DUALITY IN MATHEMATICAL PROGRAMMING

THESIS SUBMITTED FOR THE
DEGREE OF

Doctor of Philosophy

IN

STATISTICS

BY

MASHOOB MASOODI

POST GRADUATE DEPARTMENT OF STATISTICS
FACULTY OF PHYSICAL AND MATERIAL SCIENCES
UNIVERSITY OF KASHMIR
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<u>Certificate</u>

This is to certify that the work embodied in this thesis entitled **"ON DUALITY IN MATHEMATICAL PROGRAMMING"** is the original work carried out by **Mrs. Mashoob Masoodi** under our supervision and is suitable for the award of the degree of **Doctor of Philosophy** in Statistics.

The thesis has reached the standard fulfilling the requirements of regulations relating to the degree. The results contained in the thesis have not been submitted earlier to this or any other university or institute for the award of degree or diploma.

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ABSTRACT

In this thesis entitled, **"Duality in Mathematical Programming",** the emphasis is given on formulation and conceptualization of the concepts of second-order duality, second-order mixed duality, second-order symmetric duality in a variety of nondifferentiable nonlinear programming under suitable second-order convexity/second-order invexity and generalized second-order convexity / generalized second-order invexity. Throughout the thesis nondifferentiablity occurs due to square root function and support functions. A support function which is more general than square root of a positive definite quadratic form. This thesis also addresses second-order duality in variational problems under suitable second-order invexity/second-order generalized invexity. The duality results obtained for the variational problems are shown to be a dynamic generalization for thesis of nonlinear programming problem.

The thesis spreads over seven chapters.

CHAPTER – 1 is an introductory one. It offers a brief survey of related work and the summary of the research work reported in the thesis. The chapter is followed by the summary of the thesis.

CHAPTER – 2 consists of two sections, 2.1 and 2.2. The Section 2.1 deals with the second-order duality in nonlinear programming containing support functions. In this section formulations of Wolfe and Mond-Weir type duals to a nondifferentiable mathematical programming are presented and various appropriate duality theorems are validated. In the subsection 2.1.5 various special cases are also derived. In the section 2.2 mixed type second-order duality in order to combine the dual models of previous section is studied, as

it is noticed that the concept of mixed duality seems to be interesting and useful both from theoretical and algorithmic point of view.

CHAPTER – 3 is focused on the nondifferentiable multiobjective second-order duality. This section presents pair of Wolfe and Mond-Weir type multiobjective symmetric dual programs. For each pair, various duality theorems namely weak, strong and converse type duality are established under suitable second-order convexity. The subsection 3.1.2 and 3.2.2 incorporate self duality for both the pairs.

CHAPTER – 4 studies second-order symmetric duality in mathematical rogramming over cones. The subsection 4.1.3 deals with second-order symmetric and self duality for the programming problems containing support functions. The subsection 4.1.4 provides maxmin symmetric and self duality. The subsection 4.1.5 deduces some special cases.

CHAPTER – 5 The purpose of chapter 5 is to present multiobjective version of second-order mixed and self duality in traditional mathematical programming with a single objective. In addition to validation of various duality theorems under suitable second-order convexity/ generalized second-order convexity, in the subsection 5.1.4 self-duality theorem is also validated for the pair of dual programs under additional restrictions on the kernel function that appears in the formulations of the problems.

CHAPTER – 6 presents a study of second-order duality in variational problems and gives a formulation of Mond-Weir type second-order dual problem which allows weakening of second-order invexity/second-order pseudoinvexity of Wolfe type second-order dual in variational problems. Second-order invexity and generalized invexity functions are introduced. Using these second-order invexity and generalized invexity, various duality theorems are established in the subsection 6.1.3. The subsection 6.1.4 gives the second-order dual problem with natural boundary conditions when the fixed point boundary conditions are ignored. The subsection 6.1.5 points out a close relationship between the results established in this chapter with those of second-order duality nonlinear programming.

CHAPTER – 7 is devoted to the study of second-order duality for a class of nondifferentiable variational problems in which nondifferentiablity occurs due to the presence of square root of a quadratic form and support functions. In the section 7.1 variational problem containing square root of quadratic form is considered .The nondifferentiable term occurs in the integrant of the objective functional. A Wolfe type second-order dual variational problem is formulated. In subsection 7.1.1 various duality theorems are proved under second-order pseudoinvexity assumptions. The subsection 7.1.2 gives a pair of second-order Wolfe type variational problems with natural boundary conditions and in subsection 7.1.3 points out a close relationship between the results established in this chapter with those of second-order duality nonlinear programming. In the section 7.2 a second-order dual problem is formulated for a wider class of continuous programming problem in which both objective and constrained functions contain support functions. In the section 7.2.2 under second-order invexity and second-order pseudoinvexity, weak, strong and converse duality theorems are established for the pair of dual problems. In section 7.2.3, special cases are deduced and a pair of dual continuous problems with natural boundary values is constructed in the section 7.2.4. A close relationship between duality results of our problems and those of the corresponding (static) nonlinear programming problem with support functions is briefly outlined in the section 7.2.5.

The subject matter of the present thesis is published/under publication in the form of the following research papers written by author:

RESEARCH PUBLICATIONS

1) Second-order Symmetric and Maxmin Symmetric Duality with Cone Constraints, *International Journal of Operations Research,* Vol. 4 (2007), No. 4, pp 199 – 205.
2) On Nondifferentiable Multiobjective Second-order Symmetric Duality, *International Journal of Operations Research,*Vol. 5, (2008), No. 2, pp 91-98.
3) Non-Differentiable Second-order Self Symmetric Dual Multi-objective Programs, *Journal of Applied Mathematics and Informatics,* Vol. 26(2008), No. 3 - 4, pp 549 – 561.
4) Second-order Duality in Mathematical Programming with Support Functions, *Journal of Informatics and Mathematical Sciences* Vol.1, (2009), issue 2 and 3 pp.183-197.
5) Second-order Duality for Variational Problems, *European Journal of Pure and Applied Mathematics*, Vol 2, No. 2 (2009), pp 278- 295.
6) Mixed Type Second-order Duality with Support Function, *Journal of Applied Mathematics and Informatics,* Vol.27, (2009) No.5-6, pp. 1981-1995.
7) Mixed Type Second-order Symmetric Duality in Multiobjective Programming, *Journal of Informatics and Mathematical Sciences,* Vol.1, (2009), issue 2 and 3 pp.165-182.
8) Second-order Duality for a Class of Nondifferentiable Continuous Programming Problems, *to appear in European Journal of Pure and Applied Mathematics,* 2010.
9) Second-order Duality for Continuous Programming Containing Support Functions, *to appear in Applied Mathematics*.

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"And my affair I leave it to Allah. Verily, Allah is the All-Seer of (His) slaves."

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Mashoob Masoodi

Dedicated To My Parents

Since the fabric of the universe is most perfect and the work of the most wise creator, nothing at all takes place in the Universe in which some rule of the maximum or minimum does not appear.

Leonhard Euler

C O N T E N T S

1.1 GENERAL INTRODUCTION

Mathematical programming earned a status of scientific field in its own right during late 1940's and since then it has undergone significant development. It is now regarded as one of the most vital and exciting part of modern mathematics having applications in various scientific disciplines such as, engineering economics and natural sciences. A very common example of mathematical programming model appears in determining minimum weight design of structure subject to constraints on stress and deflection.

A general mathematical programming problem (MPP) can be stated as:

(MP) Optimize (minimize/maximize) $f(x)$

Subject to

$$
\begin{aligned}
&g_i(x) \le 0 \qquad (i = 1, 2, \ldots, m),\\
&h_j(x) = 0 \qquad (j = 1, 2, \ldots, k),\\
&x \in X
\end{aligned}
$$

where

i) $x=(x_1,x_2,...,x_n)^T$ is the vector of unknown decision variables and

ii) $f, g_i(i=1,...,m),\ \ h_j(j=1,...,k)$ are the real-valued functions of n real variables $x_1,...,x_n$ and $X \subseteq R^n$.In this formulation, the function f is called the objective function, the constraints. $g_i(x)\le 0,\ \ i=1,2,3,...,m$ are referred to as an inequality constraints, the constraints $h_j(x)=0,\ \ j=1,2,3,...,k$ are called equality constraints. The inclusion $x \in X$ is known as abstract constraints.

If the objective and constraint functions are differentiable then we describe above problem as differentiable programming problem. If the objective and inequality constraints are affine functions and X is convex set, then the above problem is known as convex programming problem.

If all the functions in the problem (MP) are linear then it is called linear programming problem (LPP). Dantzig developed his famous Simplex technique for solving linear programming models during the mid 1940's, though initially applied for warfare planning. Its elegance drove many scientists to solve linear programming models arising in a variety of contexts such as economics, business and engineering sciences. If the objective function and atleast one of the constraint or both are nonlinear functions in the mathematical programming problem, then the problem is termed as nonlinear programming problem, which was first introduced by R. Courant in 1943. It is the most general programming problem and other problems can be treated as special cases of the nonlinear programming problem (NLPP). Nonlinear programming plays a significant role in management science, engineering, economics, system analysis physical sciences and other areas. Some methods for solving nonlinear programming problem were discussed by Avriel [4] and Zangwill [99].

The pioneer work by Kuhn Tucker in 1951 on necessary and sufficient conditions for the optimal solution laid the foundation for the researchers to work on the nonlinear system. In 1957, the emergence of dynamic programming by Bellman brought a revolution in the subject and consequently, linear and non-linear systems have been studied simultaneously. It is disappointing to note that possibly no universal technique has been established for nonlinear system as yet.

Optimality conditions and duality have played a vital role in the progress of mathematical programming. Fritz John [59] was the first to derive necessary optimality conditions for constrained optimization problem using a Lagrange multiplier rule. Later, Kuhn and Tucker [62] established necessary optimality conditions for the existence of an optimal solution under certain constraint qualification in 1951.It was revealed afterwards that W.Karush [60] had presented way back in 1939 without imposing any constraint qualification; thus the Kuhn-Tucker conditions are now known as Karush-Kuhn-Tucker optimality conditions. Abadie [1] established a regularity condition that enabled him to derive Karush-Kuhn-Tucker conditions and Fritz John optimality conditions. Subsequently, Mangasarian and see Formovitz [68] generalized Fritz John optimality conditions which have not only laid down the foundation for many computational techniques in mathematical programming, but also are responsible for development of duality theory to a great deal. The inception of the duality theory in linear programming may be traced to the classical minimax theorem of

Von Neumann [87] and was explicitly incorporated by Gale, Kuhn and Tucker [43].Since then, it has become one of the most widely used and investigated area of mathematical programming. An extensive use of duality in mathematical programming has been made for many theoretical and computational developments in mathematical programming itself and in other fields which include engineering, operations research, economics and mathematical science.

The principle of duality connects two programs, one of which is called the primal problem and the other is called the dual problem, in such a way that the existence of an optimal solution to one of them guarantees an optimal solution to other. If the primal problem is constraint minimization (or maximization), the dual is the constrained maximization (minimization) problem. The duality results have proved to be very useful in the development of numerical algorithms for solving certain classes of optimization problem. The existence of duality theory in nonlinear programming problem helps to develop numerical algorithm as it provides suitable stopping rules for primal and dual problems. A nonlinear programming problem and its dual are said to be symmetric if dual of the dual is the original problems.

Multiobjective optimization is the art of detecting and making good compromises. It is based upon the fact that most real-world decisions are compromises between partially conflicting objectives that cannot easily be offset against each other. Thus, one is forced to look for possible compromises and finally decide which one to implement. So, the final decision in multiobjective optimization is always with a person-the decision maker.

The first notion of optimality in this setting is popularly known as Pareto-optimality and is still the most widely used. In Pareto optimality every feasible alternative that is not dominated by any other in terms of the component wise partial order is considered to be optimal. Hence each solution is considered optimal that is not definitely worse than another. Thus, multiobjective optimization does not yield a single or a set of equally good answers, but rather suggests a range of potentially very different answers.

A general multiobjective programming problem (MOPP) can be expressed as:

(MP): Optimize (minimize/maximize) $\left(f^1(x), f^2(x), \ldots, f^p(x)\right)$

Subject to

$$\begin{aligned} & g_i(x) \geq 0 \qquad (i = 1, 2, \ldots, m), \\ & h_j(x) = 0 \qquad (i = 1, 2, \ldots, k), \\ & x \in X. \end{aligned}$$

where $x \in R^n$, $f, g_i(i = 1, \ldots, m)$, $h_j(j = 1, \ldots, k)$ and X are described earlier.

Duality for continuous programming problems has been studied by many researchers. Mond and Hanson [77] were the first to consider a class of constrained variational problems and dealt with duality aspect of such problem, where the dual problem was the first-order dual. Later, a number of researchers have derived duality theorems for different forms of continuous programming or control problems, notably, Chandra, Craven and Husain [19], Bector, Chandra and Husain [13], Mond and Husain [78] and Chen [26,27] .

Mond and Hanson [78] formulated the following pair of dual variational problems:

Primal Problem:

Minimize $\int_I f(t,x,\dot{x})\,dt$

Subject to

$$x(a)=\alpha,\quad x(b)=\beta,$$

$$g(t,x,\dot{x})\leq 0,\qquad t\in I.$$

Dual problem:

Maximize $\int_a^b \left\{ f(t,u(t),\dot{u}(t)) + y(t)^T g(t,u(t),\dot{u}(t)) \right\} dt$

Subject to

$$u(a)=\alpha,\quad u(b)=\beta,$$

$$f_u(t,u(t),\dot{u}(t)) + y(t)^T g_u(t,u(t),\dot{u}(t))$$
$$-D\left[f_{\dot{u}}(t,u(t),\dot{u}(t)) + y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t)) \right] = 0,\quad t\in I$$
$$y(t)\geq 0,\qquad t\in I$$

where

i) $I=[a,b]$, a real interval and

ii) $f: I\times R^n\times R^n \to R,\ g: I\times R^n\times R^n \to R^m$ are continuously differentiable and $y: I\to R^m$ is piecewise smooth functions.

Second-order duality in mathematical programming has been extensively investigated in the literature. A second-order dual formulation for a non-linear programming problem was introduced by Mangasarian [67]. Later Mond [70] established various duality theorems under a condition which is called "Second order convexity". This condition is much simpler than that used by Mangasarian [66]. In [84], Mond and Weir reconstructed the second-order duals and higher order dual models to drive usual duality results. It is remarked here that second-order dual to a mathematical programming problem presents a tighter bound and because of which it enjoys computational advantage over a first order dual. Chen [27] was the first to identify second-order dual for a constrained variational problem and established various duality results under an involved invexity- like assumptions.

This thesis is a reflection of above narrated brief survey of literature. The main contribution of this thesis is to study duality and multiobjective duality including self and symmetric duality for a variety of mathematical programming problems confined to nondifferentiable nonlinear programming with square root of certain quadratic form and support functions which generally arise in various contexts such as in models representing oscillation of mechanical system and portfolio selection. This thesis is also devoted to study second-order duality in nonlinear programming and variational problems.

1.2 PRE-REQUISITES

1.2.1 Notations

R^n **=** n-dimensional Euclidean space,

R_+^n = The non-negative orthand in R^n**,**

A^T= Transpose of the matrix A,

$$x^T e = \sum_{1=1}^{m} x_i,\ x \in R^m\ ,\ e = (1,1,...,1) \in R^m.$$

Let θ be a numerical function defined on an open set Γ in R^n, then $\nabla f(\bar{x})$ denotes the gradient of θ at $\bar{x}$, that is

$$\nabla f(\bar{x}) = \left[\frac{\partial f(\bar{x})}{\partial x^1}, \ldots, \frac{\partial f(\bar{x})}{\partial x^2}\right]^T$$

Let ϕ be a real valued twice continuously differentiable function defined on an open set contained in $R^n \times R^m$. Then $\nabla_x \phi(x, y)$ and $\nabla_y \phi(x, y)$ denote the gradient (column) vector of ϕ with respect to x and y respectively i.e.,

$$\nabla_x \phi(\bar{x}, \bar{y}) = \left(\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \ldots, \frac{\partial \phi}{\partial x^n} \right)^T_{(\bar{x}, \bar{y})}$$

$$\nabla_y \phi(\bar{x}, \bar{y}) = \left(\frac{\partial \phi}{\partial y^1}, \frac{\partial \phi}{\partial y^2}, \ldots, \frac{\partial \phi}{\partial y^n} \right)^T_{(\bar{x}, \bar{y})}$$

Further, $\nabla^2_{xx} \phi(\bar{x}, \bar{y})$ and $\nabla^2_{xy} \phi(\bar{x}, \bar{y})$ denote respectively the $(n \times n)$ and $(n \times m)$ matrices of second- order partial derivative i.e.,

$$\nabla^2_{xx} \phi(\bar{x}, \bar{y}) = \left(\frac{\partial^2 \phi}{\partial x^i x^j} \right)_{(\bar{x}, \bar{y})}$$

$$\nabla^2_{xy} \phi(\bar{x}, \bar{y}) = \left(\frac{\partial^2 \phi}{\partial x^i x^j} \right)_{(\bar{x}, \bar{y})}$$

The symbols $\nabla^2_{yy} \phi(\bar{x}, \bar{y})$ and $\nabla^2_{yx} \phi(\bar{x}, \bar{y})$ are similarly defined. However, at certain places, to make the meaning of the context more clear, the subscript of ∇ and ∇^2 are taken as the variable with respect to which the function is being differentiated.

1.2.2 Definitions

Definition 1.1: Let X ⊆ R^n be an open and convex set and *f*: X⟶R be differentiable. Then we define f to be

1. **Convex**, if for all $x_1, x_2 \in X$,

$$f(x_1) - f(x_2) \geq (x_1 - x_2)^T \nabla f(x_2)$$

2. **Strict convex**, if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$

$$f(x_1) - f(x_2) > (x_1 - x_2)^T \nabla f(x_2)$$

3. **Quasi convex**, if for all $x_1, x_2 \in X$,

$$f(x_1) \leq f(x_2) \Rightarrow (x_1 - x_2)^T \nabla f(x_2) \leq 0$$

4. **Psedoconvex**, if for all $x_1, x_2 \in X$,

$$(x_1 - x_2)^T \nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2)$$

5. **Strictly pseudoconvex**, if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$

$$(x_1 - x_2)^T \nabla f(x_2) \geq 0 \Rightarrow f(x_1) > f(x_2)$$

6. **Invex,** if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$f(x_1) - f(x_2) \geq \eta^T(x_1, x_2) \nabla f(x_2)$$

7. **Pseudoinvex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$\eta^T(x_1, x_2) \nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2)$$

8. **Quasi-invex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$f(x_1) \leq f(x_2) \Rightarrow \eta^T(x_1, x_2) \nabla f(x_2) \leq 0.$$

Definition 1.2: Let f be a real valued twice differentiable function defined on an open set $X \subseteq R^n$, then f is said to be

1. **Second-order convex,** if for all $x, p, u \in R^n$

$$f(x) - f(u) \geq (x-u)^T \left[\nabla f(u) + \nabla^2 f(u) p\right] - \tfrac{1}{2} p^T \nabla^2 f(u) p.$$

2. **Second-order concave**, if for all $x, p, u \in R^n$

$$f(x) - f(u) \leq (x-u)^T \left[\nabla f(u) + \nabla^2 f(u) p\right] - \tfrac{1}{2} p^T \nabla^2 f(u) p.$$

3. **Second-order pseudoconvex**, if for all $x, p, u \in R^n$

$$(x-u)^T\left[\nabla f(u)+\nabla^2 f(u)p\right] \geq 0 \Rightarrow f(x) \geq f(u) - \tfrac{1}{2} p^T \nabla^2 f(u)p$$

4. **Second-order pseudoconcave**, if for all $x, p, u \in R^n$

$$(x-u)^T\left[\nabla f(u)+\nabla^2 f(u)p\right] \leq 0 \Rightarrow f(x) \leq f(u) - \tfrac{1}{2} p^T \nabla^2 f(u)p$$

5. **Second-order quasiconvex**, if for all $x, p, u \in R^n$

$$f(x) - f(u) + \tfrac{1}{2} p^T \nabla^2 f(u)p \leq 0 \Rightarrow (x-u)^T\left[\nabla f(u)+\nabla^2 f(u)p\right] \leq 0.$$

6. **Second-order quasiconcave**, if for all $x, p, u \in R^n$

$$f(x) - f(u) + \tfrac{1}{2} p^T \nabla^2 f(u)p \geq 0 \Rightarrow (x-u)^T\left[\nabla f(u)+\nabla^2 f(u)p\right] \geq 0.$$

7. **Second-order invex,** if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$f(x) - f(u) \geq \eta^T(x,u)\left[\nabla f(u)+\nabla^2 f(u)p\right] - \tfrac{1}{2} p^T \nabla^2 f(u)p.$$

8. **Second-order incave**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$f(x) - f(u) \leq \eta^T(x,u)\left[\nabla f(u)+\nabla^2 f(u)p\right] - \tfrac{1}{2} p^T \nabla^2 f(u)p.$$

9. **Second-order pseudoinvex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$\eta^T(x,u)\left[\nabla f(u)+\nabla^2 f(u)p\right] \geq 0 \Rightarrow f(x) \geq f(u) - \tfrac{1}{2} p^T \nabla^2 f(u)p\,.$$

10. **Second-order pseudoincave**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x, u \in X$

$$\eta^T(x,u)\left[\nabla f(u)+\nabla^2 f(u)p\right] \leq 0 \Rightarrow f(x) \leq f(u) - \tfrac{1}{2} p^T \nabla^2 f(u)p\,.$$

11. **Second-order quasi-invex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x,u \in X$

$$f(x) - f(u) + \tfrac{1}{2} p^T \nabla^2 f(u) p \leq 0 \Rightarrow \eta^T(x,u)\left[\nabla f(u) + \nabla^2 f(u) p\right] \leq 0.$$

12. **Second-order quasi-incave**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x,u \in X$

$$f(x) - f(u) + \tfrac{1}{2} p^T \nabla^2 f(u) p \geq 0 \Rightarrow \eta^T(x,u)\left[\nabla f(u) + \nabla^2 f(u) p\right] \geq 0.$$

Clearly, a differentiable convex, pseudoconvex, quasiconvex function is invex, pseudoinvex or quasi-invex respectively with $\eta(x_1, x_2)^T = (x_1 - x_2)$.Further we define *f* to be concave, strictly concave pseudoconcave, quasiconcave, strictly pseudo convex on X according as $-f$ is convex, strictly convex, quasi convex, pseudoconvex, strictly pseudoconvex.

Definition 1.3: Let *C* be compact convex set in R^n. The support function of *C* is defined by

$$s(x \mid C) = \max\{x^T y : y \in C\}.$$

Definition 1.4: Let $f : R^n \to R$ be a convex function, then a subgradient of *f* at a point $x \in R^n$ is a vector $\xi \in R^n$ satisfying

$$f(y) \geq f(x) + \xi(y - x), \ \forall \ y \in R^n$$

The set of all subgradients of at $x \in R^n$ is called subdifferential of *f* at *x* is denoted by $\partial f(x)$.

Definition 1.5: Let Γ be a nonempty of R^n

i) The set Γ is called cone if

$$x \in \Gamma, \lambda \geq 0 \Rightarrow \lambda, x \in \Gamma$$

ii) A cone $\Gamma \in R^n$ be a convex if

$$x + y \in \Gamma \text{ for all } x, y \in \Gamma$$

iii) A cone $\Gamma \in R^n$ be a convex cone. Then Γ^* defined as

$$\Gamma^* = \left\{ z \in R^n \middle| : z^T x \leq 0, \ for \ \ all \ \ x \in \Gamma \right\}$$

is called the polar cone of Γ

Consider the following multiobjective programming problem:

(VP): Minimize $\phi(z) = \left(\phi_1(x), \phi_2(x), ..., \phi_p(x)\right)$

Subject to

$$h_j(x) \leq 0, j = 1, 2, ...n$$

Definition 1.6: A feasible point $\bar{x}$ is said to be a weak minimum of (VP), if there does not exist any $x \in X_0$ such that $\phi(x) < \phi(\bar{x})$.

A feasible point $\bar{x}$ is said to be efficient solution of (VP), if there does not exist any feasible *x* such that $\phi(x) \leq \phi(\bar{x})$.

An efficient solution of (VP) is obviously a weak minimum to (VP).

A feasible point $\bar{x}$ is said to be properly efficient solution of (VP), if it is an efficient solution of (VP) and if there exists a scalar $M > 0$ such that for each *i* and $x \in X_0$ satisfying $\phi_i(x) < \phi_i(\bar{x})$, we have

$$\frac{\phi_i(\bar{x}) - \phi_i(x)}{(\phi_j(x) - \phi_j(\bar{x})} \leq M \ ,$$

for some *j*, satisfying $\phi_j(x) > \phi_j(\bar{x})$.

An efficient point $\bar{x} \in X$ that is not properly efficient is said to be improperly efficient. Then $\bar{x}$ is improperly efficient means that every scale M > 0 (no matter how large), then point $x \in$X and *i* such that $\psi^i(x) < \psi^i(\bar{x})$ and

$$\frac{\psi^i(\bar{x}) - \psi^i(x)}{\psi^j(x) - \psi^j(\bar{x})} > M,$$

for all *j* satisfying $\psi^j(x) > \psi^j(\bar{x})$

Definition 1.7: A function $f : R^n \times R^n \to R$ is said to be skew-symmetric if

$$f(x, y) = -f(y, x)$$

There are a number of constraint qualifications [67], which are required to be satisfied by the constraints, while establishing the necessary optimality criteria to ensure that certain Lagrange multipliers exist and are non-zero. Here we describe only four of them for completeness of notations.

i) **Slater's constraint Qualification:** Let X^0 be a convex set in R^n. The m-dimensional convex vector function *g* on X^0 which defines the convex feasible region $X = \{x : x \in X^o, g(x) \leq 0\}$ is said to satisfy Slater's constraint qualification on X^0 if there exist an $\bar{x} \in X^o$ such that $g(\bar{x}) \leq 0$.

ii) **The Kuhn Tucker Constraint Qualification:** Let X^0 be an open set in R^n. Let *g* be m-dimensional vector function on X^0 and let $X = \{x : x \in X^o, g(x) \leq 0\}$. Then the constraints are said to satisfy Kuhn Tucker constraint qualification at $\bar{x} \in X$,if g is differentiable at $\bar{x}$ and if

$$\left.\begin{array}{l} y \in R^n \\ \\ \\ \\ \\ \nabla g_i(\bar{x})\, y \leq 0 \end{array}\right\rangle \Rightarrow \left\langle \begin{array}{l} \text{There exists a n-dimentional vector function e} \\ \text{in the interval } [0,1] \text{ such that} \\ (a)\quad e(0) = \bar{x} \\ (b)\quad e(\tau) \in X \ \ for\ \ 0 \leq \tau \leq 1 \\ (c)\quad e \text{ is differentiable } at\ \tau = 0 \text{ and} \\ \qquad \dfrac{d\,e(0)}{d\tau} = \lambda y \text{ for some } \lambda > 0. \end{array}\right.$$

where $I = \{i | g_i(\bar{x}) = 0\}$.

iii) **The reverse convex constraint qualification:** Let X^0 be an open set in R^n .Let g be m-dimensional vector function defined on X^0 and let $X = \{x : x \in X^o, g(x) \leq 0\}$, g is said to satisfy the reverse constraint qualification at $\bar{x} \in X$,if g is differentiable at $\bar{x}$ and if for each $i \in I$ either g_i is concave at $\bar{x}$ or g_i is linear on R^n ,where $I = \{i | g_i(\bar{x}) = 0\}$.

iv) **Linear independence constraint qualification:** The condition that the vectors $\nabla g_i(x_0), \ldots\ldots, \nabla g_m(x_0)$ are linearly independent and is often referred to as linearly independence constraint qualification.

1.3 REVIEW OF THE RELATED WORK

1.3.1 Duality in Mathematical Programming

Nonlinear Programming

Consider the nonlinear programming problem:

(P): Minimize $f(x)$

Subject to

$$h_j(x) \le 0, \quad (j = 1, 2, \ldots, m)$$

where $f : R^n \to R$ and $h_j : R^n \to R$, $(j = 1, 2, \ldots, m)$ are continuously differentiable. The following problem:

(WD): Maximize $f(x) + y^T h(x)$

Subject to

$$\nabla\left(f(x) + y^T h(x)\right) = 0,$$

$$y \ge 0,\ y \in R^m$$

is known as the Wolfe [98] type dual for the problem (P). Mangasarian [67] explained by means of an example that certain duality theorems may not be valid if the objective or the constraint function is a generalized convex function. This motivated Mond and Weir [82] to introduce a different dual for (P) as

(MWD): Maximize $f(x)$

Subject to

$$\nabla f(x) + \nabla y^T h(x) = 0.$$

$$y^T h(x) \ge 0$$

$$y \ge 0,\ y \in R^m$$

and they proved various duality theorems under pseudoconvexity of f and quasiconvexity of $y^T h(\cdot)$ for all feasible solution of (P) and (MWD).

Later Weir and Mond [95] derived sufficiency of Fritz John optimality criteria under pseudoconvexity of the objective and quasiconvexity or semi-strict convexity of constraint functions. They formulated the following dual using Fritz John optimality conditions instead of Karush-Kuhn-Tucker optimality conditions and proved various duality theorems–thus the requirement of constraint qualification is eliminated.

(FrD): Maximize $f(x)$

Subject to

$$y_\circ \nabla f(x) + \nabla y^T h(x) = 0 .$$

$$y^T h(x) \geq 0$$

$$(y_\circ, y) \geq 0, (y_\circ, y) \neq 0$$

Duality in Nondifferentiable Mathematical Programming

Mond [72] considered the following class of nondifferentiable mathematical programming problems:

(NP): Minimize $f(x) + \left(x^T Bx\right)^{\frac{1}{2}}$

Subject to

$$h_j(x) \leq 0, \quad j = 1, 2, \ldots, m ,$$

where f and h_j , $j = 1, 2, \ldots, m$ are twice differentiable function from R^n to R and B is an n x n positive semidefinite (symmetric) matrix. It is assumed that the functions f and h_j , $j = 1, 2, \ldots, m$ are convex functions. They established a duality theorem between (NP) and the following problem

(ND): Maximize $f(u) + y^T h(u) - u^T \nabla\left[f(u) + y^T h(u)\right]$

Subject to

$$\nabla f(u) + \nabla y^T h(u) + Bw = 0,$$

$$w^T Bw \leq 1$$

$$y \geq 0.$$

Further on the lines of Mond and Weir [82], Chandra, Craven and Mond [23] introduced another dual program:

(NWD): Maximize $\quad f(u) - u^T \nabla\left[f(u) + y^T h(u) \right]$

Subject to

$$\nabla f(u) + \nabla y^T h(u) + Bw = 0,$$

$$y^T h(u) \geq 0,$$

$$w^T Bw \leq 1,$$

$$y \geq 0.$$

and established duality theorems by assuming the function $f(\cdot) + (\cdot)^T Bw$ to be pseudoconvex and $y^T h(\cdot)$ to be quasiconvex for all feasible solutions of (NP) and (NWD).

Later, Mond and Schechter [79] replaced the square root term by the norm term and considered the nondifferentiable nonlinear programming problems as:

(NP)$_1$: Minimize $\quad f(x) + \|S_x\|_p$

Subject to

$$h_j(x) \leq 0, \quad j = 1, 2, \ldots, m$$

Here f and h_j , $(j = 1, 2, \ldots, m)$ are twice differentiable function from R^n to R. The dual for (NP)$_1$ is the problem:

(ND)$_1$: Maximize $\quad f(u) + y^T h(u) - u^T S^T v$

Subject to

$$\nabla f(u) + \nabla y^T h(u) + S^T v = 0,$$

$$\|v\|_q \leq 1,$$

$$y \geq 0.$$

where p and q are conjugate exponents.

Later Schechter [89] replaced the norm term or the square root term by a more general function as the support function of a compact set. The problem considered by Schechter [89] is:

(NP)$_2$: Minimize $\quad f(x) + S(x|C)$

Subject to

$$h_j(x) \leq 0, \quad j = 1, 2, \ldots, m,$$

where f and h_j , $(j = 1, 2, \ldots, m)$ are twice differentiable function from R^n to R and $S(x|C)$ is a support function of a compact convex set $C \subseteq R^n$. Using the subdifferential of the support function of $S(x|C)$, the dual of (NP)$_2$ is the problem:

(ND)$_2$: Maximize $\quad f(u) + w^T u + y^T h(u)$

Subject to

$$\nabla f(u) + \nabla y^T h(u) + w = 0,$$

$$y \geq 0, \; w \in C.$$

Duality in Multiobjective Mathematical Programming

For multiobjective programming problem, we shall follow the following conventions for vectors in R^n

$$x < y, \qquad \Leftrightarrow \qquad x_i < y_i, \qquad i = 1, 2, \ldots, n.$$

$$x \leqq y, \quad \Leftrightarrow \quad x_i \leqq y_i, \quad i = 1, 2, \ldots, n.$$

$$x \leq y, \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \ldots, n, \text{ but } x \neq y$$

$x \not\leq y$, is the negation of $x \leq y$.

Consider the multiobjective programming problem:

(VP): V- Minimize $F(x) = \left(f_1(x), f_2(x), ..., f_p(x) \right)$

Subject to

$$h_j(\bar{x}) \leqq 0, \ (j = 1, 2, ..., m)$$

Here $X \subseteq R^n$ is an open and convex set and f_i and h_j are differentiable functions where $f_i : X \to R$, $i = 1, 2, \ldots, p$ and $h_j : X \to R$, $j = 1, 2, \ldots, m$. Here the symbol "V-Min" stands for vector minimization and minimality is taken in terms of either "*efficient points*" or "*properly efficient points*" given by Koopman [61] and Geoffrion [44] respectively.

Geoffrion [44] considered the following single objective minimization problems for fixed $\lambda \in R^p$:

(VP)$_\lambda$: Minimize $\sum_{i=1}^{p} \lambda_i f_i(x)$

Subject to

$$h_j(\bar{x}) \leqq 0, \ (j = 1, 2, ..., m),$$

and prove the following lemma connecting (VP) and $(VP)_\lambda$.

Lemma 1.1

(i) Let $\lambda_i > 0$, $(i = 1, 2, ..., p)$, $\sum_{i=1}^{p} \lambda_i = 1$ be fixed. If $\bar{x}$ is optimal for $(VP)_\lambda$, then $\bar{x}$ is properly efficient for (VP).

(ii) Let f_i and h_j be convex functions Then $\bar{x}$ is properly efficient for (VP) iff $\bar{x}$ is optimal for are differentiable functions $(VP)_\lambda$ for some $\lambda_i > 0$, $\sum_{i=1}^{p} \lambda_i = 1$ $(i = 1, 2, ..., p)$.

If f_i and h_j are differentiable convex functions then $(VP)_\lambda$ is a convex programming problem. Therefore in relation to $(VP)_\lambda$ consider the scalar maximization problem:

$(VD)_\lambda$: Maximize $\lambda^T f(x) + y^T h(x) = \lambda^T \left(f(x) + y^T h(x) \right)$

Subject to

$$\nabla \left(\lambda^T f(x) + y^T h(x) \right) = 0$$

$$\lambda \in \Lambda^+ \text{ , } y \geqq 0,$$

where $e = (1, 1, \ldots, 1) \in R^p$ and $\Lambda^+ = \left\{ \lambda \in R^P : \lambda > 0 \text{ , } \lambda^T e = 1 \right\}$.

Now as $(VD)_\lambda$ is a dual program of $(VP)_\lambda$,Weir [94] considered the following vector optimization problem in relation to (VP) as

(DV): Maximize $\left(f(x) + y^T h(x) \right) e$

Subject to

$$\nabla \left(w^T f(x) + y^T h(x) \right) = 0$$

$$w \in \Lambda^+ \text{ , } y \geqq 0,$$

where $e = (1, 1, \ldots, 1) \in R^p$

They termed (DV) as the dual of (VP) and proved various duality theorems between (VP) and (DV) under the assumption that f and g are convex functions.

Further for the purpose of weakening the convexity requirements on objective and constraint functions, Weir [94] introduced another dual program (DV1)

(DV1): Maximize $f(x)$

Subject to

$$\nabla\left(w^T f(x) + y^T h(x)\right) = 0$$

$$y^T h(x) \geq 0$$

$$w \in \Lambda^+ \text{ , } y \geqq 0,$$

And various duality theorems are proved by assuming the function f to be pseudo convex and $y^T h$ to be quasiconvex for all feasible solutions of (VP) and (DV1).

1.3.2 Symmetric Duality in Mathematical Programming

Symmetric Duality in Differentiable Mathematical Programming

Consider a function $f(x, y)$ which is differentiable in $x \in R^m$ and $y \in R^m$. Dantzig *et al* [38] introduced the following pair of problems:

(SP): Minimize $f(x, y) - y^T \nabla_y f(x, y)$

Subject to

$$\nabla_y f(x, y) \leq 0$$

$$(x, y) \geq 0.$$

(SD): Maximize $f(x,y)-x^T\nabla_x f(x,y)$

Subject to

$$\nabla_x f(x,y)\geq 0$$

$$(x,y)\geq 0.$$

and proved the existence of a common optimal solution to the primal (SP) and (SD), when (i) an optimal solution of $(x_\circ, y_\circ)$ to the primal (SP) exists (ii) f is convex in x for each y, concave in y for each x and (iii) f, twice differentiable, has the property that at $(x_\circ, y_\circ)$ its matrix of second partials with respect to y is negative definite.

Mond [71] further gave the following formulation of symmetric dual programming problems:

(MSP): Maximize $f(x,y)-y^T\nabla_y f(x,y)$

Subject to

$$\nabla_y f(x,y)\leq 0$$

$$x\geq 0.$$

(MSD): Maximize $f(x,y)-x^T\nabla_x f(x,y)$

Subject to

$$\nabla_x f(x,y)\geq 0$$

$$y\geq 0.$$

It may be remarked here that in [38], the constraints of both (SP) and (SD) include $x\geq 0,\ y\geq 0$, but in [71] only $x\geq 0$ is required in the primal and only $y\geq 0$ in the dual.

Later Mond and Weir [82] gave the following pair of symmetric dual nonlinear programming problems which allows the weakening of the convexity-concavity assumptions to pseudoconvexity-pseudoconcavity.

(M-WSP): Minimize $f(x,y)$

Subject to

$$\nabla_y f(x,y) \leq 0$$

$$y^T \nabla_y f(x,y) \geq 0,$$

$$x \geq 0.$$

(M-WSD): Maximize $f(x,y)$

Subject to

$$\nabla_x f(x,y) \leq 0$$

$$x^T \nabla_x f(x,y) \leq 0,$$

$$y \geq 0.$$

Symmetric Duality in Nondifferentiable Mathematical Programming

Let $f(x,y)$ be a real valued continuously differentiable in $x \in R^m$ and $y \in R^m$. Chandra and Husain [21] introduced pair of symmetric dual nondifferentiable programs and proved duality results assuming convexity-concavity conditions on the kernel function $f(x,y)$:

(NP): Minimize $f(x,y) - y^T \nabla_y f(x,y) + \left(x^T Bx\right)^{\frac{1}{2}}$

Subject to

$$-\nabla_y f(x,y) + Cw \geq 0 \text{ ,}$$

$$w^T Cw \leq 1,$$

$$(x,y) \geq 0.$$

(ND): Maximize $f(x,y) - x^T \nabla_x f(x,y) - \left(y^T C y\right)^{\frac{1}{2}}$

Subject to

$$-\nabla_x f(x,y) - Bz \leq 0$$

$$z^T C z \leq 1,$$

$$(x,y) \geq 0.$$

where B and C are n x m and m x m positive semidefinite matrices.

Further on the lines of Mond and Weir [82], Chandra, Craven and Mond [23] presented another pair of symmetric dual nondifferentiable programs by weakening the convexity conditions on the kernel function $f(x,y)$ to the pseudoconvexity and pseudoconcavity. The problems considered in [23] are:

(PS): Minimum $f(x,y) + \left(x^T B x\right)^{\frac{1}{2}} - y^T C z$

Subject to

$$\nabla_y f(x,y) - Cz \leq 0,$$

$$y^T \left[\nabla_y f(x,y) - Cz\right] \geq 0$$

$$z^T C z \leq 1,$$

$$x \geq 0.$$

(DS): Maximum $f(x,y) + \left(y^T C y\right)^{\frac{1}{2}} - x^T B w$

Subject to

$$\nabla_x f(x,y) + Bw \geq 0,$$

$$x^T \left[\nabla_x f(x,y) + Bw\right] \leq 0,$$

$$w^T B w \leq 1,$$

$$y \geq 0.$$

Following Balas [5] and Kumar, Husain and Chandra [63], Gulati, Husain and Izhar [45] formulated two distinct pairs of nondifferentiable symmetric dual minimax mixed integer programs:

(MPS): $\underset{x^1}{\text{Max}}\ \underset{x^2,y}{\text{Min}}\ F(x,y) = f(x,y) - (y^2)^T \nabla_{y^2} f(x,y) + \left((x^2)^T Bx^2\right)^{\frac{1}{2}}$

Subject to

$$\nabla_{y^2} f(x,y) - Cw \leq 0,$$

$$w^T Cw \leq 1,$$

$$x^2 \geq 0,$$

$$x^1 \in U,\ y^1 \in V.$$

(MDS): $\underset{y^1}{\text{Max}}\ \underset{x,y^2}{\text{Min}}\ \text{G}(x,y) = f(x,y) - (x^2)^T \nabla_{x^2} f(x,y) + \left((y^2)^T Cy^2\right)^{\frac{1}{2}}$

Subject to

$$\nabla_{x^2} f(x,y) + Bz \geq 0,$$

$$z^T Bz \leq 1,$$

$$y^2 \geq 0,$$

$$x^1 \in U,\ y^1 \in V.$$

and

(SP): $\underset{x^1}{\text{Max}}\ \underset{x^2,y}{\text{Min}}\ L(x,y) = f(x,y) - (y^2)^T Cw + \left((x^2)^T Bx^2\right)^{\frac{1}{2}}$

Subject to

$$\nabla_{y^2} f(x,y) - Cw \leq 0,$$

$$(y^2)^T \left(\nabla_{y^2} f(x,y) - Cw\right) \geq 0,$$

$$w^T Cw \leq 1,$$

$x^2 \geq 0,$

$x^1 \in U, y^1 \in V.$

(SD): $\underset{y^1}{\text{Min}}\ \underset{x,y^2}{\text{Max}}\ H(x,y) = f(x,y) + (x^2)^T Bz - \left((y^2)^T Cy^2\right)^{\frac{1}{2}}$

Subject to

$\nabla_{x^2} f(x,y) + Bz \geq 0,$

$(x^2)^T (\nabla_{x^2} f(x,y) + Bz) \leq 0,$

$z^T Bz \leq 1,$

$y^2 \geq 0,$

$x^1 \in U, y^1 \in V.$

Subsequently Mond and Schechter [81] introduced the following pair of symmetric dual programs one of which is Wolfe [98] type and another is Mond and Weir [82] type.

(P): Minimum $f(x,y) - y^T \nabla_2 f(x,y) + S(x|C_1)$

Subject to

$\nabla_2 f(x,y) - z \leq 0$ **,**

$z \in C_2,\ x \geq 0.$

(D): Maximum $f(u,v) - u^T \nabla_1 f(u,v) + S(v|C_2)$

Subject to

$\nabla_1 f(u,v) + w \geq 0$ **,**

$w \in C_1,\ v \geq 0.$ and

(P1): Minimum $f(x,y) - y^T z + S(x|C_1)$

Subject to

$$\nabla_2 f(x,y) - z \leq 0,$$

$$y^T\left(\nabla_2 f(x,y) - z\right) \geq 0,$$

$$z \in C_2,\ x \geq 0.$$

(D1): Maximum $\quad f(u,v) + u^T w + S\left(v \middle| C_2\right)$

Subject to

$$\nabla_1 f(u,v) + w \geq 0,$$

$$u^T\left(\nabla_1 f(u,v) + w\right) \leq 0,$$

$$w \in C_1,\ v \geq 0.$$

Symmetric Duality in Multiobjective Programming

Mond and Weir [83] discussed symmetric duality in multiobjective programming by considering the following pair of programs:

(PS): Minimum $\quad f(x,y) - \left(y^T \nabla_2 \lambda^T f(x,y)\right)e$

Subject to

$$\nabla_2 \lambda^T f(x,y) \leqq 0,$$

$$x \geqq 0,\ \lambda \in \Lambda^+$$

(DS): Maximum $\quad f(x,y) - \left(x^T \nabla_1 \lambda^T f(x,y)\right)e$

Subject to

$$\nabla_1 \lambda^T f(x,y) \geqq 0,$$

$$y \geqq 0,\ \lambda \in \Lambda^+$$

Where $f : R^n \times R^m \to R^p$, $e = (1,1,\ldots,1) \in R^p$ and $\Lambda^+ = \{\lambda \in R^P : \lambda > 0 , \lambda^T e = 1\}$ and proved the symmetric duality theorem under the convexity – concavity assumptions on $f(x,y)$. Here the minimization/ maximization is taken in the sense of proper efficiency as given by Geoffrion [44].

Further on the lines of scalar case (Mond and Weir [82]) also considered another pair of symmetric dual programs and proved symmetric duality results under weaker conditions of pseudoconvexity-pseudoconcavity:

(PS1): Minimum $f(x,y)$

Subject to

$$\nabla_2 \lambda^T f(x,y) \leqq 0$$

$$y^T \nabla_2 \lambda^T f(x,y) \geqq 0$$

$$x \geqq 0 \text{ , } \lambda \in \Lambda^+$$

(DS1): Maximum $f(x,y) - \left(x^T \nabla_1 \lambda^T f(x,y)\right) e$

Subject to

$$\nabla_1 \lambda^T f(x,y) \geqq 0 \text{ ,}$$

$$x^T \nabla_1 \lambda^T f(x,y) \leqq 0,$$

$$y \geqq 0 \text{ , } \lambda \in \Lambda^+.$$

Later Chandra and D.Prasad [24] introduced following pair of multiobjective programs by associating a vector valued infinite game.

(PS*): Minimum $f(x,y)-\left(y^T\nabla_2\mu^T f(x,y)\right)e$

Subject to

$$\nabla_2\mu^T f(x,y)\leqq 0,$$

$$x\geqq 0\ ,\ \mu\in\Lambda^+.$$

(DS*): Maximum $f(x,y)-\left(x^T\nabla_1\lambda^T f(x,y)\right)e$

Subject to

$$\nabla_1\lambda^T f(x,y)\geqq 0\ ,$$

$$y\geqq 0\ ,\ \lambda\in\Lambda^+$$

Here it may be noted that not the same λ is appearing in (PS*) and (DS*) and this creates certain difficulties which are also discussed in [24].

1.3.3 Second-Order Duality in Mathematical Programming

We consider the following nonlinear programming problem:

(NP): Minimum $f(x)$

Subject to

$$g(x)\leq 0$$

where $x\in R^n$, f and g are twice differentiable functions from R^n and R and R^m, respectively.

Mangasarian [66] formulated the Wolfe [98] type second-order dual of (NP).

(ND-1): Maximum $\left[f(u)+y^T g(u)\right]-\frac{1}{2}p^T\nabla^2\left[f(u)+y^T g(u)\right]$

Subject to

$$\nabla\left[f(u)+y^T g(u)\right]+\nabla^2\left[f(u)+y^T g(u)\right]p=0,$$

$$y\geq 0$$

where $p\in R^n$ and for any function $\phi:R^n\to R$, the symbol $\nabla^2\phi(x)$ designates $n\times n$ symmetric matrix of second-order partial derivatives. Mangasarian [66] established usual duality theorems between (NP) and (ND-1) under the assumptions that are involved and rather difficult to verify.

1.3.4 Second-Order Symmetric Duality in Mathematical Programming

Mangasarian [66] was the first introduced the concept of second-order duality. Later Mond [70] constructed the following pair of second-order symmetric dual problems:

(PP): Minimum $f(x,y)-y^T\left(\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\right)-\frac{1}{2}p^T\nabla_y^2 f(x,y)p$

Subject to

$$\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\leq 0,$$

$$x\geq 0.$$

(DD): Maximum $f(x,y)-x^T\left(\nabla_x f(x,y)+\nabla_x^2 f(x,y)q\right)-\frac{1}{2}q^T\nabla_x^2 f(x,y)q$

Subject to

$$\nabla_x f(x,y)+\nabla_x^2 f(x,y)q\geq 0,$$

$$y\geq 0.$$

He studied appropriate duality theorem between (PP) and (DD) under the second-order concavity assumptions on the kernel function $f(x,y)$. Further Bector and Chandra [10] introduced another pair of second-order symmetric dual nonlinear programs on the lines of Mond and Weir [82] and studied duality under weaker generalized convexity assumptions.

1.3.5 Variational Problems

A variational problem can be considered as a particular of an optimal control problem in which the control function is a derivative of a state function.

A variational problem is of the form:

(VP): Minimize $\int_I f(t,x,\dot{x})\,dt$

Subject to

$$x(a)=\alpha\ ,\ x(b)=\beta$$

$$g(t,x,\dot{x})\leq 0, \qquad t\in I,$$

$$x\in C(I,R^n).$$

where $I=[a,b]$ is a real time interval, $\dot{x}$ denotes derivative of x with respect to t, $f:I\times R^n\times R^n\to R$ and $g:I\times R^n\times R^n\to R$ are continuously differentiable functions with respect to each of their arguments; $C(I,R^n)$ is the space of continuously differentiable functions $x:I\to R^n$, and is equipped with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$ where the differentiation operator D is given by $u=Dx\Leftrightarrow x(t)=\alpha+\int_a^t u(s)\,ds$ except at discontinuities.

The following necessary conditions for the existence for (VP) are derived by Valentine [93].

Theorem 1.1.2: For every minimizing arc $x = x(t)$ of the problem (VP), there exists a function of the form

$$H = \lambda_\circ f(t, x, \dot{x}) - \lambda(t)^T g(t, x, \dot{x})$$

Such that

$$H_{\dot{x}} = \frac{d}{dt} H_x$$

$$\lambda(t)^T g(t, x, \dot{x}) = 0$$

$$(\lambda_\circ, \lambda(t)) \geq 0 \ , \ (\lambda_\circ, \lambda(t)) \neq 0 \ , \ t \in I$$

hold throughout I (except at corners of x° where $H_{\dot{x}} = \frac{d}{dt} H_x$, holds for unique right and left limits). Here $\lambda_\circ$ is constant and $\lambda(\cdot)$ is continuous except possibly for values of t corresponding to corners of x°.

In [77] Mond and Hanson studied Wolfe type duality for variational problems (VP) while in [93] they investigated Wolfe type duality symmetric duality for the variational problems (VP). Later Bector, Chandra and Husain [13] studied Mond-Weir type non-symmetric as symmetric continuous programs which are variational problems.

1.3.6 Second-Order Duality for Variational Problems

A second-order dual to a mathematical programming problem presents a tighter bound and hence it enjoys computational advantage over a first order dual. Motivated with this remark Chen has identified second-order dual. The following is the Wolfe type dual to the above problem:

Maximize: $$\int_a^b \Big\{ f\left(t,u(t),\dot{u}(t)\right)+\alpha(t)^T g\left(t,u(t),\dot{u}(t)\right)$$

$$\frac{1}{2}\beta(t)^T \Big[f_{uu}\left(t,u(t),\dot{u}(t)\right)+\left(g_u\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_u$$

$$-2D\left(f_{u\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(g_u\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_{\dot{u}}\right)$$

$$+D^2\left(f_{\dot{u}\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_{\dot{u}}\right)\Big]\beta(t)\Big\}dt$$

Subject to

$$u(a)=0=u(b), \quad \dot{u}(a)=0=\dot{u}(b)$$

$$f_u\left(t,u(t),\dot{u}(t)\right)+g_u\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)$$

$$-D\left[f_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)+g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right]$$

$$+\Big[f_{uu}\left(t,u(t),\dot{u}(t)\right)+\left(g_u\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_u$$

$$-2D\left(f_{u\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(g_u\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_{\dot{u}}\right)$$

$$+D^2\left(f_{\dot{u}\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_{\dot{u}}\right)\Big]\beta(t)=0,$$

$$t\in I,$$

$$\alpha(t)\in R_+^m \ , \ \beta(t)\in R^n \ , t\in I \ ,$$

Let

$$H = f_{uu}\left(t,u(t),\dot{u}(t)\right)+\left(g_u\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_u$$

$$-2D\left(f_{u\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(g_u\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_{\dot{u}}\right)$$

$$+D^2\left(f_{\dot{u}\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)^T \alpha(t)\right)_{\dot{u}}\right).$$

The above problem can now be written as

(VD): Maximize $$\int_a^b \Big\{ f\left(t,u(t),\dot{u}(t)\right)+\alpha(t)^T g\left(t,u(t),\dot{u}(t)\right)$$

$$-\frac{1}{2}\beta(t)^T H\left(t,u(t),\dot{u}(t),\alpha(t),\beta(t)\right)\Big\}dt$$

Subject to

$$u(a)=0=u(b), \quad \dot{u}(a)=0=\dot{u}(b)$$

$$\begin{aligned} & f_u\left(t,u(t),\dot{u}(t)\right)+g_u\left(t,u(t),\dot{u}(t)\right)^T\alpha(t) \\ & \quad -D\left[f_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)+g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)^T\alpha(t)\right] \\ & \quad +H\left(t,u(t),\dot{u}(t)\right)\alpha(t)\beta(t)=0, \quad t\in I \end{aligned}$$

$$\alpha(t)\in R_+^m \ , \ \beta(t)\in R^n \ , t\in I$$

For the above dual pair of problem, Chen [27] established the following weak duality under somewhat strange conditions , strong and converse duality theorems.

Theorem 1.1.3 (Weak Duality): Let $x(t)\in X$ be a primal feasible solution of (VP) and $(u(t),y(t),\beta(t))$ be a dual feasible solution of (VD).If $\int_a^b f(t,.,.)dt$ and $\int_a^b \alpha(t)^T\times g(t,.,.)dt$ are invex in x and $\dot{x}$ on I with respect to the same $\eta: I\times R^n\times R^n\to R^n$ satisfying $\eta=0$ at t =1and $t = b$, then there exist $k\left(t,y(t),\dot{y}(t),\alpha(t)\right)>0$ and $K\left(t,y(t),\dot{y}(t),\alpha(t)\right)>0$ such that the following conditions hold:

$$\beta(t)^T H\left(t,y(t),\dot{y}(t),\alpha(t)\right)\geq k\left(t,y(t),\dot{y}(t),\alpha(t)\right)\left\|\beta(t)^2\right\|, \ t\in I$$

$$\left\|H\left(t,y(t),\dot{y}(t),\alpha(t)\right)\right\|\leq K\left(t,y(t),\dot{y}(t),\alpha(t)\right), \ t\in I$$

$$\frac{1}{2}\|\beta(t)\|\geq\left\|\eta\left(t,y(t),\dot{y}(t)\right)\right\|\frac{K\left(t,y(t),\dot{y}(t),\alpha(t)\right)}{k\left(t,y(t),\dot{y}(t),\alpha(t)\right)}, \ t\in I.$$

Then the following inequality holds between the primal (VP) and dual (VD) objective functions:

$$\begin{aligned} \int_I f\left(t,x(t),\dot{x}(t)\right)dt \geq \int_I \Big[& f\left(t,u(t),\dot{u}(t)\right)+\alpha(t)^T g\left(t,y(t),\dot{y}(t)\right) \\ & -\frac{1}{2}\beta(t)^T H\left(t,y(t),\dot{y}(t),\alpha(t)\right)\beta(t)\Big] \end{aligned}$$

Theorem 1.1.4 (Strong Duality): If $\bar{x}(t) \in X$ is a local (or global) optimal solution of (VP) and some piecewise smooth function $v : I \to R^n$ and Slater condition holds, then there exists a piecewise smooth $\bar{\alpha} : I \to R^n$ such that $\left(\bar{x}(t), \bar{\alpha}(t), \bar{\beta}(t) = 0\right)$ is a feasible solution of (VD),and the two objective values are equal. Furthermore, if the invexity-like requirements among with additional conditions in Theorem 1.1.3 hold then $\left(\bar{x}(t), \bar{\alpha}(t), \bar{\beta}(t) = 0\right)$ is an optimal solution of (VD).

Theorem 1.1.5 (Converse Duality): Suppose that *f* and *g* are thrice continuously differentiable. Let $\left(\bar{x}(t), \bar{y}(t), \bar{\beta}(t)\right)$ be a local (or global) optimal solution of (VD), if the following conditions hold:

(i): H is nonsingular at $\left(\bar{x}(t), \bar{y}(t), \bar{\beta}(t)\right)$:

(ii): $$\left[r(t)^T H\left(t, \bar{y}(t), \dot{\bar{y}}(t), \bar{\alpha}(t) r(t)\right)\right]_x - D\left[r(t)^T H\left(t, \bar{y}(t), \dot{\bar{y}}(t), \bar{\alpha}(t) r(t)\right)\right]_{\dot{x}} = 0$$

$$\Rightarrow r(t) = 0, \quad for \ all \ r(t) \in X, t \in I$$

Then $\bar{y}(t)$ is a feasible solution of (VP), $\bar{\alpha}(t)^T g\left(t, \bar{y}(t), \dot{\bar{y}}(t)\right) = 0,$ and the two objective functions are equal. In addition, if the conditions in Theorem 1.1.3, then $\bar{y}(t)$ is an optimal solution of (VP).

1.4 SUMMARY OF THE THESIS

The results obtained in this thesis are presented in chapters 2-6. The results of this thesis are briefly summarized chapter wise.

CHAPTER 2

Chapter 2 is divided in two sections, section 2.1 and section 2.2. In section 2.1, we consider the following nondifferentiable nonlinear problem with support functions.

(NP): Min $f(x)+S(x/C)$

Subject to,

$$g_j(x)+S(x/D_j)\le 0,(j=1,2...m)$$

For this problem, we construct the following Wolfe and Mond-Weir type second-order dual.

(WD): Max $f(u)+z^T u+\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T(u)\right)-\frac{1}{2}p^T\nabla^2\left(f(u)+y^T g(u)\right)p$

Subject to,

$$\nabla(f(u)+z^T u)+\sum_{j=1}^{m} y_j\nabla\left(g_j(u)+w_j\right)+\nabla^2\left(f(u)+y^T g(u)\right)p=0,$$

$$y\ge 0,$$

$$z\in C, w_j\in D_j,\quad (j=1,2.,,,m).$$

Mond-Weir type second-order dual for the problem (NP).

(SM-WD): Max $f(u)+z^T u-\frac{1}{2}p^T\nabla^2\left(f(u)\right)p$

Subject to,

$$\nabla f(u)+z+\sum_{j=1}^{m} y_j\left(\nabla g_j(u)+w_j\right)+\nabla^2\left(f(u)+y^T g(u)\right)p=0,$$

$$\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T u\right)-\tfrac{1}{2}\, p^T\nabla^2\left(y^T g(u)\right)p \geq 0,$$

$$y \geq 0,$$

$$z \in C, w_j \in D, \quad \forall j = 1,2,...m$$

For the pair of Wolfe type second-order dual problem (NP) and (WD) usual duality theorems are validated under second-order convexity, and for the pair of second-order Mond-Weir problem (NP) and (M-WD),various duality theorems are validated under second-order generalized convexity. Special cases are also deduced. In section 2.2, mixed type second-order dual to the non-differentiable problem containing support functions is formulated and duality theorems are proved under generalized second-order convexity conditions. Special cases are also studied.

Mixed type second-order dual to the problem (NP) is formulated as:

(Mix SD):

Maximize $f(u)+u^T z+\sum_{i\in I_0} y_i\left(g_i(u)+u^T w_i\right) -\frac{1}{2}\nabla^2 p^T\left[\, f(u)+\sum_{i\in I_0} y_i g_i(u)\right]$ p

Subject to,

$$\nabla f(u)+z+\sum_{i=1}^{m} y_i\left(\nabla g_i(u)+w_i\right)+\nabla^2\left(f(u)+y^T g(u)\right)p = 0 \qquad (2)$$

$$\sum_{i\in I_\alpha} y_i\left(g_i(u)+u^T w_i\right)-\frac{1}{2}\, p^T\nabla^2\left(\sum_{i\in I_\alpha} y_i g_i(u)\right)\mathrm{p} \geq 0, \alpha=1,2,\ldots,r. \qquad (3)$$

$$y \geq 0 \qquad (4)$$

$$z \in C, w_i \in D_i,\ i = 1, 2,\ldots,m. \qquad (5)$$

where

(i) $I_\alpha \subseteq M = \{1,2,\ldots,m\}$, $\alpha = 0, 1,2,\ldots,r$ with $\bigcup_{i=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$ if $\alpha \neq \beta$.

(ii) $u \in R^n,\ p \in R^n \ and \ y \in R^m$.

CHAPTER 3

This chapter deals with second-order symmetric duality for non-differentiable multiobjective programming problems. It consists of two sections,3.1 and 3.2.In section 3.1 following Wolfe type non-differentiable multiobjective second-order symmetric dual problems are formulated and for this pair of problem weak, strong and self duality theorems are established under suitable convexity conditions.

Primal (SWP): Minimize $F(x,y,z,p) = F_i(x,y,z_1,p),\ldots F_k(x,y,z_k,p)$

Subject to,

$$\sum_{i=1}^{k} \lambda_i \left(\nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y)p\right) \leqq 0$$

$$z_i \in D_i, i = 1,2,\ldots,k$$

$$x \geqq 0$$

$$\lambda \in \wedge^+$$

Wolfe type dual to the problem (SWP) is:

Dual (SWD): Minimize $G(u,v,w,q) = G_1(u,v,w_1,q),\ldots G_k(u,v,w_k,q)$

Subject to,

$$\sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u,v) - w_i + \nabla_1^2 f_i(u,v)q\right) \geqq 0$$

$$w_i \in C_i, i = 1,2,\ldots,k$$

$$v \geqq 0$$

$$\lambda \in \wedge^{+}$$

where

i. $F_i(x, y, z_i, p) = f_i(x, y) + s(x \mid C_i) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x, y) p$

$- y^T \sum_{i=1}^{k} \lambda_i \left(\nabla_2 f_i(x, y) - z_i + \nabla_2^2 f_i(x, y) p \right)$

ii. $G_i(u, v, w_i q) = f_i(u, v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q$

$- u^T \sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u, v) - w_i + \nabla_1^2 f_i(u, v) q \right)$, and

iii. For each i, $s(x \mid C_i)$ and $s(v \mid D_i)$ represent support functions of compact convex sets C_i and D_i in R^n and R^m, respectively.

iv. $w = (w_1, \ldots w_K)$ with $w_i \in C_i$ and $z = (z_1, \ldots z_K)$ for each {i = 1,2,…, k}

v. $\wedge^{+} = \left\{ \lambda \in R^k \mid \lambda = (\lambda_i, \ldots \lambda_k), \lambda > 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$

In section 3.2 following Mond-Weir type nondifferentiable multiobjective second-order symmetric dual problem are formulated to the problem (SWP)

(SVD): Maximize $G(u, v, w, q) = (G_1(u, v, w_1, q), \ldots, G_k(u, v, w_k, q))$

Subject to,

$$\sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u, v) + w_i + \nabla_1^2 f_i(u, v) q \right) \geqq 0,$$

$$u^T \sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v) q \right) \leqq 0,$$

$$\lambda > 0,$$

$$v \geq 0, w_i \in C_i, \; i = 1, 2, \ldots, k.$$

where

(i) $$F_i(x, y, z_i, p) = f_i(x, y) + s(x \mid C_i) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x, y) p$$

$$G_i(u, v, w_i, q) = f_i(u, v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q$$

(ii) $w = (w_1, \ldots, w_k)$ with $w_i \in C_i$ for $i \in \{1, 2, \ldots, k\}$,

$z = (z_1, \ldots, z_k)$ with $z_i \in D_i$ for $i \in \{1, 2, \ldots, k\}$, and

$\lambda = (\lambda_1, \ldots, \lambda_k)^T$ with $\lambda_i \in R$ for $i \in \{1, 2, \ldots, k\}$; and

(iii) for each $i \in \{1, 2, \ldots, k\}$, $s(x \mid C_i)$ and $s(y \mid D_i)$ represent support functions of compact convex set C_i in R^n and compact convex set D_i in R^m, respectively.

For this pair of problems weak, strong and self dually theorems are established under suitable second-order generalized convexity conditions. Some additional restriction is assumed to validate self duality theorem.

CHAPTER 4

In this chapter following pair of second-order symmetric dual programs with cone constraint is formulated.

(SP): Minimize $G(x, y, p) = f(x, y) - y^T (\nabla_y f(x, y) + \nabla_y^2 f(x, y) p)$

$$-\frac{1}{2} p^T \nabla_y^2 f(x, y) p$$

Subject to,

$$-\nabla_y f(x,y) - \nabla_y^2 f(x,y) p \in C_2^*$$

$$(x,y) \in C_1 \times C_2$$

(SD): Maximize $H(x,y,q) = f(x,y) - x^T\left(\nabla_x f(x,y) + \nabla_x^2 f(x,y) q\right)$

$$-\frac{1}{2} q^T \nabla_x^2 f(x,y) q$$

Subject to,

$$\nabla_x f(x,y) + \nabla_x^2 f(x,y) q \in C_1^*$$

$$(x,y) \in C_1 \times C_2$$

where

(i) $f : C_1 \times C_2 \to R$ is a twice differentiable function,

(ii) C_1 and C_2 are closed convex cones with nonempty interior in R^n and R^m, respectively;

(iii) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively.

For this pair of problems various duality theorems including self duality theorems are proved under second-order convexity □ second-order concavity. In section 4.4 following pair of second-order mixed integer symmetric and self duality is investigated.

Primal Problem

(MSP): $\underset{x^1}{\text{Max}} \; \underset{x^2,y,s}{\text{Min}} \; \phi(x,y,s) = f(x,y) - (y^2)^T \left(\nabla_{y^2} f(x,y) + \nabla_{y^2}^2 f(x,y) s\right)$

$$-\frac{1}{2} s^T \nabla_{y^2}^2 f(x,y) s$$

Subject to,

$$-\nabla_{y^2} f(x,y) - \nabla^2_{y^2} f(x,y)s \in K_2^*$$

$$x^1 \in U, (x^2, y) \in K_1 \times T \text{ .}$$

and

Dual Problem

(MSD): $$\underset{y^1}{Min}\underset{x,y^2,r}{Max}\,\psi(x.y,r) = f(x,y) - (x^2)^T \left(\nabla_{x^2} f(x,y) + \nabla^2_{x^2} f(x,y) r\right) f(x,y)$$
$$-\frac{1}{2}(r^T)^T \nabla^2_{x^2} f(x,y) r$$

Subject to,

$$\nabla_{x^2} f(x,y) + \nabla^2_{x^2} f(x,y) r \in K_1^*$$

$$y^1 \in V, (x, y^2) \in S \times K_2$$

where $s \in R^{m-m_1}$ and $r \in R^{n-n_1}$.

Finally in this chapter, special cases are generated.

CHAPTER 5

In this chapter following pair of mixed type multiobjective second-order symmetric dual problems is formulated.

Primal Problem:

(SMP): Minimize $F(x^1, x^2, y^1, y^2, p, r)$
$$= \left(F_1(x^1, x^2, y^1, y^2, p, r), \ldots, F_k(x^1, x^2, y^1, y^2, p, r)\right)$$

Subject to,

$$\nabla_{y^1}(\lambda^T f)(x^1, y^1) + \nabla^2_{y^1}(\lambda^T f)(x^1, y^1)\, p \leqq 0\,,$$

$$\nabla_{y^2}(\lambda^T g)(x^2,y^2)+\nabla^2_{y^2}(\lambda^T g)(x^2,y^2)\ \mathrm{r} \leqq 0,$$

$$(y^2)^T\left[\nabla_{y^2}(\lambda^T g)(x^2,y^2)+\nabla^2_{y^2}(\lambda^T g)(x^2,y^2)\ \mathrm{r}\right] \geqq 0,$$

$$x^1, x^2 \geqq 0,$$

$$\lambda \in \Lambda^+.$$

Dual Problem:

(SMD): Max $\quad G(u^1,u^2,v^1,v^2,q,s)=\left(G_1(u^1,u^2,v^1,v^2,q,s),\ldots,G_k(u^1,u^2,v^1,v^2,q,s)\right)$

Subject to,

$$\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)\ \mathrm{q} \geqq 0,$$

$$\nabla_{x^2}(\lambda^T g)(u^2,v^2)+\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)\ \mathrm{s} \geqq 0,$$

$$\left(u^2\right)^T\left[\nabla_{x^2}(\lambda^T g)(u^2,v^2)+\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)\ \mathrm{s}\right] \leqq 0,$$

$$\mathrm{v}^1, \mathrm{v}^2 \geqq 0,$$

$$\lambda \in \Lambda^+.$$

where

(i) $\quad F_i(x^1,x^2,y^1,y^2,p,r)= f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1} f_i(x^1,y^1)p$

$$-(y^1)^T\left\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p\right\}$$

$$+g_i(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}g_i(x^2,y^2)r$$

(ii) $\quad G_i(u^1,u^2,v^1,v^2,q,s)= f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1} f_i(u^1,v^1)q$

$$-(u^1)^T\left\{\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\right\}$$

$$+g_i(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}g_1(u^2,v^2)s$$

(iii) $p\in R^{|K_1|}, r\in R^{|K_2|}, q\in R^{|J_1|} s\in R^{|J_2|}$, and $\lambda = (\lambda_1,\ldots,\lambda_k)^T$ with $\lambda_1 \in R$, i =1,2,…,k.

(iv) $\Lambda^+ = \left\{\lambda \in R^k \mid \lambda > 0, \sum_{i=1}^{k}\lambda_1\right\}$

For this pair of problem, weak, strong and converse duality theorems are validated under second-order convexity - second-order concavity of the kernel function appearing in the primal and dual programs. Under additional conditions on the kernel constituting the objective and constraint functions, these programs are shown to be self dual. This formulation of the programs not only generalizes mixed type first order symmetric multiobjective duality results but also unifies the pair of Wolfe and Mond-Weir type second-order symmetric multiobjective programs.

CHAPTER 6

In this chapter, we have constructed Mond -Weir type second-order dual to the variational problem and derive usual duality results under second-order pseudo-invexity and second-order quasi-invexity assumptions.These models allows the weakening of the invexity assumption required for Wolfe type dual models of Chen [27].The following is the pair of Mond-Weir dual models:

(P): Minimize $\int_I f(t,x,\dot{x})\,dt$

Subject to

$$x(a)=0=x(b),$$

$$g(t,x,\dot{x}) \leq 0, \qquad t \in I,$$

(D): Maximize $\int_I \left\{ f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F(t)\beta(t) \right\} dt$

Subject to

$$u(a) = 0 = u(b)$$

$$f_u + y(t)^T g_u - D\left(f_{\dot{u}} + y(t)^T g_{\dot{u}} \right) + (F+H)\beta(t) = 0 \text{ , } t \in I$$

$$\int_I \left\{ y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t) \right\} dt \geq 0, \qquad y(t) \geq 0, \ t \in I$$

where

$F = f_{uu} - Df_{u\dot{u}} + D^2 f_{\dot{u}\dot{u}}$ and $H = \left(y(t)^T g_u \right)_u - D\left(y(t)^T g_u \right)_{\dot{u}} + D^2\left(y(t)^T g_{\dot{u}} \right)_{\dot{u}}$ and define $D = \frac{d}{dt}$ as defined earlier.

If f and g are independent of t then $F = f_{uu}$ and $H = \left(y^T g_u \right)_u$ and consequently (D) will reduce to the second-order dual problem introduced in [11].

CHAPTER 7

This chapter consist of two main sections 7.1 and 7.2. In 7.1 we consider the following class of nondifferentiable continuous programming problem (CP^+):

(CP^+): Minimize $\int_I \left\{ f(t,x(t),\dot{x}(t)) + \left(x(t)^T B(t)x(t) \right)^{1/2} \right\} dt$

Subject to

$$x(a) = 0 = x(b),$$

$$g\left(t, x(t), \dot{x}(t)\right) \leq 0, \qquad t \in I$$

Analogously to the second-order dual problem introduced by Mangasarian [66] for a nonlinear programming problem, we consider the following second-order dual continuous programming problem (CD^+) for (CP^+).

(CD^+): Maximize $$\int_I \left\{ f\left(t, u(t), \dot{u}(t)\right) + u(t)^T B(t) z(t) + y(t)^T g\left(t, u(t), \dot{u}(t)\right) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt$$

Subject to

$$u(a) = 0 = u(b)$$

$$f\left(t, u(t), \dot{u}(t)\right) + u(t)^T B(t) z(t) + y(t)^T g\left(t, u(t), \dot{u}(t)\right) - D\left(f_{\dot{u}}\left(t, u(t), \dot{u}(t)\right) + y(t)^T g_{\dot{u}}\left(t, u(t), \dot{u}(t)\right)\right) + H(t) p(t) = 0, \ t \in I$$

$$z(t)^T B(t) z(t) \leq 1, \quad t \in I, \qquad y(t) \geq 0, \ t \in I,$$

where

$$H(t) = f_{uu}(t, u, \dot{u}) + \left(y(t)^T g_u(t, u, \dot{u})\right)_u - 2D\left[f_{u\dot{u}}(t, u, \dot{u}) + \left(y(t)^T g_u(t, u, \dot{u})\right)_{\dot{u}} \right] + D^2 \left[f_{u\dot{u}}(t, u, \dot{u}) + \left(y(t)^T g_{\dot{u}}(t, u, \dot{u})\right)_{\dot{u}} \right]$$

For this pair of problem we have established usual duality results under second-order pseudoinvexity as an continuous – time version of second-order pseudo invexity for static case. Problems with natural boundary are formulated and special cases are obtained.

In section 7.2 we have studied Wolfe type second-order duality for a wider class of nondifferentiable continuous programming problems in which support functions occur. The dual models are given below. For this pair of dual models various duality results are derived under second-order invexity and second-order pseudoinvexity.

Consider the following nondifferentiable continuous programming problem with support functions of Husain and Jabeen [52]:

(CP₊): Minimize $\int_I \left\{ f(t,x,\dot{x}) + S\left(x(t) \mid K\right)\right\} dt$

Subject to

$$x(a) = 0 = x(b),$$

$$g^j(t,x,\dot{x}) + S\left(x(t) \mid C^j\right) \leq 0,\ j = 1,2...m, \qquad t \in I,$$

where, f and g are continuously differentiable and each $C^j, j = 1,2...m$ is a compact convex set in R^n. The following problem is formulated as Wolfe type dual for the Problem (CP₊):

(CD₊): Maximize

$$\int_I \left\{ f(t,u,\dot{u}) + \bar{u}(t)^T z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt$$

Subject to

$$u(a) = 0 = u(b)$$

$$f_u(t,u,\dot{u}) + z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g_u^j(t,u,\dot{u}) + w^j(t) \right)$$
$$-D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) + H(t) p(t) = 0,\ t \in I$$

$$z(t) \in K, w^j(t) \in C^j, t \in I, j = 1,2...m.$$

$$y(t) \geq 0, \qquad t \in I.$$

2.0 INTRODUCTORY REMARKS

Many authors have studied duality for a class of nonlinear programming problems in which the objective function contains a differentiable convex function along with either a positive homogenous function or the sum of positive homogenous functions, e.g., Sinha [91], Zhang and Mond [101], Mond [72,73], Chandra and Gulati [20] and Mond and Schechter [80,81]. These authors have introduced the square root of positive semidefinite quadratic form $\left(x^T Bx\right)^{1/2}$ or a norm term of the type $\|Px\|$ as a positive homogenous function. The popularity of this kind of problem stems from the fact that even though the objective function and /or constraint functions are nondifferentiable, the dual problem comes out to be a differentiable problem and hence is more amenable to handle from the computational point of view. As demonstrated by Sinha [91], these problems have applications in the modeling of certain stochastic programming problem. While most of these studies have considered only the Wolfe type of dual, Chandra *et al* [23] studied duality for such problems in the spirit of Mond and Weir [82] in order to relax convexity conditions assumed in aforecited references.

Mangasarian [66] was the first to identify a second-order dual formulation for non-linear programs under the assumptions that are complicated and somewhat difficult to verify. Mond [70] introduced the concept of second-order convex functions (named as bonvex functions by Bector and Chandra [11]) and studied second-order duality for nonlinear programs.

This chapter has two sections 2.1 and 2.2. The purpose of the section 2.1 is to formulate Wolfe and Mond-Weir type second-order duals for a nonlinear programming problem in which the objective and the constraint functions contains a term of a support function and establish various duality results for each pair of dual problems. It is pointed out that duality results obtained in [50] become special cases of our results. In section 2.2 we present a mixed type second-order dual to the non differentiable program which combines Wolfe and Mond–Weir second-order duals considered in section 2.1.It is also pointed out that first-order mixed type duality results proved in section 2.1 are special cases of our results. It is also indicated that the duality results studied by Zhang and Mond [101] becomes special cases of our results if the support function is the objective is replaced by square root of positive semi definite quadratic form and the support functions that appear in the constraints are suppressed.

2.1 SECOND-ORDER DUALITY IN MATHEMATICAL PROGRAMMING WITH SUPPORT FUNCTIONS

2.1.1 Pre-requisites

Let $f : R^n \to R$ and $g_j : R^n \to R, (j = 1, 2.,,, m)$ be subdifferentiable functions. Let C be a compact convex set in R^n. Then consider the following nonlinear programming problem:

(P): Minimum $f(x)$

Subject to

$$g_j(x) \le 0, \qquad (j = 1, 2, ...m)$$

$$x \in C$$

The following lemmas relating to (P) will be used here:

Lemma 2.1.1 [91]: If $\bar{x}$ is an optimal solution for (P), then there exist $\lambda \in R_+$ and $\mu \in R_+^m$, such that

$$0 \in \lambda \partial f(\bar{x}) + \sum_{j=1}^{m} \mu_j \partial g_j(\bar{x}) + N_C(\bar{x}),$$

$$\lambda + \sum_{j=1}^{m} \mu_j > 0,$$

$$\mu_j g_j(\bar{x}) = 0, \qquad j = 1, 2, ...m.$$

Lemma 2.1.2 [91]: If $\bar{x}$ is an optimal solution for (P), and a Slater's constraint qualification holds for (P), then there exist non negative constants $\mu_j\ (j = 1, 2, \ldots, m)$, such that

$$0 \in \partial f(x) + \sum_{j=1}^{m} \mu_j \partial g_j(\bar{x}) + N_C(\bar{x}),$$

$$\mu_j g_j(\bar{x}) = 0, \qquad j = 1, 2, ...m.$$

It is to be noted that under the conditions of convexity on the functions f and $g_j, (j = 1, 2, \ldots, m)$, these necessary conditions are also sufficient for the optimality of $\bar{x}$ for (P).

2.1.2 Nondifferentiable Programming Problem Containing Support Functions and Duality

Let $f : R^n \to R$ and $g_j : R^n \to R,\ (j = 1, 2, ..., n)$ be twice differentiable functions. Let C and $D_j,\ (j = 1, 2, \ldots, m)$ be compact convex sets in R^n.

We consider the following nondifferentiable nonlinear programming problem:

(NP): Minimum $f(x)+S(x/C)$

Subject to

$$g_j(x)+S(x/D_j)\le 0, \qquad (j=1,2...m) \tag{2.1}$$

In studying duality for (NP) certain optimality conditions in the non-smooth setting will be required. These conditions which can be derived from [91] along with the application of Lemma 2.1.1 and Lemma 2.1.2 are given below:

Theorem 2.1.1: If $\bar{x}$ is an optimal solution for (NP), then there exists $\bar{\alpha}\in R, \bar{z}\in C, \bar{y}\in R^m$ and $\bar{w}_j\in D_j,(j=1,2,...,m)$ such that

$$\bar{\alpha}(\nabla f(\bar{x})+\bar{z})+\sum_{j=1}^{m}\bar{y}_j(\nabla g_j(\bar{x})+\bar{w}_j)=0,$$

$$\sum_{j=1}^{m}\bar{y}_j(\nabla g_j(\bar{x})+\bar{w}_j^T(\bar{x}))=0,$$

$$\bar{z}^T(\bar{x})=S(\bar{x}/C) \text{ and } \bar{w}_j^T(\bar{x})=S(\bar{x}/D_j), \qquad \forall j=1,2,..m$$

$$(\bar{\alpha},\bar{y})\ge 0,(\bar{\alpha},\bar{y})\ne 0.$$

When a suitable constraint qualification holds for (NP) the above Fritz John optimality conditions reduces to the Karush-Kuhn-Tucker optimality conditions, as this asserts positiveness of the multiplier $\bar{\alpha}$ associated with the objective function.

2.1.3 Wolfe Type Duality

Consider the following nonlinear program, which will be proved to be a dual program to (NP)

(WD): Max $\quad f(u)+z^T u+\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T(u)\right)-\frac{1}{2}p^T\nabla^2\left(f(u)+y^T g(u)\right)p$

Subject to,

$$\nabla(f(u)+z^T u)+\sum_{j=1}^{m} y_j\nabla\left(g_j(u)+w_j\right)+\nabla^2\left(f(u)+y^T g(u)\right)p=0, \tag{2.2}$$

$$y\geq 0, \tag{2.3}$$

$$z\in C, w_j\in D_j,\ (j=1,2.,,,m). \tag{2.4}$$

Theorem 2.1.2 (Weak Duality): Let x be feasible for (NP) and $(u,z,y,p,w_1,w_2,...w_m)$ be feasible for (WD) and let for all feasible $(x,z,y,p,w_1,w_2,...,w_m)$, $f(\cdot)$ and $g_j(\cdot)$, $(j=1,2,...,m)$ be second-order convex, then

$$f(x)+S(x/C)\geq f(u)+z^T u+\sum_{j=1}^{m} y_j\left(g_j(u)+w_j^T(u)\right)-\frac{1}{2}p^T\nabla^2\left(f(u)+y^T g(u)\right)p.$$

i.e., inf.(NP) ≥ sup. (WD)

Proof**:** Let x be feasible for (NP) and $(u,z,y,p,w_1,w_2,...w_m)$ be feasible for (WD), therefore, from second-order convexity of $f(\cdot)$ and $g_j(\cdot)$, $(j=1,2.,,,m)$ we have

$$\left(f(x)+y^T g(x)+\sum_{j=1}^{m} y_j w_j^T x\right)-\left(f(u)+y^T g(u)+\sum_{j=1}^{m} y_j w_j^T u\right)\geq$$

$$\sum_{j=1}^{m} y_j w_j^T(x-u)-\frac{1}{2}p^T\nabla^2\left(f(u)+y^T g(u)p\right)+$$

$$(x-u)\left[\left(\nabla f(u)+\nabla y^T g(u)+\nabla^2\left(f(u)+y^T g(u)\right)p\right)\right] \tag{2.5}$$

Now from the dual feasibility, we have

$$(x-u)\left(\nabla f(u)+\nabla y^T g(u)+\nabla^2\left(f(u)+y^T g(u)\right)p\right)=$$
$$-(x-u)z-\sum_{j=1}^{m} y_j w_j^T(x-u) \tag{2.6}$$

Therefore from (2.5) and (2.6) we get,

$$\left(f(x)+y^T g(x)+\sum_{j=1}^{m} y_j w_j^T x \right)-\left(f(u)+y^T g(u)+\sum_{j=1}^{m} y_j w_j^T u \right)\geq$$

$$-(x-u)^T z-\tfrac{1}{2} p^T \nabla^2\left(f(u)+y^T g(u)\right)p$$

i.e.,

$$\left(f(x)+z^T x\right)-\left(f(u)+z^T u+y^T g(u)+\sum_{j=1}^{m} y_j w_j^T u-1/2p^T \nabla^2\left(f(u)+y^T g(u)\right)p \right)\geq$$

$$\left(-y^T g(x)-\sum_{j=1}^{m} y_j w_j^T x \right)$$

but $S(x/C)\geq z^T x$, whenever $z\in C$ and $S(\bar{x}/D_j)\geq w_j^T x,$ whenever $w_j \in D_j$.

which implies that

$$0\geq g_j(x)+S(\bar{x}/D_j)\geq g_j(x)+w_j^T x.$$

$$0\geq y_j (g_j + S(\bar{x}/D_j)$$

$$0\geq \sum y_j g_j(x)+\sum y_j w_j^T x = y^T g(x)+\sum_{j=1}^{m} y_j w_j^T x$$

$$y^T g(x)+\sum_{j=1}^{m} y_j w_j^T x\leq 0$$

As $y\geq 0,$ we get $\left(-y^T g(x)-\sum_{j=1}^{m} y_j w_j^T x \right)\geq 0.$

Hence

$$\left(f(x)+z^T x\right)\geq\left(f(u)+z^T u+y^T g(u)+\sum_{j=1}^{m} y_j w_j^T u-1/2p^T \nabla^2\left(f(u)+y^T g(u)\right)p \right)$$

Inf. (NP) $\geq$ sup. (WD).

Corollary 2.1.1: Let $\bar{x}$ be feasible for (NP) and $(\bar{u},\bar{z},\bar{y},\bar{p},\bar{w}_1,\bar{w}_2,...\bar{w}_m)$ is feasible for (WD) with corresponding objective functions being equal. Let the hypotheses of Theorem 2.1.2 hold. Then $\bar{x}$ is optimal for (NP) and $(\bar{u},\bar{z},\bar{y},\bar{p},\bar{w}_1,\bar{w}_2,...\bar{w}_m)$ is optimal for (WD).

Theorem 2.1.3 (Strong Duality): Let $\bar{x}$ be optimal for (NP) and the suitable constraint qualification [68] hold. Then there exists $\bar{z}\in C,\bar{y}\in R^m,\bar{w}_j\in D_j,(j=1,2,...m)$such that $(\bar{x},\bar{z},\bar{y},\bar{p}=0,\bar{w}_1,\bar{w}_2,...\bar{w}_m)$ is feasible for (WD) and the objective function values of (NP) and (WD) are equal. Further if the hypothesis of Theorem 2.1.2 hold then $(\bar{x},\bar{z},\bar{y},\bar{p}=0,\bar{w}_1,\bar{w}_2,...\bar{w}_m)$ is an optimal solution for (WD).

***Proof*:** Since $\bar{x}$ be an optimal solution for (NP) and a suitable constraint qualification [68] holds for (NP), then there exists $\bar{z}\in C,\bar{y}\in R^m_+,\bar{w}_j\in D_j,(j=1,2,...m)$such that

$$\nabla f(\bar{x})+\bar{z}+\sum_{j=1}^{m}\bar{y}_j\left(\nabla g_j(\bar{x})+\bar{w}_j\right)=0,$$

$$\sum_{j=1}^{m}\bar{y}_j\left(g_j(\bar{x})+\bar{w}_j^T\bar{x}\right)=0,$$

$$\bar{z}^T\bar{x}=S(\bar{x}/C),\text{ and }\bar{w}_j{}^T\bar{x}=S(\bar{x}/D_j),\quad\forall j=1,2...m.$$

Hence $(\bar{x},\bar{z},\bar{y},\bar{p}=0,\bar{w}_1,\bar{w}_2,...\bar{w}_m)$ is feasible for (WD) and

$$f(\bar{x})+\bar{z}^T\bar{x}+\sum_{j=1}^{m}\bar{y}_j\left(g_j(\bar{x})+\bar{w}_j^T\right)=f(\bar{x})+S(\bar{x}/C),.$$

That is, the objective function values of (NP) and (WD) are equal. Remainder of the proof now immediately follows from Corollary 2.1.1.

Theorem 2.1.4 (Converse Duality): Let $(\bar{u}, \bar{z}, \bar{y}, \bar{p}, \bar{w}_1, \bar{w}_2, \ldots \bar{w}_m)$ is optimal for (WD) and the Hessian matrix $\nabla^2\left(f(\bar{u}) + \sum_{j=1}^{m} y_j g_j(\bar{u})\right)$ be non-singular and $\nabla^2\left(\nabla^2 f(\bar{u}) + \nabla^2 \sum_{j=1}^{m} y_j g_j(\bar{u})\right)$ be either positive or negative definite. Then $\sum_{j=1}^{m} y_j g_j(\bar{u}) + S(\bar{u}/D_j) = 0,$ and $\bar{u}$ is feasible for (NP) and the objective function values of (NP) and (WD) are equal. Further if the hypotheses of Theorem 2.1.2 hold then $\bar{u}$ is an optimal for (NP).

***Proof*:** First we rewrite problem (WD) in the form of (P), for this let $q = (u, z, y, p, w_1, w_2, \ldots w_m) \in \boldsymbol{R}^{(3+m)n+m}$ and

$$\text{F}(q) = \left(f(\bar{u}) + \bar{z}^T(\bar{u})\right) + \sum_{j=1}^{m} \bar{y}_j\left(g_j(\bar{u}) + \bar{w}_j^T \bar{u}\right) - 1/2\bar{p}^T \nabla^2\left(f(\bar{u}) + y^T g(\bar{u})\right)\bar{p},$$

$$\text{G}(q) = \left(\nabla f(\bar{u}) + \bar{z}\right) + \sum_{j=1}^{m} \bar{y}_j\left(\nabla g_j(\bar{u}) + \bar{w}_j\right) + \nabla^2\left(f(\bar{u}) + \bar{y}^T g(\bar{u})\right)\bar{p},$$

$$\text{H}(q) = -y.$$

Let the set S be defined by S= $\{q : q = (u, z, y, p, w_1, w_2, \ldots w_m),\ z \in C,\ w_j \in D_j,\ \forall j = 1,2,\ldots m\}$, then problem (WD) may be rewritten as follows:

Maximum $\quad$ F (q)

Subject to

$$\text{G}(q) = 0,$$

$$\text{H}(q) \le 0,$$

$$q \in S.$$

As $\bar{q} = (\bar{u}, \bar{z}, \bar{y}, \bar{p}, \bar{w}_1, \bar{w}_2, \ldots \bar{w}_m)$ is optimal for (WD), from Lemma 2.1.1, there exist constants $\alpha \ge 0, \mu_j \ge 0, j = 1,2,\ldots m$ and $\lambda_i, \boldsymbol{i} = 1,2,\ldots \boldsymbol{n},$ not all zero, and the normal cone to S at $\bar{q}$ as $N_S(\bar{q})$ such that

$$-\left[\alpha\left(\nabla f(\bar{u})+\bar{z}\right)+\sum_{j=1}^{m}\bar{y}_j\left(\nabla g_j(\bar{u})+\bar{w}_j\right)-\tfrac{1}{2}\nabla\bar{p}^T\nabla^2\left(f(\bar{u})+\bar{y}^T g(\bar{u})\right)\bar{p}\right]+$$

$$\left(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u})\right)\lambda+\lambda\nabla\left(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u})\right)\bar{p}=0 \quad (2.7)$$

$$-\alpha(-\left(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u})\right)\bar{p})+\lambda\left(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u})\right)=0 \quad (2.8)$$

$$-\alpha\left(g_j(\bar{u})+\bar{w}_j^T\bar{u}-\tfrac{1}{2}\bar{p}^T\nabla^2 g_j(\bar{u})\bar{p}\right)+\ \lambda\left(\nabla g_j(\bar{u})+\bar{w}_j+\nabla^2 g_j(\bar{u})\bar{p}\right)$$

$$-\mu_j=0, \qquad \forall j=1,2,\ldots m \quad (2.9)$$

$$-\alpha\bar{u}+\lambda\in N_C(\bar{z}), \quad (2.10)$$

$$-\alpha\bar{u}\bar{y}_j+\lambda\bar{y}_j\in N_{D_j}(\bar{w}_j), \quad (2.11)$$

$$\mu_j y_j=0,\ \forall j=1,2,\ldots m \quad (2.12)$$

From (2.8),we have,

$$(\alpha p+\lambda)\left(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u})\right)=0$$

But from non-singularity of the matrix $\left(\nabla^2 f(\bar{u})+\nabla^2 y^T g(\bar{u})\right)$ we have $(\alpha p+\lambda)$ = 0. If possible, let $\alpha=0,$ then $\lambda=0.$ From these values, (2.9) implies $\mu_j=0,\ \forall j=1,2,\ldots m$, which makes all the multipliers equal to zero. Since this cannot happen as it contradicts $(\alpha,\lambda,\mu)\neq 0$. So we must have $\alpha\neq 0,$ so $\alpha>0.$ Using the equality constraint of the dual problem in equation (2.7) we have,

$$\alpha\left[\left(\nabla^2\left(f(\bar{u})+\bar{y}^T g(\bar{u})\right)p\right)-\frac{1}{2}\bar{p}^T\nabla^2\left(f(\bar{u})+\bar{y}^T g(\bar{u})\right)\bar{p}\right]$$

$$+\ \left(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u})\right)\lambda+\lambda\nabla\left(\nabla^2 f(\bar{u})+\nabla^2\bar{y}^T g(\bar{u})\right)\bar{p}=0$$

This can be written as

$$(\alpha p+\lambda)\left(\nabla^2\left(f(\bar{u})+y^T g(\bar{u})\right)p\right)+\left(\lambda-\frac{\alpha p}{2}\right)\nabla\left(\nabla^2\left(f(\bar{u})+y^T g(\bar{u})\right)\bar{p}\right)=0$$

This along with $\alpha p+\lambda=0$ yields,

$$\frac{\alpha p}{2}\nabla\left(\nabla^2\left(f(\bar{u})+\bar{y}^T g(\bar{u})\right)\bar{p}\right)=0$$

Because of positiveness of α .This equation is simplified as

$$p^T\nabla\left(\nabla^2\left(f(\bar{u})+\bar{y}^T g(\bar{u})\right)\bar{p}\right)=0$$

which by the condition of $\nabla\left(\nabla^2\left(f(\bar{u})+y^T g(\bar{u})\right)\right)$to be either positive or negative definite implies $p=0$. Now $(\alpha p+\lambda)$ = 0, hence $\lambda=0$ .Then equation (2.9) implies that

$$\begin{aligned}&(-\alpha+\lambda)\{g_j(\bar{u})+w_j\}+\left(-\frac{\alpha p}{2}+\lambda\right)\nabla^2 g_j(\bar{u})\,p-\mu_j\\&\qquad-\alpha\left(\nabla g_j(\bar{u})+w_j\right)+0=\mu_j\\&\nabla g_j(\bar{u})+w_j=-\frac{\mu_j}{\alpha}\le 0,\\&g_j(\bar{u})+\bar{w}_j{}^T\bar{u}\le 0\,,\forall j=1,2,\dots m.\end{aligned}$$

Now from (2.10) and (2.11) we have $\bar{u}\in N_C(\bar{z})$ and $\bar{u}\in N_{D_j}(\bar{w}_j)$ so that $\bar{z}^T\bar{u}=S(\bar{u}/C)$ and $\bar{w}_j{}^T\bar{u}=S(\bar{u}/D_j)$, $\forall j=1,2,\dots m$.Hence

$$g_j(\bar{u})+\bar{w}_j^T\bar{u}=g_j(\bar{u})+S(\bar{u}/D_j)\le 0,\ \forall j=1,2,\dots m$$

which implies that $\bar{u}$ is feasible for problem (NP).Also from (2.9) and (2.12) we get

$$\bar{y}_j\left(g_j(\bar{u})+\bar{w}_j^T\bar{u}\right)=0,\quad j=1,2,\dots m.$$

Therefore,

$$\left(f(\bar{u})+\bar{z}^T\bar{u}\right)+\sum_{j=1}^{m}\bar{y}_j\left(g_j(\bar{u})+\bar{w}_j^T\bar{u}\right)-1/2\bar{p}^T\nabla^2\left(f(\bar{u})+\bar{y}^Tg(\bar{u})\right)\bar{p}=f(\bar{u})+S(\bar{u}/C).$$

This by Corollary 2.1.1 implies that $\bar{u}$ is optimal for (NP).

2.1.4 Mond - Weir Type Duality

We state the following problem as a Mond-Weir type second-order dual for the problem (NP).

(SMWD): Maximum $f(u)+z^Tu-\frac{1}{2}p^T\nabla^2(f(u))p$

Subject to

$$\nabla f(u)+z+\sum_{j=1}^{m}y_j\left(\nabla g_j(u)+w_j\right)+\nabla^2\left(f(u)+y^Tg(u)\right)p=0, \tag{2.13}$$

$$\sum_{j=1}^{m}y_j\left(g_j(u)+w_j^Tu\right)-\frac{1}{2}p^T\nabla^2\left(y^Tg(u)\right)p\geq 0, \tag{2.14}$$

$$y\geq 0, \tag{2.15}$$

$$z\in C, w_j\in D, \quad \forall j=1,2,...m \tag{2.16}$$

Theorem 2.1.5 (Weak Duality): Let x be feasible for (NP) and $(u,z,y,p,w_1,w_2,...w_m)$ be feasible for (SMWD) and let for all feasible $(x,u,z,y,p,w_1,w_2,...w_m)$ to (NP) and (SMWD), $f(\cdot)+(\cdot)^Tz$ is second-order pseudoconvex and $\sum_{j=1}^{m}y_j\left(g_j(\cdot)+(\cdot)^Tw_j\right)$ is second-order quasiconvex, then

$$f(x)+S(x/D_j)\geq f(u)+z^Tu-\frac{1}{2}p^T\nabla^2f(u)p.$$

***Proof*:** By the primal feasibility of x and dual feasibility of $(u,z,y,p,w_1,w_2,...w_m)$, we have

$$\sum_{j=1}^{m} y_j \left(g_j(x) + S\left(x/D_j\right)\right) \leq \sum_{j=1}^{m} y_j \left(g_j(x) + w_j^T u\right) - \frac{1}{2} p^T \nabla^2 \left(y^T g(u)\right) p.$$

This in view of $w_j^T x \leq S\left(x/D_j\right), \quad \forall j = 1,2,...m$, gives,

$$\sum_{j=1}^{m} y_j \left(g_j(x) + w_j^T x\right) \leq \sum_{j=1}^{m} y_j \left(g_j(x) + w_j^T u\right) - \frac{1}{2} p^T \nabla^2 \left(y^T g(u)\right) p \quad (2.17)$$

Because of second-order quasiconvexity of $\sum_{j=1}^{m} y_j \left(g_j(\cdot) + (\cdot)^T w_j\right)$, (2.17) yields,

$$(x-u)^T \left(\sum_{j=1}^{m} y_j \left(\nabla g_j(u) + w_j\right) + \nabla^2 \left(y^T g(u)\right) p \right) \leq 0$$

This is conjunction with (2.13), we get,

$$(x-u)^T \left(\nabla f(u) + z + \nabla^2 \left(\ f(u)\right) p\right) \geq 0,$$

which by second-order pseudoconvexity of $f(\cdot) + (\cdot)^T z$ gives,

$$f(x) + z^T x \geq f(u) + z^T u - \frac{1}{2} p^T \nabla^2 f(u) p.$$

Since $z^T x \leq S(x/C)$, as earlier, we have,

$$f(x) + S(x/C) \geq f(u) + z^T u - \frac{1}{2} p^T \nabla^2 f(u) p.$$

Corollary 2.1.2: Let $\bar{x}$ be feasible for (NP) and $(\bar{u}, \bar{z}, \bar{y}, \bar{p}, \bar{w}_1, \bar{w}_2, ...\bar{w}_m)$ is feasible for (SMWD) with corresponding objective function being equal. Let the hypotheses of Theorem 2.1.5 hold. Then $\bar{x}$ is optimal for (NP) and $(\bar{u}, \bar{z}, \bar{y}, \bar{p}, \bar{w}_1, \bar{w}_2, ...\bar{w}_m)$ is optimal for (SMWD).

Theorem 2.1.6 (Strong Duality): Let $\bar{x}$ be optimal for (NP) and the suitable constraint qualification holds for (NP). Then there exists $\bar{z} \in C, \bar{y} \in R^m, \bar{w}_j \in D_j, (j = 1,2,...m)$ such that $(\bar{x}, \bar{z}, \bar{y}, \bar{p} = 0, \bar{w}_1, \bar{w}_2, ...\bar{w}_m)$ is feasible for (SMWD) and the objective function values of (NP) and (MWD) are equal. Further if the hypotheses of Theorem 2.1.5 hold then $(\bar{x}, \bar{z}, \bar{y}, \bar{p} = 0, \bar{w}_1, \bar{w}_2, \ldots \bar{w}_m)$ is optimal for (SMWD).

***Proof*:** Since $\bar{x}$ be optimal for (NP) and the suitable constraint qualification holds for (NP), then there exists $\bar{z} \in C, \bar{y} \in R_+^m$, $\bar{w}_j \in D_j, (j = 1, 2, \ldots m)$ such that

$$\nabla f(\bar{x}) + \bar{z} + \sum_{j=1}^{m} \bar{y}_j \left(\nabla g_j(\bar{x}) + \bar{w}_j\right) = 0,$$

$$\sum_{j=1}^{m} \bar{y}_j \left(g_j(\bar{x}) + \bar{w}_j^T \bar{x}\right) = 0,$$

$$\bar{z}^T \bar{x} = S(\bar{x}/C), \text{ and } \bar{w}_j{}^T \bar{x} = S(\bar{x}/D_j), \forall j = 1, 2 \ldots m$$

Hence $(\bar{x}, \bar{z}, \bar{y}, \bar{p} = 0, \bar{w}_1, \bar{w}_2, \ldots \bar{w}_m)$ is feasible for (MWD) and

$$f(\bar{x}) + \bar{z}^T \bar{x} - \frac{1}{2} \bar{p}^T \nabla^2 f(\bar{x}) \bar{p} = f(\bar{x}) + S(x/C)..$$

Therefore the objective function values of (NP) and (SMWD) are equal. Rest of the proof now follows from Corollary 2.1.2.

Theorem 2.1.7 (Converse Duality): Let $(\bar{x}, \bar{z}, \bar{y}, \bar{w}, \bar{p})$ be an optimal solution to (SMWD) at which

(H$_1$): (a) the n×n Hessian matrix $\nabla^2\left(\sum_{j=1}^{m} \bar{y}_j g_j(\bar{x})\right)$ is positive definite and

$$\bar{p}^T \sum_{j=1}^{m} \bar{y}_j \left(g_j(\bar{x}) + \bar{w}_j\right) \geq 0 \quad \text{or}$$

(b) the Hessian matrix $\nabla^2\left(\bar{y}_j{}^T g_j(\bar{x})\right)$ is negative definite and $\bar{p}^T \nabla \sum_{j=1}^{m} \bar{y}_j \left(\nabla g_j(\bar{x}) + \bar{w}_j\right) \leq 0$, and

(H$_2$): the set $\{[\nabla^2 f(\bar{x})]_i\ ,\ [\nabla^2(\bar{y}g(\bar{x}))]_i \quad |\ i=1,2,...,n\ \}$, of vectors is linearly independent, where $[\nabla^2 f(\bar{x})]_i$ is the i^{th} row of $[\nabla^2 f(\bar{x})]$ and $[\nabla^2(\bar{y}^T g(\bar{x}))]_i$ is i^{th} row of the matrix $[\nabla^2(\bar{y}^T g(\bar{x}))]$

(H$_3$): the vectors $\sum_{j=1}^{m}\bar{y}_j(g_j(\bar{x})+\bar{w}_j)\neq 0$

If, for all feasible (x, z, y, u, w_1, w_2, ...,w_m, p), $f(\cdot)+(\cdot)^T z$ is second-order pseudo convex and $\sum_{j=1}^{m}\bar{y}_j(g_j(\cdot)+(\cdot)^T w_j)$ is second-order quasi convex, then $\bar{x}$ is an optimal solution of the problem (NP).

***Proof*:** Since $(\bar{x},\bar{z},\bar{y},\bar{w})$, where $\bar{w}=(\bar{w}_1,\bar{w}_2,...\bar{w}_m)$ is an optimal solution of (SM-WD), by generalized Fritz John necessary optimality conditions [68], there exists, $\alpha\in R$, $\beta\in R^n$, $\theta\in R$, and $\mu\in R^m$,such that

$$\begin{aligned}&\alpha\left\{-\left(f(\bar{x})+\bar{z}\right)+\frac{1}{2}\bar{p}^T\nabla\left[\nabla^2\left(f(\bar{x})\right)\bar{p}\right]\right\}\\&\quad+\beta^T\left\{\nabla^2\left(f(\bar{x})+\bar{y}^Tg(\bar{x})\right)+\nabla\left(\nabla^2\left(f(\bar{x})+\bar{y}^Tg(\bar{x})\right)\bar{p}\right)\right\}\\&\quad-\theta\left\{\sum_{j=1}^{m}\bar{y}_j(\nabla g_j(\bar{x})+\bar{w}_j)-\frac{1}{2}\bar{p}^T\nabla\left[\left(\nabla^2\left(\bar{y}^Tg(\bar{x})\right)\right)\bar{p}\right]\right\}=0\end{aligned}\tag{2.18}$$

$$\begin{aligned}&\beta\left\{\nabla\left(g_j(\bar{x})+\bar{w}_j\right)+\nabla^2g_j(\bar{x})\bar{p}\right\}\\&\quad-\theta^T\left\{g_j(\bar{x})+\bar{x}_j{}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2g_j(\bar{x})\bar{p}\right\}-\mu_j=0,\ j=1(1)m\end{aligned}\tag{2.19}$$

$$\left(\alpha\bar{p}+\beta\right)^T\nabla f(\bar{x})\ +\left(\theta\bar{p}+\beta\right)^T\nabla^2\left(\bar{y}g(\bar{x})\right)=0\tag{2.20}$$

$$\theta\left\{\sum_{j=1}^{m}\ \bar{y}_j(g_j(\bar{x})+\bar{x}_j{}^T\bar{w}_j)-\frac{1}{2}\bar{p}^T\nabla^2\left(\bar{y}_j(g_j(\bar{x})\right)\bar{p}\right\}=0\ ,\tag{2.21}$$

$$\mu^T\bar{y}=0\tag{2.22}$$

$$-\alpha\bar{x}+\beta \in N_c(\bar{z}) \tag{2.23}$$

$$(\beta-\theta)\bar{y}_j\,\bar{x} \quad \in N_{D_j}(\bar{w}_j)\,, \quad j=1(1)m \tag{2.24}$$

$$(\alpha,\theta,\mu)\geq 0 \tag{2.25}$$

$$(\alpha,\beta,\theta,\mu)\neq 0 \tag{2.26}$$

The relation (2.20), in view of assumption (A_2) yields,

$$\alpha\,\bar{p}+\beta=0\,, \quad \text{and} \qquad \theta\,\bar{p}+\beta=0 \tag{2.27}$$

Multiplying (2.19) by $\bar{y}_j$, and summing over j , we get,

$$\beta^T\left\{\sum_{j=1}^{m} \bar{y}_j\left(\nabla\left(g_j(\bar{x})+\bar{w}_j\right)+\nabla^2\left(\bar{y}^T g\ (\bar{x})\right)\bar{p}\right)\right\}$$

$$-\theta\left\{\sum_{i=1}^{m} \bar{y}_j\left(g_j(\bar{x})+\bar{x}_j{}^T\bar{w}_j\right)-\frac{1}{2}\bar{p}\nabla^2\left(\bar{y}^T g(\bar{x})\right)\bar{p}\right\}=0 \quad , \quad j=1(1)m \tag{2.28}$$

Using (2.21) in the above relation, we get,

$$\beta\left\{\sum_{j=1}^{m} \bar{y}_j\left(\nabla\left(g_j(\bar{x})+\bar{w}_j\right)+\nabla^2\bar{y}^T g\ (\bar{x})\bar{p}\right)\right\}=0 \tag{2.29}$$

The relation (2.18) together with the equality constraint of the dual, yields,

$$(\alpha-\theta)\left\{\sum_{j=1}^{m} \bar{y}_j\left(\nabla\left(g_j(\bar{x})+\bar{w}_j\right)\right)\right\}+(\alpha\bar{p}+\beta)^T\left[\nabla^2 f(\bar{x})+\nabla\left(\nabla^2 f(\bar{x})\bar{p}\right)\right]$$

$$+(\beta+\alpha\bar{p})^T\left[\nabla^2(\bar{y}g(\bar{x}))+\nabla\left(\nabla^2(\bar{y}g(\bar{x}))\bar{p}\right)\right]$$

$$+\frac{1}{2}(\alpha\bar{p})^T\nabla\left(\nabla^2 f(\bar{x})\bar{p}\right)-(\alpha\bar{p})^T\nabla\left(\nabla^2 f(\bar{x})\bar{p}\right)$$

$$+\left(\frac{\theta\bar{p}}{2}\right)^T\nabla\left(\nabla^2(\bar{y}g(\bar{x}))\bar{p}\right)-\alpha p\nabla\left(\nabla^2(\bar{y}g(\bar{x}))\bar{p}\right)=0$$

Using (2.27) in this equation, we have,

$$(\alpha-\theta)\left\{\sum_{j=1}^{m}\bar{y}_j\left(\nabla\left(g_j(\bar{x})+\bar{w}_j\right)\right)\right\}-$$

$$\left(\frac{\beta}{2}\right)^T\left(\nabla\left(\nabla^2\left(\bar{y}^T f(\bar{x})\,p\right)\right)+\nabla\left(\nabla^2\left(\bar{y}^T g(\bar{x})\right)\bar{p}\right)\right)=0 \qquad (2.30)$$

If $(\alpha,\theta)=0$, then (2.27) implies $\beta=0$ and $\mu=0$ from (2.19) consequently we get $(\alpha,\beta,\theta,\mu)=0$ contradicting (2.26).Thus, $(\alpha,\theta)\neq 0$,this implies that at least one of these multipliers α and θ must be positive. We claim $\bar{p}=0$.Suppose that $\bar{p}\neq 0$, then (2.27) yields,

$$(\alpha-\theta)\bar{p}=0$$

This implies $\alpha=\theta$>0.So from (2.29) along with (2.27), we have,

$$\bar{p}^T\left\{\sum_{j=1}^{m}\bar{y}_j\left(\nabla\left(g_j(\bar{x})+\bar{w}_j\right)+\nabla^2\left(\bar{y}^T g\ (\bar{x})\right)\bar{p}\right)\right\}=0 \qquad (2.31)$$

Since $\nabla^2\left(\sum_{j=1}^{m}\bar{y}_j g_j(\bar{x})\right)$is positive definite, i.e. $\bar{p}^T\nabla^2\left(\sum_{j=1}^{m}\bar{y}_j g_j(\bar{x})\right)\bar{p}>0$and $\bar{p}^T\sum_{j=1}^{m}\bar{y}_j\left(g_j(\bar{x})+\bar{w}_j\right)\geq 0$, we have

$$\bar{p}^T\left\{\sum_{j=1}^{m}\bar{y}_j\left(\nabla\left(g_j(\bar{x})+\bar{w}_j\right)+\nabla^2\left(\bar{y}^T g\ (\bar{x})\right)\bar{p}\right)\right\}>0.$$

This is contradicted by (2.31). Hence $\bar{p}=0$.By this, (2.27) implies $\beta=0$.

From (2.19), we have,

$$\Rightarrow \quad g_j(\bar{x})+\bar{w}_j{}^T\bar{x}=-\frac{\mu_j}{\theta}\leq 0\ , \qquad j=1,2,\ldots m \qquad (2.32)$$

From (2.24), we have,

$$\bar{x}^T\bar{w}_j=S\left(\bar{x}\middle|D_j\right)\ , \qquad j=1,2,\ldots,m$$

Using this in (2.32), we obtain,

$$\Rightarrow \qquad g_j(\bar{x}) + S\left(\bar{x}\,\middle|\,D_j\right) \le 0 \qquad\qquad j = 1, 2, \ldots, m$$

This implies $\bar{x}$ is feasible for (NP).

Multiplying (2.32) by $\bar{y}_i$ and adding over i, we have,

$$\sum_{j=1}^{m} \bar{y}_j \left(g(\bar{x}) + \bar{w}_j \bar{x} \right) = 0 \tag{2.33}$$

Now consider

$$\left(f(\bar{x}) + \bar{x}^T \bar{z} \right) - \frac{1}{2} \bar{p}^T \left[\nabla^2 \left(f(\bar{x}) \right) \bar{p} \right] = f(\bar{x}) + \bar{x}^T \bar{z}$$

Using $p = 0$,from (2.23), we have,

$$\bar{x}^T \bar{z} = S(\bar{x}|C)$$

Thus,

$$\left(f(\bar{x}) + \bar{x}\bar{z} \right) - \frac{1}{2} \bar{p}^T \left[\nabla^2 \left(f(\bar{x}) \right) \bar{p} \right] = f(\bar{x}) + S(\bar{x}|C) \tag{2.34}$$

If for all feasible ($x, z, y, u, w_1, w_2, \ldots w_m, p$), $f(\cdot) + (\cdot)^T z$ is second-order pseudo convex and $\sum_{j-1}^{m} \bar{y}_j \left(g_j(\cdot) + (\cdot)^T w_j \right)$, is second-order quasi convex, by Theorem 2.1.5 ,then $\bar{x}$ is an optimal solution of the problem (NP).

2.1.5 Special Cases

Now for p = 0, the dual program (WD) and (MD), becomes the Wolfe and Mond-Weir type programs for (NP) studied by Husain *et al* [50]

(WD): Maximum $\quad \left(f(u) + z^T u \right) + \sum_{j=1}^{m} y_j \left(g_j(u) + w_j^T u \right)$

Subject to

$$\left(\nabla f(u) + z \right) + \sum_{j=1}^{m} y_j \left(\nabla g_j(u) + w_j \right) = 0$$

$$y \ge 0,$$

$$z \in C, w_j \in D_{j,}\, j = 1,2,\ldots m.$$

(MD): Maximum $\left(f(u)+z^T u\right)$

Subject to

$$\left(\nabla f(u)+z\right)+\sum_{j=1}^{m} y_j\left(\nabla g_j(u)+w_j\right)=0$$

$$y \geq 0,$$

$$z \in C, w_j \in D_{j,}\, j=1,2,\ldots m.$$

2.2 MIXED TYPE SECOND-ORDER DUALITY WITH SUPPORT FUNCTIONS

2.2.1 Mixed Second-Order Type Duality

We propose the following mixed type second-order dual type to the problem (NP) which combines both Wolfe and Mond -Weir type dual models, considered in the previous section.

(Mix SD): Maximize $f(u)+u^T z+\sum_{i\in I_0} y_i\left(g_i(u)+u^T w_i\right)$

$$-\frac{1}{2}\nabla^2 p^T\left[f(u)+\sum_{i\in I_0} y_i g_i(u)\right]p$$

Subject to

$$\nabla f(u)+z+\sum_{i=1}^{m} y_i\left(\nabla g_i(u)+w_i\right)+\nabla^2\left(f(u)+y^T g(u)\right)p=0 \tag{2.35}$$

$$\sum_{i\in I_\alpha} y_i\left(g_i(u)+u^T w_i\right)-\frac{1}{2}p^T\nabla^2\left(\sum_{i\in I_\alpha} y_i g_i(u)\right)p \geq 0,\ \alpha=1,2,\ldots,r. \tag{2.36}$$

$$y \geq 0 \tag{2.37}$$

$$z \in C, w_i \in D_i,\ i=1,2,\ldots,m. \tag{2.38}$$

where

1. $I_\alpha \subseteq M=\{1,2,\ldots,m\}$, $\alpha = 0, 1,2,\ldots,r$ with $\bigcup_{i=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta=\phi$ if $\alpha \neq \beta$.

2. $u\in R^n,\ p\in R^n$ *and* $y\in R^m$.

Theorem 2.2.1 (Weak Duality): Let x be feasible for (NP) and (u, y, z, p, $w_1 \ldots w_m$) feasible for (MixSD). If for all feasible (x, u, y, z, $w_i \ldots, w_m$), $f(\cdot)+(\cdot)^T z+\sum_{i\in I_0} y_i(g_i(\cdot)+(\cdot)^T w_i)$ is second-order pseudoinvex and $\sum_{i\in I_\alpha} y_i(g_i(\cdot)+(\cdot)^T w_i)$, α = 1, 2, …, r is second-order quasi-invex with respect to the same η, then

$$\inf (\mathrm{NP}) \geq \sup (\mathrm{Mix\ SD}).$$

Proof: Since x is feasible for (NP) and (x, y, z, w, … , w_m) feasible for (Mix SD), we have, in view of $x^T w_i \leq S(x \mid D_i)$ where $w_i \in D_i$, $i = 1,2,\ldots,m$ and for $\alpha = 1, 2, \ldots , r$.

$$\sum_{i\in I_\alpha} y_i\left(g_i(x)+S(\mathrm{x} \mid \mathrm{Di})\right) \leq \sum_{i\in I_\alpha} y_i\left(g_i(x)+x^T w_i\right)$$

$$\leq 0 \leq \sum_{i\in I_\alpha} y_i\left(g_i(u)+u^T w_i\right) - \frac{1}{2} p^T \nabla^2\left(\sum_{i\in I_\alpha} y_i g_i(u)\right) p$$

By second-order quasi-invexity of $\sum_{i\in I_\alpha} y_i(g_i(\cdot)+(\cdot)^T w_i)$, $\alpha = 1, 2, \ldots , r,$ it follows that

$$\eta^T(x,u)\left(\nabla\left(\sum_{i\in I_\alpha} y_i\left(g_i(u)+u^T w_i\right)\right)+\nabla^2\left(\sum_{i\in I_\alpha} y_i g_i(u)\right) p\right) \leq 0, \ \alpha=1,2,\ldots,r$$

Hence

$$\eta^T(x,u)\left(\nabla\left(\sum_{i\in M-I_0} y_i\left(g_i(u)+u^T w_i\right)\right)+\nabla^2\left(\sum_{i\in M-I_0} y_i g_i(u)\right) p\right) \leq 0.$$

Thus from (2.35), this yields

$$\eta^T(x,u)\left(\nabla\left(f(u)+u^T z\right)+\sum_{i\in I_0} y_i \nabla\left(g_i(u)+u^T w_i\right)+\nabla^2\left(f(u)+\sum_{i\in I_0} y_i g_i(u)\right) p\right) \geq 0$$

Since $f(\cdot)+(\cdot)^T z+\sum_{i\in I_0} y_i\left(g_i(\cdot)+(\cdot)^T w_i\right)$ is second-order pseudoinvex, this implies,

$$f(x)+x^T z+\sum_{i\in I_0} y_i\left(g_i(x)+x^T w_i\right)\geq f(u)+u^T z+\sum_{i\in I_0} y_i\left(g_i(u)+u^T w_i\right)$$
$$-\frac{1}{2}p^T\nabla^2\left(f(u)+\sum_{i=I_0} y, g,(u)\right)p$$

Since $x^Tz \leq S(x|C)$, $x^Tw_i \leq S(x|D_i)$, $i \in I_0$ and $g_i(x) + S(x|D_i) \leq 0$, together with $y \leq 0$, for $i \in I_0$, the above inequality gives,

$$f(x)+S(x/C)\geq f(u)+u^T z+\sum_{i=I_0} y_i\left(g_i(u)+u^T w_i\right)$$
$$-\frac{1}{2}p^T\nabla^2\left(f(u)+\sum_{i=I_0} y_i g_i(u)\right)p$$

That is,

$$\text{Inf. (NP)} \geq \text{sup.(MixSD)}.$$

Theorem 2.2.2 (Strong Duality): If $\bar{x}$ is an optimal solution (NP) and Slater's constraint qualification [67] is satisfied at $\bar{x}$, then there exists $\bar{y}\in R^m$ with $\bar{y}=(\bar{y}_1,\bar{y}_2,...,\bar{y}_m)$, $\bar{z}\in C$ and $\bar{w}_i\in D_i$, $i = 1, 2, \ldots , m$ such that $(\bar{x},\bar{y},\bar{z},\bar{w}_1,\bar{w}_2,...,\bar{w}_m,p=0)$ is feasible for (MixSD) and the corresponding values of (NP) and (MixSD) are equal.

If also, $f(\cdot)+(\cdot)^T z+\sum_{i\in I_0} y_i\left(g_i(\cdot)+(\cdot)^T w_i\right)$ is second-order pseudo-invex for $z \in C$ and $w_i \in D_i$, $i \in I_0$ and $\sum_{i\in I_\alpha} y_i\left(g_i(\cdot)+(\cdot)^T w_i\right)$ for $w_i \in D_i$, $i \in I_\alpha$, $\alpha = 1, 2, \ldots, r$ is second-order quasi-invex with respect to the same η, then $(\bar{x},\bar{y},\bar{z},\bar{w}_1,\bar{w}_2,...,\bar{w}_m,p=0)$ is an optimal solution of (Mix SD).

***Proof*:** Since $\bar{x}$ is an optimal solution to the problem (NP) and the Slater's constraint qualification is satisfied at $\bar{x}$, then from Theorem 2.2.1, there exist $\bar{y}\in R^m$, $\bar{z}\in C$ and $\bar{w}_i\in D_i$, $i = 1, 2, \ldots m$ such that

$$\nabla\left(f(\bar{x})+\bar{x}^T\bar{z}\right)+\sum_{i\in 1} y_i\nabla\left(g_i(\bar{x})+\bar{x}_i{}^T\bar{w}_i\right)=0$$

$$\sum_{i\in 1} y_i\left(g_i(\bar{x})+\bar{x}_i{}^T w_i\right)=0$$

$$\bar{x}^T\bar{z}=S(\bar{x}/C)$$

$$\bar{x}_i{}^T\bar{w}_i=S(\bar{x}/D_i),\qquad i=1,2,\ldots,m$$

$$\bar{z}\in C,\ \bar{w}_i\in D_i,\qquad i=1,2,\ldots,m$$

$$\bar{y}\geq 0$$

The relation $\sum_{i\in I} y_i\left(g_i(\bar{x})+\bar{x}_i{}^T\bar{w}_i\right)=0$ implies $\sum_{i\in I_0}\bar{y}_i\left(g_i(\bar{x})+\bar{x}_i{}^T\bar{w}_i\right)=0$ and $\sum_{i\in I_\alpha}\bar{y}_i\left(g_i(\bar{x})+\bar{x}_i{}^T\bar{w}_i\right)=0$, $\alpha=1,2,\ldots,r$ Consequently, it implies that $(\bar{x},\bar{y},\bar{z},\bar{w}_1,\ldots,\bar{w}_m,\bar{p}=0)$ is feasible for (Mix SD) and the corresponding values of (NP) and (MixSD) are equal. If $f(\cdot)+(\cdot)^T z+\sum_{i\in I_0} y_i\left(g_i(\cdot)+(\cdot)^T w_i\right)$ is pseudoinvex, for all $z \in C$ and $w_i \in D_i$, $i = 1,2,\ldots,m$ and $\sum_{i\in I_\alpha} y_i\left(g_i(\cdot)+(\cdot)^T w_i\right)$ is second-order quasi-convex for $i \in I_\alpha$, $\alpha = 1,2,\ldots,r$, then from Theorem 2.2.1 $(\bar{x},\bar{y},\bar{z},\bar{w}_1,\ldots,\bar{w}_m,\bar{p}=0)$ must be an optimal solution of (MixSD).

We shall prove a Mangasarian type [68] strict converse duality theorem for (MixSD) to (NP).

Theorem 2.2.3 (Strict Converse Duality): Let $\bar{x}$ be an optimal solution of (NP) at which Slater's constraint qualification is satisfied. If $(\hat{x},\hat{y},\hat{p},\hat{z},\hat{w})$ is an optimal solution of (MixSD), where $\hat{w}=(\hat{w}_1,\ldots,\hat{w}_m)$ and $f(\cdot)+(\cdot)^T\hat{z}+\sum_{i\in I_0}\hat{y}_i\left(g_i(\cdot)+(\cdot)^T\hat{w}_i\right)$ is second-order strictly pseudoinvex at $\hat{x}$ and $i \in I_0$, $\sum_{i\in I_\alpha} \hat{y}_i \left(g_i(\cdot) + (\cdot)^T \hat{w}_i\right)$, α = 1, 2, …, r is second-order quasi-invex at $\hat{x}$ with respect to the same η, then $\bar{x} = \hat{x}$, i.e. $\hat{x}$ is an optimal solution of (NP).

Proof: We shall assume that $\hat{x} \neq \bar{x}$ and exhibit a contradiction. Since $\bar{x}$ is an optimal solution of (NP) at which Slater's qualification is satisfied, it follows from Theorem 2.2.1 that there exists $\bar{y} \in R^m$, $\hat{z} \in C$ and $\hat{w}_i \in D_i$, i = 1, 2,…,m such that $(\hat{x}, \hat{y}, \hat{z}, \hat{w}_1, ..., \hat{w}_m, \hat{p} = 0)$ is optimal for (MixSD). Hence

$$
\begin{aligned}
f(\bar{x}) + S(\bar{x}/C) &= f(\bar{x}) + \bar{x}^T \hat{z} + \sum_{i\in I_0} \bar{y}_i \left(g_i(\bar{x}) + \bar{x}^T \hat{w}_i\right) \\
&\quad - \frac{1}{2}\hat{p}\nabla^2\left(f(\hat{x}) + \sum_{i\in I_0} \bar{y}_i \left(g_i(\hat{x})\right)\right)\hat{p} \\
&= f(\hat{x}) + \hat{x}^T \hat{z} + \sum_{i\in I_0} \hat{y}_i \left(g_i(\hat{x}) + \hat{x}^T \hat{w}_i\right) \\
&\quad - \frac{1}{2}\hat{p}\nabla^2\left(f(\hat{x}) + \sum_{i\in I_\circ} \hat{y}_i g_i(\hat{x})\right)\hat{p} \qquad (2.39)
\end{aligned}
$$

Since $\bar{x}$ is feasible for (NP) and $(\hat{x}, \hat{y}, \hat{z}, \hat{w}_1, ..., \hat{w}_m, \hat{p})$ $\mathrm{i} \in \mathrm{I}_\alpha$ is feasible for (MixSD), we have,

$$
\sum_{i\in I_\alpha} \hat{y}_i \left(g_i(\hat{x}) + \hat{x}\hat{w}_i\right) \leq \sum_{i\in I_\alpha} \hat{y}_i \left(g_i(\hat{x}) + \hat{x}\hat{w}_i\right) - \frac{1}{2}\hat{p}\nabla^2\left(\sum_{i\in I_\alpha} \hat{y}_i g_i(\hat{x})\right)\hat{p}
$$

By second-order quasi-invexity of $\sum_{i\in I_\alpha} \hat{y}_i \left(g_i(\cdot) + (\cdot)^T \hat{w}_i\right)$, this yields,

$$
\eta^T(\bar{x}, \hat{x})\left[\sum_{i\in I_\alpha} \nabla\hat{y}_i \left(g_i(\hat{x}) + \hat{x}\hat{w}_i\right) + \nabla^2 \sum_{i\in I_\alpha} \hat{y}_i g_i(\hat{x})\hat{p}\right] \leq 0 \qquad (2.40)
$$

Because $(\hat{x}, \hat{y}, \hat{p}, \hat{z}, \hat{w})$ is feasible, we have ,

$$
\nabla\left(f(\hat{x}) + \hat{x}^T \hat{z}\right) + \sum_{i=1}^{m} \hat{y}_i \nabla\left(g_i(\hat{x}) + \hat{x}\hat{w}_i\right) + \nabla^2\left(\sum_{i=1}^{m} \hat{y}_i g_i(\hat{x})\right)\hat{p} = 0
$$

From this equation, it implies,

$$\sum_{i\in I_\alpha}\hat{y}_i\nabla\left(g_i(\hat{x})+\hat{x}\hat{w}_i\right)+\nabla^2\left(\sum_{i\in I_\alpha}\hat{y}_i\left(g_i(\hat{x})\right)\right)\hat{p}$$

$$=-\left[\nabla\left(f(\hat{x})+\hat{x}^T\hat{z}\right)+\sum_{i\in I_0}\hat{y}_i\nabla\left(g_i(\hat{x})+\hat{x}\hat{w}_i\right)+\nabla^2\left(\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\right)\hat{p}\right]=0$$

Using this in (2.40), we obtain,

$$\eta^T(\bar{x},\hat{x})\left[\nabla\left(f(\hat{x})+\hat{x}^T\hat{z}\right)+\sum_{i\in I_0}\hat{y}_i\nabla\left(g_i(\hat{x})+\hat{x}\hat{w}_i\right)+\nabla^2\left(\sum_{i\in I_\alpha}\hat{y}_i g_i(\hat{x})\right)\hat{p}\right]\geq 0$$

This, because of second-order strict pseudo-invexity of

$\sum_{i\in I_\alpha} y_i\left(g_i(\cdot)+(\cdot)\hat{w}_i\right)$ implies,

$$f(\bar{x})+\bar{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i\left(g_i(\bar{x})+\bar{x}^T\hat{w}_i\right)\geq f(\hat{x})+\hat{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i\left(g_i(\hat{x})+\hat{x}^T\hat{w}_i\right)$$

$$-\frac{1}{2}\hat{p}^T\nabla^2\left(f(\hat{x})+\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\right)\hat{p}$$

Since $\bar{x}^T\hat{z}=S\left(\bar{x}/C\right)$ and $\bar{x}^T\hat{w}_i=S\left(\bar{x}/D_i\right)$, i = 1,2,…,m, this implies,

$$f(\bar{x})+S\left(\bar{x}/C\right)+\sum_{i\in I_0}\hat{y}_i\left(g_i(\bar{x})+S\left(\bar{x}/D_i\right)\right)\geq$$

$$f(\hat{x})+\hat{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i\left(g_i(\hat{x})+\hat{x}^T\hat{w}_i\right)-\frac{1}{2}\hat{p}^T\nabla^2\left(f(\hat{x})+\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\right)\hat{p} \tag{2.41}$$

Since $\hat{y}_i\geq 0$ and $g_i(\bar{x})+S\left(\bar{x}/D_i\right)\leq 0$ for all $i\in\{1,2,\ldots,m\}$, hence $\hat{y}_i\left(g_i(\bar{x})+S\left(\bar{x}/D_i\right)\right)\leq 0$, $\forall\ i\in I_0$. Thus from the inequality (2.41), we have,

$$f(\bar{x})+S(\bar{x}/C)\geq f(\hat{x})+\hat{x}^T\hat{z}+\sum_{i\in I_0}\hat{y}_i\left(g_i(\hat{x})+\hat{x}^T\hat{w}_i\right)$$
$$-\frac{1}{2}\hat{p}^T\nabla^2\left(f(\hat{x})+\sum_{i\in I_0}\hat{y}_i g_i(\hat{x})\right)\hat{p}\,.$$

This ensues a contradiction to (2.39). Hence $\hat{x}=\bar{x}$, i.e., $\hat{x}$ is an optimal solution of (NP).

Theorem 2.2.4 (Converse Duality): Let $(\bar{x},\bar{y},\bar{w},\bar{p})$ be an optimal solution to (MixSD) at which

(A$_1$): for all $\alpha=1,2,...r,$ either

a) The $n\times n$ Hessian matrix $\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)$ is positive definite and $\bar{p}^T\nabla\sum_{i\in I_\alpha}\bar{y}_i\left(g_i(\bar{x})+\bar{x}^t\bar{w}_i\right)\geq 0$ or

b) $\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)$ is negative definite and $\bar{p}^T\nabla\sum_{i\in I_\alpha}\bar{y}_i\left(g_i(\bar{x})+\bar{x}^t\bar{w}_i\right)\leq 0$

(A$_2$): the set of vectors

$$\left\{\left[\nabla^2\left(f(\bar{x})-\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\right)\right]_j,\left[\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\right]_j\right\},\ j=1,2,...n,\ \alpha=1,2,...r,$$

are linearly independent.

where $\left[\nabla^2\left(f(\bar{x})-\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\right)\right]_j$ is jth row of the matrix $\left[\nabla^2\left(f(\bar{x})-\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\right]$ and $\left[\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\right]_j$ is jth row of the matrix $\left[\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\right]$.

(A$_3$): the vectors $\left\{\sum_{i\in I_\alpha}\bar{y}_i\left(\nabla g_i(\bar{x})+\bar{w}_i\right)\right\},\alpha=1,2,...r,$ are linearly independent.

If for all feasible $(x,z,y,u,w_1,w_2,\ldots w_m,p)$, $f(\cdot)+(\cdot)^T+\sum_{i\in I_0} y_i\left(g_i(\cdot)+(\cdot)^T w_i\right)$ is second-order pseudoinvex and $\sum_{i\in I_\alpha} y_i\left(g_i(\cdot)+(\cdot)^T w_i\right), \alpha=1,2,\ldots r,$ is second-order quasi-invex with respect to same η, then $\bar{x}$ is an optimal solution of the problem (NP).

***Proof*:** Since $(\bar{x},\bar{z},\bar{y},\bar{w},\bar{p})$, where $\bar{w}=(\bar{w}_1,\bar{w}_2,\ldots\bar{w}_m)$ is an optimal solution of (MixSD), by generalized Fritz John necessary optimality conditions [68], there exists, $\tau_0\in R,\ \theta\in R^n, \tau_\alpha\in R,\ \ \alpha=1,2,\ldots r,\ \beta\in R,$ and $\mu\in R^m$, such that

$$\tau_0\left\{-\left(\nabla f(\bar{x})+\bar{z}\right)-\sum_{i\in I_o}\bar{y}_i(\nabla g_i(\bar{x})+\bar{w}_i)+\frac{1}{2}\bar{p}\nabla\left[\nabla^2\left(f(\bar{x})+\sum_{i\in I_o}\bar{y}_i g_i(\bar{x})\right)\bar{p}\right]\right\}$$
$$+\theta\left\{\nabla^2\left(f(\bar{x})+\bar{y}^T g(\bar{x})\right)+\nabla\left(\nabla^2\left(f(\bar{x})+\bar{y}^T g(\bar{x})\right)\bar{p}\right)\right\}$$
$$+\sum_{\alpha=1}^{r}\tau_\alpha\left\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g_i(\bar{x})+\bar{w})-\frac{1}{2}\bar{p}^T\nabla\left[\left(\nabla^2\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\bar{p}\right]\right\}=0 \tag{2.42}$$

$$\tau_0\left\{g_i(\bar{x})+\bar{x}^T\bar{w}_i-\frac{1}{2}\bar{p}^T\nabla^2 g_i(\bar{x})\bar{p}\right\}$$
$$+\theta^T\left\{\nabla g_i(\bar{x})+\bar{w}_i+\nabla^2 g_i(\bar{x})\bar{p}\right\}+\mu_i=0\quad,\quad i\in I_0 \tag{2.43}$$

$$\tau_\alpha\left\{g_i(\bar{x})+\bar{x}^T\bar{w}_i-\frac{1}{2}\bar{p}^T\nabla^2 g_i(\bar{x})\bar{p}\right\}$$
$$+\theta^T\left\{\left(\nabla g_i(\bar{x})+\bar{w}_i+\nabla^2 g_i(\bar{x})\bar{p}\right)\right\}+\mu_i=0\quad,\ i\in I_0\ \ ,\alpha=1,2,\ldots,r \tag{2.44}$$

$$\left(\tau_0\bar{p}+\theta\right)^T\left\{\nabla^2\left(f(\bar{x})-\sum_{i\in I_0}\bar{y}_i g(\bar{x})\right)\right\}$$
$$+\sum_{\alpha=1}^{r}\left(\tau_\alpha\bar{p}+\theta\right)^T\left\{\nabla^2\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right\}=0 \tag{2.45}$$

$$\tau_\alpha\left\{\sum_{i\in I_\alpha}\bar{y}_i(g_i(\bar{x})+\bar{x}^T\bar{w}_i)-\frac{1}{2}\bar{p}\nabla^2\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\bar{p}\right\}=0\ ,\quad i\in I_0\ ,\alpha=1,2,....,r$$

(2.46)

$$\tau_0\bar{p}+\theta\in N_c(\bar{z}) \tag{2.47}$$

$$\left(\tau_0\bar{x}+\theta\right)y_i\in N_{D_i}(\bar{w})\ ,\ i\in I_0 \tag{2.48}$$

$$\left(\tau_0\bar{x}+\theta\right)y_i\in N_{D_i}(\bar{w})\ ,\ i\in I_\alpha,\alpha=1,2,...,r \tag{2.49}$$

$$\mu^T y=0 \tag{2.50}$$

$$\left(\tau_0,\tau_1,...\tau_r,\mu\right)\geq 0 \tag{2.51}$$

$$\left(\tau_0,\tau_1,...\tau_r,\theta,\mu\right)\neq 0 \tag{2.52}$$

The relation (2.45), in view of assumption (A_2) yields,

$$\tau_\alpha\bar{p}+\theta=0\,,\qquad \alpha=0,1,2,...r \tag{2.53}$$

Multiplying (2.44) by $\bar{y}_i, i\in I_\alpha, \alpha=1,2,...r,$ and summing with respect to $i\in I_\alpha, \alpha=1,2,...r,$ we get,

$$\tau_\alpha\left\{\sum_{i\in I_\alpha}\bar{y}_i(g_i(\bar{x})+\bar{x}^T\bar{w}_i)-\frac{1}{2}\bar{p}\nabla^2\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\bar{p}\right\}+$$

$$\theta^T\left\{\sum_{i\in I_\alpha}\bar{y}_i\left(\nabla g_i(\bar{x})+\bar{w}_i+\nabla^2\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\bar{p}\right)\right\}=0\quad ,\alpha=1,2,....,r$$

Using (2.46) ,we get,

$$\theta^T\left\{\sum_{i\in I_\alpha}\bar{y}_i\left(\nabla g_i(\bar{x})+\bar{w}_i+\nabla^2\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\bar{p}\right)\right\}=0\quad ,\alpha=1,2,....,r \tag{2.54}$$

By using the equality constraint of the dual in (2.42), we get,

$$\left(\tau_\alpha\bar{p}+\theta\right)^T\left\{\nabla^2\left(f(\bar{x})-\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\right)+\nabla\left[\nabla^2\left(f(\bar{x})+\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\right)\right]\bar{p}\right\}$$

$$+\sum_{\alpha=1}^{r}\left(\tau_\alpha\bar{p}+\theta\right)^T\left\{\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)+\nabla\left(\nabla^2\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\bar{p}\right\}$$

$$+\tau_0\left\{\nabla\sum_{i\in M-I_0}\bar{y}_i(g_i(\bar{x})+\bar{x}^T\bar{w}_i)+\nabla^2\sum_{i\in M-I_0}\bar{y}_i g_i(\bar{x})\bar{p}\right\}$$

$$-\frac{1}{2}\tau_0\bar{p}^T\left\{\nabla\left[\nabla^2\left(f(\bar{x})+\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\right)\right]\bar{p}\right\}$$

$$+\sum_{\alpha=1}^{r}\tau_\alpha\left\{\nabla\sum_{i\in I_\alpha}\bar{y}_i(g_i(\bar{x})+\bar{x}^T\bar{w}_i)+\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\bar{p}\right\}$$

$$+\sum_{\alpha=1}^{r}\frac{1}{2}\tau_\alpha\bar{p}^T\left\{\nabla\left[\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\right]\bar{p}\right\}=0$$

From (2.53), it implies,

$$\sum_{\alpha=1}^{r}(\tau_\alpha-\tau_0)\left\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g_i(\bar{x})+\bar{w}_i)+\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\bar{p}\right\}$$

$$+\frac{1}{2}\theta^T\left\{\nabla\left[\nabla^2\left(f(\bar{x})+\sum_{i\in I_0}\bar{y}_i g_i(\bar{x})\right)\right]\bar{p}+\nabla\left[\nabla^2\left(\sum_{i\in M-I_0}\bar{y}_i g_i(\bar{x})\right)\right]\bar{p}\right\}=0$$

This implies,

$$\sum_{\alpha=1}^{r}(\tau_\alpha-\tau_0)\left\{\sum_{i\in I_\alpha}\bar{y}_i(\nabla g_i(\bar{x})+\bar{w}_i)+\nabla^2\left(\sum_{i\in I_\alpha}\bar{y}_i g_i(\bar{x})\right)\bar{p}\right\}$$

$$+\frac{1}{2}\theta^T\left\{\nabla\left(\nabla^2\left(f(\bar{x})+\bar{y}^T g(\bar{x})\right)\bar{p}\right)\right\}=0 \qquad (2.55)$$

Assume that $\tau_\alpha=0$, for all $\alpha\in\{0,1,2,...r\}$. Then $\theta=0$ from (2.53), and from (2.44) $\mu=0$, Then $(\tau_0,\tau_1,...\tau_r,\theta,)=0$ which contradicts the Fritz John condition (2.52).Thus there exists an $\alpha\in\{0,1,2,...r\}$ such that $\tau_\alpha>0$.

The relation (2.53) can be rewritten as

$$\tau_0\bar{p}+\theta=0,\qquad \tau_\alpha\bar{p}+\theta=0,\qquad \alpha=1,2,...r$$

Which implies,

$$(\tau_0-\tau_\alpha)\bar{p}=0 \qquad (2.56)$$

We claim $\bar{p}=0$ Suppose that $\bar{p}\neq 0$, then (2.56) yields,

$$\tau_0 = \tau_\alpha, \alpha = 1,2,...r$$

Consequently we have,

$$\theta = -\tau_0 \bar{p}$$

Using this in (2.54), we obtain,

$$-\tau_0 \bar{p}\left\{\sum_{i\in I_\alpha} \bar{y}_i \left(\nabla g(\bar{x}) + \bar{w}_i\right) + \nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\bar{p}\right\} = 0$$

$$\Rightarrow \qquad \bar{p}\left\{\sum_{i\in I_\alpha} \bar{y}_i \left(\nabla g(\bar{x}) + \bar{w}_i\right) + \nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\bar{p}\right\} = 0 \tag{2.57}$$

From the assumption (A_1) i.e. for $\alpha = 1,2,...r$,

$$\bar{p}\sum_{i\in I_\alpha} \bar{y}_i \left(g(\bar{x}) + \bar{w}_i\right) \geq 0$$

$$\bar{p}\nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\bar{p} \geq 0 \ ,$$

$$\Rightarrow \qquad \bar{p}\sum_{i\in I_\alpha} \bar{y}_i \left(g(\bar{x}) + \bar{w}_i\right) + \bar{p}^T \nabla^2\left(\sum_{i\in I_\alpha} \bar{y}_i g_i(\bar{x})\right)\bar{p} \neq 0$$

This is contradicted by (2.56). Hence $\bar{p} = 0$.

Using $\bar{p} = 0$ in (2.55),we have,

$$\sum_{\alpha=1}^{r} \left(\tau_\alpha - \tau_0\right)\left\{\sum_{i\in I_\alpha} \bar{y}_i (\nabla g_i(\bar{x}) + \bar{w}_i)\right\} = 0$$

By (A_3), this implies,

$$\tau_0 = \tau_\alpha > 0, \qquad \alpha = 1,2,...r$$

Since $\theta = 0$, (2.43) and (2.44) implies,

$$\tau_0\left(g_i(\bar{x}) + \bar{x}_i^T \bar{w}_i\right) + \mu_i = 0 \ , \qquad i \in I_0$$

$$g_i(\bar{x}) + \bar{x}_i^T \bar{w} = -\frac{\mu_i}{\tau_0} \leq 0 \ , \quad i \in I_0 \, ,$$

$$\tau_\alpha\left(g_i(\bar{x}) + \bar{x}_i^T \bar{w}\right) + \mu_i = 0 \ , \qquad i \in I_\alpha \, , \ \ \alpha = 1,2,...,r$$

Comparing these, we have,

$$g_i(\bar{x}) + \bar{x}_i^T \bar{w} = -\frac{\mu_i}{\tau_\alpha} \le 0 \ , \quad i \in I_0,\ i \in I_\alpha\ , \ \ \alpha = 1, 2, ..., r \qquad (2.58)$$

From (2.48) and (2.49), we have,

$$\bar{x}^T \bar{w}_i = S\left(\bar{x} \middle| D_i\right) \ , \quad i \in I_0, i \in I_\alpha\ , \ \alpha = 0, 1, 2, ..., r$$

The relation (2.58) along with this implies,

$$g_i(\bar{x}) + S\left(\bar{x} \middle| D_i\right) \le 0, \qquad i = 1, 2, ..., m$$

This shows that $\bar{x}$ is feasible for (NP)

Multiplying (2.58) by $\bar{y}_i, i \in I_0$,and $\bar{y}_i, i \in I_\alpha\ \alpha = 1,2,...r,$ and adding and using $\mu^T y = 0$,

$$\sum_{i \in I_0} \bar{y}_i \left(g(\bar{x}) + \bar{w}_i \bar{x} \right) = 0 \qquad (2.59)$$

$$\sum_{i \in I_\alpha} \bar{y}_i \left(g(\bar{x}) + \bar{w}_i \bar{x} \right) = 0 \qquad (2.60)$$

$$\left(f(\bar{x}) + \bar{x}^T \bar{z} \right) - \sum_{i \in I_0} \bar{y}_i (g_i(\bar{x}) + \bar{w}_i^{\ T} \bar{x}) - \frac{1}{2} \bar{p}^T \left[\nabla^2 \left(f(\bar{x}) + \sum_{i \in I_o} \bar{y}_i g_i(\bar{x}) \right) \bar{p} \right]$$

$$= f(\bar{x}) + \bar{x}^T \bar{z} \qquad \text{(using } p = 0 \text{ and (2.59))}$$

$$= f(\bar{x}) + S(\bar{x}|C)\ , \qquad \text{by (2.47)}$$

If, for all feasible$(\bar{x}, \bar{z}, \bar{u}, \bar{w}_1 ... \bar{w}_m, \bar{p})$, $f(\cdot) + (\cdot)^T + \sum_{i \in I_0} y_i \left(g_i(\cdot) + (\cdot)^T w_i \right)$ is second-order pseudoinvex and $\sum_{i \in I_\alpha} y_i \left(g_i(\cdot) + (\cdot)^T w_i \right), \alpha = 1,2,...r,$ is second-order quasi-invex for $z \in C$ and $w_i \in D_i$ with respect to same η, by Theorem 2.2.1**,** then $\bar{x}$ is an optimal solution of the problem (NP).

2.2.2 Special Cases

If p = 0, the mixed type dual (MixSD) to the following to the following first order mixed type dual formulated in [51].

(Mix SD): Maximize $f(u)+u^T z+\sum_{i\in I_0} y_i\left(g_i(u)+u^T w_i\right)$

Subject to

$$\left(\nabla f(u)+u^T z\right)+\sum_{i=1}^{m} y_i\left(\nabla g_i(u)+u^T w_i\right) = 0$$

$$\sum_{i\in I_\alpha} y_i\left(g_i(u)+u^T w_i\right)\geq 0, \quad \alpha=1,2,\ldots,r.$$

$$y\geq 0$$

$$z\in C, w_i\in D_i, \ i=1, 2,\ldots,m.$$

where $I_\alpha \subseteq M = \{1,2,\ldots,m\}$, $\alpha = 0, 1,2,\ldots, r$ with $\bigcup_{i=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$ if $\alpha\neq\beta$.

As discussed in [31], we may write for positive semi definite matrix B, $S(x|C)=\left(x^T Bx\right)^{\frac{1}{2}}$ by taking $C=\left\{By \middle| y^T By\leq 1\right\}$. If the support function appearing in the constraints suppressed but the support function in the objective function of (NP) is retained and replaced by $\left(x^T Bx\right)^{\frac{1}{2}}$, then we have the following pair of problems treated by Zhang and Mond [101] and re-examined Zhang and Yang for correcting the converse duality theorem proved in [102].

(P): Minimize $f(x)+\left(x^T Bx\right)^{\frac{1}{2}}$

Subject to

$$g(x)\leq 0,$$

(SD): Maximize $f(u)-\sum_{i\in I_0} y_i g_i(u)+u^T z$

$$-\frac{1}{2}\nabla^2 p^T\left[f(u)+\sum_{i\in I_0} y_i g_i(u)\right]p$$

Subject to

$$\nabla f(u)-y^T g(u)+z+\nabla^2\left(f(u)+y^T g(u)\right)p=0$$

$$\sum_{i\in I_\alpha} y_i\left(g_i(u)+u^T w_i\right)-\frac{1}{2}p^T\nabla^2\left(\sum_{i=I_\alpha} y_i g_i(u)\right)p\ge 0,\ \alpha=1,2,\ldots,r,$$

$$w^T z\le 1$$

$$y\ge 0$$

where $I_\alpha \subseteq M = \{1,2,\ldots,m\}$, $\alpha = 0,\ 1,2,\ldots,\ r$ with $\bigcup_{i=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta = \phi$ if $\alpha \neq \beta$.

3.0 INTRODUCTORY REMARKS

Following Dorn [41], first order symmetric and self duality results in mathematical programming have been derived by a number of authors, notably, Dantzig *et al* [38] Mond [71], Bazaraa and Goode [8], Mond and Weir [82]. Later Weir and Mond [97] discussed symmetric duality in multiobjective programming by using the concept proper efficiency. Chandra and Prasad [24] presented a pair of multiobjective programming problem by associating a vector valued infinite game to this pair. Gulati, Husain and Ahmed [46] also established duality results for multiobjective symmetric dual problem without non-negativity constraints.

Mond [70] was the first to study Wolfe type second-order symmetric duality bonvexity – boncavity. Subsequently, Bector and Chandra [10] established second-order symmetric and self duality results for a pair of non-linear programs under pseudobonvexity – pseudoboncavity condition. Devi [40] formulated a pair of second-order symmetric dual programs and established corresponding duality results involving η-bonvex functions and Mishra [69] extended the results of [40] to multiobjective nonlinear programming. Recently, Suneja *et al* [92] presented a pair of Mond-Weir type multiobjective second-order symmetric and self dual program without nonnegativity constraint and proved various duality results under bonvexity and pseudobonvexity.

This chapter consists of two sections 3.1 and 3.2.In section 3.1 a pair of Wolfe type second-order multiobjective nonlinear programming problems containing support functions is formulated and usual duality results are proved under convexity-concavity assumption on functions involved in its formulation. Self duality for this pair is also investigated under the additional condition on the kernel function. In section 3.2 a pair of Mond-Weir type symmetric dual is formulated in order to relax convexity-concavity to pseudoconvexity-pseudoconcavity. Self duality for this pair is studied under additional condition.Special cases is also generated.

3.1 NONDIFFERENTIABLE MULTIOBJECTIVE SECOND-ORDER WOLFE TYPE SYMMETRIC DUAL PROGRAMS

3.1.1 Second-Order Multiobjective Symmetric Duality

In this section, we consider a pair of second-order Wolfe type non-differentiable multiobjective symmetric dual programs and validate weak, strong and converse duality theorems.

We have taken the auxiliary vectors *p* and *q* same throughout the formulations of two problems because it seems more natural than different p's and q's in [92].

Consider the following two programs:

Primal Program:

(SWP): Minimize $F(x,y,z,p)=F_i\left(x,y,z_1,p\right),\ldots F_k\left(x,y,z_k,p\right)$

Subject to

$$\sum_{i=1}^{k}\lambda_i\left(\nabla_2 f_i(x,y)-z_i+\nabla_2^2 f_i(x,y)p\right)\leqq 0 \tag{3.1}$$

$$z_i\in D_i, i=1,2,\ldots,k \tag{3.2}$$

$$x\geqq 0 \tag{3.3}$$

$$\lambda = \in \wedge^{+} \tag{3.4}$$

and

Dual Program:

(SWD): Minimize $\mathrm{G}\left(u,v,w,q\right)=G_1\left(u,v,w_1,q\right),\ldots G_k\left(u,v,w_k,q\right)$

Subject to

$$\sum_{i=1}^{k}\lambda_i\left(\nabla_1 f_i(u,v)-w_i+\nabla_1^2 f_i(u,v)q\right)\geqq 0 \tag{3.5}$$

$$w_i\in C_i, i=1,2,\ldots,k \tag{3.6}$$

$$v\geqq 0 \tag{3.7}$$

$$\lambda = \in \wedge^{+} \tag{3.8}$$

where

i. $$F_i(x,y,z_i,p)=f_i(x,y)+s(x\mid C_i)-y^Tz_i-\frac{1}{2}p^T\nabla_2^2 f_i(x,y)p$$
$$-y^T\sum_{i=1}^{k}\lambda_i\left(\nabla_2 f_i(x,y)-z_i+\nabla_2^2 f_i(x,y)p\right)$$

ii. $$G_i(u,v,w_iq)=f_i(u,v)-s(v\mid D_i)+u^Tw_i-\frac{1}{2}q^T\nabla_1^2 f_i(u,v)q$$
$$-u^T\sum_{i=1}^{k}\lambda_i\left(\nabla_1 f_i\left(u,v\right)+w_i+\nabla_1^2 f_i\left(u,v\right)q\right),$$ and

iii. for each i, $s\left(x\mid C_i\right)$ and $s(v\mid D_i)$ represent support functions of compact convex sets C_i and D_i in R^n and R^m, respectively.

iv. $w = (w_1, \ldots w_K)$ with $w_i \in C_i$ and $z = (z_1, \ldots z_K)$ for each {i = 1,2,…,k}

v. $\wedge^+ = \left\{ \lambda \in R^k \mid \lambda = (\lambda_i, \ldots \lambda_k), \lambda > 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$

Theorem 3.1.1 (Weak Duality): Let (x, y, λ, z, p) satisfies the constraints of (SWD) of (u, v, λ, w, q) satisfies the constraints of (SWD). If for each $i \in \{1, 2, \ldots, k\}$, $f_i(., y)$ is bonvex at x for fixed y and $f_i(x, .)$ be boncave at y for fixed x for feasible $(x, y, u, v, \lambda, p, q, z, w)$ then

$$F(x, y, z, p) \not\leq G(u, v, w, q).$$

***Proof*:** By bonvexity of $f_i(., y)$ for fixed y at u, we have.

$$f_i(x,v) - f_i(u,v) \geqq (x-u)^T \left[\nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v) q \right] - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q \tag{3.9}$$

and by boncavity of $f_i(x, .)$ for fixed x at v, we have,

$$f_i(x,v) - f_i(x,y) \leqq (v-y)^T \left[\nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y) p \right] - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p \tag{3.10}$$

Multiplying (3.10) by (– 1) and adding the resulting the inequality to (3.9), we obtain,

$$\left[f_i(x,y) - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p - y^T \left\{ \nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y) p \right\} \right]$$

$$- \left[f_i(u,v) - \frac{1}{2} q^T \nabla_2^2 f_i(u,v) q - u^T \left\{ \nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v) q \right\} \right]$$

$$\geq x^T \left\{ \nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v) q \right\} - v^T \left\{ \nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y) p \right\}.$$

or

$$\left[f_i(x,y) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p - y^T \left\{ \nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y) p \right\} \right]$$

$$- \left[f_i(u,v) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q - u^T \left\{ \nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v) q \right\} \right]$$

$$\geq x^T \left\{ \nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v) q \right\} - v^T \left\{ \nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y) p \right\}.$$

Multiplying this by $\lambda_i > 0, i \in \{1, 2 \ldots, k\}$ and summing and using $\sum_{i=1}^{k} \lambda_i = 1$ we have.

$$\sum_{i=1}^{k} \lambda_i \left[f_i(x,y) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p - y^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y) p \right\} \right]$$

$$- \sum_{i=1}^{k} \lambda_i \left[f_i(u,v) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q - u^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v) q \right\} \right]$$

$$\geq x^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v) q \right\} - v^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_2 f_i(x,y) + \nabla_2^2 f_i(x,y) p \right\}.$$

Using (3.1) with (3.7) and (3.5) with (3.3), this inequality becomes

$$\sum_{i=1}^{k} \lambda_i \left[f_i(x,y) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p - y^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y) p \right\} \right]$$

$$- \sum_{i=1}^{k} \lambda_i \left[f_i(u,v) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q - u^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v) q \right\} \right]$$

$$\geqq - \sum_{i=1}^{k} \lambda_i \left(x^T w_i \right) - \sum_{i=1}^{k} \lambda_i \left(v^T z_i \right)$$

Since $-s(x \mid C_i) \leqq -x^T w_i$ for $w_i \in C_i$ and $-s(v \mid D_i) \leqq -v^T z_i$, $i = 1, 2, \ldots, k$, therefore, this inequality reduces to

$$\sum_{i=1}^{k} \lambda_i \left[f_i(x,y) + s(x \mid C_i) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p - y^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_2 f_i(x,y) - z_i + \nabla_2^2 f_i(x,y) p \right\} \right]$$

$$\geqq \sum_{i=1}^{k} \lambda_i \left[f_i(u,v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q - u^T \sum_{i=1}^{k} \lambda_i \left\{ \nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v) q \right\} \right]$$

i.e., $$\sum_{i=1}^{k} \lambda_i F_i(x, y, z_i, p) \geqq \sum_{i=1}^{k} \lambda_i G_i(u, v, w_i, q)$$

or $\qquad \lambda^T F(x,y,z,p) \geqq \lambda^T G(u,v,w,q)$.

Thus,

$$F(x,y,z,p) \nleq G(u,v,w,q)$$

Theorem 3.1.2 (Strong Duality): Let for each $i \in \{1,2,\ldots k\}$, f_i be thrice differentiable on $R^n \times R^n$. Let $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ be properly efficient solution of (SWP); for $\lambda \leq \bar{\lambda}$ in (SWP) and assume that

(A_1): the set $\{\nabla_2^2 f_1(\bar{x},\bar{y}), \nabla_2^2 f_2(\bar{x},\bar{y}), \ldots \nabla_2^2 f_k(\bar{x},\bar{y}),\}$ is linearly independent.

(A_2): the set $\nabla_2(\nabla_2^2(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p})$ is positive or negative definite.

(A_3): the set $\{\nabla_2 f_1(\bar{x},\bar{y}) + \bar{w}_1 + \nabla_2^2 f_1(\bar{x},\bar{y})\bar{p}, \ldots, \nabla_2 f_k(\bar{x},\bar{y}) + \bar{w}_k + \nabla_2^2 f_k(\bar{x},\bar{y})\bar{p}\}$ is linearly independent.

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0)$ is feasible solution of (SWD) and F$(\bar{x},\bar{y},\bar{z},\bar{p}) = G(\bar{x},\bar{y},\bar{w},\bar{q})$

Moreover, if the hypotheses of Theorem 3.1.1 are satisfied for all feasible solution of (SWP) and (SWD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q})$ is properly efficient solution for (SWD).

***Proof*:** Since $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a properly efficient solution of (SWP), then it is also weak minimum. Hence there exist $\alpha \in R^n$ with $\alpha = (\alpha_1,\ldots\alpha_k)$, $\beta \in R^m$, $\eta \in R^k$ and $\mu \in R^k$ with $\mu = (\mu_1,\ldots\mu_k)$ and $\theta \in C_i$, $i = 1,2,\ldots,k$ such that the following Fritz John optimality conditions [68] are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$:

$$\sum_{i=1}^{k} \alpha_i \left(\nabla_1 f_i(\bar{x},\bar{y}) + \theta_i\right) + \sum_{i=1}^{k} \left(\beta - (\alpha^t e)\bar{y}\right)^T \bar{\lambda}_i \nabla_{21} f_i(\bar{x},\bar{y}) +$$

$$\sum_{\lambda=1}^{k} \left\{\left(\beta - (\alpha^T e)\bar{y}\right)^T \bar{\lambda}_i - \frac{\alpha_i \bar{p}}{2}\right\}^T \nabla_1\left(\nabla_2^2 f_i(\bar{x},\bar{y})\, p\right) = \eta \qquad (3.11)$$

$$\sum_{i=1}^{k}\left(\alpha_i-\left(\alpha^T e\right)\lambda_i\right)^T\left(\nabla_2 f_i(\bar{x},\bar{y})-\bar{z}_i\right)+\sum_{i=1}^{k}\left\{\left(\beta-\left(\alpha^T e\right)\left(\bar{y}+\bar{p}\right)^T\right)\lambda_i\nabla^2{}_2 f_i(\bar{x},\bar{y})\right\}$$

$$+\sum_{\lambda=1}^{k}\left\{\left(\beta-\left(\alpha^T e\right)\bar{y}\right)\bar{\lambda}_i-\frac{\alpha_i\bar{p}}{2}\right\}^T\nabla_2\left[\nabla_2^2 f_i\left(\left(\bar{x},\bar{y}\right)\bar{p}\right)=0\right] \tag{3.12}$$

$$\left\{\left(\beta-\left(\alpha^T e\right)\bar{y}\right)\bar{\lambda}_i-\alpha_i\bar{p}\right\}^T\nabla_2^2 f_i(\bar{x},\bar{y})=0 \tag{3.13}$$

$$\left(\beta-\left(\alpha^T e\right)\bar{y}\right)^T\left(\nabla_2 f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_2^2 f_i(\bar{x},\bar{y})\bar{p}\right)-\mu_i=0,\ \ i=1,2,\ldots,k \tag{3.14}$$

$$-\alpha_i\bar{y}+\left(\beta-\left(\alpha^T e\right)\bar{y}\right)^T\lambda_i\in N_{D_i}(\bar{z}_i),\ i=1,2,\ldots,k \tag{3.15}$$

$$\beta^T\sum_{i=1}^{k}\bar{\lambda}_i\left\{\nabla_2 f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_2^2 f_i(\bar{x},\bar{y})\bar{p}\right\}=0 \tag{3.16}$$

$$\eta^T\bar{x}=0 \tag{3.17}$$

$$\bar{\lambda}^T\mu=0 \tag{3.18}$$

$$(\alpha,\beta,\eta,\mu)\geqq 0 \tag{3.19}$$

$$(\alpha,\beta,\eta,\mu)\neq 0 \tag{3.20}$$

Since $\lambda>0$ and $\mu\geqq 0$, (3.18) implies, $\mu=0$.

In view of the assumption (A_1), (3.13) yields,

$$\left(\beta-\left(\alpha^T e\right)\bar{y}\right)\lambda_i=\alpha_i\bar{p},\quad i=1,2,\ldots,k \tag{3.21}$$

Using (3.21) in (3.12), we have,

$$\sum_{i=1}^{k}\left(\alpha_i-\left(\alpha^T e\right)\bar{\lambda}_i\right)\left\{\left(\nabla_2 f_i(\bar{x},\bar{y})-\bar{z}_i\right)+\nabla_2^2 f_i\left(\left(\bar{x},\bar{y}\right)\bar{p}\right)\right\}$$

$$+\frac{1}{2}\left(\beta-\left(\alpha^T e\right)\bar{y}\right)^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_2\left(\nabla_2^2 f(\bar{x},\bar{y})\bar{p}\right)=0 \tag{3.22}$$

Post multiplying (3.22) by $\left(\beta-\left(\alpha^T e\right)\bar{y}\right)$ and the using (3.14) with $\mu_i=0$, we obtain,

$$\left(\beta-\left(\alpha^T e\right)\bar{y}\right)^T\sum_{i=1}^{k}\bar{\lambda}_i\nabla_2\left(\nabla_2^2\bar{\lambda}_i f(\bar{x},\bar{y})\bar{p}\right)\left(\beta-\left(\alpha^T e\right)\bar{y}\right)=0$$

Which because of the condition (A_2) implies,

$$\left(\beta-\left(\alpha^T e\right)\bar{y}\right)=0 \tag{3.23}$$

Using (3.23) in (3.22), we have,

$$\sum_{i=1}^{k}\left(\alpha_i-\left(\alpha^T e\right)\bar{\lambda}_i\right)\left(\nabla_2 f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_2^2 f_i(\bar{x},\bar{y})\bar{p}\right)=0$$

This, in view of (A$_3$), gives,

$$\alpha_i-\left(\alpha^T e\right)\bar{\lambda}_i=0 \;\; i=1,2,\ldots,k. \tag{3.24}$$

If $\alpha_i=0$, $i=1,2,\ldots k,$ then from (3.23) and (3.11) imply $\beta=0$ and $\eta=0$, respectively. Consequently, we get (α,β,μ,η) = 0, contradicting (3.20).

Hence $\alpha_i>0$. Then from (3.21) together with (3.23), we have,

$$\bar{p}=0 \tag{3.25}$$

Using (3.23) and (3.25) in (3.11), we have,

$$\sum_{i=1}^{k}\alpha_i\left(\nabla_1 f_i(\bar{x},\bar{y})+\theta_i\right)=\eta$$

Which by (3.24) implies,

$$\left(\alpha^T e\right)\sum_{i=1}^{k}\bar{\lambda}_i\left(\nabla_1 f_i(\bar{x},\bar{y})+\theta_i\right)=\eta$$

This with (3.17) and (3.19) respectively gives,

$$\sum_{i=1}^{k}\bar{\lambda}_i\left(\nabla_1 f_i(\bar{x},\bar{y})+\theta_i\right)=0$$

Which, because of (3.19) and (3.17) along respectively yields,

$$\sum_{i=1}^{k}\bar{\lambda}_i\left(\nabla_1 f_i(\bar{x},\bar{y})+\theta_i\right)\geqq 0 \tag{3.26}$$

and

$$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i\left(\nabla_1 f_i(\bar{x},\bar{y})+\theta_i\right)=0 \tag{3.27}$$

From (3.23), we have,

$$\bar{y} \geqq 0 \tag{3.28}$$

From (3.16), (3.27) and (3.28), we obtain $\left(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q}=0\right)=$ $\left(\bar{x}, \bar{y}, \bar{\lambda}, \theta, \bar{q}=0\right)$

Where $\theta=\left(\theta_i, \ldots, \theta_k\right)$ is feasible for (SWD). From (3.16) together with (3.23)

$$\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i\left(\nabla_2 f_i(\bar{x}, \bar{y})-\bar{z}_i+\nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p}\right)=0 \tag{3.29}$$

From (3.15) along with (3.23) and $\alpha_i>0$, it implies for each $i \in\{1,2, \ldots, k\}$

$$\bar{y} \in N_{D_i}\left(\bar{z}_i\right) \quad \text{giving} \quad \bar{y}^T \bar{z}_i \leqq s\left(y \mid D_i\right) \tag{3.30}$$

From (3.16), (3.27), (3.29) and (3.30) along with $\bar{p}=\bar{w}=\bar{q}$, it implies, for each $i \in\{1,2, \ldots k\}$,

$$\begin{aligned}
& f_i(\bar{x}, \bar{y})+s(\bar{x} \mid C_i)-\bar{y}^T \bar{z}_i-\frac{1}{2} p^T \nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p} \\
& \qquad -\bar{y}^T \sum_{i=1}^{k} \lambda_i\left(\nabla_2 f_i(\bar{x}, \bar{y})-\bar{z}_i+\nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p}\right) \\
& = f_i(\bar{x}, \bar{y})-s(\bar{y} \mid D_i)+x^T \bar{w}_i-\frac{1}{2} q^T \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q} \\
& \qquad -\bar{x}^T \sum_{i=1}^{k} \lambda_i\left(\nabla_2 f_i(\bar{x}, \bar{y})-\bar{w}_i-\nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q}\right)
\end{aligned}$$

for each $i \in\{1,2, \ldots, k\}$,

$$F_i\left(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}\right)=G_i\left(\bar{x}, \bar{y}, \bar{w}_i, \bar{q}\right) \tag{3.31}$$

This implies,

$$F\left(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}\right)=G\left(\bar{x}, \bar{y}, \bar{w}_i, \bar{q}\right)$$

That is, the objective values of (SWP) and (SWD) are equal.

Now, we shall show the proper efficiency of $(\bar{x},\bar{y},\bar{w},\bar{\lambda},\bar{q})$ for (SWD) by exhibiting a contradiction. If $(\bar{x},\bar{y},\bar{z},\bar{q})$ is not efficient for (SWD) such that.

$$G(\bar{x},\bar{y},\bar{z},\bar{q}) \leq G_1(\bar{u},\bar{v},\bar{w},\bar{q}),$$

Which because of (3.31) yields,

$$G_i(\bar{u},\bar{v},\bar{w},\bar{q}) \geqq F_i(\bar{x},\bar{y},\bar{z},\bar{q})$$

This contradicts Theorem 3.1.1

If $(\bar{x},\bar{y},\bar{z},\bar{p})$ were improperly efficient solution of (SWD)$_{\bar{\lambda}}$, then for some feasible $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{q}) \in Z$ and some i

$$G_i(\bar{u},\bar{v},\bar{w}_i,\bar{q}) > G_i(\bar{x},\bar{y},\bar{z}_i,\bar{q})$$

and

$$G_i(\bar{u},\bar{v},\bar{w}_i,\bar{q}) - G_i(\bar{x},\bar{y},\bar{z}_i,\bar{q}) > M\left(G_j(\bar{u},\bar{v},\bar{w}_j,\bar{q}) - G_j(\bar{x},\bar{y},\bar{z}_i,\bar{q})\right)$$

For any M > 0 and all j satisfying.

$$G_j(\bar{x},\bar{y},\bar{z}_j,\bar{q}) > G_j(\bar{u},\bar{v},\bar{w}_j,\bar{q}).$$

This means $G_i(\bar{u},\bar{v},\bar{w}_i,\bar{q}) - G_i(\bar{x},\bar{y},\bar{z}_i,\bar{q})$ is finite for all $j \neq i$. Since $\bar{\lambda}_i > o$, for all $i \in \{1,2,\ldots,k\}$

$$\bar{\lambda}_i G_i(\bar{u},\bar{v},\bar{w}_i,\bar{q}) - \bar{\lambda}_i G_i(\bar{x},\bar{y},\bar{z}_i,\bar{q})$$
$$> \sum_{j \neq i = 1}^{k} \bar{\lambda}_j G_j(\bar{u},\bar{v},\bar{w}_j,\bar{q}) - \sum_{j \neq i = 1}^{k} \bar{\lambda}_j G_j(\bar{x},\bar{y},\bar{z}_i,\bar{q})$$

i.e.,

$$\sum_{i=1}^{k} \bar{\lambda}_i G_i(\bar{u},\bar{v},\bar{w}_i,\bar{q}) > \sum_{i=1}^{k} \bar{\lambda}_i G_i(\bar{x},\bar{y},\bar{z}_i,\bar{q})$$

This along with (3.31) implies,

$$\sum_{i=1}^{k}\bar{\lambda}_i G_i(\bar{u},\bar{v},\bar{w}_i,\bar{q}) > \sum_{i=1}^{k}\bar{\lambda}_i F_i(\bar{x},\bar{y},\bar{z}_i,\bar{p})$$

i.e.,

$$\bar{\lambda}^T G(\bar{u},\bar{v},\bar{w},\bar{q}) > \bar{\lambda}^T F(\bar{x},\bar{y},\bar{z}_i,\bar{p})$$

This again leads to a contradiction to Theorem 3.1.1.Hence the theorem is fully validated.

Theorem 3.1.3 (Converse Duality): Let for each $i \in \{1,2,\ldots,k\}$, f_i be thrice differentiable on $R^n \times R^n$. Let $(\bar{x},\bar{y},\bar{z},\bar{w},\bar{q})$ be a proper efficient solution of (SWD); fix $\lambda = \bar{\lambda}$ in (SWP) and assume that

(C_1): the set $\{\nabla_1^2 f_i(\bar{x},\bar{y}),\ldots,\nabla_1^2 f_k(\bar{x},\bar{y})\}$ is linearly independent.

(C_2): the matrix $\nabla_1(\nabla_1^2(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{q})$ is positive or negative definite, and

(C_3): the set $\{\nabla_1 f_1(\bar{x},\bar{y}) + \bar{w}_i + \nabla_1^2 f_1(\bar{x},\bar{y})\bar{q},\ldots,\nabla_1 f_k(\bar{x},\bar{y}) + \bar{w}_k + \nabla_1^2 f_k(\bar{x},\bar{y})\bar{q}\}$ is linearly independent.

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p}=0)$ is feasible solution of (SWP) and

$$F(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p}) = G(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{q}).$$

Moreover, if the hypotheses of theorem are satisfied for all feasible of (SWP) and (SWD).Then $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a properly efficient solution of (SWP).

***Proof*:** It follows exactly on the lines of Theorem 3.1.2.

3.1.2 Second-Order Multiobjective Self Duality

A mathematical program is said to be self dual, if it is formally identical with its dual, that is, if the dual is recast in the form of the primal. The new program so retained is the same as the primal. In general the programs (SWP) and (SWD) are not self dual without an added restriction on $f_i(x,y)$ with $x \in R^n$ and $y \in R^n$ for $i \in \{1,2,\ldots,k\}$.

We describe (SWP) and (SWD) as the dual programs if the conclusions of Theorem 3.1.2 holds.

Theorem 3.1.4 (Self Duality): If the kernel $f_i(x,y)$ with $f_i : R^n \times R^n \to R$ for $i = 1,2,\ldots,k$ is skew symmetric and $C_i = D_i$ for all $i \in \{1,2,\ldots,k\}$., then (SWP) is self dual and $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a joint properly efficient solution then so is $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ and

$$F(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p}) = G(\bar{x},\bar{y},\bar{w},\bar{q})$$

***Proof*:** Rewriting the dual program in primal form, we have

(SWP-1): Minimize $-G(x,y,w,q) = (-G_i(x,y,w,q),\ldots -G_k(x,y,w_k,q))$

Subject to

$$-\sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(\bar{x},\bar{y}) + w_i + \nabla_1^2 f_i(\bar{x},\bar{y})\bar{q}\right) \leqq 0$$

$$y \geq 0$$

$$\lambda \in \wedge^+$$

$$w_i \in C_i, \quad i = 1,2,\ldots k$$

Where

$$-G(\bar{x},\bar{y},\bar{w},\bar{q}) = -f_i(\bar{x},\bar{y}) + \bar{x}^T w_i + s(\bar{y} \mid D_i) + \frac{1}{2}\bar{q}^T \nabla_1 f_i(\bar{x},\bar{y})\bar{q}$$

$$+ x^T \sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(\bar{x},\bar{y}) + w_i + \nabla_1^2 f_i(\bar{x},\bar{y})\bar{q}\right)$$

Since each f_i is a skew symmetric, $\nabla_1 f_i(x,y) = -\nabla_2 f_i(y,x), \nabla_1^2 f_i(x,y)$ $= -\nabla_2^2 f_i(y,x)$ for all $i \in \{1,2,\ldots,k\}$, and $k \in R^n$ and $y \in R^n$. Hence the dual program (SWD – 1) can be written as

(SWD-1): Minimize $G(y,x,w,q) = (G_i(y,x,w,q),\ldots G_k(y,x,w,q))$

Subject to

$$\sum_{i=1}^{k} \lambda_i \left(\nabla_2 f_i(y,x) + z_i + \nabla_2^2 f_i(y,x) q \right) \leqq 0$$

$$y \geqq 0$$

$$z_i \in C_i$$

$$\lambda \in \wedge^{+}$$

Where

$$G_i(y,x,w,q) = f_i(y,x) + s(y \mid C_i) + y^T z_i - \frac{1}{2} q^T \nabla_2^2 f_i(y,x) q$$
$$- x^T \sum_{i=1}^{k} \lambda_i \left(\nabla_2 f_i(y,x) + z_i + \nabla_2^2 f_i(y,x) q \right)$$

This show that the program (SWP – 1) is just the primal program (SWP).

Thus $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q})$ optimal for (SWP) implies $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{w}, \bar{q})$ optimal for (SWD).By an analogous argument, $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$optimal for (SWP) implies $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p})$ optimal for (SWD).

If (SWP) and (SWD) are dual program and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is jointly optimal,

Then

$$0 = \bar{x}^T \sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(\bar{x}, \bar{y}) + \bar{w}_i + \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q} \right)$$

$$= \bar{y}^T \sum_{i=1}^{k} \lambda_i \left(\nabla_2 f_i(\bar{x}, \bar{y}) + \bar{z}_i + \nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p} \right)$$

and $\bar{p} = \bar{q} = 0.$

The objective values of the programs (SWP) and (SWD) at$(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$,

$$F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = G_i(\bar{x}, \bar{y}, \bar{w}, \bar{q}) = f_i(\bar{x}, \bar{y}), i = 1,2,\ldots,k. \qquad (3.32)$$

Since $\left(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p}\right)$ is also a joint optimal solution, one can similarly show that

$$0 = \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_1^T \left(\nabla_1 f_i\left(\bar{y}, \bar{x}\right) + \bar{z}_i + \nabla_1^2 f_i\left(\bar{y}, \bar{x}\right)\bar{p}\right)$$

$$= \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i \left(\nabla_2 f_i\left(\bar{y}, \bar{x}\right) + \bar{w}_i + \nabla_2^2 f_i\left(\bar{y}, \bar{x}\right)q\right)$$

and $\quad \bar{p} = \bar{q} = 0.$

The objective value of (SWP) and (SWD) at $\left(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p}\right)$ becomes

$$F_i\left(\bar{y}, \bar{x}, \bar{z}_i, \bar{p}\right) = G_i\left(\bar{y}, \bar{x}, \bar{w}_i, \bar{q}\right) = f_i\left(\bar{y}, \bar{x}\right), i = 1,2,\ldots,k. \tag{3.33}$$

From (3.32) and (3.33), it implies for each $i \in \{1,2,\ldots,k\}$,

$$F_i\left(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}\right) = G_i\left(\bar{y}, \bar{x}, \bar{z}_i, \bar{p}\right) = f_i\left(\bar{x}, \bar{y}\right) = f_i\left(\bar{y}, \bar{x}\right) = -f_i\left(\bar{x}, \bar{y}\right)$$

Therefore,

$$F_i\left(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}\right) = f_i\left(\bar{x}, \bar{y}\right) = 0, i = 1,2,\ldots,k.$$

This implies,

$$F_i\left(\bar{x}, \bar{y}, \bar{z}, \bar{p}\right) = 0$$

3.2 NONDIFFERENTIABLE MULTIOBJECTIVE SECOND-ORDER MOND-WEIR TYPE SYMMETRIC DUAL PROGRAMS

3.2.1 Second-Order Multiobjective Symmetric Duality

Consider the following pair of nondifferentiable second-order symmetric dual programs:

(SVP): Minimize $F(x, y, z, p) = (F_1(x, y, z, p), \ldots, F_k(x, y, z_k, p))$

Subject to

$$\sum_{i=1}^{k} \lambda_i \left(\nabla_2 f_i(x, y) - z_i + \nabla_2^2 f_i(x, y)p\right) \leqq 0, \tag{3.34}$$

$$y^T\sum_{i=1}^{k}\lambda_i\left(\nabla_2 f_i(x,y)-z_i+\nabla_2^2 f_i(x,y)p\right)\geqq 0, \tag{3.35}$$

$$\lambda>0, \tag{3.36}$$

$$x\geq 0, z_i\in D_i,\ i=1,2,...,k \tag{3.37}$$

and

(SVD): Maximize $G(u,v,w,q)=(G_1(u,v,w_1,q),...,G_k(u,v,w_k,q))$

Subject to

$$\sum_{i=1}^{k}\lambda_i\left(\nabla_1 f_i(u,v)+w_i+\nabla_1^2 f_i(u,v)q\right)\geqq 0, \tag{3.38}$$

$$u^T\sum_{i=1}^{k}\lambda_i\left(\nabla_1 f_i(u,v)+w_i+\nabla_1^2 f_i(u,v)q\right)\leqq 0, \tag{3.39}$$

$$\lambda>0, \tag{3.40}$$

$$v\geq 0, w_i\in C_i,\ i=1,2,...,k. \tag{3.41}$$

where

(i) $F_i(x,y,z_i,p)=f_i(x,y)+s(x\,|\,C_i)-y^Tz_i-\dfrac{1}{2}p^T\nabla_2^2 f_i(x,y)p$

$G_i(u,v,w_i,q)=f_i(u,v)-s(v\,|\,D_i)+u^Tw_i-\dfrac{1}{2}q^T\nabla_1^2 f_i(u,v)q$

(ii) $w=(w_1,...,w_k)$ with $w_i\in C_i$ for $i\in\{1,2,...,k\}$, $z=(z_1,...,z_k)$ with $z_i\in D_i$ for $i\in\{1,2,...,k\}$, and $\lambda=(\lambda_1,...,\lambda_k)^T$ with $\lambda_i\in R$ for $i\in\{1,2,...,k\}$; and

(iii) for each $i\in\{1,2,...,k\}$, $s(x\,|\,C_i)$ and $s(y\,|\,D_i)$ represent support functions of compact convex set C_i in R^n and compact convex set D_i in R^m, respectively.

It is to be remarked here that unlike the formulation of the Mond-Weir type second-order symmetric dual programs in [92], here we have chosen for each $i\in\{1,2,...,k\}$, $p_i=p\in R^m$ and $q_i=q\in R^n$ as this choice seems to be in conformity with the analysis for identification of second-order dual in nonlinear programming by Mangasarian [66].

Theorem 3.2.1 (Weak Duality): For feasible solutions (x, y, λ, z, p) and (u, v, λ, w, q) for the programs (SVP) and (SVD), let $\sum_{i=1}^{k} \lambda_i (f_i(\cdot, y) + (\cdot)^T w_i)$, for each $w_i \in C_i$, $i \in \{1,2,...,k\}$ be pseudobonvex at u for fixed y and $\sum_{i=1}^{k} \lambda_i (f_i(x, \cdot) + (\cdot)^T z_i)$, for each $z_i \in D_i$, $i \in \{1,2,...,k\}$ be pseudoboncave at y. Then

$$F(x, y, \lambda, z, p) \not\leq G(u, v, \lambda, w, q) .$$

***Proof*:** By multiplying (3.38) by x^T and subtracting (3.39), we have

$$(x-u)^T \sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u,v) + w_i + \nabla_1^2 f_i(u,v) q \right) \geqq 0 .$$

This, because of pseudobonvexity of $\sum_{i=1}^{k} \lambda_i \left(f_i(\cdot, y) + (\cdot)^T w_i \right)$, implies

$$\sum_{i=1}^{k} \lambda_i \left(f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i + \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q \right) \geqq 0 . \tag{3.42}$$

From (3.34), (3.35) and $v \geq 0$, we have,

$$(v-y)^T \sum_{i=1}^{k} \lambda_i \left(\nabla_2 f(x,y) - z_i + \nabla_2^2 f(x,y) p \right) \leqq 0 .$$

By pseudoboncavity of $\sum_{i=1}^{k} \lambda_i \left(f_i(x, \cdot) + (\cdot)^T z_i \right)$, from this we get,

$$\sum_{i=1}^{k} \lambda_i \left(- f_i(x,v) + v^T z_i + f_i(x,y) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p \right) \geqq 0 . \tag{3.43}$$

On adding (3.42) and (3.43), we have

$$\sum_{i=1}^{k} \lambda_i \left(f_i(x,y) + x^T w_i - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p \right)$$
$$- \sum_{i=1}^{k} \lambda_i \left(f_i(u,v) + u^T w_i - v^T z_i - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q \right) \geqq 0 .$$

Since for each $w_i \in C_i$, $x^T w_i \leq s(x|C_i)$ and each $z_i \in D_i$, $v^T z_i \leq s(v|D_i)$, the above inequality gives,

$$\sum_{i=1}^{k} \lambda_i \left(f_i(x,y) + s(x|C_i) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x,y) p \right)$$

$$\geqq \sum_{i=1}^{k} \lambda_i \left(f_i(u,v) - s(v|D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q \right)$$

or

$$\sum_{i=1}^{k} \lambda_i F_i(x,y,z_i,p) \geqq \sum_{i=1}^{k} \lambda_i G_i(u,v,w_i,q)$$

That is,

$$F(x,y,z,p) \geqq G(u,v,w,q)$$

This implies

$$F(x,y,z,p) \nleq G(u,v,w,q).$$

Theorem 3.2.2 (Strong Duality): Let $f_i, (i=1,2,...,k)$ be thrice differentiable on $R^n \times R^m$. Let $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ be a properly efficient solution of (SVP); fix $\lambda = \bar{\lambda}$ in (SVD) and assume that

(H1): The set $\{\nabla_2^2 f_1, ..., \nabla_2^2 f_k\}$ is linearly independent,

(H2): $\nabla_2(\nabla_2^2(\lambda^T f)\bar{p})$ is positive or negative definite, and,

(H3): The set $\{\nabla_2 f_1 - \bar{z} + \nabla_2^2 f_1 \bar{p}, ..., \nabla_2 f_k - \bar{z}_k + \nabla_2^2 f_k \bar{p}\}$ is linearly independent.

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0)$ is feasible for (SVD) and $F(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p}) = G(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q})$.

Moreover, if the hypotheses of Theorem 3.2.1 are satisfied for all feasible solutions of (SVP) and (SVD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q})$ is a properly efficient solution of (SVD).

***Proof*:** Since $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \bar{p})$ is a properly efficient solution of (SVP), it is weak minimum of (SVP). Hence there exists $\alpha \in R^n$, $\beta \in R^m$, $\mu \in R^k$, $\eta \in R^k$, $\gamma \in R^k$ and $\theta_i \in R^n$, $(i = 1, 2, ..., k)$ such that the following Fritz John optimality condition [68] are satisfied at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \bar{p})$, (suppressing the arguments):

$$\sum_{i=1}^{k} \alpha_i (\nabla_1 f_i + \theta_i) + \sum_{i=1}^{k} \bar{\lambda}_i (\beta - \gamma \bar{y})^T \nabla_1 \nabla_2^2 f_i$$
$$+ \sum_{i=1}^{k} \left\{ (\beta - \gamma \bar{y}) \bar{\lambda}_i - \frac{\alpha_i \bar{p}}{2} \right\}^T \nabla_1 (\nabla_1^2 f_i \bar{p}) = \eta \tag{3.44}$$

$$\sum_{i=1}^{k} (\alpha_i - \gamma \bar{\lambda}_i)(\nabla_2 f_i - \bar{z}_i) + \sum_{i=1}^{k} (\beta - \gamma \bar{y} - \gamma \bar{p}) \bar{\lambda} \nabla_2^2 f_i$$
$$+ \sum_{i=1}^{k} \left\{ (\beta - \gamma \bar{y}) \bar{\lambda}_i - \frac{\alpha_i \bar{p}}{2} \right\}^T \nabla_2 (\nabla_2^2 f_i \bar{p}) = 0 \tag{3.45}$$

$$\sum_{i=1}^{k} \{ (\beta - \gamma \bar{y}) \bar{\lambda}_i - \alpha_i \bar{p} \}^T \nabla_2^2 f_i = 0 \tag{3.46}$$

$$(\beta - \gamma \bar{y})^T \{ \nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p} \} - \mu_i = 0 \tag{3.47}$$

$$\alpha_1 \bar{y} + (\beta - \gamma \bar{y}) \lambda_1 \in N_{D_1}(\bar{z}_i), \ \ i = 1, 2, ..., k, \tag{3.48}$$

$$\theta_i \in C_i, \ \theta_i^T x = s(\bar{x} \mid C_i), \ \ i = 1, 2, ..., k, \tag{3.49}$$

$$\beta^T \sum_{k=1}^{k} \bar{\lambda}_1 (\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0, \tag{3.50}$$

$$\gamma \bar{y}^T \sum_{k=1}^{k} \bar{\lambda}_1 (\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0, \tag{3.51}$$

$$\mu^T \bar{\lambda} = 0, \tag{3.52}$$

$$\eta^T \bar{x} = 0, \tag{3.53}$$

$$(\alpha, \beta, \gamma, \mu, \eta) \geqq 0, \tag{3.54}$$

$$(\alpha, \beta, \gamma, \mu, \eta) \neq 0 \tag{3.55}$$

Since $\bar{\lambda} > 0$, from (3.52), it follows that $\mu = 0$. Consequently, from (3.47), we obtain,

$$(\beta - \gamma\bar{y})^T (\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0 \tag{3.56}$$

In view of (H_1), (3.46) yields,

$$(\beta - \gamma\bar{y})\bar{\lambda}_i = \alpha_i \bar{p},\ i = 1, 2, ..., k. \tag{3.57}$$

Using (3.57) in (3.45), we have,

$$\sum_{i=1}^{k} (\alpha_i - \gamma\bar{\lambda}_i)\left\{(\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p})\right\} + \frac{1}{2}\sum_{i=1}^{k} \bar{\lambda}_i \nabla_2 (\nabla_2^2 f_i \bar{p})(\beta - \gamma\bar{y}) = 0 \tag{3.58}$$

Pre-multiplying (3.58) by $(\beta - \gamma\bar{y})^T$ and then using (3.56), we get,

$$(\beta - \gamma\bar{y})^T \nabla_2 (\nabla_2 (\bar{\lambda}^T f)\bar{p}(\beta - \gamma\bar{y}) = 0 .$$

In view of (H_3), this yields,

$$\beta - \gamma\bar{y} = 0 . \tag{3.59}$$

Using (3.59) in (3.58), we obtain,

$$\sum_{i=1}^{k} (\alpha_1 - \gamma\bar{\lambda}_i)(\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0$$

This, because of (H_3), implies,

$$\alpha_i - \gamma\bar{\lambda}_i = 0,\ i = 1, 2, ..., k . \tag{3.60}$$

If $\gamma = 0$, from (3.44), (3.59) and (3.60), we have $\eta = 0$, $\beta = 0$ and $\alpha = 0$ respectively. Hence $(\alpha, \beta, \gamma, \mu, \eta) = 0$, contradicting (3.55). Thus $\gamma > 0$ and from (3.60), it implies $\alpha_i > 0$, $(i = 1, 2, ..., k)$. From (3.57) along with (3.59), we have $\bar{p} = 0$. Consequently from (3.44) together with (3.59) and (3.54), we obtain,

$$\sum_{i=1}^{k} \alpha_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = \eta$$

By (3.60), it implies,

$$\gamma \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 \; f_i \left(\bar{x}, \bar{y}\right) + \theta_i) \;\; = \eta$$

Which from (3.53) and (3.54) along implies,

$$\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 \; f \left(\bar{x}, \bar{y}\right)_i + \theta_i) \geqq 0 \tag{3.61}$$

and

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 \; f_i \left(\bar{x}, \bar{y}\right) + \theta_i) = 0 \tag{3.62}$$

From (3.49) and (3.59) respectively we have,

$$w_i \in C_i, \; i = 1, 2, ..., k, \; y \geq 0 \tag{3.63}$$

From (3.62) and (3.63), it from that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q} = 0) = (\bar{x}, \bar{y}, \bar{\lambda}, \bar{\theta}, \bar{q} = 0)$ where $\theta = (\theta_1, ..., \theta_k)$ is feasible for (SVD).

From (3.48) along with (3.59) and $\alpha_i > 0$, it implies $\bar{y} \in N_{D_i}(\bar{z}_i)$, $i \in \{1, 2, ..., k\}$;

and this gives,

$$\bar{y}^T \bar{z}_i \leq s(\bar{y} \mid D_i), \; i \in \{1, 2, ..., k\} \tag{3.64}$$

Now, using (3.50), (3.62) and (3.64) along with $\bar{p} = \bar{w} = \bar{q}$, we have

$$f_i(\bar{x}, \bar{y}) + s(\bar{x} \mid C_i) - \bar{y}^T \bar{z}_i - \frac{1}{2} \bar{p}^T \nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p}$$
$$= f_i(\bar{x}, \bar{y}) + s(y \mid D_i) - \bar{x}^T \bar{w}_i - \frac{1}{2} \bar{q}^T \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q}$$

for $\quad i \in \{1, 2, ..., k\}$

or

$$F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = G_i(\bar{x}, \bar{y}, \bar{w}_i, \bar{q}) \text{ for each } i \in \{1, 2, ..., k\}$$

This implies

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{w}, \bar{q}) \text{ for each } i \in \{1, 2, ..., k\} \tag{3.65}$$

We claim that $(\bar{x},\bar{y},\bar{w},\bar{q})$ is efficient for (SVD). If this would not be the case, then there would exist a feasible solution $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{q})$ of (SVD) such that

$$G(\bar{x},\bar{y},\bar{w},\bar{q}) \le G(\bar{u},\bar{v},\bar{w},\bar{q}),$$

Which by (3.65) gives

$$F(\bar{x},\bar{y},\bar{z},\bar{p}) \le G(\bar{u},\bar{v},\bar{w},\bar{q})$$

This is a contradiction to Theorem 3.2.1.

If $(\bar{x},\bar{y},\bar{w},\bar{q})$ were improperly efficient for (SVD), then for some feasible $(u,v,\bar{\lambda},w,q)$ of (SVD) and some i

$$\left(f_i(u,v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(\bar{x},\bar{y})\bar{q} \right) \\ - \left(f_i(\bar{x},\bar{y}) - s(\bar{y} \mid D_i) + \bar{x}^T \bar{w}_i - \frac{1}{2} q^T \nabla_1^2 f_i(\bar{x},\bar{y})\bar{q} \right) > M,$$

for any $M > 0$. Using (3.65), we have,

$$\left[f_i(u,v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u,v) q \right] \\ - \left[f_i(\bar{x},\bar{y}) + s(\bar{x} \mid C_i) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(\bar{x},\bar{y})\bar{p} \right] > M.$$

i.e.

$$G_i(u,v,w_i,q) - F_i(\bar{x},\bar{y},\bar{z}_i,\bar{p}) > M$$

and for any $\lambda > 0$, this yields,

$$\sum_{i=1}^{k} \bar{\lambda}_i G_i(u,v,w_i,q) > \sum_{i=1}^{k} \bar{\lambda}_i F_i(\bar{x},\bar{y},\bar{z}_i,\bar{p})$$

i.e.,

$$\bar{\lambda}^T G(u,v,w_i,q) > \bar{\lambda}^T F(\bar{x},\bar{y},\bar{z}_i,\bar{p}).$$

This again contradicts Theorem 3.2.1.

Theorem 3.2.3 (Converse Duality): Let f_i for $i \in \{1,2,...,k\}$ be thrice differentiable on $R^n \times R^n$. Let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q})$ be properly efficient of (SVD); fix $\lambda = \bar{\lambda}$ in (SVP) and assume that

(C1): the set $\{\nabla_1^2 f_1,...,\nabla_1^2 f_k\}$ is linearly independent

(C2): the set $\{\nabla_1^2 f_1 + \bar{w}_i + \nabla_1^2 f_1 \bar{q},...,\nabla_1^2 f_k + \bar{w}_k + \nabla_1^2 f_k \bar{q}\}$ is linearly independent, and

(C3): $\nabla_1(\nabla_1^2(\lambda^T f)\bar{q})$ is positive or negative definite.

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p}=0)$ is feasible of (SVP), and $F(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p}) = G(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q})$.

Moreover, if the hypotheses of Theorem 3.2.1 are satisfied for all feasible solution of (SVP) and (SVD), then $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a properly efficient of (SVP).

***Proof*:** It follows on the lines of Theorem 3.2.2.

3.2.2 Second-Order Multiobjective Self Duality

In this section, we now prove the following self duality theorem for the primal (SVP) and the dual (SVD). We describe (SVP) and (SVD) as the dual programs if the conclusions of Theorem 3.2.2 hold.

Theorem 3.2.4 (Self Duality): Let for $i \in \{1,2,...,k\}$, f_i be skew symmetric and $C_i = D_i$. Then (SVP) is self dual. If also (SVP) and (SVD) are dual programs, and $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a joint optimal solution, then so is $(\bar{y},\bar{x},\bar{\lambda},\bar{z},\bar{p})$ and

$$F(\bar{x},\bar{y},\bar{z},\bar{p}) = 0.$$

***Proof*:** Recasting the dual (SVD) as a minimization program, we have

Minimize $\left(-f_1(x,y)+s(y\,|\,D_1)-x^Tw_i+\frac{1}{2}q^T\nabla_1^2 f_1(x,y),\ldots\right.$

$$\left.-f_k(x,y)+s(y\,|\,D_k)-x^Tw_k+\frac{1}{2}q^T\nabla_1^2 f_k(x,y)q\right)$$

Subject to

$$-\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(x,y)+w_i-\nabla_1^2 f_i(x,y)q)\le 0$$

$$-x^T\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(x,y)+w_i-\nabla_1^2 f_i(x,y)q)\ge 0$$

$$\lambda>0, w_i\in C_i,\ i=1,2,\ldots,k,$$

$$y\geqq 0.$$

Since f_i is skew symmetric, therefore, for each $i\in\{1,2,\ldots,k\}$, $f_i(x,y)=-f_i(y,x)$, $\nabla_1 f_i(x,y)=-\nabla_2 f_i(y,x)$ and $\nabla_1^2 f_i(x,y)=-\nabla_2^2 f_i(y,x)$.

Therefore, the above program become,

Minimize $\left(f_1(y,x)-s(y\,|\,D_1)-x^Tw_i-\frac{1}{2}q^T\nabla_1^2 f_1(y,x),\ldots\right.$

$$\left.+f_k(y,x)-s(y\,|\,D_k)-x^Tw_k-\frac{1}{2}q^T\nabla_1^2 f_k(y,x)q\right)$$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(y,x)+w_i+\nabla_2^2 f_i(y,x)q)\leqq 0$$

$$x^T\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(y,x)+w_i+\nabla_2^2 f_i(y,x)q)\geqq 0$$

$$\lambda>0, w_i\in D_i,\ i=1,2,\ldots,k,$$

$$y\geqq 0.$$

This is just (SVP).Thus $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q})$ optimal for (SVP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{z},\bar{q})$ optimal for (SVD).By a similar argument, $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ optimal for (SVP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{z},\bar{p})$ optimal for (SVD).

If (SVP) and (SVD) are dual programs and $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is jointly optimal, then by Theorem 3.2.2, we have for each $i\in\{1,2,\ldots,k\}$,

$$s(\bar{x} \mid C_i) - \bar{y}^T \bar{z}_i = -s(\bar{y} \mid D_i) + \bar{x}^T \bar{w}_i \quad \text{and} \quad \bar{p} = \bar{q} = 0. \qquad (3.66)$$

For joint optimal solution $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$, we have for each $i \in \{1, 2, ..., k\}$

$$F_i(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_i, \bar{p}) = f_i(\bar{x}, \bar{y}) + s(\bar{x} \mid C_i) - \bar{y}^T \bar{z}_i - \frac{1}{2} \bar{p}^T \nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p}$$

$$= f_i(\bar{x}, \bar{y}) - s(\bar{y} \mid D_i) + \bar{x}^T \bar{w}_i - \frac{1}{2} \bar{q}^T \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q}$$

$$= G_i(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}_i, \bar{q}).$$

This, in view of (3.66) yields,

$$F_i(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_i, \bar{p}) = G_i(\bar{x}, \bar{y}, \bar{\lambda}, w_i, \bar{q}) = f_i(\bar{x}, \bar{y}) \text{ for } i \in \{1, 2, ..., k\}. \qquad (3.67)$$

Since $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}_i, \bar{p})$ is also a joint optimal solution, one can show, in a similar manner, that

$$F_i(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_i, \bar{p}) = f_i(\bar{y}, \bar{x}) = G_i(\bar{y}, \bar{x}, \bar{\lambda}, \bar{w}_i, \bar{q}) \text{ for } i \in \{1, 2, ..., k\}. \qquad (3.68)$$

From (3.67) and (3.68), we have,

$$F_i(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_i, \bar{p}) = f_i(\bar{x}, \bar{y}) = f_i(y, x) = -f_i(x, y) \quad \text{for} \quad i \in \{1, 2, ..., k\}.$$

Therefore, for each $i \in \{1, 2, ..., k\}$.

$$F_i(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}_i, \bar{p}) = 0 \text{ for each } i \in \{1, 2, ..., k\}.$$

That is,

$$F(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p}) = 0.$$

3.2.3 Special Cases

If we choose $C_i = \{0\}$ and $D_i = \{0\}$ for each $i \in \{1, 2, ..., k\}$ and p_i corresponding to each f_i instead of having $p = p_i$, for each $i \in \{1, 2, ..., k\}$ in the primal (SVP) and q_i corresponding to each f_i in the dual (SVD) instead of having $q = q_i$ for each $i \in \{1, 2, ..., k\}$, then these programs reduce to the following programs without non-negativity constraints, studied by Suneja *et al* [92]:

Primal (SVP): Minimize $F(x,y,p)=(F_1(x,y,p),...,F_k(x,y,p_k))$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(x,y)+\nabla_2^2 f_i(x,y)p_i)\leqq 0$$

$$y^T\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(x,y)+\nabla_2^2 f_i(x,y)p_i)\geqq 0$$

$$\lambda>0$$

and

Dual (SVD): Maximize $G(u,v,q)=(G_1(u,v,q_1),...,G_k(u,v,q_k))$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(u,v)+\nabla_1^2 f_i(u,v)q_i)\geqq 0$$

$$u^T\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(u,v)+\nabla_1^2 f_i(u,v)q_i)\leqq 0$$

$$\lambda>0$$

where for each $i\in\{1,2,...,k\}$

$$F_i(x,y,p_i)=f_i(x,y)-\frac{1}{2}p_i^T\nabla_2 f_i(x,y)p_i,$$

$$G_i(u,v,q_i)=f_i(u,v)-\frac{1}{2}q_i^T\nabla_1 f_i(u,v)q_i,$$

where $p=(p_1,...,p_k)$, $p_i\in R^m$ and $q=(q_1,...,q_k)$ with $q_i\in R^n$, $\lambda=(\lambda_1,...,\lambda_k)^T$ with $\lambda_i\in R$.

If only $p=q=0$, then our programs reduce to the following pair of first order Mond-Weir type symmetric dual programs.

Primal (VP): Minimize $F(x,y,z)=(F_1(x,y,z_1),...,F_k(x,y,z_k))$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_2^2 f_i(x,y)-z_i) \leqq 0$$

$$y^T\sum_{i=1}^{k}\lambda_i(\nabla_2^2 f_i(x,y)-z_i) \geqq 0$$

$$x \geqq 0, \quad \lambda > 0$$

$$z_i \in D_i, \quad i=1,2,...,k \text{ ,}$$

and

Dual (VD): Maximize $G(u,v,w)=(G_1(u,v,w_1),...,G_k(u,v,w_k))$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_1^2 f_i(u,v)+w_i) \geqq 0$$

$$u^T\sum_{i=1}^{k}\lambda_i(\nabla_1^2 f_i(u,v)+w_i) \leqq 0$$

$$y \geqq 0, \quad \lambda > 0$$

$$w_i \in C_i, \quad i=1,2,...,k$$

where

$$F_i(x,y,z_i)= f_i(x,y)+s(x\,|\,C_i)-y^z z_i$$

and

$$G_i(u,v,w_i)= f_i(u,v)-s(v\,|\,C_i)+u^T w_i \text{ .}$$

For these programs, the duality and self duality results easily follow.

4.1 INTRODUCTORY REMARKS

Mond [70] initiated second-order symmetric duality of Wolfe type in nonlinear programming and also indicated possible computational advantages of second-order dual over the first order dual. Later, Bector and Chandra [10] presented a pair of Mond-Weir type second-order dual programs and proved weak, strong and self duality theorems under pseudobonvexity – pseudoboncarity. Devi [40] constructed a pair of second-order symmetric dual programs over cones and studied duality for the same; but this formulation of second-order symmetric dual programs seems quite strange and apparently different from the traditional Wolfe type second-order symmetric dual programs of Mond [70] as well as Mond-Weir type second-order symmetric dual programs formulated by Bector and Chandra [10].

In [5] Balas presented a pair of Wolfe type first order minimax mixed integer symmetric dual programs as a generalization of the results of Dantzig *et al.* [38], while Kumar [63] and Husian and Chandra [21] dealt with Mond-Weir type first order maximin mixed integer symmetric dual programs. Later, Gulati and Ahmed [47] formulated second-order maximin mixed integer symmetric dual programs and proved various duality theorems including self duality theorem.

In this chapter, we formulate Wolfe type second-order dual programs with cone constraints and prove weak, strong, converse and self duality theorems under bonvexity – boncavity condition. Further, we generalize these Wolfe type dual programs to maximin second-order dual programs by constraining some of the components of the two variables of the programs belong to arbitrary sets of integers of these programs also, symmetric as well as self duality is incorporated. Particular cases are generated from our results.

4.2 Pre-requisites

For the results in this chapter, we shall require the Fritz John type necessary optimality conditions derived by Bazaraa and Goode [8] and which are embodied in the following proposition.

***Proposition* 4.1:** Let X be a convex set with nonempty interior in R^n and C be a closed convex cone in R^m. Let F be real valued function and G be a vector valued function, both defined on X.

Consider the problem:

(P_0): Minimize $F(z)$

Subject to

$$G(z) \in C \text{ and } z \in X$$

If z solves the problem (P_0), then there exist $\alpha_0 \in R$ and $\delta \in C^*$ such that

$$[\alpha_0 \nabla F(z_0) + \nabla \delta^T G(z_0)]^T (z - z_0) \geq 0 \text{ for all } z \in X,$$

$$\delta^T G(z_0) = 0,$$

$$(\alpha_0, \delta) \geq 0,$$

$$(\alpha_0, \delta) \neq 0.$$

The following concept of separability (Balas [5]) is also needed in the subsequent analysis of this research.

***Definition* 4.3:** Let $s^1, s^2, ..., s^p$ be elements of an elementary vector space. A real valued function $H_0(s^1, s^2, ..., s^p)$ will be called separable with respect to s^1 if there exist real-valued function $H_1(s^1)$ (independent of $s^2, ..., s^p$) and $H_2(s^2, ..., s^p)$ (independent of s^1), such that

$$H_0(s^1, s^2, ..., s^p) = H_1(s^1) + H_2(s^2, ..., s^p).$$

4.3 Formulation of the Problems

In this section, we formulate a pair of second-order symmetric dual nonlinear programs with cone constraints and establish appropriate duality theorems.

Consider the following two programs:

Primal Problem

(SP): Minimize $G(x, y, p) = f(x, y) - y^T(\nabla_y f(x, y) + \nabla_y^2 f(x, y) p)$

$$- \frac{1}{2} p^T \nabla_y^2 f(x, y) p$$

Subject to

$$-\nabla_y f(x, y) - \nabla_y^2 f(x, y) p \in C_2^* \tag{4.1}$$

$$(x, y) \in C_1 \times C_2 \tag{4.2}$$

and

Dual Problem

(SD): Maximize $H(x,y,q) = f(x,y) - x^T\left(\nabla_x f(x,y) + \nabla_x^2 f(x,y)q\right)$

$$-\frac{1}{2}q^T\nabla_x^2 f(x,y)q$$

Subject to

$$\nabla_x f(x,y) + \nabla_x^2 f(x,y)q \in C_1^* \tag{4.3}$$

$$(x,y) \in C_1 \times C_2 \tag{4.4}$$

where

(i) $f : C_1 \times C_2 \to R$ is a twice differentiable function,

(ii) C_1 and C_2 are closed convex cones with nonempty interior in R^n and R^m, respectively;

(iii) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively.

Theorem 4.1 (Weak Duality): Let (x,y,p) and (u,v,q) be feasible solutions of (SP) and (SD) respectively. Assume that $f(\cdot,y)$ is bonvex with respect to x for fixed y and $f(x,\cdot)$ is boncave with respect to y for fixed x for all feasible (x,y,p,u,v,q).

Then

inf.(SP) ≥ sup.(SD).

Proof: By bonvexity of $f(\cdot,y)$, we have,

$$f(x,v) - f(u,v) \geq (x-u)^T[\nabla_x f(u,v) + \nabla_x^2 f(u,v)q] - \frac{1}{2}q^T\nabla_x^2 f(u,v)q \tag{4.5}$$

and by boncavity of $f(x,\cdot)$, we have,

$$f(x,v) - f(x,y) \leq (v-y)^T[\nabla_y f(x,y) + \nabla_y^2 f(x,y)p] - \frac{1}{2}p^T\nabla_y^2 f(x,y)p \tag{4.6}$$

Multiplying (4.6) by (–1) and adding the resulting inequality to (4.5), we obtain,

$$\left[f(x,v) - y^T \left(\nabla_y f(x,y) + \nabla_y^2 f(x,y)p\right) - \frac{1}{2} p^T \nabla_y^2 f(x,y)p \right]$$

$$- \left[f(u,v) - u^T \left(\nabla_x f(u,v) + \nabla_x^2 f(u,v)q\right) - \frac{1}{2} q^T \nabla_x^2 f(u,v)q \right]$$

$$\geq x^T \left[\nabla_x f(u,v) + \nabla_x^2 f(u,v)q\right] - v^T \left[\nabla_y f(x,y) + \nabla_y^2 f(x,y)p\right]. \tag{4.7}$$

Now since $x \in C_1$ and $\nabla_x f(u,v) + \nabla_x^2 f(u,v)q \in C_1^*$, we have,

$$x^T \left[\nabla_x f(u,v) + \nabla_x^2 f(u,v)q\right] \geq 0. \tag{4.8}$$

and since $v \in C_2$ and $-[\nabla_y f(x,y) + \nabla_y^2 f(x,y)] \in C_2^*$, we have,

$$-v^T [\nabla_y f(x,y) + \nabla_y^2 f(x,y)p] \geq 0. \tag{4.9}$$

The inequality (4.7) together with (4.8) and (4.9), yields,

$$f(x,y) - y^T [\nabla_y f(x,y) + \nabla_y^2 f(x,y)p] - \frac{1}{2} p^T \nabla_y^2 f(x,y)p$$

$$\geq f(u,v) - u^T [\nabla_x f(u,v) + \nabla_x^2 f(u,v)q] - \frac{1}{2} q^T \nabla_x^2 f(u,v)q$$

This implies,

inf.(SP) $\geq$ sup.(SD).

Theorem 4.2 (Strong Duality): Let $(\bar{x}, \bar{y}, \bar{p})$ be an optimal solution of (SP).Also let

(A_1): the matrix $\nabla_y^2 f(\bar{x}, \bar{y})$ is non singular, and

(A_2): $\nabla_y(\nabla_y^2 f(\bar{x}, \bar{y})\bar{p})$ be negative definite.

Then $(\bar{x}, \bar{y}, \bar{q} = 0)$ is feasible for (SD) and the objective values of the programs (SP) and (SD) are equal. Moreover, if the requirements of Theorem 4.1 are fulfilled, then $(\bar{x}, \bar{y}, \bar{q})$ is an optimal solution of (SD).

***Proof*:** We use Proposition 4.1 to prove this theorem. Here $z=(x,y,p)$, $\bar{z}=(\bar{x},\bar{y},\bar{p})$, $x\in C_1$, $p\in R^m$ and $y\in C_2$

$$F(\bar{z})=f(\bar{x},\bar{y})-\bar{y}^T\left(\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\right)-\frac{1}{2}\bar{p}^T\nabla_y^2 f(\bar{x},\bar{y})\bar{p}$$

$$G(\bar{z})=-\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p} \text{ and } C=C_2^*$$

Since $(\bar{x},\bar{y},\bar{p})$ is an optimal solution of (SP), by Proposition 4.1, there exist $\alpha\in R$ and $\beta\in C_2^*$ such that

$$\left[\alpha\nabla_x f(\bar{x},\bar{y})-(\alpha\bar{y}+\beta)\nabla_x\nabla_y f(\bar{x},\bar{y})-\left(\alpha\bar{y}+\frac{\alpha\bar{p}}{2}+\beta\right)\nabla_x\nabla_y^2 f(\bar{x},\bar{y})\,\bar{p}\right](x-\bar{x})$$
$$-\left[(\alpha\bar{y}+\alpha\bar{p}+\beta)\nabla_y^2 f(\bar{x},\bar{y})+\left(\alpha\bar{y}+\frac{\alpha\bar{p}}{2}+\beta\right)\nabla_x\nabla_y^2 f(\bar{x},\bar{y})\,\bar{p}\right](y-\bar{y})\geq 0 \tag{4.10}$$

$$(\alpha y+\alpha p+\beta)\nabla_y^2 f(\bar{x},\bar{y})=0 \tag{4.11}$$

$$\beta^T\left[\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\right]=0, \tag{4.12}$$

$$(\alpha,\beta)\geq 0, \tag{4.13}$$

$$(\alpha,\beta)\neq 0. \tag{4.14}$$

The relation (4.11), in view of the hypothesis (A_1), gives,

$$\beta=-\alpha(\bar{y}+\bar{p}). \tag{4.15}$$

It follows that $\alpha\neq 0$, for if $\alpha=0$, (4.15) implies $\beta=0$. Hence $(\alpha,\beta)=0$ contradicts (4.14). Thus $\alpha>0$.

Now putting $\bar{x}=x$ and using (4.15) in (4.10), we obtain,

$$\left(\frac{\alpha\bar{p}}{2}\right)^T\left[\nabla_y\left(\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\right)\right](y-\bar{y})\geq 0,\text{ for all } y\in C_2.$$

Putting $y=\bar{p}+\bar{y}$ and using $\alpha>0$, from the above inequality

$$p^T[\nabla_y(\nabla_y^2 f(\bar{x},\bar{y})\bar{p})]\bar{p}\geq 0$$

Which, because of (A_2), yields,

$$\bar{p}=0 \tag{4.16}$$

Using (4.15) and (4.16) along with $\alpha>0$ in (4.10),we have,

$$\nabla_x f(\bar{x},\bar{y})(x-\bar{x})\geq 0, \text{ for all } x\in C_{1.} \tag{4.17}$$

Since C_1 is closed convex cone, therefore, for each $x\in C_1$ and $\bar{x}\in C_1$, it implies $x+\bar{x}\in C_1$. Now, replacing x by $x+\bar{x}$ in (4.17), we have,

$$x^T(\nabla_x f(\bar{x},\bar{y})+\nabla_x^2 f(\bar{x},\bar{y})\cdot 0)\geq 0 \tag{4.18}$$

This implies,

$$\nabla_x f(\bar{x},\bar{y})+\nabla_x^2 f(\bar{x},\bar{y})\cdot 0\in C_1^*.$$

Thus $(\bar{x},\bar{y},\bar{q}=0)$ is feasible for (SD).

Putting $x=0$ in (4.17) and $x=\bar{x}$ in (4.18), we have respectively,

$$\bar{x}^T(\nabla_x f(\bar{x},\bar{y})+\nabla_x^2(\bar{x},\bar{y})\cdot 0)\leq 0$$

and

$$\bar{x}^T(\nabla_x f(\bar{x},\bar{y})+\nabla_x^2(\bar{x},\bar{y})\cdot 0)\geq 0.$$

These together implies,

$$\bar{x}^T(\nabla_x f(\bar{x},\bar{y})+\nabla_x^2(\bar{x},\bar{y})\cdot 0)=0 \tag{4.19}$$

Using $\beta=\alpha\bar{y}$ and $\bar{p}=0$ along with $\alpha>0$ in (4.12), we have,

$$\bar{y}^T(\nabla_y f(\bar{x},\bar{y})+\nabla_y^2(\bar{x},\bar{y})\cdot 0)=0 \tag{4.20}$$

Consequently, we obviously have,

$$G(\bar{x},\bar{y},\bar{p}) = f(\bar{x},\bar{y}) - \bar{y}^T(\nabla_y f(\bar{x},\bar{y}) + \nabla_y^2(\bar{x},\bar{y})\bar{p}) - \frac{1}{2}\bar{p}^T\nabla_y^2 f(\bar{x},\bar{y})\bar{p}$$

$$= f(\bar{x},\bar{y}) - \bar{x}^T(\nabla_x f(\bar{x},\bar{y}) + \nabla_x^2(\bar{x},\bar{y})\bar{q}) - \frac{1}{2}\bar{q}^T\nabla_x^2 f(\bar{x},\bar{y})\bar{q}$$

$$= H(\bar{x},\bar{y},\bar{q}).$$

That is, the objective values of (SP) and (SD) are equal. By Theorem 4.1, the optimality of $(\bar{x},\bar{y},\bar{z})$ for (SD) follows.

We will only state a converse duality theorem (Theorem 4.3) as the proof of this theorem would follow analogously to that of Theorem 4.2.

Theorem 4.3 (Converse Duality): Let $(\bar{x},\bar{y},\bar{q})$ be an optimal solution of (SD). Also let

(C_1): the matrix $\nabla_x^2 f(\bar{x},\bar{y})$ is nonsingular, and

(C_2): $\nabla_x(\nabla_x^2 f(\bar{x},\bar{y})\bar{q})$ be a positive definite.

Then $(\bar{x},\bar{y},\bar{p}=0)$ is feasible for (SP) and the objective values of (SP) and (SD) are equal. Furthermore, if the hypothesis of Theorem 4.1 are met, then $(\bar{x},\bar{y},\bar{p})$ is an optimal solution of (SP).

Theorem 4.4 (Self Duality): Let $f: R^n \times R^m \to R$ be skew symmetric and $C_1 = C_2$, then (SP) is self dual. Furthermore, if (SP) and (SD) are dual programs and $(\bar{x},\bar{y},\bar{s})$ is an optimal solution for (SP), then $(\bar{x},\bar{y},\bar{p}=0)$ and $(\bar{y},\bar{x},\bar{q}=0)$ are optimal solutions for (SP) and (SD), and

$$G(\bar{x},\bar{y},\bar{p}) = 0 = H(\bar{x},\bar{y},\bar{q}).$$

Proof: Recasting the problem (SD) as a minimization problem, we have

(SD)$_1$: Minimize $-\left\{f(x,y)-x^T\left(\nabla_x f(x,y)+\nabla_x^2 f(x,y)q\right)-\frac{1}{2}q^T\nabla_x^2 f(x,y)q\right\}$

Subject to

$$\nabla_x f(x,y)+\nabla_x^2 f(x,y)q\in C_1^*$$

$$(x,y)\in C_1\times C_2 .$$

Since f is skew symmetric,

$$\nabla_x f(x,y)=-\nabla_y f(y,x) \text{ and } \nabla_x^2 f(x,y)=-\nabla_y^2 f(y,x);$$

and $C_1=C_2$, the problem (SD)$_1$ becomes,

Minimize $\left\{f(y,x)-x^T\left(\nabla_y f(y,x)+\nabla_y^2 f(y,x)q\right)-\frac{1}{2}q^T\nabla_y^2 f(y,x)q\right\}$

Subject to

$$-\nabla_y f(y,x)-\nabla_y^2 f(y,x)q\in C_2^*$$

$$(x,y)\in C_1\times C_2$$

which is just the primal problem (SP).Thus (SP) is self dual. Hence if $(\bar{x},\bar{y},\bar{q})$ is an optimal solution for (SP), then and conversely. Also, $G(\bar{x},\bar{y},\bar{p})=H(\bar{x},\bar{y},\bar{q})$.

Now we shall show that $G(\bar{x},\bar{y},\bar{p})=0$.

$$G(\bar{x},\bar{y},\bar{p})=f(\bar{x},\bar{y})-\bar{y}^T\left(\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\right)-\frac{1}{2}\bar{p}^T\nabla_y^2 f(\bar{x},\bar{y})\bar{p} \tag{4.21}$$

Since $\bar{y}\in C_2$ and $-\nabla_y f(\bar{x},\bar{y})-\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\in C_2^*$, therefore, we have

$$-\bar{y}^T(\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p})\ge 0 . \tag{4.22}$$

Using (4.22) in (4.21), we have,

$$G(\bar{x},\bar{y},\bar{p})\ge f(\bar{x},\bar{y})-\frac{1}{2}\bar{p}^T\nabla_y^2 f(\bar{x},\bar{y})\bar{p} .$$

Using the conclusion $\bar{p}=0$ of Theorem 4.2, we get

$$G(\bar{x},\bar{y},\bar{p}) \geq f(\bar{x},\bar{y}) . \tag{4.23}$$

Similarly, in view of $\bar{x} \in C_1$ together with $\nabla_x f(\bar{x},\bar{y}) + \nabla_x^2 f(\bar{x},\bar{y})\bar{q} \in C_1^*$, and $\bar{q}=0$, we have,

$$H(\bar{x},\bar{y},\bar{q}) \leq f(\bar{x},\bar{y}) . \tag{4.24}$$

By Theorem 4.2, we have,

$$f(\bar{x},\bar{y}) \leq G(\bar{x},\bar{y},\bar{p}) = H(\bar{x},\bar{y},\bar{q}) \leq f(\bar{x},\bar{y}) .$$

This implies,

$$G(\bar{x},\bar{y},\bar{p}) = H(\bar{y},\bar{x},\bar{q}) = f(\bar{x},\bar{y}) = f(y,x) = -f(x,y) .$$

Consequently, we have,

$$G(\bar{x},\bar{y},\bar{p}) = 0 .$$

4.4 Maxmin Symmetric and Self Duality

Let U and V be two arbitrary sets of integers in R^{n_1} and R^{m_1} respectively. Let K_1 and K_2 be closed convex cones with nonempty interiors in R^{n-n_1}, and R^{m-m_1}, respectively. Let $f(x,y)$ be a real valued function defined on a open set in $R^n \times R^m$ containing $S \times T$ where $S = U \times K_1$ and $T = V \times K_2$. Let K_i^*, $(i=1,2)$ be the polars of K_i.

We consider the following pair of nonlinear mixed integer programs:

Primal Problem

(MSP): $\underset{x^1}{\text{Max}}\ \underset{x^2,y,s}{\text{Min}}\ \phi(x,y,s) = f(x,y) - (y^2)^T \left(\nabla_{y^2} f(x,y) + \nabla_{y^2}^2 f(x,y)s \right)$

$$-\frac{1}{2} s^T \nabla_{y^2}^2 f(x,y)s$$

Subject to

$$-\nabla_{y^2} f(x,y) - \nabla^2_{y^2} f(x,y)s \in K_2^*$$

$$x^1 \in U, (x^2, y) \in K_1 \times T \text{ .}$$

and

Dual Problem

(MSD): $$\underset{y^1}{Min}\underset{x,y^2,r}{Max}\,\psi(x.y,r) = f(x,y) - (x^2)^T \left(\nabla_{x^2} f(x,y) + \nabla^2_{x^2} f(x,y)r\right) f(x,y)$$
$$-\frac{1}{2}(r^T)^T \nabla^2_{x^2} f(x,y)r$$

Subject to

$$\nabla_{x^2} f(x,y) + \nabla^2_{x^2} f(x,y)r \in K_1^*$$

$$y^1 \in V, (x, y^2) \in S \times K_2$$

where $s \in R^{m-m_1}$ and $r \in R^{n-n_1}$.

Also their feasible solutions will be denoted by

$$A = \left\{(x,y,s) \mid x^1 \in U, (x^2, y) \in K_1 \times T, \nabla_{x^2} f(x,y) + \nabla^2_{x^2} f(x,y)r \in K_1^*\right\}$$

$$B = \{x,y,r\} \mid y^1 \in V, \left(x, y^2\right) \in S \times K_2, -\nabla_{y^2} f(x,y) - \nabla^2_{y^2} f(x,y)s \in K_2^*\big\}.$$

Theorem 4.5 (Symmetric Duality): Let $(\bar{x}, \bar{y}, \bar{s})$ be an optimal solution of (MSP). Also, Let

(i) $f(x,y)$ be separable with respect to x^1 or y^1,

(ii) $f(x,y)$ be bonvex in x^2 for every (x^1, y), and boncave in y^2 for every (x, y^1).

(iii) $f(x,y)$ be thrice differentiable in x^2 and y^2,

(iv) $\nabla^2_{y^2} f(x,y)$ is non singular, and

(v) $\nabla_{y^2}(\nabla^2_{y^2} f(\bar{x},\bar{y})\bar{s})$ is negative definite.

Then

(a) $\bar{s} = 0$

(b) $(x^2)^T \nabla_{x^2} f(\bar{x}, \bar{y}) = 0$

(c) $\phi(\bar{x}, \bar{y}, \bar{s} = 0) = \psi(\bar{x}, \bar{y}, \bar{r} = 0)$, and

(d) $(\bar{x}, \bar{y}, \bar{r})$ is an optimal solution of (MSD)

***Proof*:** Let $Z = \underset{x^1}{Max}\, \underset{x^2,y,s}{Min} \{\phi(x,y,s) : (x,y,s) \in A\}$

and

$$W = \underset{y^1}{\text{Min}}\, \underset{x,y^2,r}{\text{Max}} \{\psi(x,y,r) : (x,y,r) \in B\}$$

Since $f(x,y)$ is separable with respect to x^1 or y^1 (say, with respect to x^1), it follows that

$$f(x,y) = f^1(x^1) + f^2(x^2, y). \tag{4.25}$$

Therefore, $\nabla_{y^2} f(x,y) = \nabla_{y^2} f^2(x^2, y)$ and $\nabla^2_{y^2} f(x,y) = \nabla^2_{y^2} f^2(x^2, y)$.

Now Z can be rewritten as

$$Z = \underset{x^1}{\text{Max}}\ \underset{x^2,y,s}{\text{Min}} \Big\{ f^1(x^1) + f^2(x^2,y) - (y^2)^T(\nabla_{y^2} f^2(x^2,y) + \nabla^2_{y^2} f^2(x^2,y)s) - \frac{1}{2} s^T \nabla^2_{y^2} f^2(x^2,y)s \Big\}$$

Subject to

$$-\nabla_{y^2} f^2(x^2,y) - \nabla^2_{y^2} f^2(x^2,y)s \in K_2^*$$

$$(x^2, y^2) \in K_1 K_2, x^1 \in U \text{ and } y^1 \in V$$

$$= \underset{x^1}{\text{Max}}\ \underset{y^1}{\text{Min}}\ \underset{x^2,y^2,s}{\text{Min}} \Big\{ f^1(x^1) + f^2(x^2,y) - (y^2)^T(\nabla_{y^2} f^2(x^2,y) + \nabla^2_{y^2} f^2(x^2,y)s) - \frac{1}{2} s^T \nabla^2_{y^2} f^2(x^2,y)s \Big\},$$

or

$$Z = \underset{x^1}{Max}\,\underset{y^1}{Min}\left\{ f^1\left(x^1\right) + \Theta^1\left(y^1\right) \,\middle|\, x^1 \in U, y^1 \in V \right\} \tag{4.26}$$

where

(MPS)₀: $$\Theta^1(y^1) = \underset{x^2, y^2, s}{\text{Min}} \left\{ f^2(x^2, y) - (y^2)^T (\nabla_{y^2} f^2(x^2, y) + \nabla^2_{y^2} f^2(x^2, y)s) - \frac{1}{2} s^T \nabla^2_{y^2} f^2(x^2, y)s \right\}$$

Subject to $-\nabla_{y^2} f^2(x^2, y) - \nabla^2_{y^2} f^2(x^2, y)s \in K_2^*$

$(x^2, y^2) \in K_1 \times K_2$.

Similarly,

$$W = \underset{y^1}{Min}\,\underset{x^1}{Max}\left\{ f^1\left(x^1\right) + \Theta^2\left(y^1\right) \,\middle|\, x^1 \in U, y^1 \in V \right\} \tag{4.27}$$

where

(MSD)₀: $$\Theta^2\left(y^1\right) = \underset{x^2, y^2 . r}{Min} \left\{ f^2\left(x^2, y\right) - \left(x^2\right)^T \left(\nabla_{x^2} f^2\left(x^2, y\right) + \nabla^2_{x^2} f^2\left(x^2, y\right)r\right) - \frac{1}{2} r^T \nabla^2_{x^2} f^2(x^2, y)r \right\}$$

Subject to $\nabla_{x^2} f^2(x^2, y) + \nabla^2_{x^2} f^2(x^2, y)r \in K_1^*$

$(x^2, y^2) \in K_1 \times K_2$.

For any given y^1, the program $(\text{MPS})_0$ and $(\text{MPD})_0$ are a pair of second-order symmetric dual nonlinear program involving cone treated in the proceeding section and hence in view of assumptions (ii)-(v), Theorem 4.2 becomes applicable.

Therefore, for $y^1 = \bar{y}^1$,we have,

$$\bar{s} = 0, (\bar{x}^2)^T \nabla_{x^2} f^2(\bar{x}^2, \bar{y}) = 0 \tag{4.28}$$

and

$$\Theta^1(\bar{y}^1) = \Theta^2(\bar{y}^1) \tag{4.29}$$

It remains to show that $(\bar{x},\bar{y},\bar{r}=0)$ is optimal for (MSD). If this is not the case, there exists $y^{*1}\in V$ such that $\Theta^2(y^{*1})<\Theta^2(\bar{y}^1)$. But then, in view of the assumptions (iv) and (v) , we have

$$\Theta^1(\bar{y}^1)=\Theta^2(\bar{y}^1)>\Theta^2(y^{*1})=\Theta^1(y^{*1}),$$

which contradicts the optimality of $(\bar{x}^2,\bar{y}^2,\bar{s}=0)$ for (MSP). Hence $(\bar{x},\bar{y},\bar{r}=0)$ is an optimal solution for (MSD).

Also, (4.25) and (4.28) prove (b), whereas $\phi(\bar{x},\bar{y},\bar{s}=0)=\psi(\bar{x},\bar{y},\bar{r}=0)$ follows form (4.26), (4.27) and (4.29).

As earlier, here to, the converse duality theorem (Theorem 4.6) will be merely stated.

Theorem 4.6 (Converse Duality): Let $(\bar{x},\bar{y},\bar{r})$ be an optimal solution of (MSD), also let

(i) $f(x,y)$ be separable with respect to x^1 and y^1

(ii) $f(\cdot,y)$ be bonvex in x^2 for every (x^1,y) ,and boncave in y^2 for every (x,y^1) ,

(iii) $f(x,y)$ be thrice differentiable in x^2 and y^2 ,

(iv) $\nabla^2_{x^2} f(\bar{x},\bar{y})$ is non singular

(v) $\nabla_{x^2}(\nabla^2_{x^2} f(\bar{x},\bar{y})\bar{r})$ is positive definite.

Then

(e) $\bar{r}=0$

(f) $(y^2)^T\nabla_{y^2} f(\bar{x},\bar{y})=0$

(g) $\phi(\bar{x},\bar{y},\bar{s}=0)=\psi(\bar{x},\bar{y},\bar{r}=0)$ and

(h) $(\bar{x},\bar{y},\bar{s})$ is an optimal solution of (MSP).

Theorem 4.7 (Self Duality): Let $f: R^n \times R^m \to R$ be skew symmetric. Then (MSP) is self dual. Further, if (MSP) and (MSD) are dual programs and $(\bar{x},\bar{y},\bar{s})$ is an optimal solution for (MSP), then $(\bar{x},\bar{y},\bar{s}=0)$ and $(\bar{x},\bar{y},\bar{r}=0)$ are optimal solution for (MSP) and (MSD) respectively, and

$$\phi(\bar{x},\bar{y},\bar{s})=0=\psi(\bar{x},\bar{y},\bar{r}).$$

***Proof*:** The proof follows along the lines of Theorem 4.4.

4.5 Special Cases

If $C_1 = R_+^n$ and $C_2 = R_+^m$ where R_+^n and R_+^m are nonnegative orthants in R^n and R^m, Then the problems (SP) and (SD) will reduce to the following problems treated by Mond [70] :

Primal (P): Minimize
$$G_0(x,y,p)= f(x,y)-y^T\nabla_y f(x,y)+\nabla_y^2 f(x,y)p) -\frac{1}{2}p^T\nabla_y^2 f(x,y)p$$

Subject to

$$\nabla_y f(x,y)+\nabla_y^2 f(x,y)p \le 0,$$

$$x \ge 0, y \ge 0.$$

and

Dual (D): Maximize
$$H_0(x,y,q)= f(x,y)-x^T\nabla_x f(x,y)+\nabla_x^2 f(x,y)q) -\frac{1}{2}q^T\nabla_x^2 f(x,y)q$$

Subject to

$$\nabla_x f(x,y)+\nabla_x^2 f(x,y)q \ge 0$$

$$x \ge 0, y \ge 0.$$

It is to be remarked that $y \geq 0$ and $x \geq 0$ can be deleted respectively from the problems (P) and (D) as these constraints are not essential.

If only p and q are required the zero vectors, then our problem (SP) and (SD) become the following (first order) symmetric dual programs over cones studied by Bazaraa and Goode [8]:

Primal (P_0): Minimize $f(x,y) - y^T \nabla_y f(x,y)$

Subject to

$$-\nabla_y f(x,y) \in C_2^*,$$

$$(x,y) \in C_1 \times C_2$$

Dual (D_0): Maximize $f(x,y) - x^T \nabla_x f(x,y)$

Subject to

$$-\nabla_x f(x,y) \in C_1^*,$$

$$(x,y) \in C_1 \times C_2$$

Finally, if U and V are empty sets and $p = s$ and $r = q$, Then (MSP) and (MSD) will become, the problems (SP) and (SD) considered in Section 4.3.

5.1 INTRODUCTORY REMARKS

Chandra, Husain and Abha [22] presented a new symmetric dual formulation (called mixed symmetric dual formulation) for a class of nonlinear programming problem and derived various duality results. Their mixed formulation unifies the Wolfe [98] and Mond-Weir type [71] symmetric dual formulations respectively, incorporated by Dantzig *et al.* [38] and Mond-Weir [71].

Recently Suneja *et al.* [92] studied Mond-Weir type second-order symmetric duality in multiobjective programming by establishing usual duality theorems under η-bonvexity and η-boncavity assumptions. They also proved self duality theorems under skew symmetry of the kernel function that occur in the formulation of the problems. In [92] each component of the multiobjective dual models involves different auxiliary variables p_i and q_i $i = 1, 2, \ldots, k$, disagreeing with the formulation of second-order dual model having single auxiliary variable p, presented by Mangasarian [66].

The purpose of this chapter is to present multiobjective version of the second-order mixed symmetric and self duality in traditional mathematical programming with a single objective treated by Husain and Abha [49]. This formulation of the problems considers the same auxiliary variable p in the primal and the same auxiliary variable q in the dual, which is the conformity with the Mangasarian's [66] formulation. Obviously, our formulation unifies Wolfe and Mond-Weir type symmetric second-order dual models which are not studied in the literature. In addition to validation of various duality theorems under suitable second-order convexity/ generalized second-order convexity, an attempt is also made to identify self duality for this pair of programs under additional restrictions on the kernel functions involved.

5.2 Pre-requisites and Definitions

Let R^n denoted the n-dimensional Euclidean space. The following ordering relations in R^n are recalled for our use. If x, y $\in R^n$, then

$x < y \quad \Leftrightarrow \quad x_i < y_i$, (i = 1,2, …, n)

$x \leqq y \quad \Leftrightarrow \quad x_i \leq y_i$, (i = 1,2, …, n)

$x \leq y \quad \Leftrightarrow \quad x_i \leq y_i$, (i = 1,2, …, n), but $x \neq y$

$x \nleq$ y is the negation of $x \leq y$.

For $x, y \in R, x \leq y \, and \, x < y$ have the usual meaning.

Let ϕ (x,y) be twice differentiable real-valued function defined on $R^n \times R^n$. Let $\nabla_x \phi(\bar{x}, \bar{y})$ and $\nabla_y \phi(\bar{x}, \bar{y})$ denote the gradient vectors with respect to x and y, respectively evaluated at $(\bar{x}, \bar{y})$. Also let $\nabla_x^2 \phi(\bar{x}, \bar{y})$ and $\nabla_y^2 \phi(\bar{x}, \bar{y})$ debits the Hessian matrix of second-order partial derivatives of ϕ with respect to x and y, respectively evaluated at $(\bar{x},\bar{y})$. The symbols $\nabla_{xx}\phi(\bar{x},\bar{y})$ and $\nabla_{yy}\phi(\bar{x},\bar{y})$ are similarly defined. The symbols $\nabla_y\left(\nabla_x^2\phi(\bar{x},\bar{y})q\right)$ and $\nabla_x\left(\nabla_y^2\phi(\bar{x},\bar{y})p\right)$ denote the matrices whose $(i,j)^{\text{th}}$ elements are respectively given as $\frac{\partial}{\partial y_i}\left(\nabla_x^2\phi(\bar{x},\bar{y})q\right)_j$, with $q\in R^n$ and $\frac{\partial}{\partial x_i}\left(\nabla_y^2\phi(\bar{x},\bar{y})p\right)_j$ with $p\in R^m$.

***Definition* 5.1:** The function ϕ is said to be bonvex in first variable x at $u\in R^m$, if for all $v\in R^n$, $q\in R^n$, $x\in R^n$ and for fixed y.

$$\phi(x,v)-\phi(u,v) \geqq (x-u)^T\left[\nabla_x\phi(u,v)+\nabla_x^2(u,v)q\right]-\frac{1}{2}q^T\nabla_x^2\phi(u,v)q$$

and $\phi(x,y)$ is used to be boncave in the second variable y at v, if for all $u\in R^m$, $p_i\in R^m$, $y\in R^m$ and for fixed $x\in R^n$,

$$\phi(x,v)-f(x,y)\leqq(v-y)^T\left[\nabla_y\phi(x,y)+\nabla_y^2\phi(x,y)p\right]-\frac{1}{2}p^T\nabla_y^2\phi(x,y)p$$

***Definition* 5.2:** The function ϕ is said to be pseudobonvex in the first variable x at $u\in R^n$, if for all v$\in R^n$, $q_i\in R^n$ and $x\in R^n$ and for fixed y,

$$(x-u)^T\left[\nabla_x\phi(u,v)+\nabla_x^2\phi(u,v)q\right]\geqq 0$$

$$\Rightarrow \quad \phi(x,v)\geqq\phi(u,v)-\frac{1}{2}q^T\nabla_x^2\phi(u,v)q$$

and ϕ is said to be pseudoboncave in the second variable y at v$\in$R^n, if for all $u\in R^m$, $p\in R^m$ and $y\in R^m$ and for fixed $x\in R^n$

$$(v-y)^T\left[\nabla_y\phi(x,y)+\nabla_y^2\phi(x,y)p\right]\leqq 0$$

$$\Rightarrow \quad \phi(x,v)\leqq\phi(x,y)-\frac{1}{2}p^T\nabla_y^2\phi(x,y)p$$

5.3 Mixed Type Second-Order Multiobjective Duality

For $N = \{1,2,\ldots, n\}$ and $M\{1,2,\ldots, m\}$, let $J_1 \subseteq N$ and $K_1 \subseteq M$ and $J_2 = N\backslash J_1$ and $K_2 = M\backslash K_1$. Let $|J_1|$denote the number of elements in the subset J_1. The other symbols $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in R^{|J_1|}$ and $x^2 \in R^{|J_2|}$, then any $x \in R$ can be written as $x = (x^1, x^2)$. Similarly for $y^1 \in R^{|K_1|}$ and $y^2 \in R^{|K_2|}$. can be written as $y = (y^1, y^2)$. Let $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ be twice differentiable functions. It is to be noticed here that if J_1 is an empty set, the $J_2 = N$, $|J_1| = 0$ and $|J_2| = N$. Then $R^{|J_1|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be the zero-dimensional and $|K_1|$-dimensional vectors respectively. Similarly we can describe the cases K_1 an empty set, K_2 an empty set and J_2, as an empty set.

We now introduce the following pair of nonlinear programs and study its second-order symmetric duality by the following theorems:

Primal Problem:

(SMP): Minimize $F(x^1, x^2, y^1, y^2, p, r)$

$$= \left(F_1(x^1, x^2, y^1, y^2, p, r), \ldots, F_k(x^1, x^2, y^1, y^2, p, r)\right)$$

Subject to

$$\nabla_{y^1}(\lambda^T f)(x^1, y^1) + \nabla^2_{y^1}(\lambda^T f)(x^1, y^1)\, p \leqq 0, \tag{5.1}$$

$$\nabla_{y^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{y^2}(\lambda^T g)(x^2, y^2)\, r \leqq 0, \tag{5.2}$$

$$(y^2)^T \left[\nabla_{y^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{y^2}(\lambda^T g)(x^2, y^2)\, r\right] \geqq 0, \tag{5.3}$$

$$x^1, x^2 \geqq 0, \tag{5.4}$$

$$\lambda \in \Lambda^+. \tag{5.5}$$

Dual Problem:

(SMD): Maximize $G(u^1, u^2, v^1, v^2, q, s)$

$$=\left(G_1(u^1,u^2,v^1,v^2,q,s),...,G_k(u^1,u^2,v^1,v^2,q,s)\right)$$

Subject to

$$\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)\, q \geqq 0, \quad (5.6)$$

$$\nabla_{x^2}(\lambda^T g)(u^2,v^2)+\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)\, s \geqq 0, \quad (5.7)$$

$$\left(u^2\right)^T\left[\nabla_{x^2}(\lambda^T g)(u^2,v^2)+\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)\, s\right] \leqq 0, \quad (5.8)$$

$$v^1, v^2 \geqq 0, \quad (5.9)$$

$$\lambda \in \Lambda^+. \quad (5.10)$$

where

(i) $$F_i(x^1,x^2,y^1,y^2,p,r) = f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1}f_i(x^1,y^1)p$$
$$-(y^1)^T\left\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p\right\}$$
$$+g_i(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}g_i(x^2,y^2)r$$

(ii) $$G_i(u^1,u^2,v^1,v^2,q,s) = f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1}f_i(u^1,v^1)q$$
$$-(u^1)^T\left\{\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\right\}$$
$$+g_i(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}g_1(u^2,v^2)s$$

(iii) $p \in R^{|K_1|}, r \in R^{|K_2|}, q \in R^{|J_1|} s \in R^{|J_2|}$, and $\lambda=(\lambda_1,\ldots,\lambda_k)^T$ with $\lambda_1 \in R$, $i=1,2,\ldots,k$.

(iv) $$\Lambda^+ = \left\{\lambda \in R^k \mid \lambda > 0, \sum_{i=1}^{k}\lambda_i = 1\right\}$$

***Theorem* 5.1 (*Weak duality*):** For $(x^1, x^2, y^1, y^2, \lambda, p, r)$ be feasible for (SMP) and $(u^1, u^2, v^1, v^2, \lambda, q, s)$ feasible for (SMD), let

(i) for each $i \in \{1,2,\ldots,k\}$; $f_1(.,y^1)$ be bonvex at u^1 for fixed y^1 and $f_i(x^1,.)$ be boncave at y^1 for fixed x^1, and

(ii) $\lambda^T g(.,y^2)$ be pseudoconvex at u^2 for fixed y^2, and $\lambda^T g(x^2,.)$ be pseudoboncave at y^2 for fixed x^2.

Then $F(x^1,x^2,y^1,y^2,p,r) \nleq G(u^1,u^2,v^1,v^2,q,s)$

***Proof*:** By the bonvexity-boncavity of f_i, $i \in \{1,2,\dots,k\}$,

$$f_i(x^1,v^1)-f_i(u^1,v^1) \geqq (x^1-u^1)^T\left[\nabla_{x^1} f_i(u^1,v^1)+\nabla^2_{x^1} f_i(u^1,v^1)q\right] -\frac{1}{2}q^T\nabla^2_{x^1} f_i(u^1,v^1)q \tag{5.11}$$

and

$$f_i(x^1,v^1)-f_i(x^1,y^1)^T \leqq (v^1-y^1)\left[\nabla_{y^1} f_i(x^1,y^1)+\nabla^2_{y^1} f_i(x^1,y^1)p\right] -\frac{1}{2}p^T\nabla^2_{y^1} f_i(x^1,y^1)p \tag{5.12}$$

Multiplying (5.12) by (-1) and adding resulting inequality to (5.11), we have,

$$\begin{aligned} &f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1} f_i(x^1,y^1)p-(y^1)^T\left\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p\right\} \\ &-\left[f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1} f_i(u^1,v^1)q-(u^1)^T\{\nabla_{x^1} f_i(u^1,v^1)+\nabla^2_{x^1} f_i(u^1,v^1)q\}\right] \\ &\geqq (x^1)^T\left\{\nabla^1_{x^1} f_i(u^1,v^1)+\nabla^2_{x^1} f_i(u^1,v^1)q\right\}-(v^1)^T\left\{\nabla_{y^1} f_i(x^1,y^1)+\nabla^2_{y^1} f_i(x^1,y^1)p\right\} \end{aligned}$$

Using (5.5) and (5.10), this inequality becomes,

$$\begin{aligned} &\sum_{i=1}^{k}\lambda_i\left[f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1} f_i(x^1,y^1)p-(y^1)^T\left\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p\right\}\right] \\ &-\sum_{i=1}^{k}\lambda_i\left[f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1} f_i(u^1,v^1)q-(u^1)^T\left\{\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\right\}\right] \\ &\geqq \left(x^1\right)^T\left\{\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\right\} \\ &\qquad -(v^1)^T\left\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p\right\} \end{aligned}$$

This, in view of (5.6) with (5.4), and (5.1) with (5.9), yields,

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1}f_i(x^1,y^1)p-(y^1)^T\left\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)_i(x^1,y^1)p\right\}\right]$$

$$\geqq\sum_{i=1}^{k}\lambda_i\left[f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1}f_i(u^1,v^1)q-(u^1)^T\left\{\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\right\}\right] \tag{5.13}$$

From (5.4), (5.7) and (5.8), we have,

$$(x^2-u^2)^T\left[\nabla_{x^2}(\lambda^T g)(u^2,v^2)+\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)s\right]\geqq 0$$

Also from (5.9), (5.2) and (5.3), we have,

$$(v^2-y^2)^T\left[\nabla_{y^2}(\lambda^T g)(x^2,y^2)+\nabla^2_{y^2}(\lambda^T g)(x^2,y^2)r\right]\leqq 0$$

By pseudobonvexity of $\lambda^T g(.,y^2)$ at u^2, we have,

$$\lambda^T g(x^2,v^2)\geqq\left[(\lambda^T g)(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)s\right] \tag{5.14}$$

and by pseudoboncavity $\lambda^T g(x^2,.)$ at y^2, we have,

$$\lambda^T g(x^2,v^2)\leqq \lambda^T g(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}(\lambda^T g)(x^2,y^2)r \tag{5.15}$$

From (5.14) and (5.15), we have,

$$\lambda^T g(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}(\lambda^T g)(x^2,y^2)r\geqq\lambda^T g(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}(\lambda^T g)(x^2,v^2)s \tag{5.16}$$

Combing (5.13) and (5.16), we have,

$$\sum_{i=1}^{k}\lambda_i\left[f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1}f_i(x^1,y^1)p-(y^1)^T\left\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p\right\}\right.$$
$$\left.+g_i(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}\lambda^T g_i(x^2,y^2)r\right]$$

$$\geqq\sum_{i=1}^{k}\lambda_i\left[f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1}f_i(u^1,v^1)q-(u^1)^T\left\{\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\right\}\right.$$
$$\left.+\lambda^T g_i(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}g_i(u^2,v^2)s\right]$$

or

$$\sum_{i=1}^{k}\lambda_i F_i(x^1,x^2,y^1,y^2,p,r) \geqq \sum_{i=1}^{k}\lambda_i G_i(u^1,u^2,v^1,v^2,q,s)$$

or

$$\lambda^T F(x^1,x^2,y^1,y^2,p,r) \geqq \lambda^T G(u^1,u^2,v^1,v^2,q,s)$$

This implies

$$F(x^1,x^2,y^1,y^2,p,r) \not\leq G(u^1,u^2,v^1,v^2,q,s)$$

Theorem 5.2 (Strong Duality): Let for each $i\in\{1,2,\ldots,k\}$, f_i be thrice differentiable on $R^n\times R^m$. Let $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})$ be a properly efficient solution of (SMP); fix $\lambda=\bar{\lambda}$. Assume that

(A$_1$): the set $(\nabla^2_{y^1} f_1,\nabla^2_{y^1} f_2,\ldots,\nabla^2_{y^1} f_k)$ is linearly independent,

(A$_2$): the set $(\nabla^2_{y^2} g_1,\nabla^2_{y^2} g_2,\ldots,\nabla^2_{y^2} g_k)$ is linearly independent,

(A$_3$): both the Hessian matrices $\nabla_{y^1}\left(\nabla^2_{y^1}\left(\bar{\lambda}^T f\right)p\right)$ and $\nabla_{y^2}\left(\nabla^2_{y^2}\left(\bar{\lambda}^T g\right)r\right)$, are either positive or negative definite,

(A$_4$): the set $(\nabla_{y^2} g_1+\nabla^2_{y^2} g_1\bar{r},\nabla_{y^2} g_2+\nabla^2_{y^2} g_2\bar{r},\ldots,\nabla_{y^2} g_k+\nabla^2_{y^2} g_k\bar{r})$ is linearly independent and

(A$_5$): the set $\{\nabla_{y^1} f_1+\nabla^2_{y^1} f_1\bar{p},\nabla_{y^1} f_2+\nabla^2_{y^1} f_2\bar{p},\ldots,\nabla_{y^1} f_k+\nabla^2_{y^1} f_k\bar{p}\}$ is linearly independent.

where $f_1=f_1(\bar{x}^1,\bar{y}^1),\ \mathrm{g}_1=g_1(\bar{x}^1,\bar{y}^1), i=1,2,\ldots,k.$ Then $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q}=0,\bar{s}=0)$ is feasible for (SMD) and $F(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})=G(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q},\bar{s})$.

***Proof*:** Since $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})$ is a properly efficient solution of (SMP), it is also a weak minimum. Hence there exist $\alpha\in R^k$, with $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_k)$ and $\mu\in R^k$, $\beta\in R^{|K_1|},\theta\in R^{|K_2|},\delta^1\in R^{|J_1|},\delta^2\in R^{|J_2|}$ and $\eta\in R$

such that the following Fritz John optimality condition [68] are satisfied at $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r})$

$$\nabla_{x^1}(\alpha^T f) + \left(\beta - (\alpha^T e)\bar{y}^1\right)^T \nabla_{y^1 x^1}(\lambda^T f)$$
$$+ \sum_{i=1}^{k} \left\{ \left(\beta - (\alpha^T e)\bar{y}^1\right)\bar{\lambda}_1 - \frac{\alpha_1 \bar{p}}{2} \right\} \nabla_{x^1}(\nabla^2_{y^1} f_1 \bar{p}) = \delta^1 \tag{5.17}$$

$$\nabla_{x^2}(\alpha^T g) + (\theta - \eta\bar{y}^2)\nabla_{y^2 x^2}(\lambda^T g)$$
$$+ \sum_{i=1}^{k} \left\{ (\theta - \eta\bar{y})\bar{\lambda}_1 - \frac{\alpha_1 \bar{r}}{2} \right\} \nabla_{x^2}(\nabla^2_{y^2} g_1 \bar{r}) = \delta^2 \tag{5.18}$$

$$\nabla_{y^1}\left(\alpha - (\alpha^T e)\bar{\lambda}\right)^{\mathrm{T}} f + \left(\beta - (\alpha^T e)\bar{y}^1 - (\alpha^T e)\bar{p}\right)\nabla^2_{y^1}(\bar{\lambda}^T f)$$
$$+ \sum_{i=1}^{k} \left\{ \left(\beta - (\alpha^T e)\bar{y}^1\right)\bar{\lambda}_1 - \frac{\alpha_1 \bar{p}}{2} \right\} \nabla_{y^1}(\nabla^2_{y^1} f_1 \bar{p}) = 0 \tag{5.19}$$

$$\nabla_{y^2}(\alpha - \eta\bar{\lambda})^{\mathrm{T}} \mathrm{g} + (\theta - \eta\bar{y}^2 - \eta\bar{r})^T \nabla^2_{y^2}(\bar{\lambda}^T g)$$
$$+ \left\{ \left(\theta - \eta\bar{y}^2\right)\bar{\lambda}_1 - \frac{\alpha^T \bar{r}}{2} \right\} \nabla_{y^2}(\nabla^2_{y^2} g\bar{r}) = 0 \tag{5.20}$$

$$\sum_{i=1}^{k}((\beta - (\alpha^T e)\bar{y}^1)\bar{\lambda}_1 - \alpha_1\bar{p})\nabla^2_{y^1} f_1 = 0 \tag{5.21}$$

$$\sum_{i=1}^{k}((\theta - \eta\bar{y}^2)\bar{\lambda}_1 - \alpha_1\bar{r})\nabla^2_{y^2} g_1 = 0 \tag{5.22}$$

$$(\beta - (\alpha^T e)\bar{y}^1)^T \left[\nabla_{y^1} f_1 + \nabla^2_{y^1} f\bar{p}\right] + (\theta - \eta\bar{y}^2)^T \left[\nabla_{y^2} g + \nabla^2_{y^2} g\bar{r}\right] + \mu = 0 \tag{5.23}$$

$$\beta\left[\nabla_{y^1}\left(\bar{\lambda}^T f_1\right) + \nabla^2_{y^1}\left(\bar{\lambda}^T f\right)\bar{p}\right] = 0 \tag{5.24}$$

$$\theta^T\left[\nabla_{y^2}\left(\bar{\lambda}^T g\right) + \nabla^2_{y^2}\left(\bar{\lambda}^T g\right)\bar{r}\right] = 0 \tag{5.25}$$

$$\eta(\bar{y}^2)^T\left[\nabla_{y^2}\left(\bar{\lambda}^T g\right) + \nabla^2_{y^2}\left(\bar{\lambda}^T g\right)\bar{r}\right] = 0 \tag{5.26}$$

$$\delta^1\bar{x}^1 = 0 \tag{5.27}$$

$$\delta^2\bar{x}^2 = 0 \tag{5.28}$$

$$\mu^T \bar{\lambda} = 0 \tag{5.29}$$

$$(\alpha, \beta, \theta, \eta, \delta^1, \delta^2, \mu) \geq 0 \tag{5.30}$$

$$(\alpha, \beta, \theta, \eta, \delta^1, \delta^2, \bar{\lambda}, \mu) \neq 0 \tag{5.31}$$

Since $\bar{\lambda} > 0$, from (5.29), we have,

$$\mu = 0 \tag{5.32}$$

From (5.21) along with the assumption (A_1) and (5.22) along with the assumption (A_2), we obtain,

$$(\beta - (\alpha^T e)\bar{y}^1)\bar{\lambda}_1 = \alpha_1 \bar{p}, i = 1, 2, ..., k \tag{5.33}$$

and

$$(\theta - \eta \bar{y}^2)\bar{\lambda}_1 = \alpha_1 \bar{r}, i = 1, 2, ..., k. \tag{5.34}$$

Multiplying (5.23) by $\bar{\lambda}$ and using (5.29), we get,

$$\begin{aligned}(\beta - (\alpha^T e)\bar{y}^1)^T (\nabla_{y^1}(\bar{\lambda}^T f) + \nabla^2_{y^1}(\bar{\lambda}^T f)\bar{p})\\ +(\theta^T - \eta \bar{y}^2)[\nabla_{y^2}(\bar{\lambda}^T g) + \nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r}] = 0\end{aligned} \tag{5.35}$$

From (5.25) and (5.26), we have,

$$(\theta^T - \eta \bar{y}^2)[\nabla_{y^2}(\bar{\lambda}^T g) + \nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r}] = 0 \tag{5.36}$$

i.e.,

$$(\theta^T - \eta \bar{y}^2)[\nabla_{y^2}(\eta \bar{\lambda})^T g + \nabla^2_{y^2}(\eta \bar{\lambda})^T g \bar{r}] = 0 \tag{5.37}$$

Using (5.36) i.e. in (5.35), we have,

$$(\beta - (\alpha^T e)\bar{y}^1)^T (\nabla_{y^1}(\bar{\lambda}^T f) + \nabla_{y^1}(\bar{\lambda}^T f)\bar{p}) = 0 \tag{5.38}$$

i.e.,

$$(\beta-(\alpha^T e)\bar{y}^1)^T(\nabla_{y^1}((\alpha^T e)\bar{\lambda}^T f)+\nabla^2_{y^1}((\alpha^T e)\bar{\lambda}^T f)=0 \qquad (5.39)$$

Using (5.33) in (5.19) and (5.34) in (5.20), we obtain,

$$(\alpha-(\alpha^T e)\bar{\lambda})^T(\nabla_{y^1} f+\nabla^2_{y^1} f\,\bar{p})+\frac{1}{2}(\beta-(\alpha^T e)\bar{y}^1)^T\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda} f)\bar{p})=0 \qquad (5.40)$$

$$(\alpha-\eta^T\bar{\lambda})^T(\nabla_{y^2} g+\nabla^2_{y^2} g\bar{r})+\frac{1}{2}(\theta-\eta\bar{y}^2)\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda} g)\bar{r})=0 \qquad (5.41)$$

On multiplying (5.40) by $(\beta-(\alpha^T e)\bar{y}^1)$ and (5.41) by $(\theta-\eta\bar{\lambda})^T$ and then adding, we obtain,

$$\begin{aligned}(\beta-(\alpha^T e)\bar{y}^1)&\left\{\nabla_{y^1}(\alpha-(\alpha^T e)\bar{\lambda})^T f+\nabla^2_{y^1}((\alpha-(\alpha^T e)\bar{\lambda})^T f\,\bar{p}\right\}\\&+(\theta-\eta^T\bar{y}^2)^T\left\{\nabla_{y^2}\left(\alpha-\eta^T\bar{\lambda}\right)g+(\nabla^2_{y^2}(\alpha-\eta^T\bar{\lambda})g\bar{r}\right\}\\&+\frac{1}{2}(\beta-(\alpha^T e)\bar{y}^1)^T\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda}^T f)\bar{p})+(\beta-(\alpha^T e)\bar{y}^1)\\&+\frac{1}{2}(\theta-\eta\bar{y}^2)^T\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r})+(\theta-\eta\bar{y}^2)=0\end{aligned} \qquad (5.42)$$

Using (5.32) and then multiply (5.23) by α, we have,

$$\begin{aligned}(\beta-(\alpha^T e)\bar{y}^1)&\left\{\nabla_{y^1}(\alpha-(\alpha^T e)\bar{\lambda})^T f+\nabla^2_{y^1}((\alpha-(\alpha^T e)\bar{\lambda})^T f\,\bar{p}\right\}\\&+\left(\theta-\eta\bar{y}^2\right)^T\left\{\nabla_{y^2}\left(\alpha^T g\right)+\nabla^2_{y^2}\left(\alpha^T g\right)\bar{r}\right\}=0\end{aligned}$$

Summing (5.37) and (5.39) from this inequality ,we have,

$$\begin{aligned}(\beta-(\alpha^T e)\bar{y}^1)&\left\{\nabla_{y^1}(\alpha-(\alpha^T e)\bar{\lambda})^T f+\nabla^2_{y^1}((\alpha-(\alpha^T e)\bar{\lambda})^T f\,\bar{p}\right\}\\&+(\theta-\eta^T\bar{y}^2)^T\left\{\nabla_{y^2}\left(\alpha-\eta^T\bar{\lambda}\right)g+(\nabla^2_{y^2}(\alpha-\eta^T\bar{\lambda})g\bar{r}\right\}=0\end{aligned} \qquad (5.43)$$

Using (5.43) in (5.42), we have,

$$(\beta-(\alpha^T e)\bar{y}^1)\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda}^T f)\bar{p})^T(\beta-(\alpha^T e)\bar{y}^1)$$
$$+(\theta-\eta\bar{y}^2)^T\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r})^T(\theta-\eta\bar{y}^2)=0 \qquad (5.44)$$

But by the assumption (A_3), we have,

$$(\beta-(\alpha^T e)\bar{y}^1)^T\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda} f)\bar{p})(\beta-(\alpha^T e)\bar{y}^1)=0$$

and

$$(\theta-\eta\bar{y}^2)^T\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r})^T(\theta-\eta\bar{y}^2)=0$$

Which respectively gives,

$$\beta-(\alpha^T e)\bar{y}^1=0 \qquad (5.45)$$

and

$$\theta-\eta\bar{y}^2=0 \qquad (5.46)$$

From (5.40) together with (5.45), we have,

$$(\alpha-(\alpha^T e)\bar{\lambda}^1)^T(\nabla_{y^1} f+\nabla^2_{y^1} f\ \bar{p})=0$$

Which because (A_5),gives,

$$\alpha-(\alpha^T e)\bar{\lambda}^1=0 \qquad (5.47)$$

The relation (5.41) together with (5.46) ,gives,

$$(\alpha-\eta\bar{\lambda}^1)^T\{\nabla_{y^2} g+\nabla^2_{y^2} g\bar{r}\}=0$$

Which because of (A_4) implies,

$$\alpha-\eta\bar{\lambda}=0 \qquad (5.48)$$

If possible, let $\eta = 0$. Then from (5.48), we have $\alpha = 0$ and from (5.45) and (5.46) we have $\theta = 0 = \beta$. From (5.17) and (5.18), we get $\delta^1 = 0$ and $\delta^2 = 0$. Contradicting (5.31).

Hence η > 0.From (5.48) we have $\alpha > 0$. From (5.45) and (5.46) we obtain,

$$\bar{y}^1 \geqq 0, \;\; \bar{y}^2 \geqq 0. \tag{5.49}$$

From (5.17) along with (5.33)and (5.47), we get,

$$\nabla_{x^1}(\bar{\lambda}^T f) = \delta^1.$$

This along with (5.30)) and (5.27), we obtain,

$$\nabla_{x^1}(\bar{\lambda}^T f) \geq 0. \tag{5.50}$$

and

$$(\bar{x}^1)^T \nabla_{x^1}(\bar{\lambda}^T f) = 0. \tag{5.51}$$

From (5.18) along with (5.34) and (5.47), yields,

$$\nabla_{x^2}(\bar{\lambda}^T g) = \delta^2.$$

This along with (5.30) and (5.28) ,yields,

$$\nabla_{x^2}(\bar{\lambda}^T g) \geq 0 \tag{5.52}$$

and

$$(\bar{x}^2)^T \nabla_{x^2}(\bar{\lambda}^T g) = 0. \tag{5.53}$$

From (5.33) along with (5.45) and $\alpha_1 > 0$, and from (5.34) along with (5.46) $\alpha_1 > 0$, respectively, we have,

$$\bar{p} = 0 = \bar{r}.$$

From (5.49), (5.50), (5.52) and (5.53), it implies that $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, q = 0, s = 0)$ is feasible for (SMD).

From (5.24) along with (5.45) and $\alpha > 0$ and (5.26) with $\eta > 0$, we have respectively,

$$(\bar{y}^1)^T(\nabla_{y^1}(\bar{\lambda}^T f)+\nabla^2_{y^1}(\bar{\lambda}^T f)\bar{p})=0 \tag{5.54}$$

and

$$(\bar{y}^2)^T(\nabla_{y^2}(\bar{\lambda}^T g)+\nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r})=0 \tag{5.55}$$

Consider

$$\begin{aligned} F_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}) = {} & f_i(\bar{x}^1,\bar{y}^1)-\frac{1}{2}\bar{p}\nabla^2_{y^1} f_i(\bar{x}^1,\bar{y}^1)\bar{p} \\ & -\left(\bar{y}^1\right)^T\{\nabla_{y^1}(\bar{\lambda} f)^T(\bar{x}^1,\bar{y}^1)+\nabla^2_{y^1}(\bar{\lambda} f)^T(\bar{x}^1,\bar{y}^1)\bar{p}\} \\ & +g_i\left(\bar{x}^2,\bar{y}^2\right)-\frac{1}{2}\bar{r}_i^T\nabla^2_{y^2} g_i(\bar{x}^2,\bar{y}^2)\bar{r} \end{aligned}$$

This, along with (5.54) and $\bar{p}=0=\bar{r}$, becomes

$$F_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}) = f_i(\bar{x}^1,\bar{y}^1)+g_i(\bar{x}^2,\bar{y}^2),\ \mathrm{i}=1,2,\ldots,\mathrm{k} \tag{5.56}$$

Again consider,

$$\begin{aligned} G_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q},\bar{s}) = {} & f_i(\bar{x}^1,\bar{y}^1)-\frac{1}{2}\bar{q}\nabla^2_{x^1} f_i(\bar{x}^1,\bar{y}^1)\bar{q} \\ & -\left(\bar{x}^1\right)^T\{\nabla_{x^1}(\bar{\lambda} f)^T(\bar{x}^1,\bar{y}^1)+\nabla^2_{x^1}(\bar{\lambda} f)^T(\bar{x}^1,\bar{y}^1)\bar{q}\} \\ & +g_i(\bar{x}^2,\bar{y}^2)-\frac{1}{2}\bar{s}^T\nabla^2_{x^2} g_i(\bar{x}^2,\bar{y}^2)\bar{s} \end{aligned}$$

This along with (5.51) and $\bar{q}=\bar{s}=0$, becomes

$$G_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q},\bar{s}) = f_i(\bar{x}^1,\bar{y}^1)+g_i(\bar{x}^2,\bar{y}^2),\ \ \mathrm{i}=1,2,\ldots,\mathrm{k} \tag{5.57}$$

From (5.56) and (5.57), we have,

$$F_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}) = G_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q},\bar{s}),\ \text{ for all } \mathrm{i}\in\{1,2,\ldots,\mathrm{k}\}$$

This implies ,

$$F_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}) = G_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q},\bar{s})\,. \tag{5.58}$$

That is, the objective values of (SMP) and (SMD) are equal.

If $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ is not efficient, then there exists $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s})$ such that

$$G(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) \geq G(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$$

Which because of (5.58) gives,

$$G(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) \geq F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}).$$

This contradicts the weak duality (Theorem 5.1).

If $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ were improperly efficient, then for some feasible $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s})$ and some i

$$G_i(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) - F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) > M$$

and so is

$$\bar{\lambda}^T G_i(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) > \bar{\lambda}^T F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}).$$

This again contradicts Theorem 5.1. Hence $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ is, indeed, a properly efficient solution of (SMD).

We shall merely state the following converse duality as its proof is immediate due to symmetry of the formulation of the problem (SMP) and (SMD).

Theorem 5.3 (Converse Duality): Let for each $i \in \{1,2,\ldots,k\}$, f_i be thrice differentiable on $R^n \times R^m$. Let $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ be a properly efficient solution of (SMD); fix $\lambda = \bar{\lambda}$. Assume that

(C_1): the set $\nabla^2_{x^1} f_1, \nabla^2_{x^1} f_2, \ldots, \nabla^2_{x^1} f_k$ is linearly independent,

(C_2): the set $\nabla^2_{x^2} g_1, \nabla^2_{x^2} g_2, \ldots, \nabla^2_{x^2} g_k$ is linearly independent, (C$_3$): both the Hessian matrices $\nabla_{x^1}(\nabla^2_{x^1}(\lambda^T f)\bar{q})$ and $\nabla_{x^2}(\nabla^2_{x^2}(\lambda^T g)\bar{r})$ are either positive or negative definite,

(C$_4$): the set $\{\nabla_{x^1} f_1 + \nabla^2_{x^1} f_1\bar{q}, \nabla_{x^1} f_2 + \nabla^2_{x^1} f_2\bar{q}, ..., \nabla_{x^1} f_k + \nabla^2_{x^1} f_k\bar{q}\}$ is linearly independent; and

(C$_5$): the set $\{\nabla_{x^2} g_1 + \nabla^2_{x^2} g_1\bar{s}, \nabla_{x^2} g_2 + \nabla^2_{x^2} g_2\bar{s}, ..., \nabla_{x^2} g_k + \nabla^2_{x^2} g_k\bar{s}\}$ is linearly independent.

where

$$f_i = f_i(\bar{x}^1, \bar{y}^1),\ \ f_i = f_i(\bar{x}^1, \bar{y}^1),\ i = 1, 2, ..., k.$$

Then $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p} = 0, \bar{r} = 0)$ is feasible for (SMP) and $F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = G(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$.

Moreover, if the hypothesis of Theorem 5.1 are satisfied for all feasible solutions of (SMP) and (SMD), then $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r})$ is a properly efficient solution of (SMP).

5.4 Mixed Type Second-Order Multiobjective Self Duality

In this section, we now prove the following self-duality Theorem. A mathematical program is said to be self-dual, if it is formally identical with its dual, that is, if the dual is recast in the form of the primal, the new program so obtained is the same as the primal. In general the program (SMP) and (SMD) are not self dual without added restriction on $f_i(x,y)$ and $f_i(y,x)$, $i \in \{1,2,\ldots,k\}$. The functions $f_i : R^{|J_1|} \times R^{|J_2|} \to R$ and $g_i : R^{|J_1|} \times R^{|J_2|} \to R$, $i \in \{1,2,\ldots,k\}$, is the skew symmetric if for all x, $y \in R^n$, $f_i(x^1,y^1) = -f_i(y^1,x^1)$, $i \in \{1,2,\ldots,k\}$ and $g_i(x^2, y^2) = -g_i(y^2,x^2)$.

We describe the programs (SMP) and (SMD) as dual program if the conclusion of Theorem 5.2 hold.

Theorem 5.4 (Self Duality): If the kernel function $f_i(\bar{x}^1, \bar{y}^1)$ and $g_i(\bar{x}^2, \bar{y}^2)$ for $i \in \{1,2,\ldots,k\}$ are skew symmetric, then (SMP) is self-dual. If also (SMP) and (SMD) are dual program, and $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r})$ is a joint optimal solution, then so is $(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda}, \bar{p}, \bar{r})$ and $F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = 0.$

***Proof*:** Consider (SMP) and note that (SMD) can be written:

Minimize $-G(x^1, x^2, y^1, y^2, q, s)$

$$= \left(-G_i(x^1, x^2, y^1, y^2, q, s), \ldots, -G_k(x^1, x^2, y^1, y^2, q, s)\right)$$

Subject to

$$-(\nabla_{x^1}(\lambda^T f)(x^1, y^1) + \nabla_{x^1}(\lambda^T f)(x^1, y^1)q) \leqq 0,$$

$$-(\nabla_{x^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{x^2}(\lambda^T g)(x^2, y^2)s) \leqq 0,$$

$$-(x^2)^T(\nabla_{x^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{x^2}(\lambda^T g)(x^2, y^2)s) \geqq 0,$$

$$y^1, y^2 \geqq 0$$

$$^*\lambda \in \Lambda^+$$

where

$$G_i(x^1, x^2, y^1, y^2, \lambda, q, s) = f_i(x^1, y^1) - \frac{1}{2} q^T \nabla^2_{x^1} f_i(x^1, y^1) q$$
$$-(x^1)^T(\nabla_{x^1}(\lambda^T f)(x^1, y^1) + \nabla^2_{x^1}(\lambda^T f)(x^1, y^1)q)$$
$$+ g_i(x^2, y^2) - \frac{1}{2} s^T \nabla^2_{x^2} g_i(x^2, y^2) s$$

Since for each $i \in \{1,2,\ldots,k\}$ f_i and g_i are skew symmetric, $\nabla_{x^1} f_i(x^1, y^1) = -\nabla_{y^1} f_i(y^1, x^1),\ \nabla_{x^2} g_i(x^2, y^2) = -\nabla_{y^2} f_i(y^2, x^2),$

$\nabla_{x^2}(\lambda^T g)(x^2,y^2) = -\nabla_{y^2}(\lambda^T g)(y^2,x^2)$, $\nabla^2_{x^2}(\lambda^T g)(x^2,y^2) = -\nabla^2_{y^2}(\lambda^T g)(y^2,x^2)$, $\nabla^2_{x^1}(\lambda^T f)(x^1,y^1) = -\nabla^2_{y^1}(\lambda^T f)(y^1,x^1)$, $\nabla_{x^1}(\lambda^T f)(x^1,y^1) = -\nabla_{y^1}(\lambda^T f)(y^1,x^1)$, and program (SMD) becomes

$$\text{Minimize } -G(y^1,y^2,x^1,x^2,q,s)$$
$$= \left(G_1(y^1,y^2,x^1,x^2,q,s),\ldots,G_k(y^1,y^2,x^1,x^2,q,s)\right)$$

Subject to

$$(\nabla_{y^1}(\lambda^T f)(y^1,x^1) + \nabla^2_{y^1}(\lambda^T f)(y^1,x^1)q) \leqq 0\,,$$

$$(\nabla_{y^2}(\lambda^T g)(y^2,x^2) + \nabla^2_{y^2}(\lambda^T g)(y^2,x^2)s) \leqq 0\,,$$

$$(x^2)^T(\nabla_{y^2}(\lambda^T g)(y^2,x^2) + \nabla^2_{y^2}(\lambda^T g)(y^2,x^2)s) \geqq 0\,,$$

$$y^1, y^2 \geqq 0$$

$$\lambda \in \Lambda^+$$

where

$$G_i(y^1,y^2,x^1,x^2,\lambda,q,s) = f_i(y^1,x^1) + \frac{1}{2}q^T\nabla^2_{y^1} f_i(y^1,x^1)q$$
$$-(x^1)^T\{\nabla_{y^1}(\lambda^T f)^T(y^1,x^1) + \nabla^2_{y^1}(\lambda^T f)(y^1,x^1)q\}$$
$$+g_i(y^2,x^2) - \frac{1}{2}s^T\nabla^2_{x^2} g_i(y^2,x^2)s, \quad \mathrm{i} = 1,2,\ldots,\mathrm{k}$$

This is just (SMP).

Thus $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q},\bar{r})$ optimal for (SMD) implies $(\bar{y}^1,\bar{y}^2,\bar{x}^1,\bar{x}^2,\bar{\lambda},\bar{q},\bar{s})$ optimal for (SMP). By an analogous argument, $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})$ optimal for (SMP) implies $(\bar{y}^1,\bar{y}^2,\bar{x}^1,\bar{x}^2,\bar{\lambda},\bar{p},\bar{s})$ optimal for (SMD).

If (SMP) and (SMD) are dual programs and $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r})$ is jointly optimal, then

$$(\bar{x}^1)^T (\nabla_{x^1} (\bar{\lambda}^T f)(\bar{x}^1, \bar{y}^1) + \nabla^2_{x^1} (\bar{\lambda}^T g)(\bar{x}^2, \bar{y}^1) q) = 0,$$

$$(\bar{x}^2)^T (\nabla_{x^2} (\lambda^T g)(\bar{x}^2, \bar{y}^2) + \nabla^2_{x^2} (\bar{\lambda}^T g)(\bar{x}^2, \bar{y}^2) r) = 0,$$

$$(\bar{y}^1)^T (\nabla_{y^2} (\bar{\lambda}^T f)(\bar{x}^1, \bar{y}^1) + \nabla^2_{y^1} (\bar{\lambda}^T f)(\bar{x}^1, \bar{y}^1) \bar{p}) = 0,$$

$$(\bar{y}^2)^T (\nabla_{y^2} (\bar{\lambda}^T g)(\bar{x}^2, \bar{y}^2) + \nabla^2_{y^2} (\bar{\lambda}^T g)(\bar{x}^2, \bar{y}^2) \bar{s}) = 0$$

The objective values of (SMP) and (SMD) at $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r})$ in view of the above relation, becomes for each i $\in \{1,2,\ldots,k\}$,

$$F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2) = G_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s}).$$

Since $(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda}, \bar{p}, \bar{r})$ is also a joint optimal solution, it can be similarly shown that

$$\bar{q} = \bar{s} = 0, \ (\bar{x}^1)^T \{\nabla_{y^1} (\bar{\lambda}^T f)(\bar{y}^1, \bar{x}^1) + \nabla^2_{y^1} (\bar{\lambda}^T f)(\bar{y}^1, \bar{x}^1) \bar{q}\} = 0$$

and the objective value of (SMP) and (SMD) at $(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda}, \bar{p}, \bar{r})$ can be given as

$$\begin{aligned} F_i(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda}, \bar{p}, \bar{r}) &= G_i(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda}, \bar{p}, \bar{r}) \\ &= f_i(\bar{y}^1, \bar{x}^1) + g_i(\bar{y}^2, \bar{x}^2), i \in \{1, 2, ..., k\} \\ &= -\left(f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2) \right), i \in \{1, 2, ..., k\} \end{aligned}$$

By Theorem 2, we have,

$$\begin{aligned} F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) &= G_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) \\ &= f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2), i \in \{1, 2, ..., k\} \end{aligned}$$

Therefore,

$$F_i(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})=0,\ i\in\{1,2,...,k\}$$

This implies

$$F(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})=0.$$

5.5 **Special Cases**

If $k = 1$, $\lambda = 1$, $f_i = f$ and $g_i = g$, the second-order symmetric multiobjective dual programs (SMP) and (SMD) to the following program, studied by Husain and Abha [49]:

Primal Program

(SP): Minimize $F(x^1,y^1,y^1,y^2,p,r)=f(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1}f(\bar{x}^1,\bar{y}^1)p$

$$+(y^1)^T\{\nabla_{y^1}f(x^1,y^1)+\nabla^2_{y^1}f(x^1,y^1)p\}$$

$$+g(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}g(x^2,y^2)r$$

Subject to

$$\nabla_{y^1}f(x^1,y^1)+\nabla^2_{y^1}f(x^1,y^1)p \leqq 0,$$

$$\nabla_{y^2}g(x^2,y^2)+\nabla^2_{y^2}g(x^2,y^2)r \leqq 0$$

$$(y^2)^T\{\nabla_{y^2}g(x^2,y^2)+\nabla^2_{y^2}g(x^2,y^2)r \geqq 0$$

$$x^1,x^2 \geqq 0,$$

Dual Program

(SD): Maximize $G(u^1,u^2,v^1,v^2,q,s)=f(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1}f(u^1,v^1)q$

$$+(u^1)^T\{\nabla_{x^1}f(u^1,v^1)+\nabla^2_{x^1}f(u^1,v^1)q\}$$

$$+g(u^2,v^2)-\frac{1}{2}s^T\nabla^2{}_{y^2}g(u^2,v^2)s$$

Subject to

$$\nabla_{x^1} f(u^1, v^1) + \nabla^2_{x^1} f(u^1, v^1) q \geqq 0,$$

$$\nabla_{x^2} g(u^2, v^2) + \nabla^2_{x^2} g(u^2, v^2) s \geqq 0,$$

$$(u^2)^T \{\nabla_{x^2} g(u^2, v^2) + \nabla^2_{x^2} g(u^2, v^2) s\} \leqq 0,$$

$$v^1, v^2 \geqq 0.$$

If $J_2 = \phi$ and $K_2 = \phi$, the programs (SMP) and (SMD) reduce to the following pair of Wolfe type second-order multiobjective dual programs which are not explicitly studied in the literature

Primal Program:

(SWP): Minimize $F^1(x^1, y^1, p) = (F^1_1(x^1, y^1, p), \ldots, F^1_k(x^1, y^1, p))$

Subject to

$$\nabla_{y^1} (\lambda^T f)(x^1, y^1) + \nabla^2_{y^1} (\lambda^T f)(x^1, y^1) p \leqq 0,$$

$$x^1 \geq 0,$$

$$\lambda \in \Lambda^+$$

Dual program:

(SWD): Minimize $G^1(u^1, v^1, q) = (G^1_1(u^1, v^1, q), \ldots, G^1_k(u^1, v^1, q))$

Subject to:

$$\nabla_{x^1} (\lambda^T f)(u^1, v^1) + \nabla^2_{x^1} (\lambda^T f)(u^1, v^1) q \geqq 0,$$

$$y^1 \geq 0,$$

$$\lambda \in \Lambda^+,$$

where, for each $i \in \{1,2,\ldots,k\}$,

i) $$F^1_i(x^1, y^1, p) = f_i(x^1, y^2) - (y^T)[\nabla_{y^1} (\lambda^T f)(x^1, y^1) + \nabla^2_{y^1} (\lambda^T f)(x^1, y^1) p]$$

$$-\frac{1}{2} p^T \nabla^2_{y^1} f_i(x^1, y^1) p$$

ii) $$G_i^1(u^1,v^1,q) = f_i(u^1,v^1) - (u^T)[\nabla_{x^1}(\lambda^T f)(u^1,v^1) + \nabla_{x^1}^2(\lambda^T f)(u^1,v^1)q]$$
$$-\frac{1}{2}q^T\nabla_{y^1}^2 f_i(u^1,v^1)q.$$

If $J_1 \neq \phi$ and $K_1 = \phi$, the programs (SMP) and (SMD) become the Mond-Weir second-order multiobjective dual program which are reported in mathematical programming.

Primal Program

(SMWP): Minimize $F^2(x^2,y^2,r) = \left(F_1^2(x^2,y^2,r),...,F_k^2(x^2,y^2,r)\right)$

Subject to

$$\nabla_{y^2}(\lambda^T g)(x^2,y^2) + \nabla_{y^2}^2(\lambda^T g)(x^2,y^2)\ \mathrm{r} \leqq 0$$

$$(y^2)^T\left[\nabla_{y^2}(\lambda^T g)(x^2,y^2) + \nabla_{y^2}^2(\lambda^T g)(x^2,y^2)\ \mathrm{r}\right] \geqq 0$$

$$x^2 \geqq 0$$

$$\lambda > 0$$

Dual Program

(SMWD): Maximize $G^2(u^2,v^2,s) = \left(G_1^2(u^2,v^2,s),...,G_k^2(u^2,v^2,s)\right)$

Subject to

$$\nabla_{x^2}(\lambda^T g)(u^2,v^2) + \nabla_{x^2}^2(\lambda^T g)(u^2,v^2)\ \mathrm{s} \geqq 0$$

$$\left(u^2\right)^T\left[\nabla_{x^2}(\lambda^T g)(u^2,v^2) + \nabla_{x^2}^2(\lambda^T g)(u^2,v^2)\ \mathrm{s}\right] \leqq 0$$

$$v^2 \geqq 0$$

$$\lambda > 0$$

where , for each $i \in \{1,2,\ldots,k\}$,

i) $$F_i(x^2, y^2, r) = g_i(x^2, y^2) - \frac{1}{2} r^T \nabla^2_{y^2} g_i(x^2, y^2) r$$

ii) $$G_i(u^2, v^2, s) = g_i(u^2, v^2) - \frac{1}{2} s^T \nabla^2_{x^2} g_i(u^2, v^2) s$$

If $p = q = s = r = 0$, then the programs (SMP) and (SMD) reduce to the mixed type first-order symmetric multiobjective programs studied by Bector, Chandra and Abha [12].

6.1 INTRODUCTORY REMARKS

The calculus of variation has been one of the prominent branches of analysis, for more than two centuries. It is a tool of great power that can be used to wide variety of problems, in pure mathematics. It can also be used to express basic principles of mathematical physics in forms of utmost simplicity and elegance. Hanson [48] pointed out that some of the duality results in the mathematical programming have the analogues in calculus of variations. Exploring this relationship between mathematical programming and classical calculus of variation, Mond and Hanson [77] formulated a constrained variational problem as mathematical programming problem in abstract space and using Valentine [93] optimality conditions for the same, presented its Wolfe dual variational problem for validating various duality results under usual convexity. Later Bector, Chandra and Husain [13] studied Mond-Weir type duality for the problem of Mond and Hanson [77] for relaxing its convexity requirements. In [19] Chandra, Craven and Husain studied optimality and duality for a class of nondifferentiable variational problems in which the integrand of the objective functional contains a term of a square root of the quadratic form, while in [52], Husain and Jabeen studied optimality criteria and duality for variational problems in which integrand of objective and constraint functions contains terms of support functions.

Second-order duality in mathematical programming has been extensively studied in recent years. Mangasarian [66] was the first to identify a second-order dual formulation for non-linear programs under the assumptions that are complicated and somewhat difficult to verify. Mond [70] introduced the concept of second-order convex functions (named as bonvex functions by Bector and Chandra [11]) and studied second-order duality for nonlinear programs.

Recently Chen [27] is the first to identify second-order duality in variational problems. He studied usual duality results under invexity assumptions on the functions that occur in the formulation of the problem along with some strange assumptions. Mond [70] has pointed out that the second-order dual for a nonlinear programming gives a tighter bound and has computational advantage over a first order dual. Motivated with this of Mond [70] in this exposition, we construct Mond -Weir type second-order dual to the variational problem and derive usual duality results under second-order pseudo- invexity and second-order quasi-invexity assumptions.

The relationship of our results to second-order duality results in nonlinear programming reported in [11] is indicated. In essence it is shown that our duality results can be viewed as dynamic generalizations of corresponding (static) duality theorems of nonlinear programming already in the literature.

6.2 Pre-requisites and Definitions

Let $I=[a,b]$ be a real interval, $f: I\times R^n\times R^n \to R$ and $g: I\times R^n\times R^n \to R^m$ be twice continuously differentiable functions. In order to consider $f\left(t,x(t),\dot{x}(t)\right)$, where $x: I\to R^n$ is differentiable with derivative $\dot{x}$, denoted by f_x *and* $f_{\dot{x}}$ the partial derivative of f with respect to x *and* $\dot{x}$, respectively, that is,

$$f_x=\begin{pmatrix}\dfrac{\partial f}{\partial x^1}\\ \dfrac{\partial f}{\partial x^2}\\ \vdots\\ \dfrac{\partial f}{\partial x^n}\end{pmatrix},\qquad f_{\dot{x}}=\begin{pmatrix}\dfrac{\partial f}{\partial \dot{x}^1}\\ \dfrac{\partial f}{\partial \dot{x}^2}\\ \vdots\\ \dfrac{\partial f}{\partial \dot{x}^n}\end{pmatrix};$$

denote by f_{xx} the Hessian matrix of f with respect to x, that is,

$$f_{xx}=\begin{pmatrix}\dfrac{\partial^2 f}{\partial x^1\partial x^1} & \dfrac{\partial^2 f}{\partial x^1\partial x^2} & \cdots & \dfrac{\partial^2 f}{\partial x^1\partial x^n}\\ \dfrac{\partial^2 f}{\partial x^2\partial x^1} & \dfrac{\partial^2 f}{\partial x^2\partial x^2} & \cdots & \dfrac{\partial^2 f}{\partial x^2\partial x^n}\\ \vdots & \vdots & \ddots & \vdots\\ \dfrac{\partial^2 f}{\partial x^n\partial x^1} & \dfrac{\partial^2 f}{\partial x^n\partial x^2} & \cdots & \dfrac{\partial^2 f}{\partial x^n\partial x^n}\end{pmatrix}_{n\times n}$$

It is obvious that f_{xx} is a symmetric $n\times n$ matrix. Denote by g_x the $m\times n$ matrix with respect to x, that is,

$$g_x=\begin{pmatrix}\dfrac{\partial g_1}{\partial x^1} & \dfrac{\partial g_1}{\partial x^2} & \cdots & \dfrac{\partial g_1}{\partial x^n}\\ \dfrac{\partial g_2}{\partial x^1} & \dfrac{\partial g_2}{\partial x^2} & \cdots & \dfrac{\partial g_2}{\partial x^n}\\ \vdots & \vdots & \ddots & \vdots\\ \dfrac{\partial g_m}{\partial x^1} & \dfrac{\partial g_m}{\partial x^2} & \cdots & \dfrac{\partial g_m}{\partial x^n}\end{pmatrix}_{m\times n}$$

Similarly $f_{\dot{x}}$, $f_{\dot{x}x}$, $f_{x\dot{x}}$ *and* $g_{\dot{x}}$ can be defined.

Denote by X, the space of piecewise smooth functions $x : I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$, where the differentiation operator D is given by

$$u = D\,x \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\,ds,$$

where α is given boundary value; thus $\frac{d}{dt} = D$ except at discontinuities.

We introduce the following definitions which are needed for the duality results to hold.

Definition 6.1 (Second-order Invexity): If there exists a vector function $\eta = \eta(t, x, \bar{x}) \in R^n$ where $\eta : I \times R^n \times R^n \to R^n$ and with $\eta = 0$ at *t* = *a* and *t* = *b*, such that for the functional $\int_I \phi(t, x, \dot{x})\,dt$ where $\phi : I \times R^n \times R^n \to R$ satisfies

$$\int_I \phi(t, x, \dot{x})\,dt - \int_I \left\{ \phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t) \right\} dt$$
$$\geq \int_I \left\{ \eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t) \right\} dt,$$

then $\int_I \phi(t, x, \dot{x})\,dt$ is second-order invex with respect to η where $G = \phi_{xx} - D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}}$ and $\beta \in C(I, R^n)$, the space of continuous *n*-dimensional vector function. The function β is analogous to the auxiliary vector *p* in [11].

Definition 6.2 (Second-order Pseudoinvex): If the functional $\int_I \phi(t, x, \dot{x})\,dt$ satisfies

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\} dt \geq 0 \;\Rightarrow$$

$$\int_I \phi(t,x,\dot{x})\,dt \geq \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\} dt,$$

then $\int_I \phi(t,x,\dot{x})\,dt$ is said to be second-order pseudoinvex with respect to η.

Definition 6.3 (Second-Order Quasi-invex): If the functional $\int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\int_I \phi(t,x,\dot{x})\,dt \leq \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\} dt, \Rightarrow$$

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\} dt \leq 0,$$

then $\int_I \phi(t,x,\dot{x})\,dt$ is said to be second-order quasi-invex with respect to η.

If ϕ does not depend on *t*, then the above definitions reduce to those given in [11] for static cases.

Consider the following constrained variational problem:

(VP): Minimize $\int_I f(t,x,\dot{x})\,dt$

Subject to

$$x(a) = 0 = x(b),$$

$$g(t,x,\dot{x}) \leq 0, \quad t \in I,$$

$$h(t,x,\dot{x}) = 0, \quad t \in I,$$

where $f: I \times R^n \times R^n \to R$, $g: I \times R^n \times R^n \to R^m$ and $h: I \times R^n \times R^n \to R^k$ are continuously differentiable.

Proposition 6.1 [3] (Fritz-John Conditions): If (VP) attains a local (or) global minimum at $x = \bar{x} \in X$ then there exist Lagrange multiplier $\tau \in R$, $z : I \to R^k$ and piecewise smooth $y : I \to R^m$ such that

$$\tau f_x\left(t,\bar{x},\dot{\bar{x}}\right) + y(t)^T g_x\left(t,\bar{x},\dot{\bar{x}}\right) + z(t)^T h_x\left(t,\bar{x},\dot{\bar{x}}\right)$$
$$- D\left[f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right) + y(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right) + z(t)^T h_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right] = 0, \quad t \in I,$$

$$y(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right) = 0, \quad t \in I ,$$

$$\left(\tau, y(t)\right) \geq 0, \quad t \in I ,$$

$$\left(\tau, y(t), z(t)\right) \neq 0, \quad t \in I .$$

The Fritz John necessary conditions for (VP), become the Karush-Kuhn-Tucker conditions [66] if $\tau = 1$. If $\tau = 1$, the solution $\bar{x}$ is said to be normal.

6.3 Second-Order Duality

Consider the following variational problem (CP) by ignoring the equality constraint of (VP):

(CP): Minimize $\int_I f\left(t, x, \dot{x}\right) dt$

Subject to

$$x(a) = 0 = x(b), \tag{6.1}$$

$$g\left(t, x, \dot{x}\right) \leq 0, \qquad t \in I, \tag{6.2}$$

Chen [27] presented the following Wolfe type second-order dual problem for (CP) analogous to that for nonlinear programming by Mangasarian [66] and established various duality results under somewhat strange invexity-like conditions.

Maximize: $\int_a^b \Big\{ f\left(t,u(t),\dot u(t)\right)+\alpha(t)^T g\left(t,u(t),\dot u(t)\right)$

$$\frac{1}{2}\beta(t)^T\Big[f_{uu}\left(t,u(t),\dot u(t)\right)+\left(g_u\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_u$$
$$-2D\left(f_{u\dot u}\left(t,u(t),\dot u(t)\right)+\left(g_u\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_{\dot u}\right)$$
$$+D^2\left(f_{\dot u\dot u}\left(t,u(t),\dot u(t)\right)+\left(g_{\dot u}\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_{\dot u}\right)\Big]\beta(t)\Big\}dt$$

Subject to

$$u(a)=0=u(b),\quad \dot u(a)=0=\dot u(b)$$

$$f_u\left(t,u(t),\dot u(t)\right)+g_u\left(t,u(t),\dot u(t)\right)^T\alpha(t)$$
$$-D\left[f_{\dot u}\left(t,u(t),\dot u(t)\right)+g_{\dot u}\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right]$$
$$+\Big[f_{uu}\left(t,u(t),\dot u(t)\right)+\left(g_u\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_u$$
$$-2D\left(f_{u\dot u}\left(t,u(t),\dot u(t)\right)+\left(g_u\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_{\dot u}\right)$$
$$+D^2\left(f_{\dot u\dot u}\left(t,u(t),\dot u(t)\right)+\left(g_{\dot u}\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_{\dot u}\right)\Big]\beta(t)=0,$$

$$t\in I,$$

$$\alpha(t)\in R_+^m\ ,\ \beta(t)\in R^n\ ,t\in I\ ,$$

where R_+^m designates the non-negative orthant of the Euclidean space R^n.

Let

$$H = f_{uu}\left(t,u(t),\dot u(t)\right)+\left(g_u\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_u$$
$$-2D\left(f_{u\dot u}\left(t,u(t),\dot u(t)\right)+\left(g_u\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_{\dot u}\right)$$
$$+D^2\left(f_{\dot u\dot u}\left(t,u(t),\dot u(t)\right)+\left(g_{\dot u}\left(t,u(t),\dot u(t)\right)^T\alpha(t)\right)_{\dot u}\right).$$

Then the above dual problem can be expressed in a much simpler form which is given below.

(VD): Maximize $\int_a^b \Big\{ f\left(t,u(t),\dot{u}(t)\right)+\alpha(t)^T g\left(t,u(t),\dot{u}(t)\right)$

$$-\frac{1}{2}\beta(t)^T H\left(t,u(t),\dot{u}(t),\alpha(t),\beta(t)\right)\Big\}dt$$

Subject to

$$u(a)=0=u(b), \quad \dot{u}(a)=0=\dot{u}(b)$$

$$f_u\left(t,u(t),\dot{u}(t)\right)+g_u\left(t,u(t),\dot{u}(t)\right)^T\alpha(t)$$
$$-D\left[f_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)+g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)^T\alpha(t)\right]$$
$$+H\left(t,u(t),\dot{u}(t)\right)\alpha(t)\beta(t)=0, \;\; t\in I$$

$$\alpha(t)\in R_+^m \;,\;\; \beta(t)\in R^n \;\;, t\in I$$

It is remarked here that if f and g are independent of t, then (VD) becomes second-order dual problem studied by Mangasarian [66].

Now we present the following Mond –Weir type second-order dual (CD) in the spirit of [11] to relax second-order invexity requirements and establish various duality results between the problems (CP) and (CD) under generalized second-order invexity hypothesis.

(CD): Maximize $\int_I \left\{ f(t,u,\dot{u})-\frac{1}{2}\beta(t)^T F\beta(t)\right\}dt$

Subject to

$$u(a)=0=u(b) \tag{6.3}$$

$$f_u+y(t)^T g_u-D\left(f_{\dot{u}}+y(t)^T g_{\dot{u}}\right)+(F+H)\beta(t)=0 \;,\; t\in I \tag{6.4}$$

$$\int_I \left\{y(t)^T g(t,u,\dot{u})-\frac{1}{2}\beta(t)^T H\beta(t)\right\}dt\geq 0, \tag{6.5}$$

$$y(t)\geq 0,\; t\in I \tag{6.6}$$

where

$$F = f_{uu} - Df_{u\dot{u}} + D^2 f_{\dot{u}\dot{u}} \text{ and } H = \left(y(t)^T g_u\right)_u - D\left(y(t)^T g_u\right)_{\dot{u}}$$

$$+D^2\left(y(t)^T g_{\dot{u}}\right)_{\dot{u}} \text{ and define } D = \frac{d}{dt} \text{ as defined earlier.}$$

If f and g are independent of t then $F = f_{uu}$ and $H = \left(y^T g_u\right)_u$ and consequently (CD) will reduce to the second-order dual problem introduced in [11].

Theorem 6.1 (Weak Duality): Let $x(t) \in X$ be a feasible solution of (CP) and $(u(t), y(t), \beta(t))$ be feasible solution of (CD).If $\int_I f(t,.,.)dt$ be second-order pseudoinvex and $\int_I y(t)^T g(t,.,.)dt$ be second-order quasi-invex with respect to the same $\eta : I \times R^n \times R^n \to R^n$ satisfying $\eta = 0$ at $t = a$ and $t = b$, then

$$\int_I f(t,x,\dot{x})dt \geq \int_I \left\{ f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F\beta(t) \right\} dt$$

Proof: The relations, $g(t,x,\dot{x}) \leq 0$, $y(t) \geq 0$, $t \in I$ and (6.5) imply

$$\int_I y(t)^T g(t,x,\dot{x})dt \leq \int_I \left\{ y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t) \right\} dt,$$

This, because of second-order quasi-invexity of $\int_I y(t)^T g(t,.,.)dt$, implies that,

$$\int_I \left\{ \eta^T \left(y(t)^T g_u\right) + (D\eta)^T \left(y(t)^T g_{\dot{u}}\right) + \eta^T H\beta(t) \right\} dt \leq 0$$

i.e.,
$$\int_I \eta^T \left(y(t)^T g_u\right) dt + \int_I (D\eta)^T \left(y(t)^T g_{\dot{u}}\right) dt + \int_I \eta^T H\beta(t) dt \leq 0$$

This, by integration by parts, this inequality yields,

$$\int_I \eta^T \left(y(t)^T g_u\right) dt + \eta y(t)^T g_{\dot{u}} \Big|_a^b - \int_I \eta^T D\left(y(t)^T g_{\dot{u}}\right) dt + \int_I \eta^T H\beta(t) dt \leq 0$$

Using $\eta = 0$ at $t = a$ and $t = b$ in the above inequality, we obtain,

$$\int_I \eta \left[\left(y(t)^T g_u \right) - D\left(y(t)^T g_{\dot{u}} \right) + \eta^T H \beta(t) \right] dt \le 0 ,$$

Using (6.4), this gives,

$$\int_I \left[\eta^T \left(f_u - Df_{\dot{u}} \right) + \eta^T F \beta(t) \right] dt \ge 0 .$$

Integrating by parts, gives,

$$\int_I \left(\eta^T f_u + \left(D\eta \right)^T f_{\dot{u}} + \eta^T F \beta(t) dt \right) \ge 0 .$$

This, in view of second-order pseudoinvexity of $\int_I f\left(t,.,.\right) dt$ implies,

$$\int_I f\left(t, x, \dot{x}\right) dt \ge \int_I \left\{ f\left(t, u, \dot{u}\right) - \frac{1}{2} \beta\left(t\right)^T F \beta\left(t\right) \right\} dt .$$

This implies,

infimum(CP) $\ge$ supremum(CD).

Theorem 6.2 (Strong Duality): If $\bar{x}(t) \in X$ is an optimal solution of (CP) and meets the normality conditions, then there exists a piece wise smooth $\bar{y} : R \to R^m$ such that $\left(\bar{x}(t), \bar{y}(t), \beta(t) = 0 \right)$ is a feasible for (CD) and the two objective values are equal. Furthermore, if the hypothesis of Theorem 6.1 holds, then $\left(\bar{x}(t), \bar{y}(t), \beta(t) \right)$ is an optimal solution for (CD).

***Proof*:** From Proposition 6.1, there exists a piece wise smooth function $\bar{y} : R \to R^m$ satisfying the following conditions:

$$\left(f_x\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{y}(t)^T g_x\left(t, \bar{x}, \dot{\bar{x}}\right) \right) - D\left(f_{\dot{x}}\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{y}(t)^T g_{\dot{x}}\left(t, \bar{x}, \dot{\bar{x}}\right) \right) = 0, \quad t \in I$$

i.e.,

$$\left(f_x\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{y}\left(t\right)^T g_x\left(t, \bar{x}, \dot{\bar{x}}\right) \right) - D\left(f_{\dot{x}}\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{y}\left(t\right)^T g_{\dot{x}}\left(t, \bar{x}, \dot{\bar{x}}\right) \right)$$
$$+ (F + H)\beta(t) = 0, \qquad (6.7)$$

where

$$\beta(t) = 0, \ \ t \in I$$

$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) = 0$$

i.e., $$\int_I \left\{ \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2} \beta(t)^T H \beta(t) \right\} dt = 0, \ \ where \ \ \beta(t) = 0, \ \ t \in I \tag{6.8}$$

$$\bar{y}(t) \geq 0 \, , \ \ t \in I \tag{6.9}$$

From (6.7), (6.8) and (6.9), it implies that $(\bar{x}(t), \bar{y}(t), \beta(t) = 0)$ is feasible for (CD) and the objective value of (CP) and (CD) are equal. The optimality of $(\bar{x}(t), \bar{y}(t), \beta(t))$ follows by an application of Theorem 6.1.

Theorem 6.3 (Converse Duality): Suppose that f and g are thrice continuously differentiable. Let $(\bar{x}(t), \bar{y}(t), \beta(t))$ be an optimal solution of (CD) at which

(A_1): the Hessian matrices F and H are not the multiple of each other.

(A_2): $y(t)^T g_x - Dy(t)^T g_{\dot{x}} \neq 0,$

(A_3): i) $\int_I \beta(t)^T \left(y(t)^T g_x - Dy(t)^T g_{\dot{x}} \right) dt \geq 0$ and $\int_I \beta(t)^T H \beta(t) dt > 0$

or

ii) $\int_I \beta(t)^T \left(y(t)^T g_x - Dy(t)^T g_{\dot{x}} \right) dt \leq 0$ and $\int_I \beta(t)^T H \beta(t) dt < 0$

If, for all feasible $(x(t), y(t), \beta(t))$, $\int_I f(t,.,.) dt$ be second-order pseudoinvex and $\int_I y(t)^T g(t,.,.) dt$ be second-order quasi-invex with respect to the same η, then $\bar{x}(t)$ is an optimal solution of (P).

Proof: Since $\left(\bar{x}(t),\bar{y}(t),\beta(t)\right)$ is an optimal solution for (CD), by proposition 6.1, there exist Lagrange multiplier $\alpha\in R$, and piece wise smooth $\lambda: I\to R^n,\ \gamma\in R\ \ and\ \ \mu: I\to R^m$ such that Fritz John conditions hold at $\left(\bar{x}(t),\bar{y}(t),\beta(t)\right)$:

$$-\alpha\left[\left(f_x-\frac{1}{2}\left(\beta(t)^T F\beta(t)\right)_x\right)-D\left(f_{\dot{x}}-\frac{1}{2}\left(\beta(t)^T F\beta(t)\right)_{\dot{x}}\right)\right]$$
$$+\lambda(t)^T\left\{f_{xx}+\left(y(t)^T g_x\right)_x-D\left(f_{\dot{x}x}+\left(y(t)^T g_{\dot{x}}\right)_x\right)+\left((F+H)\beta(t)\right)_x\right.$$
$$\left.-D\left(f_{x\dot{x}}+\left(y(t)^T g_x\right)_{\dot{x}}\right)-\left(f_{\dot{x}\dot{x}}+\left(y(t)^T g_{\dot{x}}\right)_{\dot{x}}\right)+\left((F+H)\beta(t)\right)_{\dot{x}}\right\}$$
$$+\gamma\left\{y(t)^T g_x-\frac{1}{2}\left(\beta(t)^T F\beta(t)\right)_x-D\left(y(t)^T g_{\dot{x}}-\frac{1}{2}\left(\beta(t)^T F\beta(t)\right)_{\dot{x}}\right)\right\}=0,\ t\in I$$

(6.10)

$$\left(\lambda(t)+\alpha\beta(t)\right)F+\left(\lambda(t)+\gamma\beta(t)\right)H=0,\quad t\in I \tag{6.11}$$

$$\lambda(t)^T\left[g_{jx}-Dg_{j\dot{x}}+\left(g_{jxx}-Dg_{jx\dot{x}}+D^2 g_{\dot{x}\dot{x}}\right)\beta(t)\right]$$
$$+\gamma\left[g_j+\frac{1}{2}\beta(t)^T\left(g_{jxx}-Dg_{jx\dot{x}}+D^2 g_{\dot{x}\dot{x}}\right)\beta(t)\right]+\mu_j(t)=0,\ t\in I$$

(6.12)

$$\left(f_x+y(t)^T g_{\dot{x}}\right)-D\left(f_{\dot{x}}+y(t)^T g_{\dot{x}}\right)+(F+H)\beta(t)=0,\qquad t\in I$$

(6.13)

$$\gamma\int_I\left\{y(t)^T g-\frac{1}{2}\beta(t)H\beta(t)\right\}dt=0,\qquad t\in I \tag{6.14}$$

$$\mu^T(t)\bar{y}(t)=0,\qquad t\in I \tag{6.15}$$

$$\left(\alpha,\gamma,\mu(t)\right)\geq 0,\qquad t\in I \tag{6.16}$$

$$\left(\alpha,\gamma,\lambda(t),\mu(t)\right)\neq 0,\qquad t\in I \tag{6.17}$$

In view of hypothesis (A_1), the equation (6.11) yields,

$$\left.\begin{array}{ll}\lambda(t)+\alpha\beta(t)=0, & t\in I\\ \lambda(t)+\gamma\beta(t)=0, & t\in I\end{array}\right\} \tag{6.18}$$

Multiplying (6.12) by $y_j(t)$ and summing over j, we have,

$$\begin{aligned}&\lambda(t)^T\left[y(t)^T g_x-D\left(y(t)^T g_{\dot{x}}\right)+\left(\left(y(t)^T g_x\right)_x-D\left(y(t)^T g_x\right)_{\dot{x}}+D^2\left(y(t)^T g_{\dot{x}}\right)_{\dot{x}}\right)\beta(t)\right]\\ &\qquad -\gamma\left[y(t)^T g_x-\frac{1}{2}\beta(t)^T\left(\left(y(t)^T g_x\right)_x-D\left(y(t)^T g_x\right)_{\dot{x}}+D^2\left(y(t)^T g_{\dot{x}}\right)_{\dot{x}}\right)\beta(t)\right]\\ &\qquad\qquad +\mu^T(t)y(t)=0,\end{aligned}$$

Using (6.15) and then integrating, we have,

$$\begin{aligned}&\int_I \lambda(t)^T\left\{y(t)^T g_x-D\left(y(t)^T g_{\dot{x}}\right)+H\beta(t)\right\}dt\\ &\qquad -\gamma\int_I\left\{y(t)^T g_x-\frac{1}{2}\beta(t)^T H\beta(t)\right\}dt=0\end{aligned}$$

This, because of (6.14), yields,

$$\int_I \lambda(t)^T\left\{y(t)^T g_x-D\left(y(t)^T g_{\dot{x}}\right)+H\beta(t)\right\}dt=0 \tag{6.19}$$

If $(\alpha,\gamma)=0$ *i.e* $\alpha=0=\gamma$, then (6.18), implies $\lambda(t)=0,\ t\in I$ and $\mu(t)=0$ from (6.12).

Thus, we have,

$$(\alpha,\gamma,\lambda(t),\mu(t))=0.$$

This contradicts (6.17). Hence

$$(\alpha,\gamma)\neq 0 \quad i.e. \quad \alpha>0 \ \ or\ \ \gamma>0.$$

We claim $\beta(t)=0,\ t\in I$. Suppose that $\beta(t)\neq 0,\ t\in I$.

From (6.18) we have,

$$(\alpha-\gamma)\beta(t)=0$$

implying $\alpha=\gamma>0$. Using (6.18) in (6.19), we have,

$$\int_I \alpha\beta(t)^T\left\{y(t)^T g_x - D\left(y(t)^T g_{\dot{x}}\right)+H\beta(t)\right\}dt=0$$

implies $$\int_I \beta(t)^T\left\{y(t)^T g_x - D\left(y(t)^T g_{\dot{x}}\right)\right\}dt+\int_I \beta(t)^T H\beta(t)dt=0 \quad (6.20)$$

In view of the hypothesis (A_3) i.e.,

$$\int_I\left\{\beta(t)^T y(t)^T g_x - D\left(y(t)^T g_{\dot{x}}\right)\right\}dt\geq 0$$

and $$\int_I \beta(t)^T H\beta(t)dt>0.$$

We have,

$$\int_I \beta(t)^T\left\{y(t)^T g_x - D\left(y(t)^T g_{\dot{x}}\right)+H\beta(t)\right\}dt>0$$

This contradicts (6.20). Hence $\beta(t)=0,\ t\in I$. Consequently (6.18) implies $\lambda(t)=0,\ t\in I$

From (6.10), we have ,

$$-\alpha(f_x-Df_{\dot{x}})+\gamma\left(y(t)^T g_x - Dy(t)^T g_{\dot{x}}\right)=0 \quad (6.21)$$

Also from (6.4), we have,

$$(f_x-Df_{\dot{x}})=-\left(y(t)^T g_x - D\left(y(t)^T g_{\dot{x}}\right)\right)$$

Using this in (6.21), we have,

$$(\alpha-\gamma)\left(y(t)^T g_x - Dy(t)^T g_{\dot{x}}\right)=0$$

In view of the hypothesis (A_2), this gives,

$$\alpha = \gamma > 0.$$

From (6.12) ,we have,

$$\gamma g_j + \mu_j(t) = 0$$

Because $\gamma > 0$, this gives,

$$g_j = -\frac{\mu_j(t)}{\gamma} \leq 0$$

$$g\left(t, \bar{x}, \dot{\bar{x}}\right) \leq 0$$

$\bar{x}$ is feasible to (CP). In view of $\beta(t) = 0,\ t \in I$ gives the equality of two objective values follows. The optimality of $\bar{x}$ for (CP) follows from Theorem 6.1.

6.4 Natural Boundary Values

In this section, we formulate dual variational problem with natural boundary values rather than fixed end points.

(CP_0): Minimize $\int_I f(t, x, \dot{x}) dt$

Subject to

$$g(t, x, \dot{x}) \leq 0, \qquad t \in I$$

(CD_0): Maximize $\int_I \left\{ F(t, x, \dot{x}) - \frac{1}{2} \beta(t)^T F \beta(t) \right\} dt$

Subject to

$$f_x + y(t)^T g_x - D\left(f_{\dot{x}} + y(t)^T g_{\dot{x}}\right) + (F + H)\beta(t) = 0 \ ,\ t \in I$$

$$y(t) \geq 0,\ \ t \in I$$

$$y(t)^T g_{\dot{x}}\Big|_{t=a} = 0,$$

$$y(t)^T g_{\dot{x}}\Big|_{t=b} = 0,$$

We shall not repeat the proofs of Theorem 6.1-6.3, as these follow on the lines of the analysis given in [11].

6.5 Nonlinear Programming

If all functions in (CP_0) and (CD_0) are independent of t, then these problems will reduce to following pair of dual problems, treated by Bector and Chandra [11].

(P_1): Minimize $f(x)$

Subject to

$$g(x) \leq 0,$$

(D_1): Maximize $f(x) - \frac{1}{2} p^T \nabla^2 f(x) p$

Subject to

$$\nabla\left(f + y^T g\right) + \nabla^2\left(f + y^T g\right) p = 0$$

$$y^T g(x) - \frac{1}{2} p^T \nabla^2\left(y^T g(x)\right) p \geq 0$$

$$y \geq 0$$

Where $f_x(x) = \nabla f(x),\ y^T g(x) = \nabla\left(y^T g\right),\ f_{xx}(x) = \nabla^2 f(x)$

$$\nabla^2\left(y^T g(x)\right) = \left(y^T g_x\right)_x \ \textit{and}\ \beta = p$$

7.0. INTRODUCTORY REMARKS

The purpose of this chapter is to study second-order duality for two classes of nondifferentiable continuous programming problem. This chapter comprises two sections 7.1 and 7.2 addressing second-order duality for one having nondifferentiability due square root of certain quadratic form and other containing support functions. The popularity of this type of problems seems to originate from the fact that, even though the objective function and or / constraint functions are non-smooth, a simple representation of the dual problem may be found. The theory of non-smooth mathematical programming deals with more general type of functions by means of generalized sub-differentials. However, square root of positive semi-definite quadratic forms and support functions are amongst few cases of the nondifferentiable functions for which one can write down the sub-or quasi-differentials explicitly. Here, various duality theorems for this pair of Wolfe type dual problems for which each class of problems are validated under second-order pseudoinvexity condition. A pair of Wolfe type dual variational problems with natural boundary values rather than fixed end points is presented and the proofs of its duality results are indicated. It is also shown that our second-order duality results can be considered as dynamic generalizations of corresponding (Static) second-order duality results established for nondifferentiable nonlinear programming problem, considered by Zhang and Mond [101].

7.1 SECOND-ORDER DUALITY FOR A CLASS OF NONDIFFERENTIABLE CONTINUOUS PROGRAMMING PROBLEMS

In this section, we formulate a Wolfe type second-order dual associated with a class of nondifferentiable continuous programming problems with square root of certain quadratic form appearing in the objective functional. Under the second-order pseudo-invexity, various duality theorems are validated for this pair of dual problems. A pair of dual problems with natural boundary values is constructed and the proofs of its various duality results are merely indicated. Further, it is shown that our results can be viewed as dynamic generalizations of corresponding (static) second –order duality theorems for a class of nondifferentiable nonlinear programming problems existing in the literature.

Consider the following class of nondifferentiable continuous programming problem studied in [19]:

(P^+): Minimize $\int_I \left\{ f\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B(t) x(t)\right)^{1/2} \right\} dt$

Subject to

$$x(a) = 0 = x(b),$$

$$g\left(t, x(t), \dot{x}(t)\right) \leq 0, \quad t \in I$$

$$h\left(t, x(t), \dot{x}(t)\right) = 0, \quad t \in I$$

where,

i) *f*, *g* and *h* are twice differentiable functions from $I \times R^n \times R^n$ into R, R^m *and* R^k respectively ,and

ii) $B(t)$ is a positive semi definite $n \times n$ matrix with $B(\cdot)$ continuous on I .

The following proposition gives the Fritz John optimality conditions which are derived by Chandra, Craven and Husain [19].

Proposition 7.1.1 (Fritz-John Conditions): If (P^+) attains a local minimum at $\bar{x} \in X$ and if $h_x(\cdot, \bar{x}(\cdot), \dot{\bar{x}}(\cdot))$ maps X onto a closed subspace of $C(I, R^p)$, then there exist Lagrange multipliers $\tau \in R_+$, piecewise smooth $\bar{y} : I \to R^m$ and $\bar{\lambda} : I \to R^k$,not all zero, and also piecewise smooth $\bar{z} : I \to R^n$ satisfying ,for all $t \in I$,

$$\begin{aligned}
&\tau f_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{z}(t)^T B(t) + \bar{y}(t)^T g_x(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{\mu}(t)^T h_x(t, \bar{x}(t), \dot{\bar{x}}(t)) \\
&= D\left[\tau f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{y}(t)^T g_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{\mu}(t)^T h_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))\right] \quad t \in I,
\end{aligned}$$

$$\bar{y}(t)^T g(t, \bar{x}(t), \dot{\bar{x}}(t)) = 0, \qquad t \in I$$

$$\bar{z}(t)^T B(t) \bar{z}(t) \leq 1, \quad t \in I$$

$$\bar{x}(t)^T B(t) \bar{z}(t) = \left(\bar{x}(t)^T B(t) \bar{x}(t)\right)^{1/2}, t \in I$$

If $h_x(\cdot, \bar{x}(\cdot), \dot{\bar{x}}(\cdot))$ is subjective, then τ and $\bar{y}$ are not both zero.

The following Schwartz inequality has been used in obtaining the above optimality conditions and will also be needed in the forthcoming analysis.

Lemma 7.1.1 (Schwartz inequality): It states that

$$x(t)^T B(t) z(t) \le \left(x(t)^T B(t) x(t)\right)^{1/2} \left(z(t)^T B(t) z(t)\right)^{1/2}, \ t \in I \quad (7.1)$$

with equality in (7.1) if (and only if)

$$B(t)\left(x(t) - q(t) z(t)\right) = 0 \text{ for some } q(t) \in R.$$

Remark 7.1.1: The Fritz John necessary optimality conditions in Proposition 7.1.1 for (P^+), become the Karush-Kuhn-Tucker type optimality conditions if $\tau = 1$. It suffices for $\tau = 1$, that the following Slater's condition holds:

$$g\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) + g_x\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) v(t) + g_{\dot{x}}\left(t, \bar{x}(t), \dot{\bar{x}}(t)\right) \dot{v}(t) < 0,$$

$$v(t) \in X \text{ and all } t \in I.$$

7.1.1 Second-Order Duality

Consider the following continuous programming problem (CP) by ignoring the equality constraint, $h\left(t, x(t), \dot{x}(t)\right) = 0, \ t \in I,$ in the problem (P^+):

(CP^+): Minimize $\int_I \left\{ f\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B(t) x(t)\right)^{1/2} \right\} dt$

Subject to

$$x(a) = 0 = x(b), \quad (7.2)$$

$$g\left(t, x(t), \dot{x}(t)\right) \le 0, \qquad t \in I \quad (7.3)$$

Analogously to the second-order dual problem introduced by Mangasarian [66] for a nonlinear programming problem, we consider the following second-order dual continuous programming problem (CD^+) for (CP^+).

(CD$^+$): Maximize $\int_I \Big\{ f\left(t,u(t),\dot{u}(t)\right) + u(t)^T B(t) z(t) + y(t)^T g\left(t,u(t),\dot{u}(t)\right)$

$$-\frac{1}{2} p(t)^T H(t) p(t) \Big\} dt$$

Subject to

$$u(a) = 0 = u(b) \tag{7.4}$$

$$\begin{aligned} &f\left(t,u(t),\dot{u}(t)\right) + u(t)^T B(t) z(t) + y(t)^T g\left(t,u(t),\dot{u}(t)\right) \\ &\quad -D\Big(f_{\dot{u}}\left(t,u(t),\dot{u}(t)\right) + y(t)^T g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)\Big) \\ &\quad +H(t) p(t) = 0\ ,\ t \in I \end{aligned} \tag{7.5}$$

$$z(t)^T B(t) z(t) \le 1, \quad t \in I, \tag{7.6}$$

$$y(t) \ge 0,\ \ t \in I, \tag{7.7}$$

where

$$\begin{aligned} H(t) = f_{uu}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u})\right)_u - 2D\left[f_{u\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u})\right)_{\dot{u}} \right] \\ + D^2\left[f_{u\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_{\dot{u}}(t,u,\dot{u})\right)_{\dot{u}} \right] \end{aligned}$$

Theorem 7.1.1 (Weak Duality): Let $x(t) \in X$ be a feasible solution of (CP$^+$) and $\left(u(t), y(t), z(t)\right)$ be a feasible solution of (CD$^+$). If $\int_I \Big\{ f(t,.,.) + (\cdot)^T B(t) z(t) + y(t)^T g(t,.,.) \Big\} dt$ is second-order pseudoinvex with respect to $\eta = \eta(t,x,u)$, then

inf. (CP$^+$) $\ge$ sup. (CD$^+$).

Proof: From (7.5), we have,

$$\begin{aligned} &\int_I \eta^T \Big\{ f_u\left(t,u(t),\dot{u}(t)\right) + B(t) z(t) + y(t)^T g_u\left(t,u(t),\dot{u}(t)\right) \\ &\qquad - D\Big(f_{\dot{u}}\left(t,u(t),\dot{u}(t)\right) + y(t)^T g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)\Big)\Big\} dt + \int_I \eta^T H(t) p(t) dt \end{aligned}$$

$$= \int_I \Big[\eta^T \Big\{ f_u\big(t,u(t),\dot{u}(t)\big) + B(t)z(t) + y(t)^T g_u\big(t,u(t),\dot{u}(t)\big)$$

$$+ (D\eta)^T \Big(f_{\dot{u}}\big(t,u(t),\dot{u}(t)\big) + y(t)^T g_{\dot{u}}\big(t,u(t),\dot{u}(t)\big)\Big) + \eta^T H(t)p(t) \Big\} \Big] dt$$

$$- \eta^T \Big(f_{\dot{u}}\big(t,u(t),\dot{u}(t)\big) + y(t)^T g_{\dot{u}}\big(t,u(t),\dot{u}(t)\big)\Big)\Big|_{t=a}^{t=b}$$

(by integration by part)

Using the boundary conditions (7.2) and (7.4), we have,

$$\int_I \Big[\eta^T \Big\{ f_u\big(t,u(t),\dot{u}(t)\big) + B(t)z(t) + y(t)^T g_u\big(t,u(t),\dot{u}(t)\big)$$

$$+ (D\eta)^T \Big(f_{\dot{u}}\big(t,u(t),\dot{u}(t)\big) + y(t)^T g_{\dot{u}}\big(t,u(t),\dot{u}(t)\big)\Big) + \eta^T H(t)p(t) \Big\} \Big] dt = 0$$

This, in view of second-order pseudoinvexity of $\int_I \big\{ f(t,.,.) + (\cdot)^T B(t)z(t) + y(t)^T g(t,.,.) \big\} dt$, yields,

$$\int_I \big\{ f(t,x,\dot{x}) + x(t)^T B(t)z(t) + y(t)^T g(t,x,\dot{x}) \big\} dt$$

$$\geq \int_I \Big\{ f(t,u,\dot{u}) + u(t)^T B(t)z(t) + y(t)^T g(t,u,\dot{u}) - \frac{1}{2} p(t)^T H(t)p(t) \Big\} dt$$

Because of Schwartz inequality (7.1) along with (7.5), (7.6) and (7.2), this implies,

$$\int_I \Big\{ f\big(t,x(t),\dot{x}(t)\big) + \big(x(t)^T B(t)x(t)\big)^{1/2} \Big\} dt$$

$$\geq \int_I \Big\{ f(t,u,\dot{u}) + u(t)^T B(t)z(t) + y(t)^T g(t,u,\dot{u}) - \frac{1}{2} p(t)^T H(t)p(t) \Big\} dt,$$

yielding,

$$\inf(\mathrm{CP}^+) \geq \sup(\mathrm{CD}^+).$$

Theorem 7.1.2 (Strong Duality): If $\bar{x}(t) \in X$ is an optimal solution of (CP$^+$) and is also normal, then there exist piecewise smooth function $y: I \to R^m$ and $z: I \to R^n$ such that $\big(\bar{x}(t), \bar{y}(t), \bar{z}(t), p(t) = 0\big)$ is a feasible solution of (CD$^+$) and the two objective values are equal. Furthermore, if the hypotheses of Theorem 7.1.1 hold, then $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),p(t)\right)$ is an optimal of (CD$^+$).

***Proof*:** From Proposition 7.1.1, there exist a piecewise smooth function $\bar{y}:I\to R^m$ and $\bar{z}:I\to R^n$ such that

$$\left(f_x\left(t,\bar{x},\dot{\bar{x}}\right)+B(t)\bar{z}(t)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)\right)$$
$$-D\left(f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{y}(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right)=0,\quad t\in I$$

$$\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)=0,\qquad t\in I$$

$$\bar{x}(t)^T B(t)\bar{z}(t)=\left(\bar{x}(t)^T B(t)x(t)\right)^{1/2},\quad t\in I$$

$$\left(\bar{z}(t)^T B(t)\bar{z}(t)\right)\le 1,\qquad t\in I,$$

$$y(t)\ge 0,\qquad t\in I$$

Hence $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{p}(t)=0\right)$ satisfies the constraints of (CD$^+$) and the objective values are equal. Furthermore, for every feasible solution $\left(u(t),y(t),z(t),p(t)\right)$, from the above conditions we have,

$$\int_I\left\{f\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{x}(t)^T B(t)\bar{z}(t)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)-\frac{1}{2}\bar{p}(t)^T H(t)\bar{p}(t)\right\}dt$$
$$=\int_I\left\{f\left(t,\bar{x},\dot{\bar{x}}\right)+\left(\bar{x}(t)^T B(t)\bar{x}(t)\right)^{\frac{1}{2}}\right\}dt$$

Using

$$\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)=0\ ,\bar{x}(t)^T B(t)\bar{z}(t)=\left(\bar{x}(t)^T B(t)x(t)\right)^{1/2}\ \ and\ \ \bar{p}(t)=0,t\in I\,)$$
$$\ge\int_I\left\{f\left(t,u,\dot{u}\right)+u(t)^T B(t)z(t)+y(t)^T g\left(t,u,\dot{u}\right)-\frac{1}{2}p(t)^T H(t)p(t)\right\}dt$$

So, $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{p}(t)\right)$is an optimal solution of (CD+).

Theorem 7.1.3 (Converse Duality): Assume that *f and* g are thrice continuously differentiable and $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{p}(t)\right)$ be an optimal solution of (CD+) .Let the following conditions hold:

(i): The Hessian matrix H(t) is non-singular, and

(ii): $\left(\psi(t)^T H(t)\psi(t)\right)_x - D\left(\psi(t)^T H(t)\psi(t)\right)_{\dot{x}} + 2\psi(t)^T D\left(H(t)\psi(t)\right)_{\dot{x}} = 0$

$\Rightarrow \psi(t) = 0, \quad t \in I$

Then $x(t)$is feasible for (CP+), $\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right) = 0$,$t \in I$. In addition, if the hypotheses in Theorem 7.1.1 hold, then $\bar{x}(t)$ is an optimal solution.

Proof: Since $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{p}(t)\right)$ is an optimal solution for (CD+), by Proposition 7.1.1, there exist Lagrange multiplier$\tau \in R$, and piecewise smooth $\theta : I \to R^n$, $\mu : I \to R^m$ and $\alpha : I \to R^n$ such that following conditions hold at the feasible point of (CD+).

$$\tau\Big[\left(f_x + B(t)\bar{z}(t) + \bar{y}(t)g_x\right) - \frac{1}{2}\left(p(t)^T B(t)p(t)\right)_x$$
$$-D\left\{f_{\dot{x}} + \bar{y}(t)g_{\dot{x}} - \frac{1}{2}\left(p(t)^T B(t)p(t)\right)_{\dot{x}}\right\}\Big]$$
$$+\theta(t)^T\Big[f_{xx} + \left(y(t)^T g_x\right)_x - D\left(f_{\dot{x}x} + \left(y(t)^T g_{\dot{x}}\right)_x\right) + \left(H(t)p(t)\right)_x$$
$$-D\left\{\left(f_{x\dot{x}} + \left(y(t)^T g_x\right)_{\dot{x}}\right) - D\left(f_{\dot{x}\dot{x}} + \left(y(t)^T g_{\dot{x}}\right)_{\dot{x}}\right) + \left(H(t)p(t)\right)_{\dot{x}}\right\}\Big] = 0,\ t \in I \tag{7.8}$$

$$\tau\left(g^j - \frac{1}{2}p(t)^T g^j_{xx} p(t)\right) + \theta(t)^T\left(g^j_{xx} - 2Dg^j_{x\dot{x}} + D^2 g^j_{\dot{x}\dot{x}}\right)p(t)$$
$$+\mu^j(t) = 0, t \in I,\ j = 1(1)m \tag{7.9}$$

$$\tau\bar{x}(t)^T B(t)+\theta(t)B(t)-2\alpha(t)B(t)z(t)=0 \tag{7.10}$$

$$\left(\theta(t)-\tau p(t)\right)H(t)=0, \qquad t\in I \tag{7.11}$$

$$f_x+B(t)z(t)+\bar{y}(t)^T g_x-D\left(f_{\dot{x}}+\bar{y}(t)g_{\dot{x}}\right)+H(t)p(t)=0, \quad t\in I \tag{7.12}$$

$$\alpha(t)\left(1-\bar{z}(t)^T B(t)\bar{z}(t)\right)=0, \qquad t\in I \tag{7.13}$$

$$\bar{\mu}(t)^T\bar{y}(t)=0, \quad t\in I \tag{7.14}$$

$$\left(\tau,\alpha(t),\mu(t)\right)\geq 0, \qquad t\in I \tag{7.15}$$

$$\left(\tau,\alpha(t),\mu(t),\theta(t)\right)\neq 0 \qquad t\in I \tag{7.16}$$

By singularity of H (t), (7.11) yields,

$$\theta(t)+\tau\bar{p}(t)=0, \qquad t\in I \tag{7.17}$$

If $\tau=0$, (7.17) implies $\theta(t)=0,\ t\in I$. From (7.9), we have $\mu(t)=0,\ t\in I$. The relation (7.10) together with (7.13) gives $\alpha(t)=0.$ Hence $\left(\tau,\alpha(t),\theta(t),\mu(t)\right)=0,\ t\in I$, contradicting (7.16). Consequently $\tau>0$. From (7.17) and $\tau>0$, (7.8) becomes,

$$\begin{aligned}
&\left(f_x+B(t)\bar{z}(t)+\bar{y}(t)g_x\right)-\frac{1}{2}\left(\bar{p}(t)^T B(t)\bar{p}(t)\right)_x \\
&-D\left(f_{\dot{x}}+\bar{y}(t)g_{\dot{x}}-\frac{1}{2}\left(\bar{p}(t)^T B(t)\bar{p}(t)\right)_{\dot{x}}\right) \\
&+\Big[f_{xx}+\left(\bar{y}(t)^T g_x\right)_x-D\left(f_{\dot{x}x}+\left(\bar{y}(t)^T g_{\dot{x}}\right)_x\right)+\left(H(t)\bar{p}(t)\right)_x \\
&-D\left\{\left(f_{x\dot{x}}+\left(\bar{y}(t)^T g_x\right)_{\dot{x}}\right)-D\left(f_{\dot{x}\dot{x}}+\left(\bar{y}(t)^T g_{\dot{x}}\right)_{\dot{x}}\right)+\left(H(t)\bar{p}(t)\right)_{\dot{x}}\right\}\Big]\bar{p}(t)=0, \quad t\in I
\end{aligned}$$

Using the expression of H (t) and (7.12), this gives,

$$\bar{p}(t)^T H(t)\bar{p}(t)+D\left(\bar{p}(t)^T H(t)\bar{p}(t)\right)_{\dot{x}}-2\bar{p}(t)^T D\left(H(t)\bar{p}(t)\right)_{\dot{x}}=0,\ t\in I,$$

which, because of the hypothesis (ii) implies $\bar{p}(t)=0,\ t\in I$. From (7.9), we have,

$$\tau g^{j}+\mu^{j}(t)=0,\ t\in I\ ,\ j=1,2...m \tag{7.18}$$

This, because of $\tau>0$, yields,

$$g^{j}\left(t,\bar{x},\dot{\bar{x}}\right)\leq 0\ ,t\in I$$

The relation (7.18) along with (7.14) and $\tau>0$ gives,

$$\bar{y}(t)^{T} g\left(t,\bar{x},\dot{\bar{x}}\right)=0\ ,t\in I \tag{7.19}$$

Using $\theta(t)=0,\ t\in I$ and $\tau>0$, (7.10) yields,

$$B(t)\bar{x}(t)^{T}=2\left(\frac{\alpha(t)}{\tau}\right)B(t)\bar{z}(t),t\in I, \tag{7.20}$$

Which is the required condition for the equality in Schwartz inequality, i.e.,

$$\bar{x}(t)^{T} B(t)\bar{z}(t)=\left(\bar{x}(t)^{T} B(t)\bar{x}(t)\right)^{\frac{1}{2}}\left(\bar{z}(t)^{T} B(t)\bar{z}(t)\right)^{\frac{1}{2}} \tag{7.21}$$

If $\alpha(t)>0,\ t\in I$, (7.13) gives, $\bar{z}(t)^{T} B(t)\bar{z}(t)=1$, and so (7.20) implies,

$$\bar{x}(t)^{T} B(t)\bar{z}(t)=\left(\bar{x}(t)^{T} B(t)\bar{x}(t)\right)^{\frac{1}{2}},t\in I$$

If $\alpha(t)=0,\ t\in I$, (7.20) implies, $B(t)\bar{x}(t)=0,t\in I$. So we still get

$$\bar{x}(t)^{T} B(t)\bar{z}(t)=\left(\bar{x}(t)^{T} B(t)\bar{x}(t)\right)^{\frac{1}{2}},t\in I \tag{7.22}$$

Therefore, from (7.19), (7.22) and $\bar{p}(t)=0,\ t\in I$, we have

$$\int_I\left\{f\left(t,\bar{x},\dot{\bar{x}}\right)+\left(\bar{x}(t)^{T} B(t)\bar{x}(t)\right)^{\frac{1}{2}}\right\}dt=$$
$$\int_I\left\{f\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{x}(t)^{T} B(t)\bar{z}(t)+\bar{y}(t)^{T} g\left(t,\bar{x},\dot{\bar{x}}\right)-\frac{1}{2}\bar{p}(t)^{T} H(t)\bar{p}(t)\right\}dt$$

This, by the application of Theorem 7.1.1 yields the optimality of $\bar{x}(t)$ for (CP^{+}).

7.1.2. Natural Boundary Values

In this section, we formulate a pair of nondifferentiable dual variational problems with natural boundary values rather than fixed end points:

(CP$_0$): Minimize $\int_I \left\{ f(t,x,\dot{x}) + \left(\bar{x}(t)^T B(t) \bar{x}(t)\right)^{\frac{1}{2}} \right\} dt$

Subject to

$$g(t,x,\dot{x}) \leq 0, \qquad t \in I \ .$$

(CD$_0$): Maximize $\int_I \left\{ f(t,x(t),\dot{x}(t)) + x(t)^T B(t) z(t) + y(t)^T g(t,x(t),\dot{x}(t)) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt$

Subject to

$$\left(f_x(t,x,\dot{x}) + B(t)\bar{z}(t) + \bar{y}(t)^T g_x(t,x,\dot{x}) \right)$$
$$-D\left(f_{\dot{x}}(t,x,\dot{x}) + \bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x}) \right) + H(t)p(t) = 0, \quad t \in I$$

$$z(t)^T B(t) z(t) \leq 1, \qquad t \in I$$

$$y(t) \geq 0, \qquad t \in I$$

$$f_{\dot{x}}(t,x,\dot{x}) + \bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x})\Big|_{t=a} = 0,$$

$$f_{\dot{x}}(t,x,\dot{x}) + \bar{y}(t)^T g_{\dot{x}}(t,x,\dot{x})\Big|_{t=b} = 0,$$

We shall not repeat the proofs of Theorem 7.1.1-7.1.3, as these follow on the lines of the analysis of the preceding section with slight modifications.

7.1.3. Non-differentiable Nonlinear Programming Problems

If all functions in the problems (CP_0) and (CD_0) are independent of t and $b-a=1$, then these problems will reduce to following nondifferentiable dual variational problems, treated by Zhang and Mond [101].

(NP): Minimize $\quad f(x)+\left(x^T Bx\right)^{1/2}$

Subject to

$$g(x)\le 0,$$

(ND): Maximize $\quad f(x)+x^T Bz+y^T g(x)-\frac{1}{2}p^T\nabla^2\left(f(x)+y^T g(x)\right)p$

Subject to

$$\nabla\left(f(x)+x^T Bz+y^T g(x)\right)+\nabla^2\left(f(x)+y^T g(x)\right)p=0$$

$$z^T Bz\le 1, y\ge 0$$

where $\quad \nabla\left(f(x)+x^T Bz+y^T g(x)\right)=f_x(x)+Bz+y^T g_x(x)$

and $\quad \nabla^2\left(f(x)+y^T g(x)\right)=f_{xx}(x)+\left(y^T g_x(x)\right)_{\dot{x}}$

7.2 SECOND-ORDER DUALITY FOR CONTINUOUS PROGRAMMING CONTAINING SUPPORT FUNCTIONS

In this section, a second-order dual problem is formulated for a more general class of continuous programming problem in which both objective and constrained function contain support functions, hence it is nondifferentiable. Under second-order invexity and second-order pseudoinvexity, weak, strong and converse duality theorems are established for this pair of dual problems. Special cases are deduced and a pair of dual continuous problems with natural boundary values is constructed. A close relationship between duality results of our problems and those of the corresponding (static) nonlinear programming problem with support functions is briefly outlined.

7.2.1. Pre-requisites

Consider the following nondifferentiable continuous programming problem with support functions of Husain and Jabeen [52]:

(CP+): Minimize $\int_I \left\{ f\left(t,x,\dot{x}\right) + S\left(x(t) \mid K\right) \right\} dt$

Subject to

$$x(a) = 0 = x(b), \qquad (7.23)$$

$$g^j\left(t,x,\dot{x}\right) + S\left(x(t) \mid C^j\right) \leq 0,\ j = 1,2...m, \quad t \in I, \qquad (7.24)$$

where, f and g are continuously differentiable and each C^j ,(j=1,2….m) is a compact convex set in R^n .Husain and Zamrooda [52] derived the following optimality conditions for (CP+):

Lemma 7.2.1 (Fritz-John Necessary optimality Conditions): If the problem (CP+) attains a minimum at $x = \bar{x} \in X$, there exist $r \in R$ and piecewise smooth function $\bar{y} : I \to R^m$ with $\bar{y}(t) = \left(\bar{y}^1(t), \bar{y}^2(t), ... \bar{y}^m(t)\right)$, $\bar{z} : I \to R^n$ and $w^j : I \to R^n,\ j = 1,2...m$,such that

$$r\left[f_x\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{z}(t)\right] + \sum_{j=1}^{m} \bar{y}^j(t)\left[g^j\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{w}^j(t)\right]$$
$$= D\left[rf_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{y}(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right], \quad t \in I$$

$$\sum_{j=1}^{m} \bar{y}^j(t)\left[g^j\left(t,\bar{x},\dot{\bar{x}}\right) + \bar{x}(t)^T \bar{w}^j(t)\right] = 0 \ ,t \in I$$

$$\bar{x}(t)^T \bar{z}(t) = S(\bar{x}(t) \mid K), \quad t \in I$$

$$\bar{x}(t)^T \bar{w}^j(t) = S(\bar{x}(t) \mid C^j), \ j = 1, 2 \ldots m, \ t \in I$$

$$\bar{z}(t) \in K, w^j(t) \in C^j, \ j = 1, 2 \ldots m, \ t \in I$$

$$(r, \bar{y}(t)) \geq 0, \quad t \in I$$

$$(r, \bar{y}(t)) \neq 0, \quad t \in I$$

The minimum $\bar{x}$ of (CP_+) may be described as normal if $\bar{r} = 1$ so that the Fritz John optimality conditions reduce to Karush-Kuhn-Tucker optimality conditions. It suffices for $\bar{r} = 1$ that Slater's condition holds at $\bar{x}$.

Now we review some well known facts about a support function for easy reference.

Let Γ be a compact set in R^n, then the support function of Γ is defined by

$$S(x(t)|\Gamma) = \max\left\{x(t)^T v(t) : v(t) \in \Gamma, t \in I\right\}$$

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis i.e., there exist $z(t) \in R^n$, $t \in I$, such that

$$S(y(t)|\Gamma) - S(x(t)|\Gamma) \geq (y(t) - x(t))^T z(t)$$

From [81], the subdifferential of $S(x(t)|\Gamma)$ is given by

$$\partial S(x(t)|\Gamma) = \left\{z(t) \in \Gamma, \ t \in I \text{ such that} \left| x(t)^T z(t) = S(x(t)|\Gamma) \right.\right\}.$$

For any set $A \subset R^n$, the normal cone to A at a point $x(t) \in A$ is defined by

$$N_A(x(t)) = \left\{ y(t) \in R^n \middle| y(t)\left(z(t) - x(t)\right) \le 0, \forall z(t) \in A \right\}$$

It can be verified that for a compact convex set B, $y(t) \in N_B(x(t))$ if and only if

$$S\left(y(t)\middle| B\right) = x^T(t)\,y(t)\ ,\ t \in I$$

7.2.2 Second-Order Duality

The following problem is formulated as Wolfe type dual for the Problem (CP_+):

(CD_+): Maximize
$$\int_I \left\{ f(t,u,\dot{u}) + \bar{u}(t)^T z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) - \frac{1}{2} p(t)^T H(t)\, p(t) \right\} dt$$

Subject to

$$u(a) = 0 = u(b) \tag{7.25}$$

$$f_u(t,u,\dot{u}) + z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g_u^j(t,u,\dot{u}) + w^j(t) \right) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) + H(t)\,p(t) = 0\ ,\ t \in I \tag{7.26}$$

$$z(t) \in K, w^j(t) \in C^j, t \in I, j = 1,2...m. \tag{7.27}$$

$$y(t) \ge 0, t \in I. \tag{7.28}$$

If $p(t) = 0,\ t \in I$, the above dual becomes the dual of the problem studied in [52].

***Theorem 7.2.1 (Weak Duality)*:** Let $x(t) \in X$ be a feasible solution of (CP$_+$) and $\left(u(t), y(t), z(t), w^1(t), w^2(t), ..., w^m(t), p(t)\right)$ be feasible solution for (CD$_+$). If for all feasible $\left(x(t), u(t), y(t), z(t), w^1(t), w^2(t), ..., w^m(t), p(t)\right)$ and with respect to $\eta = \eta(t, x, u)$

i) $\int_I \left\{ f(t,.,.) + (\cdot)^T z(t) \right\} dt$ and $\sum_{j=1}^{m} \int_I \left\{ y^j(t) \left(g^j(t,\cdot,\cdot) + (\cdot) w^j(t) \right) \right\} dt$ second-order invex .

or

ii) $\int_I \left\{ f(t,.,.) + (\cdot)^T z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,\cdot,\cdot) + (\cdot) w^j(t) \right) \right\} dt$ is second-order pseudoinvex .

then

inf (CP$_+$) $\geq$ sup (CD$_+$).

Proof:(i) $\int_I \left\{ f(t, x, \dot{x}) + S\left(x(t) \mid K\right) \right\} dt$

$$-\int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt$$

$$\geq \int_I \left\{ f(t,x,\dot{x}) + x(t)^T z(t) \right\} dt - \int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) \right\} dt$$
$$- \sum_{j=1}^{m} \int_I y^j(t) \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) dt + \int_I \frac{1}{2} p(t)^T H(t) p(t) dt$$

$$\geq \int_I \left[\eta^T \left\{ f_u(t,u,\dot{u}) + z(t) + + (D\eta)^T f_{\dot{u}}(t,u,\dot{u}) + \eta^T F(t) p(t) \right\} dt \right]$$
$$- \int_I p(t)^T F(t) p(t) dt - \sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) dt$$
$$+ \int_I \frac{1}{2} p(t)^T H(t) p(t) dt,$$

where $F(t) = f_{xx} - 2Df_{x\dot{x}} + D^2 f_{\dot{x}\dot{x}}$ and using the second-order invexity of $\int_I \left\{ f(t,.,.) + (\cdot)^T z(t) \right\} dt$

$$= \int_I \left[\eta^T \left\{ f_u(t,u,\dot{u}) + z(t) + +Df_{\dot{u}}(t,u,\dot{u}) + F(t)\,p(t) \right\} dt \right] + \eta^T f_{\dot{u}}(t,u,\dot{u}) \Big|_{t=a}^{t=b}$$

$$-\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) dt - \int_I \frac{1}{2} p(t)^T F(t)\,p(t) dt + \int_I \frac{1}{2} p(t)^T H(t)\,p(t) dt$$

(by Integrating by parts)

$$= -\int_I \eta^T \left[\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j_{\dot{u}}(t,u,\dot{u}) + w^j(t) \right) - D\left(y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) - G(t)\,p(t) \right] dt$$

$$-\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) dt$$

$$-\int_I \frac{1}{2} p(t)^T F(t)\,p(t) dt + \int_I \frac{1}{2} p(t)^T H(t)\,p(t) dt$$

$$= -\int_I \eta^T \left[\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j_{\dot{u}}(t,u,\dot{u}) + w^j(t) \right) + (D\eta)^T y(t)^T g_{\dot{u}} + G(t)\,p(t) G(t)\,p(t) \right] dt$$

$$-\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t) \right) dt$$

$$-\int_I \frac{1}{2} p(t)^T F(t)\,p(t) dt + \int_I \frac{1}{2} p(t)^T H(t)\,p(t) dt$$

$$\geq -\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,x,\dot{x}) + u(t)^T w^j(t) \right) dt - \int_I \frac{1}{2} p(t)^T G(t)\,p(t) dt$$

$$-\int_I \frac{1}{2} p(t)^T F(t)\,p(t) dt + \int_I \frac{1}{2} p(t)^T H(t)\,p(t) dt$$

$$\geq -\sum_{j=1}^{m} \int_I y^j(t)^T \left(g^j(t,x,\dot{x}) + S\left(x(t) \mid C^j \right) \right) dt \geq 0$$

This implies ,

$$\int_I \left(f\left(t,x,\dot{x}\right) + S\left(x\left(t\right) \mid K\right)\right) dt$$

$$\geq \int_I \left\{ f\left(t,u,\dot{u}\right) + u\left(t\right)^T z\left(t\right) + \sum_{j=1}^{m} y^j\left(t\right)\left(g^j\left(t,x,\dot{x}\right) + u\left(t\right)^T w^j\left(t\right)\right) - \frac{1}{2} p\left(t\right)^T H\left(t\right) p\left(t\right)\right\} dt$$

yielding,

inf (CP_+) ≥ sup (CD_+).

(ii) From (7.26), we have

$$0 = \int_I \Big[\eta^T \Big\{ f_u\left(t,u,\dot{u}\right) + z\left(t\right) + \sum_{j=1}^{m} y^j\left(t\right)^T \left(g^j\left(t,u,\dot{u}\right) + u\left(t\right)^T w^j\left(t\right)\right) - D\left(f_{\dot{u}}\left(t,u,\dot{u}\right) + y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u}\right)\right) + H\left(t\right) p\left(t\right)\Big\} dt\Big]$$

$$= \int_I \Big[\eta^T \Big\{ f_u\left(t,u,\dot{u}\right) + z\left(t\right) + \sum_{j=1}^{m} y^j\left(t\right)^T \left(g_u^j\left(t,u,\dot{u}\right) + w^j\left(t\right)\right) + \left(D\eta\right)^T \left(f_{\dot{u}}\left(t,u,\dot{u}\right) + y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u}\right)\right) + \eta^T H\left(t\right) p\left(t\right)\Big\} dt - \eta^T \left(f_{\dot{u}} + y g_{\dot{u}}\right)\Big|_{t=a}^{t=b}\Big]$$

(by Integrating by parts)

Using boundary conditions (7.23) and (7.25)

$$\int_I \Big[\eta^T \Big\{ f_u\left(t,u,\dot{u}\right) + z\left(t\right) + \sum_{j=1}^{m} y^j\left(t\right)\left(g_u^j\left(t,u,\dot{u}\right) + w^j\left(t\right)\right)\Big\} + \left(D\eta\right)^T \left(f_{\dot{u}}\left(t,u,\dot{u}\right) + y\left(t\right)^T g_{\dot{u}}\left(t,u,\dot{u}\right)\right) + \eta^T H\left(t\right) p\left(t\right)\Big] dt = 0$$

This, in view of second-order pseudo-invexity of

$$\int_I \left\{ f\left(t,.,.\right) + \left(\cdot\right)^T z\left(t\right) + \sum_{j=1}^{m} y^j\left(t\right)^T \left(g^j\left(t,\cdot,\cdot\right) + \left(\cdot\right) w^j\left(t\right)\right)\right\} dt$$

yields,

$$\int_I \left\{ f(t,x,\dot{x}) + x(t)^T z(t) + \sum_{j=1}^{m} y^j(t)\left(g^j(t,x,\dot{x}) + x(t)^T w^j(t)\right) \right\} dt$$

$$\geq \int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j=1}^{m} y^j(t)\left(g^j(t,u,\dot{u}) + u(t)^T w^j(t)\right) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt$$

$$\Rightarrow \int_I \left\{ f(t,x,\dot{x}) + S\left(x(t) \mid K\right) + \sum_{j=1}^{m} y^j(t)\left(g^j(t,x,\dot{x}) + S\left(x(t) \mid C^j\right)\right) \right\} dt$$

$$\geq \int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,u,\dot{u}) + u(t)^T w^j(t)\right) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt$$

Using (7.24) and (7.28) together with $x(t)^T z(t) \leq S\left(x(t) \mid K\right)$ and $x(t)^T w^j(t) \leq S\left(x(t) \mid C^j\right), t \in I, j = 1,2,...m$

This gives,

$$\int_I \left\{ f(t,x,\dot{x}) + S\left(x(t) \mid K\right) \right\} dt$$

$$\geq \int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j=1}^{m} y^j(t)\left(g^j(t,u,\dot{u}) + u(t)^T w^j(t)\right) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt.$$

That is,

inf (CP_+) $\geq$ sup (CD_+).

***Theorem 7.2.2 (Strong Duality)*:** If $\bar{x}(t) \in X$ is a local (or global) optimal solution of (CP_+) and is also normal, then there exist piece wise smooth factor $y : I \to R^m$, $z : I \to R^n$ and $w^j : I \to R^n (j = 1,2,...m)$ such that $\left(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{w}^1(t), \bar{w}^2(t)..., \bar{w}^m(t), p(t) = 0\right)$ is a feasible solution of (CD₊) and the two objective values are equal. Furthermore, hypotheses of Theorem 7.2.1 hold, then $\left(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{w}^1(t), \bar{w}^2(t) ..., \bar{w}^m(t), p(t)\right)$ is an optimal solution of (CD₊).

***Proof*:** From Lemma 7.2.1, there exist piecewise smooth function $y : I \to R^m$, $z : I \to R^n$ and $w^j : I \to R^n (j = 1, 2, ...m)$ satisfying $f_x\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{z}(t)$

$$+\sum_{j=i}^{m} \bar{y}^j(t)^T \left(g_x^j\left(t, \bar{x}, \dot{\bar{x}}\right) + w^j\right) - D\left(f_{\dot{x}}\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{y}(t)^T g_{\dot{x}}\left(t, \bar{x}, \dot{\bar{x}}\right)\right) = 0, \quad t \in I$$

$$\sum_{j=i}^{m} \bar{y}^j(t)^T \left(g_x^j\left(t, \bar{x}, \dot{\bar{x}}\right) + w^j\right) = 0, \ t \in I$$

$$\bar{x}(t)^T \bar{z}(t) = S\left(\bar{x}(t) \mid K\right), \quad t \in I$$

$$\bar{x}(t)^T \bar{w}^j(t) = S\left(\bar{x}(t) \mid C^j\right), \ j = 1, 2...m, \ t \in I$$

$$\bar{z}(t) \in K, w^j(t) \in C^j, \ j = 1, 2...m, \ t \in I$$

$$\bar{y}(t) \geq 0, \quad t \in I$$

Hence $\left(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{w}^1(t), \bar{w}^2(t) ..., \bar{w}^m(t), p(t) = 0\right)$ satisfies the constraints of (CD₊) and

$$\int_I \left\{ f\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{u}(t)^T \bar{z}(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{x}(t)^T \bar{w}^j(t)\right) \right.$$

$$\left. - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt = \int_I \left\{ f\left(t, \bar{x}, \dot{\bar{x}}\right) + S\left(x(t) / K\right) \right\} dt \quad (7.29)$$

That is, the objective values are equal. Furthermore, for every feasible solution, we have from (7.29)

$$\int_I \left\{ f\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{u}(t)^T \bar{z}(t)+\sum_{j=1}^{m} y^j(t)^T \left(g^j\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{x}(t)^T \bar{w}^j(t)\right) -\frac{1}{2}p(t)^T H(t)\,p(t)\right\}dt$$

$$\geq \int_I \left\{ f\left(t,u,\dot{u}\right)+u(t)^T z(t)+\sum_{j=1}^{m} y^j(t)^T \left(g^j\left(t,u,\dot{u}\right)+u(t)^T w^j(t)\right) -\frac{1}{2}p(t)^T H(t)\,p(t)\right\}dt$$

So, $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{w}^1(t),\bar{w}^2(t)...,\bar{w}^m(t)\right)$ is optimal for (CD_+).

Theorem 7.2.3 (Converse Duality)**:** Let *f and g* are thrice continuously differentiable and $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{w}^1(t),\bar{w}^2(t)...,\bar{w}^m(t),p(t)\right)$ be an optimal solution of (CD_+) .If the following conditions hold:

(A_1): The Hessian matrix H(t) is non-singular, and

(A_2):

$$\left(\psi(t)^T H(t)\psi(t)\right)_x - D\left(\psi(t)^T H(t)\psi(t)\right)_{\dot{x}} +2\psi(t)D\left(H(t)\psi(t)\right)_{\dot{x}}=0, t\in I$$

$$\Rightarrow \psi(t)=0, \quad t\in I$$

Then $\bar{x}(t)$ is feasible solution of (CP_+), then $\sum_{j=1}^{m}\bar{y}^j(t)\left(g^j\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{x}(t)^T \bar{w}^j(t)\right)=0, \quad t\in I$. In addition, if the hypotheses in Theorem 7.2.1 hold, then $\bar{x}(t)$ is an optimal solution of (CP_+).

Proof: Since $\left(\bar{x}(t),\bar{y}(t),\bar{z}(t),\bar{w}^1(t),\bar{w}^2(t)...,\bar{w}^m(t),p(t)\right)$ is an optimal solution for (CD_+), then there exist piece wise smooth $\theta: I\to R^n$ and $\mu: I\to R^m$ such that following conditions [81] are satisfied.

$$\tau\left[\left(f_x\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{z}(t)+\sum_{j=1}^{m}\bar{y}^j(t)\left(g_x\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{w}^j(t)\right)\right)-\frac{1}{2}\left(p(t)^T H(t)p(t)\right)_x\right.$$

$$\left.-D\left(f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)-\frac{1}{2}\left(p(t)^T H(t)p(t)\right)_{\dot{x}}\right)\right]$$

$$+\theta(t)^T\left\{f_{xx}\left(t,\bar{x},\dot{\bar{x}}\right)+\left(y(t)^T g_x\right)_x\right.$$

$$-D\left(f_{\dot{x}x}\left(t,\bar{x},\dot{\bar{x}}\right)+\left(y(t)^T g_{\dot{x}}\right)_x\right)$$

$$+\left(H(t)p(t)\right)_x-D\left(f_{x\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+\left(y(t)^T g_x\right)_{\dot{x}}\right)$$

$$\left.-D\left(f_{\dot{x}\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+\left(y(t)^T g_{\dot{x}}\right)_{\dot{x}}\right)+\left(H(t)p(t)\right)_{\dot{x}}\right\}=0,$$

$$t\in I \qquad (7.30)$$

$$\tau\left(g^j\left(t,\bar{x},\dot{\bar{x}}\right)+\left(\bar{x}(t)\bar{w}^j(t)\right)+\frac{1}{2}p(t)^T g^j_{xx}p(t)\right)$$

$$+\theta(t)^T\left(g^j_{xx}-2Dg^j_{x\dot{x}}+D^2g^j_{\dot{x}\dot{x}}\right)p(t)+\mu^j(t)=0, t\in I, I=1,2,\ldots m \qquad (7.31)$$

$$\left(f_x\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{z}(t)+\sum_{j=1}^{m}\bar{y}^j(t)\left(g^j\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{w}^j(t)\right)\right)$$

$$-D\left(f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{y}(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right)+\left(H(t)\bar{p}(t)\right)=0, \quad t\in I \qquad (7.32)$$

$$\tau\bar{x}(t)^T+\theta(t)\in N_K\left(z(t)\right) \qquad (7.33)$$

$$\tau\bar{x}(t)^T y^j(t)+\theta(t)y^j(t)\in N_{C_j}\left(w^j(t)\right), j=1,2\ldots m \qquad (7.34)$$

$$\left(\theta(t)-\tau\bar{p}(t)\right)H(t)=0, \qquad t\in I \qquad (7.35)$$

$$\bar{\mu}(t)^T\bar{y}(t)=0, \quad t\in I \qquad (7.36)$$

$$\left(\tau,\mu(t)\right)\geq 0, \qquad t\in I \qquad (7.37)$$

$$\left(\tau,\mu(t),\theta(t)\right)\neq 0 \qquad t\in I \qquad (7.38)$$

By the singularity of H (t), (7.35) implies,

$$\theta(t)+\tau\bar{p}(t)=0, \qquad t\in I \tag{7.39}$$

If $\tau=0$, then $\theta(t)=0,\ t\in I$ and so $\bar{\mu}(t)=0,\ t\in I$. This contradicts (7.38), Hence $\tau>0$.

$$\begin{aligned} & f_x+z(t)+\sum_{j=1}^{m}\bar{y}^j(t)\left(g_x^j+w^j(t)\right)-\frac{1}{2}\left(p(t)^T H(t)p(t)\right)_x \\ & -D\left(f_{\dot{x}}+\bar{y}(t)^T g_{\dot{x}}-\frac{1}{2}\left(\bar{p}(t)^T B(t)\bar{p}(t)\right)_{\dot{x}}\right) \\ & +\theta(t)^T\left\{f_{xx}+\left(y(t)^T g_x\right)_x-D\left(f_{\dot{x}x}+\left(y(t)^T g_{\dot{x}}\right)_x\right)+\left(H(t)p(t)\right)_x\right. \\ & -D\left(f_{x\dot{x}}+\left(y(t)^T g_x\right)_{\dot{x}}\right)-D\left(f_{\dot{x}\dot{x}}(t,\bar{x},\dot{\bar{x}})+\left(y(t)^T g_{\dot{x}}\right)_{\dot{x}}\right) \\ & \left.+\left(H(t)p(t)\right)_{\dot{x}}\right\}=0, \qquad t\in I \end{aligned} \tag{7.40}$$

Using the expression of H (t) and (7.40), this gives,

$$\begin{aligned} & \bar{p}(t)^T H(t)\bar{p}(t)+D\left(\bar{p}(t)^T H(t)\bar{p}(t)\right)_{\dot{x}} \\ & -2\bar{p}(t)^T D\left(H(t)\bar{p}(t)\right)_{\dot{x}}=0,\ t\in I \end{aligned}$$

This, in view of the hypothesis (A_2) implies,

$$\bar{p}(t)=0,\ t\in I \tag{7.41}$$

The relations (7.33) and (7.34) imply

$$\bar{x}(t)^T\in N_K\left(z(t)\right) \quad \textit{and} \qquad \bar{x}(t)^T\in N_{C_j}\left(w^j(t)\right),\ j=1,2...m$$

This respectively yields,

$$\bar{x}(t)^T\bar{z}(t)=S\left(\bar{x}(t)\,|\,K\right), \quad t\in I$$

and $$\bar{x}(t)^T\bar{w}^j(t)=S\left(\bar{x}(t)\,|\,C^j\right),\ j=1,2...m,\ t\in I$$

The relation (7.31) with $p(t)=0,\ \ t\in I$ and (7.36) ,gives,

$$\sum_{j=1}^{m} \bar{y}^j(t)\left(g^j\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{x}(t)\bar{w}^j(t)\right) = 0, \ t \in I \qquad (7.42)$$

The relation (7.31) with $p(t) = 0, \ t \in I$, $\mu^j(t) \geq 0, \ t \in I$ and $\bar{x}(t)^T \bar{z}(t) = S\left(\bar{x}(t) \mid K\right), \ t \in I$

yields

$$g^j\left(t, \bar{x}, \dot{\bar{x}}\right) + S\left(\bar{x}(t) \mid C^j\right) \leq 0, \ j = 1, 2...m, \ t \in I$$

That is, $\bar{x}$ is feasible to (CP).

Now, in view of $\bar{x}(t)^T \bar{z}(t) = S\left(\bar{x}(t) \middle| K\right)$, $t \in I$ and (7.42), we have

$$\int_I \left\{ f\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{x}(t)^T \bar{z}(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j\left(t, \bar{x}, \dot{\bar{x}}\right) + \bar{x}(t)^T \bar{w}^j(t)\right) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt = \int_I f\left(t, \bar{x}, \dot{\bar{x}}\right) + S\left(\bar{x}(t) \mid K\right) dt$$

This, along with the hypotheses of Theorem 7.2.1, yields that $\bar{x}(t)$ is an optimal solution of (CP_+).

7.2.3 Special Cases

Let for $t \in I, B(t)$ be a positive semi-definite matrix and continuous on I. Then

$$\left(x(t)^T B(t) x(t)\right)^{1/2} = S\left(x(t) \mid K\right), t \in I$$

where $\qquad K = \left\{ B(t) z(t) \middle| z(t)^T B(t) z(t) \leq 1, t \in I \right\}$

Replacing $S\left(x(t) \mid K\right)$ by $\left(x(t)^T B(t) x(t)\right)^{1/2}$ and suppressing each $S\left(x(t) \mid C^j\right)$ from the constraints of (CP_+), we have following problems treated in the previous chapter.

(CP$_2$): Minimize $\int_I \left\{ f(t,x,\dot{x}) + \left(x(t)^T B(t) x(t) \right)^{1/2} \right\} dt$

Subject to

$$x(a) = 0 = x(b),$$

$$g(t,x,\dot{x}) \leq 0, \qquad t \in I$$

(CD): Maximize $\int_I \left\{ f(t,u,\dot{u}) + u(t)^T B(t) z(t) + y(t)^T g(t,u,\dot{u}) - \frac{1}{2} p(t)^T H(t) p(t) \right\} dt$

Subject to

$$u(a) = 0 = u(b)$$

$$f(t,u,\dot{u}) + u(t)^T B(t) z(t) + y(t)^T g_u(t,u,\dot{u}) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) + H(t) p(t) = 0, \; t \in I$$

$$z(t)^T B(t) z(t) \leq 1, \quad t \in I$$

$$y(t) \geq 0,$$

7.2.4 Problems With Natural Boundary Values

In this section, we formulate a pair of nondifferentiable dual variational problems with natural boundary values rather than fixed end points. The proof for duality theorems for this pair of dual problems is omitted, as they follow immediately on the basis of analysis of the preceding section and, of course, slight modifications are needed. The problems are:

(CP$_0$): Minimize $\int_I \left\{ f(t,x,\dot{x}) + S\left(\bar{x}(t) \mid K \right) \right\} dt$

Subject to

$$g(t,x,\dot{x}) + S\left(\bar{x}(t) \mid C^j \right) \leq 0, \qquad t \in I, \; j = 1, 2 \ldots, m$$

(CD_0): Maximize $$\int_I \left\{ f(t,x,\dot{x}) + x(t)^T z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(t,x,\dot{x}) + x(t)^T w^j(t) \right) - \frac{1}{2} p(t)^T H(t)\, p(t) \right\} dt$$

Subject to

$$f_x(t,x,\dot{x}) + z(t) + \sum_{j=1}^{m} y^j(t)\left(g_x^j(t,x,\dot{x}) + w^j(t) \right)$$

$$-D\left(f_{\dot{x}}(t,x,\dot{x}) + y(t)^T g_{\dot{x}}(t,x,\dot{x}) \right) + \left(H(t)\, \bar{p}(t) \right) = 0, \quad t \in I$$

$$z(t) \in K, w^j(t) \in C^j, \; j = 1,2 \ldots m, t \in I$$

$$y(t) \geq 0, \quad t \in I$$

$$f_{\dot{x}}(t,x,\dot{x}) + y(t)^T g_{\dot{x}}(t,x,\dot{x}) \Big|_{t=a} = 0,$$

$$f_{\dot{x}}(t,x,\dot{x}) + y(t)^T g_{\dot{x}}(t,x,\dot{x}) \Big|_{t=b} = 0.$$

7.2.5 Nonlinear Programming Problems

If all functions in the problems (CP_0) and (CD_0) are independent of t, then these problems will reduce to following nonlinear programming problems studied earlier.

(CP_1): Minimize $f(x) + S(\bar{x}(t) \mid K)$

Subject to

$$g^j(x) + S\left(\bar{x}(t) \mid C^j\right) \leq 0, \; j = 1,2 \ldots m$$

(CD_1): Maximize $f(u) + u^T z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g^j(u) + u^T w^j(t) \right) - \frac{1}{2} p^T H p$

Subject to

$$f(u) + z(t) + \sum_{j=1}^{m} y^j(t)^T \left(g_u^j(u) + w^j(t) \right) + Hp = 0'$$

$$z \in K, \quad w^j \in C^j, \; j = 1,2 \ldots m.,$$

where $H = f_{uu}(u) + y^T g_{uu}(u).$

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