CERTIFICATE

Certified that the thesis entitled "Mark sequences in digraphs" being resubmitted by Uma Tul Samee, in partial fulfillment of the requirements for the award of Doctor of Philosophy in Mathematics, is her own work carried out by her under my supervision and guidance. The content of this thesis, in full or in parts, has not been submitted to any Institute or University for the award of any degree or diploma.

Dr. T. A. Chishti
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ABSTRACT

In Chapter 1, we present a brief introduction of digraphs and some definitions. Chapter 2 is a review of scores in tournaments and oriented graphs. Also we have obtained several new results on oriented graph scores and we have given a new proof of Avery's theorem on oriented graph scores. In chapter 3, we have introduced the concept of marks in multidigraphs, non-negative integers attached to the vertices of multidigraphs. We have obtained several necessary and sufficient conditions for sequences of non-negative integers to be mark sequences of some r-digraphs. We have derived stronger inequalities for these marks. Further we have characterized uniquely mark sequences in r-digraphs. This concept of marks has been extended to bipartite multidigraphs and multipartite multidigraphs in chapter 4. There we have obtained characterizations for mark sequences in these types of multidigraphs and we have given algorithms for constructing corresponding multidigraphs. Chapter 5 deals with imbalances and imbalance sequences in digraphs. We have generalized the concept of imbalances to oriented bipartite graphs and have obtained criteria for a pair of integers to be the pair of imbalance sequences of some oriented bipartite graph. We have shown the existence of an oriented bipartite graph whose imbalance set is the given set of integers.

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[2]. T. A. Chishti and U. Samee, Mark sequences in bipartite multidigraphs and constructions, Acta Univ. Sapientiae, Mathematica, **4, 2** (2012) 38-48.

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CONTENTS

CHAPTER 1

Introduction

1.1 Background

The theory of digraphs (or directed graphs) is one of the richest theories in Graph Theory and has developed enormously within the last three decades. There is an extensive literature on digraphs (more than 3000 papers). Many of these papers contain, not only interesting theoretical results, but also important algorithms as well as applications. The earlier work for digraphs can be found in Chartrand [10], Harary et. al [27], Chartrand and Lesniak-Foster [9] and Behzad, Chartrand [8]. The recent book on digraphs is by Jorgen Bang-Jensen, Gregory Gutin [30]. There are numerous applications of directed graphs in many areas of science and technology. Algorithms on (directed) graphs often play an important role in problems arising in several areas, including computer science and operations research. Secondly, many problems on (directed) graphs are inherently algorithmic.

The concept of degrees and degree sequences in graphs has been extended to digraphs in many ways, like outdegrees, indegrees, scores, imbalances and marks as seen in the present work. This concept of attaching a non-negative integer to the vertices of a digraph is interesting for research as it finds applications in many ways like in the investigation of the structure of the digraphs and also in the ranking of objects. Ranking of objects is a typical practical problem. One of the popular ranking methods is the pairwise comparison of the objects. Many authors describe different applications: e.g., biological, chemical, network modeling, economical, human relation modeling, and sport applications.

The tournament theory is one of the interesting areas of research in digraphs, and an earlier collection of results in tournaments is given by Moon [38]. One of the important aspects of tournaments is the score structure in which much work has been done and some of the results can be seen in the survey article by Reid [52]. Other classes of tournaments are bipartite tournaments and k-partite tournaments which were studied by Beineke

[12], Beineke and Moon [13], Merajuddin [35] and Moon [37]. The score sequence problem of an r-tournament and the score sequence pair problem of an (r_{11}, r_{12}, r_{22})-tournament are applied to the theoretical framework of the communication network central technique.

We mention here some definitions which have been used throughout this dissertation. The other definitions are given in the thesis wherever required.

1.2 Basic Definitions

Definition 1.2.1. *Digraph (or directed graph).* A digraph is a pair (V, A), where V is a nonempty set of objects called vertices and A is a subset of $V^{(2)}$, (the set of ordered pairs of distinct elements of V). The elements of A are called arcs of D.

Definition 1.2.2. *Multidigraph.* A multidigraph D is a pair (V, A), where V is a nonempty set of vertices and A is a multiset of arcs (directed edges) of $V^{(2)}$. The number of times an arc occurs in D is called its multiplicity and arcs with multiplicity greater than one are called multiple arcs.

Definition 1.2.3. *General digraph.* A general digraph D is a pair (V, A), where V is a non empty set of vertices and A is a multiset of arcs, being a multisubset of $V^{(2)}$. An arc of the form uu, where $u \in V$, is called the loop of D, and arcs which are not loops are called the proper arcs. The number of times a loop occurs is called its multiplicity. A loop with multiplicity greater than one is called a multiple loop. An arc $(u, v) \in A$ is represented by $u \to v$. In this case u is said to be adjacent to v, and v is said to be adjacent from u.

Definition 1.2.4. *Subdigraph of a digraph.* Let $D = (V, A)$ be a digraph, $H = (U, B)$ is the subdigraph of D whenever $U \subseteq V$ and $B \subseteq A$. If $U = V$ the subdigraph is said to be spanning.

Definition 1.2.5. *Underlying graph of a digraph D.* The underlying graph of a digraph $D = (V, A)$ is obtained by removing all directions from the arcs of D and replacing any resulting pair of parallel edges by a single edge. Equivalently, the underlying graph of a digraph D is obtained by replacing each arc (u, v) or a symmetric pair of arcs (u, v) and (v, u) by the edge uv.

Definition 1.2.6. *Outdegree and indegree.* In a digraph $D = (V, A)$, the outdegree of a vertex v is the number of vertices to which the vertex v is adjacent, it is denoted by $d^+(v)$ or d_v^+. Similarly the indegree of a vertex v in a digraph D is the number of vertices from which v is adjacent and it is denoted by $d^-(v)$ or d_v^-. The total degree or (simply) degree of v is $d_v = d_v^+ + d_v^-$. If $d_v = k$ for every $v \in V$, then D is said to be k-regular digraph. If for every $v \in V$, $d_v^+ = d_v^-$, the digraph is said to be an isograph, or diregular or a balanced digraph. A vertex v for which $d_v^+ = d_v^- = 0$, is called an isolate. A vertex v is called a transmitter, or receiver accordingly as $d_v^+ > 0$, $d_v^- = 0$, or $d_v^+ = 0$, $d_v^- > 0$. A vertex v is called a carrier if $d_v^+ = d_v^- = 1$.

Definition 1.2.7. *Complete symmetric digraph.* A digraph D is said to be complete symmetric, if both $uv \in A$ and $vu \in A$ for all $u, v \in V$. Clearly this corresponds to K_n, where $\mid V \mid= n$, and is denoted by K_n^*.

Definition 1.2.8. Two digraphs D_1 and D_2 are said to be *isomorphic* if their underlying graphs are isomorphic and the direction of arcs are same and we write $D_1 \cong D_2$.

Definition 1.2.9. *Complement of a Digraph.* The complement of digraph $D = (V, A)$ is denoted by $\bar{D}$. It has a vertex set V and $uv \in \bar{A}$ if and only if $uv \notin A$. $\bar{D}$ is the relative complement of D in K_n^*, where K_n^* is a complete symmetric digraph, and $\mid V \mid= n$.

Definition 1.2.10. *Converse of a digraph.* The converse of a digraph D is the digraph D' with vertex set V and $uv \in A'$ if and only if $vu \in A$ that is, the arc set A' is obtained by reversing the direction of each arc of D. Clearly, $(D')' = D'' = D$.

Definition 1.2.11. *Self complementary digraph.* A digraph D is said to be self complementary if $D \cong \bar{D}$, and D is said to be self converse if $D \cong D'$. A digraph is said to self dual if $D \cong \bar{D} \cong D'$.

Definition 1.2.12. *Directed Walk.* A directed walk in a digraph D is a sequence $v_0 a_1 v_1 a_2 \cdots a_k v_k$, where $v_i \in V$ and $a_i \in A$ are such that $a_i = v_{i-1} v_i$ for $1 \le i \le k$ and no arc being repeated. As there is only one arc of the form

v_iv_j, the walk can also be represented by the vertex sequence $v_0v_1\cdots v_k$. The number of occurrences of arcs on a walk is the length of the walk. So the length of the above walk is k. A vertex may appear more than once in a walk. If $v_0 \neq v_k$, the walk is open, and if $v_0 = v_k$ the walk is closed. A walk is spanning if $V = v_0v_1\cdots v_k$.

Definition 1.2.13. A *semiwalk* is a sequence $v_0a_1v_1a_2\cdots a_kv_k$, with $v_i \in V$ and $a_i \in A$ such that either $a_i = v_{i-1}v_i$ or v_iv_{i-1}, $1 \leq i \leq k$ and no arc being repeated. The length of the above semiwalk is k. If $v_0 \neq v_k$, the semiwalk is open. If $v_0 = v_k$, the semiwalk is closed.

Definition 1.2.14. *Directed Path.* A directed path is an open walk in which no vertex is repeated. A directed cycle is a closed walk in which no vertex is repeated. A digraph is acyclic if it has no cycles. If no vertex is repeated in an open(closed) semiwalk, it is called a semipath(semicycle).

Definition 1.2.15. *Joining and Reaching.* In a digraph D, a vertex u is said to be joined to a vertex v, if there is a semipath from u to v. A vertex u is said to be reachable from a vertex v, if there is a path from v to u. A vertex v is called a source of D, if every vertex is reachable from v and v is called a sink of D, if v is reachable from every other vertex.

Definition 1.2.16. *Connectedness in digraphs.* A digraph D is said to be strongly connected or strong if every two of its distinct vertices u and v are such that u is reachable from v and v is reachable from u. A digraph is unilaterally connected or unilateral if either u is reachable from v or v is reachable from u, and is weakly connected or weak if u and v are joined by a semipath. A digraph is said to be disconnected if it is not even weak. A digraph is said to be strictly weak if it is weak but not unilateral. It is strictly unilateral if it is unilateral but not strong.

Definition 1.2.17. *Oriented graph.* An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops.

CHAPTER 2

On scores in tournaments and oriented graphs

In this Chapter, we report the results available in literature on score sequences in tournaments and oriented graphs. We obtain many new results on score sequences in oriented graphs. Also we give a new proof for Avery's theorem on oriented graph scores.

2.1 Introduction

Definition 2.1.1. A tournament $T = (V, A)$ is a complete oriented graph with vertex set $V(T) = V = \{v_1, v_2 ... v_n\}$ and arc set A, that is for any pair of vertices v_i, v_j either (v_i, v_j) is an arc or (v_j, v_i) is an arc, but not both. In other words, a tournament is an orientation of a complete simple graph. The score of a vertex v_i is denoted by s_{v_i} (or simply by s_i), is the outdegree of v_i. Clearly, $0 \leq s_{v_i} \leq n-1$. The sequence $S = [s_1, s_2, \cdots, s_n]$ in non-decreasing order is the score sequence or the score structure of a tournament T. A sequence S of non-negative integers in non-decreasing order is said to be realizable if there exists a tournament with score sequence S.

A tournament can be considered as the result of a competition where n participants play each other once that cannot end in a tie and score one point for each win. Player v_i is represented in the tournament by vertex v_i and an arc from v_i to v_j means that v_i defeats v_j. The player v_i obtains a total score s_{v_i} points in the competition, and the vertex scores can be ordered to obtain the score sequence of the tournament. If there is an arc from a vertex x to vertex y, then we say x dominates y and we write $x \to y$ or $x(1-0)y$.

Definition 2.1.2. A triple in a tournament is an induced subtournament with three vertices. For any three vertices x, y and z, the triple of the form $x(1-0)y(1-0)z(0-1)x$ is said to be transitive, while as the triple of the form $x(1-0)y(1-0)z(1-0)x$ is said to be intransitive. A tournament is said to be transitive if all its triples are transitive. Also, a regular tournament on n vertices (n odd) is one whose all vertices have scores $\frac{n-1}{2}$.

2.2 Score in tournaments

In this section we present the characterizations for sequences on non-negative integers to be score sequences of tournaments. Landau [31] in 1953 gave the following necessary and sufficient conditions for the non-negative integers in non-decreasing order to be the score sequences of a tournament.

Theorem 2.2.1. *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a tournament if and only if for each $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} s_i \geq \binom{|I|}{2}, \tag{2.1}$$

with equality when $|I| = n$, where $|I|$ is the cardinality of the set I.

Since $s_1 \leq \cdots \leq s_n$, the inequality (2.1), called Landau inequalities, are equivalent to $\sum\limits_{i}^{k} s_i \geq \binom{k}{2}$, for $k = 1, 2, \cdots, n-1$, and equality for $k = n$.

There are now several proofs of this fundamental result in tournament theory, clever arguments involving gymastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization, a constructive argument utilizing network flows, another one involving systems of distinct representatives. Landau's original proof appeared in 1953 [31], Matrix considerations by Fulkerson [23] (1960) led to a proof, discussed by Brauldi and Ryser [17] in (1991). Berge [14] in (1960) gave a network flow proof and Alway [3] in (1962) gave another proof. A constructive proof via matrices by Fulkerson [24] (1965), proof of Ryser (1964) appears in the monograph of Moon (1968). An inductive proof was given by Brauer, Gentry and Shaw [15] (1968). The proof of Mahmoodian [33] given in (1978) appears in the textbook by Behzad, Chartrand and Lesnik-Foster [8](1979). A proof by contradiction was given by Thomassen [58] (1981) and was adopted by Chartrand and Lesniak [20] in subsequent revisions of their 1979 textbook, starting with their 1986 revision. A nice proof was given by Bang and Sharp [7](1979) using systems of distinct representatives. Three years later in 1982, Achutan, Rao and Ramachandra-Rao [1] obtained a proof as result of some slightly more general work. Bryant [19] (1987) gave a proof via a slightly different use of distinct representatives. Partially ordered sets were employed in a proof by Aigner [2] in 1984 and described by Li [32] in 1986 (his version appeared in 1989). Two proofs of sufficiency appeared in a paper by Griggs and Reid [26] (1996) one a direct proof and the second is self contained. Again two proofs appeared in 2009 one by Brauldi and Kiernan [18] using Rado's theorem from Matroid theory, and another inductive proof by Holshouser and Reiter [29] (2009). More recently Santana and Reid [55] (2012) have given a new proof in the vein of the two proofs by Griggs and Reid (1996).

The following is the recursive method to determine whether or not a sequence is the score sequence of some tournament. It also provides an algorithm to construct the corresponding tournament.

Theorem 2.2.2. *Let S be a sequence of n non-negative integers not exceeding $n-1$, and let S' be obtained from S by deleting one entry s_k and reducing $n-1-s_k$ largest entries by one. Then S is the score sequence of some tournament if and only if S' is the score sequence.*

Brauldi and Shen [16] obtained stronger inequalities for scores in tournaments. These inequalities are individually stronger than Landau's inequalities, although collectively the two sets of inequalities are equivalent.

Theorem 2.2.3. *A sequence $S = [s_i]_1^n$ of non-negative integers in non-deceasing order is a score sequence of a tournament if and only if for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} s_i \geq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{2} \binom{|I|}{2} \tag{2.2}$$

with equality when $|I| = n$.

It can be seen that equality can occur oftenly in (2.2), for example, equality hold for regular tournaments of odd order n whenever $|I| = k$ and $I = \{n-k+1, \cdots, n\}$. Further Theorem 2.2.3 is best possible in the sense that, for any real $\epsilon > 0$, the inequality

$$\sum_{i\in I} s_i \geq (\frac{1}{2}+\epsilon)\sum_{i\in I}(i-1) + (\frac{1}{2}-\epsilon)\binom{|I|}{2} \tag{2.3}$$

fails for some I and some tournaments, for example, regular tournaments. Brauldi and Shen [16] further observed that while an equality appears in (2.3), there are implications concerning the strong connectedness and regularity of every tournament with the score sequence S. Brauldi and Shen also obtained the upper bounds for scores in tournaments.

Theorem 2.2.4. *A sequence* $S = [s_i]_1^n$ *of non-negative integers in non-deceasing order is a score sequence of a tournament if and only if for each subset* $I \subseteq [n] = \{1, 2, \cdots, n\}$,

$$\sum_{i\in I} s_i \leq \frac{1}{2}\sum_{i\in I}(i-1) + \frac{1}{4}|I|(2n-|I|-1),$$

with equality when $|I| = n$.

Brauldi and Shen also obtained the lower bounds for scores in tournaments.

Theorem 2.2.5. *A sequence* $S = [s_i]_1^n$ *of non-negative integers in non-deceasing order is a score sequence of a tournament if and only if for each subset* $I \subseteq [n] = \{1, 2, \cdots, n\}$,

$$\sum_{i\in I} s_i \geq \frac{1}{2}\sum_{i\in I}(i-1) + \frac{1}{2}\binom{|I|}{2},$$

with equality when $|I| = n$.

Definition 2.2.6. A score sequence is simple (uniquely realizable) if it belongs to exactly one tournament.

Avery [4] observed that the score sequence S is simple if and only if every strong component of S is simple. Further a strong score sequence is simple if and only if it is one of [0], [1,1,1], [1,1,2,2], or [2,2,2,2,2]. The following characterization of simple score sequences in tournaments is due to Avery [4].

Theorem 2.2.7. *The score sequence S is simple if and only if every strong component of S is one of [0], [1,1,1], [1,1,2,2], or [2,2,2,2,2].*

Definition 2.2.8. A tournament T is called self converse if $T \cong T'$, where T' is the converse of T obtained by reversing the orientations of all arcs of T. Transitive tournaments are examples of self-converse tournaments.

Eplett [22] characterized self converse score sequences in tournaments.

Theorem 2.2.9. *A sequence $S = [s_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of a self-converse tournament if and only if for each $1 \leq k \leq n$,*

$$\sum_{i}^{k} s_i \geq \binom{k}{2}$$

with equality when $k = n$, and for $1 \leq i \leq n$,

$$s_i + s_{n+1-i} = n - 1.$$

Definition 2.2.10. A bipartite tournament is a complete oriented bipartite graph. A bipartite tournament T is a directed graph whose vertex set is the union of two disjoint nonempty sets X and Y, and whose arc set comprises exactly one of the pairs (x, y) or (y, x) for each $x \in X$ and each $y \in Y$. Bipartite tournaments are bipartite analogues of tournaments. The score of a vertex is its outdegree. There are two sequences (lists of scores) one for each set and are called as the pair of score lists. If $|X| = m$ and $|Y| = n$, it is

mXn bipartite tournament. A bipartite tournament is reducible if there is a nonempty proper subset of its vertex set to which there are no arcs from the other vertices, otherwise irreducible. The property of irreducibility is equivalent to having all pairs of vertices mutually reachable or to being strongly connected.

A bipartite tournament represents the outcomes of a competition between two groups of participants, each participant of one group competing with every participant of the other group.

The following recursive characterization is due to Gale [25].

Theorem 2.2.11. *The lists of non-negative integers $A = [a_1, a_2, \cdots, a_m]$ and $B = [b_1, b_2, \cdots, b_n]$ in non-decreasing order are the score lists of some bipartite tournament if and only if the lists $A' = [a_1, a_2, \cdots, a_{m-1}]$ and $B' = [b_1, b_2, \cdots, b_{a_m}, b_{a_m+1} - 1, \cdots, b_n - 1]$ are the score lists.*

Beineke and Moon [11] showed that if two bipartite tournaments have the same score lists then one can be transformed to another.

Theorem 2.2.12. *If two bipartite tournaments have the same score lists, then each can be transformed into the other by successively reversing the arcs of 4-cycles.*

Analogous to Landau's theorem, Moon [36] was the first to establish the following result for scores in bipartite tournaments.

Theorem 2.2.13. *A pair of lists A and B of non-negative integers in non-decreasing order are the score lists of some bipartite tournament if and only if for $1 \le i \le m$ and $1 \le j \le n$,*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \ge kl \qquad (2.4)$$

with equality when $k = m$ and $l = n$.

The realizations are irreducible if and only if $a_1 > 0$ *and* $b_1 > 0$ *and the inequalities (2.4) are all strict except* $k = m$ *and* $l = n$.

The following characterization of bipartite score lists is due to Ryser [53].

Theorem 2.2.14. *A pair of lists* A *and* B *of non-negative integers with* A *in non-increasing order are the score lists of some bipartite tournament if and only if for* $1 \leq k \leq m$,

$$\sum_{i=1}^{k} a_i \leq \sum_{j=1}^{n} \quad min \quad (k, m - b_j) \tag{2.5}$$

with equality when $k = m$.

The realizations are irreducible if and only if $0 < b_j < m$ *for each* j *and the inequalities (2.5) are all strict except* $k = m$.

Let $A = [a_1, a_2, \cdots, a_m]$ and $B = [b_1, b_2, \cdots, b_n]$ be two lists of integers. Let $\bar{A} = [n - a_1, n - a_2, \cdots, n - a_m]$ and $\bar{B} = [m - b_1, m - b_2, \cdots, m - b_n]$.

Definition 2.2.15. If a pair (A, B) is realizable and all is realizations are isomorphic, then (A, B) is said to be uniquely realizable. The pair $A = [1, 1, \cdots, 1] = [1^m]$ and $B = [b_1, b_2, \cdots, b_n]$ is uniquely realizable. (A, B) is uniquely realizable if and only if $(\bar{A}, B)$ is uniquely realizable. Since decomposition into irreducible components is determined by the lists, so only irreducible bipartite tournaments are considered for unique realizability.

Bagga and Beineke [6] characterized uniquely realizable score lists in bipartite tournaments.

Theorem 2.2.16. *An irreducible pair* (A, B) *of score lists is uniquely realizable if and only if one of the following holds.*
I (wlog) $A = [1^m]$ *and* B *is arbitrary*
$\bar{I}$ *(dual of I)* $A = [(n-1)^m]$ *and* B *is arbitrary*
II (wlog) $A = [1^{m-1}, a]$ *and* $B = [b^n]$
$\bar{I}I$ *dual of II*

III (wlog) $A = [1, a^{m-1}]$ *and* $B = [2^n]$
$I\bar{I}I$ *dual of III*

Definition 2.2.17. An r-tournament is a complete oriented multigraph in which there are exactly r arcs between every two vertices. The score of a vertex in an r-tournament is the outdegree of that vertex and the scores listed in non-decreasing order is the score sequence.

Takahashi [56] has considered several variations of the score sequence problem of an r-tournament and has given efficient algorithms.

Definition 2.2.18. A directed graph D is said to be an (r_{11}, r_{12}, r_{22})-tournament if the vertex set of D is partitioned into two disjoint sets A and B such that there are r_{11} directed arcs between every pair of vertices in A, r_{22} directed arcs between every pair of vertices in B, and r_{12} directed arcs between each vertex of A and each vertex of B. The score of the vertex is the outdegree of the vertex.

Let T be an (r_{11}, r_{12}, r_{22})-tournament with parts $U = \{u_1, u_2, \cdots, u_m\}$ and $V = \{v_1, v_2, \cdots, v_n\}$. Let $a(u_i)$ or a_i be the score of a vertex u_i, $1 \le i \le m$ and $b(v_j)$ or b_j be the score of a vertex v_j, $1 \le j \le n$. The sequences $A = [a_1, a_2, \cdots, a_m]$ and $B = [b_1, b_2, \cdots, b_n]$ is called the score sequence pair of (r_{11}, r_{12}, r_{22})-tournament and is denoted by $[A, B]$. Takahashi, Watanabe and Yoshimura [57] have characterized the score sequence pair of (r_{11}, r_{12}, r_{22})-tournament and have also given an algorithm for determining in linear time whether a pair of two non-negative integer sequences is realizable or not.

2.3 Scores in oriented graphs

Definition 2.3.1. An oriented graph D is a digraph with no symmetric pairs of directed arcs and with no loops. In D, let d_i^+ and d_i^- be the outdegree and indegree of the vertex v_i. Define the score a_{v_i} or simply a_i of a vertex v_i as follows.

$$a_i = n - 1 + d_i^+ - d_i^-.$$

Evidently, $0 \leq a_i \leq 2n-2$. The list of scores $[a_i]_1^n$ in non-decreasing or non-increasing order is the called the score sequence of D.

One of the interpretations of an oriented graph is the result of a round robin competition in which the participants play each other exactly once, ties (draws) are allowed, that is, the participants play each other once with an arc from u to v if and only u defeats v. A player receives two points for each win and one point for each tie. The total points received by a participant v_i is a_i.

Let d_i^+, d_i^- and d_i^* respectively be outdegree, indegree and non-arcs incident at v_i. Then

$$d_i^+ + d_i^- + d_i^* = n - 1 = a_i - d_i^+ + d_i^-$$

or,

$$a_i = 2d_i^+ + d_i^*$$

This shows that $a_i = n - 1 + d_i^+ - d_i^- = 2(wins) + (draws)$.

Avery [5] extended Landau's theorem on tournament scores to oriented graph scores.

Theorem 2.3.2.(Avery) *A sequence $A = [a_i]_1^n$ of non-negative integers in non-deceasing order is a score sequence of an oriented graph if and only if for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} a_i \geq k(k-1)$$

with equality when $|I| = n$.

Avery's theorem on oriented graph scores can be restated in the following. We give here the proof which appeared in Pirzada, Merajuddin and

Samee [47].

Theorem 2.3.2.(Avery) *A sequence* $A = [a_i]_1^n$ *of non-negative integers in non-deceasing order is a score sequence of an oriented graph if and only if for* $1 \le k \le n-1$,

$$\sum_{i=1}^{k} a_i \ge k(k-1) \tag{2.6}$$

with equality when $k = n$.

Proof. Necessity. Let $[a_i]_1^n$ be a score sequence of some oriented graph D. Let W be the oriented subgraph induced by any k vertices $w_1, w_2, \cdots, w_k$ of D. Let α denote the number of arcs of D that start in W and end outside W and let β denote the number of arcs of D that start outside of W and end in W. Clearly, $\beta \le k(n-k)$.

Thus, $\sum\limits_{i=1}^{k} a_{w_i} = \sum\limits_{i=1}^{k}(n-1+d_D^+(w_i)-d_D^-(w_i)) = nk-k+\sum\limits_{i=1}^{k} d_D^+(w_i) - \sum\limits_{i=1}^{k} d_D^-(w_i) = nk-k+[\sum\limits_{i=1}^{k} d_W^+(w_i)+\alpha]-[\sum\limits_{i=1}^{k} d_W^-(w_i)+\beta] = nk-k+(number\ of\ arcs\ of\ W)+\alpha-(number\ of\ arcs\ of\ W)-\beta$.

Therefore, $\sum\limits_{i=1}^{k} a_{w_i} = nk-k+\alpha-\beta$.

Hence, $\sum\limits_{i=1}^{k} a_{w_i} \ge nk-k-\beta \ge nk-k-k(n-k) = k(k-1)$.

Sufficiency. Let n denote the least integer so that there is a non-decreasing sequence of non-negative integers satisfying conditions (2.6) that is not a score sequence of any oriented graph. Among all such sequences of length n, pick one, denoted by $A = [a_i]_1^n$, in which the smallest term a_1 is as small as possible.

We consider two cases, (a) equality in (2.6) holds for some $k < n$ and (b) each inequality in (2.6) is strict for all $k < n$.

Case(a). Assume $k(k < n)$ is the smallest integer such that

$$\sum_{i=1}^{k} a_i = k(k-1).$$

Clearly the sequence $[a_1, a_2, \cdots, a_k]$ satisfies conditions (2.6) and is a sequence of length less than n. Therefore by the given assumption $[a_1, a_2, \cdots, a_k]$ is a score sequence of some oriented graph, say D_1.

Now, $\sum_{i=1}^{p}(a_{k+i} - 2k) = \sum_{i=1}^{p+k} a_i - \sum_{i=1}^{k} a_i - 2pk \geq (p+k)(p+k-1) - k(k-1) - 2pk = p(p-1)$, for each p, $1 \leq p \leq n-k$, with equality when $p = n-k$. Since $p < n$, the minimality of n implies that the sequence $[a_{k+1} - 2k, a_{k+2} - 2k, \cdots, a_n - 2k]$ is the score sequence of some oriented graph D_2. The oriented graph D of order n, consisting of disjoint copies of D_1 and D_2, such that there is an arc from each vertex of D_2 to every vertex of D_1, has score sequence $a = [a_i]_1^n$, a contradiction.

Case(b). Assume that each inequality in conditions (2.6) is strict for all $k < n$. Clearly, $a_1 > 0$. Consider the sequence $A' = [a'_i]_1^n$, where $a'_i = a_i - 1$, or $a_i + 1$, or a_i according as $i = 1$, or $i = n$, or otherwise.

Then, $\sum_{i=1}^{k} a'_i = (\sum_{i=1}^{k} a_i) - 1 > k(k-1) - 1$, for all k, $1 \leq k < n$.

Therefore, $\sum_{i=1}^{k} a'_i \geq k(k-1)$, for all k, $1 \leq k < n$.

Also, $\sum_{i=1}^{n} a'_i = (\sum_{i=1}^{n} a_i) - 1 + 1 = n(n-1)$.

Thus the sequence $A' = [a'_i]_1^n$ satisfies conditions (2.6) and therefore is a score sequence of some oriented graph D. Let u and v, respectively denote the vertices with score $a'_1 = a_1 - 1$ and $a'_n = a_n + 1$. If in D either $v(1-0)u$, or $v(0-0)u$, then transforming them respectively to $v(0-0)u$, or $v(0-1)u$, we get an oriented graph with score sequence A, a contradiction.

Now let $u(1-0)v$. We claim that there exists at least one vertex w so that the triple formed by the vertices u, v and w is intransitive, that is, of the form $u(1-0)v(1-0)w(1-0)u$, or $u(1-0)v(1-0)w(0-0)u$, or $u(1-0)v(0-0)w(1-0)u$. Assume to the contrary that for each vertex $w \in V - \{u, v\}$, the triple formed by the vertices u, v and w are transitive, that is, of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(1-0)u$, or $u(1-0)v(0-1)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$. Then in all such cases, $d^+(u) > d^+(v)$ and $d^-(u) < d^-(v)$. This shows that $a_u > a_v$. This proves the claim.

Hence transforming the intransitive triples respectively to $u(1-0)v(0-0)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, we obtain an oriented graph with score sequence A. This contradicts the assumption. □

A constructive proof of Avery's theorem can be seen in Pirzada, Merajuddin and Samee [47]. The following results appear in Pirzada, Merajuddin and Samee [48].

Theorem 2.3.3. *A sequence $A = [a_i]_1^n$ of non-negative integers with $a_1 \le a_2 \le \cdots < a_k = a_{k+1} = \cdots = a_{k+m-1} < a_{k+m} \le a_{k+m+1} \le \cdots \le a_n$ and let $A' = [a'_i]_1^n$ where $a'_i = a_i - 1, a_i + 1, a_i$ according as $i = k$, or $i = k + m - 1$ or otherwise. Then A is a score sequence of some oriented graph if and only if A' is a score sequence of an oriented graph.*

Proof. Clearly, $k \ge 1$ and $m \ge 2$, so that either $k + m - 1 = n$, or $a_k = a_{k+1} = \cdots = a_{k+m-1} < a_{k+m}$. For $1 \le i \le n$, $A' = [a'_i]_1^n$ where $a'_i = a_i - 1, a_i + 1, a_i$ according as $i = k$, or $i = k + m - 1$ or otherwise. Obviously, $a'_1 \le a'_2 \le \cdots \le a'_n$.

Let A' be the score sequence of some oriented graph D' of order n in which vertex v'_i has score a'_i, $1 \le i \le n$. Then $a'_{k+m-1} = a'_k + 2$. If either $v'_{k+m-1}(1-0)v'_k$, or $v'_{k+m-1}(0-0)v'_k$ then making respectively, the transformation $v'_{k+m-1}(0-0)v'_k$, or $v'_k(1-0)v'_{k+m-1}$, gives an oriented graph of order n with score sequence A.

If $v'_k(1-0)v'_{k+m-1}$, since $a'_k \le a'_{k+m-1}$ there exists at least one vertex v'_j in $V' - \{v'_k, v'_{k+m-1}\}$ such that triple formed by v'_k, v'_{k+m-1} and v'_j is transitive and of the form $v'_k(1-0)v'_{k+m-1}(1-0)v'_j(1-0)v'_k$ or $v'_k(1-0)v'_{k+m-1}(1-0)v'_j(0-0)v'_k$ or $v'_k(1-0)v'_{k+m-1}(0-0)v'_j(1-0)v'_k$. These can be transformed respectively to $v'_k(1-0)v'_{k+m-1}(0-0)v'_j(0-0)v'_k$ or $v'_k(1-0)v'_{k+m-1}(1-0)v'_j(0-1)v'_k$ or $v'_k(1-0)v'_{k+m-1}(0-1)v'_j(0-0)v'_k$, and we obtain an oriented graph of order n with score sequence A.

If for every vertex $v'_j \in V' - \{v'_k, v'_{k+m-1}\}$ the triple formed by v'_k, v'_{k+m-1} and v'_j is transitive, we again get a contradiction.

Now, let A be the score sequence of some oriented graph D of order n in which vertex v_i has score a_i, $1 \le i \le n$. We have $a_{k+m-1} = a_k$. If either $v_k(1-0)v_{k+m-1}$, or $v_k(0-0)v_{k+m-1}$, then making respectively, the transformation $v_k(0-0)v_{k+m-1}$, or $v_k(0-1)v_{k+m-1}$, gives an oriented graph of order n with score sequence A'. If $v_{k+m-1}(1-0)v_k$, we claim that there exists at least one vertex $v_j \in V - \{v_{k+m-1}, v_k\}$ such that the triple formed by the vertices v_{k+m-1}, v_k and v_j is intransitive, and of the form $v_{k+m-1}(1-0)v_k(1-0)v_j(1-0)v_{k+m-1}$, or $v_{k+m-1}(1-0)v_k(1-0)v_j(0-0)v_{k+m-1}$, or $v_{k+m-1}(1-0)v_k(0-0)v_j(1-0)v_{k+m-1}$. These can be transformed respectively to $v_{k+m-1}(1-0)v_k(0-0)v_j(0-0)v_{k+m-1}$, or $v_{k+m-1}(1-0)v_k(0-0)v_j(0-1)v_{k+m-1}$, or $v_{k+m-1}(1-0)v_k(0-1)v_j(0-0)v_{k+m-1}$ and we obtain an oriented graph of order n with score sequence A'.

In case for every vertex $v_j \in V - \{v_k, v_{k+m-1}\}$, then the triple formed by v_{k+m-1}, v_k and v_j is transitive, we again get a contradiction. Thus A' is a score sequence if and only if A is a score sequence. $\square$

Theorem 2.3.4. *Let $A = [a_i]_1^n$ be a sequence of non-negative integers in non-decreasing order with at least two odd terms a_k and a_m(say) with $a_k < a_m$ and let $A' = [a'_i]_1^n$ with $a'_i = a_i - 1,$ or $a_i + 1,$ or a_i according as $i = k$ or $i = k + m - 1$ or otherwise. Then A is a score sequence if and only if A' is a score sequence.*

Proof. Let a_k be the lowest odd term, and a_m be the greatest odd term and let $A' = [a'_1, a'_2, \cdots, a'_n]$, where $a'_i = a_i - 1,$ *or* $a_i + 1,$ *or* a_i according as $i = k$ or $i = k + m - 1$ or otherwise. Clearly, $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Let A' be the score sequence of some oriented graph D' of order n in which vertex v'_i has score $a'_i,\ 1 \leq i \leq n$. Then, $a'_m \geq a'_k + 2$ with equality appearing when the two odd terms are same. Therefore, it follows by the argument used in Theorem 2.3.3, that is the score sequence of some oriented graph D of order n in which vertex v_i has score a_i, $1 \leq i \leq n$. We have $a_m \geq a_k$. The equality appears when the two odd terms are same, and in this case A' is a score sequence of some oriented graph of order n, again by Theorem 2.3.3. If $a_m > a_k$, then $a_m \geq a_k + 2$, since $a_m = a_k + 1$ implies that one of a_k or a_m is even, which contradicts the choice of a_k and a_m. Thus, by using again the argument as in Theorem 2.3.3, it follows that A' is a score sequence of some oriented graph of order n. $\square$

Lemma 2.3.5. *(a) Let A and A' be given as in Theorem 2.3.3. Then A satisfies (2.6) if and only if A' satisfies (2.6).*
(b) Let A and A' be given as in Theorem 2.3.4. Then A satisfies (2.6) if and only if A' satisfies (2.6).

Proof(a). If A satisfies (2.6), then $\sum_{i=1}^{j} a'_i = \sum_{i=1}^{j} a_i$, or $\sum_{i=1}^{k-1} a_i + (a_k - 1) + \sum_{i=k+1}^{j} a_i$, or $\sum_{i=1}^{k-1} a_i + (a_k - 1) + \sum_{i=k+1}^{k+m-2} a_i + (a_{k+m-1} + 1) + \sum_{i=k+m}^{j} a_i$ according as $j \leq k-1$, or $k \leq j \leq k+m-2$, or $j \geq k+m-1$ respectively.
If $j \leq k-1$ and $j \geq k+m-1$, then $\sum_{i=1}^{j} a'_i \geq j(j-1)$.
If $k \leq j \leq k+m-2$, claim $\sum_{i=1}^{j} a_i > j(j-1)$, for $k \leq j \leq k+m-2$.
Assume to the contrary, that for some j, $k \leq j < k+m-2$, $\sum_{i=1}^{j} a_i \leq j(j-1)$.
For (2.6), we have $\sum_{i=1}^{j} a_i \geq j(j-1)$.
Combining the two, we obtain $\sum_{i=1}^{j} a_i = j(j-1)$.
Therefore, again by (2.6), we have $a_{j+1} + j(j-1) = a_{j+1} + \sum_{i=1}^{j} a_i = \sum_{i=1}^{j+1} a_i \geq j(j+1) = j(j-1+2) = j(j-1) + 2j$.
That is, $a_{j+1} \geq 2j$. Also, $a_j = a_{j+1}$ implies that $a_j \geq 2j$.
Thus, $\sum_{i=1}^{j} a_i = \sum_{i=1}^{j-1} a_i + a_j \geq (j-1)(j-2) + 2j = j(j-1) - (j-1) + 2j$.
Therefore $\sum_{i=1}^{j} a_i \geq j(j-1) + 2 > j(j-1)$, contradicting the assumption.
Hence,

$$\sum_{i=1}^{j} a_i > j(j-1), \text{for } k \leq j \leq k+m-2. \tag{2.7}$$

Thus, when $k \leq j \leq k+m-2$, using (2.7), we obtain $\sum_{i=1}^{j} a'_i = \sum_{i=1}^{j} a_i - 1 > j(j-1)$.
Therefore in all cases A' satisfies (2.6). Now, if A' satisfies (2.6), it can be easily seen that A also satisfies (2.6).
Proof of (b) follows similarly. □

Now we give a direct proof for the sufficiency of Avery's theorem 2.3.2.

Proof of Theorem 2.3.2. Sufficiency. Let the sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order satisfy (2.6). Clearly, the sequence

$A_n = [0, 2, 4, \cdots, 2n-2]$ satisfies (2.6),since it is the score sequence of the transitive tournament of order n. Now, if any sequence $A \neq A_n$ satisfies (2.6), then $a_1 \geq 0$ and $a_n \leq 2n-2$. We claim that A contains either (a) a repeated term, or (b) at least two odd terms, or both (a) and (b). To verify the claim, suppose that there is no repeated term. If at least one term is odd, then a parity argument shows that there are at least two odd terms. So assume that all terms are even. Therefore, $a_1 \geq 0$, $a_2 > a_1$, and a_2 is even imply that $a_2 \geq 2$. And $a_2 \geq 2$, $a_3 > a_2$, and a_3 is even imply that $a_3 \geq 4$. Inductively, $a_i \geq 2(i-1)$, for all $1 \leq i \leq n$. Thus, $n(n-1) = \sum\limits_{i=1}^{n} a_i \geq 2\sum\limits_{i=1}^{n}(i-1) = n(n-1)$. This implies that equality holds throughout. Thus, $a_i = 2(i-1)$, for all $1 \leq i \leq n$, and $A = A_n$, a contradiction. Consequently, if there is no repeated term, then at least two terms are odd.

We produce a new sequence A' from A which also satisfies (2.6), A' is closer to A_n than A, and A' is a score sequence if and only if A is a score sequence. When A contains a repeated term, reduce the first occurrence of that repeated term in A by one and increase the last occurrence of that repeated term by one to form A'. If A contains at least two odd terms, reduce the first odd term by one and increase the last odd term by one to form A'. The process is repeated until the sequence A_n is obtained. Let the total order on the non-negative integer sequences be defined by $X = [x_1, x_2, \cdots, x_n] \preceq Y = [y_1, y_2, \cdots, y_n]$ if either $X = Y$, or $x_i < y_i$ for some i, $1 \leq i \leq n$, and $x_{i+1} = y_{i+1}, \cdots, x_n = y_n$. Clearly, $\preceq$ is reflexive, antisymmetric and satisfies comparability. We write $X \prec Y$, if $X \prec Y$ but $X \neq Y$. For any sequence $A \neq A_n$, satisfies (2.6), $A \prec A_n$, where $A_n = [0, 2, 4, \cdots, 2n-2]$, the score sequence of a transitive tournament of order n. Thus, we have shown that for any sequence A satisfies (2.6), we can form another sequence A' satisfying (2.6)(By Lemma 2.3.5) such that $A \prec A'$, and A is a score sequence if and only if A' is a score sequence (By Theorem 2.3.3 and 2.3.4). Therefore, by the repeated application of this transformation, starting from the original sequence satisfying (2.6), we reach A_n. Hence A is a score sequence. $\square$

A recursive characterization of score sequences in oriented graphs also appears in Avery [5].

Theorem 2.3.6 (Avery) *Let A be a sequence of integers between 0 and $2n-2$ inclusive and let A' be obtained from A by deleting the greatest entry $2n-2-r$ say, and reducing each of the greatest r remaining entries in A by one. Then A is a score sequence if and only if A' is a score sequence.*

Theorem 2.3.6 provides an algorithm for determining whether a given non-decreasing sequence A of non-negative integers is a score sequence of an oriented graph and for constructing a corresponding oriented graph.

Pirzada, Merajuddin, Samee [47] obtained the following stronger inequalities for oriented graph scores.

Theorem 2.3.7. *A sequence $A = [a_i]_1^n$ of non-negative integers in non-deceasing order is a score sequence of an oriented graph if and only if for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} a_i \geq \sum_{i \in I} (i-1) + \binom{|I|}{2} \tag{2.8}$$

with equality when $|I| = n$.

Proof. Sufficiency. Let the sequence $A = [a_i]_1^n$ of non-negative integers satisfy (2.8).

Now, for any $I \subseteq [n]$,

$$\textstyle\sum_{i \in I}(i-1) \geq \sum_{i=1}^{|I|}(i-1) = \binom{|I|}{2}.$$

Therefore inequalities (2.8) give

$$\textstyle\sum_{i \in I} a_i \geq \binom{|I|}{2} + \binom{|I|}{2} = 2\binom{|I|}{2}.$$

This shows that inequalities (2.8) imply inequalities (2.6). Thus A is a score sequence.

Necessity. Assume $A = [a_i]_1^n$ is a score sequence of some oriented graph.

For any subset $I \subseteq [n]$, define

$$f(I) = \sum_{i \in I} a_i - \sum_{i \in I}(i-1) - \binom{|I|}{2}.$$

Consider all subsets that minimize the function f. Among all such subsets that minimize the function f, choose one, say I, of the smallest cardinality. Claim $I = \{i : 1 \leq i \leq |I|\}$. If not, then there exists $i \notin I$ and $j \in I$ such that $j = i + 1$. Then, $a_i \leq a_j$.

For $j \in I$, we have $f(I) = \sum_{t \in I} a_t - \sum_{t \in I}(t-1) - \binom{|I|}{2} = \sum_{t \in I, t \neq j} a_t - (\sum_{t \in I, t \neq j}(t-1) + (j-1)) - \binom{|I|}{2}$.

Therefore, $f(I) - f(I - j) = a_j - (j + |I| - 2)$.

Since $f(I) - f(I - j) < 0$, so $a_j - (j + |I| - 2) < 0$.

Again, $f(I \cup \{i\}) = \sum_{t \in I} a_t + a_i - (\sum_{t \in I}(t-1) + (i-1)) - \binom{|I|+1}{2}$.

So, $f(I \cup \{i\}) - f(I) = a_i - (i-1) - |I|$.

As $f(I \cup \{i\}) - f(I) \geq 0$, therefore $a_i - (i-1) - |I| \geq 0$.

Thus, $a_j < j + |I| - 2$ and $a_i < i + |I| - 1$.

Therefore, $i + |I| - 1 \leq a_i \leq a_j < j = |I| - 2$ and this gives $i + |I| - 1 < i + |I| - 1$, since $j = i + 1$. This is a contradiction and the claim is proved.

Hence, $f(I) = \sum_{i=1}^{|I|} a_i - \sum_{i=1}^{|I|}(i-1) - \binom{|I|}{2} = \sum_{i=1}^{|I|} a_i - \binom{|I|}{2} - \binom{|I|}{2} \geq -2\binom{|I|}{2} - 2\binom{|I|}{2} = 0$. $\square$

Equality in (2.8) occurs, for example, in the transitive tournament of order n with score sequence $[0, 2, \cdots, 2n-2]$ and in regular tournaments of order $2m+1$ with score sequence $[2m, 2m, \cdots, 2m]$. Also (2.8) is best possible in a certain sense since for any real $\epsilon > 0$, the inequality

$$\sum_{i \in I} a_i \geq (1+\epsilon) \sum_{i \in I}(i-1) + \binom{|I|}{2} \tag{2.9}$$

fails for some subsets I involving some oriented graphs in which the outdegree equals the indegree at each vertex.

Pirzada, Merajuddin, Samee [47] obtained the following upper bound for oriented graph scores.

Theorem 2.3.8. *A sequence $A = [a_i]_1^n$ of non-negative integers in non-decreasing order is a score sequence of an oriented graph if and only if for each subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i \in I} a_i \leq \sum_{i \in I} (i-1) + \frac{1}{2}|I|(2n - |I| - 1)$$

with equality when $|I| = n$.

Proof. A is a score sequence if and only if for every $I \subseteq [n] = \{1, 2, \cdots, n\}$ and $J = [n] - 1$,

$$\textstyle\sum_{i \in I} a_i + \sum_{i \in J} a_i = 2\binom{|I|}{2}$$

and

$$\textstyle\sum_{i \in J} a_i \geq \sum_{i \in J}(i-1) + \binom{|J|}{2},$$

or, if and only if $\sum_{i \in I} a_i = 2\binom{|I|}{2} - \sum_{i \in J} a_i \leq 2\binom{|I|}{2} - (\sum_{i \in J}(i-1) + \binom{|J|}{2}) = 2\binom{|I|}{2} - [\frac{n(n-1)}{2} - \sum_{i \in I}(i-1) + \binom{n-|I|}{2}] = \frac{n(n-1)}{2} + \sum_{i \in I}(i-1) - \binom{n-|I|}{2}] = \sum_{i \in I}(i-1) + \frac{n(n-1)}{2} - \frac{(n-|I|)(n-|I|-1)}{2} = \sum_{i \in I}(i-1) + \frac{1}{2}|I|(2n - |I| - 1)$, (because $\sum_{i \in I}(i-1) + \sum_{i \in J}(i-1) = \frac{n(n-1)}{2}$ and $|I| + |J| = n$). $\square$

Definition 2.3.9. A score sequence is said to be simple if it belongs to exactly one oriented graph. An oriented graph D is reducible if it is possible to partition its vertices into two nonempty sets V_1 and V_2 in such a way that every vertex of V_2 is adjacent to all vertices of V_1. If this is not possible D is irreducible. Let $D_1, D_2, \cdots, D_k$ be irreducible oriented graphs and let $D = [D_1, D_2, \cdots, D_k]$ denote the oriented graph having all arcs of D_i, $1 \leq i \leq k$, and every vertex of D_j is adjacent to all vertices of D_i with $1 \leq i < j \leq k$. $D_1, D_2, \cdots, D_k$ are called irreducible components of D. A score sequence A is said to be irreducible if all the oriented graphs D with score sequence A are irreducible.

We note that the score sequence A is irreducible if and only if the inequalities in Avery's theorem are strict for all $1 \leq k \leq n-1$. A is irreducible if D is irreducible and the irreducible components of A are the score sequences of the irreducible components of D. Pirzada [40] showed that [0] and [1,1] are the only irreducible score sequences that are simple. Thus the score sequence A of an oriented graph is simple if and only if every irreducible component of A is one of $[0]$ or $[1, 1]$.

CHAPTER 3

Marks in digraphs

In this chapter we introduce the concept of marks, non-negative integers attached to the vertices of an r-digraph. We obtain several necessary and sufficient conditions for the sequence of non-negative integers to be the mark sequence of r-digraphs. These conditions provide algorithms for constructing corresponding r-digraphs. We obtain stronger inequalities for marks in digraphs. We characterize irreducible and uniquely realizable mark sequences in r-digraphs.

3.1 Introduction

We start with the following definition of a multidigraph.

Definition 3.1.1. An r-digraph (or multidigraph) is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. An r-digraph D is complete if there are exactly r arcs between every pair of vertices of D. In an r-digraph D, if there are exactly r arcs which are parallel, then D is called an r-tournament. A double tournament can be treated as a tournament whose arcs have been duplicated.

Let D be an r-digraph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote respectively the outdegree and indegree, of a vertex v_i.

Definition 3.1.2. The mark (or r-score) p_{v_i} (or simply p_i) of v_i is defined as

$$p_i = r(n-1) + d^+_{v_i} - d^-_{v_i}.$$

Note $0 \leq p_{v_i} \leq 2r(n-1)$. The sequence $P = [p_i]_1^n$ in non-decreasing order is called the mark sequence of D. A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is said to be realizable if there exists an r-digraph whose mark sequence is P. Clearly 1-digraph is an oriented graph and a complete 1-digraph is a tournament.

Definition 3.1.3. A regular r-digraph on n vertices is one whose all vertices have marks $r(n-1)$. The converse D' of an r-digraph D is obtained by reversing each arc of D.

An r-digraph can be interpreted as the result of a competition in which the participants play each other at most r times, with an arc from u to v if and only if u defeats v. A player receives two points for each win, and one point for each tie, that is the case in which two players do not play one another or the competition between the players yields no result. With this marking system, player v receives a total of p_v points.

Between any two vertices u and v in an r-digraph, we have $u(x-y)v$, where $0 \leq x \leq r$, $0 \leq y \leq r$ and $0 \leq x+y \leq r$. In particular, we have one of the following possibilities between any two vertices u and v in a 2-digraph. (i) Exactly two arcs directed from u to v, and no arc directed from v to u, and this is denoted by $u(2-0)v$. (ii) Exactly one arc from u to v, and exactly one arc from v to u, and this is denoted by $u(1-1)v$. (iii) Exactly one arc from u to v, and no arc from v to u. This is denoted by $u(1-0)v$. (iv) No arcs from u to v, and no arc from v to u, and is denoted by $u(0-0)v$.

An r-triple in an r-digraph is an induced r-subdigraph with three vertices and is of the form $u(x_1-x_2)v(y_1-y_2)w(z_1-z_2)u$, where for $i=1,2$, we have $0 \leq x_i, y_i, z_i \leq r$ and $0 \leq \sum_1^2 x_i, \sum_1^2 y_i, \sum_1^2 z_i \leq r$. Further, in an r-digraph, an oriented triple (1-triple) is an induced 1-subdigraph with three vertices. An oriented triple is said to be transitive if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. An r-triple is said to be transitive if it contains only transitive 1-triples and an r-digraph is said to be transitive if every of its r-triples is transitive. In particular, a triple C in a 2-digraph is transitive if every oriented triple of C is transitive.

3.2 Characterization of mark sequences

The following result can be easily established.

Lemma 3.2.1. *If D and D' are two r-digraphs with the same mark sequence, then D can be transformed to D' by successively transforming (i) appropriate oriented triples in one of the following ways,*
either (a) by changing the intransitive oriented triple $u(1-0)v(1-0)w(1-0)u$, to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$, which has the same mark sequence, or vice versa,
or (b) by changing an intransitive oriented triple $u(1-0)v(1-0)w(0-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$, which has the same mark sequence, or vice versa.
or (ii) by changing a double $u(1-1)v$ to a double $u(0-0)v$ which has the same mark sequence, or vice versa.

As an application of Lemma 3.2.1, we have the following observation.

Lemma 3.2.2. *Among all r-digraphs with a given mark sequence those with the fewest arcs are transitive.*

We have the following results on marks in 2-digraphs.

Theorem 3.2.3. *Let $[p_i]_{i=1}^n$ be a sequence of non-negative integers with $p_1 \le p_2 \le \cdots \le p_k = p_{k+1} = \cdots = p_{k+m-1} < p_{k+m} \le p_{k+m+1} \le \cdots \le p_n$ and let $P' = [p'_i]_1^n$ with*

$$P'_i = \begin{cases} p_i - 1, & \text{for } i = k, \\ p_i + 1, & \text{for } i = k+m-1, \\ p_i, & \text{otherwise.} \end{cases}$$

Then P is a mark sequence of a 2-digraph if and only if P' is a mark sequence of a 2-digraph.

Proof. Clearly, $k \ge 1$ and $m \ge 2$, and that either $k+m-1 = n$, or $p_k = p_{k+1} = \cdots = p_{k+m-1} < p_{k+m}$. Now, P' is defined as (for $1 \le i \le n$),

$$P_i' = \begin{cases} p_i - 1, & \text{for } i = k, \\ p_i + 1, & \text{for } i = k + m - 1, \\ p_i, & \text{otherwise.} \end{cases}$$

Clearly, $p_1' \leq p_2' \leq \cdots \leq p_n'$.

Let P' be a mark sequence of some 2-digraph D' with n vertices in which vertex v_i' has mark p_i', $1 \leq i \leq n$. We denote v_{k+m-1}' by v_j'. Then $p_j' = p_k' + 2$. If in D', $v_j'(2-0)v_k'$, or $v_j'(1-1)v_k'$, or $v_j'(1-0)v_k'$, or $v_j'(0-1)v_k'$, or $v_j'(0-0)v_k'$, then transforming these respectively to $v_j'(1-0)v_k'$, or $v_j'(0-1)v_k'$, or $v_j'(0-0)v_k'$, or $v_j'(0-2)v_k'$, or $v_j'(0-1)v_k'$, we obtain a 2-digraph D with mark sequence P.

If $v_j'(0-2)v_k'$, claim that there exists at least one vertex w' in $W' = V' - \{v_j', v_k'\}$ such that the 2-triple C formed by v_k', v_j' and w' contains at least one intransitive 1-triple of the form $v_k'(1-0)v_j'(1-0)w'(1-0)v_k'$, or $v_k'(1-0)v_j'(0-0)w'(1-0)v_k'$, or $v_k'(1-0)v_j'(1-0)w'(0-0)v_k'$, which can be transformed respectively to $v_k'(0-0)v_j'(1-0)w'(0-0)v_k'$, or $v_k'(0-0)v_j'(0-0)w'(0-1)v_k'$, or $v_k'(0-0)v_j'(0-1)w'(0-0)v_k'$ with marks remaining unchanged.

Assume that this is not true, so that for every vertex $w' \in W'$, the 2-triple C formed by v_k', v_j' and w' contains only transitive 1-triples of the form
(i) $v_k'(1-0)v_j'(1-0)w'(0-1)v_k'$, (ii) $v_k'(1-0)v_j'(0-1)w'(1-0)v_k'$
(iii) $v_k'(1-0)v_j'(0-1)w'(0-1)v_k'$, (iv) $v_k'(1-0)v_j'(0-0)w'(0-1)v_k'$
(v) $v_k'(1-0)v_j'(0-1)w'(0-0)v_k'$, (vi) $v_k'(1-0)v_j'(0-0)w'(0-0)v_k'$.

If at least one among (i)-(vi) appears in C, then clearly $p_j' < p_k'$ since number of arcs directed away from v_j' is less than those directed away from v_k', and number of arcs directed towards v_j' is greater than those directed towards v_k'. So, we get a contradiction.

If (i) appears for every vertex w' in W, so that 2-triple C formed by v_k', v_j' and w' is of the form $v_k'(2-0)v_j'(2-0)w'(0-1)v_k'$, then

$$p_j' = 2n - 2 + d^+_{v_j'} - d^-_{v_j'} = 2n - 2 + 2(n-2) - 2 = 4n - 8,$$

and

$$p_k' = 2n - 2 + d^+_{v_k'} - d^-_{v_k'} = 2n - 2 + 2(n-2) - 2 = 3n - 2,$$

Therefore, $p_j' = p_k' + n - 6$.

For each $n \neq 8$, clearly $p_j' \neq p_k' + 2$, a contradiction.

If $n = 8$, $p'_j = p'_k + 2$, but then for any w',$v'_k(2-0)v'_j(2-0)w'(0-1)v'_k$ can be transformed to $v'_k(1-0)v'_j(1-0)w'(0-2)v'_k$, and the marks remaining unchanged.

If (ii) appears for every vertex w' in W, so that the 2-triple C formed is of the form $v'_k(2-0)v'_j(1-0)w'(2-0)v'_k$, then $p'_j = n-2$ and $p'_k = 4$.

Thus, $p'_j = p'_k + n - 6$.

For each n, $n \neq 8$, clearly $p'_j \neq p'_k + 2$, a contradiction. For $n = 8$, $p'_j = p'_k + 2$, but then for some $'w$, $v'_k(2-0)v'_j(0-1)w'(2-0)v'_k$ can be transformed to $v'_k(1-0)v'_j(0-2)w'(1-0)v'_k$, with marks remaining unchanged.

Hence in all cases, we obtain $v'_k(1-0)v'_j$, and marks remaining unchanged. Then, transforming $v'_k(1-0)v'_j$ to $v'_k(2-0)v'_j$, we get a 2-digraph D with mark sequence P.

Now, let P be a mark sequence of some 2-digraph D with n vertices in which vertex v_i has mark p_i , $1 \leq i \leq n$. Then, $p_j = p_k$. We denote v_{k+m-1} by v_j. If in D, either $v_j(0-2)v_k$, or $v_j(1-1)v_k$, or $v_j(1-0)v_k$, or $v_j(0-1)v_k$, or $v_j(0-0)v_k$, then transforming these respectively to $v_j(0-1)v_k$, or $v_j(1-0)v_k$, or $v_j(2-0)v_k$, or $v_j(1-1)v_k$, or $v_j(1-0)v_k$, we get a 2-digraph with mark sequence P'.

If $v_j(2-0)v_k$, we claim that there exists at least one vertex w in $W = V - \{v_j, v_k\}$ such that the 2-triple C formed by the vertices v_j, v_k and w contains at least one intransitive 1-triple of the form $v_j(1-0)v_k(1-0)w(1-0)v_j$, $v_j(1-0)v_k(1-0)w(0-0)v_j$, or $v_j(1-0)v_k(0-0)w(1-0)v_j$, which can be transformed respectively to $v_j(0-0)v_k(0-0)w(0-0)v_j$, $v_j(0-0)v_k(0-0)w(0-1)v_j$, or $v_j(0-0)v_k(0-1)w(0-0)v_j$ with the marks remaining same.

Assume that this is not true, so that for every vertex $w \in W$, the 2-triple C formed by v_j, v_k and w contains only transitive 1-triples of the form (i) $v_j(1-0)v_k(1-0)w(0-1)v_j$, (ii) $v_j(1-0)v_k(0-1)w(1-0)v_j$, (iii) $v_j(1-0)v_k(0-1)w(0-1)v_j$, (iv) $v_j(1-0)v_k(0-1)w(0-0)v_j$, (v) $v_j(1-0)v_k(0-0)w(0-1)v_j$, (vi) $v_j(1-0)v_k(0-0)w(0-0)v_j$.

If at least one among (i)-(vi) appear in C, then clearly $p_j > p_k$, since in each case the number of arcs directed away from v_j is greater than those directed away from v_k, and the number of arcs directed towards v_j is less than those directed towards v_k. Therefore, we get a contradiction.

If (i) appears for every vertex w in W so that C is of the form $v_j(2-$

$0)v_k(2-0)w(0-1)v_j$, then

$$p_j = 2n-2+d^+_{v_j} - d^-_{v_j} = 2n-2+2+n-2 = 3n-2$$

and

$$p_k = 2n-2+d^+_{v_j} - d^-_{v_j} = 2n-2+2(n-2)-2 = 4n-8.$$

For every $n \neq 6$, $p_j \neq p_k$, again a contradiction. If $n = 6$, we have $p_j = p_k$. But then for any w, $v_j(2-0)v_k(2-0)w(0-1)v_j$ can be transformed to $v_j(1-0)v_k(1-0)w(0-2)v_j$ with the marks remaining unchanged.

If (ii) appears for every vertex w in W so that C is of the form $v_j(2-0)v_k(0-1)w(2-0)v_j$, then $p_j = 2n-2+2-2(n-2) = 4$, and $p_k = 2n-2-2-(n-2) = n-2$. Clearly, for every $n \neq 6$, $p_j \neq p_k$, and we get a contradiction. For $n = 6$, we get $p_j = p_k$. But then for any w, $v_j(2-0)v_k(0-1)w(2-0)v_j$ can be transformed to $v_j(1-0)v_k(0-2)w(1-0)v_j$ with the marks unchanged.

Thus in all cases, we have $v'_j(0-1)v'_k$, and transforming it to $v'_j(0-2)v'_k$, we obtain a 2-digraph D with mark sequence P. □

Theorem 3.2.4. *Let $P = [p_i]_1^n$ be a sequence of non-negative integers in non-decreasing order with at least two terms p_t and p_r such that $1 \leq p_r - p_t \leq 3$ and let $P' = [p'_i]_1^n$ with*

$$P'_i = \begin{cases} p_i - 1, & \textit{for } i = t, \\ p_i + 1, & \textit{for } i = r, \\ p_i, & \textit{otherwise.} \end{cases}$$

Then P is a mark sequence of a 2-digraph if and only if P' is a mark sequence of a 2-digraph.

Proof. Let the sequence P contain at least two terms p_t and p_r such that $1 \leq p_r - p_t \leq 3$, where without loss of generality, we may assume that $p_{t-1} < p_t$ and $p_r < p_{r+1}$. If, (i) $p_{t-q-1} < p_{t-q} = \cdots = p_{t-1} = p_t$, we take $1 \leq p_r - p_{t-q} \leq 3$, or (ii) $p_r = p_{r+1} = \cdots = p_{r+m} < p_{r+m+1}$, we take $1 \leq p_{r+m} - p_t \leq 3$, or if both (i) and (ii), we take $1 \leq p_{r+m} - p_{t-q} \leq 3$. As P' is defined as (for $1 \leq i \leq n$),

$$P_i' = \begin{cases} p_i - 1, & \text{for } i = t, \\ p_i + 1, & \text{for } i = r, \\ p_i, & \text{otherwise.} \end{cases}$$

Therefore, $p_1' \leq p_2' \leq \cdots \leq p_n'$.

Let P' be a mark sequence of some 2-digraph D' in which vertex v_i' has mark p_i', $1 \leq i \leq n$. Then $3 \leq p_r' - p_t' \leq 5$. If in D', $v_r'(2-0)v_t'$, or $v_r'(1-1)v_t'$, or $v_r'(1-0)v_t'$, or $v_r'(0-1)v_t'$, or $v_r'(0-0)v_t'$, transforming these respectively to $v_r'(1-0)v_t'$, or $v_r'(0-1)v_t'$, or $v_r'(1-1)v_t'$, or $v_r'(0-2)v_t'$, or $v_r'(0-1)v_t'$ we obtain a 2-digraph with mark sequence P.

If $v_r'(0-2)v_t'$, we claim that there exists at least one vertex w' in $W' = V' - \{v_r', v_t'\}$ such that the 2-triple C formed by the vertices v_r', v_t' and w' contains at least one intransitive 1-triple of the form $v_t'(1-0)v_r'(1-0)w'(1-0)v_t'$, or $v_t'(1-0)v_r'(1-0)w'(0-0)v_t'$, or $v_t'(1-0)v_r'(0-0)w'(1-0)v_t'$, which can be transformed respectively to $v_t'(0-0)v_r'(0-0)w'(0-0)v_t'$, or $v_t'(0-0)v_r'(0-0)w'(0-1)v_t'$, or $v_t'(0-0)v_r'(0-1)w'(0-0)v_t'$, with marks remaining unchanged.

Assume that this is not true, so that for every vertexw' in W', the 2-triple C formed by v_r', v_t' and w' contains only transitive 1-triples of the form (i) $v_t'(1-0)v_r'(1-0)w'(0-1)v_t'$, (ii) $v_t'(1-0)v_r'(0-1)w'(1-0)v_t'$, (iii) $v_t'(1-0)v_r'(0-1)w'(0-1)v_t'$, (iv) $v_t'(1-0)v_r'(0-0)w'(0-1)v_t'$, (v) $v_t'(1-0)v_r'(0-1)w'(0-0)v_t'$, (vi) $v_t'(1-0)v_r'(0-0)w'(0-0)v_t'$.

If at least one among (i)-(vi) appear in C, then $p_r' < p_t' + 3$, since the number of arcs directed away from v_r' is less than those directed away from v_t', and the number of arcs directed towards v_r' is greater then those directed towards v_t'. This is a contradiction.

If (i) appears for every vertex w' in W' so that C is of the form $v_t'(2-0)v_r'(2-0)w'(0-1)v_t'$, then

$$p_r' = 2n - 2 + d^+_{v_r'} - d^-_{v_r'} = 2n - 2 + 2(n-2) - 2 = 4n - 8,$$

and

$$p_t' = 2n - 2 + d^+_{v_r'} - d^-_{v_r'} = 2n - 2 + 2 + n - 2 = 3n - 2.$$

Therefore, $p_r' - p_t' = n - 6$.

Clearly, for $n \neq 9, 10, 11$, we have $6 \leq p_r' - p_t' \leq 2$, which is a contradiction.

For $n = 9, 10, 11, p'_r - p'_t = 3, 4, 5$. But then for any w', $v'_t(2-0)v'_r(2-0)w'(0-1)v'_t$ can be transformed to $v'_t(1-0)v'_r(1-0)w'(0-2)v'_t$ without changing the marks.

If (ii) appears for every vertex w' in W' so that C is of the form $v'_t(2-0)v'_r(0-1)w'(2-0)v'_t$, then $p'_r = 2n-2-2-(n-2) = n-2$, and $p'_t = 2n-2+2-2(n-2) = 4$.

Therefore, $p'_r - p'_t = n-6$.

For $n \neq 9, 10, 11$, clearly $6 \leq p'_r - p'_t \leq 2$, a contradiction.

For $n = 9, 10, 11$, we get $p'_r - p'_t = 3, 4, 5$. But then for any w', $v'_t(2-0)v'_r(0-1)w'(2-0)v'_t$ can be transformed to $v'_t(1-0)v'_r(0-2)w'(1-0)v'_t$ with the marks remaining unchanged.

Hence in all cases, we have $v'_t(1-0)v'_r$, and then transforming it to $v'_t(2-0)v'_r$, we obtain a 2-digraph D with mark sequence P.

Conversely, let P be a mark sequence of some 2-digraph D in which vertex v_i has mark p_i, $1 \leq i \leq n$. Then, $1 \leq p_r - p_t \leq 3$. If in D, either $v_t(2-0)v_r$, or $v_t(1-1)v_r$, or $v_t(1-0)v_r$, or $v_t(0-1)v_r$, or $v_t(0-0)v_r$, then transforming them respectively to $v_t(1-0)v_r$, or $v_t(0-1)v_r$, or $v_t(1-1)v_r$, or $v_t(0-2)v_r$, or $v_t(0-1)v_r$, we get a 2-digraph with mark sequence P'.

If in D, $v_t(0-2)v_r$, we claim that there exists at least one vertex w in $W = V - \{v_r, v_t\}$ such that the 2-triple C formed by the vertices v_r, v_t and w contains at least one intransitive 1-triple of the form $v_r(1-0)v_t(1-0)w(1-0)v_r$, $v_r(1-0)v_t(1-0)w(0-0)v_r$, or $v_r(1-0)v_t(0-0)w(1-0)v_r$. Then these can be respectively transformed to $v_r(0-0)v_t(0-0)w(0-0)v_r$, or $v_r(0-0)v_t(0-0)w(0-1)v_r$, or $v_r(0-0)v_t(0-1)w(0-0)v_r$ with the marks remaining same.

If this is not true, then for every vertex w in W, the 2-triple C formed by v_r, v_t and w contains only transitive 1-triples of the form (i) $v_r(1-0)v_t(1-0)w(0-1)v_r$, or (ii) $v_r(1-0)v_t(0-1)w(1-0)v_r$, or (iii) $v_r(1-0)v_t(0-1)w(0-1)v_r$, or (iv) $v_r(1-0)v_t(0-1)w(0-0)v_r$, or (v) $v_r(1-0)v_t(0-0)w(0-1)v_r$, or (vi)$v_r(1-0)v_t(0-0)w(0-0)v_r$.

If at least one among (i) - (vi) appear in C, clearly $p_r > p_t + 3$, since outgoing arcs from v_r is greater than those going out of v_t, and incoming arcs to v_t is greater than those of v_r. Thus, we get a contradiction.

If (i) appears for every vertex w in W, so that C is of the form $v_r(2-$

$0)v_t(2-0)w(0-1)v_r$, then $p_t = 2n-2+2(n-2)-2 = 4n-8$, and $p_r = 2n-2+2+(n-2) = 3n-2$.

Therefore, $p_r - p_t = 6-n$.

Clearly, for $n \leq 3,4,5$, we have $p_r - p_t \geq 4$, or $p_r - p_t \leq 0$ which is a contradiction.

For $n = 3,4,5$, we obtain $1 \leq p_r - p_t \leq 3$. But then $v_r(2-0)v_t(2-0)w(0-1)v_r$, can be transformed to $v_r(1-0)v_t(1-0)w(0-2)v_r$, with marks remaining unchanged.

If (ii) appears for every vertex w in W, so that C is of the form $v_r(2-0)v_t(0-1)w(2-0)v_r$, then $p_t = 2n-2-(n-2)-2 = n-2$, and $p_r = 2n-2+2-2(n-2) = 4$.

So, $p_r - p_t = 6-n$.

For $n \leq 3,4,5$, we have $p_r - p_t \geq 4$, or $p_r - p_t \leq 0$, which is a contradiction. For $n = 3,4,5$, we obtain $1 \leq p_r - p_t \leq 3$, but then we can transform $v_r(2-0)v_t(0-1)w(2-0)v_r$ to $v_r(1-0)v_t(0-2)w(1-0)v_r$ with the marks remaining unchanged.

Hence in all cases, we have $v_r(1-0)v_t$, and finally transforming it to $v_r(2-0)v_t$, we obtain a 2-digraph D' with mark sequence P'. $\square$.

An analogous result to Landau$'$s theorem on tournament scores is the following characterization of marks in 2-digraphs by Pirzada and Samee [42].

Theorem 3.2.7. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of a 2-digraph if and only*

$$\sum_{i=1}^{k} p_i \geq 2k(k-1), \tag{3.1}$$

for $1 \leq k \leq n$, with equality when $k = n$.

Theorem 3.2.8. *Let P and P' be given as in Theorem 3.2.3. Then P satisfies (3.1) if and only if P' satisfies (3.1).*

Proof. If P satisfies (3.1),then

$$\sum_{i=1}^{j} p'_i =$$

$$\begin{cases} \sum_{i=1}^{j} p_i, & \text{for } j \leq k-1, \\ \sum_{i=1}^{k-1} p_i + (p_k - 1) + \sum_{i=k+1}^{j} p_i, & \text{for } k \leq j \leq k+m-2, \\ \sum_{i=1}^{k-1} p_i + (p_k - 1) + \sum_{i=k+1}^{k+m-2} p_i + (p_{k+m-1} + 1) + \sum_{i=k+m}^{j} p_i, & \text{for } j \geq k+m-1. \end{cases}$$

When $j \leq k-1$ and $j \geq k+m-1$, we observe that $\sum_{i=1}^{j} p'_i \geq 2j(j-1)$. When $k \leq j \leq k+m-2$, we show that $\sum_{i=1}^{j} p_i > 2j(j-1)$, $k \leq j \leq k+m-2$.

Assume to the contrary, that for some j, $k \leq j \leq k+m-2$, $\sum_{i=1}^{j} p_i \leq 2j(j-1)$. From conditions (3.1), we have, $\sum_{i=1}^{j} p_i \geq 2j(j-1)$. Combining the two, we get $\sum_{i=1}^{j} p_i = 2j(j-1)$. Therefore, again by (3.1), we have

$$p_{j+1} + 2j(j-1) = p_{j+1} + \sum_{i=1}^{j} p_i = \sum_{i=1}^{j+1} p_i \geq 2j(j+1) = 2j(j-1) + 4j.$$

Therefore, $p_{j+1} \geq 4j$. Also, $p_j = p_{j+1}$ gives $p_j \geq 4j$. Thus,

$$\sum_{i=1}^{j} p_i = \sum_{i=1}^{j-1} p_i + p_j \geq 2(j-1)(j-2) + 4j = 2j(j-1) + 4 > 2j(j-1),$$

which contradicts our assumption. Thus, we have

$$\sum_{i=1}^{j} p'_i = \sum_{i=1}^{j} p_{i-1} > 2j(j-1) - 1 \geq 2j(j-1).$$

Hence, in all cases, P' satisfies (3.1).

If P' satisfies (3.1) then it is easy to see that P also satisfies (3.1). $\square$

Lemma 3.2.9. *Let P and P' be given as in Theorem 3.2.4. Then P satisfies (3.1) if and only if P' satisfies (3.1).*

Proof. If P satisfies (3.1), then

$$\sum_{i=1}^{j} p'_i =$$

$$\begin{cases} \sum_{i=1}^{j} p_i, & \text{for } j \leq t-1, \\ \sum_{i=1}^{t-1} p_i + (p_t - 1) + \sum_{i=t+1}^{r-1} p_i, & \text{for } t \leq j \leq r-1, \\ \sum_{i=1}^{t-1} p_i + (p_t - 1) + \sum_{i=t+1}^{r-1} p_i + (p_r + 1) + \sum_{i=r+1}^{j} p_i, & \text{for } j \geq r. \end{cases}$$

For $j \leq t-1$ and $j \geq r$, clearly, $\sum_{i=1}^{j} p'_i \geq 2j(j-1)$. For $t \leq j \leq r-1$, claim, $\sum_{i=1}^{j} p_i > 2j(j-1)$. If not, let for some j, $t \leq j \leq r-1$, $\sum_{i=1}^{j} p_i \leq 2j(j-1)$. From conditions (3.1), we have $\sum_{i=1}^{j} p_i \geq 2j(j-1)$. Combining the two, we get $\sum_{i=1}^{j} p_i = 2j(j-1)$. Again by (3.1), we have

$$p_{j+1} + 2j(j-1) = p_{j+1} + \sum_{i=1}^{j} p_i = \sum_{i=1}^{j+1} p_i \geq 2j(j+1) = 2j(j-1) + 4j.$$

Thus, $p_{j+1} \geq 4j$. Now, $1 \leq p_r - p_t \leq 3$, so that $p_t = p_r - x$, where $1 \leq x \leq 3$. If p_t and p_r are consecutive terms, then $j = t$ and $j+1 = t+1 = r$. Therefore, $p_r = p_{t+1} \geq 4t$ so that $p_t \geq 4t - x$. Now,

$$\sum_{i=1}^{t} p_i = \sum_{i=1}^{t-1} p_i + p_t \geq 2(t-1)(t-2) + p_t \geq 2(t-1)(t-2) + 4t - x = 2t(t-1) + 4 - x > 2t(t-1),$$

as $1 \leq x \leq 3$, and thus contradicts the assumption. If $p_{t-1} < p_t = p_{t+1} = \cdots = p_j = p_{j+1} = \cdots = p_{r-1} < p_r$, then $p_t = 4t$, so that

$$\sum_{i=1}^{t} p_i = \sum_{i=1}^{t-1} p_i + p_t \geq 2(t-1)(t-2) + 4t = 2t(t-1) + 4 > 2t(t-1)$$

again a contradiction. Thus the claim is proved.

Therefore, $\sum_{i=1}^{j} p'_i = \sum_{i=1}^{j} p_i > 2j(j-1) - 1 \geq 2j(j-1)$.

If P' satisfies (3.1), then P also satisfies (3.1). □

Proof of Theorem 3.2.7. Necessity. Let D be a 2-digraph with mark sequence$[p_i]_1^n$. Let W be the 2-subdigraph induced by any set of k vertices $w_1, w_2, \cdots, w_k$ of D. Let α denote the number of arcs of D that start in

W and end outside W, and let β denote the number of arcs of D that start outside of W and end in W. Note that each vertex w in W, and for every vertex v of D not in W, there are at most two arcs from v to w, so that $\beta \leq 2k(n-k)$. Therefore, we have $\beta \leq 2nk - 2k^2$. Then, $\sum\limits_{i=1}^{k} p_{w_i} = \sum\limits_{i=1}^{k}(2n-2+d_D^+(w_i)-d_D^-(w_i)) = 2nk-2k+\sum\limits_{i=1}^{k} d_D^+(w_i) - \sum\limits_{i=1}^{k} d_D^-(w_i) = 2nk-2k+[\sum\limits_{i=1}^{k} d_W^+(w_i)+\alpha] - [\sum\limits_{i=1}^{k} d_W^-(w_i)+\beta] = 2nk-2k+$(number of arcs of W)$+\alpha-$(number of arcs of W)$-\beta$. Therefore,

$$\sum_{i=1}^{k} p_{w_i} = 2nk - 2k + \alpha - \beta \tag{3.2}$$

Now, from (3.2) we have,

$$\sum_{i=1}^{k} p_{w_i} \geq 2nk - 2k - \beta \geq 2nk - 2k - 2nk + 2k^2 = 2k(k-1).$$

Applying this result to the k vertices with marks $p_1, p_2, \cdots, p_k$ yields the desired inequality. If $k = n$, then $\alpha = \beta = 0$, and the required equality follows from Equation (3.2).

Sufficiency. Clearly, the sequence $P_n = [0, 4, 8, \cdots, 4n-4]$ satisfies conditions (3.2) as it is the mark sequence of the transitive double tournament. In a sequence $P \neq P_n$, satisfying (3.1), we have $p_1 \geq 0$ and $p_n \leq 4n-4$. We claim that P contains either (a) a repeated term, or (b) at least two terms, say p_r and p_t such that $1 \leq p_r - p_t \leq 3$, or both (a) and (b). To verify the claim, suppose that there is no repeated term. Then, $p_1 < p_2 < \cdots < p_n$. If there is no consecutive pair $p_i < p_{i+1}$ for which $1 \leq p_{i+1} \leq p_i \leq 3$, then $p_{i+1} - p_i \geq 4$, for all $1 \leq i \leq n$. Since $p_1 \geq 0$, $p_2 \geq 4$, $p_3 \geq 8, \cdots$, $p_n \geq 4(n-1)$. Thus, by (3.1)

$$2n(n-1) = \sum_{i=1}^{n} p_i \geq 4\sum_{i=1}^{n-1} i = 4\frac{(n-1)n}{2} = 2n(n-1).$$

Thus there is equality throughout. This implies that $p_i = 4(i-1)$, and that $P = P_n$, a contradiction.

In case of (a), when P has a repeated term, reduce its first occurrence by one, and increase its last occurrence by one to form P', and in case of (b) when P contains at least two terms, say p_r and p_t with $1 \leq p_r - p_t \leq 3$, reduce p_t by one and increase p_r by one to form P'. The process of applying (a), or (b), or both is repeated (using Theorem 3.2.3 and Theorem 3.2.4) till we get the sequence P_n. Let the total order on the non-negative integer sequences of length n be defined by $X = [x_1, x_2, \cdots, x_n] \preceq Y = [y_1, y_2, \cdots, y_n]$ if either $X = Y$, or for some i, $1 \leq i \leq n$, $x_n = y_n$, $x_{n-1} = y_{n-1}$,$\cdots$,$x_{i+1} = y_{i+1}$, $x_i < y_i$. Clearly,$\preceq$ is reflexive, antisymmetric, transitive, and satisfies comparability, we write $X \prec Y$ if $X \preceq Y$ but $X \neq Y$. For any sequence $P \neq P_n$, satisfying (1), we form another sequence P' satisfying (3.1) such that $P \prec P'$, and P is mark sequence if and only if P' is a mark sequence. Therefore, by repeated application of this transformation, starting from the original sequence satisfying (3.1), we reach P_n. Hence P is a mark sequence.$\square$

The following is the combinatorial criteria for sequences of non-negative integers to be the mark sequence of an r-digraph. One proof of this characterization can be seen in Pirzada [43] and the other proof uses networks and flows has appeared in Pirzada and Samee [42].

Theorem 3.2.10. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of a r-digraph if and only*

$$\sum_{i=1}^{k} p_i \geq rk(k-1), \tag{3.3}$$

for $1 \leq k \leq n$, with equality when $k = n$.

Proof. (i) SUFFICIENCY. Let $q_i = p_i - r(n-1)$. Then $\sum_{i=1}^{n} q_i = 0$, and we may assume that $q_1 \leq q_2 \leq \cdots \leq q_k < 0 \leq q_{k+1} \leq \cdots \leq q_n$.

Construct a network with vertex set $\{s, v_1, v_2, \ldots, v_n, t\}$ of cardinality $n+2$ as follows.

1. There are arcs (s, v_i), $1 \leq i \leq k$ from the source s to vertex v_i. The arc (s, v_i) has capacity $-q_i$, $1 \leq i \leq k$.

2. Arcs (v_i, t) from v_i to the sink t, $r+1 \leq i \leq n$. The arc (v_i, t) has capacity $-q_i$.

3. For each pair v_i, v_j of distinct vertices ($i \neq j$), we have one arc from v_i to v_j and one arc from v_j to v_i, each with capacity r.

It is easy to check that a r-digraph with mark sequence $[p_i]_i^n$ can be obtained from an integral flow of value $-\sum_{i=1}^k q_i = \sum_{i=k+1}^n q_i$ by reducing the flow on cycles of length 2 until one of the two edges has flow value zero.

In view of the max-flow-min-cut-Theorem, it suffices to check that each cut has capacity at least $\sum_{i=k+1}^n q_i$.

We thus assume that $\{s\} \cup C$ is a cut, $C \subseteq \{v_1, v_2, \ldots, v_n\}$, $|C| = t$, and that $|C \cap \{v_1, v_2, \ldots, v_k\}| = a$ and $|C \cap \{v_{k+1}, v_{k+2}, \ldots, v_n\}| = b = t - a$.

For its capacity, we have the following estimate.

$$\begin{aligned}
\operatorname{cap}(\{s\} \cup C) &= \sum_{i: i \leq k, v_i \notin C} -q_i + \sum_{i: i > k, v_i \in C} q_i + t(n-t) \cdot r \\
&\geq -\sum_{i=a+1}^{k} q_i + \sum_{i=k+1}^{k+b} q_i + t(n-t) \cdot r.
\end{aligned}$$

This expression is bounded from below by $-\sum_{i=1}^k q_i = \sum_{i=k+1}^n q_i$ if and only if

$$\sum_{i=1}^{a} q_i + \sum_{i=k+1}^{k+b} q_i + t(n-t) \cdot r \geq 0,$$

if and only if

$$\sum_{i=1}^{a} p_i + \sum_{i=k+1}^{k+b} p_i + t(n-t) \cdot r \geq t \cdot r(n-1)$$

(since $p_i = r(n-1) + q_i$), if and only if

$$\sum_{i=1}^{a} p_i + \sum_{i=k+1}^{k+b} p_i \geq rt(t-1).$$

This latter inequality is certainly implied by the inequality

$$\sum_{i=1}^{t} p_i \geq rt(t-1),$$

since the p_i are non-decreasing.

(ii) NECESSITY. Follows from the construction in (i) if we use the cuts $\{s\} \cup \{v_1, v_2, \ldots, v_t\}$, $1 \leq t \leq n$. $\Box$

Now we give two recursive characterizations for mark sequences in r-digraphs.

Theorem 3.2.11. *Let $P = [p_i]_1^n$ be a sequence of non-negative integers in non-decreasing order, where for each i, $0 \leq p_i \leq 2r(n-1)$. Let P' be obtained from P by deleting the greatest entry $p_n (= 2r(n-1) - k$, say) and (a) if $k \leq n-1$, reducing the k greatest remaining entries by one each, or (b) if $k > n-1$, reducing the $k-(n-1)$ greatest entries by two each and the remaining $2n-2-k$ entries by one. Then P is the mark sequence of some r-digraph if and only if P' (arranged in non-decreasing order) is the mark sequence of some r-digraph.*

Proof. Let P' be a mark sequence of some r-digraph D'. If P' is obtained from P as in (a), then an r-digraph D with mark sequence P is obtained by adding a vertex v in D' such that $v((r-1)-0)v_i$ for those vertices v_i in D' with mark $v_i = p_i - 1$, and $v(r-0)v_i$ for those vertices v_i in D' with mark $v_i = p_i$. If P' is obtained from P as in (b), then again an r-digraph D with mark sequence P is obtained by adding a vertex v in D' such that $v((r-1)-1)v_i$ for those vertices v_i in D, with mark $v_i = p_i - 2$ and $v((r-1)-0)v_i$ for those vertices v_i in D' with mark $v_i = p_i - 1$.

Conversely, let P be the mark sequence of some r-digraph D. We assume D is transitive, if not D becomes transitive by using Lemma 3.2.1. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of D, and let $p_n = 2r(n-1) - k$. If $k \leq n-1$, construct D such that $v_n((r-1)-0)v_i$ for all i, $n-k \leq i \leq n-1$, and $v_n(r-0)v_j$ for all j, $1 \leq j \leq n-k-1$. Clearly, $D - v_n$ realizes P' (arranged in non-decreasing order). If $k > n-1$, construct D such that $v_n((r-1)-1)v_i$ for all i, $2n-k-1 \leq i \leq n-1$, and $v_n((r-1)-0)v_j$ for all j, $1 \leq j \leq 2n-k-2$. Then again, $D - v_n$ realizes P' (arranged in non-decreasing order). $\Box$

Theorem 3.2.11 provides an algorithm for determining whether a given non-decreasing sequence of non-negative integers is a mark sequence and for constructing a corresponding r-digraph. At each stage, we form P' according to Theorem 3.2.11 such that P' is in non-decreasing order. If $p_n = 2r(n-1) - k$, deleting p_n and performing (a) or (b) of Theorem 3.2.11 according as $k \leq n-1$ or $k > n-1$, we get $P' = [p'_1, p'_2, \cdots, p'_{n-1}]$. If the mark of vertex v_i was decreased by one in the process, then the construction yielded $v_n\,((r-1)-0)\,v_i$ and if it was decreased by two, then the construction yielded $v_n\,((r-1)-1)\,v_i$. For a vertex v_j whose mark remained unchanged, the construction yielded $v_n\,(r-0)\,v_j$. If this process is applied recursively, then it tests whether or not P is a mark sequence, and if P is a mark sequence the corresponding r-digraph with mark sequence P is constructed.

Theorem 3.2.12. *Let $P = [p_i]_1^n$ be a sequence of non-negative integers in non-decreasing order, where for each i, $0 \leq p_i \leq 2r(n-1)$. Let P' be obtained from P by deleting the greatest entry $p_n (= 2r(n-1) - k$, say) and (a) if k is even, say $k = 2t$, reducing the t greatest remaining entries by two each, or (b) if k is odd, say $k = 2t+1$, reducing the t greatest remaining entries by two and reducing the greatest among the remaining entries by one. Then P is the mark sequence of some r-digraph if and only if P' (arranged in non-decreasing order) is the mark sequence of some r-digraph.*

Proof. This can be proved by using the same argument as in the proof of Theorem 3.2.11.

Theorem 3.2.12 also provides an algorithm for determining whether a given non-decreasing sequence of non-negative integers is a mark sequence and for constructing a corresponding r-digraph.

Definition 3.2.13. In a 2-digraph, the set of distinct marks of the vertices is called its mark set. For example, the 2-digraph D with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$, and arcs as $v_5(1-0)v_4$, $v_5(1-0)v_3$, $v_4(2-0)v_3$, $v_4(1-0)v_1$, $v_3(2-0)v_2$, $v_2(1-0)v_1$ has mark sequence [6,7,7,10,10] and mark set $\{6, 7, 10\}$.

The following existence result for mark sets in 2-digraphs is due to

Pirzada and Naikoo [51].

Theorem 3.2.14. *Let $P = \{p_1, p_2, \cdots, p_n\}$ be the set of non-negative even integers in decreasing order and for all even g, $2 \leq g \leq n$,*

$$p_g > 2(p_{g-1} - p_{g-2} + \cdots - p_2 + p_1 + 1),$$

and for all odd h, $3 \leq h \leq n$,

$$p_h > 2(p_{h-1} - p_{h-2} + \cdots + p_2 - p_1 - 1).$$

Then there is a 2-digraph with mark set P.

In Theorem 3.2.14, if $p_g \leq 2(p_{g-1} - p_{g-2} + \cdots - p_2 + p_1 + 1)$, for some even g, $2 \leq g \leq n$, then the existence of a 2-digraph with mark set P is not always true. To see this, let $g = n = 2$ and $p_1 = 2$, $p_2 = 4$. If there is a 2-digraph with mark set $P = \{2, 4\}$, there exist positive integers n_1 and n_2 such that $2n_1 + 4n_2 = 2(n_1 + n_2)(n_1 + n_2 - 1) < 2(n_1 + 2n_2 = 2n_1 + 4n_2$ (since n_1 and n_2 are positive integers), implying $2n_1 + 4n_2 < 2n_1 + 4n_2$, which is impossible. Similarly, in Theorem 3.2.14, if $p_h \leq 2(p_{h-1} - p_{h-2} + \cdots + p_2 - p_1 - 1)$ for some odd h, $3 \leq h \leq n$, the existence of a 2-digraph with mark set P is not always true.

Further we note that in general, every set of odd positive integers is not the mark set of any 2-digraph. For example, there is no 2-digraph with mark set $P = \{1, 5\}$. For if $P = \{1, 5\}$ is a mark set of some 2-digraph, there exist vertices v_1 and v_2 with $p_{v_1} = 1$ and $p_{v_2} = 5$ such that $v_1(0-1)v_2$. Since $p_{v_2} = 5$, there exists another vertex v_3 with $p_{v_3} = 5$ such that $v_3(2-0)v_1$ and either $v_3(0-1)v_2$ or $v_3(0-2)v_2$. If $v_3(0-1)v_2$, then $p_{v_2} \geq 6$, or if $v_3(0-2)v_2$ then $p_{v_2} \geq 6$, both cases lead to a contradiction.

The above facts imply that in general, every set of non-negative integers is not a mark set. The next result [51] provides a construction of a 2-digraph on kr vertices using a 2-digraph on r vertices.

Theorem 3.2.15. *Let D be a 2-digraph on r vertices with mark set $\{p_1, p_2, \cdots, p_n\}$. Then for each $k \geq 1$, there exists a 2-digraph on kr vertices with mark set*

$\{p_1 + 2(k-1)r, p_2 + 2(k-1)r, \cdots, p_n + 2(k-1)r\}$.

3.3 Stronger inequalities on marks of r-digraphs

The following result gives a lower bound for $\sum\limits_{i\in I} p_i$.

Theorem 3.3.1. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of an r-digraph if and only if for every subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i\in I} p_i \geq r\sum_{i\in I}(i-1) + r\binom{|I|}{2} \tag{3.4}$$

with equality when $|I| = n$.

Proof. Sufficiency. Let the sequence $P = [p_i]_1^n$ of non-negative integers in nondecreasing order satisfy equation (3.4). Now, for any $I \subseteq [n]$, we have

$$\sum_{i\in I}(i-1) \geq \sum_{i=1}^{|I|}(i-1) = \binom{|I|}{2}.$$

Therefore, from equation (3.4), we have

$$\sum_{i\in I} p_i \geq r\sum_{i\in I}(i-1) + r\binom{|I|}{2} \geq r\binom{|I|}{2} + r\binom{|I|}{2} = 2r\binom{|I|}{2}.$$

Hence, by Theorem 3.2.3, P is a mark sequence.

Necessity. Assume that $P = [p_i]_1^n$ is a mark sequence of some r-digraph. For any subset $I \subseteq [n]$, define

$$f(I) = \sum_{i\in I} p_i - r\sum_{i\in I}(i-1) - r\binom{|I|}{2}.$$

Claim $I = \{i : 1 \leq i \leq |I|\}$. If not, then there exists $i \notin I$ and $j \in I$ such that $j = i+1$. So, $p_i \leq p_j$. For $j \in I$, we have

$$f(I) = \sum_{t\in I} p_t - r\sum_{t\in I}(t-1) - r\binom{|I|}{2},$$

$$f(I-j) = \sum_{t\in I, j\notin I} p_t + p_j - r\left(\sum_{t\in I, j\notin I}(t-1)+(j-1)\right) - r\binom{|I|-1}{2}.$$

Therefore

$$\begin{aligned} f(I) - f(I-\{j\}) &= p_j - r(j-1) - r\binom{|I|}{2} + r\binom{|I|-1}{2} \\ &= p_j - r(j-1) - r(|I|-1) \\ &= p_j - r(j+|I|-2). \end{aligned}$$

Since $f(I) - f(I-\{j\}) < 0$, so $p_j - r(j+|I|-2) < 0$. Again $f(I \cup \{i\}) = \sum_{t\in I} p_t + p_i - r\left(\sum_{t\in I}(t-1)+(i-1)\right) - r\binom{|I|+1}{2}$. So $f(I\cup\{i\}) - f(I) = p_i - r(i-1) - r\binom{|I|+1}{2} + r\binom{|I|}{2} = p_i - r(i+|I|-1)$. As $f(I\cup\{i\}) - f(I) \geq 0$, therefore $p_i - r(i+|I|)-1) \geq 0$. Thus $p_j < r(j+|I|-2)$ and $p_i \geq r(i+|I|-1)$. Therefore $r(i+|I|-1) \leq p_i \leq p_j < r(j+|I|-2)$. Since $j = i+1$, therefore $r(i+|I|-1) < r(i+1+|I|-2)$. That is, $r(i+|I|-1) < r(i+|I|-1)$, which is a contradiction. Hence

$$\begin{aligned} f(I) &= \sum_{i=1}^{|I|} p_i - r\sum_{i=1}^{|I|}(i-1) - r\binom{|I|}{2} \\ &= \sum_{i=1}^{|I|} p_i - r\binom{|I|}{2} - r\binom{|I|}{2} \\ &\geq r|I|(|I|-1) - 2r\binom{|I|}{2} = 0. \end{aligned}$$

(by Theorem 3.2.3)

Thus $\sum_{i\in I} p_i - r\sum_{i\in I}(i-1) - r\binom{|I|}{2} \geq 0$, that is, $\sum_{i\in I} p_i \geq r\sum_{i\in I}(i-1) + r\binom{|I|}{2}$. This proves the necessity. □

We note that equality can occur often in Equation (3.4). For example, in the transitive r-digraph of order n with mark sequence $[0, 2r, 4r, \cdots, 2r(n-1)]$, and in the regular r-digraph of order n with mark sequence $[r(n-1), r(n-1), \cdots, r(n-1)]$. We further observe that Theorem 3.3.1 is best possible, since for any real $\epsilon > 0$, the inequality

$$\sum_{i\in I} p_i \geq (1+\epsilon)r\sum_{i\in I}(i-1) + (1-\epsilon)r\binom{|I|}{2}$$

fails for some I, and some r-digraphs. This can been seen, for example, in the transitive r-digraph of order n with mark sequence $[0, 2r, 4r, \cdots, 2r(n-1)]$, and in the regular r-digraph of order n with mark sequence $[r(n-1), r(n-1), \cdots, r(n-1)]$.

The next result gives a set of upper bounds for $\sum_{i\in I} p_i$ and is equivalent to the set of lower bounds for $\sum_{i\in I} p_i$ in Theorem 3.3.1.

Theorem 3.3.2. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of an r-digraph if and only if for every subset $I \subseteq [n] = \{1, 2, \cdots, n\}$,*

$$\sum_{i\in I} p_i \le r\sum_{i\in I}(i-1) + \frac{1}{2}r|I|(2n - |I| - 1),$$

with equality when $|I| = n$.

Proof. We have $[n] = \{1, 2, \cdots, n\}$. Let $J = [n] - I$, so that $I + J = [n]$ and $|J| + |I| = n$. Therefore, by Theorem 3.3.1, P is a mark sequence if and only if

$$\sum_{i\in[n]} p_i = rn(n-1) \quad \text{and} \quad \sum_{i\in J} p_i \ge r\sum_{i\in J}(i-1) + r\binom{|J|}{2}$$

if and only if

$$\sum_{i\in I} p_i + \sum_{i\in J} p_i = rn(n-1) \ \text{ and } \ \sum_{i\in J} p_i \ge r\sum_{i\in J}(i-1) + r\binom{|J|}{2}$$

if and only if

$$\begin{aligned}\sum_{i\in I} p_i &= rn(n-1) - \sum_{i\in J} p_i \\ &\le rn(n-1) - r\sum_{i\in J}(i-1) - r\binom{|J|}{2} \\ &= rn(n-1) - \left(r\frac{n(n-1)}{2} - r\sum_{i\in I}(i-1)\right) - r\binom{n-|I|}{2}\end{aligned}$$

(because $r\sum_{i\in I}(i-1)+r\sum_{i\in J}(i-1)=r\binom{n}{2}$ and $|I|+|J|=n$)
Thus

$$\begin{aligned}\sum_{i\in I} p_i &= rn(n-1) - r\frac{n(n-1)}{2} + r\sum_{i\in I}(i-1) - \frac{r}{2}(n-|I|)(n-|I|-1)\\ &= r\sum_{i\in I}(i-1) + \frac{r}{2}|I|(2n-|I|-1),\end{aligned}$$

which proves the result.

We now have the following results.

Theorem 3.3.3. *If $P=[p_i]_1^n$ is a mark sequence of an r-digraph, then for each i, $r(i-1)\le p_i\le r(n+i-2)$.*
Proof. Let $I=\{i\}$ in Theorem 3.3.1 and Theorem 3.3.2. Then

$$\sum_{i\in I} p_i \ge r\sum_{i\in I}(i-1) + r\binom{|I|}{2}$$

implies that $p_i\ge r(i-1)$, and

$$\sum_{i\in I} p_i \le r\sum_{i\in I}(i-1) + \frac{r}{2}|I|(2n-|I|-1)$$

implies that $p_i\le r(n+i-2)$. Therefore

$$r(i-1)\le p_i\le r(n+i-2).$$

Theorem 3.3.4. *Let $P=[p_i]_1^n$ be a mark sequence of an r-digraph. If*

$$\sum_{i\in I} p_i = r\sum_{i\in I}(i-1) + r\binom{|I|}{2},$$

for some $I\subseteq[n]$, then one of the following holds.
(a) $I=[1,|I|]$ and $\sum_{i=1}^{|I|} p_i = r|I|(|I|-1)$.
(b) $I=[t,t+|I|-1]$ for some t, $2\le t\le n-|I|+1$,

$$\sum_{i=1}^{t+|I|-1} p_i = r(t+|I|-1)(t+|I|-2)$$

and $p_i = r(t+|I|-2)$ *for all* $i \leq t+|I|-1$.

(c) $I = [1,m] \cup [m+t, t+|I|-1]$ *for some* m *and* t *such that* $1 \leq m \leq |I|-1$ *and* $2 \leq t \leq n-|I|+1$, $\sum_{i=1}^{m} p_i = rm(m-1)$, $\sum_{i=1}^{t+|I|-1} p_i = r(t+|I|-1)(t+|I|-2)$ *and* $p_i = r(m+t+|I|-2)$ *for all* i , $m+1 \leq i \leq t+|I|-1$.

An application of Holder's theorem gives the inequalities of the sum of the squares of marks.

Theorem 3.3.5. *If* $P = [p_i]_1^n$ *is a mark sequence of an* r*-digraph, then*

(a) $\sum_{i=1}^{t} p_i^2 \geq \sum_{i=1}^{t}(2rt-2r-p_i)^2$, *for* $1 \leq t \leq n$, *with equality when* $t = n$.

(b) For $1 < g < \infty$, $\frac{1}{g} + \frac{1}{h} = 1$, $\sum_{i=1}^{t} p_i^g \geq t(rt-r)^g$, *where* $1 \leq t \leq n$, *with equality when* $t = n$ *and* $p_1 = p_2 = \cdots = p_t$.

Proof (a). By Theorem 3.2.3, we have for $1 \leq t \leq n$ with equality when $t = n$,

$$rt(t-1) \leq \sum_{i=1}^{t} p_i,$$

or

$$\sum_{i=1}^{t} p_i^2 + 2(2rt-2r)rt(t-1) \leq \sum_{i=1}^{t} p_i^2 + 2(2rt-2r)\sum_{i=1}^{t} p_i,$$

or

$$\sum_{i=1}^{t} p_i^2 + t(2rt-2r)^2 - 2(2rt-2r)\sum_{i=1}^{t} p_i \leq \sum_{i=1}^{t} p_i^2,$$

or

$$p_1^2+\cdots+p_t^2+\underbrace{(2rt-2r)^2+\cdots+(2rt-2r)^2}_{k-\text{times}}-2(2rt-2r)p_1-\cdots-2(2rt-2r)p_t \leq \sum_{i=1}^{t} p_i^2,$$

or

$$(2rt-2r-p_1)^2+\cdots+(2rt-2r-p_t)^2 \leq \sum_{i=1}^{t} p_i^2,$$

or

$$\sum_{i=1}^{t}(2rt-2r-p_i)^2 \leq \sum_{i=1}^{t} p_i^2.$$

(b) Again, by Theorem 3.2.3, we have for $1 \leq t \leq n$ with equality when $t = n$,

$$rt(t-1) \leq \sum_{i=1}^{t} p_i = \sum_{i=1}^{t}(p_t)(1) \leq \left(\sum_{i=1}^{t} p_i^g\right)^{\frac{1}{g}} \left(\sum_{i=1}^{t} 1^h\right)^{\frac{1}{h}}$$

and $p_1 = p_2 = \cdots = p_t$, (by Holders inequality). Therefore

$$rt(t-1) \leq \sum_{i=1}^{t} p_i = \left(\sum_{i=1}^{k} p_i^g\right)^{\frac{1}{g}} t^{\frac{1}{h}},$$

and $p_1 = p_2 = \cdots = p_t$. That is,

$$rt^{1-\frac{1}{h}}(t-1) \leq \left(\sum_{i=1}^{t} p_i^g\right)^{\frac{1}{g}},$$

and $p_1 = p_2 = \cdots = p_t$.

Hence

$$\sum_{i=1}^{t} p_i^g \geq t(rt-r)^g,$$

for $1 \leq t \leq n$ with equality when t=n, and p_1=$p_2 = \cdots = p_t$, since $\frac{1}{g} + \frac{1}{h} = 1$.

Given an r-digraph on n vertices, the following result provides the existence of an r-digraph with more vertices.

Theorem 3.3.6. *Let D be an r-digraph on n vertices with mark sequence $[p_i]_1^n$. Then, for each $t \geq 1$, there exists an r-digraph on tn vertices with mark sequence $[p_i + r(t-1)n]_1^{tn}$.*

Proof. For each $i, 1 \leq i \leq t$, let D^i be a copy of D with n vertices. Define an r-digraph D_1 as

$$D_1 = D^1 \cup D^2 \cup \cdots \cup D^t,$$

such that vertices and arcs of D_1 are that of D^i, and let there be no arc between the vertices of D^i and D^j $(i \neq j)$. Then D_1 is an r-digraph on tn vertices with mark sequence $[p_i + r(t-1)n]_1^{tn}$.

3.4 Uniquely realizable mark sequences

Definition 3.4.1. An r-digraph is reducible if it is possible to partition its vertices into two non empty sets V_1 and V_2 in such a way that there are exactly r arcs directed from every vertex of V_2 to each vertex of V_1, and there is no arc from any vertex of V_1 to any vertex of V_2. If D_1 and D_2 are r-digraphs having respectively vertex sets V_1 and V_2, then the r-digraph D consisting of all arcs of D_1 and all arcs of D_2, and exactly r arcs directed from each vertex of D_2 to every vertex of D_1, we denote it by $D = [D_1, D_2]$. If this is not possible the r-digraph is said to be irreducible.

Let $D_1, D_2, \cdots, D_h$ be irreducible r-digraphs with disjoint vertex sets. Then $D = [D_1, D_2, \cdots, D_h]$ is the r-digraph having all arcs of D_i, $1 \leq i \leq h$, and exactly r arcs from each vertex of D_j to every vertex of D_i, $1 \leq i < j \leq h$. we say $D_1, D_2, \cdots, D_h$ are the irreducible components of D, and such a decomposition is called the irreducible decomposition of D.

Definition 3.4.2. A mark sequence P is said to be irreducible if all the r-digraphs D with mark sequence P are irreducible.

The following result characterizes irreducible r-digraphs.

Theorem 3.4.3. *If D is a connected r-digraph with mark sequence $P = [p_i]_1^n$, then D is irreducible if and only if for $k = 1, 2, \cdots, n-1$, $\sum_{i=1}^{k} p_i > rk(k-1)$ and $\sum_{i=1}^{n} p_i = rn(n-1)$.*

Proof. Let D be a connected, irreducible k-digraph having mark sequence $P = [p_i]_1^n$. $\sum_{i=1}^{k} p_i > rk(k-1)$ holds, since it has already been established for any r-digraph. Also $\sum_{i=1}^{n} p_i = rn(n-1)$ implies that for any integer $t < n$, the r-subdigraph D' induced by any set of t vertices has a sum of marks in D' equal to $kt(t-1)$. Since D is irreducible, therefore either there is an arc from at least one of these t vertices to at least one of the other $n-t$ vertices, or there is exactly one arc from at least one of the other $n-t$ vertices to at least one vertex in D'. Therefore, for $1 \leq t < n-1$,

$$\sum_{i=1}^{t} p_i \geq kt(t-1)+1 > kt(t-1).$$

For the converse, suppose the given conditions hold. It follows that there exists an r-digraph with mark sequence $P = [p_i]_1^n$. Assume such an r-digraph is reducible, and let $D = [D_1, D_2, \ldots, D_h]$ be the irreducible component decomposition of D. Since there are exactly r arcs from every vertex of D_j to each vertex of D_i, $1 \leq i < j \leq h$, D is evidently connected. If m is the number of vertices in D_1, then $m < n$, and $\sum_{i=1}^{m} p_i = km(m-1)$, which is a contradiction to the given hypothesis. Hence, D is irreducible. □

We note that a disconnected r-digraph is always irreducible, since if D_1 and D_2 are the components of D, then there are no arcs between the vertices of D_1 and D_2.

As a consequence of Theorem 3.4.3, we have the following result which characterizes the irreducible components of an r-digraph.

Theorem 3.4.4. *If D is an r-digraph with mark sequence $P = [p_i]_1^n$, and $\sum\limits_{i=1}^{k} p_i = rk(k-1)$, $\sum\limits_{i=1}^{t} p_i = rt(t-1)$ and $\sum\limits_{i=1}^{q} p_i > rq(q-1)$, for $k+1 \leq q \leq t-1$, $0 \leq k < t \leq n$, then the r-subdigraph induced by the vertices $v_{k+1}, v_{k+2}, \cdots, v_t$ is an irreducible component of D with mark sequence $P = [p_i - rk]_{k+1}^t$.*

The mark sequence P is irreducible if D is irreducible and the irreducible components of P are the mark sequences of the irreducible components of D. That is, if $D_1, D_2, \cdots, D_h$ is the irreducible component decomposition of an r-digraph D with mark sequence P, then the irreducible components P_i of P are the mark sequences of the r-subdigraphs induced by the vertices of D_i, $1 \leq i \leq h$. Theorem 3.4.4 shows that the irreducible components of P are determined by the successive values of k ($1 \leq k \leq n$) for which

$$\sum_{i=1}^{k} p_i = rk(k-1).$$

This is illustrated by the following examples of 2-digraphs.

(i) Let $P = [1, 3, 9, 12, 15, 20]$. Equation (3.2.1) is satisfied for $k = 2, 5, 6$. Therefore, the irreducible components of P are $[0]$, $[1, 4, 7]$, $[0]$ in ascending order.

(ii) Let $P = [0, 5, 8, 11, 17, 19]$. Here Equation (3.2.1) is satisfied for $k = 1, 4, 6$. Therefore, the irreducible components of P are $[0]$, $[1, 4, 7]$ and $[1, 3]$ in ascending order.

Definition 3.4.5. A mark sequence is uniquely realizable if it belongs to exactly one r-digraph.

We have the following observation.

Theorem 3.4.6. *The mark sequence P of an r-digraph D is uniquely realizable if and only if every irreducible component of P is uniquely realizable.*

The following result determines which irreducible mark sequences in 2-digraphs are uniquely realizable.

Theorem 3.4.7. *The only irreducible mark sequences in 2-digraphs that are uniquely realizable are* $[0]$ *and* $[1, 3]$.

Proof. Let P be an irreducible mark sequence, and let D with vertex set V be a 2-digraph having mark sequence P. Then D is irreducible. Therefore, D cannot be partitioned into 2-subdigraphs D_1, D_2, ..., D_k such that there are exactly two arcs from every vertex of D_α to each vertex of D_β, $1 \leq \beta < \alpha \leq k$. First assume D has $n \geq 3$ vertices. Let $W = \{w_1, w_2, \ldots, w_r\}$ and $U = \{u_1, u_2, \ldots, u_s\}$ respectively be any two disjoint subsets of V such that $r + s = n$. Since D is irreducible, (1) there do not exist exactly two arcs from every w_i $(1 \leq i \leq r)$ to each u_j $(1 \leq\ j \leq s)$, and (2) there do not exist exactly two arcs from every u_j $(1 \leq j \leq s)$ to each w_i $(1 \leq i \leq s)$. First of all we consider Case (1), and then Case (2) follows by using the same argument as in (1).

Case (1). There exists at least one vertex, say w_1, in W, and at least one vertex, say u_1, in U such that either (a) $w_1(1-1)u$, or (b) $w_1(0-2)u_1$, or (c) $w_1(1-0)u_1$, or (d) $w_1(0-1)u_1$, or (e) $w_1(0-0)u_1$.

Assume $w_i(2-0)u_j$ for each i $(1 \leq i \leq r)$ and j $(1 \leq \; j \leq s)$, except for $i = j = 1$.

If in D, either (a) $w_1(1-1)u_1$, or (e) $w_1(0-0)u_1$, then transforming them respectively to $w_1(0-0)u_1$, or $w_1(1-1)u_1$, gives a 2-digraph D' with the same mark sequence. In both cases, D and D' have different number of arcs, and thus are non-isomorphic.

(b) Let $w_1(0-2)u_1$. Since there are only six possibilities between w_1 and w_i, therefore, for any other vertex w_i in W we have one of the following cases:

(i) $w_1(2-0)w_i(2-0)u_1(2-0)w_1$, (ii) $w_1(1-1)w_i(2-0)u_1(2-0)w_1$, (iii) $w_1(1-0)w_i(2-0)u_1(2-0)w_1$, (iv) $w_1(0-1)w_i(2-0)u_1(2-0)w_1$, (v) $w_1(0-0)w_i(2-0)u_1(2-0)w_1$, (vi) $w_1(0-2)w_i(2-0)u_1(2-0)w_1$.

Transforming (i)–(v) respectively to $w_1(1-0)w_i(1-0)u_1(1-0)w_1$, $w_1(0-1)w_i(1-0)u_1(1-0)w_1$, $w_1(0-0)w_i(1-0)u_1(1-0)w_1$, $w_1(0-2)w_i(1-0)u_1(1-0)w_1$, $w_1(0-1)w_i(1-0)u_1(1-0)w_1$, gives a 2-digraph with the same mark sequence. In all these five cases, D and D' have different number of arcs, and thus are non-isomorphic.

If (vi) occurs in D, and also $w_q(2-0)w_i$ for $1 \leq i < q \leq r$, then the 2-digraph D is reducible with irreducible components D_1, D_2, $\ldots$, D_r respectively having vertex sets $V_1 = \{u_1, u_2, \ldots, u_s, w_1\}$, $V_2 = \{w_2\}$, $V_3 = \{w_3\}$, $\ldots$, $V_k = \{w_r\}$.

Also for any vertex u_j in U, since there are only six possibilities between u_1 and u_j, we have one of the following cases:

(vii) $w_1(0-2)u_1(0-2)u_j(0-2)w_1$, (viii) $w_1(0-2)u_1(1-1)u_j(0-2)w_1$, (ix) $w_1(0-2)u_1(1-0)u_j(0-2)w_1$, (x) $w_1(0-2)u_1(0-1)u_j(0-2)w_1$, (xi) $w_1(0-2)u_1(0-0)u_j(0-2)w_1$, (xii) $w_1(0-2)u_1(2-0)u_j(0-2)w_1$.

If any one of (vii)–(xi) appears in D, then making respectively the transformations $w_1(0-1)u_1(0-1)u_j(0-1)w_1$, $w_1(0-1)u_1(1-0)u_j(0-1)w_1$, $w_1(0-1)u_1(2-0)u_j(0-1)w_1$, $w_1(0-1)u_1(1-1)u_j(0-1)w_1$, $w_1(0-1)u_1(1-0)u_j(0-1)w_1$, we get a 2-digraph with the same mark sequence, but the numbers of arcs in D and D' are different, and thus D and D' are non-isomorphic.

If (xii) and any of (i)–(v) appear simultaneously, then there exists a 2-digraph D' with the same mark sequence, but D and D' have different numbers of arcs. Thus, D and D' are non-isomorphic.

If (vi) and (xii) appear simultaneously, and also $w_q(2-0)w_i$ for all $1 \leq i < q \leq r$, then D is reducible with the irreducible components D_1, D_2, $\dots$, D_r having vertex sets $V_1 = \{u_1, u_2, \dots, u_s, w_1\}$, $V_2 = \{w_2\}$, $V_3 = \{w_3\}$, $\dots$, $V_r = \{w_r\}$ respectively.

(c) Let $w_1(1-0)u_1$. For any vertex w_i in W, since there are only six possibilities between w_1 and w_i, we have one of the following cases:

(i) $w_1(2-0)w_i(2-0)u_1(0-1)w_1$, (ii) $w_1(1-1)w_i(2-0)u_1(0-1)w_1$, (iii) $w_1(1-0)w_i(2-0)u_1(0-1)w_1$, (iv) $w_1(0-1)w_i(2-0)u_1(0-1)w_1$, (v) $w_1(0-0)w_i(2-0)u_1(0-1)w_1$, (vi) $w_1(0-2)w_i(2-0)u_1(0-1)w_1$.

For (i)–(v) making respectively the transformations $w_1(1-0)w_i(1-0)u_1(0-2)w_1$, $w_1(0-1)w_i(1-0)u_1(0-2)w_1$, $w_1(0-1)w_i(1-0)u_1(0-2)w_1$, $w_1(1-1)w_i(1-0)u_1(2-0)w_1$, $w_1(0-1)w_i(1-0)u_1(2-1)w_1$, we obtain a 2-digraph D' with the same mark sequence, but the numbers of arcs in D and D' are not equal. Thus, D and D' are non-isomorphic.

Now, for any other vertex u_j in U, there are only six possibilities between u_1 and u_j, and we have one of the following cases:

(vii) $w_1(1-0)u_1(0-2)u_j(0-2)w_1$, (viii) $w_1(1-0)u_1(1-1)u_j(0-2)w_1$, (ix) $w_1(1-0)u_1(1-0)u_j(0-2)w_1$, (x) $w_1(1-0)u_1(0-1)u_j(0-2)w_1$, (xi) $w_1(1-0)u_1(0-0)u_j(0-2)w_1$, (xii) $w_1(1-0)u_1(2-0)u_j(0-2)w_1$.

If any one of (vii)–(xi) appears, then making respectively the transformations $w_1(2-0)u_1(0-1)u_j(0-1)w_1$, $w_1(2-0)u_1(1-0)u_j(0-1)w_1$, $w_1(2-0)u_1(2-0)u_j(0-1)w_1$, $w_1(2-0)u_1(1-1)u_j(0-1)w_1$, $w_1(2-0)u_1(1-0)u_j(0-1)w_1$, we get a 2-digraph D' with the same mark sequence, but D and D' have different numbers of arcs. Thus, D and D' are non-isomorphic.

If (xii) and one of (i)-(v) appears simultaneously, we once again arrive to the conclusion that there exists a 2-digraph D' with the mark sequence P, but D and D' are non-isomorphic.

Thus, we are left with the case when (vi) and (xii) appear simultaneously, and also $w_q(2-0)w_i$ for all $1 \leq i < q \leq r$. But, then D is reducible having the irreducible components D_1, D_2, $\dots$, D_r with vertex sets

$V_1 = \{u_1, u_2, \ldots, u_s, w_1\}$, $V_2 = \{w_2\}$, $\ldots$, $V_r = \{w_r\}$ respectively.

(d) Let $w_1(0-1)u_1$. Since there are only six possibilities between w_1 and w_i, therefore for any other vertex w_i in W, we have one of the following cases:

(i) $w_1(2-1)w_i(2-0)u_1(1-0)w_1$, (ii) $w_1(1-1)w_i(2-0)u_1(1-0)w_1$, (iii) $w_1(1-0)w_i(2-0)u_1(1-0)w_1$, (iv) $w_1(0-1)w_i(2-0)u_1(1-0)w_1$, (v) $w_1(0-0)w_i(2-0)u_1(1-0)w_1$, (vi) $w_1(0-2)w_i(2-0)u_1(1-0)w_1$.

If any one of (i)–(v) appears, then making respectively the transformations $w_1(1-0)w_i(1-0)u_1(0-0)w_1$, $w_1(0-1)w_i(1-0)u_1(0-0)w_1$, $w_1(0-0)w_i(1-0)u_1(0-0)w_1$, $w_1(0-2)w_i(1-0)u_1(0-0)w_1$, $w_1(0-1)w_i(1-0)u_1(0-0)w_1$, gives a 2-digraph D' with the same mark sequence, but the numbers of arcs in D and D' are different so that D and D' are non-isomorphic.

If (vi) appears in D, and also if $w_q(2-0)w_i$ for all $1 \leq i < q \leq r$, then D becomes reducible.

Now, for any other vertex u_j in U, there are only six possibilities between u_1 and u_j, and we have one of the following cases:

(vii) $w_1(0-1)u_1(0-2)u_j(0-2)w_1$, (viii) $w_1(0-1)u_1(1-1)u_j(0-2)w_1$, (ix) $w_1(0-1)u_1(1-0)u_j(0-2)w_1$, (x) $w_1(0-1)u_1(0-1)u_j(0-2)w_1$, (ix) $w_1(0-1)u_1(0-0)u_j(0-2)w_1$, (xii) $w_1(0-1)u_1(2-0)u_j(0-2)w_1$.

If any one of (vii)–(xi) appears in D, then making respectively the transformations $w_1(0-0)u_1(0-1)u_j(0-1)w_1$, $w_1(0-0)u_1(1-0)u_j(0-1)w_1$, $w_1(0-0)u_1(2-0)u_j(0-1)w_1$, $w_1(0-0)u_1(0-0)u_j(0-1)w_1$, $w_1(0-0)u_1(1-0)u_j(0-1)w_1$, gives a 2-digraph D' with the same mark sequence, but the numbers of arcs in D and D' are different so that D is not isomorphic to D'.

If (xii) and any one of (i)–(v) appear simultaneously, then once again there exists a 2-digraph D' with the same mark sequence, but D and D' have different numbers of arcs so that D and D' are non-isomorphic.

If (vi) and (xii) appear simultaneously, and also $w_q(2-0)w_i$ for all $1 \leq i < q \leq r$, then D is reducible.

Now, let D have exactly two vertices say u and v. The only irreducible mark sequences realizing D are $[2,2]$, and $[1,3]$. Obviously the sequence $[2,2]$ has two non-isomorphic realizations namely $u(0-0)v$ and $u(1-1)v$, and $[1,3]$

has the unique realization $u(0-1)v$. Thus $P = [1, 3]$ is uniquely realizable.

If D has only one vertex, then $P = [0]$, which evidently is uniquely realizable. $\square$

On combining Theorem 3.4.6 and 3.4.7, we have the following result for 2-digraphs.

Theorem 3.4.8. *The mark sequence P of a 2-digraph is uniquely realizable if and only if every irreducible component of P is of the form* $[0]$ *and* $[1, 3]$.

We observe that in the mark sequence $P = [4i-4]_1^n$ every irreducible component is $[0]$ and thus P is uniquely realizable. Therefore the mark sequence P of an r-digraph is uniquely realizable if and only if every irreducible component of P is of the form $[0]$ and $[1, 2k-1]$.

CHAPTER 4

Marks in bipartite multidigraphs

In this chapter, we extend the concept of marks to bipartite multidigraphs and multipartite multidigraphs. We obtain necessary and sufficient conditions for a pair of sequences of non-negative integers to be mark sequences of some bipartite multidigraph. These characterizations give algorithms for constructing the corresponding bipartite multidigraphs. We provide analogous characterizations for multipartite multidigraphs.

4.1 Introduction

A bipartite r-digraph is an orientation of a bipartite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. So bipartite 1-digraph is an oriented bipartite graph and a complete bipartite 1-digraph is a bipartite tournament. Let $D(X, Y)$ be a bipartite r-digraph with $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. For any vertex v_i in $D(X, Y)$, let $d^+_{v_i}$ and $d^-_{v_i}$ be the outdegree and indegree, respectively, of v_i. Define p_{x_i} (or simply p_i) $= rn + d^+_{x_i} - d^-_{x_i}$ and q_{y_j} (or simply q_j)$= rm + d^+_{y_j} - d^-_{y_j}$ as the marks (or r-scores) of x_i in X and y_j in Y respectively. Clearly, $0 \le p_{x_i} \le 2rn$ and $0 \le q_{y_j} \le 2rm$. Then the sequences $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ in non-decreasing order are called the mark sequencesof $D(X, Y)$.

A bipartite r-digraph can be interpreted as the result of a competition between two teams in which each player of one team plays with every player of the other team atmost r times in which ties(draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player x_i (respectively y_j) receives a total of p_{x_i} (respectively q_{y_j}) points. The sequences P and Q of non-negative integers in non-decreasing order are said to be realizable if there exists a bipartite r-digraph with mark sequences P and Q.

In a bipartite r-digraph $D(X,Y)$, if there are a_1 arcs directed from a vertex $x \in X$ to a vertex $y \in Y$ and a_2 arcs directed from vertex y to vertex x, with $0 \leq a_1, a_2 \leq r$ and $0 \leq a_1 + a_2 \leq r$, we denote it by $x(a_1 - a_2)y$. For example, if there are exactly r arcs directed from $x \in X$ to $y \in Y$ and no arc directed from y to x, and this is denoted by $x(r-0)y$, and if there is no arc directed from x to y and no arc directed from y to x, this is denoted by $x(0-0)y$.

An oriented tetra in a bipartite r-digraph is an induced 1-subdigraph with two vertices from each part. Define oriented tetras of the form $x(1-0)y(1-0)x'\ (1-0)y'(1-0)x$ and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ to be of α-type and all other oriented tetras to be of β-type. A bipartite r-digraph is said to be of α-type or β-type according as all of its oriented tetras are of α-type or β-type respectively. We assume, without loss of generality, that β-type bipartite r-digraphs have no pair of symmetric arcs because symmetric arcs $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, can be transformed to $x(0-0)y$ with the same marks. A transmitter is a vertex with indegree zero.

4.2 Characterization of marks in bipartite multidigraphs

The work in this section has appeared in Chishti and Samee [21]. We start with the following observation.

Lemma 4.2.1. Among all bipartite r-digraphs with given mark sequences, those with the fewest arcs are of β-type.

Proof. Let $D(X,Y)$ be a bipartite r-digraph with mark sequences P and Q. Assume $D(X,Y)$ is not of β-type. Then $D(X,Y)$ has an oriented tetra of α-type, that is, $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ or $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ where $x, x' \in X$ and $y, y' \in Y$. Since $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-0)x$ with the same mark sequences and four arcs fewer, and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-1)x$ with the same mark sequences and two arcs fewer, therefore, in both cases we obtain a bipartite r-digraph having same mark sequences P and Q with fewer arcs. Note that if there are symmetric arcs between x and y, that is $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, then these can be transformed to $x(0-0)y$ with the same mark sequences and a arcs fewer. Hence the result follows.

Lemma 4.2.2. Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be mark sequences of a β-type bipartite r-digraph. Then either the vertex with mark p_m, or the vertex with mark q_n, or both can act as transmitters.

We know if $P = [p_1, p_2, \cdots, p_m]$ and $Q = [q_1, q_2, \cdots, q_n]$ are mark sequences of a bipartite r-digraph, then $p_i \leq 2rn$ and $q_j \leq 2rm$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. We have the following observation.

Lemma 4.2.3. If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, 0]$ with each $p_i = rn$ are mark sequences of some bipartite r-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0]$ are also mark sequences of some bipartite r-digraph.

We now have some observations about bipartite 2-digraphs, as these will be required in application of Theorem 4.2.11.

Lemma 4.2.4. If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, q_n]$ with $4n - p_m = 3$ and $q_n \geq 3$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, q_n - 3]$ are also mark sequences of some bipartite 2-digraph.

Proof. Let P and Q as given above be mark sequences of bipartite 2-digraph D with parts $X = \{x_1, x_2, \cdots, x_{m-1}, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_{n-1}, y_n\}$. Since $4n - p_m = 3$ and $3 \leq q_n \leq 4m$, therefore in D necessarily $x_m(2-0)y_i$, for all $1 \leq i \leq n-1$. Also $y_n(1-0)x_m$, because if $y_n(0-0)x_m$, or $y_n(0-2)x_m$,or $y_n(0-1)x_m$, then in all these cases $p_{x_m} \geq 4(n-1) + 2$, a contradiction to our assumption. Also $y_n(2-0)x_m$ is not possible because in that case $p_{x_m} = 4(n-1) < 4n - 3$.

Now delete x_m, obviously this keeps marks of y_1,y_2, $\cdots$,y_{n-1} as zeros and reduces mark of y_n by 3, and we obtain a bipartite 2-digraph with mark sequences $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, q_n - 3]$, as required.

Lemma 4.2.5. If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, q_n]$ with $4n - p_m = 4$ and $q_n \geq 4$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, q_n - 4]$ are also mark sequences of some bipartite 2-digraph.

Proof. Let P and Q as given above be mark sequences of bipartite 2-digraph D with parts $X = \{x_1, x_2, \cdots, x_{m-1}, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_{n-1}, y_n\}$. Since $4n - p_m = 4$ and $4 \leq q_n \leq 4m$, therefore in D necessarily $x_m(2-0)y_i$, for all $1 \leq i \leq n-1$. Also $y_n(2-0)x_m$, because if $y_n(0-0)x_m$, or $y_n(1-0)x_m$, or $y_n(0-2)x_m$,or $y_n(0-1)x_m$, then in all these cases $p_{x_m} \geq 4(n-1)+1$, a contradiction to our assumption.

Now delete x_m, obviously this keeps marks of y_1,y_2, $\cdots$,y_{n-1} as zeros and reduces mark of y_n by 4, and we obtain a bipartite 2-digraph with mark sequences $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, q_n - 4]$, as required.

Lemma 4.2.6. If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, q_n]$ with $4n - p_m = 4$ and $q_n \geq 3$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, q_n - 3]$ are also mark sequences of some bipartite 2-digraph.

Proof. The proof follows by using the same argument as in Lemma 4.2.5.

Lemma 4.2.7. If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, 1, 3]$ with $4n - p_m = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.

Lemma 4.2.8. If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, 1, 1, 2]$ with $4n - p_m = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.

Lemma 4.2.9. If $P = [p_1, p_2, \cdots, p_{m-1}, p_m]$ and $Q = [0, 0, \cdots, 0, 1, 1, 1, 1]$ with $4n - p_m = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \cdots, p_{m-1}]$ and $Q' = [0, 0, \cdots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.

Remarks 4.2.10. We note that the sequences of non-negative integers $[p_1]$ and $[q_1, q_2, \cdots, q_n]$, with $p_1 + q_1 + q_2 + \cdots + q_n = 2rn$, are always mark sequences of some bipartite r-digraph. We observe that the bipartite r-digraph $D(X, Y)$, with vertex sets $X = \{x_1\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, where for q_i even, say $2t$, we have $x_1((r-t)-t)y_i$ and for q_i odd, say $2t+1$, we have $x_1((r-t-1)-t)y_i$, has mark sequences $[p_1]$ and $[q_1, q_2, \cdots, q_n]$. Also we note that the sequences $[0]$ and $[2r, 2r, \cdots, 2r]$ are mark sequences of some bipartite r-digraph.

The next result provides a useful recursive test whether or not a pair of sequences is realizable.

Theorem 4.2.11. Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order with $p_m \geq q_n$ and $rn \leq p_m \leq 2rn$.
(A) If $q_n \leq 2r(m-1)+1$, let P' be obtained from P by deleting one entry p_m, and Q' be obtained as follows.

For $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, reducing $[2r-(i-1)]n - p_m$ largest entries of Q by i each, and reducing $p_m - (2r-i)n$ next largest entries by $i-1$ each.
(B) In case $q_n > 2r(m-1)+1$, say $q_n = 2r(m-1)+1+h$, where $1 \leq h \leq r-1$, then let P' be obtained from P by deleting one entry p_m, and Q' be obtained from Q by reducing the entry q_n by $h+1$.

Then P and Q are the mark sequences of some bipartite r-digraph if and only if P' and Q' (arranged in non-decreasing order) are the mark sequences of some bipartite r-digraph.
Proof. Let P' and Q' be the mark sequences of some bipartite r-digraph $D'(X', Y')$. First suppose Q' is obtained from Q as in A. Construct a bipartite r-digraph $D(X, Y)$ as follows. Let $X = X' \cup x$, $Y = Y'$ with $X' \cap x = \phi$. Let $x((r-i)-0)y$ for those vertices y of Y' whose marks are reduced by i in going from P to P' and Q to Q', and $x(r-0)y$ for those vertices y of Y' whose marks are not reduced in going from P to P' and Q to Q'. Then $D(X, Y)$ is the bipartite r-digraph with mark sequences P and Q. Now, if Q' is obtained from Q as in B, then construct a bipartite r-digraph $D(X, Y)$

as follows. Let $X = X' \cup x$, $Y = Y'$ with $X' \cap x = \phi$. Let $x((r-h-1)-0)y$ for that vertex y of Y' whose marks are reduced by h in going from P and Q to P' and Q'. Then $D(X,Y)$ is the bipartite r-digraph with mark sequences P and Q.

Conversely, suppose P and Q be the mark sequences of a bipartite r-digraph $D(X,Y)$. Without loss of generality, we choose $D(X,Y)$ to be of β-type. Then by Lemma 4.2.2, any of the vertex $x \in X$ or $y \in Y$ with mark p_m or q_n respectively can be a transmitter. Let the vertex $x \in X$ with mark p_m be a transmitter. Clearly, $p_m \geq rn$ and because if $p_m < rn$, then by deleting p_m we have to reduce more than n entries from Q, which is absurd.
(A) Now $q_n \leq 2r(m-1)+1$ because if $q_n > 2r(m-1)+1$, then on reduction $q'_n = q_n - 1 > 2r(m-1)+1-1 = 2r(m-1)$, which is impossible.

Let $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, let V be the set of $[2r-(i-1)]n - p_m$ vertices of largest marks in Y, and let W be the set of $p_m - (2r-i)n$ vertices of next largest marks in Y and let $Z = Y - \{V, W\}$. Construct $D(X,Y)$ such that $x((r-i)-0)v$ for all $v \in V$, $x((r-i-1)-0)w$ for all $w \in W$ and $x(r-0)z$ for all $z \in Z$. Clearly, $D(X,Y) - x$ realizes P' and Q' (arranged in non-decreasing order).
(B) Now in D, let $q_n > 2r(m-1)+1$, say $q_n = 2r(m-1)+1+h$, where $1 \leq h \leq r-1$. This means $y_n(r-0)x_i$, for all $1 \leq i \leq m-1$. Since x_m is a transmitter, so there cannot be an arc from y_n to x_m. Therefore $x_m((r-h-1)-0)y_n$, since y_n needs $h+1$ more marks. Now delete x_m, it will decrease the mark of y_n by $h+1$, and the resulting bipartite r-digraph will have mark sequences P' and Q' as desired.

Theorem 4.2.11 provides an algorithm of checking whether or not the sequences P and Q of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding bipartite r-digraph. Let $P = [p_1, p_2, \cdots, p_m]$ and $Q = [q_1, q_2, \cdots, q_n]$, where $p_m \geq q_n$, $rn \leq p_m \leq 2rn$ and $q_n \leq 2r(m-1)+1$, be the mark sequences of a bipartite r-digraph with parts $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ respectively. Deleting p_m and performing A of Theorem 4.2.11 if $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, we get $Q' = [q'_1, q'_2, \cdots, q'_n]$. If the marks of the vertices y_j were decreased by i in this process, then the construction yielded

$x_m((r-i)-0)y_j$, if these were decreased by $i-1$, then the construction yielded $x_m((r-i+1)-0)y_j$. If we perform B of Theorem 4.2.11, the mark of y_n was decreased by $h+1$, the construction yielded $x_m((r-h-1)-0)y_n$. For vertices y_j whose marks remained unchanged, the construction yielded $x_m(r-0)y_j$. Note that if the conditions $p_m \geq rn$ does not hold, then we delete q_n for which the conditions get satisfied and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not P and Q are the mark sequences, and if P and Q are the mark sequences, then a bipartite r-digraph with mark sequences P and Q is constructed.

We illustrate this reduction and the resulting construction with the following examples.

Example 4.2.12. Consider the two sequences of non-negative integers given by $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$. We check whether or not P and Q are mark sequences of some bipartite 3-digraph.

1. $P = [14, 14, 15]$, $Q = [6, 6, 8, 9]$

We delete 15. Clearly $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 15 \geq (2r-i)n = (2.3-3)4 = 12$. So reduce $[2r-(i-1)]n-p_m = [2.3-(3-1]4-15 = 16-15 = 1$ largest entry of Q by $i = 3$ and $p_m-(2r-i)n = 15-(2.3-3)4 = 15-12 = 3$ next largest entries of Q by $i-1 = 3-1 = 2$ each, we get $P_1 = [14, 14]$, $Q_1 = [4, 4, 6, 6]$, and arcs are defined as $x_3(0-0)y_4$, $x_3(1-0)y_3$, $x_3(1-0)y_2$, $x_3(1-0)y_1$.

2. $P_1 = [14, 14]$, $Q_1 = [4, 4, 6, 6]$

We delete 14. Here $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 14 \geq (2r-i)n = (2.3-3)4 = 12$. Reduce $[2r-(i-1)]n-p_m = [2.3-(3-1]4-14 = 16-14 = 2$ largest entries of Q_1 by $i = 3$ and $p_m-(2r-i)n = 14-(2.3-3)4 = 14-12 = 2$ next largest entries of Q_1 by $i-1 = 3-1 = 2$ each, we get $P_2 = [14]$, $Q_2 = [2, 2, 3, 3]$, and arcs are defined as $x_2(0-0)y_4$, $x_2(0-0)y_3$, $x_2(1-0)y_2$, $x_2(1-0)y_1$.

3. $P_2 = [14]$, $Q_2 = [2, 2, 3, 3]$

We delete 14. Here $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 14 \geq (2r-i)n = (2.3-3)4 = 12$. Reduce $[2r-(i-1)]n-p_m = [2.3-(3-1]4-14 = 16-14 = 2$

largest entries of Q_2 by $i = 3$ and $p_m-(2r-i)n = 14-(2.3-3)4 = 14-12 = 2$ next largest entries of Q_2 by $i - 1 = 3 - 1 = 2$ each, we get $P_3 = \phi$, $Q_3 = [0, 0, 0, 0]$, and arcs are defined as $x_1(0 - 0)y_4$, $x_1(0 - 0)y_3$, $x_1(1 - 0)y_2$, $x_1(1 - 0)y_1$.

The resulting bipartite 3-digraph has mark sequences $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$ with vertex sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ and arcs as $x_3(0 - 0)y_4$, $x_3(1 - 0)y_3$, $x_3(1 - 0)y_2$, $x_3(1 - 0)y_1$, $x_2(0 - 0)y_4$, $x_2(0 - 0)y_3$, $x_2(1 - 0)y_2$, $x_2(1 - 0)y_1$, $x_1(0 - 0)y_4$, $x_1(0 - 0)y_3$, $x_1(1 - 0)y_2$, $x_1(1 - 0)y_1$.

Example 4.2.13. Consider the two sequences of non-negative integers given by $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$. We check whether or not P and Q are mark sequences of some bipartite 4-digraph.

1. $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$

We delete 24. Here $[2r - (i - 1)]n = [2.4 - (1 - 1)]3 = 24$, so reduce $[2r-(i-1)]n-p_m = [2.4-(1-1]3-24 = 24-24 = 0$ largest entries of Q by $i = 1$, and obviously we reduce $p_m-(2r-i)n = 24-(2.4-1)3 = 24-21 = 3$ next largest entries of Q by $i - 1 = 1 - 1 = 0$ each, we get $P_1 = [13, 16, 22]$ and $Q_1 = [5, 6, 10]$, and arcs are $x_4(4 - 0)y_3$, $x_4(4 - 0)y_2$, $x_4(4 - 0)y_1$.

2. $P_1 = [13, 16, 22]$ and $Q_1 = [5, 6, 10]$

We delete 22. Here $[2r-(i-1)]n = [2.4-(1-1)]3 = 24 \geq 22 \geq (2r-i)n = (2.4-1)3 = 21$. Reduce $[2r-(i-1)]n-p_m = [2.4-(1-1]3-22 = 24-22 = 2$ largest entries of Q_1 by $i = 1$ and $p_m-(2r-i)n = 22-(2.4-1)3 = 22-21 = 1$ next largest entries of Q_1 by $i - 1 = 1 - 1 = 0$ each, we get $P_2 = [13, 16]$, $Q_2 = [5, 5, 9]$, and arcs are defined as $x_3(3 - 0)y_3$, $x_3(3 - 0)y_2$, $x_3(4 - 0)y_1$.

3. $P_2 = [13, 16]$, $Q_2 = [5, 5, 9]$

We delete 16. Here $[2r-(i-1)]n = [2.4-(3-1)]3 = 18 \geq 16 \geq (2r-i)n = (2.4-3)3 = 15$. Reduce $[2r-(i-1)]n-p_m = [2.4-(3-1]3-16 = 18-16 = 2$ largest entries of Q_2 by $i = 3$ and $p_m-(2r-i)n = 16-(2.4-3)3 = 16-15 = 1$ next largest entry of Q_2 by $i-1 = 3-1 = 2$, we get $P_3 = [13]$, $Q_3 = [3, 2, 6]$, and arcs are defined as $x_2(3 - 0)y_3$, $x_2(3 - 0)y_2$, $x_2(2 - 0)y_1$.

4. $P_3 = [13]$, $Q_3 = [3, 2, 6]$. Here $13 + 3 + 2 + 6 = 24$ which is same as $2rn = 2.4.3 = 24$. Thus by the argument as discussed in the remarks, P_3 and Q_3 are mark sequences of some bipartite 4-digraph. Here arcs are $x_1(1-3)y_3$,

$x_1(3-1)y_2$, $x_1(2-1)y_1$.

The resulting bipartite 4-digraph with mark sequences $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$ has vertex sets $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3\}$ and arcs as $x_4(4-0)y_3$, $x_4(4-0)y_2$, $x_4(4-0)y_1$, $x_3(3-0)y_3$, $x_3(3-0)y_2$, $x_3(4-0)y_1$, $x_2(3-0)y_3$, $x_2(3-0)y_2$, $x_2(2-0)y_1$, $x_1(1-3)y_3$, $x_1(3-1)y_2$, $x_1(2-1)y_1$.

Now we give a combinatorial criterion for determining whether the sequences of non-negative integers are realizable as marks. This is analogous to Landau's theorem [31] on tournament scores and similar to the result by Beineke and Moon [11] on bipartite tournament scores.

Theorem 4.2.14. Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then P and Q are the mark sequences of some bipartite r-digraph if and only if

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j \geq 2rfg, \tag{4.1}$$

for $1 \leq f \leq m$ and $1 \leq g \leq n$, with equality when $f = m$ and $g = n$.

Proof. The necessity of the condition follows from the fact that the sub-bipartite r-digraph induced by f vertices from the first part and g vertices from the second part has a sum of marks $2rfg$.

For sufficiency, assume that $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order satisfying conditions (4.1) but are not mark sequences of any bipartite r-digraph. Let these sequences be chosen in such a way that m and n are the smallest possible and p_1 is the least with that choice of m and n. We consider the following two cases.

Case(a). Suppose the equality in (4.1) holds for some $f \leq m$ and $g \leq n$, so that

$$\textstyle\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j = 2rfg.$$

By the minimality of m and n, $P_1 = [p_i]_1^f$ and $Q_1 = [q_j]_1^g$ are the mark sequences of some bipartite r-digraph $D_1(X_1, Y_1)$. Let $P_2 = [p_{f+1} - 2rg, p_{f+2} - 2rg, \cdots, p_m - 2rg]$ and $Q_2 = [q_{g+1} - 2rf, q_{g+2} - 2rf, \cdots, q_n - 2rf]$.

Consider the sum

$$\sum_{i=1}^{s}(p_{f+i}-2rg)+\sum_{j=1}^{t}(q_{g+j}-2rf)=\sum_{i=1}^{f+s}p_i+\sum_{j=1}^{g+t}q_j-\left(\sum_{i=1}^{f}p_i+\sum_{j=1}^{g}q_j\right)$$
$$\begin{aligned}&-2rsg-2rtf\\&\geq 2r(f+s)(g+t)-2rfg-2rsg-2rtf\\&=2r(fg+ft+sg+st-fg-sg-tf)\\&=2rst,\end{aligned}$$

for $1 \leq s \leq m-f$ and $1 \leq t \leq n-g$, with equality when $s = m-f$ and $t = n-g$. Thus, by the minimality of m and n, the sequences P_2 and Q_2 form the mark sequences of some bipartite r-digraph $D_2(X_2, Y_2)$. Now construct a new bipartite r-digraph $D(X, Y)$ as follows.

Let $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ with $X_1 \cap X_2 = \phi$, $Y_1 \cap Y_2 = \phi$. Let $x_2(r-0)y_1$ and $y_2(r-0)x_1$ for all $x_i \in X_i, y_i \in Y_i$, where $1 \leq i \leq 2$, so that we get the bipartite r-digraph $D(X, Y)$ with mark sequences P and Q, which is a contradiction.

Case (b). Suppose the strict inequality holds in (4.1) for some $f \neq m$ and $g \neq n$. Also, assume that $p_1 > 0$. Let $P_1 = [p_1 - 1, p_2, \cdots, p_{m-1}, p_m + 1]$ and $Q_1 = [q_1, q_2, \cdots, q_n]$. Clearly, P_1 and Q_1 satisfy the conditions (2.1). Thus, by the minimality of p_1, the sequences P_1 and Q_1 are the mark sequences of some bipartite r-digraph $D_1(X_1, Y_1)$. Let $p_{x_1} = p_1 - 1$ and $p_{x_m} = p_m + 1$. Since $p_{x_m} > p_1 + 1$, therefore there exists a vertex $y \in Y_1$ such that $x_m(1-0)y(1-0)x_1$, or $x_m(0-0)y(1-0)x_1$, or $x_m(1-0)y(0-0)x_1$, or $x_m(0-0)y(0-0)x_1$, is an induced sub-bipartite 1-digraph in $D_1(X_1, Y_1)$, and if these are changed to $x_m(0-0)y(0-0)x_1$, or $x_m(0-1)y(0-0)x_1$, or $x_m(0-0)y(0-1)x_1$, or $x_m(0-1)y(0-1)x_1$ respectively, the result is a bipartite r-digraph with mark sequences P and Q, which is a contradiction. Hence the result follows.

4.3 Marks in multipartite multidigraphs

A k-partite 2-digraph (or briefly multipartite 2-digraph(M2D))is an orientation of a k-partite multigraph that is without loops and contains at most 2 edges between any pair of vertices from distinct parts. So k-partite 1-digraph is an oriented k-partite graph, and a complete k-partite 1-digraph is a k-partite tournament. Let $D = D(X_1, X_2, \cdots, X_k)$ be an M2D with parts $X_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$, $1 \leq i \leq k$. Let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$, $1 \leq j \leq n_i$, be respectively the outdegree and indegree of a vertex $x_{ij} \in X_i$. Define $p_{x_{ij}}$(or simply p_{ij}) $= 2\left(\sum_{t=1,t\neq i}^{k} n_t\right) + d^+_{x_{ij}} - d^-_{x_{ij}}$ as the mark (or 2-score) of x_{ij}. Clearly, $0 \leq p_{x_{ij}} \leq 4\sum_{t=1,t\neq i}^{k} n_t$. Then the k sequences $P_i = [p_{ij}]_1^{n_i}$, $1 \leq i \leq k$, in non-decreasing order are called the mark sequences of D.

An M2D can be interpreted as a result of a competition among k teams in which each player of one team plays with every player of the other $k-1$ teams at most 2 times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player x_{ij} receives a total of $p_{x_{ij}}$ points. The k sequences of non-negative integers p_i, $1 \leq i \leq k$, in non-decreasing order are said to be realizable if there exists an M2D with mark sequences P_i.

For two vertices x_{ij} in X_i and x_{st} in X_s, $i \neq s$ in an M2D $D(X_1, X_2, ..., X_k)$, we have one of the following six possibilities. (i) exactly two arcs directed from x_{ij} to x_{st} and no arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(2-0)x_{st}$, (ii) exactly two arcs directed from x_{st} to x_{ij} and no arc directed from x_{ij} to x_{st}, this is denoted by $x_{ij}(0-2)x_{st}$, (iii) exactly one arc directed from x_{ij} to x_{st} and exactly one arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(1-1)x_{st}$, and is called a pair of symmetric arcs between x_{ij} and x_{st}, (iv) exactly one arc directed from x_{ij} to x_{st} and no arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(1-0)x_{st}$, (v) exactly one arc directed from x_{st} to x_{ij} and no arc directed from x_{ij} to x_{st}, this is denoted by $x_{ij}(0-1)x_{st}$, (vi) no arc directed from x_{ij} to x_{st} and no arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(0-0)x_{st}$.

A triple in M2D (k-partite 2-digraph) ($k \geq 3$) is an induced 2-subdigraph of three vertices with exactly one vertex from one part, and is of the form $x_{ij}(a_1-a_2)x_{mn}(b_1-b_2)x_{st}(c_1-c_2)x_{ij}$, ($i \neq m \neq s$, $1 \leq j \leq n_i$, $1 \leq n \leq n_m$, $1 \leq t \leq n_s$), where for $1 \leq g \leq 2$, $0 \leq a_g \leq 2$, $0 \leq b_g \leq 2$, $0 \leq c_g \leq 2$ and $0 \leq \sum_{g=1}^{2} a_g \leq 2$, $0 \leq \sum_{g=1}^{2} b_g \leq 2$, $0 \leq \sum_{g=1}^{2} c_g \leq 2$. An oriented triple in M2D is an induced 1-subdigraph of three vertices with exactly one vertex from one part. An oriented triple is said to be transitive if it is of the form $x_{ij}(1-0)x_{mn}(1-0)x_{st}(0-1)x_{ij}$, or $x_{ij}(1-0)x_{mn}(0-1)x_{st}(0-0)x_{ij}$, or $x_{ij}(1-0)x_{mn}(0-0)x_{st}(0-1)x_{ij}$, or $x_{ij}(1-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}$, or $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}$, otherwise it is intransitive. An M2D is said to be transitive if every of its oriented triple is transitive. In particular, a triple C in M2D is transitive if every oriented triple of C is transitive.

Through out this section we discuss k-partite 2-digraphs, with $k \geq 3$, except at few places where we require bipartite 2-digraphs. We know if $P = [p_1, p_2, \cdots, p_l]$ and $Q = [q_1, q_2, \cdots, q_m]$ are mark sequences of a bipartite 2-digraph, then $p_i \leq 4m$, $1 \leq i \leq l$ and $q_j \leq 4l$, $1 \leq j \leq m$. Also the sequences of non-negative integers $[p_1]$ and $[q_1, q_2, \cdots, q_m]$, with $p_1 + q_1 + q_2 + \cdots + q_m = 4m$ are always mark sequences of some bipartite 2-digraph. Obviously the sequences $[0]$ and $[4, 4, \cdots, 4]$ are the mark sequences of a bipartite 2-digraph.

We have the following observation about k-partite 2-digraphs, $k \geq 3$.

Lemma 4.3.1. Let D and D' be two M2D's with the same mark sequences. Then D can be transformed to D' by successively transforming (i) appropriate oriented triples formed by vertices $x_{ij} \in X_i$, $x_{mn} \in X_m$ and $x_{st} \in X_s$, $i \neq m \neq s$, in one of the following ways:
either (a) by changing an intransitive oriented triple $x_{ij}(1-0)x_{mn}(1-0)x_{st}(1-0)x_{ij}$ to a transitive oriented triple $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}$, which has same mark sequences, or vice versa,
or (b) by changing an intransitive oriented triple $x_{ij}(1-0)x_{mn}(1-0)x_{st}(0-0)x_{ij}$ to a transitive oriented triple $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-1)x_{ij}$, which has same mark sequences, or vice versa,
or (ii) by changing the symmetric arcs $x_{ij}(1-1)x_{mn}$ to $x_{ij}(0-0)x_{mn}$, which has same mark sequences, or vice versa.

Proof. Let P_i be mark sequences of an M2D D whose parts are X_i, $1 \leq i \leq k$. Suppose D' be an M2D with parts X'_i, $1 \leq i \leq k$. To prove the result it is sufficient to show that D' can be obtained from D by transforming oriented triples in any one of the ways as given in i(a) or i(b) or by changing the arcs as given in (ii).

We fix n_i for $2 \leq i \leq k$ and use induction on n_1. For $n_1 = 1$, $n_2 = 1, \cdots,$ $n_k = 1$ and $k = 3$ the result is obvious. Assume that the result is true when there are fewer than n_1 vertices in the first part. Let $j_2, j_3, \cdots, j_k$ be such that for $m_2, m_3, \cdots, m_k$, $1 \leq j_i < m_i \leq n_i$ $(2 \leq i \leq k)$, the corresponding arcs have same orientations in D and D'. For $j_2, j_3, \cdots, j_k$, $2 \leq i, p, q \leq k$, $p \neq q$, the oriented triples are of the form

(I) $x_{1n_1}(1-0)x_{ij_p}(1-0)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_p}(0-0)x'_{ij_q}$

(II)$x_{1n_1}(0-0)x_{ij_p}(0-1)x_{ij_q}$ and $x'_{1n_1}(1-0)x'_{ij_p}(0-0)x'_{ij_q}$

(III)$x_{1n_1}(1-0)x_{ij_p}(0-0)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_p}(0-1)x'_{ij_q}$

(IV)$x_{1n_1}(1-0)x_{ij_p}$ and $x'_{1n_1}(0-0)x'_{ij_p}$

Case (I). Since x_{1n_1} and x'_{1n_1} have equal marks, therefore $x_{1n_1}(0-1)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$, or $x_{1n_1}(0-0)x_{ij_q}$ and $x'_{1n_1}(1-0)x'_{ij_q}$. Thus there is an oriented triple $x_{1n_1}(1-0)x_{ij_p}(1-0)x_{ij_q}(1-0)x_{1n_1}$, or $x_{1n_1}(1-0)x_{ij_p}(1-0)x_{ij_q}(0-0)x_{1n_1}$ in D and corresponding to these $x'_{1n_1}(0-0)x'_{ij_p}(0-0)x'_{ij_q}(0-0)x'_{1n_1}$, or $x'_{1n_1}(0-0)x'_{ij_p}(0-0)x'_{ij_q}(0-1)x'_{1n_1}$ respectively is an oriented triple in D'.

Case II. Since x_{1n_1} and x'_{1n_1} have equal marks, so $x_{1n_1}(1-0)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$ and thus there is an oriented triple $x_{1n_1}(0-0)x_{ij_p}(0-1)x_{ij_q}(0-1)x_{1n_1}$ in D and corresponding to this $x'_{1n_1}(1-0)x'_{ij_p}(0-0)x'_{ij_q}(0-0)x'_{1n_1}$ is an oriented triple in D'.

Case III. Since x_{1n_1} and x'_{1n_1} have equal marks, so $x_{1n_1}(0-1)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$ and thus there is an oriented triple $x_{1n_1}(1-0)x_{ij_p}(0-0)x_{ij_q}(1-0)x_{1n_1}$ in D and corresponding to this $x'_{1n_1}(0-0)x'_{ij_p}(0-1)x'_{ij_q}(0-0)x'_{1n_1}$ is an oriented triple in D'.

Case IV. Since x_{1n_1} and x'_{1n_1} have equal marks, so $x_{1n_1}(1-1)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$.

Thus it follows from (I)-(IV) that there is an M2D that can be obtained from D by any one of the transformations i(a) or i(b) or (ii) with mark sequences remaining unchanged. Hence the result follows by induction. $\square$

Lemma 4.3.1 leads to the following observation.

Corollary 4.3.2. Among all M2D's with given mark sequences those with the fewest arcs are transitive.

A transmitter is a vertex with indegree zero. We assume without loss of generality that transitive M2D's have no arcs of the form $x(1-1)y$, as they can be transformed to $x(0-0)y$ with same marks. This implies that in a transitive M2D with mark sequences $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, any of the vertex with mark p_{in_i} can act as transmitter.

Let $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, be k sequences of non-negative integers in non-decreasing order with $p_{1n_1} \ge p_{in_i}$,

$$2\sum_{t=2}^{k} n_t \le p_{1n_1} \le 4\sum_{t=2}^{k} n_t \quad and \quad 0 \le p_{in_i} \le 4\left(\sum_{t=2, t\ne i}^{k} n_t\right) - 3$$

for all $2 \le i \le k$. Let P_1' be obtained from P_1 by deleting one entry p_{1n_1}, and let $P_2', P_3', \cdots, P_k'$ be obtained as follows.
(A)(i). If $p_{1n_1} \ge 3\sum_{t=2}^{k} n_t$, then reducing $4\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ largest entries of $P_2, P_3, \cdots, P_k$ by one each,
or**(ii).** If $p_{1n_1} < 3\sum_{t=2}^{k} n_t$, then reducing $3\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ largest entries of $P_2, P_3, \cdots, P_k$ by two each, and $p_{1n_1} - 2\left(\sum_{t=2}^{k} n_t\right)$ remaining entries by one each.
(B). In case any one of $p_{in_i} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$, $2 \le i \le k$, say for instance $p_{jn_j} = 4\sum_{t=2}^{k} n_t - 2$, then also $p_{1n_1} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$ as $p_{1n_1} \ge p_{in_i}$. In this case we reduce p_{jn_j} by two.

The next result provides a useful recursive test whether the sequences of non-negative integers form the mark sequences of some M2D.

Theorem 4.3.3. P_i are the mark sequences of some M2D if and only if P_i' (arranged in non-decreasing order) as obtained in (A) or (B) are the mark sequences of some M2D.
Proof. Let P_i', $1 \le i \le k$, be the mark sequences of some M2D $D'(X_1', X_2', \cdots, X_k')$. First assume $P_2', P_3', \cdots, P_k'$ be obtained from $P_2, P_3, \cdots, P_k$ as in (A)(i). Construct an M2D $D(X_1, X_2, \cdots, X_k)$ as follows. Let $X_1 = X_1' \cup \{x\}, X_i = X_i'$, $2 \le i \le k$, with $X_1' \cap \{x\} = \phi$. Let $x(1-0)y$ for those vertices y of

$X_2^{/}, X_3', \cdots X_k'$ whose marks are reduced by one in going from P_i to P_i', and $x(2-0)y$ for those vertices y of $X_2', X_3', \cdots, X_k'$ whose marks are not reduced in going from P_i to P_i', $1 \leq i \leq k$. Then $D(X_1, X_2, \cdots, X_k)$ is M2D with mark sequences P_i, $1 \leq i \leq k$.

Now, if P_2', P_3',$\cdots$, P_k' are obtained from P_2, P_3,$\cdots$, P_k as in (A)(ii), then construct an M2D $D(X_1, X_2, \cdots, X_k)$ as follows. Let $X_1 = X_1' \cup \{x\}$, $X_i = X_i'$, $2 \leq i \leq k$, with $X_1' \cap \{x\} = \phi$. Let $x(1-0)y$ for those vertices y of $X_2', X_3', \cdots, X_k'$ whose marks are reduced by one in going from P_i to P_i', and $x(1-1)y$ for those vertices y of $X_2', X_3', \cdots, X_k'$ whose marks are reduced by two in going from P_i to P_i', $1 \leq i \leq k$. For (B), we take $x(1-1)y$ for those vertices y of $X_2', X_3', \cdots, X_k'$ whose marks are reduced by two in going from P_i to P_i', $1 \leq i \leq k$. Then $D(X_1, X_2, \cdots, X_k)$ is M2D with mark sequences P_i, $1 \leq i \leq k$.

Conversely, suppose P_i be mark sequences of some M2D $D(X_1, X_2, \cdots, X_k)$, $1 \leq i \leq k$. Now any of the vertex $x_{in_i} \in X_i$ with mark p_{in_i}, $1 \leq i \leq k$, can act as a transmitter. Clearly for (i) and (ii) $p_{1n_1} \geq 2\sum_{t=2}^{k} n_t$ and $p_{in_i} \leq 4\sum_{t=1,t\neq i}^{k} n_t - 3$ for all $2 \leq i \leq k$, because if $p_{1n_1} \leq 2\sum_{t=2}^{k} n_t$, then by deleting p_{1n_1} we have to reduce more than $\sum_{t=2}^{k} n_t$ entries from $P_2, P_3, \cdots, P_k$, which is absurd.
(i) If $p_{1n_1} \geq 3\sum_{t=2}^{k} n_t$, let X be the set of $4\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ vertices of largest marks in $X_2, X_3, \cdots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. In case X does not contain all $4\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ vertices of largest marks, we can bring them to X by using Lemma 4.3.1. Construct $D(X_1, X_2, \cdots, X_k)$ such that $x_{1n_1}(1-0)x$ for all x in X and $x_{1n_1}(2-0)y$ for all y in Y. Clearly, $D(X_1, X_2, \cdots, X_k) - \{x_{1n_1}\}$ realizes $P_1', P_2', \cdots, P_k'$.
(ii) If $p_{1n_1} < 3\sum_{t=2}^{k} n_t$, let X be the set of $3\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ vertices of largest marks in $X_2, X_3, \cdots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. Construct $D(X_1, X_2, \cdots, X_k)$ such that $x_{1n_1}(1-1)x$ for all x in X and $x_{1n_1}(1-0)y$ for all y in Y. Then again $D(X_1, X_2, \cdots, X_k) - \{x_{1n_1}\}$ realizes $P_1', P_2', \cdots, P_k'$.
(B) If for instance $p_{jn_j} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$, then necessarily $p_{1n_1} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$ so that $x_{1n_1}(0-0)x_{jn_j}$ or $x_{1n_1}(1-1)x_{jn_j}$. Clearly, $D(X_1, X_2, \cdots, X_k) - \{x_{1n_1}\}$ realizes $P_1', P_2', \cdots, P_k'$. $\square$

Theorem 4.3.3 provides an algorithm for determining whether or not the k sequences P_i, $1 \leq i \leq k$, of non-negative integers in non-decreasing order are mark sequences, and for constructing a corresponding M2D. Let $P_i = [p_{i1}, p_{i2}, \cdots, p_{in_i}]$, $1 \leq i \leq k$, with (a) $p_{1n_1} \geq 2\sum_{t=2}^{k} n_t$, (b) $p_{in_i} \leq 4\left(\sum_{t=1,t\neq i}^{k} n_t\right) - 2$ for all $2 \leq i \leq k$, be mark sequences of an M2D with parts $X_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$, $1 \leq i \leq k$. Deleting p_{1n_1} and performing A(i) or A(ii), or B of Theorem 4.3.3 according as $p_{1n_1} \geq 3\sum_{t=2}^{k} n_t$ or $p_{1n_1} < 3\sum_{t=2}^{k} n_t$, or any one of $p_{in_i} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$, $2 \leq i \leq k$, we obtain $P_2', P_3', \cdots, P_k'$. If the marks of the vertices x_{ij} were decreased by one in this process, then the construction yielded $x_{1n_1}(1-0)x_{ij}$, and if these were decreased by two, then the construction yielded $x_{1n_1}(1-1)x_{ij}$. For vertices x_{st} whose marks remained unchanged, the construction yielded $x_{1n_1}(2-0)x_{st}$. Note that if any of the conditions A or B does not hold, then we delete p_{in_i} for that i for which the conditions get satisfied, and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not P_i are mark sequences, and if P_i are mark sequences, then an M2D with mark sequences P_i, $1 \leq i \leq k$ is constructed. During the application of Theorem 4.3.3, the algorithm may reach a stage where we get just two sequences, and it is not possible to apply Theorem 4.3.3, in those cases we apply Lemma 4.2.3 to Lemma 4.2.9 by choosing $r = 2$.

We illustrate this reduction and the resulting construction with the following examples.

Example 4.3.4. Consider the five sequences of non-negative integers as follows: $P_1 = [15, 16, 21]$, $P_2 = [16, 20]$, $P_3 = [15, 20]$, $P_4 = [17, 19]$, $P_5 = [16, 17]$.

1. [15,16], [15,18], [14,18], [16,17], [15,16] $x_{13}(0-0)x_{22}$, $x_{13}(0-0)x_{32}$, $x_{13}(0-0)x_{42}$, $x_{13}(1-0)x_{21}$, $x_{13}(1-0)x_{31}$, $x_{13}(1-0)x_{41}$, $x_{13}(1-0)x_{51}$, $x_{13}(1-0)x_{52}$
2. [15], [13,16], [12,16], [14,15], [13,14] $x_{12}(0-0)x_{21}$, $x_{12}(0-0)x_{22}$, $x_{12}(0-0)x_{31}$, $x_{12}(0-0)x_{32}$, $x_{12}(0-0)x_{41}$, $x_{12}(0-0)x_{42}$, $x_{12}(0-0)x_{51}$, $x_{12}(1-0)x_{52}$
3. [13], [13], [11,14], [12,13], [12,12] $x_{22}(0-0)x_{32}$, $x_{22}(0-0)x_{11}$, $x_{22}(0-0)x_{42}$, $x_{22}(0-0)x_{41}$, $x_{22}(0-0)x_{52}$, $x_{22}(1-0)x_{31}$, $x_{22}(1-0)x_{51}$
4. [11], [11], [11], [10,11], [11,11] $x_{32}(0-0)x_{11}$, $x_{32}(0-0)x_{21}$, $x_{32}(0-0)x_{42}$, $x_{32}(0-0)x_{41}$, $x_{32}(1-0)x_{51}$, $x_{32}(1-0)x_{52}$

5. [9], [9], [9], [10], [9,10] $x_{42}(0-0)x_{11}$, $x_{42}(0-0)x_{21}$, $x_{42}(0-0)x_{31}$, $x_{42}(0-0)x_{52}$, $x_{42}(1-0)x_{51}$

6. [7], [8], [8], [8], [9]

   $x_{51}(0-0)x_{41}$, $x_{51}(0-0)x_{11}$, $x_{51}(1-0)x_{21}$, $x_{51}(1-0)x_{31}$

7. [5], ϕ, [6], [6], [7], $x_{21}(0-0)x_{52}$, $x_{21}(0-0)x_{31}$, $x_{21}(0-0)x_{41}$, $x_{21}(0-0)x_{11}$

8. [3], ϕ, ϕ, [4], [5], $x_{31}(0-0)x_{52}$, $x_{31}(0-0)x_{41}$, $x_{31}(0-0)x_{11}$

9. [1], ϕ, ϕ, ϕ, [3], $x_{41}(0-0)x_{52}$, $x_{41}(0-0)x_{11}$

10. [0], ϕ, ϕ, ϕ, ϕ, $x_{52}(0-0)x_{11}$

The resulting 5-partite 2-digraph has mark sequences $P_1 = [15, 16, 21]$, $P_2 = [16, 20]$, $P_3 = [15, 20]$, $P_4 = [17, 19]$, $P_5 = [16, 17]$ with vertex sets $X_1 = \{x_{11}, x_{12}, x_{13}\}$, $X_2 = \{x_{21}, x_{22}\}$, $X_3 = \{x_{31}, x_{32}\}$, $X_4 = \{x_{41}, x_{42}\}$, $X_5 = \{x_{51}, x_{52}\}$, and arcs as $x_{13}(0-0)x_{22}$, $x_{13}(0-0)x_{32}$, $x_{13}(0-0)x_{42}$, $x_{13}(1-0)x_{21}$, $x_{13}(1-0)x_{31}$, $x_{13}(1-0)x_{41}$, $x_{13}(1-0)x_{51}$, $x_{13}(1-0)x_{52}$, $x_{12}(0-0)x_{21}$, $x_{12}(0-0)x_{22}$, $x_{12}(0-0)x_{31}$, $x_{12}(0-0)x_{32}$, $x_{12}(0-0)x_{41}$, $x_{12}(0-0)x_{42}$, $x_{12}(0-0)x_{51}$, $x_{12}(1-0)x_{52}$, $x_{22}(0-0)x_{32}$, $x_{22}(0-0)x_{11}$, $x_{22}(0-0)x_{42}$, $x_{22}(0-0)x_{41}$, $x_{22}(0-0)x_{52}$, $x_{22}(1-0)x_{31}$, $x_{22}(1-0)x_{51}$, $x_{32}(0-0)x_{11}$, $x_{32}(0-0)x_{21}$, $x_{32}(0-0)x_{42}$, $x_{32}(0-0)x_{41}$, $x_{32}(1-0)x_{51}$, $x_{32}(1-0)x_{52}$, $x_{42}(0-0)x_{11}$, $x_{42}(0-0)x_{21}$, $x_{42}(0-0)x_{31}$, $x_{42}(0-0)x_{52}$, $x_{42}(1-0)x_{51}$, $x_{51}(0-0)x_{41}$, $x_{51}(0-0)x_{11}$, $x_{51}(1-0)x_{21}$, $x_{51}(1-0)x_{31}$, $x_{21}(0-0)x_{52}$, $x_{21}(0-0)x_{31}$, $x_{21}(0-0)x_{41}$, $x_{21}(0-0)x_{11}$, $x_{31}(0-0)x_{52}$, $x_{31}(0-0)x_{41}$, $x_{31}(0-0)x_{11}$, $x_{41}(0-0)x_{52}$, $x_{41}(0-0)x_{11}$, $x_{52}(0-0)x_{11}$

Example 4.3.5. Consider the three sequences of non-negative integers as follows: $P_1 = [12, 18]$, $P_2 = [1, 2, 3]$, $P_3 = [10, 18]$.

1. [12], [1,2,3], [10,16]

   $x_{12}(2-0)x_{21}$, $x_{12}(2-0)x_{22}$, $x_{12}(2-0)x_{23}$, $x_{12}(2-0)x_{31}$, $x_{12}(0-0)x_{32}$

2. [12], [1,2,3], [10]

   $x_{32}(2-0)x_{11}$, $x_{32}(2-0)x_{21}$, $x_{32}(2-0)x_{22}$, $x_{32}(2-0)x_{23}$

3. ϕ, [1,2,1], [8]

   $x_{11}(2-0)x_{21}$, $x_{11}(2-0)x_{22}$, $x_{11}(0-0)x_{23}$, $x_{11}(0-0)x_{31}$

3. ϕ, [0,0,0], ϕ

   $x_{31}(1-0)x_{21}$, $x_{31}(0-0)x_{22}$, $x_{31}(1-0)x_{23}$

The resulting 3-partite 2-digraph has mark sequences $P_1 = [12, 18]$, $P_2 = [1, 2, 3]$, $P_3 = [10, 18]$ and vertex sets $X_1 = \{x_{11}, x_{12}\}$, $X_2 = \{x_{21}, x_{22}, x_{23}\}$,

$X_3 = \{x_{31}, x_{32}\}$ and arcs $x_{12}(2-0)x_{21}$, $x_{12}(2-0)x_{22}$, $x_{12}(2-0)x_{23}$, $x_{12}(2-0)x_{31}$, $x_{12}(0-0)x_{32}$, $x_{32}(2-0)x_{11}$, $x_{32}(2-0)x_{21}$, $x_{32}(2-0)x_{22}$, $x_{32}(2-0)x_{23}$, $x_{11}(2-0)x_{21}$, $x_{11}(2-0)x_{22}$, $x_{11}(0-0)x_{23}$, $x_{11}(0-0)x_{31}$, $x_{31}(1-0)x_{21}$, $x_{31}(0-0)x_{22}$, $x_{31}(1-0)x_{23}$.

The next result gives a combinatorial criterion for determining whether k sequences of non-negative integers in non-decreasing order are realizable as marks.

Theorem 4.3.6. Let $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, be k sequences of non-negative integers in non-decreasing order. Then, P_i are the mark sequences of some M2D if and only if

$$\sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} \ge 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j, \tag{4.2}$$

for all sequences of k integers s_i, $1 \le s_i \le n_i$, with equality when $s_i = n_i$ for all i.

Proof. A sub k-partite 2-digraph induced by s_i vertices for $1 \le i \le k$, $1 \le s_i \le n_i$, has a sum of marks $4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j$. This proves the necessity.

For sufficiency, let $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, be the sequences of non-negative integers in non-decreasing order satisfying conditions (4.2) but are not the mark sequences of any M2D. Let these sequences be chosen in such a way that n_i, $1 \le i \le k$, be smallest possible and p_{11} is the least with that choice of n_i. We consider the following two cases.

Case (i). Assume equality in (4.2) holds for some $s_j \le n_j$, $1 \le j \le k-1$, $s_k < n_k$, so that

$$\sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} = 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j.$$

By the minimality of n_i, $1 \le i \le k$, the sequences $P_i = [P_{i1}, P_{i2}, \cdots, P_{is_i}]$ are mark sequences of some M2D $D'(X'_1, X'_2, \cdots, X'_k)$.

For $1 \le i \le k$, define

$$P''_i = \left[\left(p_{i(s_i+1)} - 4\sum_{t=1, t\ne i}^{k} s_t\right), \ \left(p_{i(s_i+2)} - 4\sum_{t=1, t\ne i}^{k} s_t\right), \cdots, \left(p_{i(n_i)} - 4\sum_{t=1, t\ne i}^{k} s_t\right)\right].$$

Now consider the sum

$$\sum_{i=1}^{k}\sum_{j=1}^{f_i}[p_{i(s_i+j)} - 4\sum_{t=1,t\neq i}^{k} s_t]$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{f_i} p_{i(s_i+j)} - 4\sum_{i=1}^{k}\sum_{j=1}^{f_i}\sum_{t=1,t\neq i}^{k} s_t$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{f_i+s_i} p_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} - 4\sum_{i=1}^{k}\sum_{j=1}^{f_i}\sum_{i=1}^{k} s_t + 4\sum_{i=1}^{k}\sum_{j=1}^{f_i} s_i$$

$$\geq 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} [(s_i+f_i)(s_j+f_j)] - 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j - 4\sum_{i=1}^{k} f_i \sum_{t=1}^{k} s_t + 4\sum_{i=1}^{k} f_i s_i$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} (s_i s_j + s_i f_j + f_i s_j + f_i f_j) - 2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j$$

$$- 4\sum_{i=1}^{k}\sum_{t=1}^{k} f_i s_t + 4\sum_{i=1}^{k} f_i s_i$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j + 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} (s_i f_j + f_i s_j) + 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_i f_j$$

$$- 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j - 4\sum_{i=1}^{k}\sum_{t=1}^{k} f_i s_t + 4\sum_{i=1}^{k} f_i s_i$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_i f_j$$

$$+ 4\sum_{i=1}^{k-1}[(s_i f_{i+1}) + f_i s_{i+1}) + (s_i f_{i+2}) + f_i s_{i+2}) + \cdots + (s_i f_k) + f_i s_k)]$$

$$- 4\sum_{i=1}^{k}(f_i s_1 + f_i s_2 + \cdots + f_i s_k) + 4(f_1 s_1 + f_2 s_2 + \cdots + f_k s_k)$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_i f_j$$

$$+ 4\{[(s_1 f_2 + f_1 s_2) + (s_1 f_3 + f_1 s_3) + \cdots + (s_1 f_k + f_1 s_k)]$$

$$+ [(s_2 f_3 + f_2 s_3) + (s_2 f_4 + f_2 s_4) + \cdots + (s_2 f_k + f_2 s_k)]$$

$$+ \cdots + [(s_{k-1} f_k + f_{k-1} s_k)]\}$$

$$
\begin{aligned}
&- 4[(f_1s_1 + f_1s_2 + \cdots + f_1s_k) + (f_2s_1 + f_2s_2 + \cdots + f_2s_k)\\
&+ \cdots + (f_ks_1 + f_ks_2 + \cdots + f_ks_k)]\\
&+ 4(f_1s_1 + f_2s_2 + \cdots + f_ks_k)\\
&= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_if_j,
\end{aligned}
$$

for $1 \le f_i \le n_i - s_i$ with equality when $f_i = n_i - s_i$ for all i, $1 \le i \le k$. Then by minimality of n_i, $1 \le i \le k$, the sequences P_i'' form the mark sequences of some M2D $D''(X_1'', X_2'', \cdots, X_k'')$.

Now construct a new M2D $D(X_1, X_2, \cdots, X_k)$ as follows. Let

$$X_1 = X_1' \cup X_1'', X_2 = X_2' \cup X_2'', \cdots, X_k = X_k' \cup X_k''$$

with $X_i' \cap X_i'' = \phi$.

Let

$$x_i''(2-0)x_1', x_i''(2-0)x_2', \cdots, x_i''(2-0)x_{i-1}', x_i''(2-0)x_{i+1}', \cdots, x_i''(2-0)x_k',$$

for all x_i'' in X_i'' and for all x_i' in X_i', $1 \le i \le k$. Then clearly $D(X_1, X_2, \cdots, X_k)$ is an M2D with mark sequences P_i, $1 \le i \le k$, which is a contradiction.

Case (ii). Assume strict inequality in (4.2) holds for some $s_i \ne n_i$, $1 \le i \le k$. Let $P_1' = [p_{11} - 1, p_{12}, \cdots, p_{1n_1-1}, p_{1n_1} + 1]$ and $P_j' = [p_{j1}, p_{j2}, \cdots, p_{jn_j}]$ for all j, $2 \le j \le k$. Clearly the sequences P_i', $1 \le i \le k$, satisfy conditions (4.2). Therefore by the minimality of p_{11}, the sequences P_i', $1 \le i \le k$, are mark sequences of some M2D $D'(X_1', X_2', \cdots, X_k')$. Let $p_{x_{11}} = p_{11} - 1$ and $p_{x_{1n_1}} = p_{1n_1} + 1$. Since $p_{x_{1n_1}} > p_{x_{11}} + 1$, there exists a vertex x_{ij} in X_i, $2 \le i \le k$, $1 \le j \le n_i$, such that $x_{1n_1}(1-0)x_{ij}(1-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(1-0)x_{11}$, or $x_{1n_1}(1-0)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(0-0)x_{11}$ in $D'(X_1', X_2', \cdots, X_k')$, and if these are changed to $x_{1n_1}(0-0)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-1)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(0-1)x_{11}$, or $x_{1n_1}(0-1)x_{ij}(0-1)x_{11}$ respectively, the result is an M2D with mark sequences P_i, $1 \le i \le k$, which is again a contradiction. Hence the result follows. □

Definition 4.3.7. A k-partite r-digraph (or briefly multipartite multidigraph(MMD))is an orientation of a k-partite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. So, k-partite 1-digraph is an oriented k-partite graph, and a complete k-partite 1-digraph is a k-partite tournament. Let $D = D(X_1, X_2, \cdots, X_k)$ be a multipartite multidigraph with parts $X_i = \{x_{i1}, x_{i2}, \cdots, x_{in_i}\}$, $1 \leq i \leq k$. Let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$, $1 \leq j \leq n_i$, be respectively the outdegree and indegree of a vertex $x_{ij} \in X_i$. Define $p_{x_{ij}}$(or simply p_{ij}) $= r\left(\sum_{t=1,t\neq i}^{k} n_t\right) + d^+_{x_{ij}} - d^-_{x_{ij}}$ as the mark (or r-score) of x_{ij}. Clearly, $0 \leq p_{x_{ij}} \leq 2r\sum_{t=1,t\neq i}^{k} n_t$. Then the k sequences $p_i = [p_{ij}]_1^{n_i}$, $1 \leq i \leq k$, in non-decreasing order are called the mark sequences of D.

An MMD can be interpreted as a result of a competition among k teams in which each player of one team plays with every player of the other $k-1$ teams at most r times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player x_{ij} receives a total of $p_{x_{ij}}$ points. The k sequences of non-negative integers p_i, $1 \leq i \leq k$, in non-decreasing order are said to be realizable if there exists an MMD with mark sequences P_i. All the results on multipartite 2-digraphs can be extended to MMD. The following is the combinatorial characterization for mark sequences in MMD. We prove it here in a different way.

Theorem 4.3.8. Let $P_i = [p_{ij}]_1^{n_i}$, $1 \leq i \leq k$, be k sequences of non-negative integers in non-decreasing order. Then, P_i are the mark sequences of some MMD if and only if

$$\sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} \geq 2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j, \tag{4.3}$$

for all sequences of k integers s_i, $1 \leq s_i \leq n_i$, with equality when $s_i = n_i$ for all i.

Proof. A sub k-partite r-digraph induced by s_i vertices for $1 \leq i \leq k$, $1 \leq s_i \leq n_i$, has a sum of marks $2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j$. This proves the necessity.

For sufficiency, let $P_i = [p_{ij}]_1^{n_i}$, $1 \leq i \leq k$, be the sequences of non-negative integers in non-decreasing order satisfying conditions (4.3) but are not the mark sequences of any MMD. Let these sequences be chosen in such a way that n_i, $1 \leq i \leq k$, be smallest possible and p_{11} is the least with that choice of n_i. We consider the following two cases.

Case (i). Assume equality in (4.3) holds for some $s_j \leq n_j$, $1 \leq j \leq k-1$, $s_k < n_k$, so that

$$\sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} = 2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j.$$

By the minimality of n_i, $1 \leq i \leq k$, the sequences $P_i = [P_{i1}, P_{i2}, \cdots, P_{is_i}]$ are mark sequences of some MMD $D'(X_1', X_2', \cdots, X_k')$.

Define

$$P_i'' = \left[p_{i(s_i+1)} - 2r\sum_{t=1,t\neq i}^{k} s_t, p_{i(s_i+2)} - 2r\sum_{t=1,t\neq i}^{k} s_t, \cdots, p_{i(n_i)} - 2r\sum_{t=1,t\neq i}^{k} s_t\right],$$

$1 \leq i \leq k$.

Now consider the sum

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{f_i}(p_{i(s_i+j)} - 2r\sum_{t=1,t\neq i}^{k} s_t) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{f_i} p_{i(s_i+j)} - 2r\sum_{i=1}^{k}\sum_{j=1}^{f_i}\sum_{t=1,t\neq i}^{k} s_t \\
&= \sum_{i=1}^{k}\sum_{j=1}^{f_i+s_i} p_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} - 2r\sum_{i=1}^{k}\sum_{j=1}^{f_i}\sum_{i=1}^{k} s_t + 2r\sum_{i=1}^{k}\sum_{j=1}^{f_i} s_i \\
&\geq 2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}[(s_i+f_i)(s_j+f_j)] - 2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j \\
&- 2r\sum_{i=1}^{k} f_i\sum_{t=1}^{k} s_t + 2r\sum_{i=1}^{k} f_i s_i
\end{aligned}$$

$$
\begin{aligned}
\sum_{i=1}^{k}\sum_{j=1}^{f_i}(p_{i(s_i+j)} &= 2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(s_is_j+s_if_j+f_is_j+f_if_j-2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}s_is_j\\
&-2r\sum_{i=1}^{k}\sum_{t=1}^{k}f_is_t+2r\sum_{i=1}^{k}f_is_i\\
&=2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}s_is_j+2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(s_if_j+f_is_j)+2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}f_if_j\\
&-2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}s_is_j-2r\sum_{i=1}^{k}\sum_{t=1}^{k}f_is_t+2r\sum_{i=1}^{k}f_is_i\\
&=2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}f_if_j\\
&+2r\sum_{i=1}^{k-1}[(s_if_{i+1})+f_is_{i+1})+(s_if_{i+2})+f_is_{i+2})+\cdots+(s_if_k)+f_is_k)]\\
&-2r\sum_{i=1}^{k}(f_is_1+f_is_2+\cdots+f_is_k)+2r(f_1s_1+f_2s_2+\cdots+f_ks_k)\\
&=2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}f_if_j\\
&+2r\{[(s_1f_2+f_1s_2)+(s_1f_3+f_1s_3)+\cdots+(s_1f_k+f_1s_k)]\\
&+[(s_2f_3+f_2s_3)+(s_2f_4+f_2s_4)+\cdots+(s_2f_k+f_2s_k)]\\
&+\cdots+[(s_{k-1}f_k+f_{k-1}s_k)]\}\\
&-2r[(f_1s_1+f_1s_2+\cdots+f_1s_k)+(f_2s_1+f_2s_2+\cdots+f_2s_k)\\
&+\cdots+(f_ks_1+f_ks_2+\cdots+f_ks_k)]\\
&+2r(f_1s_1+f_2s_2+\cdots+f_ks_k)\\
&=2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}f_if_j,
\end{aligned}
$$

for $1 \le f_i \le n_i - s_i$ with equality when $f_i = n_i - s_i$ for all i, $1 \le i \le k$. Then by minimality of n_i, $1 \le i \le k$, the sequences P_i'' form the mark sequences of some MMD $D''(X_1'', X_2'', \cdots, X_k'')$.

Now construct a new MMD $D(X_1, X_2, \cdots, X_k)$ as follows. Let

$$X_1 = X_1' \cup X_1'', X_2 = X_2' \cup X_2'', \cdots, X_k = X_k' \cup X_k''$$

with $X_i' \cap X_i'' = \phi$.

Let

$$x_i''(r-0)x_1', x_i''(r-0)x_2', \cdots, x_i''(r-0)x_{i-1}', x_i''(r-0)x_{i+1}', \cdots, x_i''(r-0)x_k',$$

for all x_i'' in X_i'' and for all x_i' in X_i', $1 \leq i \leq k$. Then clearly $D(X_1, X_2, \cdots, X_k)$ is an MMD with mark sequences P_i, $1 \leq i \leq k$, which is a contradiction.

Case (ii). Assume strict inequality in (4.3) holds for some $s_i \neq n_i$, $1 \leq i \leq k$. Let

$$P_1' = [p_{11} - 1, p_{12}, \cdots, p_{1n_1-1}, p_{1n_1} + 1]$$

and

$$P_j' = [p_{j1}, p_{j2}, \cdots, p_{jn_j}]$$

for all j, $2 \leq j \leq k$. Clearly the sequences P_i', $1 \leq i \leq k$, satisfy conditions (4.3). Therefore by the minimality of p_{11}, the sequences P_i', $1 \leq i \leq k$, are mark sequences of some MMD $D'(X_1', X_2', \cdots, X_k')$. Let

$$p_{x_{11}} = p_{11} - 1$$

and

$$p_{x_{1n_1}} = p_{1n_1} + 1.$$

Since

$$p_{x_{1n_1}} > p_{x_{11}} + 1,$$

there exists a vertex x_{ij} in X_i, $2 \leq i \leq k$, $1 \leq j \leq n_i$, such that $x_{1n_1}(1-0)x_{ij}(1-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(1-0)x_{11}$, or $x_{1n_1}(1-0)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(0-0)x_{11}$ in $D'(X_1', X_2', \cdots, X_k')$, and if these are changed to $x_{1n_1}(0-0)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-1)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(0-1)x_{11}$, or $x_{1n_1}(0-1)x_{ij}(0-1)x_{11}$ respectively, the result is an MMD with mark sequences P_i, $1 \leq i \leq k$, which is again a contradiction. Hence the result follows. □

CHAPTER 5

Imbalances in digraphs

In this chapter, we study imbalances and imbalance sequences in digraphs. We extend this concept of imbalances to oriented bipartite graphs. We provide necessary and sufficient conditions for sequences of integers to be imbalance sequences of some oriented bipartite graphs. We show the existence of an oriented bipartite graph with given imbalance set.

5.1 Introduction

Definition 5.1.1. The imbalance of a vertex v_i in a digraph as b_{v_i} (or simply b_i)$= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \cdots, f_n]$ with $f_1 \geq f_2 \geq \cdots \geq f_n$ is feasible if it has sum zero and satisfies $\sum_{i=1}^{k} f_i \leq k(n-k)$, for $1 \leq k < n$.

The following result [39] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 5.1.2. *A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ is an imbalance sequence of a simple digraph if and only if $\sum_{i=1}^{k} b_i \leq k(n-k)$, for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Corollary 5.1.3. *A sequence of integers* $B = [b_1, b_2, \cdots, b_n]$ *with* $b_1 \leq b_2 \leq$

$\cdots \leq b_n$ *is an imbalance sequence of a simple digraph if and only if*

$$\sum_{i=1}^{k} b_i \geq k(k-n),$$

for $1 \leq k < n$ *with equality when* $k = n$.

Pirzada [45] obtained the following result on imbalances in simple directed graphs.

Theorem 5.1.4. *If* $B = [b_1, b_2, \cdots, b_n]$ *is an imbalance sequence of a simple directed graph with* $b_1 \geq b_2 \geq \cdots \geq b_n$, *then* $\sum_{i=1}^{k} b_i^2 \leq \sum_{i=1}^{k}(2n - 2k - b_i)^2$, *for* $1 \leq k < n$ *with equality when* $k = n$.

Definition 5.1.5. The set of distinct imbalances of the vertices in an oriented graph is called its imbalance set.

The following result due to Pirzada [45] gives the existence of an oriented graph with a given imbalance set.

Theorem 5.1.6. *Let* $P = \{p_1, p_2, \cdots, p_m\}$ *and* $Q = \{-q_1, -q_2, \cdots, -q_n\}$ *where* $p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n$ *are positive integers such that* $p_1 < p_2 < \cdots < p_m$ *and* $q_1 < q_2 < \cdots < q_n$. *Then there exists an oriented graph with imbalance set* $P \cup Q$.

5.2 Imbalance sequences in multidigraphs

Define b_{v_i} (or simply b_i) $= d_{v_i}^{+} - d_{v_i}^{-}$ as imbalance of v_i. Clearly, $-r(n-1) \leq b_{v_i} \leq r(n-1)$. The imbalance sequence of D is formed by listing the vertex imbalances in non-decreasing order. Let u and v be distinct vertices in D. If there are f arcs directed from u to v and g arcs directed from v to u, we denote this by $u(f-g)v$, where $0 \leq f, g, f+g \leq r$.

The work of this section has appeared in [49]. The following observation can be easily established and is analogues to Theorem 2.2 of Avery[1].

Lemma 5.2.1. *If D_1 and D_2 are two multi digraphs with same imbalance sequence, then D_1 can be transformed to D_2 by successively transforming (i) appropriate oriented triples in one of the following ways,*
either (a) by changing the intransitive oriented triple $u(1-0)v(1-0)w(1-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$, which has the same imbalance sequence or vice versa,
or (b) by changing the intransitive oriented triple $u(1-0)v(1-0)w(0-0)u$ to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$, which has the same imbalance sequence or vice versa;
or (ii) by changing a double $u(1-1)$ to a double $u(0-0)$, which has the same imbalance sequence or vice versa.

The above observations lead to the following result.

Theorem 5.2.2. *Among all multidigraphs with given imbalance sequence, those with the fewest arcs are transitive.*
Proof. Let B be an imbalance sequence and let D be a realization of B that is not transitive. Then D contains an intransitive oriented triple. If it is of the form $u(1-0)v(1-0)w(1-0)u$, it can be transformed by operation i(a) of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$ with the same imbalance sequence and three arcs fewer. If D contains an intransitive oriented triple of the form $u(1-0)v(1-0)w(0-0)u$, it can be transformed by operation i(b) of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$ with the same imbalance sequence but one arc fewer. In case D contains both types of intransitive oriented triples, they can be transformed to transitive ones with certainly lesser arcs. If in D there is a double $u(1-1)$, by operation (ii) of Lemma 5.2.1, it can be transformed to $u(0-0)$, with same imbalance sequence but two arcs fewer. □

The next result gives necessary and sufficient conditions for a sequence of integers to be the imbalance sequence of some multi digraph.

Theorem 5.2.3. *A sequence $B = [b_1, b_2, \cdots, b_n]$ of integers in non-decreasing order is an imbalance sequence of a multi digraph if and only if for $1 \leq k \leq n$*

$$\sum_{i=1}^{k} b_i \geq rk(k-n), \tag{5.1}$$

with equality when $k = n$.

Proof. Necessity. A multi subdigraph induced by k vertices has a sum of imbalances $rk(k-n)$.

Sufficiency. Assume that $B = [b_1, b_2, \cdots, b_n]$ be the sequence of integers in non-decreasing order satisfying conditions (5.1) but is not the imbalance sequence of any multi digraph. Let this sequence be chosen in such a way that n is the smallest possible and b_1 is the least with that choice of n. We consider the following two cases.

Case(i). Suppose equality in (5.1) holds for some $k \leq n$, so that

$$\sum_{i=1}^{k} b_i = rk(k-n),$$

for $1 \leq k < n$.

By minimality of n, $B_1 = [b_1, b_2, \cdots, b_k]$ is the imbalance sequence of some multi digraph D_1 with vertex set, say V_1. Let $B_2 = [b_{k+1}, b_{k+2}, \cdots, b_n]$. Consider,

$$\begin{aligned}
\sum_{i=1}^{f} b_{k+i} &= \sum_{i=1}^{k+f} b_i - \sum_{i=1}^{k} b_i \\
&\geq r(k+f)[(k+f)-n] - rk(k-n) \\
&= r(k^2 + kf - kn + fk + f^2 - fn - k^2 + kn) \\
&\geq r(f^2 - fn) \\
&= rf(f-n),
\end{aligned}$$

for $1 \leq f \leq n-k$, with equality when $f = n-k$. Therefore, by the minimality for n, the sequence B_2 forms the imbalance sequence of some multi digraph D_2 with vertex set, say V_2. Construct a new multi digraph D with vertex set as follows.

Let $V = V_1 \cup V_2$ with, $V_1 \cap V_2 = \phi$ and the arc set containing those

arcs which are in D_1 and D_2. Then we obtain the multi digraph D with the imbalance sequence B, which is a contradiction.
Case (ii). Suppose that the strict inequality holds in (5.1) for some $k < n$, so that

$$\sum_{i=1}^{k} b_i > rk(k-n),$$

for $1 \leq k < n$. Let $B_1 = [b_1 - 1, b_2, \cdots, b_{n-1}, b_n + 1]$, so that B_1 satisfy the conditions (1). Thus by the minimality of b_1, the sequences B_1 is the imbalances sequence of some multi digraph D_1 with vertex set, say V_1). Let $b_{v_1} = b_1 - 1$ and $b_{v_n} = b_n + 1$. Since $b_{v_n} > b_{v_1} + 1$, there exists a vertex $v_p \in V_1$ such that $v_n(0-0)v_p(1-0)v_1$, or $v_n(1-0)v_p(0-0)v_1$, or $v_n(1-0)v_p(1-0)v_1$, or $v_n(0-0)v_p(0-0)v_1$, and if these are changed to $v_n(0-1)v_p(0-0)v_1$, or $v_n(0-0)v_p(0-1)v_1$, or $v_n(0-0)v_p(0-0)v_1$, or $v_n(0-1)v_p(0-1)v_1$ respectively, the result is a multi digraph with imbalances sequence B, which is again a contradiction. This proves the result. $\square$

On arranging the imbalance sequence in non-increasing order, we have the following observation.

Corollary 5.2.4. *A sequence $B = [b_1, b_2, \cdots, b_n]$ of integers with $b_1 \geq b_2 \geq \cdots \geq b_n$ is an imbalance sequence of a multi digraph if and only if*

$$\sum_{i=1}^{k} b_i \leq rk(n-k),$$

for $1 \leq k \leq n$, with equality when $k = n$.

The converse of a multidigraph D is a multidigraph D', obtained by reversing orientations of all arcs of D. If $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \leq b_2 \leq \cdots \leq b_n$ is an imbalance sequence of a multi digraph D, then $B' = [-b_n, -b_{n-1}, \cdots, -b_1]$.

The next result gives lower and upper bounds for the imbalance b_i of a vertex v_i in a multidigraph D.

Theorem 5.2.5. *If $B = [b_1, b_2, \cdots, b_n]$ is an imbalance sequence of a multi-digraph D, then for each i*

$$r(i-n) \leq b_i \leq r(i-1).$$

Proof. Assume to the contrary that $b_i < r(i-n)$, so that for $k < i$,

$$b_k \leq b_i < r(i-n).$$

That is, $b_1 < r(i-n), b_2 < r(i-n), \cdots, b_i < r(i-n)$.
Adding these inequalities, we get $\sum_{k=1}^{i} b_k < ri(i-n)$, which contradicts Theorem 5.2.3. Therefore, $r(i-n) \leq b_i$.

The second inequality is dual to the first. In the converse multi digraph with imbalance sequence $B = [b'_1, b'_2, \cdots, b'_n]$ we have, by the first inequality

$$b'_{n-i+1} \geq r[(n-i+1)-n] = r(-i+1).$$

Since $b_i = -b'_{n-i+1}$, therefore

$$b_i \leq -r(-i+1) = r(i-1).$$

Hence, $b_i \leq r(i-1)$, completing the proof. □

Now we obtain the following inequalities for imbalances in multidigraphs.

Theorem 5.2.6. *If $B = [b_1, b_2, \cdots, b_n]$ is an imbalance sequence of a multi digraph with $b_1 \geq b_2 \geq \cdots \geq b_n$, then*

$$\sum_{i=1}^{k} b_i^2 \leq \sum_{i=1}^{k} (2rn - 2rk - b_i)^2,$$

for $1 \leq k \leq n$ with equality when $k = n$.
Proof. By Corollary 5.2.4, we have for $1 \leq k \leq n$ with equality when $k = n$

$$rk(n-k) \geq \sum_{i=1}^{k} b_i,$$

or

$$\sum_{i=1}^{k} b_i^2 + 2(2rn-2rk)rk(n-k) \geq \sum_{i=1}^{k} b_i^2 + 2(2rn-2rk)\sum_{i=1}^{k} b_i,$$

or

$$\sum_{i=1}^{k} b_i^2 + k(2rn-2rk)^2 - 2(2rn-2rk)\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b_i^2,$$

or

$$\begin{aligned} b_1^2 + b_2^2 + \cdots + b_k^2 + (2rn-2rk)^2 + (2rn-2rk)^2 + \cdots + (2rn-2rk)^2 \\ - 2(2rn-2rk)b_1 - 2(2rn-2rk)b_2 - \cdots - 2(2rn-2rk)b_k \\ \geq \sum_{i=1}^{k} b_i^2, \end{aligned}$$

or

$$\sum_{i=1}^{k}(2rn-2rk-b_i)^2 \geq \sum_{i=1}^{k} b_i^2.$$

□

The set of distinct imbalances of vertices in a multi digraph is called its imbalance set. the following result gives the existence of a multidigraph with a given imbalance set.

Theorem 5.2.7. *If $P = \{p_1, p_2, \cdots, p_m\}$ and $Q = \{-q_1, -q_2, \cdots, -q_n\}$ where $p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n$ are positive integers such that $p_1 < p_2 < \cdots < p_m$ and $q_1 < q_2 < \cdots < q_n$ and $(p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n) = t$, $1 \leq t \leq r$, then there exists a multidigraph with imbalance set $P \cup Q$. Note that$(p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n)$ denotes the greatest common divisor of $p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n$.*

Proof. Since $(p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n) = t$, $1 \leq t \leq r$, there exist positive integers $f_1, f_2, \cdots, f_m$ and $g_1, g_2, \cdots, g_n$ with $f_1 < f_2 < \cdots < f_m$ and $g_1 < g_2 < \cdots < g_n$ such that $p_i = tf_i$ for $1 \leq i \leq m$ and $q_i = tg_i$ for $1 \leq j \leq n$.

We construct a multi digraph D with vertex set V as follows. Let

$$V = X_1^1 \cup X_2^1 \cup \cdots \cup X_m^1 \cup X_1^2 \cup X_1^3 \cup \cdots \cup X_1^n \cup Y_1^1 \cup Y_2^1 \cup \cdots \cup Y_m^1 \cup Y_1^2 \cup Y_1^3 \cup \cdots \cup Y_1^n,$$

with $X_i^j \cap X_k^l = \phi$, $Y_i^j \cap Y_k^l = \phi$, $X_i^j \cap Y_k^l = \phi$ and

$|X_i^1| = g_1$, for all $1 \leq i \leq m$, $|X_1^i| = g_i$, for all $2 \leq i \leq n$,

$|Y_i^1| = f_i$, for all $1 \leq i \leq m$, $|Y_1^i| = f_1$, for all $2 \leq i \leq n$.

Let there be t arcs directed from every vertex of X_i^1 to each vertex of Y_i^1, for all $1 \leq i \leq m$ and let there be t arcs directed from every vertex of X_1^i to each vertex of Y_1^i, for all $2 \leq i \leq n$ so that we obtain the multi digraph D with imbalances of vertices as under.

For $1 \leq i \leq m$, for all $x_i^1 \in X_i^1$

$$b_{x_i^1} = t|Y_i^1| - 0 = tf_i = p_i,$$

for $2 \leq i \leq n$, for all $x_1^i \in X_1^i$

$$b_{x_1^i} = t|Y_1^i| - 0 = tf_1 = p_1,$$

for $1 \leq i \leq m$, for all $y_i^1 \in Y_i^1$

$$b_{y_i^1} = 0 - t|X_i^1| = -tg_i = -q_i,$$

and for $2 \leq i \leq n$, for all $y_1^i \in Y_1^i$

$$b_{y_1^i} = 0 - t|X_1^i| = -tg_i = -q_i.$$

Therefore imbalance set of D is $P \cup Q$. □

5.3 Imbalances in oriented bipartite digraphs

Defintion 5.3.1. An oriented bipartite graph is the result of assigning a direction to each edge of a simple bipartite graph. Let $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$ be the parts of an oriented bipartite graph $D(U, V)$. For any vertex x in $D(U, V)$, let d_x^+ and d_x^- denote the outdegree and indegree of x. Define a_{u_i}(or simply a_i) $= d_{u_i}^+ - d_{u_i}^-$ and b_{v_j}(or simply b_j) $= d_{v_j}^+ - d_{v_j}^-$ respectively, as imbalances of the vertices u_i in U and v_j in V. The sequences $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ in non-decreasing order is a pair of imbalance sequences of $D(U, V)$.

In any oriented bipartite graph $D(U, V)$, we have one of the following possibilities between a vertex u in U and a vertex v in V. (i) An arc directed from u to v, denoted by $u(1-0)v$, or (ii) An arc directed from v to u, denoted by $u(0-1)v$, or (iii) There is no arc from u to v and there is no arc from v to u and this is denoted by $u(0-0)v$.

A tetra in an oriented bipartite graph is an induced sub-oriented graph with two vertices from each part. Define tetras of the form $u_1(1-0)v_1(1-0)u_2(1-0)v_2(1-0)u_1$ and $u_1(1-0)v_1(1-0)u_2(1-0)v_2(0-0)u_1$ to be of α-type, and all other tetras to be of β-type. An oriented bipartite graph is said to be of α-type or β-type according as all of its tetras are of α-type or β-type respectively.

Some results on oriented bipartite graphs can be found in [2,4]. The results of this section have appeared in Chishti and Samee [54]. The following observation is an immediate consequence of above definitions and facts.

Theorem 5.3.2. *Among all oriented bipartite graphs with given imbalance sequence, those with the fewest arcs are of β-type.*

A transmitter is a vertex with indegree zero. In a β-type oriented bipartite graph with imbalance sequences $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$, either the vertex with imbalance a_p, or the vertex with imbalance b_q, or both may act as transmitters.

The next result provides a useful recursive test whether the given sequences are the imbalance sequences of an oriented bipartite graph.

Theorem 5.3.2. *Let $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ be the sequences of integers in non-decreasing order with $a_p > 0$, $a_p \leq q$ and $b_q \leq p$. Let A' be obtained from A by deleting one entry a_p, and B' be obtained from B by increasing a_p smallest entries of B by 1 each. Then A and B are the imbalance sequences of some oriented bipartite graph if and only if A' and B' are the imbalance sequences.*

Proof. Let A' and B' be the imbalance sequences of some oriented bipartite graph D' with parts U' and V'. Then an oriented bipartite graph D with imbalance sequences A and B can be obtained by adding a transmitter u_p in U' such that $u_p(1-0)v_i$ for those vertices v_i in V' whose imbalances are increased by 1 in going from A and B to A' and B'.

Conversely, suppose A and B be the imbalance sequences of an oriented bipartite graph D with parts U and V. Without loss of generality, we chose D to be of β-type. Then there is a vertex u_p in U with imbalance a_p(or a vertex v_q in V with imbalance b_q, or both u_p and v_q) which is a transmitter. Let the vertex u_p in U with imbalance a_p be a transmitter. Clearly, $d^+_{u_p} \geq 0$ and $d^-_{u_p} = 0$ so that $a_p = d^+_{u_p} - d^-_{u_p} \geq 0$. Also, $d^+_{v_q} \leq p$ and $d^-_{v_q} \geq 0$ so that $b_q = d^+_{v_q} - d^-_{v_q} \leq p$.

Let V_1 be the set of a_p vertices of smallest imbalances in V, and let $W = V - V_1$. Construct D such that $u_p(1-0)v_i$ for all $v_i \in V_i$. Clearly, $D - \{u_p\}$ realizes A' and B'. □

Theorem 5.3.2 provides an algorithm for determining whether the two sequences of integers in non-decreasing order are the imbalance sequences, and for constructing a corresponding oriented bipartite graph. Suppose $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ are imbalance sequences of an oriented bipartite graph with parts $U = \{u_1, u_2, \cdots, u_p\}$ and $V = \{v_1, v_2, \cdots, v_q\}$, where $a_p > 0$, $a_p \leq q$ and $b_q \leq p$. Deleting a_p, and increasing a_p smallest entries of B by 1 each to form $B' = [b'_1, b'_2, \cdots, b'_q]$. Then arcs are defined by $u_p(1-0)v_j$ for which $b'_{v_j} = b_{v_j} + 1$. Now, if the condition $a_p > 0$ does not hold, then we delete b_q(obviously $b_q > 0$), and increase b_q smallest entries of A by 1 each to form $A' = [a'_1, a'_2, \cdots, a'_p]$. In this case, arcs are defined by $v_q(1-0)u_i$ for which $a'_{u_i} = a_{u_i} + 1$. If this method is applied successively, then (i) it tests whether A and B are the imbalance sequences and, if A and B are the imbalance sequences (ii) an oriented bipartite graph $D(U, V)$ with imbalance sequences A and B is constructed.

Example 5.3.3. We illustrate this reduction and the resulting construction as follows, beginning with sequences A_1 and B_1

$A_1 = [-3, 1, 2, 2]$ $\quad B_1 = [-3, -1, 0, 1, 1]$

$A_2 = [-3, 1, 2]$ $\quad B_2 = [-2, 0, 0, 1, 1]$ $\quad u_4(1-0)v_1,\ u_4(1-0)v_2$

$A_3 = [-3, 1]$ $\quad B_3 = [-1, 1, 0, 1, 1]$ $\quad u_3(1-0)v_1,\ u_3(1-0)v_2$

or $A_3 = [-3, 1]$ $B_3 = [-1, 0, 1, 1, 1]$
$A_4 = [-3]$ $B_4 = [0, 0, 1, 1, 1]$ $u_2(1-0)v_1$
$A_5 = [-2]$ $B_5 = [0, 0, 1, 1]$ $v_5(1-0)u_1$
$A_6 = [-1]$ $B_6 = [0, 0, 1]$ $v_4(1-0)u_1$
$A_7 = [0]$ $B_7 = [0, 0]$ $v_2(1-0)u_1$

Obviously, an oriented bipartite graph D with parts $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4, v_5\}$ in which $u_4(1-0)v_1,\ u_4(1-0)v_2,\ u_3(1-0)v_1,\ u_3(1-0)v_2,\ u_2(1-0)v_1,\ v_5(1-0)u_1,\ v_4(1-0)u_1,\ v_2(1-0)u_1$ are arcs has imbalance sequences $[-3, 1, 2, 2]$ and $[-3, -1, 0, 1, 1]$.

The following result is a combinatorial criterion for determining whether the sequences are realizable as imbalances.

Theorem 5.3.4. *Two sequences $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ of integers in non-decreasing order are the imbalance sequences of some oriented bipartite graph if and only if*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \geq 2kl - kq - lp, \tag{5.2}$$

for $1 \leq k \leq p$, $1 \leq l \leq q$ with equality when $k = p$ and $l = q$.

Proof. The necessity follows from the fact that an oriented sub-bipartite graph induced by k vertices from the first part and l vertices from the second part has a sum of imbalances $2kl - kq - lp$.

For sufficiency, assume that $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ are the sequences of integers in non-decreasing order satisfying conditions (5.2) but are not the imbalance sequences of any oriented bipartite graph. Let these sequences be chosen in such a way that p and q are the smallest possible and a_1 is the least with that choice of p and q. We consider the following two cases.

Case(i). Suppose equality in (5.2) holds for some $k \leq p$ and $l < q$, so that

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j = 2kl - kq - lp.$$

By the minimality of p and q, $A = [a_1, a_2, \cdots, a_p]$ and $B = [b_1, b_2, \cdots, b_q]$ are the imbalance sequences of some oriented bipartite graph $D_1(U_1, V_1)$. Let $A_2 = [a_{k+1}, a_{k+2}, \cdots, a_p]$ and $B_2 = [b_{l+1}, b_{l+2}, \cdots, b_q]$.

Now,

$$\begin{aligned}
\sum_{i=1}^{f} a_{k+i} + \sum_{j=1}^{g} b_{l+j} &= \sum_{i=1}^{k+f} a_i + \sum_{j=1}^{l+g} b_j - (\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j) \\
&\geq 2(k+f)(l+g) - (k+f)q - (l+g)p - 2kl + kq + lp \\
&= 2kl + 2kg + 2fl + 2fg - kq - fq - lp - gp - 2kl + kq + lp \\
&= 2fg - fq - gp + 2kg + 2fl \\
&\geq 2fg - fq - gp,
\end{aligned}$$

for $1 \leq f \leq p-k$ and $1 \leq g \leq q-l$, with equality when $f = p-k$ and $g = q-l$. So, by the minimality for p and q, the sequences A_2 and B_2 form the imbalance sequences of some oriented bipartite graph $D_2(U_2, V_2)$. Now construct a new oriented bipartite graph $D(U, V)$ as follows.

Let $U = U_1 \cup U_2$, $V = V_1 \cup V_2$ with $U_1 \cap U_2 = \phi$, $V_1 \cap V_2 = \phi$ and the arc set containing those arcs which are between U_1 and V_1 and between U_2 and V_2. Then we obtain an oriented bipartite graph $D(U, V)$ with the imbalance sequences A and B, which is a contradiction.

Case (ii). Suppose that the strict inequality holds in (5.2) for some $k \neq p$ and $l \neq q$. Let $A_1 = [a_1 - 1, a_2, \cdots, a_{p-1}, a_p]$ and $B_1 = [b_1, b_2, \cdots, b_q]$, so that A_1 and B_1 satisfy the conditions (1). Thus by the minimality of a_1, the sequences A_1 and B_1 are the imbalances sequences of some oriented bipartite graph $D_1(U_1, V_1)$. Let $a_{u_1} = a_1 - 1$ and $a_{u_p} = a_p + 1$. Since $a_{u_p} > a_{u_1} + 1$, therefore there exists a vertex $v_1 \in V_1$ such that $u_p(0-0)v_1(1-0)u_1$, or $u_p(1-0)v_1(0-0)u_1$, or $u_p(1-0)v_1(1-0)u_1$, or $u_p(0-0)v_1(0-0)u_1$, in $D_1(U_1, V_1)$ and if these are changed to $u_p(0-1)v_1(0-0)u_1$, or $u_p(0-0)v_1(0-1)u_1$, or $u_p(0-0)v_1(0-0)u_1$, or $u_p(0-1)v_1(0-1)u_1$ respectively, the result is an oriented bipartite graph with imbalances sequences A and B, which is a contradiction. This proves the result. □

Definition 5.3.5. The set of distinct imbalances of the vertices in an oriented bipartite graph is called its imbalance set.

Finally, we give the existence of an oriented bipartite graph with a given imbalance set.

Theorem 5.3.6. *Let* $A = [a_1, a_2, \cdots, a_n]$ *and* $B = [-b_1, -b_2, \cdots, -b_n]$, *where* $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ *are positive integers with* $a_1 < a_2 < \cdots < a_n$ *and* $b_1 < b_2 < \cdots < b_n$. *Then there exists an oriented bipartite graph with imbalance set* $A \cup B$.

Proof. Construct an oriented bipartite graph $D(U, V)$ as follows. Let $U = U_1 \cup U_2 \cup \cdots \cup U_n$, $V = V_1 \cup V_2 \cup \cdots \cup V_n$ with $U_i \cap U_j = \phi(i \neq j)$, $V_i \cap V_j = \phi(i \neq j)$, $|U_i| = b_i$ for all i, $1 \leq i \leq n$ and $|V_j| = a_j$ for all j, $1 \leq j \leq n$. Let there be an arc from every vertex of U_i to each vertex of V_i for all i, $1 \leq i \leq n$, so that we obtain the oriented bipartite graph $D(U, V)$ with the imbalances of vertices as follows. For $1 \leq i, j \leq n$, $a_{u_i} = |V_i| - 0 = a_i$, for all $u_i \in U_i$ and $b_{v_j} = 0 - |U_j| = -b_j$, for all $v_i \in V_i$. Therefore the imbalance set of $D(U, V)$ is $A \cup B$.

Obviously the oriented bipartite graph constructed above is not connected. In order to see the existence of oriented bipartite graph, whose underlying graph is connected, we proceed as under.

Taking $U_i = \{u_1, u_2, \cdots, u_{b_i}\}$ and $V_j = \{v_1, v_2, \cdots, v_{a_j}\}$, and let there be an arc from each vertex of U_i to every vertex of V_j except the arcs between u_{b_i} and v_{a_j}, that is $u_{b_i}(0-0)v_{a_j}$, $1 \leq i \leq n$ and $1 \leq j \leq n$. We take $u_{b_1}(0-0)v_{a_2}$, $u_{b_2}(0-0)v_{a_3}$, and so on $u_{b_{(n-1)}}(0-0)v_{a_n}$, $u_{b_n}(0-0)v_{a_1}$. The underlying graph of this oriented bipartite graph is connected. □

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