Self Duality*): Let f^i and g^i, $i = 1, 2, \ldots, p$, be skew symmetric. Then the problem (Mix SP) is self dual. If the problems (Mix SP) and (Mix SD) are dual problems and $\left(\bar{x}^1(t), \bar{x}^2(t), \bar{y}(t), \bar{y}^2(t), \bar{\lambda}\right)$ is a joint optimal solution of (Mix SP) and (Mix SD), then so is $\left(\bar{y}(t), \bar{y}^2(t), \bar{x}^1(t), \bar{x}^2(t), \bar{\lambda}\right)$**,** and the common functional value is zero, i.e.

Minimum **(Mix SP)** = $\int_I \left\{ f\left(x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g\left(x^2, \dot{x}^2, y^2, \dot{y}^2\right) \right\} dt = 0$

Proof: By skew symmetric of f^i and g^i, we have

$$f^i_{x^1}\left(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)\right) = -f^i_{y^1}\left(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t)\right)$$

$$g^i_{x^2}\left(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)\right) = -g^i_{\dot{y}^2}\left(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t)\right)$$

$$f^i_{y^1}\left(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)\right) = -f^i_{x^1}\left(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t)\right)$$

$$g^i_{y^2}\left(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)\right) = -g^i_{\dot{x}^2}\left(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t)\right)$$

$$f^i_{x^1}\left(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)\right) = -f^i_{\dot{y}^1}\left(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t)\right)$$

$$g^i_{\dot{x}^2}\left(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)\right) = -g^i_{\dot{y}^2}\left(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t)\right)$$

$$f^i_{y^1}\left(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)\right) = -f^i_{\dot{x}^1}\left(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t)\right)$$

$$g^i_{\dot{y}^2}\left(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)\right) = -g^i_{\dot{x}^2}\left(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t)\right)$$

Recasting the dual problem (Mix SD) as a minimization problem and using the above relations, we have

(Mix SD1): Minimize $-\int_I \left\{ f\left(t, y^1, \dot{y}^1, x^1, \dot{x}^1\right) + g\left(t, y^2, \dot{y}^2, x^2, \dot{x}^2\right)\right.$

$$-x^1(t)^T\left(\lambda^T f_{x^1}\left(t,y^1,\dot{y}^1,x^1,\dot{x}^1\right)\right.$$
$$\left.\left.-D\lambda^T f_{\dot{x}^1}\left(t,y^1,\dot{y}^1,x^1,\dot{x}^1\right)\right)e\right\}dt$$

Subject to

$$x^1(a)=0=x^1(b)\;,\quad y^1(a)=0=y^1(b)$$

$$x^2(a)=0=x^2(b)\;,\quad y^2(a)=0=y^2(b)$$

$$\lambda^T f_{x^1}\left(t,y^1,\dot{y}^1,x^1,\dot{x}^1\right)-D\lambda^T f_{\dot{x}^1}\left(t,y^1,\dot{y}^1,x^1,\dot{x}^1\right)\leqq 0\;,\; t\in I$$

$$\lambda^T g_{x^2}\left(t,y^2,\dot{y}^2,x^2,\dot{x}^2\right)-D\lambda^T g_{\dot{x}^2}\left(t,y^2,\dot{y}^2,x^2,\dot{x}^2\right)\leqq 0\;,\; t\in I$$

$$\int_I x^2(t)^T\left(\lambda^T g_{x^2}\left(t,y^2,\dot{y}^2,x^2,\dot{x}^2\right)\right.$$
$$\left.-D\lambda^T g_{\dot{x}^1}\left(t,y^2,\dot{y}^2,x^2,\dot{x}^2\right)\right)dt\geqq 0$$

$$\lambda\in\Lambda^+$$

This shows that the problem (Mix SD_1) is just the primal problem (Mix SP). Therefore, $\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t),\bar{\lambda}\right)$ is an optimal solution of (Mix SD) implies that $\left(\bar{y}^1(t),\bar{y}^2(t),\bar{x}^1(t),\bar{x}^2(t),\bar{\lambda}\right)$ is an optimal solution for (Mix SP), and by symmetric duality also for (Mix SD).

Now from (6.55),

Minimum **(Mix SP)** = $\int_I\left\{f\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+g\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right\}dt$

Correspondingly with the solution $\left(\bar{y}(t),\bar{y}^2(t),\bar{x}^1(t),\bar{x}^2(t),\bar{\lambda}\right)$**,** we have,

Minimum **(Mix SP)** = $\int_I\left\{f\left(t,y^1,\dot{y}^1,x^1,\dot{x}^1\right)+g\left(t,y^2,\dot{y}^2,x^2,\dot{x}^2\right)\right\}dt$

By the skew symmetric of f^i and g^i, we have,

Minimum **(Mix SP)** = $\int_I\left\{f\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+g\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right\}dt$

$$=\int_I\left\{f\left(t,y^1,\dot{y}^1,x^1,\dot{x}^1\right)+g\left(t,y^2,\dot{y}^2,x^2,\dot{x}^2\right)\right\}dt$$

$$=-\int_I\left\{f\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+g\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right\}dt$$

this yields,

$$\text{Minimum } \textbf{(Mix SP)} = \int_I \left\{ f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \right\} dt = 0$$

This accomplishes the proof of the theorem.

6.1.6 Natural Boundary Values

The pair of mixed symmetric multiobjective variational problem with natural boundary values rather than fixed points may be formulated as,

Primal (Mix SP$_0$):

$$\begin{aligned} \text{Minimize } \int_I \Big\{ & f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \\ & - y^1(t)^T \left(\lambda^T f_{y^1}\left(x^1, \dot{x}^1, y^1, \dot{y}^1\right) \right. \\ & \left. - D\lambda^T f_{\dot{y}^1}\left(x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right) e \Big\} dt \end{aligned}$$

Subject to

$$\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \leqq 0,$$

$$\lambda^T g_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) - D\lambda^T g_{\dot{y}^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \leqq 0,$$

$$\begin{aligned} \int_I y^2(t)^T & \left(\lambda^T g_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \right. \\ & \left. - D\lambda^T g_{\dot{y}^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\right) \geqq 0, \end{aligned}$$

$$\lambda^T f_{y^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\Big|_{t=a} = 0, \quad \lambda^T f_{\dot{y}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\Big|_{t=b} = 0,$$

$$\lambda^T g_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\Big|_{t=a} = 0, \quad \lambda^T g_{\dot{y}^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\Big|_{t=b} = 0,$$

$$\lambda \in \Lambda^+.$$

Dual (Mix SD$_0$):

$$\begin{aligned} \text{Maximize } \int_I \Big\{ & f\left(u^1, \dot{u}^1, v^1, \dot{v}^1\right) + g\left(u^2, \dot{u}^2, v^2, \dot{v}^2\right) \\ & - u^1(t)^T \left(\lambda^T f_{u^1}\left(u^1, \dot{u}^1, v^1, \dot{v}^1\right) \right. \\ & \left. - D\lambda^T f_{\dot{u}^1}\left(u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right) e \Big\} dt \end{aligned}$$

Subject to

$$\lambda^T f_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \geqq 0,$$

$$\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-D\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\geqq 0,$$

$$\int_I u^2(t)^T\left(\lambda^T g_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right.$$
$$\left.-D\lambda^T g_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right)dt\leqq 0,$$

$$\lambda^T f_{\dot{x}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\Big|_{t=a}=0\ ,\ \lambda^T f_{\dot{x}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\Big|_{t=b}=0\ ,$$

$$\lambda^T g_{\dot{x}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\Big|_{t=a}=0\ ,\ \lambda^T g_{\dot{x}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\Big|_{t=b}=0,$$

$$\lambda\in\Lambda^+$$

For these problems, Theorem 6.1- 6.3 will remain true except that some slight modifications in the arguments for these theorems are to be indicated.

6.1.7 Nonlinear Programming

If the time dependency of (Mix SP) and (Mix SD) is removed and $b-a=1$, we obtain following pair of static mixed type multiobjective dual problems studied by Bector, Chandra and Abha [12].

Primal (Mix SP$_1$):

Minimize $f\left(x^1,y^1\right)+g\left(x^2,y^2\right)-\left(y^1\right)^T\left(\lambda^T f_{y^1}\left(x^1,y^1\right)\right)$

Subject to

$$\lambda^T f_{y^1}\left(x^1,y^1\right)\leqq 0\ ,$$

$$\lambda^T g_{y^2}\left(x^2,y^2\right)\leqq 0,$$

$$y^2(t)^T\left(\lambda^T g_{y^2}\left(x^2,y^2\right)\right)\geqq 0,$$

$$\lambda\in\Lambda^+.$$

Dual (Mix SD1):

Maximize $f\left(u^1,v^1\right)+g\left(u^2,v^2\right)-u^1(t)^T\left(\lambda^T f_{u^1}\left(u^1,v^1\right)\right)$

Subject to

$$\lambda^T f_{u^1}\left(u^1,v^1\right)\geqq 0\ ,$$

$$\lambda^T g_{u^2}\left(u^2,v^2\right)\geqq 0,$$

$$\left(u^2\right)^T\left(\lambda^T g_{u^2}\left(u^2,v^2\right) \leqq 0\right. ,$$

$$\lambda \in \Lambda^+ .$$

Chapter-7

SYMMETRIC DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS CONTAINING SUPPORT FUNCTIONS

7.1 SYMMETRIC DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS

7.1.1 Introductory Remarks

This chapter is comprises of two main sections, 7.1 and 7.2. The purpose of the section 7.1 is to present pairs of Wolfe and Mond-Weir type symmetric dual multiobjective variational problems containing support functions in order to extend the results of the chapter 6 to nondifferentiable cases and hence study symmetric and self duality for these pairs of nondifferentiable multiobjective variational problems. The problems with natural boundary values are formulated in the subsection 7.1.5 and it is also pointed out that our results can be considered as dynamic generalizations of corresponding (static) symmetric duality results of multiobjective nonlinear programming problems involving support functions. The section 7.2 of this chapter deals with the unification of the formulations of the pairs of Wolfe and Mond-Weir type symmetric dual multiobjective variational problems involving support functions treated in section 7.1 and study symmetric and self duality for these pairs of nondifferentiable dual variational problems under appropriate convexity assumptions. Our duality results reported in this research extends the results of chapter 6 to nondifferentiable setting by introducing support functions.

7.1.2 Wolfe Type Symmetric Duality

In this section, we present the following pair of Wolfe type symmetric dual multiobjective variational problems containing support functions:

(SWP): Minimize: $\int_I \left(H^1, H^2, ..., H^p\right) dt$

Subject to:

$$x(a) = 0 = x(b), \tag{7.1}$$

$$y(a) = 0 = y(b), \tag{7.2}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_y^i(t, x, \dot{x}, y, \dot{y}) - z^i(t) - Df_{\dot{y}}^i(t, x, \dot{x}, y, \dot{y}) \right) \leqq 0 \text{ , } t \in I \tag{7.3}$$

$$z^i(t) \in C^i \text{ , } i = 1, ..., p, \quad t \in I \tag{7.4}$$

$$x(t) \geqq 0 \text{ , } t \in I \tag{7.5}$$

$$\lambda \in \Lambda^+ = \left\{ \lambda \in R^p \,\middle|\, \lambda > 0,\ \lambda^T e = 1, e = (1, 1, \ldots, 1)^T \in R^p \right\} \tag{7.6}$$

where,

1. $H^i = f^i(t, x, \dot{x}, y, \dot{y}) + s\left(x(t) \middle| C^i\right)$

$$- y(t)^T \sum_{i=1}^{p} \lambda^i \left(f_y^i(t, x, \dot{x}, y, \dot{y}) + z^i(t) - Df_{\dot{y}}^i(t, x, \dot{x}, y, \dot{y}) \right) - y(t)^T z(t)$$

2. $f^i : I \times R^n \times R^n \to R, (i = 1, 2, \ldots, p)$, is continuously differentiable function.

(SWD): Maximize: $\int_I \left(G^1, G^2, ..., G^p\right) dt$

Subject to:

$$u(a) = 0 = u(b), \tag{7.7}$$

$$v(a) = 0 = v(b), \tag{7.8}$$

$$\sum_{i=1}^{p} \lambda^i \left(f_u^i(t, u, \dot{u}, v, \dot{v}) + \omega^i(t) - Df_{\dot{u}}^i(t, u, \dot{u}, v, \dot{v}) \right) \geqq 0 \text{ , } t \in I \text{ ,} \tag{7.9}$$

$$\omega^i(t) \in K^i \ , i = 1, \dots, p \quad , t \in I\ , \tag{7.10}$$

$$v(t) \geqq 0\ ,\ t \in I\ , \tag{7.11}$$

$$\lambda \in \Lambda^+\ . \tag{7.12}$$

where,

$$G^i = f^i(t,u,\dot{u},v,\dot{v}) + s\left(v(t)\middle|K^i\right)$$
$$-u(t)^T \sum_{i=1}^{p} \lambda^i \left(f_u^i(t,u,\dot{u},v,\dot{v}) + \omega^i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right) - x(t)^T \omega(t)$$

We shall prove various duality results under convexity-concavity assumptions.

Theorem 7.1 (*Weak Duality*): Let $\left(x(t), y(t), z_1(t), \dots, z_p(t), \lambda\right)$ be feasible for the (SWP) and $\left(u(t), v(t), \omega_1(t), \dots, \omega_p(t), \lambda\right)$ be feasible for the dual (SWD). Assume that for each i, $\int_I f^i(t,.,.,y,\dot{y})dt$ is convex in $(x,\dot{x})$ for fixed $(y,\dot{y})$ and $\int_I f^i(t,x,\dot{x},.,.)dt$ is concave in $(y,\dot{y})$ for fixed $(x,\dot{x})$. Then,

$$\int_I H dt \not\leq \int_I G dt$$

where,

$$H = \left(H^1, H^2, \dots, H^p\right) \text{ and } G = \left(G^1, G^2, \dots, G^p\right).$$

Proof: Using the convexity of $\int_I f^i(t,.,.,y,\dot{y})dt$ in $(x,\dot{x})$ for fixed $(y,\dot{y})$, we have

$$\int_I f^i(t,x,\dot{x},v,\dot{v})dt - \int_I f^i(t,u,\dot{u},v,\dot{v})dt$$
$$\geq \int_I \left[\left(x(t)-u(t)\right)^T f_u^i(t,u,\dot{u},v,\dot{v}) - \left(\dot{x}(t)-\dot{u}(t)\right)^T f_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right]dt$$
$$= \int_I \left[\left(x(t)-u(t)\right)^T \left\{ f_u^i(t,u,\dot{u},v,\dot{v}) - f_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right\}\right]dt$$
$$+\left(x(t)-u(t)\right)^T f_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\Big|_{t=b}^{t=a}.$$

This on using (7.1) and (7.7) gives,

$$\int_I f^i(t,x,\dot{x},v,\dot{v})\,dt-\int_I f^i(t,u,\dot{u},v,\dot{v})\,dt$$

$$\geqq \int_I\Big[\big(x(t)-u(t)\big)^T\left\{f_u^i(t,u,\dot{u},v,\dot{v})-f_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right\}\Big]dt\,. \tag{7.13}$$

Also by concavity of $\int_I f^i(t,x,\dot{x},.,.)\,dt$, we have

$$-\int_I f^i(t,x,\dot{x},v,\dot{v})\,dt-\int_I f^i(t,x,\dot{x},y,\dot{y})\,dt$$

$$\geqq -\int_I\Big[\big(v(t)-y(t)\big)^T f_y^i(t,x,\dot{x},y,\dot{y})-\big(\dot{v}(t)-\dot{y}(t)\big)^T f_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\Big]dt$$

$$=-\int_I\Big[\big(v(t)-y(t)\big)^T\left\{f_y^i(t,x,\dot{x},y,\dot{y})-f_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}\Big]dt$$

$$+\big(v(t)-y(t)\big)^T f_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\Big|_{t=b}^{t=a}$$

which by using (7.2) and (7.8) we have,

$$-\int_I f^i(t,x,\dot{x},v,\dot{v})\,dt-\int_I f^i(t,x,\dot{x},y,\dot{y})\,dt$$

$$\geqq -\int_I\Big[\big(v(t)-y(t)\big)^T\left\{f_y^i(t,x,\dot{x},y,\dot{y})-f_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}\Big]dt \tag{7.14}$$

Adding (7.13) and (7.14) we have,

$$\int_I f^i(t,x,\dot{x},y,\dot{y})\,dt-\int_I f^i(t,u,\dot{u},v,\dot{v})\,dt$$

$$\geq \int_I\Big[\big(x(t)-u(t)\big)^T\left\{f_u^i(t,u,\dot{u},v,\dot{v})-Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right\}$$

$$-\big(v(t)-y(t)\big)^T\left\{f_y^i(t,x,\dot{x},y,\dot{y})-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}\Big]dt.$$

$$=\int_I\Big[\big(x(t)\big)^T\left\{f_u^i(t,u,\dot{u},v,\dot{v})-Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right\}$$

$$-\big(u(t)\big)^T\left\{f_u^i(t,u,\dot{u},v,\dot{v})-Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right\}$$

$$-\big(v(t)\big)^T\left\{f_y^i(t,x,\dot{x},y,\dot{y})-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}$$

$$+\big(y(t)\big)^T\left\{f_y^i(t,x,\dot{x},y,\dot{y})-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}\Big]dt.$$

Multiplying this by λ^i and summing over i , $i = 1, 2, .., p$, we get,

$$\sum_{i=1}^{p} \lambda^i \int_I f^i\left(t, x, \dot{x}, y, \dot{y}\right) dt - \sum_{i=1}^{p} \lambda^i \int_I f^i\left(t, u, \dot{u}, v, \dot{v}\right) dt$$

$$\geqq \int_I \Big[\left(x(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_u^i\left(t, u, \dot{u}, v, \dot{v}\right) - Df_{\dot{u}}^i\left(t, u, \dot{u}, v, \dot{v}\right) \right\}$$

$$- \left(u(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_u^i\left(t, u, \dot{u}, v, \dot{v}\right) - Df_{\dot{u}}^i\left(t, u, \dot{u}, v, \dot{v}\right) \right\}$$

$$- \left(v(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right\}$$

$$+ \left(y(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right\} \Big] dt$$

$$= \int_I \Big[\left(x(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_u^i\left(t, u, \dot{u}, v, \dot{v}\right) + \omega_i(t) - Df_{\dot{u}}^i\left(t, u, \dot{u}, v, \dot{v}\right) \right\}$$

$$- \left(u(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_u^i\left(t, u, \dot{u}, v, \dot{v}\right) + \omega_i(t) - Df_{\dot{u}}^i\left(t, u, \dot{u}, v, \dot{v}\right) \right\}$$

$$- \sum_{i=1}^{p} \lambda^i x(t) \omega_i(t) + \sum_{i=1}^{p} \lambda^i u(t) \omega_i(t)$$

$$- \left(v(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - z_i(t) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right\}$$

$$+ \left(y(t)\right)^T \sum_{i=1}^{p} \lambda^i \left\{ f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - z_i(t) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right\}$$

$$- \sum_{i=1}^{p} \lambda^i v(t) z_i(t) + \sum_{i=1}^{p} \lambda^i y(t) z_i(t) \Big] dt.$$

Using (7.3), (7.5), (7.9) and (7.11), we have,

$$\sum_{i=1}^{p} \lambda^i \int_I \Big[f^i\left(t, x, \dot{x}, y, \dot{y}\right) - \left(y(t)\right)^T \sum_{i=1}^{p} \lambda_i \left\{ f_y^i\left(t, x, \dot{x}, y, \dot{y}\right) - z_i(t) - Df_{\dot{y}}^i\left(t, x, \dot{x}, y, \dot{y}\right) \right\}$$

$$+ \sum_{i=1}^{p} \lambda_i \left(x(t) \omega_i(t)\right) - \sum_{i=1}^{p} \lambda_i \left(y(t) z_i(t)\right)$$

$$\geqq \sum_{i=1}^{p} \lambda_i \int_I \Big[f^i\left(t, u, \dot{u}, v, \dot{v}\right) - \left(u(t)\right)^T \sum_{i=1}^{p} \lambda_i \left\{ f_u^i\left(t, u, \dot{u}, v, \dot{v}\right) + \omega_i(t) \right.$$

$$\left. - Df_{\dot{u}}^i\left(t, u, \dot{u}, v, \dot{v}\right) \right\} + u(t) \omega_i(t) - y(t) z_i(t) \Big] dt.$$

In view of $s\left(x(t) \middle| C^i\right) \geqq x(t)^T \omega^i(t)$, $i = 1, ..., p$ and $s\left(v(t) \middle| K^i\right) \geqq \left(v(t)\right)^T z^i(t)$, $i = 1, ..., p$, this yields,

$$\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i(t,x,\dot{x},y,\dot{y})+s\big(x(t)\big|C_i\big)$$

$$-\big(y(t)\big)^T\sum_{i=1}^{p}\lambda_i\left\{f_y^i(t,x,\dot{x},y,\dot{y})-z_i(t)-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}-\big(y(t)\big)^T z_i(t)\Big]dt$$

$$\geqq\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i(t,u,\dot{u},v,\dot{v})-s\big(v(t)\big|K_i(t)\big)$$

$$+\big(y(t)\big)^T\sum_{i=1}^{p}\lambda^i\left\{f_y^i(t,x,\dot{x},y,\dot{y})-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right\}$$

That is,

$$\sum_{i=1}^{p}\lambda_i\int_I H^i dt\geqq\sum_{i=1}^{p}\lambda_i\int_I G^i dt\,.$$

This yields,

$$\int_I Hdt\not\leq\int_I Gdt\,.$$

Theorem 7.2 (*Strong Duality*): Let $\left(x(t),y(t),z_1(t),...,z_p(t),\lambda\right)$ be an efficient solution of (SWP) and $\lambda=\bar{\lambda}$ be fixed in (SWD). Furthermore, assume that

(C_1): $$\Big\{\big(\phi(t)\big)^T\left(\lambda^T f_{yy}(t,x,\dot{x},y,\dot{y})-D\lambda^T f_{\dot{y}y}(t,x,\dot{x},y,\dot{y})\right)$$
$$-D\Big[\big(\phi(t)\big)^T\left(-D\lambda^T f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\Big]$$
$$+D^2\Big[\big(\phi(t)\big)^T\left(-f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\Big]\Big\}\big(\phi(t)\big)=0,t\in I\Rightarrow\phi(t)=0,t\in I$$

(C_2): $f_y^i(t,x,\dot{x},y,\dot{y})-z^i(t)-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})$, $i=1,...,p$, $t\in I$ are linearly independent.

Then $\left(x(t),y(t),\omega_1(t),...,\omega_p(t),\lambda\right)$ is feasible for (SWD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 7.1 hold, then there exists $\omega_1(t),\omega_2(t),...,\omega_p(t)$ such that $\left(u(t),v(t),\lambda,\omega_1(t),...,\omega_p(t)\right)$ $=\left(x(t),y(t),\lambda,\omega_1(t),...,\omega_p(t)\right)$ is an efficient solution of the dual (SWD).

Proof: Since $\left(x(t),y(t),z_1(t),...,z_p(t),\lambda\right)$ is efficient for the problem (SWP), it is weak minimum [35]. Hence there exists $\tau\in R^p$, $\eta\in R^m$, $\gamma\in R$,

$\theta(t): I \to R^n$ and $\alpha(t) \in R^n$ such that the following Fritz-John optimality conditions, are satisfied

$$\sum_{i=1}^{p} \tau^i \left(f_x^i(t,x,\dot{x},y,\dot{y}) + \omega^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},y,\dot{y}) \right)$$
$$+\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\left(\lambda^T f_{yx}^i(t,x,\dot{x},y,\dot{y}) - D\lambda^T f_{\dot{y}x}^i(t,x,\dot{x},y,\dot{y})\right)$$
$$+D\left[\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda_i\left(f_{y\dot{x}}^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}\dot{x}}^i(t,x,\dot{x},y,\dot{y})\right.\right.$$
$$\left.\left. - f_{\dot{y}x}^i(t,x,\dot{x},y,\dot{y})\right)\right] + D^2\left[\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda_i\left(f_{\dot{y}\dot{x}}^i(t,x,\dot{x},y,\dot{y})\right)\right] = 0,\ t \in I, \tag{7.15}$$

$$\sum_{i=1}^{p}\left(\tau^i - \left(\tau^T e\right)\lambda^i\right)\left(f_y^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right)$$
$$+\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(f_{yy}^i(t,x,\dot{x},y,\dot{y}) - Df_{\dot{y}y}^i(t,x,\dot{x},y,\dot{y})\right)$$
$$-D\left[\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(-Df_{\dot{y}\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right)\right]$$
$$+D^2\left[\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T \sum_{i=1}^{p}\lambda^i\left(-f_{\dot{y}\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right)\right] = 0,\ t \in I, \tag{7.16}$$

$$\left(\theta(t)-\left(\tau^T e\right)y(t)\right)\left(f_y^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right) - \eta = 0,\ t \in I, \tag{7.17}$$

$$x^T(t) z^i(t) = s\left(x(t)\middle| C^i\right),\ t \in I \tag{7.18}$$

$$\theta(t)\sum_{i=1}^{p}\left(f_x^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})\right) = 0,\ t \in I, \tag{7.19}$$

$$\tau^i y(t) - \left(\theta(t) - \left(\tau^T e\right)\lambda^i\right) \in N_{C^i}\left(z^i(t)\right),\ t \in I, \tag{7.20}$$

$$x^T(t)\,\alpha(t) = 0,\ t \in I \tag{7.21}$$

$$\eta^T \lambda = 0 \tag{7.22}$$

$$\left(\tau, \theta(t), \alpha(t), \eta\right) \geqq 0,\ t \in I, \tag{7.23}$$

$$\left(\tau, \theta(t), \alpha(t), \eta\right) \neq 0,\ t \in I. \tag{7.24}$$

Since $\lambda > 0$, (7.22) implies $\eta = 0$. Consequently, (7.17) reduces to

$$\left(\theta(t)-\left(\tau^T e\right)y(t)\right)\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z_i\left(t\right)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)=0, t\in I, \tag{7.25}$$

Post multiplying (7.16) by $\left(\theta(t)-\left(\tau^T e\right)y(t)\right)$, we get,

$$\begin{aligned}&\left\{\sum_{i=1}^{p}\left(\tau^i-\left(\tau^T e\right)\lambda^i\right)\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i\left(t\right)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)\right.\\&+\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\lambda^i\left(f_{yy}^i\left(t,x,\dot{x},y,\dot{y}\right)-Df_{\dot{y}y}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)\\&-D\left(\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\lambda^i\left(-Df_{\dot{y}\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)\right)\\&\left.+D^2\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\lambda^i\left(-f_{\dot{y}\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)\right\}\left(\theta(t)-\left(\tau^T e\right)y(t)\right)=0,\end{aligned} \tag{7.26}$$

Premultiplying (7.25) by $\left(\tau^i-\left(\tau^T e\right)\lambda^i\right)$ and summing over i, we have

$$\sum_{i=1}^{p}\left(\tau^i-\left(\tau^T e\right)\lambda^i\right)\left(\theta(t)-\left(\tau^T e\right)y(t)\right)\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z_i\left(t\right)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)=0, t\in I \tag{7.27}$$

Using (7.27) in (7.26) we have,

$$\begin{aligned}&\left\{\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\left(\lambda^T f_{yy}\left(t,x,\dot{x},y,\dot{y}\right)-D\lambda^T f_{\dot{y}y}\left(t,x,\dot{x},y,\dot{y}\right)\right)\right.\\&+D\left(\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\left(-D\lambda^T f_{\dot{y}\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right)\right)\right)\\&\left.+D^2\left(\theta(t)-\left(\tau^T e\right)y(t)\right)^T\sum_{i=1}^{p}\left(-\lambda^T f_{\dot{y}\dot{y}}\left(t,x,\dot{x},y,\dot{y}\right)\right)\right\}\left(\theta(t)-\left(\tau^T e\right)y(t)\right)=0, t\in I\end{aligned}$$

This in view of hypothesis (C_1) we have,

$$\phi(t)=\left(\theta(t)-\left(\tau^T e\right)y(t)\right)=0,\ t\in I. \tag{7.28}$$

Hence from (7.15) we have,

$$\sum_{i=1}^{p}\tau^i\left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right)+\omega^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)-\alpha\left(t\right)=0\quad,\ t\in I. \tag{7.29}$$

Let $\tau=0$. From (7.29) we have $\alpha(t)=0$, $\theta(t)=0$, $t\in I$.

Therefore, $\left(\tau,\theta(t),\alpha(t),\eta\right)=0$, $t\in I$, but this contradicts (7.24). Hence $\tau>0$.

From (7.16) we have,

$$\sum_{i=1}^{p}\left(\tau^i-\left(\tau^T e\right)\lambda_i\right)\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i\left(t\right)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)=0\ ,\quad t\in I\ .$$

From hypothesis (C$_2$), $\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i\left(t\right)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)$is linearly independent, hence,

$$\tau^i=\left(\tau^T e\right)\lambda^i \qquad i=1,2...,p\ . \tag{7.30}$$

From (7.29), we have,

$$\sum_{i=1}^{p}\left(\tau^T e\right)\lambda^i\left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right)+\omega^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)-\alpha\left(t\right)=0,\ t\in I$$

yielding,

$$\sum_{i=1}^{p}\lambda^i\left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right)+\omega^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)-\alpha\left(t\right)=0,\ t\in I \tag{7.31}$$

This, in view of (7.23) implies,

$$x^T\left(t\right)\sum_{i=1}^{p}\lambda^i\left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right)+\omega^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)\geqq 0,\ t\in I \tag{7.32}$$

Again (7.31) together with (7.21) gives

$$x^T\left(t\right)\sum_{i=1}^{p}\lambda^i\left(f_x^i\left(t,x,\dot{x},y,\dot{y}\right)+\omega^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)=0,\ t\in I \tag{7.33}$$

The relation (7.28) along with (7.19) yields,

$$y^T\left(t\right)\sum_{i=1}^{p}\lambda^i\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i\left(t\right)-Df_{\dot{y}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)=0\ ,t\in I \tag{7.34}$$

Now from (7.20), with $\tau^i>0$, we have,

$$y\left(t\right)\in N_{C^i}\left(z^i\left(t\right)\right)\ ,\quad i=1,...p\ ,\ t\in I \tag{7.35}$$

This implies,

$$y^T\left(t\right)z^i\left(t\right)=s\left(y\left(t\right)\middle|K^i\right)\ ,\quad t\in I \tag{7.36}$$

Also from (7.28) we have,

$$y\left(t\right)=\frac{\theta\left(t\right)}{\left(\tau^T e\right)}\geq 0\ ,t\in I \tag{7.37}$$

The relations (7.33), (7.37) and $\omega^i \in K^i$ yield that $\left(\bar{x}(t), \bar{y}(t), \bar{\omega}_1(t), ..., \bar{\omega}_p(t), \bar{\lambda}\right)$ is feasible for (SWD).

Consider,

$$
\begin{aligned}
H^i &= f^i\left(t, \bar{x}, \dot{\bar{x}}, \dot{y}, \dot{\bar{y}}\right) + s\left(\bar{x}(t) \middle| C^i\right) \\
&\quad - \bar{y}(t)^T \sum_{i=1}^{p} \lambda^i \left(f_y^i\left(t, \bar{x}, \dot{\bar{x}}, \dot{y}, \dot{\bar{y}}\right) - \bar{z}^i(t) - Df_{\dot{y}}^i\left(t, \bar{x}, \dot{\bar{x}}, \dot{y}, \dot{\bar{y}}\right)\right) - \bar{y}(t)^T \bar{z}(t) \\
&= f^i\left(t, \bar{x}, \dot{\bar{x}}, \dot{y}, \dot{\bar{y}}\right) + \bar{x}^T(t)\bar{\omega}^i(t) - s\left(\bar{y}(t) \middle| K^i\right) \\
&\quad - \bar{x}(t)^T \sum_{i=1}^{p} \lambda^i \left(f_x^i\left(t, \bar{x}, \dot{\bar{x}}, \dot{y}, \dot{\bar{y}}\right) + \bar{\omega}^i(t) - Df_{\dot{x}}^i\left(t, \bar{x}, \dot{\bar{x}}, \dot{y}, \dot{\bar{y}}\right)\right)
\end{aligned}
$$

or

$$H^i = G^i \quad , \quad i = 1, 2, ...p, \quad t \in I,$$

implying,

$$\int_I H^i dt = \int_I G^i dt \quad , \quad i = 1, 2, ...p$$

or

$$\int_I H dt = \int_I G dt$$

This by Theorem 7.1 establishes the efficiency of $\left(\bar{x}(t), \bar{y}(t), \bar{\omega}_1(t), ..., \bar{\omega}_p(t), \bar{\lambda}\right)$ for the dual problem (SWD).

Now we state the converse duality theorem whose proof follows by symmetry of the formulations of the pair of problems.

Theorem 7.3 (*Converse Duality*): Let $\left(x(t), y(t), \omega_1(t), ..., \omega_p(t), \lambda\right)$ be an efficient solution for (SWP) and $\lambda = \bar{\lambda}$ be fixed in (SWD). Furthermore, assume that

(A_1) :

$$
\begin{aligned}
&\Big\{\left(\psi(t)\right)^T \left(\lambda^T f_{xx}\left(t, x, \dot{x}, y, \dot{y}\right) - D\lambda^T f_{\dot{x}x}\left(t, x, \dot{x}, y, \dot{y}\right)\right) \\
&\quad - D\left[\left(\psi(t)\right)^T \left(-D\lambda^T f_{\dot{x}\dot{x}}\left(t, x, \dot{x}, y, \dot{y}\right)\right)\right] \\
&\quad + D^2\left[\left(\psi(t)\right)^T \left(-\lambda^T f_{\dot{x}\dot{x}}\left(t, x, \dot{x}, y, \dot{y}\right)\right)\right]\Big\}\left(\psi(t)\right) = 0, t \in I \Rightarrow \psi(t) = 0, t \in I
\end{aligned}
$$

(A$_2$): $f_x^i\left(t,x,\dot{x},y,\dot{y}\right)+\omega^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)$, $i=1,...,p$ are linearly independent.

Then $\left(x(t),y(t),z_1(t),...,z_p(t),\lambda\right)$ is feasible for (SWD) and the objective functional values are equal. If, in addition, the hypothesis of Theorem 7.1 hold, then there exist $z_1(t),z_2(t),...,z_p(t)$ such that $\left(u(t),v(t),\lambda,z_1(t),...,z_p(t)\right)=\left(x(t),y(t),\lambda,z_1(t),...,z_p(t)\right)$ is an efficient solution of dual (SWD).

7.1.3 Mond-Weir Type Duality

In this section, we present the following pair of Mond-Weir dual problems, (SM-WP) and (SM-WD):

(SM-WP): Maximize: $\int_I\left(\Phi^1,\Phi^2,...,\Phi^p\right)dt$

Subject to:

$$x\left(a\right)=0=x\left(b\right)$$
$$y\left(a\right)=0=y\left(b\right)$$

$$\sum_{i=1}^{p}\lambda^i\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)\leqq 0\ ,\ t\in I$$

$$\int_I y^T\left(t\right)\sum_{i=1}^{p}\lambda^i\left(f_y^i\left(t,x,\dot{x},y,\dot{y}\right)-z^i\left(t\right)-Df_{\dot{x}}^i\left(t,x,\dot{x},y,\dot{y}\right)\right)dt\geqq 0$$

$$z^i\left(t\right)\in C^i\ ,\ i=1,...,p,\quad t\in I$$

$$x\left(t\right)\geqq 0\ ,\ t\in I$$

$$\lambda>0$$

(SM-WD): Minimize: $\int_I\left(\psi^1,\psi^2,...,\psi^p\right)dt$

Subject to:

$$u\left(a\right)=0=u\left(b\right)$$
$$v\left(a\right)=0=v\left(b\right)$$

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i\left(t,u,\dot{u},v,\dot{v}\right)+\omega^i\left(t\right)-Df_{\dot{u}}^i\left(t,u,\dot{u},v,\dot{v}\right)\right)\geqq 0\ ,\ t\in I$$

$$\int_I y^T(t)\sum_{i=1}^{p}\lambda^i\left(f_u^i(t,u,\dot{u},v,\dot{v})+\omega^i(t)-Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right)dt \leqq 0$$

$$\omega^i(t)\in K^i \;, i=1,\dots,p,\; t\in I$$

$$v(t)\geqq 0\;,\; t\in I$$

$$\lambda>0$$

where

1. $\Phi^i = f^i(t,x,\dot{x},y,\dot{y})+s\left(x(t)\middle|C^i\right)-y(t)^T z(t) \quad ,i=1,\dots,p$
2. $\psi^i = f^i(t,u,\dot{u},v,\dot{v})-s\left(v(t)\middle|K^i\right)+u(t)^T\omega(t) \quad ,i=1,\dots,p$

The duality theorems for these problems will be merely stated below for completeness as their proofs follow on the lines of [98]:

Theorem 7.4 (*Weak Duality*): Let $\left(x(t),y(t),z_1(t),\dots,z_p(t),\lambda\right)$ be feasible for the (SM-WP) and $\left(u(t),v(t),\omega_1(t),\dots,\omega_p(t),\lambda\right)$ be feasible for the dual (SM-WD). Assume that, for each i, $\sum_{i=1}^{p}\lambda^i\int_I\left(f^i(t,.,.,y,\dot{y})dt+(.)z^i\right)dt$ is pseudoconvex in $(x,\dot{x})$ for fixed $(y,\dot{y})$ and $\sum_{i=1}^{p}\lambda^i\int_I\left(f^i(t,x,\dot{x},.,.)-(.)z^i\right)dt$ is pseudoconcave in $(y,\dot{y})$ for fixed $(x,\dot{x})$. Then,

$$\int_I\phi(t,x,\dot{x},y,\dot{y},\lambda)dt \not\leq \int_I\psi(t,x,\dot{x},y,\dot{y},\lambda)dt$$

Theorem 7.5 (*Strong Duality*): Let $\left(x(t),y(t),z_1(t),\dots,z_p(t),\lambda\right)$ be an efficient solution for (SM-WP) and $\lambda=\bar{\lambda}$ be fixed in (SM-WD). Furthermore, assume that

(H$_1$): $$-D\left[\left(\phi(t)\right)^T\left(-D\lambda^T f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\right]$$
$$\int_I\Big\{\left(\phi(t)\right)^T\left(\lambda^T f_{yy}(t,x,\dot{x},y,\dot{y})-D\lambda^T f_{\dot{y}y}(t,x,\dot{x},y,\dot{y})\right)$$
$$+D^2\left[\left(\phi(t)\right)^T\left(-f_{\dot{y}\dot{y}}(t,x,\dot{x},y,\dot{y})\right)\right]\Big\}\left(\phi(t)\right)dt=0, t\in I\Rightarrow\phi(t)=0, t\in I$$

(H$_2$): $f_y^i(t,x,\dot{x},y,\dot{y})-z^i(t)-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})$, $i=1,\dots,p$ are linearly independent.

Then $\left(x(t), y(t), \omega_1(t), ..., \omega_p(t), \lambda\right)$ is feasible for (SM-WD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 7.4 hold, then there exist $\omega_1(t), \omega_2(t), ..., \omega_p(t)$ such that $\left(u(t), v(t), \lambda, \omega_1(t), ..., \omega_p(t)\right) = \left(x(t), y(t), \lambda, \omega_1(t), ..., \omega_p(t)\right)$ is an efficient solution of dual (SM-WD).

Theorem 7.6 (*Converse Duality*): Let $\left(x(t), y(t), \omega_1(t), ..., \omega_p(t), \lambda\right)$ be an efficient solution for (SM-WP) and $\lambda = \bar{\lambda}$ be fixed in (SM-WD). Furthermore, assume that

(A₁):
$$\left\{\int_I \left(\psi(t)\right)^T \left(\lambda^T f_{xx}(t, x, \dot{x}, y, \dot{y}) - D\lambda^T f_{\dot{x}x}(t, x, \dot{x}, y, \dot{y})\right)\right.$$
$$-D\left[\left(\psi(t)\right)^T \left(-D\lambda^T f_{\dot{x}\dot{x}}(t, x, \dot{x}, y, \dot{y})\right)\right]$$
$$\left. +D^2\left[\left(\psi(t)\right)^T \left(-f_{\dot{x}\dot{x}}\right)\right]\right\}\left(\psi(t)\right) dt = 0, t \in I \Rightarrow \psi(t) = 0, t \in I$$

(A₂): $f_x^i(t, x, \dot{x}, y, \dot{y}) + z^i(t) - Df_{\dot{x}}^i(t, x, \dot{x}, y, \dot{y})$, $i = 1, ..., p$, $t \in I$ are linearly independent.

Then $\left(x(t), y(t), z_1(t), ..., z_p(t), \lambda\right)$ is feasible for (SM-WD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 7.4 hold, then there exist $z_1(t), z_2(t), ..., z_p(t)$ such that $\left(u(t), v(t), \lambda, z_1(t), ..., z_p(t)\right) = \left(x(t), y(t), \lambda, z_1(t), ..., z_p(t)\right)$ is an efficient solution of dual (SM-WD).

7.1.4 Self Duality

A problem is said to be self-dual if it is formally identical with its dual, in general, the problems (SP) and (SWD) are not formally identical if the kernel function does not possess any special characteristics. Hence, skew symmetry of each f^i is assumed in order to validate the following self-duality theorems for the two pairs of problems treated in the preceding sections.

Theorem 7.7 (*Self Duality*): Let f^i, $i = 1, 2, \ldots, p$, be skew symmetric and $C^i = K^i$ and $\omega^i(t) = z^i(t)$. Then the problem (SP) is self dual. If the problems (SWP) and (SWD) are dual problems and $\left(\bar{x}(t), \bar{y}(t), z^1(t), ..., z^p(t), \bar{\lambda}\right)$ is a joint optimal solution of (SWP) and (SWD), then so is $\left(\bar{y}(t), \bar{x}(t), z^1(t), ..., z^p(t), \bar{\lambda}\right)$**,** i.e.

Minimum **(SWP)** = $\int_I \left(f^1(t, x, \dot{x}, y, \dot{y}), f^2(t, x, \dot{x}, y, \dot{y}), \ldots, f^p(t, x, \dot{x}, y, \dot{y})\right) dt = 0$.

Proof: By skew symmetric of f^i, we have

$$f_x^i\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) = -f_y^i\left(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\right)$$
$$f_y^i\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) = -f_x^i\left(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\right)$$
$$f_{\dot{x}}^i\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) = -f_{\dot{y}}^i\left(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\right)$$
$$f_{\dot{y}}^i\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\right) = -f_{\dot{x}}^i\left(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\right)$$

Recasting the dual problem (SWD) as a minimization problem and using the above relations, we have

(SWD$_1$): Minimize $-\int_I \left(G^1, G^2, ..., G^p\right) dt$

Subject to

$$x(a) = 0 = x(b) \quad , \quad y(a) = 0 = y(b)$$

$$-\sum_{i=1}^{p} \lambda^i \left[-f_x^i(t, y, \dot{y}, x, \dot{x}) + \omega^i(t) - Df_{\dot{x}}^i(t, y, \dot{y}, x, \dot{x}) \right] \leqq 0 \ , \ t \in I,$$

$$= \sum_{i=1}^{p} \lambda^i \left[f_y^i(t, x, \dot{x}, y, \dot{y}) - \omega^i(t) - Df_{\dot{y}}^i(t, x, \dot{x}, y, \dot{y}) \right] \leqq 0 \ , \ t \in I$$

$$v(t) \geqq 0 \ , \ t \in I$$

$$\omega^i(t) \in K^i \ , i = 1, ..., p, \ \ t \in I$$

$$
\begin{aligned}
&\lambda \in \Lambda^+ \\
&-G^i = -f^i(t,x,\dot{x},y,\dot{y}) - s\left(y(t)\middle|K_i\right) - x(t)\omega_i(t) \\
&\qquad\qquad - x(t)\sum_{i=1}^{p}\lambda^i\left[f_x^i(t,x,\dot{x},y,\dot{y}) - \omega_i(t) - Df_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})\right] \\
&= f^i(t,y,\dot{y},x,\dot{x}) - s\left(y(t)\middle|K_i\right) - x(t)\omega_i(t) \\
&\qquad - x(t)\sum_{i=1}^{p}\lambda^i\left[f_y^i(t,x,\dot{x},y,\dot{y}) - \omega_i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right] \\
&= H^i\left(t,y,\dot{y},x,\dot{x},\omega_1,\dots,\omega_p,\lambda\right)
\end{aligned}
$$

Hence we have,

(SWP-1): Minimize $\int_I \left(H^1, H^2, \dots, H^p\right) dt$

Subject to

$$x(a) = 0 = x(b)\ , \quad y(a) = 0 = y(b)$$

$$\sum_{i=1}^{p}\lambda^i\left[f_y^i(t,x,\dot{x},x,\dot{y}) - z_i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right] \leqq 0\ ,\ t \in I\ ,$$

$$y(t) \geqq 0\ ,\ t \in I$$

$$z^i(t) \in C^i\ , \quad t \in I$$

$$\lambda > 0,\ \lambda^T e = 1 \quad \text{where} \quad e^T = (1,\dots\dots,1),$$

which is just the primal problem (SWP). Therefore $\left(\bar{x}(t),\bar{y}(t),\bar{z}^1(t),\dots,\bar{z}^p(t),\bar{\lambda}\right)$ is an efficient solution of dual problem implies that $\left(\bar{y}(t),\bar{x}(t),\bar{z}^1(t),\dots,\bar{z}^p(t),\bar{\lambda}\right)$ is an efficient solution of the primal. Similarly $\left(\bar{x}(t),\bar{y}(t),\bar{z}^1(t),\dots,\bar{z}^p(t),\bar{\lambda}\right)$ is an efficient solution of (SP) implies $\left(\bar{y}(t),\bar{x}(t),\bar{z}^1(t),\dots,\bar{z}^p(t),\bar{\lambda}\right)$ is an efficient solution of the dual problem (SWD). In view of (7.18), (7.33), (7.34) and (7.36), we get,

Minimum **(SWP)** =

$$\int_I \left\{ f^1\left(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{y}(t),\dot{\bar{y}}(t)\right),\dots, f^p\left(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{y}(t),\dot{\bar{y}}(t)\right)\right\} dt$$

Corresponding, to the solution $\left(\bar{y}(t),\bar{x}(t),\bar{z}^1(t),\dots,\bar{z}^p(t),\bar{\lambda}\right)$, we have, Minimum **(SWP)** =

$$\int_I \left\{ f^1\left(t, \bar{y}(t), \dot{\bar{y}}(t), \bar{x}(t), \dot{\bar{x}}(t)\right), .., f^p\left(t, \bar{y}(t), \dot{\bar{y}}(t), \bar{x}(t), \dot{\bar{x}}(t)\right) \right\} dt$$

By the skew-symmetry of each f^i

$$\int_I \left\{ f^1\left(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}\right), ..., f^p\left(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}\right) \right\} dt$$

$$= \int_I \left\{ f^1\left(t, \bar{y}, \dot{\bar{y}}, \bar{x}, \dot{\bar{x}}\right), ..., f^p\left(t, \bar{y}, \dot{\bar{y}}, \bar{x}, \dot{\bar{x}}\right) \right\} dt$$

$$= -\int_I f^1 \left\{ \left(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}\right), ..., f^p\left(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}\right) \right\} dt = 0$$

The following self duality theorem for the pair of Mond-Weir type self duality theorem will be merely stated for completeness and its proof runs parallel to that of Theorem 7.7.

Theorem 7.8 (*Self Duality*): Let $f^i, i = 1, 2, \ldots, p$, be skew symmetric and $C^i = K^i$ and $\omega^i = z^i$. Then the problem (SM-WP) is self dual. If the problems (SM-WP) and (SM-WD) are dual problems and $\left(\bar{x}(t), \bar{y}(t), z^1(t), ..., z^p(t), \bar{\lambda}\right)$ is a joint optimal solution of (SM-WP) and (SM-WD), then so is $\left(\bar{y}(t), \bar{x}(t), z^1(t), ..., z^p(t), \bar{\lambda}\right)$**,** i.e.

Minimum **(SM-WP)** = $\int_I f\left(t, x, \dot{x}, y, \dot{y}\right) dt = 0$.

7.1.5 Natural Boundary Values

The pairs of Wolfe type and Mond-Weir type symmetric multiobjective variational problem can be formulated with natural boundary values rather than fixed end points. The problems with natural boundary conditions are needed to establish well defined relationship between the pairs of continuous programming problems and nonlinear programming problems.

Following is the pair of Wolfe type symmetric dual problems with natural boundary values.

Primal (SWP_0): Maximize $\int_I \left(H^1, H^2, ..., H^p\right) dt$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left[f_y^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right] \leqq 0,$$

$$(x(t)) \geqq 0,\ t \in I$$

$$z^i(t) \in C^i, \qquad t \in I$$

$$\lambda > 0,\ \ \lambda^T e = 1,\ e^T = (1,.....,1)$$

$$f_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\Big|_{t=a} = 0,\ \ f_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\Big|_{t=b} = 0$$

Dual (SWD_0): Maximize $\int_I \left(G^1, G^2, ..., G^p\right) dt$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left[f_u^i(t,u,\dot{u},v,\dot{v}) + \omega_i(t) - Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v}) \right] \geqq 0,\ t \in I,$$

$$v(t) \geqq 0,\ t \in I$$

$$\omega^i(t) \in K^i,\ \ t \in I$$

$$\lambda > 0,\ \ \lambda^T e = 1,\ e^T = (1,.....,1)$$

$$f_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})\Big|_{t=a} = 0,\ \ f_{\dot{x}}^i(t,x,\dot{x},y,\dot{y})\Big|_{t=b} = 0$$

The duality results for each of the above pairs of dual problems can be proved easily on the lines of the proofs of the Theorems 7.1-7.8, with slight modifications in the arguments, as in Mond and Hanson [108].

Following is the pair of Mond-Weir type symmetric dual problems with natural boundary values.

Primal (SM-WP_0): Maximize $\int_I \left(\Phi^1, \Phi^2, ..., \Phi^p\right) dt$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left(f_y^i(t,x,\dot{x},y,\dot{y}) - z^i(t) - Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y}) \right) \leqq 0$$

$$,\ t \in I$$

$$\int_I y^T(t)\sum_{i=1}^{p}\lambda^i\left(f_y^i(t,x,\dot{x},y,\dot{y})-z^i(t)-Df_{\dot{y}}^i(t,x,\dot{x},y,\dot{y})\right)dt \geqq 0$$

$$z^i(t)\in C^i\ ,\ i=1,\dots,p,\quad t\in I$$

$$x(t)\geqq 0\ ,\ t\in I$$

$$\lambda>0$$

$$\lambda^T\ f_{\dot{y}}\ (t,x,\dot{x},y,\dot{y})\Big|_{t=a}=0\ ,\ \lambda^T f_{\dot{y}}\ (t,x,\dot{x},y,\dot{y})\Big|_{t=b}=0$$

Dual (SM-WD$_0$): Minimize: $\int_I (\psi^1,\psi^2,\dots,\psi^p)dt$

Subject to:

$$\sum_{i=1}^{p}\lambda^i\left(f_u^i(t,u,\dot{u},v,\dot{v})+\omega^i(t)-Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right)\geqq 0\ ,\ t\in I$$

$$y^T(t)\sum_{i=1}^{p}\lambda^i\left(f_u^i(t,u,\dot{u},v,\dot{v})+\omega^i(t)-Df_{\dot{u}}^i(t,u,\dot{u},v,\dot{v})\right)\leqq 0\ ,\ t\in I$$

$$\omega^i(t)\in K^i\ ,i=1,\dots,p\ ,\ t\in I$$

$$v(t)\geqq 0\ ,\ t\in I$$

$$\lambda\in\Lambda^+$$

$$\lambda^T f_{\dot{u}}\ (t,u,\dot{u},v,\dot{v})\Big|_{t=a}=0\ ,\ \lambda^T f_{\dot{u}}\ (t,u,\dot{u},v,\dot{v})\Big|_{t=b}=0$$

7.1.6 Nondifferentiable Multiobjective Nonlinear Programming

If the time dependency of Wolfe type symmetric pair of dual problems, (SWPo) and (SWDo) is removed and $b-a=1$, we obtain following pair of Wolfe type static nondifferentiable multiobjective dual problems with support functions which are not explicitly reported in literature.

(Primal SWP-2): Minimize $\hat{H}=\left(\hat{H}^1,\hat{H}^2,\dots,\hat{H}^p\right)$

Subject to

$$\sum_{i=1}^{p}\lambda^i\left[f_y^i(x,y)-z^i\right]\leqq 0,$$

$$y\geqq 0,\ t\in I$$

$$z^i\in K^i\ ,\ i=1,\ldots,p$$

$$\lambda\in\Lambda^+$$

where,

$$\hat{H}^i = f^i(x,y)+s\left(x\middle|C^i\right)-y^T\sum_{i=1}^{p}\lambda^i\left(f_y^i(x,y)-z^i\right)-y^Tz\ ,\quad i=1,\ldots,p$$

(Dual SWD-2): Maximize: $\hat{G}=\left(\hat{G}^1,\hat{G}^2,\ldots,\hat{G}^p\right)$

Subject to:

$$\sum_{i=1}^{p}\lambda^i\left[f_u^i(u,v)+\omega^i\right]\geqq 0,$$

$$x\geqq 0,\ t\in I$$

$$\omega^i\in C^i\ ,\quad i=1,\ldots,p$$

$$\lambda\in\Lambda^+$$

where

$$\hat{G}^i = f^i(u,v)+s\left(v\middle|K^i\right)-u^T\sum_{i=1}^{p}\lambda^i\left(f_u^i(u,v)+\omega^i\right)-x^T\omega$$

As in the case of pair of Wolfe type dual problems, the pair of Mond-Weir type dual problems (SM-WPo) and (SM-WDo) reduce to the following static counterparts in nonlinear programming.

Primal (SM-WP-2): Maximize $\hat{\Phi}^i=\left(\hat{\Phi}^1,\hat{\Phi}^2,\ldots,\hat{\Phi}^p\right)$

Subject to

$$\sum_{i=1}^{p}\lambda^i\left[f_y^i(x,y)-z^i\right]\leqq 0,$$

$$y^T\sum_{i=1}^{p}\lambda^i\left(f_y^i(x,y)-z^i\right)\geqq 0,$$

$$z^i\in C^i\ ,\ i=1,\ldots,p,$$

$$x\geqq 0\quad,\quad \lambda>0\ .$$

where

$$\hat{\Phi}^i = f^i(x,y) + s\left(x \middle| C^i\right) - y^T z$$

Dual (SM-WD-2): Maximize $\hat{\psi}^i = \left(\hat{\psi}^1, \hat{\psi}^2, \ldots, \hat{\psi}^p\right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left[f_u^i(u,v) + \omega^i \right] \geqq 0,$$

$$y^T \sum_{i=1}^{p} \lambda^i \left(f_u^i(u,v) + \omega^i \right) \leqq 0$$

$$\omega^i \in K^i \quad i = 1, \ldots, p$$

$$v \geqq 0 \quad , \quad \lambda \in \Lambda^+$$

where,

$$\hat{\psi}^i = f^i(x,y) + s\left(y \middle| K^i\right) - x^T w$$

7. 2 MIXED TYPE SYMMETRIC AND SELF DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS WITH SUPPORT FUNCTIONS

7.2.1 Statement of the Problem

In the subsection 7.2.1, we shall give notations and the formulations of the problems for studying duality subsequently.

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $J_1 \subset N, L_1 \subset M, J_2 = N \setminus J_1$ and $L_2 = M \setminus L_1$. Let $|J_1|$ denote the number of elements in the subset J_1. The other symbol $|J_2|, |L_1|$ and $|L_2|$ are similarly defined. Let $x^1 : I \to R^{|J_1|}$ and $x^2 : I \to R^{|J_2|}$, then any $x : I \to R^n$ can be written as $x = \left(x^1, x^2\right)$. Similarly for $y^1 : I \to R^{|L_1|}$ and $y^2 : I \to R^{|L_2|}$ can be written as $y = \left(y^1, y^2\right)$. Let $f : I \times R^{|J_1|} \times R^{|L_1|} \to R^p$ and $g : I \times R^{|J_2|} \times R^{|L_2|} \to R^p$ be twice continuously differentiable functions.

We state the following pair of mixed type multiobjective symmetric dual variational problems with support functions involving vector functions f and g.

(Mix SP): Minimize: $\int_I \Big(H^1\big(t, x^1, x^2, y^1, y^2, \dot{x}^1, \dot{x}^2, \dot{y}^1, \dot{y}^2, z^1, z^2, \lambda\big), \ldots,$

$$H^p\big(t, x^1, x^2, y^1, y^2, \dot{x}^1, \dot{x}^2, \dot{y}^1, \dot{y}^2, z^1, z^2, \lambda\big)\Big) dt$$

Subject to:

$$x^1(a) = 0 = x^1(b) \ , \quad y^1(a) = 0 = y^1(b) \ , \tag{7.38}$$

$$x^2(a) = 0 = x^2(b) \ , \quad y^2(a) = 0 = y^2(b) . \tag{7.39}$$

$$\sum_{i=1}^{p} \lambda^i \left[f^i_{y^1}\big(t, x^1, \dot{x}^1, y^1, \dot{y}^1\big) - z^1_i(t) - D f^i_{\dot{y}^1}\big(t, x^1, \dot{x}^1, y^1, \dot{y}^1\big) \right] \leqq 0 \ , \ t \in I \ , \tag{7.40}$$

$$\sum_{i=1}^{p} \lambda^i \left[g^i_{y^2}\big(t, x^2, \dot{x}^2, y^2, \dot{y}^2\big) - z^2_i(t) - D g^i_{\dot{y}^2}\big(t, x^2, \dot{x}^2, y^2, \dot{y}^2\big) \right] \leqq 0 \ , \ t \in I \ , \tag{7.41}$$

$$\int_I y^2(t)^T \left[\sum_{i=1}^{p} \lambda^i \Big(g^i_{y^2}\big(t, x^2, \dot{x}^2, y^2, \dot{y}^2\big) - z^2_i(t) - D g^i_{\dot{y}^2}\big(t, x^2, \dot{x}^2, y^2, \dot{y}^2\big) \Big) \right] \geqq 0 , \tag{7.42}$$

$$\big(x^1(t), x^2(t)\big) \geqq 0 \ , \ t \in I \ , \tag{7.43}$$

$$z^1_i(t) \in K^1_i \quad \text{and} \quad z^2_i(t) \in K^2_i \ , \tag{7.44}$$

$$\lambda \in \Lambda^+ \tag{7.45}$$

where

$$\begin{aligned} H^i = {} & f^i\big(t, x^1, \dot{x}^1, y^1, \dot{y}^1\big) + g^i\big(t, x^2, \dot{x}^2, y^2, \dot{y}^2\big) + s\big(x^1(t) \big| C^1_i\big) + s\big(x^2(t) \big| C^2_i\big) \\ & - y^1(t) \sum_{i=1}^{p} \lambda^i \left[f^i_{y^1}\big(t, x^1, \dot{x}^1, y^1, \dot{y}^1\big) - z^1_i(t) - D f^i_{\dot{y}^1}\big(t, x^1, \dot{x}^1, y^1, \dot{y}^1\big) \right] \\ & - z^1_i(t) y^1(t) - z^2_i(t) y^2(t) \end{aligned}$$

(Mix SD): Maximize: $\int_I \Big(G^1\big(t, u^1, u^2, v^1, v^2, \dot{u}^1, \dot{u}^2, \dot{v}^1, \dot{v}^2, w^1, w^2, \lambda\big), \ldots,$

$$G^p\big(t, u^1, u^2, v^1, v^2, \dot{u}^1, \dot{u}^2, \dot{v}^1, \dot{v}^2, w^1, w^2, \lambda\big)\Big) dt$$

Subject to:

$$u^1(a) = 0 = u^1(b) \ , \quad v^1(a) = 0 = v^1(b) \ , \tag{7.46}$$

$$u^2(a)=0=u^2(b)\ ,\quad v^2(a)=0=v^2(b). \tag{7.47}$$

$$\sum_{i=1}^{p}\lambda^i\left[f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+\omega_i^1(t)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right]\geqq 0\ ,\ t\in I\ , \tag{7.48}$$

$$\sum_{i=1}^{p}\lambda^i\left[g^i_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)+\omega_i^2(t)-Dg^i_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right]\geqq 0\ ,\ t\in I, \tag{7.49}$$

$$\int_I u^2(t)^T\left[g^i_{u^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)+\omega_i^2(t)-Dg^i_{\dot{u}^2}\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right]\leqq 0\ ,\ t\in I\ , \tag{7.50}$$

$$\left(v^1(t),v^2(t)\right)\geqq 0\ ,\ t\in I\ , \tag{7.51}$$

$$\omega_i^1(t)\in C_i^1 \quad \text{and} \quad \omega_i^2(t)\in C_i^2\quad , \tag{7.52}$$

$$\lambda\in\Lambda^+ \tag{7.53}$$

where,

$$\begin{aligned}G^i = {} & f^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+g^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\\ & +s\left(v^1(t)\middle|K_i^1\right)+s\left(v^2(t)\middle|K_i^2\right)+u^1(t)\omega_i^1(t)+u_i^2(t)\omega^2(t)\\ & -u^1(t)\sum_{i=1}^{p}\lambda^i\left[f^i_{u^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+\omega_i^1(t)-Df^i_{\dot{u}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right]\end{aligned}$$

7.2.2 Mixed Type Multiobjective Symmetric Duality

In this section, we present various duality results for a pair of mixed type multiobjective symmetric problems, (Mix SP) and (Mix SD) under pseudo-concavity-pseudo-concavity assumptions.

Theorem 7.9 (*Weak Duality*): Let $\left(x^1(t),x^2(t),y^1(t),y^2(t),z^1(t),z^2(t),\lambda\right)$ be feasible for (Mix SP) and $\left(u^1(t),u^2(t),v^1(t),v^2(t),\omega(t)^1,\omega^2(t),\lambda\right)$ be feasible for (Mix SD).

Assume that

(H$_1$): for each i $\int_I\left\{f^i\left(t,.,.,y^1(t),\dot{y}^1(t)\right)dt\right.$ be convex in $x^1,\dot{x}^1$ for fixed $y^1,\dot{y}^1$ and $\int_I\left\{f^i\left(t,x^1(t),\dot{x}^1(t),.,.\right)dt\right.$ be concave in $y^1,\dot{y}^1$ on I for fixed $x^1,\dot{x}^1$.

(H$_2$): $\sum_{i=1}^{p}\lambda_i\int_I\Big(g^i_{u^2}\left(t,.,.,y^2(t),\dot{y}^2(t)\right)+(\quad)^T\omega_i^2(t)\Big)dt$ pseudo-convex in $x^2,\dot{x}^2$ for fixed $y^2,\dot{y}^2$ and $\sum_{i=1}^{p}\lambda_i\int_I\Big(g^i_{y^2}\left(t,x^2(t),\dot{x}^2(t),.,.\right)-(\quad)^T z_i^2(t)\Big)dt$ is pseudo-concave in $y^2,\dot{y}^2$ for fixed $x^2,\dot{x}^2$.

Then,

$$\int_I H dt \not\leq \int_I G dt \,.$$

where $H = \left(H^1, H^2, ..., H^i, ..., H^p\right)$ and $G = \left(G^1, G^2, ..., G^i, ..., G^p\right)$.

Proof: Using the convexity of each $\int_I f^i\left(t, \cdot, \cdot, y, \dot{y}\right) dt$ in $\left(x, \dot{x}\right)$ for fixed $\left(y, \dot{y}\right)$, we have

$$\begin{aligned}
&\int_I f^i\left(t, x^1, \dot{x}^1, v^1, \dot{v}^1\right) dt - \int_I f^i\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) dt \\
&\geq \int_I \left[\left(x^1(t) - u^1(t)\right)^T f^i_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) + \left(\dot{x}^1(t) - \dot{u}^1(t)\right)^T f^i_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right] dt \\
&= \int_I \left[\left(x^1(t) - u^1(t)\right)^T \left\{ f^i_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D f^i_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right\}\right] dt \\
&+ \left(x^1(t) - u^1(t)\right)^T f^i_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\Big|_{t=b}^{t=a}
\end{aligned}$$

Using (7.38) and (7.46), this yields,

$$\begin{aligned}
&\int_I f^i\left(t, x^1, \dot{x}^1, v^1, \dot{v}^1\right) dt - \int_I f^i\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) dt \\
&\qquad \geq \int_I \left[\left(x^1(t) - u^1(t)\right)^T \left\{ f^i_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D f^i_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right\}\right] dt
\end{aligned} \tag{7.54}$$

Also by concavity of $\int_I \left\{ f^i\left(t, x^1(t), \dot{x}^1(t), \cdot, \cdot\right) dt\right.$, we have,

$$\begin{aligned}
&-\int_I f^i\left(t, x^1, \dot{x}^1, v^1, \dot{v}^1\right) dt - \int_I f^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) dt \\
&\geqq -\int_I \left[\left(v^1(t) - y^1(t)\right)^T f^i_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + \left(\dot{v}^1(t) - \dot{y}^1(t)\right)^T f^i_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right] dt \\
&= -\int_I \left[\left(v^1(t) - y^1(t)\right)^T \left\{ f^i_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D f^i_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right\}\right] dt \\
&+ \left(v^1(t) - y^1(t)\right)^T f^i_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\Big|_{t=b}^{t=a},
\end{aligned}$$

which by using (7.39) and (7.47), gives,

$$\begin{aligned}
&-\int_I f^i\left(t, x^1, \dot{x}^1, v^1, \dot{v}^1\right) dt - \int_I f^i\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) dt \\
&\qquad \geqq -\int_I \left[\left(v^1(t) - y^1(t)\right)^T \left\{ f^i_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D f^i_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right\}\right] dt
\end{aligned} \tag{7.55}$$

The addition of (7.54) and (7.55) implies,

$$\int_I f^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)dt-\int_I f^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt$$

$$\geq\int_I\Big[\left(x^1(t)-u^1(t)\right)^T\left\{f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}$$

$$-\left(v^1(t)-y^1(t)\right)^T\left\{f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-Df^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}\Big]dt$$

$$=\int_I\Big[\left(x^1\right)^T\left\{f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}$$

$$-\left(u^1(t)\right)^T\left\{f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}$$

$$-\left(v^1(t)\right)^T\left\{f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-Df^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}$$

$$+\left(y^1(t)\right)^T\left\{f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-Df^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}\Big]dt$$

Multiplying this by λ^i and summing over i , $i=1,2,..,p$, we get,

$$\sum_{i=1}^p\lambda_i\int_I f^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)dt-\sum_{i=1}^p\lambda_i\int_I f^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)dt$$

$$\geqq\int_I\Big[\left(x^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}$$

$$-\left(u^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}$$

$$-\left(v^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-Df^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}$$

$$+\left(y^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-Df^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}\Big]dt$$

$$=\int_I\Big[\left(x^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+\omega^1_i(t)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}$$

$$-\left(u^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{u^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+\omega^1_i(t)-Df^i_{\dot{u}^1}\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right\}$$

$$-\sum_{i=1}^p\lambda_i x^1(t)\omega^1_i(t)+\sum_{i=1}^p\lambda_i u(t)\omega^1_i(t)$$

$$-\left(v^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-z^1_i(t)-Df^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}$$

$$+\left(y^1(t)\right)^T\sum_{i=1}^p\lambda_i\left\{f^i_{y^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-z^1_i(t)-Df^i_{\dot{y}^1}\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right\}$$

$$-\sum_{i=1}^p\lambda_i v^1(t)z^1_i(t)+\sum_{i=1}^p\lambda_i y^1(t)z^1_i(t)\Big]dt$$

Using (7.40), (7.43), (7.48) and (7.51) we get,

$$\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-\left(y^1\right)^T\sum_{i=1}^{p}\lambda_i\Big\{f_{y^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)$$

$$-z_i^1(t)-Df_{\dot{y}^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\Big\}+\sum_{i=1}^{p}\lambda_i\left(x^1(t)\omega_i^1(t)\right)-\sum_{i=1}^{p}\lambda_i\left(y^1(t)z_i^1(t)\right)$$

$$\geqq\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-\left(u^1(t)\right)^T\sum_{i=1}^{p}\lambda_i\Big\{f_{u^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)$$

$$+\omega_i^1(t)-Df_{\dot{u}^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\Big\}+u^1(t)\omega_i^1(t)-y^1(t)z_i^1(t)\Big]dt$$

In view of $s\left(x^1(t)\middle|C_i^1\right)\geqq\left(x^1(t)\right)^T\omega_i^1,\ i=1,\dots,p$ and $s\left(v^1(t)\middle|K_i^1\right)\geqq\left(v^1(t)\right)^T z_i^1$, $i=1,\dots,p$ yields,

$$\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+s\left(x^1(t)\middle|C_i^1\right)-\left(y^1(t)\right)^T\sum_{i=1}^{p}\lambda_i\Big\{f_{y^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)$$

$$-z_i^1(t)-Df_{\dot{y}^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\Big\}-\left(y^1(t)\right)^T z_i^1(t)\Big]dt$$

$$\geqq\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)-s\left(v^1(t)\middle|K_i^1\right)-\left(u^1\right)^T\sum_{i=1}^{p}\lambda_i\Big\{f_{u^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)$$

$$+\omega_i^1(t)-Df_{\dot{u}^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\Big\}+u^1(t)\omega_i^1(t)\Big]dt \tag{7.56}$$

From (7.49) together with (7.43) and (7.50), we have

$$\int_I\Big[\left(x^2(t)-u^2(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-\omega_i^2(t)-Dg_{\dot{u}^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right)dt\geqq 0$$

which, on by integration by parts implies,

$$\int_I\Big[\left(x^2(t)-u^2(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-\omega_i^2(t)\right)$$

$$+\left(\dot{x}^2(t)-\dot{u}^2(t)\right)^T\sum_{i=1}^{p}\lambda_i g_{\dot{u}^1}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\Big]dt$$

$$-\left(x^2(t)-u^2(t)\right)^T\sum_{i=1}^{p}\lambda_i g_{\dot{u}^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\Big|_{t=a}^{t=b}\geqq 0$$

which by using (7.39) and (7.47), we have,

$$\int_I\Big[\left(x^2(t)-u^2(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-\omega_i^2(t)\right)$$

$$+\left(\dot{x}^2(t)-\dot{u}^2(t)\right)^T\sum_{i=1}^{p}\lambda_i g_{\dot{u}^1}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\Big]dt\geqq 0$$

By pseudo-convexity of $\sum_{i=1}^{p}\lambda_i\int_I\left(g_{u^2}^i\left(t,.,.,y^2,\dot{y}^2\right)+(.)^T\,\omega_i^2(t)\right)dt$, this yields,

$$\sum_{i=1}^{p}\lambda_i\int_I\left(g_{u^2}^i\left(t,x^2,\dot{x}^2,v^2,\dot{v}^2\right)+x^2(t)^T\,\omega_i^2(t)\right)dt$$

$$\geqq\sum_{i=1}^{p}\lambda_i\int_I\left(g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)+u^2(t)^T\,\omega_i^2(t)\right)dt \tag{7.57}$$

From (7.41) together with (7.51) and (7.42), we have

$$\int_I\left[\left(v^2(t)-y^2(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-z_i^2(t)-Dg_{\dot{y}^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)\right]dt\leqq 0$$

$$\Rightarrow\int_I\left[\left(v^2(t)-y^2(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-z_i^2(t)\right)\right.$$
$$\left.+\left(\dot{v}^2(t)-\dot{y}^2(t)\right)^T\sum_{i=1}^{p}\lambda_i g_{\dot{y}^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right]dt$$
$$-\left(\dot{v}^2(t)-\dot{y}^2(t)\right)^T\sum_{i=1}^{p}\lambda_i g_{\dot{y}^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\Big|_{t=a}^{t=b}\geqq 0$$

Which by using (7.39) and (7.47), we have,

$$\int_I\left[\left(v^2(t)-y^2(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-z_i^2(t)\right)\right.$$
$$\left.+\left(\dot{v}^2(t)-\dot{y}^2(t)\right)^T\sum_{i=1}^{p}\lambda_i g_{\dot{y}^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right]dt\leqq 0$$

By pseudo concavity of $\int_I\sum_{i=1}^{p}\lambda_i\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,.,.\right)-(.)^T\,z_i^2(t)\right)dt$, we have,

$$\sum_{i=1}^{p}\lambda_i\int_I\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,v^2,\dot{v}^2\right)-v^2(t)^T\,z_i^2(t)\right)dt$$
$$\leqq\sum_{i=1}^{p}\lambda_i\int_I\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-y^2(t)^T\,z_i^2(t)\right)dt$$

$$-\sum_{i=1}^{p}\lambda_i\int_I\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,v^2,\dot{v}^2\right)-v^2(t)^T\,z_i^2(t)\right)dt$$
$$\geqq-\sum_{i=1}^{p}\lambda_i\int_I\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-y^2(t)^T\,z_i^2(t)\right)dt \tag{7.58}$$

The relations (7.57) and (7.58) gives,

$$-\sum_{i=1}^{p}\lambda_i\int_I\left(\left(x^2(t)\right)^T\omega_i^2(t)+v^2(t)^T z_i^2(t)\right)dt$$

$$\geqq -\sum_{i=1}^{p}\lambda_i\int_I\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)-y^2(t)^T z_i^2(t)\right)dt$$

$$+\sum_{i=1}^{p}\lambda_i\int_I\left(g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-u^2(t)^T w_i^2(t)\right)dt$$

$$\sum_{i=1}^{p}\lambda_i\int_I\left(g_{y^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)+s\left(x^2(t)\middle|C_i^2\right)-y^2(t)^T z_i^2(t)\right)dt$$

$$\geqq \sum_{i=1}^{p}\lambda_i\int_I\left(g_{u^2}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-s\left(v(t)\middle|K_i^2\right)+u^2(t)^T w_i^2(t)\right)dt. \quad (7.59)$$

Combining (7.56) and (7.59), we get,

$$\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)+g^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)+s\left(x^1(t)\middle|C_i^1\right)+s\left(x^2(t)\middle|C_i^2\right)$$

$$+\left(y^1(t)\right)^T\sum_{i=1}^{p}\lambda_i\int_I\left(f_{y^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)-z_i^1(t)-Df_{\dot{y}^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right)$$

$$-y^1(t)z_i^1(t)-y^2(t)z_i^2(t)\Big]dt$$

$$\geqq\sum_{i=1}^{p}\lambda_i\int_I\Big[f^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+g^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)-s\left(v^1(t)\middle|K_i^1\right)+s\left(v^2(t)\middle|K_i^2\right)$$

$$-\left(u^1(t)\right)^T\sum_{i=1}^{p}\lambda_i\int_I\left(f_{u^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+\omega_i^1(t)-Df_{\dot{u}^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right)$$

$$+\left(u^1(t)\right)^T\omega_i^1(t)+\left(u^2(t)\right)^T\omega_i^2(t)\Big]dt.$$

That is,

$$\sum_{i=1}^{p}\lambda_i\int_I H^i dt\geqq\sum_{i=1}^{p}\lambda_i\int_I G^i dt.$$

This yields,

$$\int_I H dt\not\leq\int_I G dt.$$

Theorem 7.10: (*Strong Duality*):

Let $\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t),\bar{z}_1^1(t),\bar{z}_2^1(t),...,\bar{z}_p^1(t),\bar{z}_1^2(t),\bar{z}_2^2(t),...,z_p^2(t),\bar{\lambda}\right)$ be an efficient solution of (Mix SP). Let $\lambda=\bar{\lambda}$ be fixed in (Mix SD). Furthermore assume that,

(C$_1$):

$$\int_I \Big[\Big\{\left(\phi^1(t)\right)^T \left(\bar{\lambda}^T f_{y^1y^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right) - D\bar{\lambda}^T f_{\dot{y}^1y^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)$$
$$-D\left[\left(\phi^1(t)\right)^T\left(-D\bar{\lambda}^T f_{\dot{y}^1\dot{y}^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]$$
$$+D^2\left[\left(\phi^1(t)\right)^T\left(-\bar{\lambda}^T f_{\dot{y}^1\dot{y}^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]\Big\}\left(\phi^1(t)\right)^T\Big]dt \geqq 0$$

and

$$\int_I \Big[\Big\{\left(\phi^2(t)\right)^T \left(\bar{\lambda}^T g_{y^2y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right) - D\bar{\lambda}^T g_{y^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)$$
$$-D\phi^2(t)^T\left(-D\bar{\lambda}^T g_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)$$
$$+D^2\phi^2(t)^T\left(-\bar{\lambda}^T g_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\Big\}\phi^2(t)\Big]dt \geqq 0,$$

(C$_2$):

$$\int_I \Big\{\left(\phi^1(t)\right)^T \left(\bar{\lambda}^T f_{y^1y^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right) - D\bar{\lambda}^T f_{\dot{y}^1y^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)$$
$$-D\left[\left(\phi^1(t)\right)^T\left(-D\bar{\lambda}^T f_{\dot{y}^1\dot{y}^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]$$
$$+D^2\left[\left(\phi^1(t)\right)^T\left(-\bar{\lambda}^T f_{\dot{y}^1\dot{y}^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]\Big\}\left(\phi^1(t)\right)dt = 0, t \in I$$
$$\Rightarrow \left(\phi^1(t)\right) = 0\ ,\ t \in I$$

and

$$\int_I \Big[\Big\{\left(\phi^2(t)\right)^T \left(\bar{\lambda}^T g_{y^2y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right) - D\bar{\lambda}^T g_{y^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)$$
$$-D\phi^2(t)^T\left(-D\bar{\lambda}^T g_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)$$
$$+D^2\phi^2(t)^T\left(-\bar{\lambda}^T g_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\Big\}\phi^2(t)\Big]dt = 0, t \in I$$
$$\Rightarrow \phi^2(t) = 0, t \in I$$

(C$_3$): $g^i_{y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right) + \bar{\omega}^2_i(t) - Dg^i_{\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)$, $t \in I, i = 1,\dots,p$ are linearly independent.

Then $\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t),\bar{\omega}^1_1(t),\bar{\omega}^1_2(t),\dots,\bar{\omega}^1_p(t);\bar{\omega}^2_1(t),\bar{\omega}^2_2(t),\dots,\bar{\omega}^2_p(t),\bar{\lambda}\right)$ is feasible for (Mix SD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 7.9 hold, then there exist $\bar{\omega}_1^1(t),\bar{\omega}_2^1(t),...,\bar{\omega}_p^1(t)$; $\bar{\omega}_1^2(t),\bar{\omega}_2^2(t),...,\bar{\omega}_p^2(t)$ such that $\left(u^1(t),u^2(t),v^1(t),v^2(t),\omega_1^1(t),\omega_2^1(t),...,\omega_p^1(t);\omega_1^2(t),\omega_2^2(t),...,\omega_p^2(t),\bar{\lambda}\right)$ $=\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^1(t),\bar{\omega}_1^1(t),\bar{\omega}_2^1(t),...,\bar{\omega}_p^1(t);\bar{\omega}_1^2(t),\bar{\omega}_2^2(t),...,\bar{\omega}_p^2(t),\bar{\lambda}\right)$ is an efficient solution of dual (Mix SD).

Proof: Since $\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t),\bar{z}_1^1(t),\bar{z}_2^1(t),...,\bar{z}_p^1(t),\bar{z}_1^2(t),\bar{z}_2^2(t),...,z_p^2(t)\bar{\lambda}\right)$ is efficient, it is weak minimum, there exist $\tau\in R^p$, $\eta\in R^p$, $\gamma\in R$, $\theta^1(t):I\to R^{|K_1|}$, $\theta^2(t):I\to R^{|K_2|}$ and $\alpha(t)\in R^m$, $\beta(t)\in R^m$ such that the following Fritz-John optimality conditions, are satisfied

$$
\begin{aligned}
&\sum_{i=1}^{p}\tau^i\left(f_{x^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)+\bar{\omega}_1^i(t)-Df_{\dot{x}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\\
&\quad+\left(\theta^1(t)-\tau e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(f_{y^1x^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-Df_{\dot{y}^1x^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\\
&\quad-D\left[\left(\theta^1(t)-\tau e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(f_{y^1\dot{x}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-Df_{\dot{y}^1\dot{x}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right.\right.\\
&\quad\left.\left.-f_{\dot{y}^1x^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]+D^2\left[\left(\theta^1(t)-\tau e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\lambda_i\left(-f_{\dot{y}^1\dot{x}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]\\
&\quad-\alpha(t)=0,\ t\in I
\end{aligned}
\tag{7.60}
$$

$$
\begin{aligned}
&\sum_{i=1}^{p}\tau^i\left(g_{x^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)+\bar{\omega}^2(t)-Df_{\dot{x}^1}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\\
&\quad+\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(g_{y^2x^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-Dg_{\dot{y}^2x^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\\
&\quad-D\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(g_{y^2\dot{x}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right.\right.\\
&\quad\left.\left.-Dg_{\dot{y}^2\dot{x}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-g_{\dot{y}^2x^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right]\\
&\quad+D^2\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(-g_{\dot{y}^2\dot{x}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right]-\beta(t)=0,\ t\in I.
\end{aligned}
\tag{7.61}
$$

$$\sum_{i=1}^{p}\left(\tau^{i}-\tau^{T}e\bar{\lambda}_{i}\right)\left(f_{y^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)-\bar{z}_{i}^{1}\left(t\right)-Df_{\dot{y}^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)\right)$$
$$+\left(\theta^{1}\left(t\right)-\tau^{T}e\bar{y}^{1}\left(t\right)\right)^{T}\sum_{i=1}^{p}\bar{\lambda}_{i}\left(f_{y^{1}y^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)-Df_{\dot{y}^{1}y^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)\right)$$
$$-D\left[\left(\theta^{1}\left(t\right)-\tau^{T}e\bar{y}^{1}\left(t\right)\right)^{T}\sum_{i=1}^{p}\bar{\lambda}_{i}\left(-Df_{\dot{y}^{1}\dot{y}^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)\right)\right]$$
$$+D^{2}\left[\left(\theta^{1}\left(t\right)-\tau^{T}e\bar{y}^{1}\left(t\right)\right)^{T}\sum_{i=1}^{p}\bar{\lambda}_{i}\left(-f_{\dot{y}^{1}\dot{y}^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)\right)\right]=0\ ,\ t\in I \quad (7.62)$$

$$\sum_{i=1}^{p}\left(\tau^{i}-\gamma\bar{\lambda}_{i}\right)\left(g_{y^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)-\bar{z}_{i}^{2}\left(t\right)-Dg_{\dot{y}^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)\right)$$
$$+\left(\theta^{2}\left(t\right)-\gamma\bar{y}^{2}\left(t\right)\right)^{T}\sum_{i=1}^{p}\bar{\lambda}_{i}\left(g_{y^{2}y^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)-Dg_{\dot{y}^{2}y^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)\right)$$
$$-D\left[\left(\theta^{2}\left(t\right)-\gamma\bar{y}^{2}\left(t\right)\right)^{T}\sum_{i=1}^{p}\bar{\lambda}_{i}\left(-Dg_{\dot{y}^{2}\dot{y}^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)\right)\right]$$
$$+D^{2}\left[\left(\theta^{2}\left(t\right)-\gamma\bar{y}^{2}\left(t\right)\right)^{T}\sum_{i=1}^{p}\bar{\lambda}_{i}\left(-g_{\dot{y}^{2}\dot{y}^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)\right)\right]=0\ ,\ t\in I \quad (7.63)$$

$$\left(\theta^{1}\left(t\right)-\tau e y^{1}\left(t\right)\right)^{T}\left(f_{y^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)-z_{i}^{1}\left(t\right)-Df_{\dot{y}^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)\right)$$
$$+\left(\theta^{2}\left(t\right)-\gamma\bar{y}^{2}\left(t\right)\right)^{T}\left(g_{y^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)-z_{i}^{2}\left(t\right)-Dg_{\dot{y}^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)\right)-\eta=0\ ,\ t\in I$$
(7.64)

$$\theta^{1}\left(t\right)^{T}\sum_{i=1}^{p}\lambda_{i}\left(f_{y^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)-z_{i}^{1}\left(t\right)-Df_{\dot{y}^{1}}^{i}\left(t,\bar{x}^{1},\dot{\bar{x}}^{1},\bar{y}^{1},\dot{\bar{y}}^{1}\right)\right)=0\ ,\ t\in I \quad (7.65)$$

$$\theta^{2}\left(t\right)\sum_{i=1}^{p}\lambda_{i}\left(g_{y^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)-z_{i}^{2}\left(t\right)-Dg_{\dot{y}^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)\right)=0\ ,\ t\in I \quad (7.66)$$

$$\gamma y^{2}\left(t\right)\sum_{i=1}^{p}\lambda_{i}\left(g_{y^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)-\bar{z}_{i}^{2}\left(t\right)-Dg_{\dot{y}^{2}}^{i}\left(t,\bar{x}^{2},\dot{\bar{x}}^{2},\bar{y}^{2},\dot{\bar{y}}^{2}\right)\right)=0\ ,\ t\in I \quad (7.67)$$

$$\eta^{T}\lambda=0 \quad (7.68)$$

$$\alpha\left(t\right)^{T}\bar{x}^{1}\left(t\right)=0\ ,\ t\in I \quad (7.69)$$

$$\beta\left(t\right)^{T}\bar{x}^{2}\left(t\right)=0\ ,\ t\in I \quad (7.70)$$

$$\bar{y}^{1}\left(t\right)\in N_{K_{i}^{1}}\left(\bar{z}_{i}^{1}\left(t\right)\right)\ ,\ t\in I \quad (7.71)$$

$$-\tau_i \bar{y}^2(t)-\left(\theta^2(t)-\gamma \bar{y}^2(t)\right)\in N_{K_i^2}\left(\bar{z}_i^2(t)\right),\ t\in I \tag{7.72}$$

$$\bar{\omega}^1(t)\in C_1\ ,\ \left(\bar{\omega}^1(t)\right)^T \bar{x}^1(t)=s\left(\bar{x}^1(t)\middle|C_1\right),\ t\in I \tag{7.73}$$

$$\bar{\omega}^2(t)\in C_2\ ,\ \left(\bar{\omega}^2(t)\right)^T \bar{x}^2(t)=s\left(\bar{x}^2(t)\middle|C_2\right),\ t\in I \tag{7.74}$$

$$\left(\tau,\theta^1(t),\theta^2(t),\gamma,\alpha(t),\beta(t),\eta\right)\geqq 0\ ,\ t\in I \tag{7.75}$$

$$\left(\tau,\theta^1(t),\theta^2(t),\gamma,\alpha(t),\beta(t),\eta\right)\neq 0\ ,\ t\in I \tag{7.76}$$

Since $\lambda>0$, (7.68) implies $\eta=0$. Consequently, (7.64) reduces to

$$\begin{aligned}&\left(\theta^1(t)-\left(\tau^T e\right)\bar{y}^1(t)\right)\left(f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-\bar{z}_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\\&+\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)\left(g_{y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-\bar{z}_i^2(t)-Dg_{\dot{y}^2}^i\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\right)=0,\ t\in I\end{aligned} \tag{7.77}$$

Postmultiplying (7.62) by $\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)$, (7.63) by $\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)$ and then adding, we have,

$$\begin{aligned}&\left\{\sum_{i=1}^{p}\left(\tau^i-\tau^T e\bar{\lambda}_i\right)\left(f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-\bar{z}_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right.\\&\quad+\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(f_{y^1y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-Df_{\dot{y}^1y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\\&\quad-D\left[\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(-Df_{\dot{y}^1\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]\\&\quad\left.+D^2\left[\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(-f_{\dot{y}^1\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]\right\}\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)\\&\quad+\left\{\sum_{i=1}^{p}\left(\tau^i-\gamma\bar{\lambda}_i\right)\left(g_{y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-\bar{z}_i^2(t)-Dg_{\dot{y}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right.\\&\quad+\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(g_{y^2y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-Dg_{\dot{y}^2y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\\&\quad-D\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(-Dg_{\dot{y}^2\dot{y}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right]\\&\left.+D^2\left[\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(-g_{\dot{y}^2\dot{y}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right]\right\}\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)=0\end{aligned} \tag{7.78}$$

Multiplying (7.77) by λ^i and then using (7.66) and (7.67), we obtain,

$$\left(\theta^1(t)-\tau e\bar{y}^1(t)\right)^T \sum_{i=1}^{p} \lambda_i \left(f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right) -\bar{z}_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)=0,\ t\in I$$

$$\left(\theta^1(t)-\tau e\bar{y}^1(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right) -\bar{z}_i^1(t)-Df_{\dot{y}^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\right)\tau^T e=0,\ t\in I$$

$$\int_I \left\{\left(\theta^1(t)-\tau e\bar{y}^1(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right) - z_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\tau^T e\right\}dt=0 \quad (7.79)$$

Multiplying (7.77) by τ^i and summing over i, we have

$$\int_I \left(\theta^1(t)-\tau e\bar{y}^1(t)\right)\sum_{i=1}^{p}\tau^i\left(f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-\bar{z}_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)$$
$$+\left(\theta^2(t)-\gamma\bar{y}^2(t)\right)\sum_{i=1}^{p}\tau^i\left(g_{y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-\bar{z}_i^2(t)-Dg_{\dot{y}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)dt=0, \quad (7.80)$$

By subtraction of (7.79) and (7.80) and then using (7.66) and (7.67), we have,

$$\int_I \left(\theta^1(t)-\tau e\bar{y}^1(t)\right)\sum_{i=1}^{p}\tau^i\left(f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right) -\bar{z}_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\left(\tau^i-\left(\tau^T e\right)\lambda_i\right)$$
$$+\left(\theta^2-\gamma\bar{y}^2(t)\right)^T\sum_{i=1}^{p}\left(\tau^i-\bar{\lambda}^i\gamma\right)\left(g_{y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right) -\bar{z}_i^2-Dg_{\dot{y}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\Big\}dt=0 \quad (7.81)$$

From (7.78) and (7.81) we obtain,

$$\int_I \Bigg\{\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(f_{y^1y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-Df_{\dot{y}^1y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)$$
$$-D\left[\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(-Df_{\dot{y}^1\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]$$
$$+D^2\left[\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T\sum_{i=1}^{p}\bar{\lambda}_i\left(-f_{\dot{y}^1\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]\Bigg\}\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)dt$$

$$
\begin{aligned}
&+\int_I \Big\{ \left(\theta^2(t)-\gamma \bar{y}^2(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(g^i_{y^2y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right) - Dg^i_{\dot{y}^2y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right) \\
&-D\left[\left(\theta^2(t)-\gamma \bar{y}^2(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(-Dg^i_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right] \\
&+D^2\left[\left(\theta^2(t)-\gamma \bar{y}^2(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(-g^i_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right]\Big\} \\
&\left(\theta^2(t)-\gamma \bar{y}^2(t)\right)dt = 0
\end{aligned}
$$

In view of the hypothesis (C_1), we have

$$
\begin{aligned}
&\int_I \Big\{ \left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(f^i_{y^1y^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right) - Df^i_{\dot{y}^1y^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right) \\
&\quad -D\left[\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(-Df^i_{\dot{y}^1\dot{y}^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right] \\
&\quad +D^2\left[\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(-f^i_{\dot{y}^1\dot{y}^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right)\right]\Big\}\left(\theta^1(t)-\tau^T e\bar{y}^1(t)\right)dt = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_I \Big\{ \left(\theta^2(t)-\gamma \bar{y}^2(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(g^i_{y^2y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right) - Dg^i_{\dot{y}^2y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right) \\
&\quad -D\left[\left(\theta^2(t)-\gamma \bar{y}^2(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(-Dg^i_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right] \\
&\quad +D^2\left[\left(\theta^2(t)-\gamma \bar{y}^2(t)\right)^T \sum_{i=1}^{p} \bar{\lambda}_i \left(-g^i_{\dot{y}^2\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right]\Big\}\left(\theta^2(t)-\gamma \bar{y}^2(t)\right)dt = 0
\end{aligned}
$$

From this, in view of the hypothesis (C_2), we have,

$$\phi^1(t) = \theta^1(t) - \tau^T e\bar{y}^1(t) = 0, \quad t \in I \tag{7.82}$$

$$\phi^2(t) = \theta^2(t) - \gamma \bar{y}^2(t) = 0, \; t \in I \tag{7.83}$$

From (7.83) and (7.63),

$$\sum_{i=1}^{p}\left(\tau^i - \gamma\bar{\lambda}_i\right)^T \left(g^i_{y^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right) - \bar{z}^2_i(t) - Dg^i_{\dot{y}^2}\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right) = 0$$

This in view of hypothesis (C_3), we have

$$\tau^i = \gamma\bar{\lambda}_i \quad , \quad i = 1, \ldots, p \tag{7.84}$$

If possible, let, $\gamma = 0$. From (7.84), we have $\tau = 0$, and therefore, from (7.82)and (7.83), we have,

$$\phi^1(t) = 0,\ \phi^2(t) = 0,\ t \in I$$

Also from (7.60) and (7.61), we have $\alpha(t) = 0 = \beta(t)$, $t \in I$.

Hence $\left(\tau, \theta^1(t), \theta^2(t), \gamma, \alpha(t), \beta(t)\right) = 0$, contradicting Fritz John condition (7.76).Hence $\gamma > 0$ and consequently $\tau > 0$. From (7.60) and (7.61) along with (7.84), we obtain

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f^i_{x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) + \bar{\omega}^1_i(t) - Df^i_{\dot{x}^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right) - \alpha(t) = 0\ ,\ t \in I \tag{7.85}$$

and

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(g^i_{x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) + \bar{\omega}^2_i(t) - Dg^i_{\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right) - \beta(t) = 0\ ,\ t \in I \tag{7.86}$$

This, in view of (7.75), yields,

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(f^i_{x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) + \bar{\omega}^1_i(t) - Df^i_{\dot{x}^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right) \geqq 0\ ,\ t \in I \tag{7.87}$$

$$\sum_{i=1}^{p} \bar{\lambda}^i \left(g^i_{x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) + \bar{\omega}^2_i(t) - Dg^i_{\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right) \geqq 0\ ,\ t \in I \tag{7.88}$$

and, in view of (7.69) and (7.70) together with (7.43) gives,

$$\bar{x}^1(t)^T \sum_{i=1}^{p} \bar{\lambda}^i \left(f^i_{x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) + \bar{\omega}^1_i(t) - Df^i_{x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right) = 0,\ t \in I \tag{7.89}$$

$$\bar{x}^2(t)^T \sum_{i=1}^{p} \bar{\lambda}^i \left(g^i_{x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) + \bar{\omega}^2_i(t) - Dg^i_{\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right) = 0, t \in I \tag{7.90}$$

The relation (7.90) can be written as,

$$\int_I \bar{x}^2(t)^T \sum_{i=1}^{p} \bar{\lambda}^i \left(g^i_{x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) + \bar{\omega}^2_i(t) - Dg^i_{\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right) dt = 0 \tag{7.91}$$

From (7.82) and (7.83) we have

$$\left(\bar{y}^1(t), \bar{y}^2(t)\right) \geqq 0 \quad t \in I$$

$$\bar{y}^1(t)^T \sum_{i=1}^{p} \bar{\lambda}^i \left[f^i_{y^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) - \bar{z}^1_i(t) - Df^i_{\dot{y}^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right] = 0 \tag{7.92}$$

Also from (7.71) and (7.72), we have

$$\bar{y}^1(t)^T \bar{z}_i^1(t) = s\left(\bar{y}^1(t) \middle| K_i^1\right) \ , \ \ t \in I,\ i = 1,2,\ldots,p \tag{7.93}$$

$$\bar{y}^2(t)^T \bar{z}_i^2(t) = s\left(\bar{y}^2(t) \middle| K_i^2\right) \ , \ \ t \in I,\ i = 1,2,\ldots,p \tag{7.94}$$

Consequently, from (7.88), (7.91), (7.92) and (7.93), the feasibility of $\left(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), \ldots, \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), \ldots, \bar{z}_p^2(t), \bar{\lambda}\right)$ for (Mix SD) follows.

Consider,

$$\begin{aligned} H^i = {} & f^i\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) + g^i\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) + s\left(\bar{x}^1(t) \middle| C_i^1\right) + s\left(\bar{x}^2(t) \middle| C_i^2\right) \\ & - \bar{y}^1(t)^T \sum_{i=1}^{p} \lambda^i \left[f_{y^1}^i\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) - \bar{z}_i^1(t) - D f_{\dot{y}^1}^i\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) \right] \\ & - \bar{y}^1(t)^T \bar{z}_i^1(t) - \bar{y}^2(t)^T \bar{z}_i^2(t) \end{aligned}$$

Using (7.73), (7.74), (7.91), (7.92), (7.93) and (7.94) in proper sequence, we obtain,

$$\begin{aligned} H^i = {} & f^i\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) + g^i\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) \\ & - s\left(\bar{y}^1(t) \middle| K_i^1\right) - s\left(\bar{y}^2(t) \middle| K_i^2\right) + \bar{x}^1(t)^T \bar{\omega}_i^1(t) + \bar{x}^2(t)^T \bar{\omega}_i^2(t) \\ & - \bar{x}^1(t) \sum_{i=1}^{p} \bar{\lambda}^i \left[f_{x^1}^i\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) - \omega_i^1(t) - D f_{\dot{x}^1}^i\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) \right] \\ & = G^i \ \text{ for } \ i = 1,2,\ldots., p \end{aligned}$$

Therefore,

$$\int_I \left(H^1, H^2, \ldots, H^i, \ldots, H^p\right) dt = \int_I \left(G^1, G^2, \ldots, G^i, \ldots, G^p\right) dt$$

This proves the efficiency of $\left(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{\omega}_1^1(t), \bar{\omega}_2^1(t), \ldots, \bar{\omega}_p^1(t); \bar{\omega}_1^2(t), \bar{\omega}_2^2(t), \ldots, \bar{\omega}_p^2(t), \bar{\lambda}\right)$ of the dual problem (Mix SD) by an application of Theorem 7.9.

The proof of the following converse duality theorem follows automatically by symmetry of the formulation:

Theorem 7.11(*Converse Duality*): Let $\left(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\omega}_1^1(t), \bar{\omega}_2^1(t), ..., \bar{\omega}_p^1(t);\right.$ $\left.\bar{\omega}_1^2(t), \bar{\omega}_2^2(t), ..., \bar{\omega}_p^2(t), \bar{\lambda}\right)$ be an efficient solution of (Mix SP). Let $\lambda = \bar{\lambda}$ be fixed in (Mix SD). Furthermore assume that

(A₁):
$$\int_I \Big\{ \left(\psi^1(t)\right)^T \left(\bar{\lambda}^T f_{x^1x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) - D\bar{\lambda}^T f_{\dot{x}^1x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right)$$
$$-D\left[\left(\psi^1(t)\right)^T \left(-D\bar{\lambda}^T f_{\dot{x}^1\dot{x}^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right)\right]$$
$$+D^2\left[\left(\psi^1(t)\right)^T \left(-\bar{\lambda}^T f^i_{\dot{x}^1\dot{x}^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right)\right]\Big\} \left(\psi^1(t)\right)^T dt \geqq 0$$

and

$$\int_I \Big\{ \left(\psi^2(t)\right)^T \left(\bar{\lambda}^T g_{x^2x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) - D\bar{\lambda}^T g_{\dot{x}^2x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right)$$
$$-D\left[\left(\psi^2(t)\right)^T \left(-D\bar{\lambda}^T g_{\dot{x}^2\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right)\right]$$
$$+D^2\left[\left(\psi^2(t)\right)^T \left(-\bar{\lambda}^T g_{\dot{x}^2\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right)\right]\Big\} \left(\psi^2(t)\right)^T dt \geqq 0$$

(A₂):
$$\int_I \Big\{ \left(\psi^1(t)\right)^T \left(\bar{\lambda}^T f_{x^1x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) - D\bar{\lambda}^T f_{\dot{x}^1x^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right)$$
$$-D\left[\left(\psi^1(t)\right)^T \left(-D\bar{\lambda}^T f_{\dot{x}^1\dot{x}^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right)\right]$$
$$+D^2\left[\left(\psi^1(t)\right)^T \left(-\bar{\lambda}^T f^i_{\dot{x}^1\dot{x}^1}\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right)\right)\right]\Big\} \left(\psi^1(t)\right)^T dt = 0, t \in I$$
$$\Rightarrow \psi^1(t) = 0, t \in I$$

and

$$\int_I \Big\{ \left(\psi^2(t)\right)^T \left(\bar{\lambda}^T g_{x^2x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) - D\bar{\lambda}^T g_{\dot{x}^2x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right)$$
$$-D\left[\left(\psi^2(t)\right)^T \left(-D\bar{\lambda}^T g^i_{\dot{x}^2\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right)\right]$$
$$+D^2\left[\left(\psi^2(t)\right)^T \left(-\bar{\lambda}^T g^i_{\dot{x}^2\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)\right)\right]\Big\} \left(\psi^2(t)\right)^T dt = 0, t \in I$$
$$\Rightarrow \psi^2(t) = 0, t \in I$$

and

(A₃): $g^i_{x^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) + \bar{\omega}_i^2(t) - Dg^i_{\dot{x}^2}\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right)$, $i = 1, ..., p$ are linearly independent.

Then $\left(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), ..., \bar{z}_p^1(t), \bar{z}_1^2(t), \bar{z}_2^2(t), ..., z_p^2(t), \lambda\right)$ is feasible for (Mix SD) and the objective functional values are equal. If, in addition, the hypotheses of Theorem 7.9 hold, then there exist $\bar{z}_1^1(t), \bar{z}_2^1(t), ..., \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), ..., \bar{z}_p^2(t)$ such that

$$\left(u^1(t), u^2(t), v^1(t), v^2(t), z_1^1(t), z_2^1(t), ..., z_p^1(t); z_1^2(t), z_2^2(t), ..., z_p^2(t), \lambda\right)$$
$$= \left(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), ..., \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), ..., z_p^2(t), \bar{\lambda}\right)$$

is an efficient solution of dual (Mix SD).

7.2.3 Self Duality

A mathematical problem is said to be self-dual if it is formally identical with its dual. In general, the problems (Mix SP) and (Mix SD) cannot be formally identical if the kernel function does not owe any special characteristics. Hence skew symmetric of f^i and g^i is assumed in order to validate the following self-duality theorem for the pair of problems treated in the preceding section.

Theorem 7.12 (*Self Duality*): Let f^i and $g^i, i = 1, 2, \ldots, p$, be skew symmetric and $C_i^1 = K_i^1$ and $C_i^2 = K_i^2$ with $\omega_i^1(t) = z_i^1(t)$, $\omega_i^2(t) = z_i^2(t)$, $t \in I$. Then the problem (Mix SP) is self dual. If the problems (Mix SP) and (Mix SD) are dual problems and $\left(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t), \bar{z}_1^1(t), \bar{z}_2^1(t), ..., \bar{z}_p^1(t); \bar{z}_1^2(t), \bar{z}_2^2(t), ..., \bar{z}_p^2(t), \bar{\lambda}\right)$ is a joint optimal solution of (MixSP) and (MixSD), then so is $\left(\bar{y}^1(t), \bar{y}^2(t), \bar{x}^1(t), \bar{x}^2(t), \bar{\omega}_1^1(t), \bar{\omega}_2^1(t), ..., \bar{\omega}_p^1(t); \bar{\omega}_1^2(t), \bar{\omega}_2^2(t), ..., \bar{\omega}_p^2(t), \bar{\lambda}\right)$**,** and the common functional value is zero, i.e.

$$\text{Minimum } \textbf{(Mix SP)} = \int_I \left\{ f^i\left(t, \bar{x}^1, \dot{\bar{x}}^1, \bar{y}^1, \dot{\bar{y}}^1\right) + g^i\left(t, \bar{x}^2, \dot{\bar{x}}^2, \bar{y}^2, \dot{\bar{y}}^2\right) \right\} dt = 0$$

Proof: By skew symmetric of f^i and g^i, we have

$$f_{x^1}^i\left(t, \bar{x}^1(t), \dot{\bar{x}}^1(t), \bar{y}^1(t), \dot{\bar{y}}^1(t)\right) = -f_{y^1}^i\left(t, \bar{y}^1(t), \dot{\bar{y}}^1(t), \bar{x}^1(t), \dot{\bar{x}}(t)\right)$$

$$g^i_{x^2}\left(t,\bar{x}^2(t),\dot{\bar{x}}^2(t),\bar{y}^2(t),\dot{\bar{y}}^2(t)\right)=-g^i_{\dot{y}^2}\left(t,\bar{y}^2(t),\dot{\bar{y}}^2(t),\bar{x}^2(t),\dot{\bar{x}}^2(t)\right)$$

$$f^i_{y^1}\left(t,\bar{x}^1(t),\dot{\bar{x}}^1(t),\bar{y}^1(t),\dot{\bar{y}}^1(t)\right)=-f^i_{x^1}\left(t,\bar{y}^1(t),\dot{\bar{y}}^1(t),\bar{x}^1(t),\dot{\bar{x}}(t)\right)$$

$$g^i_{y^2}\left(t,\bar{x}^2(t),\dot{\bar{x}}^2(t),\bar{y}^2(t),\dot{\bar{y}}^2(t)\right)=-g^i_{\dot{x}^2}\left(t,\bar{y}^2(t),\dot{\bar{y}}^2(t),\bar{x}^2(t),\dot{\bar{x}}^2(t)\right)$$

$$f^i_{x^1}\left(t,\bar{x}^1(t),\dot{\bar{x}}^1(t),\bar{y}^1(t),\dot{\bar{y}}^1(t)\right)=-f^i_{\dot{y}^1}\left(t,\bar{y}^1(t),\dot{\bar{y}}^1(t),\bar{x}^1(t),\dot{\bar{x}}(t)\right)$$

$$g^i_{\dot{x}^2}\left(t,\bar{x}^2(t),\dot{\bar{x}}^2(t),\bar{y}^2(t),\dot{\bar{y}}^2(t)\right)=-g^i_{\dot{y}^2}\left(t,\bar{y}^2(t),\dot{\bar{y}}^2(t),\bar{x}^2(t),\dot{\bar{x}}^2(t)\right)$$

$$f^i_{y^1}\left(t,\bar{x}^1(t),\dot{\bar{x}}^1(t),\bar{y}^1(t),\dot{\bar{y}}^1(t)\right)=-f^i_{\dot{x}^1}\left(t,\bar{y}^1(t),\dot{\bar{y}}^1(t),\bar{x}^1(t),\dot{\bar{x}}(t)\right)$$

$$g^i_{\dot{y}^2}\left(t,\bar{x}^2(t),\dot{\bar{x}}^2(t),\bar{y}^2(t),\dot{\bar{y}}^2(t)\right)=-g^i_{\dot{x}^2}\left(t,\bar{y}^2(t),\dot{\bar{y}}^2(t),\bar{x}^2(t),\dot{\bar{x}}^2(t)\right)$$

Recasting the dual problem (Mix SD) as a minimization problem and using the above relations, we have,

Mix (SD1): Minimize $-\int_I \left(G^1,G^2,...,G^p\right)dt$

Subject to

$$x^1(a)=0=x^1(b)\quad,\quad y^1(a)=0=y^1(b)$$

$$x^2(a)=0=x^2(b)\quad,\quad y^2(a)=0=y^2(b)$$

$$\sum_{i=1}^{p}\bar{\lambda}^i\left[f^i_{y^1}\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}\right)-\bar{\omega}^1_i(t)-Df^i_{\dot{y}^1}\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}\right)\right]\leqq 0\ ,\ t\in I$$

$$=\sum_{i=1}^{p}\bar{\lambda}^i\left[g^i_{y^2}\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)-\bar{\omega}^2_i(t)-Dg^i_{\dot{y}^2}\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)\right]\leqq 0\ ,\ t\in I$$

$$\int_I \bar{u}^2(t)^T\left[g^i_{y^2}\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)-\bar{\omega}^2_i(t)-Dg^i_{\dot{y}^2}\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)\right]dt\geqq 0\ ,\ t\in I$$

$$\left(\bar{v}^1(t),\bar{v}^2(t)\right)\geqq 0\ ,\ t\in I$$

$$\bar{\omega}^1_i(t)\in C^1_i\quad\text{and}\quad\bar{\omega}^2_i(t)\in C^2_i$$

$$\bar{\lambda}>0,\ \ \lambda^T e=1\ ,\ e^T=(1,.....,1)$$

also,

$$-G^i=-f^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-g^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)$$
$$-s\left(\bar{y}^1(t)\middle|K^1_i\right)-s\left(\bar{y}^2(t)\middle|K^2_i\right)-\bar{x}^1(t)^T\,\bar{\omega}^1_i(t)-\bar{x}^2_i(t)^T\,\bar{\omega}^2(t)$$
$$+\bar{x}^1(t)^T\sum_{i=1}^{p}\bar{\lambda}^i\left[f^i_{x^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-\bar{\omega}^1_i(t)-Df^i_{\dot{x}^1}\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right]$$

$$
\begin{aligned}
&= f^i\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}\right)+g^i\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)\\
&\quad -s\left(\bar{y}^1(t)\middle|K_i^1\right)-s\left(\bar{y}^2(t)\middle|K_i^2\right)-\bar{x}^1(t)^T\,\bar{\omega}_i^1(t)-x^2(t)^T\,\omega_i^2(t)\\
&\quad -x^1(t)^T\sum_{i=1}^{p}\lambda^i\left[f_{y^1}^i\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}\right)-\bar{\omega}_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}\right)\right]\\
&= H^i\left(t,\bar{y}^1,\bar{y}^2,\dot{\bar{y}}^1,\dot{\bar{y}}^2,\bar{x}^1,\bar{x}^2,\dot{\bar{x}}^1,\dot{\bar{x}}^2,\bar{\omega}_i^1,\bar{\omega}_i^2,\bar{\lambda}\right)
\end{aligned}
$$

Hence, by using various hypotheses of this theorem, we have

(MIX SP-1): Maximum $\int_I \left(H^1,H^2,\ldots,H^p\right)dt$

Subject to

$$x^1(a)=0=x^1(b)\ ,\quad y^1(a)=0=y^1(b)$$

$$x^2(a)=0=x^2(b)\ ,\quad y^2(a)=0=y^2(b)$$

$$\sum_{i=1}^{p}\bar{\lambda}^i\left[f_{y^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)-\bar{z}_i^1(t)-Df_{\dot{y}^1}^i\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)\right]\leqq 0\ ,\ t\in I\ ,$$

$$\sum_{i=1}^{p}\bar{\lambda}^i\left[g_{y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-\bar{z}_i^2(t)-Dg_{\dot{y}^1}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right]\leqq 0\ ,\ t\in I\ ,$$

$$\int_I \bar{y}^2(t)^T\left[\sum_{i=1}^{p}\bar{\lambda}^i\left(g_{y^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)-\bar{z}_i^2(t)-Dg_{\dot{y}^2}^i\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\right)\right]\geqq 0,$$

$$\left(\bar{y}^1(t),\bar{y}^2(t)\right)\geqq 0\ ,\ t\in I$$

$$\bar{z}_i^1(t)\in C_i^1\quad\text{and}\quad \bar{z}_i^2(t)\in C_i^2\quad,\ i=1,2,\ldots,p$$

$$\bar{\lambda}>0,\ \ \lambda^T e=1\ ,\ e^T=(1,\ldots..,1)$$

which is just the primal problem (Mix SP). Therefore $\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t),\bar{z}_1^1(t),\bar{z}_2^1(t),\ldots,\bar{z}_p^1(t);\bar{z}_1^2(t),\bar{z}_2^2(t),\ldots,\bar{z}_p^2(t),\bar{\lambda}\right)$is an efficient solution of dual problem implies that $\left(\bar{y}^1(t),\bar{y}^2(t),\bar{x}^1(t),\bar{x}^2(t),\bar{z}_1^1(t),\bar{z}_2^1(t),\ldots,\bar{z}_p^1(t);\bar{z}_1^2(t),\bar{z}_2^2(t),\ldots,\bar{z}_p^2(t),\bar{\lambda}\right)$is an efficient solution of the primal.

Similarly$\left(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t),\bar{z}_1^1(t),\bar{z}_2^1(t),\ldots,\bar{z}_p^1(t);\bar{z}_1^2(t),\bar{z}_2^2(t),\ldots,\bar{z}_p^2(t),\bar{\lambda}\right)$ is an efficient solution of (MixSP) implies

$\left(\bar{y}^1(t),\bar{y}^2(t),\bar{x}^1(t),\bar{x}^2(t),\bar{z}_1^1(t),\bar{z}_2^1(t),...,\bar{z}_p^1(t);\bar{z}_1^2(t),\bar{z}_2^2(t),...,z_p^2(t),\bar{\lambda}\right)$

is an efficient solution of the dual problem (MixSD).

In view of (7.73), (7.74), (7.91), (7.93) and (7.94), we have,

Minimum **(MixSP)** $=\int_I\Big\{f^1\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)+g^1\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)$

$,...,f^p\left(t,\bar{x}^1,\dot{\bar{x}}^1,\bar{y}^1,\dot{\bar{y}}^1\right)+g^p\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\Big\}dt$

Corresponding, to the solution

$\left(\bar{y}(t),\bar{x}(t),\bar{z}_1^1(t),\bar{z}_2^1(t),...,\bar{z}_p^1(t);\bar{z}_1^2(t),\bar{z}_2^2(t),...,\bar{z}_p^2(t),\bar{\lambda}\right)$, we have,

Minimum **(Mix SP)** = $\int_I\Big\{f^1\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}^1\right)+g^1\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)$

$,...,f^p\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}^1\right)+g^p\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)\Big\}dt$

By the skew-symmetry of each f^i, we have,

$$\int_I\Big\{f^1\left(t,\bar{x}^1,\dot{\bar{x}}^2,\bar{y}^1,\dot{\bar{y}}^2\right)+g^1\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)$$

$$,...,f^p\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)+g^p\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\Big\}dt$$

$$=\int_I\Big\{f^1\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}^1\right)+g^1\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)$$

$$,...,f^p\left(t,\bar{y}^1,\dot{\bar{y}}^1,\bar{x}^1,\dot{\bar{x}}^1\right)+g^p\left(t,\bar{y}^2,\dot{\bar{y}}^2,\bar{x}^2,\dot{\bar{x}}^2\right)\Big\}dt$$

$$=-\int_I\Big\{f^1\left(t,\bar{x}^1,\dot{\bar{x}}^2,\bar{y}^1,\dot{\bar{y}}^2\right)+g^1\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)$$

$$,...,f^p\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)+g^p\left(t,\bar{x}^2,\dot{\bar{x}}^2,\bar{y}^2,\dot{\bar{y}}^2\right)\Big\}dt=0$$

7.2.4 Special Cases

If $J_2=0$, $L_2=0$, the pair of mixed type nondifferentiable symmetric dual problems (Mix SP) and (Mix SD) reduce to the Wolfe type nondifferentiable symmetric dual variational problem (WP) and (WD) studied by in Section 7.1.

If $J_1=0$, $L_1=0$, the pair of mixed type symmetric dual problem (Mix SP) and (Mix SD) reduce to the following Mond-Weir type symmetric dual variational problem (M-WP) and (M-WD) recently studied Section 7.1

If $C_i^1 = C_i^2 = \{0\}$ and $K_i^1 = K_i^2 = \{0\}, i = 1, 2..., p$, i.e., support functions are suppressed from the formulation of the dual models (Mix SP) and (Mix SD), we have the following pair of reduced dual models, studied in Chapter 6.

(Mix SP*): Minimize
$$\int_I \Big\{ f\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) + g\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) - y^1(t)^T \left(\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right) e \Big\} dt$$

Subject to

$$x^1(a) = 0 = x^1(b) \;, \quad y^1(a) = 0 = y^1(b) \;,$$
$$x^2(a) = 0 = x^2(b) \;, \quad y^2(a) = 0 = y^2(b) \;,$$
$$\lambda^T f_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) \leqq 0 \;, \; t \in I \;,$$
$$\lambda^T g_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) - D\lambda^T g_{\dot{y}^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) \leqq 0 \;, \; t \in I \;,$$
$$\int_I y^2(t)^T \left(\lambda^T g_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right), - D\lambda^T g_{\dot{y}^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\right) \geqq 0 \;,$$
$$\lambda \in \Lambda^+ .$$

(Mix SD*): Maximize
$$\int_I \Big\{ f\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) + g\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right) - u^1(t)^T \left(\lambda^T f_{y^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{y}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right)\right) e \Big\} dt$$

Subject to

$$u^1(a) = 0 = u^1(b) \;, \quad v^1(a) = 0 = v^1(b),$$
$$u^2(a) = 0 = u^2(b) \;, \quad v^2(a) = 0 = v^2(b),$$
$$\lambda^T f_{u^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) - D\lambda^T f_{\dot{u}^1}\left(t, u^1, \dot{u}^1, v^1, \dot{v}^1\right) \geqq 0 \;, \; t \in I \;,$$
$$\lambda^T g_{u^2}\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right) - D\lambda^T g_{\dot{u}^2}\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right) \geqq 0 \;, \; t \in I \;,$$
$$\int_I u^2(t)^T \left(\lambda^T g_{u^2}\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right) - D\lambda^T g_{\dot{u}^2}\left(t, u^2, \dot{u}^2, v^2, \dot{v}^2\right)\right) \geqq 0, ,$$
$$\lambda \in \Lambda^+ .$$

where $\Lambda^+ = \left\{ \lambda \in R^p \,\middle|\, \lambda > 0, \; \lambda^T e = 1, e = (1, 1, \ldots, 1)^T \in R^p \right\}$

7.2.5 Natural Boundary Values

The pairs of mixed type symmetric nondifferentiable multiobjective variational problem can be formulated with natural boundary values rather than fixed end points. The problems with natural boundary conditions are needed to establish well-defined relationship between the pairs of dual continuous programming problems and nonlinear programming problems.

Following is the pair of mixed type symmetric dual problems with natural boundary values:

Primal (Mix SP$_0$):

Maximize $\int_I \left(H^1, H^2, ..., H^p\right) dt$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left[f^i_{y^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right) - z_i^1(t) - Df^i_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\right] \leqq 0 \ , \ t \in I \ ,$$

$$\sum_{i=1}^{p} \lambda^i \left[g^i_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right) - z_i^2(t) - Dg^i_{\dot{y}^1}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\right] \leqq 0 \ , \ t \in I \ ,$$

$$\int_I y^2(t)^T \sum_{i=1}^{p} \lambda^i \Big(g^i_{y^2}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)$$
$$- z_i^2(t) - Dg^i_{\dot{y}^1}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\Big)\Big] dt \geqq 0,$$

$$\left(x^1(t), x^2(t)\right) \geqq 0 \ , \ t \in I$$

$$z_i^1(t) \in K_i^1 \quad \text{and} \quad z_i^2(t) \in K_i^2 \quad , \quad i = 1, 2, ..., p$$

$$\lambda > 0, \ \ \lambda^T e = 1 \ , \ e^T = (1, ...,1)$$

$$f^i_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\Big|_{t=a} = 0 \ , \ f^i_{\dot{y}^1}\left(t, x^1, \dot{x}^1, y^1, \dot{y}^1\right)\Big|_{t=b} = 0, i = 1, 2, ..., p,$$

$$\lambda^T g_{\dot{y}^1}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\Big|_{t=a} = 0 \ , \ \lambda^T g_{\dot{y}^1}\left(t, x^2, \dot{x}^2, y^2, \dot{y}^2\right)\Big|_{t=b} = 0$$

Dual (Mix SD$_0$):

Maximize $\int_I \left(G^1, G^2, ..., G^p\right) dt$

Subject to

$$\sum_{i=1}^{p}\lambda^i\left[f_{u^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)+\omega_i^1(t)-Df_{\dot{u}^1}^i\left(t,u^1,\dot{u}^1,v^1,\dot{v}^1\right)\right]\geqq 0\ ,\ t\in I,$$

$$\sum_{i=1}^{p}\lambda^i\left[g_{u^1}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)+\omega_i^2(t)-Dg_{\dot{u}^1}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right]\geqq 0\ ,\ t\in I,$$

$$\int_I u^2(t)^T\sum_{i=1}^{p}\lambda^i\left[g_{u^1}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)+\omega_i^2(t)\right.$$
$$\left.-Dg_{\dot{u}^1}^i\left(t,u^2,\dot{u}^2,v^2,\dot{v}^2\right)\right]dt\leqq 0,\ t\in I$$

$$\left(v^1(t),v^2(t)\right)\geqq 0\ ,\ t\in I$$

$$\omega_i^1(t)\in C_i^1\quad\text{and}\quad\omega_i^2(t)\in C_i^2,\qquad i=1,2,...,p$$

$$\lambda>0,\ \ \lambda^Te=1\ ,\ e^T=(1,.....,1)$$

$$f_{\dot{x}^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\Big|_{t=a}=0\ ,\ f_{\dot{x}^1}^i\left(t,x^1,\dot{x}^1,y^1,\dot{y}^1\right)\Big|_{t=b}=0\ ,\qquad i=1,2,...,p$$

$$\lambda^Tg_{\dot{x}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\Big|_{t=a}=0\ ,\ \lambda^Tg_{\dot{x}^2}\left(t,x^2,\dot{x}^2,y^2,\dot{y}^2\right)\Big|_{t=b}=0$$

The duality results for each of the above pairs of dual variational problems can be proved easily on the lines of the proofs of the Theorems 7.9-7.12, with slight modifications in the arguments, as in Mond and Hanson [108].

7.2.6 Multiobjective Nonlinear Programming

If the time dependency is removed from the variational problems (Mix SPo) and (Mix SDo) with natural boundary values and $b-a=1$, we obtain the following pair of static mixed type multiobjective dual problems involving support functions, which are not explicitly reported in the literature with their correct formulations..

Primal (Mix SP$_1$):

Minimize $\hat{H} = \left(\hat{H}^1, \hat{H}^2, ..., \hat{H}^p\right)$

Subject to

$$\sum_{i=1}^{p} \lambda^i \left[f_{y^1}^i \left(x^1, y^1\right) - z_i^1 \right] \leqq 0,$$

$$\sum_{i=1}^{p} \lambda^i \left[g_{y^2}^i \left(x^2, y^2\right) - z_i^2 \right] \leqq 0,$$

$$\left(y^2\right)^T \left[\sum_{i=1}^{p} \lambda^i \left(g_{y^2}^i \left(x^2, y^2\right) - z_i^2 \right) \right] \geqq 0,$$

$$z_i^1 \in K_i^1 \quad \text{and} \quad z_i^2 \in K_i^2 \ , \ i = 1, ..., p,$$

$$\lambda \in \Lambda^+,$$

$$\hat{H}^i = f^i\left(x^1, y^1\right) + g^i\left(x^2, y^2\right) + s\left(x^1 \middle| C_i^1\right) + s\left(x^2 \middle| C_i^2\right)$$
$$- y^1 \sum_{i=1}^{p} \lambda^i \left[f_{y^1}^i \left(x^1, y^1\right) - z_i^1 - Df_{\dot{y}^1}^i \left(x^1, y^1\right) \right] - z_i^1 y^1 - z_i^2 y^2$$

Dual (Mix SD1):

Maximize $\hat{G} = \left(\hat{G}^1, \hat{G}^2, ..., \hat{G}^p\right)$

Subject to:

$$\sum_{i=1}^{p} \lambda^i \left[f_{u^1}^i \left(u^1, v^1\right) + \omega_i^1 \right] \geqq 0 \ ,$$

$$\sum_{i=1}^{p} \lambda^i \left[g_{u^1}^i \left(u^2, v^2\right) + \omega_i^2 \right] \geqq 0,$$

$$\left(u^2\right)^T \sum_{i=1}^{p} \lambda^i \left[g_{u^1}^i \left(u^2, v^2\right) + \omega_i^2 \right] \leqq 0,$$

$$\omega_i^1 \in C_i^1 \quad \text{and} \quad \omega_i^2 \in C_i^2 \quad i = 1, 2, ..., p,$$

$$\lambda \in \Lambda^+.$$

where,

$$\hat{G}^i = f^i\left(x^1, y^1\right) + g^i\left(x^2, y^2\right) + s\left(v^1 \middle| K_i^1\right) + s\left(v^2 \middle| K_i^2\right)$$
$$+ u^1 \omega_i^1 + u_i^2 \omega^2 - u^1 \sum_{i=1}^{p} \lambda^i \left[f_{u^1}^i \left(x^1, y^1\right) + \omega_i^1 - Df_{\dot{u}^1}^i \left(x^1, y^1\right) \right]$$

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