Symmetric Duality in Mathematical Programming

Dissertation submitted

In partial fulfillment for the award of degree of

Master of Philosophy

In

Statistics

By

Gulzar Ahmad Shalbaf

Under the Supervision of
Professor Aquil Ahmed
Professor and Head

Post Graduate Department of Statistics
Faculty of Physical and Material Sciences
University of Kashmir
Srinagar-190006, J&K, India
NAAC Accredited Grade "A"
2013

DEPARTMENT OF STATISTICS
UNIVERSITY OF KASHMIR
(NAAC Accredited Grade "A")
SRINAGAR-190006-J&K

Dr. Aquil Ahmed (Ph.D.Roorkee)
Professor & Head

Date:

Certificate

This is to certify that the scholar **Mr. Gulzar Ahmad Shalbaf,** has carried out the present dissertation entitled **"Symmetric Duality in Mathematical Programming"** under my supervision and the work is suitable for submission for the award of degree of Master of Philosophy in Statistics. It is further certified that the work has not been submitted in part or full for the award of M.Phil or any other degree.

Professor Aquil Ahmed
Supervisor

Tel. (O):+91194-2420078 Ext-2227, 2316 Mobile: +919419009425, Tele fax: +91194-2421357,
E- mail: aqlstat@yahoo.co.in, & aquilstat@kashmiruniversity.ac.inWeb: Kashmir university.net

…Dedicated

to my

Parents…

Acknowledgement

All praise be to Allah (SWT), our Lord, Creator, Cherisher and Sustainer of this universe and salutations upon His blessed and final Messenger, our beloved Prophet Muhammad (PBUH).

At the completion of this academic venture, it gives me immense pleasure to express my heartfelt gratitude to all those who have helped and encouraged me all the way throughout the course of this study.

Words fail to express the sense of gratitude that I have towards my supervisor Professor Aquil Ahmed, Head Department of Statistics. His charismatic persona and dynamism are inspirational. I have known him as a dedicated professor and a brilliant teacher who takes keen interest in the well-being of his scholars and students. It is only due to his constant supervision, timely advice and continuous vigil over the progress of the work that the present study could be smoothly conducted and completed. I would always love to approach him with problems and enjoy the healthy criticism and warm attitude. His enthusiasm for learning and academics has always egged me on. I would like to take this opportunity to thank him from the core of my heart for his enormous contribution and effort.

I extend my gratefulness to the authorities of University of Kashmir for their cooperation and keen interest in the development of my research. Heartfelt thanks to my teachers, Dr. Anwar Hasan, Dr. M.A.K. Beig, Dr. Tariq Rashid Jan and Dr. Sheikh Parvaiz for their support during this academic exercise.

I cannot forget to express my words of thankfulness to employees of Statistics Department for their cooperation during this study.

I am highly thankful to the employees of our library for their help during the completion of this study.

It deserves a special mention to convey my heartfelt thanks to my friends and all my batch mates for their continuous encouragement and moral support.

No words can describe the unconditional and unending support and cooperation that I received from my parents and my whole family. Their contribution has been invaluable in the completion of this work. I would like to thank them from the core of my heart.

Gulzar Ahmad Shalbaf

Contents

Preface

Symmetric duality in mathematical programming is defined as a mathematical programming problem in which the dual of the dual is the original problem. In this dissertation emphasis has given on the formulation and conceptualization of the concepts of symmetric duality. An attempt has been made to investigate the properties and relations for first and second order symmetric duality problems for differentiable mathematical programs.

This dissertation is divided into four chapters.

CHAPTER- 1 includes the genesis of the problem, pre-requisites and definitions, which are used in the subsequent chapters. Linear programming, basic assumptions of linear programming, the Simplex method, fundamental properties of the solutions, non-linear programming, classification of non-linear programming, constrained algorithms, separable programming, quadratic programming, fractional programming, optimality conditions for non-Linear programming, necessary optimality conditions, Fritz-John necessary conditions and multi-objective optimization are discussed in brief.

CHAPTER-2 consists of three sections. The first section deals with the duality in linear programming. In the subsection properties of duality and the various useful aspects of duality are presented. In the second Section, duality in non-linear programming has been studies and in the last Section duality in multi objective mathematical programming is investigated. The last Section is further subdivided into six subsections namely describing non-linear multi-objective programming, Kuhn-Tucker type necessary conditions for efficiency, Fritz- John type necessary conditions for efficient, Kuhn-Tucker type sufficient conditions for properly efficient solution, Fritz-John type sufficient conditions for efficiency. An example for the illustration purpose is also presented.

CHAPTER-3 deals with the symmetric duality in mathematical programming, symmetric duality in differentiable programming, Symmetric duality in non-differentiable programming, Symmetric duality in multi-objective programming and Mond-Weir type second order multi-objective Symmetric duality are also discussed.

CHAPTER-4 consists of two sections, the first section is developed to the second order symmetric duality in mathematical programming. The second section deals with the formulation of the problems and weak duality, strong duality, converse duality and self duality relations are discussed.

Chapter 1

Introduction

1.1 General Introduction

The subject of mathematical Programming has its roots in the study of linear inequalities which paved the way for further work on the problem while the applied side of the subject started with the application of Mathematical programming to solve the problems in Economics. The subject really took off in 1947 when G.B. Dantzig invented the Simplex method for solving the linear programming problems that arose in U.S. Air Force planning problems. Mathematical programming is one of the best developed and most used branches of mathematical science which has got applications in almost every field of real life problems. Mathematical programming plays very important role in solving our real life problems. Mathematical programming consists of an objective function and a set of constraints and we try to optimize our objective function subject to the associated set of constraints. If a single objective is to be achieved while satisfying the set of constraints, we say that it is a problem of Single Objective mathematical programming problem otherwise a Multi-objective Mathematical programming problem. This is an indispensable tool of decision making for everyone whether an administrator, a Planner, an Educationist, Manager, Scientist, Health expert or even a common man. The concept of optimization is now well rooted as a principle of underlying the analysis of various complex decisions or allocation problems. It offers a certain degree of philosophical elegance that is hard to dispute and is often gives indispensable degree of operational simplicity. Using this optimization philosophy one approaches a complex decision problems involving the selection of values for a number of inter related variables, by focusing attention on a single objective or multiple objective, designed to quantify performance and measure the quality of decision. This one objective or several objectives is maximized (or minimized) depending on the formulation subject to certain constraint that may limit the selection of decisions variable values. If a single aspect of a problem can be isolated and characterized by an objective, be it profit or loss in business setting, speed or distance in a physical problem, expected return in environment or social welfare in the context of a government and planning, the optimization provides a suitable framework for analysis.

A general structure of the mathematical programming problem is as under:

Optimize $f(x)$

Subject to

$$g_i(x) \geq 0 \qquad (i = 1, 2, \ldots, m),$$

$$h_j(x) = 0 \qquad (j = 1, 2, \ldots, k),$$

$$x \in X$$

where

$x = (x_1, x_2, \ldots, x_n)^T$ is the vector of unknown decision variables and $f(x)$, g_i $(i = 1, 2, \ldots, m)$, h_j $(j = 1, 2, \ldots, k)$ are the real valued functions of n real variables $(x_1, x_2, \ldots, x_n)$ and $X \subseteq R^n$. In this formulation, the function $f(x)$ is called the objective function, the constraints $g_i(x) \geq 0$, $i = 1, 2, \ldots, m$ are referred to as an inequality constraints, the constraints $h_j(x) = 0$, $j = 1, 2, \ldots, k$ are the equality constraints. The inclusion $x \in X$ is known as abstract constraints.

Mathematical programming is concerned with the determination of a maximum or minimum of a function of several variables, which are required to satisfy a number of constraints. Problem of this kind arise in quite diverse field including engineering, management science, economic planning, technological design etc.

If all the functions in the mathematical programming problem are linear then it is called linear programming problem. If the objective function and at least one of the constraint or both are nonlinear functions in the mathematical programming problems, then the problem is termed as nonlinear programming problem.

1.2 Prerequisites and definitions

Solution

An n-tuple $(x_1,x_2,\ldots ,x_n)$ of real numbers which satisfies the constraints of a general LPP is called a solution to the General LPP.

Basic Solution

Given a system of m simultaneous linear equations in n unknowns $(m<n)$

$$Ax = b,\ x^T \epsilon\ R^n$$

Where A is an $m \times n$ matrix of rank m. Let B be any $m \times m$ sub matrix formed by m linearly independent columns of A. Then a solution obtained by setting $(n\text{-}m)$ variables not associated with the columns of B, equal to zero, and solving the resulting system, is called a basic solution to the given system of equations. The m variables, which may be all different from zero, are called basic variables. The $m \times m$ non-singular sub-matrix B is called a basis matrix with the columns of B as basis vectors. If B is a basis sub matrix chosen, then the basic solution to the system is

$$x_B = B^{-1}\, b.$$

Basic Feasible Solution

A basic feasible solution is a basic solution which satisfies the non-negativity restrictions, that is, all basic variables are non negative. Basic feasible solutions are of two types:

(a) Non-degenerate BFS

A non-degenerate basic feasible solution is a BFS which has exactly m positive x_i $(i=1,2,\ldots, m)$. In other words, all m basic variables are positive and the remaining n variables will be all zero.

(b) Degenerate BFS

A basic feasible solution is said to be degenerate, if one or more basic variables are zero valued.

Slack Variable

The non-negative variable which is added to the left hand side of the constraint to convert it into equation is known as slack variable.

Surplus Variable

The positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

Optimum Solution

Any feasible solution which optimizes (minimizes or maximizes) the objective function of a General LPP is called an optimum solution to the General LPP.

Alternative Optima

When the objective function is in parallel to a binding constraint (*i*.e. a constraint that is satisfied as an equation by the optimal solution), the objective function will assume the same optimal value at more than one solution point and hence called alternative optima.

Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints, meaning that the solution space is unbounded in at least one direction. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

Infeasible Solution

If the constraints are not satisfied simultaneously, the model has no feasible solution. If this situation happens we use artificial variables.

Convex Set, convex combination and affine combination

A subset $S \subset R^n$, is said to be convex, if for any two points x_1 and x_2 in S the line segment joining the points x_1 and x_2 is also contained in S.

In other words, a subset S $\subset R^n$ is convex, if and only if $x_1, x_2 \epsilon S \Rightarrow \lambda x_1 + (l-\lambda) x_2 \epsilon S; 0 \leq \lambda \leq 1.$

Weighted averages of the form $\lambda x_1 + (l-\lambda) x_2$, where $\lambda \epsilon [0, 1]$, are referred to as convex combinations of x_1 and x_2.inductivity, Weighted averages of the form $\sum_{j=1}^{k} \lambda_j x_j$, where $\sum_{j=1}^{k} \lambda_j = 1, \lambda_j \geq 0, j = 1,\ldots, k$, are also called convex combinations of $x_1, x_2, \ldots, x_k$. In this definition, if the non-negativity conditions on the multipliers λ_j is dropped, $j = 1,\ldots,k$, then the combination is known as affine combination.

Extreme point

An extreme point (vertex) of a convex set which does not lie on any segment joining two other points of the set. Thus, a point x of a convex set S is an extreme point of the set, if there does not exist any pair of points $x_1, x_2 \epsilon S$, such that

$$x = \lambda x_1 + (l-\lambda) x_2, \qquad 0 < \lambda < 1$$

Convex Function

A function $f(x)$ defined on a set $S \subset E_n$ is said to be convex at a point $x_0 \in S$ if

$$x_1 \in S,\ 0 \leq \lambda \leq 1 \text{ and}$$

$$\lambda x_0 + (1-\lambda) x_1 \in S \Rightarrow f(\lambda x_0 + (1-\lambda) x_1) \leq \lambda f(x_0) + (1-\lambda) f(x_1).$$

The function $f(x)$ is said to be convex on S if it is convex at every point of S. This does not assume S to be a convex set. However, if S is a convex, the $f(x)$ is convex on S if

$$x_1, x_2 \in S,\ 0 \leq \lambda \leq 1 \Rightarrow f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

Strictly Convex Function

A function $f(x)$is said to be strictly convex, defined on a set $S \subset E_n$ at $x_0 \in S$ if

$$x_1 \in S,\ 0 < \lambda < 1,\ x_1 \neq x_0 \text{ and}$$

$$\lambda x_0 + (1-\lambda) x_1 \in S \Rightarrow f(\lambda x_0 + (1-\lambda) x_1) < \lambda f(x_0) + (1-\lambda) f(x_1).$$

Concave Function

A function $f(x)$ defined on a set $S \subset E_n$ is said to be concave at a point $x_0 \in S$ if $x_1 \in S$, $0 \leq \lambda \leq 1$ and $\lambda x_0 + (1-\lambda) x_1 \in S$

$$\Rightarrow f(\lambda x_0 + (1-\lambda) x_1) \geq \lambda f(x_0) + (1-\lambda) f(x_1).$$

Strictly Concave Function

A function $f(x)$ defined on set $S \subset E_n$ is said to be strictly concave at point $x_0 \in S$ if

$$x_1 \in S,\ 0 < \lambda < 1,\ x_1 \neq x_0 \text{ and}$$

$$\lambda x_0 + (1-\lambda) x_1 \in S \Rightarrow f(\lambda x_0 + (1-\lambda) x_1) > \lambda f(x_0) + (1-\lambda) f(x_1).$$

Quasi-convex Function

A function $f(x)$ is said to be quasi-convex on a convex sets $S \subset E_n$ if for each $x_1, x_2 \in S$ such that $f(x_2) > f(x_1)$,the function $f(x)$ assumes a value no larger than $f(x_2)$ on each point in the intersection of closed line segment $[x_1, x_2 \text{ and } S]$

$$x_1, x_2 \in S, f(x_2) \geq f(x_1), 0 < \lambda < 1 \text{ and}$$

$$\lambda x_1 + (1-\lambda) x_2 \in S \Rightarrow f(\lambda x_1 + (1-\lambda) x_2) \leq max\{f(x_1), f(x_2)\}.$$

Quasi-concave Function

A function $f(x)$ is said to be quasi-concave defined on a convex subset $S \subset E_n$ if and only if

$$x_1, x_2 \in S, 0 < \lambda < 1 \text{ and}$$

$$f(x_2) \geq f(x_1) \Rightarrow f(\lambda x_1 + (1-\lambda) x_2) \geq min\{f(x_1), f(x_2)\} \geq f(x_1).$$

Strictly Quasi-convex Function

Let $f: S \rightarrow E_1$, where S is a nonempty convex set in E_n. The function f is said to be strictly quasi-convex if, for each $x_1, x_2 \in S$ with $f(x_1) \neq f(x_2)$ we have

$$f(\lambda x_1 + (1-\lambda) x_2) < max\{f(x_1), f(x_2)\} \text{ for each } \lambda \in (0,1).$$

The function f is strictly quasi-concave if $(-f)$ is strictly quasi-convex.

Strongly Quasi-convex Function

Let S be a nonempty convex set in E_n and let $f: S \rightarrow E_1$. The function f is said to be strongly quasi-convex if for each x_1, $x_2 \epsilon S$, with $x_1 \neq x_2$, we have

$$f\{\lambda x_1 + (1-\lambda) x_2\} < \text{maximum } \{f(x_1), f(x_2)\}, \quad \text{for each } \lambda \epsilon (0, 1).$$

The function f is strongly quasi-concave if $(-f)$ is strongly quasi-convex.

We can summarize

(i) Every strictly convex function is strongly quasi-convex.
(ii) Every strongly quasi-convex function is strictly quasi-convex.
(iii) Every strongly quasi-convex function quasi-convex.

Differentiable convex functions

Let S be a nonempty set in E_n, and let f:$S\rightarrow E_1$, then f is said to be differentiable at $\bar{x} \epsilon\, int\, S$ if there exists a vector $\nabla f(\bar{x})$, called the gradient vector, and a function α: $E_n \rightarrow E_1$ such that

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})^t (x-\bar{x}) + ||\, x-\bar{x}\, ||\, \alpha(\bar{x}\, ; x-\bar{x}) \quad \text{for each } x \epsilon S$$

Where $\quad lim_{x\rightarrow\bar{x}}\, \alpha(\bar{x}\, ;\, x - \bar{x}) = 0$

$\nabla f(\bar{x})$ is the n-dimensional gradient vector of f at $\bar{x}$, whose n components are the partial derivatives of f with respect to $x_1, x_2, \ldots, x_n$ evaluated at $\bar{x}$.

$$\nabla f(\bar{x}) = \left(\frac{\partial f(\bar{x})}{\partial x_1} \ldots \frac{\partial f(\bar{x})}{\partial x_n}\right)^t = (f_1(\bar{x}) \ \ldots f_n(\bar{x}))^t$$

Note: the gradient represents the steepness and direction of that slope.

Twice Differentiable Convex Functions

Let S be a nonempty set in E_n and let $f: S \rightarrow E_1$. Then f is said to twice differentiable at $\bar{x} \epsilon$ *int.* S if there exists a vector $\nabla f(\bar{x})$, and an

$n \times n$ symmetric matrix $H(\bar{x})$, called the Hessian matrix, and a function α: $E_n \rightarrow E_1$ such that

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})^t (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^t H(\bar{x})(x - \bar{x})$$
$$+ ||x - \bar{x}||^2 \alpha(\bar{x}; x - \bar{x})$$

for each $x \in S$, Where $lim_{x \rightarrow \bar{x}} \alpha(\bar{x}\,;\, x - \bar{x}) = 0$. The function f is said to be twice differentiable at each open set $S' \subseteq S$ if it is twice differentiable at each point in S'.

For twice-differential functions the Hessian matrix $H(\bar{x})$ is comprised of second-order partial derivatives $\frac{\partial^2 f(\bar{x})}{\partial x_i\, \partial x_j} = f_{ij}(\bar{x})$ for

$i = 1,2,\ldots,n,$ and is given as follows:

$$H(\bar{x}) = \begin{bmatrix} \frac{\partial^2 f(\overline{x})}{\partial {x_1}^2} & \frac{\partial^2 f(\overline{x})}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f(\overline{x})}{\partial x_1 x_n} \\ \frac{\partial^2 f(\overline{x})}{\partial x_2 x_1} & \frac{\partial^2 f(\overline{x})}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f(\overline{x})}{\partial x_2 x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f(\overline{x})}{\partial x_n x_1} & \frac{\partial^2 f(\overline{x})}{\partial x_n x_2} & \cdots & \frac{\partial^2 f(\overline{x})}{\partial {x_n}^2} \end{bmatrix} = \begin{bmatrix} f_{11}(\bar{x}) & f_{12}(\bar{x}) & \cdots & f_{1n}(\bar{x}) \\ f_{21}(\bar{x}) & f_{22}(\bar{x}) & \cdots & f_{2n}(\bar{x}) \\ \vdots & \vdots & & \vdots \\ f_{n1}(\bar{x}) & f_{n2}(\bar{x}) & \cdots & f_{nn}(\bar{x}) \end{bmatrix}$$

Let f be a twice differentiable real valued function of x and y, where $x \epsilon R^n$. Then $\nabla_x f$ and $\nabla_y f$ denote gradient vectors with respect to x and y, respectively. $\nabla_{xx} f$ and $\nabla_{yy} f$ are, respectively, the $n{\times}n$ and $m{\times}m$ symmetric Hessian matrices. $(\partial/\partial y_i)\,(\nabla_{yy} f)$ is the $m{\times}m$ matrix obtained by differentiating, the elements of $\nabla_{yy} f$ with respect to y_i and $(\nabla_{xx}\, f\,(x,y)q)_y$ denotes the matrix whose $(i,j)^{th}$ element is $(\partial/\partial y_i)\,(\nabla_{xx}\, f\,(x,y)q)_j$, where $q \epsilon R^n$.

η_1-Bonvex Function

The second order convex function is known as bonvex function. A real twice differentiable function f defined on $X \times Y$, are open sets in R^n and R^m, respectively, is said to be η_1-Bonvex in the first variable at $u \epsilon X$, if there exists a function $\eta_1 : X \times X \rightarrow R^n$ such that for $v \epsilon Y$, $q \epsilon R^n$, $x \epsilon X$,

$$f(x, v) - f(u, v) \geq \eta_1^T(x, u)[\nabla_x f(u, v) + \nabla_{xx} f(u, v)\, q] - \frac{1}{2} q^T \nabla_{xx} f(u, v)\, q$$

and $f(x, y)$ is said to be η_2-Bonvex in the first variable at $v \epsilon Y$, if there exists a function $\eta_2 : y \times y \rightarrow R^m$ such that for $u \epsilon X$, $p \epsilon R^m$, $y \epsilon Y$,

$$f(u, y) - f(u, v) \geq \eta_2^T(y, v)[\nabla_y f(u, v) + \nabla_{yy} f(u, v)\, p] - \frac{1}{2} p^T \nabla_{yy} f(u, v) p.$$

η_1- Pseudobonvex Function

A real twice differentiable function f defined on $X \times Y$ is said to be η_1- Pseudobonvex in the first variable at $u \epsilon X$, if there exists a function $\eta_1 : X \times X \rightarrow R^n$ such that for $v \epsilon Y$, $q \epsilon R^n$, $x \epsilon X$,

$$\eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v) q] \geq 0 \Rightarrow f(x,u) - f(u,v) + \frac{1}{2} q^T \nabla_{xx} f(u,v)\, q \geq 0$$

and $f(x, y)$ is said to be η_2-Bonvex in the second variable at $v \epsilon Y$, if there exists a function $\eta_2 : y \times y \rightarrow R^m$ such that for $u \epsilon X$, $p \epsilon R^m$, $y \epsilon Y$,

$$\eta_2^T(y,v)[\nabla_y f(u,v) + \nabla_{yy} f(u, v)\, p] \geq 0$$

$$\Rightarrow f(u,y) - f(u,v) + \frac{1}{2} p^T \nabla_{yy} f(u,v) p \geq 0.$$

Convex cones

A nonempty set C in E_n is called a cone with vertex zero if $x \epsilon C$ implies that $\lambda x \epsilon C$ for all $\lambda \geq 0$. If, in addition, C is convex, then C is called a convex cone.

Polar cone

Let S be a nonempty set in E_n. Then the polar cone of S, denoted by S^*, is given by $\{ p : p^t x \leq 0$ for all $x \epsilon S\}$.

Efficient Solution

A point $x_0 \in X$ is said to be an efficient solution of the vector minimum problem (VP) if there exists no other feasible point $x \in X$, such that $f(x) \leq f(x_0)$.

Weak Efficient solution

A point $x_0 \in X$ is said to be weak efficient solution for VP, if there exists no other point $x \in X$ with $f(x) \leq f(x_0)$.

It readily follows that if $x_0 \in X$ is efficient, then it is also weak efficient.

Properly Efficient Solution

An efficient solution x_0 of the vector minimum problem (VP) is said to be properly efficient solution, if there exists a scalar M > 0 such that for each $i \in \{1,2, \dots ,p\}$ and $x \in X$ satisfying

$f^i (x) < f^i (x_0)$, we have

$$\frac{(f^i (x_0) - f^i (x))}{(f^j (x) - f^j (x_0))} \leq M$$

for some j such that $(f^j (x) > f^j (x_0)$.

Improperly Efficient Solution

An efficient solution $x_0 \in X$ is said to be improperly efficient if for each scalar $M > 0$ (no matter how large) there is a point $x \in X$ and an i such that

$f^i (x) < f^i (x_0)$ and

$$\frac{f^i (x_0) - f^i (x)}{f^j (x) - f^j (x_0)} > M$$

For some j such that $(f^j (x) > f^j (x_0)$.

Attainable solutions

When we solve a multi-objective optimization problem for the case of single objective ignoring the remaining objective, then the solution that we obtain is called an attainable solution. A problem with K objectives will have at the most K attainable solutions. We may say x is an attainable solution of multi-objective problem if

(i) x is a feasible solution.

(ii) $f^j(x)$ is minimum (optimum) for at least one $j = 1, 2,..., k$.

Ideal Solution

If all the objective functions of a multi-objective problem reach their minima (maxima) at a unique point of the feasible region, then this point is called the perfect or ideal optimal solution of the problem. In other words, a point x is said to be ideal solution of a multi-objective problem if

(i) x is feasible.

(ii) $f^j(x)$ is optimal for all $j, j = 1, 2,..., K$.

1.3 Linear programming

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equations and/or inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resources such as market demand, production process and equipment, storage capacity, raw material availability, etc. By linearity is meant a mathematical expression among the variables are linear e.*g*., the expression $a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n$ is linear.

It was in 1947 that George Dantzig and his associates found out a technique for solving military planning problems while they were working on a project for U.*S*. Air Force. This technique consisted of representing the various activities of an organization as a linear programming model and arriving at the optimal programme by minimizing a linear objective function. Afterwards, Dantzig suggested this approach for solving business and industrial problems. He also developed the most powerful mathematical tool known as "simplex method" to solve linear programming problems.

1.3.1 The Mathematical Description of Linear Programming

The following problem is known as the Linear Programming Problem or simply as the Linear Program:

Optimize $Z = f(x) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to

$$a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \ (\leq, =, \geq)\ b_i,\ i = 1, \ldots, m.$$

$$x_j \ (\geq, \text{unrestricted}, \leq),\ j = 1, \ldots, n.$$

x_j are the variables of the problem and are allowed to take on any set of real values that satisfy the constraints.

c_j, b_i and a_{ij} are parameters of the problem and (along with dimensions *n*, *m* and objective/row/variable types) provide the precise description of a particular instance of the LP model we wish to solve.

c_j are the profit coefficients.

b_i are the demand coefficients.

a_{ij} are the activity coefficients or coefficients of variation.

1.3.2 Basic assumptions of linear programming problem

The following four basic assumptions are necessary for all linear programming problems:

i) **Certainty:** In all LPP'S it is assumed that all the parameters; such as availability of resources by a unit decision variable must be known and fixed. In other words, this assumption means that all the coefficients in the objective function as well as in the constraints are completely known with certainty and do not change during the period of study.

ii) **Additivity**: The variables in the objective and each constraint contribute the sum of the contributions of each variable.

iii) **Divisibility:** The variables can take on continuous values subject to the constraints.

iv) **Proportionality**: This requires the contribution of each decision variable in both the objective function and the constraints to be directly proportional to the value of the variable.

1.3.3 The Simplex Method

Linear programming problems involving two decision variables can easily be solved by graphical method. The method also provides an insight into the concepts of Simplex Method- a powerful technique to solve the linear programming problems involving three or more decision variables.

The Simplex Method, also called the 'Simplex Algorithm' is an iterative procedure for solving a linear programming problem in a finite number of steps. The method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another in such manner that the value of the objective function at the succeeding vertex is less (or more, as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution. If at any stage, the procedure leads us to a vertex which has an edge leading to infinity and if the objective function value can be further improved by moving along that edge, the simplex method tells us that there is an unbounded solution.

Any linear programming problem (LPP) given in the standard form can be solved by using the well known Simplex Procedure Dantzig,G.*B.* [19]. If the problem is not in the standard form and lacks full basis, then the problem can be solved by using Artificial Variable Technique which utilizes Big M and Two Phase Methods. The LPP which are large scale problems can be solved by using the Revised Simplex Method (Dantzig, Orden, Wolfe [19] and Dantzig [14]). The revised simplex method uses the same basic principles or technique of improving the given basic feasible solution as used in regular simplex method except the difference that the whole tableau is not calculated. The relevant information it needs to move from one basic feasible solution to another is directly generated from the original equations. The revised simplex method considerably reduces the computational work involved in the large LPP.

1.3.4 Fundamental properties of the solutions

The fundamental properties of the solutions to the L.P.P. are as under:

i) Reduction of a Feasible Solution to a Basic Feasible Solution. If an L.P.P. has a feasible solution, then it also has a basic feasible solution.

Note: There exists only finite number of basic feasible solutions to an L.P.P.

ii) Extreme Point Correspondence. A basic feasible solution to an L.P.P. must correspond to an extreme point of the set of all feasible solutions and conversely.

Note:

(a) The number of extreme points of convex set of feasible solutions is finite. It follows from the preceding property that there is only one extreme point for a given basic feasible solution and conversely. So there is one-to-one correspondence between the extreme points and the basic feasible solutions. But basic feasible solutions are finite in number, therefore, the extreme points of the convex set of feasible solutions are also finite in number.

(b) An extreme point can have at most m positive x_j's where m is the number of constraints.

iii) Fundamental Property of Linear Programming. If the feasible region of an L.P.P. is a convex polyhedron, then there exists an optimal solution to the L.P.P. and at least one basic feasible solution must be optimal.

Note: If the optimal value of Z is attained at more than one extreme points of S, then every convex combination of such extreme points also provides an optimal solution to the L.P.P.is either unique or infinite in number.

iv) Replacement of a Basis Vector. Let an L.P.P. have a basic feasible solution. If we drop of the basis vectors and introduce a non-basis vector in the basis set, then the new solution obtained is also a basic feasible solution.

v) Improved Basic Feasible Solution. Let X_B be a basic feasible solution to the L.P.P.

Maximum $Z = CX$

subject to

$$AX = b, X \geq 0.$$

Let $\hat{X}_B$ be another basic feasible solution obtained by admitting a non–basis column vector a_j in the basis, for which the net evaluation $z_j - c_j$ is negative. Then $\hat{X}_B$ is an improved basis feasible solution to the problem, that is

$$\hat{c}_B \hat{X}_B > c_B X_B.$$

Note: If $z_j - c_j = 0$ for at least one j for which $y_{ij} > 0, i = 1, 2, \ldots, m;$ then another basic feasible solution is obtained which gives an unchanged value of the objective function.

vi) Unbounded Solution. Let there exists a basic feasible solution to a given L.P.P. if for at least one j, negative, then there does not exit any optimum solution to this L.P.P.

vii) **Conditions of optimality**. A sufficient condition for a basic feasible solution to an L.P.P. to an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which the column vector $a_j \in A$ is not in the basis B.

Note: A necessary and sufficient condition for a basic feasible solution to an L.P.P. to be an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which a_j does not belong B.

1.4 Nonlinear programming

If the objective function and at least one of the constraint or both are nonlinear functions in the mathematical programming problems, then the problem is termed as nonlinear programming problem, which was first introduced by *R*. Courant in 1943. It is the most general programming problem and other problems can be treated as special cases of the nonlinear programming problem. Some methods for solving nonlinear programming problem were discussed by Avriel [2] and Zangwill [57].

The pioneer work by Kuhn Tucker in 1951 on necessary and sufficient conditions for the optimal solution laid the foundation for the researchers to work on the nonlinear system. In 1957, the emergence of dynamic programming by Bellman brought a revolution in the subject and consequently, linear and non-linear systems have been studied simultaneously. It is disappointing to note that possibly no universal technique has been established for nonlinear system as yet.

Optimality conditions and duality have played a vital role in the progress of mathematical programming. Fritz John [25] was the first to derive necessary optimality conditions for constrained optimization problem using a Lagrange multiplier rule. Later, Kuhn and Tucker established necessary optimality conditions for the existence of an optimal solution under certain constraint qualification in 195I. It was revealed afterwards that Karush. W [35] had presented way back in 1939 without imposing any constraint qualification; thus the Kuhn-Tucker conditions are now known as Karush-Kuhn-Tucker optimality conditions. Abadie [1] established a regularity condition that enabled him to derive Karush-Kuhn-Tucker conditions and Fritz John optimality conditions. Subsequently, Mangasarian and Formovitz. [40] generalized Fritz John optimality conditions which have not only laid down the foundation for many computational techniques in mathematical programming, but also are responsible for development of duality theory to a great deal. The inception of the duality theory in linear programming may be traced to the classical minimax theorem of Neumann [48] and was explicitly incorporated by Gale, Kuhn and Tucker [26]. Since then, it has become one of the most widely used and investigated area of mathematical programming. An extensive use of duality in mathematical programming has been made for many theoretical and computational developments in mathematical programming itself and in other fields which include engineering, operations research, economics and mathematical science.

A linear programming problem is expressed as

Maximize or minimize $Z = f(x_1, x_2, \ldots, x_n)$,

Subject to

$$g^1\ (x_1, x_2, \ldots, x_n) \leq, =, \geq b_1,$$

$$g^2\ (x_1, x_2, \ldots, x_n) \leq, =, \geq b_2,$$

$$. \qquad\qquad\qquad .$$

$$. \qquad\qquad\qquad .$$

$$g^m\ (x_1, x_2, \ldots, x_n) \leq, =, \geq b_m,$$

$$x_j \geq 0, j = 1, 2, \ldots, n.$$

If either the objective function and/ or one or more of the constraints are nonlinear in $x(x_1, x_2, \ldots, x_n)$, the problem is called a nonlinear programming problem.

In other words, the general nonlinear programming problem (NLPP) is to determine the n–tuple $x = (x_1, x_2, \ldots, x_n)$, so as to

Maximize or minimize $Z = f(x)$,

Subject to

$$g^i\ (x) \leq, =, \geq b_i,\ i = 1, 2, \ldots, m,$$

$$x \geq 0,$$

Where $f(x)$ or some $g^i(x)$ or both are nonlinear.

The method of solving an L.P.P.is based on the property that the optimal solution lies at one or more extreme points of the feasible region. This limits our search to corner points only and the optimal solution is obtained after a finite number of iterations as in simplex method. Unfortunately, the same is not true for nonlinear programming problems. In such problems the optimal solution can be located at any point along the boundaries of the feasible region or even within the region.

Secondly, due to nonlinearity of the objective function and constraints, it becomes difficult to distinguish between the local and global solution.

Thirdly, it is sometimes difficult to test the optimality of the nonlinear programming problems, especially when the feasible region is not convex.

Therefore, the nonlinearity of the functions makes the solution of the problem much more involved as compared to linear programming problems and there is no single algorithm like the simplex method, which can be employed to solve efficiently all nonlinear programming problems.

1.4.1 Classification of Nonlinear Programming problems

Nonlinear Programming problems can be classified as

- **i) Convex Programming Problem.**
- **ii) Non Convex Programming Problem**

i) Convex Programming Problem

Convex programming is an important and richly studied subclass of nonlinear programming. The problem of minimizing a convex function or maximizing a concave function over a feasible convex set is known as convex programming problem or convex program.

A convex programming problem has following two forms:

Minimize $f(x)$ (convex)	Maximize $f(x)$ (concave)
Subject to $x \in C$ (convex)	subject to $x \in C$ (convex)

The general convex program is of the type.

Optimize $f(x)$

Subject to

$$g_i(x) \geq b_i \text{ , } i \epsilon P$$

$$g_i(x) = b_i \text{ , } i \epsilon Q$$

$$g_i(x) \leq b_i \text{ , } i \epsilon R$$

is a convex program if,

i) f is concave for maximization, and convex for minimization.

ii) For every $i \in R$, the constraint g_i is convex and for each $i \in P$, the constraint g_i is concave.

iii) For every $i \in$ Q, the constraint g_i is linear.

ii) Non Convex Programming Problem

The mathematical programming which is not convex is called a Non-convex programming problem. A Non-convex Program encompasses all nonlinear programming problems that do not satisfy the convexity assumptions. However, even if we are successful at finding a local minimum, there is no assurance that it will also be a global minimum. Therefore, there is no algorithm that will guarantee finding an optimal solution for all such problems.

1.4.2 Constrained Algorithms

The general constrained nonlinear programming problem is defined as

Maximize (or minimize) $z = f(x)$

Subject to

$$g(x) \leq 0$$

$$x \geq 0$$

The non-negatively conditions $x \geq 0$ are part of the given constraints. Also, at least one of the functions *f(x)* and *g(x)* is nonlinear and all the functions are continuously differentiable.

No general algorithm exists for handling nonlinear models, because of the irregular behavior of the nonlinear functions. The most general result applicable to the problem is the Kuhn-Tucker [37] conditions. Unless *f(x)* and *g(x)* are well behaved functions (convexity and concavity conditions), the *K*-T theory yields only necessary conditions for optimum.

The algorithms may be classified generally as indirect and direct methods. Indirect methods solve the nonlinear problem by dealing with one or more linear problems that are based on the original program. Direct methods deals with the nonlinear problem itself.

The indirect method includes separable, quadratic, geometric and stochastic programming.

1.4.3 Separable Programming

Separable programming deals with such nonlinear programming problems in which the objective function as well as the constraints are separable. Separable programming uses the simplex method to obtain solutions to nonlinear programs where the objective function and the constraints functions can be expressed as the sum of functions, each involving, only one variable.

Separable nonlinear program (*P*) can be expressed as problem *P*

Minimize $\sum_{j=1}^{n} f_i\ (x_j)$

Subject to

$$\sum_{j=1}^{n} g_{ij}\ (x_j) \leq P_i \text{ for } i = 1, 2, \ldots, m$$

$$x_j \geq 0 \text{ for } j = 1, 2, \ldots, n.$$

Problems of these types arise in numerous applications, including econometric data fitting, electrical network analysis, design and management of water supply systems, logistics and statistics.

1.4.4 Quadratic Programming

Quadratic programming represents a special class of nonlinear programming in which the objective function is quadratic and the constraints are linear. The quadratic programming problems are computationally the least difficult to handle. For this reason, quadratic functions and program are as widely used as the linear functions and programs in modeling the optimization problems.

A quadratic program model is defined as follows:

Maximize (or minimize) $Z = c\,x + x^T\,D\,x$

Subject to

$$A\,x \leq b$$

$$x \geq 0$$

where

$$x = (x_1, x_2, \dots, x_n)^T$$

$$c = (c_1, c_2, \dots, c_n)$$

$$b = (b_1, b_2, \dots, b_n)^T$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$D = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix}$$

The function $x^T Dx$ defines a quadratic form, where D is symmetric matrix. The matrix D is negative definite if the problem is maximization, and positive definite if the problem is minimization. This means that Z is strictly convex in x for minimization and strictly concave for maximization. The constraints are assumed to be linear in the case, which guarantees a convex solution space.

1.4.5 Fractional Programming

Fractional programming is an important technique of mathematical programming. This technique is used to solve the problem of maximizing the fraction of two linear functions subject to a set of linear equalities and the non-negatively constraints. This problem can be directly solved by starting with a basic feasible solution and showing the condition for improving the current basic feasible solution. In order to test optimality of the solution we establish the optimality criterion first and ultimately, the problem can be easily solved by the method which is similar to 'Simplex Method' of linear programming.

The linear fractional programming problem plays an important role in non-linear programming. In military, programming games have this form when troops are in the field and the decision to be taken is how to distribute the fire among several possible types of targets. The fractional programming method is useful in solving the problem in economics whenever the different economic activities utilize the fixed resources in proportion to the level of their values. In financial analysis of a firm, the purpose of optimization is to find the optimum of the specific index number, usually the most favorable rates of revenues and allocation and hence playing an important role in finance.

Mathematical Formulation of Linear Fractional Programming Problem

A linear fractional program is an optimization problem of the form

Minimize $\frac{(c^T x + c_0)}{(d^T x + d_0)}$

Subject to

$A\, x \geq b$

$x \geq 0$

A problem of type LFP is also known as hyperbolic optimization problem. Here the objective function is the quotient of the two linear functions. The set

$$T = \{x\text{: } x \in R^n,\ A\, x \geq b,\ x \geq 0\}$$

is the constraint set of LFP assume that

$$d^T x + d_0 > 0 \text{ for all } x \in T$$

Using a simple technique due to Charnes and Cooper [9], we can reduce LFP to an equivalent linear program. Under transformation $y = y_0 x$.

Let $u = d^T x + d_0$, $y_0 = \frac{1}{u} > 0$, $y = y_0 x$

Then,

$$\frac{(c^T x + c_0)}{(d^T x + d_0)} = y_0\, (c^T x + c_0) = c^T y + c_o\, y_0$$

hence the program LFP becomes:

Minimize $(c^T\, y + c_o\, y_0)$

Subject to

$Ay \geq y_0 b,$

$d^T x + d_0\, y_0 = 1,$

$y \geq 0,\ y_0 > 0.$

1.4.6 Optimality conditions for nonlinear programming

A number of algorithms have been developed by the researchers, each applicable to a specific type of NLPP only. Some methods for solving nonlinear programming problem were discussed by Avriel and Zangwill. The pioneer work by Kuhn Tucker in 1951 on necessary and sufficient conditions for the optimal solution laid the foundation for the researchers to work on the nonlinear system. In 1957, the emergence of dynamic programming by Bellman brought a revolution in the subject and consequently, linear and non-linear systems have been studied simultaneously. It is disappointing to note that possibly no universal technique has been established for nonlinear system as yet.

1.4.7 Necessary Optimality Conditions

Given a point x in E_n, we wish to determine, if possible, whether or not the point is a local or a global minimum. For this purpose we need to characterize the minimum point. Fortunately the differentiability assumption of f provides a means of obtaining this characterization.

Necessary conditions for scalar convex programming were first investigated by Fritz John in 1948. He gave the characterization of optimality for scalar nonlinear program.

1.4.8 Fritz John Necessary Conditions

Let X be a nonempty open set in E_n and let $f: E_n \to E_1$ and $g_i: E_n \to E_1$ for $i= 1,2,...,m$. We consider the nonlinear problem to

Minimize $f(x)$

Subject to

$x \in X$ and

$g_i \leq 0$ for $i=1,2,...,m$.

Let $\bar{x}$ be the feasible solution, and denote $I = \{I : g_i(\bar{x}) = 0\}$.

Furthermore, suppose that f and g_i for $i \in I$ are differentiable at $\bar{x}$ and that g_i for $i \notin I$ are continuous at $\bar{x}$. If $\bar{x}$ locally solves problem, then there exists scalars u_0 and u_i for i$\in I$ such that

$$u_0 \nabla f(\bar{x}) + \sum_{i=I} u_i \nabla g_i(\bar{x}) = 0$$

$$u_0, u_i \geq 0 \text{ for } i \in I$$

$$(u_0, u_i) \neq (0, 0)$$

Where u_I is a vector whose components are u_i for i $\in$ I. furthermore, if g_i for $i \notin I$ are also differentiable at $\bar{x}$, then the foregoing conditions can be written in the following equivalent form:

$$u_0 \nabla f(\bar{x}) + \sum_{i=I} u_i \nabla g_i(\bar{x}) = 0$$

$$u_i \; g_i(\bar{x}) = 0 \text{ for } i=1,2,\ldots,m$$

$$u_0, u_i \geq 0 \text{ for } i=1,2,\ldots,m$$

$$(u_0, \boldsymbol{u}) \neq (0,= \boldsymbol{0})$$

where $\boldsymbol{u}$ is a vector whose components are u_i for $i = 1,2,\ldots,m$. In the above conditions, the scalars u_0 and u_i, $i = 1,2,\ldots,m$ are called the Lagrangian multipliers. If the Lagrangian multiplier u_0 is equal to zero, the F-J conditions do not make use of any information pertaining to the gradient of the objective function. In this case, any function can replace f and there will be no change in the above necessary condition. So the F-J conditions are of no practical value in locating an optimal point when u_0=0. In order to exclude such cases, some restrictions are imposed on the constraints. These restrictions are termed as constraint qualifications. Some of these constraint qualifications make use mostly of the differentiability of the functions defining the feasible region X.

Some of the constraint qualifications are

i) The Kuhn-Tucker Constraint Qualification.

ii) The Weak Arrow-Hurwitz-Uzawa - constraint Qualification.

iii) Abadie Constraint Qualification.

iv) Slaters Constraint Qualification.

i) The Kuhn-Tucker Constraint Qualification

The vector function g is said to satisfy the Kuhn-Tucker constraint qualification at $\bar{x} \in X$ if g is differentiable at $\bar{x}$ and if

$$\left.\begin{array}{c} y \in E_n \\ \nabla g_i(\bar{x}) \leqq 0 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l} there\ exixts\ an\ n-dimensional\ vector\ function\ e \\ \quad on\ the\ interval\ [0,1] such\ that \\ \quad a)\ e(0) = \bar{x} \\ \quad b)\ e(t) \in X\ for\ 0 \leqq t \leqq 1 \\ \quad c)\ e\ is\ differentiable\ at\ t = 0 \\ and\ \frac{d}{dt}\ e(0) = \lambda\, y\ , for\ some\ \lambda > 0 \end{array}\right.$$

Where I ={ i ϵ M: $g_i(\bar{x}) = 0$ } .

ii) Kuhn-Tucker Type Necessary Conditions

Let X be a nonempty open set in E_n and let $f : E_n \rightarrow E_1$, and $g_i: E_n \rightarrow E_1$ for *i=1,2,...,m*.

We consider the Nonlinear Problem (NLP) to

Minimize *f(x)*

subject to

$x \in X$ and

$g_i(x) \leq 0$ for $i = 1,2,...,m$.

Let $\bar{x}$ be the feasible solution, and denote $I = \{i = g_i(\bar{x}) = 0\}$. Furthermore, suppose that *f* and g_i for $i \in I$ are differentiable at $\bar{x}$ and that g_i for *i* L *I* are continuous at $\bar{x}$. Furthermost, suppose that $\nabla g_i(\bar{x})$ for $i \in$ l are linearly independent. If $\bar{x}$ locally solves Problem (NLP), then there exists scalars and u_i for $i \in I$ such that

$$\nabla f(\bar{x}) + \sum_{i=I} u_i \nabla g_i(\bar{x}) = 0$$

$$u_i \geq 0 \text{ for } i = I$$

In addition to the above assumption, if g_i for each *i* L *I* is also differentiable at $\bar{x}$, then the foregoing conditions can be written in the following equivalent form:

$$\nabla f(\bar{x}) + \sum_{i=I} u_i \nabla g_i(\bar{x}) = 0$$

$$u_i g_i(\bar{x}) = 0 \text{ for } i = 1,2,\ldots,m$$

$$u_i \geq 0 \text{ for } i = 1,2,\ldots,m$$

where u_i is the Lagrangian multiplier

The above necessary conditions hold under any constraint qualification. Kuhn-Tucker also proved that the above necessary conditions are sufficient for optimality under suitable convexity assumptions.

1.5 Multi-objective Optimization

Multi-objective optimization (also known as multi-objective programming, vector optimization, multi-criteria optimization, multi-attribute optimization or pareto optimization) is an area of multiple criteria decision making, that is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously. It is an art of detecting and making good compromises. It is based upon the fact that most real-world decisions are compromises between partially conflicting objectives that cannot easily be offset against each other. Thus, one is forced to look for possible compromises and finally decide which one to implement. So, the final decision in multi-objective optimization is always with a person-the decision maker.

The multi-objective problems on the vector minimum problems seek to obtain compromise solutions called efficient solutions by Koopmans [36]. An efficient solution is also referred to as non-inferior or non-dominated or Pareto optimal solution. The concept of efficiency has proved to be of great significance in the discussion of multi-objective programming problems. Pareto [49] began the study of efficient solutions by reducing multi-objective programming problems to the single objective one. However, the problem was first explicitly defined and studied by Kuhn and Tucker to eliminate certain anomalous efficient solutions: they also proposed a slightly restricted definition of efficiency, called proper efficiency. Later, Geoffrion [28] modified this concept and called an efficient solution to be properly efficient if the ratio of gain to loss is always finite. His work motivated many workers in this field. Iserman [34] proved that in a linear multi-objective programming problem every efficient solution is properly efficient. This result was extended by Chew & Choo [10] for pseudo linear vector maximum problems under certain boundedness assumption. Gulati & Islam [30] observed that every efficient solution of a linear vector maximum problem with non-linear constraints qualification is properly efficient.

The first notion of optimally in this setting is popularly known as Pareto-optimality and is still the most widely used. In Pareto optimality every feasible alternative that is not dominated by any other in terms of the component wise partial order is considered to be optimal. Hence each solution is considered optimal that is not definitely worse than another. Thus, multi-objective optimization does not yield a single or a set of equally good answers, but rather suggests a range of potentially very different answers.

A general multi-objective programming problem (MOPP) can be expressed as:

(MP): Optimize $f(x) = \{f_1(x), f_2(x), \ldots, f_p(x)\}$

Subject to

$$g_i(x) \geq 0 \ (i = 1,2,\ldots, m),$$

$$h_j(x) = 0 \ (j = 1,2,\ldots, k),$$

$$x \in X.$$

where

$x = (x_1, x_2, \ldots, x_n)^T$ is the vector of unknown decision variables and $f(x)$, $g_i (i=1,2,\ldots,m)$, $h_j (j=1,2,\ldots,k)$ are the real valued functions of n real variables $x_1, x_2, \ldots, x_n$ and $X \subseteq R^n$. In this formulation , the function $f(x)$ is called the objective function, the constraints $g_i(x) \geq 0$, $i=1,2,\ldots,m$ are referred to as an inequality constraints, the constraints $h_j(x)=0$, $j=1,2,\ldots,k$ are the equality constraints. The inclusion $x \in X$ is called the abstract constraints.

Isermann [34] proved that every efficient solution of a linear multi-objective programming problem is properly efficient. It is not so in nonlinear multi-objective programming. Gulati and Talaat [31] observed that under a certain constraint qualification every efficient solution of a convex multi-objective programming problem is properly efficient. We shall show efficient solution to linear multi-objective problems.

Consider the following linear multi-objective mathematical programming problem (LMOMP).

(LMOMP) Minimize $f(x) = \{f_1(x), f_2(x), \ldots, f_p(x)\}$

Subject to

$$x \in X = \{x \in S: g(x) \leqq 0\},$$

where S is an open subset of R^n, and $f{:}R^n \to R^p$ and $g{:}R^n \to R^m$ are differentiable functions on S.

The corresponding scalar programming problem (EP) is

(EP) Minimize $d\, f(x)$

Subject to

$$\left.\begin{array}{r} g(x) \leqq 0 \\ f(x) \leqq f(\bar{x}) \end{array}\right\} \quad \ldots \text{(A)}$$

$$x \in S$$

where $d > 0$ is a constant vector in R^p.

Theorem1.9.1: Let $\bar{x}$ be an efficient solution of linear multi-objective mathematical programming problem and at $\bar{x} \in X$,

i) f is convex,

ii) g_i is quasi-convex, and

iii) The system *(A)* satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$, Then $\bar{x}$ is a properly efficient solution of linear multi-objective mathematical programming problem LMOMP.

Proof: Since $\bar{x}$ is an efficient solution of LMOMP and the system *(A)* satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$, hence, there exists $\bar{u} \in R^k$ and $\bar{v} \in R^m$ such that

$$\bar{u}\,\nabla f(\bar{x}) + \bar{v}\,\nabla g(\bar{x}) = 0,$$

$$\bar{v}\,g(\bar{x}) = 0,$$

$$\bar{u} > 0, \bar{v} \geqq 0$$

Now since *f* is convex and g_I is quasi-convex at $\bar{x} \in X$, implies that $\bar{x}$ is a properly efficient solution of LMOMP.

Chapter 2

Duality in Mathematical Programming

2.1 Duality in linear programming

One of the most important discoveries in the early development of linear programming was the concept of duality and its division into important branches. The discovery disclosed the fact that every linear programming problem has associated with it another linear programming problem. The original problem is called the “primal” while the other is called its “dual”. The relationship between the ‘primal’ and the ‘dual’ problems is actually a very intimate and useful one. The optimal solution of either problem reveals information concerning the optimal solution of the other. If the optimal solution to one is known, then the optimal solution of the other is readily available. This fact is important because the situation can arise where the dual is easier to solve than the primal.

Since any LP can be written in the standard form, it has its dual. Since the dual of a LP is itself a LP, it has its dual. So we could keep on taking duals forever. The changes that occur from primal to dual is that the objective function changes from minimum to maximum and the inequalities reverse in the constraints.

In both of the primal and dual problems, the variables are non-negative and the constraints are inequalities. Such problems are called symmetric dual linear problems. In maximization problems the inequalities must be in "less than or equal to" form; while in the minimization problems they must be "greater than or equal to" form.

The general form of the primal problem with its associated dual is:

Primal (P):

Maximize $z = \sum_{j=1}^{n} c_j \ x_j$

Subject to

$$\sum_{j=1}^{n} a_{ij} \ x_j \leq b_i \ (i = 1, 2,\ldots,m)$$

$$x_j \geq 0 \ (j=1, 2,\ldots,n)$$

Dual (D):

Minimize $w = \sum_{i=1}^{m} b_i\, y_i$

Subject to

$\sum_{i=1}^{m} a_{ij}\, y_i \geq c_j \;(j=1,2, \ldots ,n)$

$y_i \geq 0 \;(i=l,2,\ldots, m)$

In matrix form:

Primal (P):

Maximize $z = c\, x$

Subject to

$A\, x \leq b$

$x \geq 0$

Where A is $(m \times n)$ matrix, b is $(m \times 1)$ column vector, c is $(1 \times n)$ row vector, x is $(n \times 1)$ column vector.

Dual (D):

Minimize $w = y\, b$

Subject to

$y\, A \geq c$

$y \geq 0$,

where y is $(1 \times m)$ row vector.

The number of constraints in the dual changes from m to n. The number of variables changes from *n* to *m*.

2.1.1 Properties of duality in linear programming

i) The dual of dual is the primal.

ii) Let x_0 be a feasible solution to the primal problem

Maximize $f(x) = c\,x$ subject to $A\,x \leq b,\ x \geq 0$

Where x^T and $c \in R^n$, $b^T \in R^m$ and A is an $m \times n$ real matrix. If W_0 be a feasible solution to the dual of the primal, namely

Minimize $g(w) = b^T w$ subject to $A^T\ w \geq c^T,\ w \geq 0$

Where $w^T \in R^m$, then $cx_0 \leq b^T w_0$.

iii) Let x_0 be a feasible solution to the primal problem

Maximize $f(x) = c\,x$ subject to $A\,x \leq b,\ x \geq 0$

And W_0 be a feasible solution to its dual:

Minimize $g(w) = b^T$w subject to $A^T\ w \geq c^T,\ w \geq 0$

Where x^T and c $\in R^n$, w^T and $b^T \in R^m$ and is $m \times n$ real matrix.

If $cx_0 = b^T\ w_0$, then both x_0 and W_0 are optimum solutions to the primal and dual respectively.

iv) Let a primal problem be

Maximize $f(x) = cx$ subject to $Ax \leq b,\ x \geq 0\ x^T$ and c $\in R^n$

And the associated dual be

Minimize $g(w) = b^T w$ subject to $A^T w \geq c^T,\ w \geq 0\ w^T$ *and* $b^T \in R^m$.

If $x_0\,(W_0)$ is an optimum solution to the primal(dual), then there exists a feasible solution $W_0(\,x_0\,)$ to the dual (primal) such that

$$c\,x_0 = b^T\,w_0.$$

v) If the primal or the dual has a finite optimum solution, then the other problem also possesses a finite optimum solution and the optimum values of the objective functions of the two problems are equal.

vi) If either the primal or the dual problem has an unbounded objective function value, then the other problem has no feasible solution.

vii) Let x_0 and w_0 be the feasible solutions to the primal $\{max.\ c^T x\!: Ax \le b,\ x \ge 0\}$ and its dual $\{min.\ b^T w\!: A^T w \ge c^T,\ w \ge 0\}$ respectively. Then, a necessary and sufficient condition for x_0 and w_0 to be optimal to their respective problems is that

$$w_0^T(b - Ax_0) = 0 \text{ and } x_0^T\,(A^T w_0 - c^T) = 0.$$

The above property is known as Complementary Slackness Property. If, in an optimal solution of a linear program, the value of the dual variable associated with a constraint is non zero, then that constraint must be satisfied with equality. Further, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.

For the primal linear program posed as a maximization problem with less than or equal to constraints, this means:

$$\text{If } \hat{y}_i > 0, \text{ then } \textstyle\sum_{j=1}^{n} a_{ij}\ \hat{x}_j = b_i\ ,$$

$$\text{If } \textstyle\sum_{j=1}^{n} a_{ij}\ \hat{x}_j < b_i\ , \text{ then } \hat{y}_i = 0$$

The property identifies a relationship between variables in one problem and associated constraints in the other problem. It says that if a variable is positive, then the associated dual constraints must be binding. It also says that if a constraint fails to bind, then the associated variable is zero. The statement really is about 'complementary slackness' in the sense that it asserts that there cannot be slack in both a constraint and the associated dual variable. The complementary slackness property is useful because of certain applications.

a) Used in finding an optimal primal solution for the given optimal dual solution and vice versa.

b) Used in verifying whether a feasible solution is optimal for the primal problem.

c) Used in investigating the general properties of the optimal solutions to primal and dual by testing the different hypothesis.

2.1.2 The various useful aspects of duality

(i) If the primal problem contains a large number of constraints and a small number of variables, the computational procedure can be reduced by converting it into dual and solve it.

(ii) Many times, the LPP requires the use of artificial variable because the LPP does not have full basis. This problem can be avoided just by writing the dual of the primal problem and as such the phase first the two phase method can be avoided.

(iii) In case, when the LPP does not provide the initial basic feasible solution, the dual may provide the basic feasible solution as of infeasible but the solution is optimal and the infeasibility solution can be forced to be feasible, keeping the solution optimal.

(iv) Calculation of the dual checks the accuracy of the primal solution.

2.2 Duality for Nonlinear Programming

The existence of duality theory in nonlinear programming problem helps to develop numerical algorithm as it provides suitable stopping rules for primal and dual problems. Duality in non-linear programming or for any mathematical programming is, generally speaking, the statement of a relationship of a certain kind between two mathematical programming problems. The relationship commonly has three aspects:

(i) One problem - the "primal" - is a constrained maximization problem,

(ii) The existence of a solution to one of these problem ensures the existence of a solution to the other, in which case their respective extreme values are equal, and

(iii) If the constraints of one problem are consistent while those of the other are not, there is a sequence of points satisfying the constraints of the first on which its objective function tends to infinity.

In non-linear duality results or in non-linear case the function f of the primal problem appears not only in the constraints of the dual, as expected, but remains involved in its objective function as well.

Consider the nonlinear programming problem.

(P): Minimize $f(x)$

Subject to

$$h_j(x) \leq 0, j = 1,2,...,m$$

Where $f : R^n \to R$ and $h_j : R^n \to R$, $j = 1,2,...,m$ are differentiable.

(WD): Maximize $f(x) + y^T\, h(x)$

Subject to

$$\nabla(f(x) + y^T\, h(x)) = 0,$$

$$y \geq 0, y \in R^m$$

is known as the Wolfe [56] type dual for the problem (*P*). Mangasarian [39] explained by means of an example that certain duality theorems may not be valid if the objective or the constraint function is a generalized convex function. This motivated Mond and Weir [46] to introduce a different dual for (*P*) as

(MWD): Maximize $f(x)$

Subject to

$$\nabla f(x) + \nabla y^T h(x) = 0.$$

$$y^T\, h(x) \geq 0$$

$$y \geq 0, y \in R^m$$

and they proved various duality theorems under pseudo-convexity of *f* and quasi-convexity of y^Th(.) for all feasible solution of (*P*) and (MWD).

Later Weir and Mond [55] derived sufficiency of Fritz John optimality criteria under pseudo-convexity of the objective and qasi-convexity or semi-strict convexity of constraint functions. They formulated the following dual using Fritz John optimality conditions instead of Karush-Kuhn-Tucker optimality conditions and proved various duality theorems-thus the requirement of constraint qualification is eliminated.

(FrD): Maximize $f(x)$

Subject to

$$y_0 \nabla f(x) + \nabla y^T h(x) = 0.$$

$$y^T h(x) \geq 0$$

$$(y_0, y) \geq 0$$

2.3 Duality in Multi-objective mathematical Programming

The theory of duality in multi-objective mathematical programming has experienced a very distinct development. Depending upon the type of the objective functions and especially, on the type of efficiency used, different concepts of duality have been studied.

The first results concerning duality in multi-objective mathematical programming was obtained by Gale, Kuhn and Tucker [26] in 1951. They established some theorems of duality in multiple objective linear programming.

Another very important approach in the theory of duality for convex optimization problems has been introduced in the beginning of the eighties. Weir [52] first introduced the duals for multi-objective mathematical programming problem in the differentiable case and then Weir and Mond [53] have weakened the initial assumptions by formulating and proving the duality also in the non-differentiable case, under generalized convexity assumptions and without requiring any constraint qualifications.

For multi-objective programming problem, we shall follow the following conventions for vectors in R^n

$x < y \Leftrightarrow x_i < y_i, i = 1,2,...,n.$

$x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1,2,...,n.$

$x \leq y \Leftrightarrow x_i \leq y_i, i = 1,2,...,n,$ but $x \neq y$

$x \nleq y$ is the negative of $x \leq y$.

Consider the multi-objective programming problem:

(VP): Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$

Subject to

$h_j(\bar{x}) \leqq 0, (j = 1, 2,...,m)$

Here $X \subseteq R^n$ is an open and convex set and f_i and h_j are differentiable functions where $f_i: X \rightarrow R$, $i = 1, 2,...,p$ and $h_j: X \rightarrow R$, $j = 1,2,...m$. Here the symbol 'VP' stands for vector minimization and minimality is taken in terms of either *"efficient points"* or *"properly efficient points"* given by Koopman [36] and Geoffrion [28] respectively.

Geoffrion [28] considered the following single objective minimization problems for fixed $\lambda \in R^p$

(VP)$_\lambda$: Minimize $\sum_{i=1}^{p} \lambda_i f_i(x)$

Subject to $h_j(\bar{x}) \leqq 0, (j = 1, 2,...,m)$

and prove the following lemma connecting (VP) and *(VP)$_\lambda$*.

Lemma 2.5.1: Let $\lambda_i > 0, (i = 1,2,...,p), \sum_{i=1}^{p} \lambda_i = 1$ be fixed. If $\bar{x}$ is optimal for (VP)$_\lambda$, then $\bar{x}$ is properly efficient for (VP).

(i) Let f_i and h_j be convex functions Then $\bar{x}$ is properly efficient for (VP) iff $\bar{x}$ is optimal for all differentiable functions (VP)$_\lambda$ for some $\lambda_i > 0, \sum_{i=1}^{p} \lambda_i = 1, (i = 1, 2,...,p)$.

If f_i and h_j are differentiable convex functions then (VP)$_\lambda$ is a convex programming problem. Therefore in relation to (VP)$_\lambda$ consider the scalar maximization problem:

(VD)$_\lambda$: Maximize $\lambda^T f(x) + y^T h(x) = \lambda^T (f(x) + y^T h(x))$

Subject to

$$\nabla(\lambda^T f(x) + y^T h(x)) = 0$$

$$\lambda \in \Lambda^+, y \geqq 0,$$

where $e = (1, 1,... ,1) \in R^p$ and $\Lambda^+ = \{ \lambda \in R^p : \lambda > 0, \lambda^T e = 1\}$.

Now as (VD)$_\lambda$ is a dual program of (VP)$_\lambda$, Weir [52] considered the following vector optimization problem in relation to (VP) as

(DV): Maximize $(f(x) + y^T h(x))e$

Subject to

$$\nabla(w^T f(x) + y^T h(x)) = 0$$

$$w \in \Lambda^+, y \geqq 0,$$

where $e = (1, 1,... ,1) \in R^p$

They termed (DV) as the dual of (VP) and proved various duality theorems between (VP) and (DV) under the assumption that f and g are convex functions.

Further for the purpose of weakening the convexity requirements on objective and constraint functions, Weir [52] introduced another dual program (DV1)

(DV1): Maximize $f(x)$

Subject to

$$\nabla(w^T f(x) + y^T h(x)) = 0$$

$$y^T h(x) \geq 0$$

$$w \in \Lambda^+, y \geqq 0,$$

And various duality theorems are proved by assuming the function f to be pseudo convex and $y^T h$ to be quasi-convex for all feasible solutions of (VP) and (DV1).

2.3.1 Non linear multi-objective mathematical programming

We consider the following nonlinear multi-objective mathematical programming problem (NMMP):

(NMMP): Minimize $f(x) = \{ f_1(x), f_2(x), \ldots, f_p(x)\}$

Subject to

$$x \in X = \{x \in R^n : g(x) \leqq 0\},$$

where f: $R^n \to R^p$ and g: $R^n \to R^m$ are differentiable

Proper efficiency in nonlinear multi-objective mathematical programming

We also state the nonlinear programming problem which is in subsequent relation with multi-objective mathematical problem.

(NLP): Minimize $d^T f(x)$, $(d > 0)$

Subject to

$$\left.\begin{array}{r} g(x) \leqq 0 \\ f(x) \leqq f(x^0) \end{array}\right\} \quad \ldots\text{(AB)}$$

where $d \epsilon R^k$ is a constant vector.

Lemma 2.6.1: $x^0 \epsilon X$ is an efficient solution of NMMP if and only if x^0 is an optimal solution of nonlinear programme.

Proof: Let $x^0 \epsilon X$ be an efficient solution of NMMP and suppose to be the contrary, that x^0 be not an optimal solution of NLP. Then there exists x^* such that

$$g(x^*) \leqq 0, -f(x^*) + f(x^0) \leqq 0$$

$$d f(x^*) > d f(x^0).$$

Since $d > 0$, the above conditions are equivalent to

$$g(x^*) \leqq 0, f(x^*) \geqq f(x^0)$$

which contradicts the fact that x^0 is an efficient solution NMMP. Conversely, let x^0 be an optimal solution of NLP and suppose to the contrary, that x^0 is not an efficient solution of NMMP. Then there exists an $x^1 \epsilon X$ such that

$$f(x^1) \geq f(x^0) \text{ and } g(x^1) \leqq 0$$

that is, x^1 is a feasible point of NLP and $d f(x^1) > d f(x^0)$ which contradicts that x^0 is an optimal solution of NLP. Hence the Lemma.

2.3.2 Kuhn-Tucker Type Necessary Conditions for Efficiency

Theorem 2.6.2: Let $x^0 \in X$ be an efficient solution of NMMP and let the system (*AB*) satisfy the Kuhn-Tucker constraint qualification at x^0. Then, there exist $u^0 \in R^k$ and $v^0 \in R^m$ such that,

$$u^{0^T} \nabla f(x^0) + v^{0^T} \nabla g(x^0) = 0,$$

$$v^{0^T} g(x^0) = 0,$$

$$u^0 > 0, v^0 > 0.$$

Proof: Since x^0 is an efficient solution of NMMP by Lemma 2.6.1, x^0 is an optimal solution of NLP. Hence by (Kuhn-Tucker stationary point necessary optimality theorem) in Mangasarian [40] there exist $v^0 \in R^m$, $w^0 \in R^k$ such that

$$d^T \nabla f(x^0) + v^{0^T} \nabla g(x^0) + w^{0^T} \nabla f(x^0) = 0,$$
$$v^{0^T} g(x^0) = 0,$$
$$v^0, \; w^0 \geqq 0.$$

Since $d > 0, \; w^0 \geqq 0$, the above conditions imply that

$$u^{0^T} \nabla f(x^0) + v^{0^T} \nabla g(x^0) = 0,$$
$$v^{0^T} g(x^0) = 0,$$
$$u^0 = (d + w^0) > 0, v^0 \geqq 0.$$

2.3.3 Fritz-John Type Necessary Conditions For Efficiency

Theorem2.6.3: Let $x^0 \in X$ be an efficient solution of NMMP. Then there exists $u^0 \in R^k$ and $w^0 \in R^m$ such that

$$u^{0^T} \nabla f(x^0) + v^{0^T} \nabla g(x^0) = 0,$$

$v^{0^T} g(x^0) = 0,$

$u^0 \geqq 0, v^0 \geqq 0, (u^0, {v^0}_p) \geq 0$

Where $P = \{i : g_i(x^0) = 0$ and g_i is not concave at $x^0\}$

Proof: Since x^0 is an efficient solution of NMMP by Lemma1, x^0 is an optimal solution of NLP. Hence by (Fritz John stationary point necessary optimality theorem) in Mangasarian [40], there exist $r^0 \in R$, $w^0 \in R^k$, $v^0 \in R^m$ such that

$$r^0 (d^T \nabla f(x^0)) + v^{0^T} \nabla g(x^0) + w^{0^T} \nabla f(x^0) = 0,$$
$$v^{0^T} g(x^0) = 0,$$
$$(r^0, v^0, w^0) \geqq 0.$$

Setting $u^0 = r^0 d + w^0$, we get

$$u^{0^T} \nabla f(x^0) + v^{0^T} \nabla g(x^0) = 0,$$
$$v^{0^T} g(x^0) = 0,$$
$$u^0 \geqq 0, v^0 \geqq 0, (u^0, {v^0}_p) \geq 0$$

2.3.4 Kuhn-Tucker Type Sufficient Conditions For Properly Efficient Solution

Theorem 2.6.4: Let f be convex and g_I be quasi-convex at $x^0 \in X$. If there exist $u^0 \in R^k$ and $v^0 \in R^m$ and $v^0 \in R^m$ satisfying

$$u^{0^T} \nabla f(x^0) + v^{0^T} \nabla g(x^0) = 0, \quad \text{... (2.6.41)}$$

$$v^{0^T} g(x^0) = 0, \quad \text{... (2.6.42)}$$

$$u^0 \geqq 0, v^0 \geqq 0, \quad \text{... (2.6.43)}$$

then x^0 is a properly efficient solution of NMMP.

Proof: Let $J = \{ i : g_i(x^0) < 0\}$. Therefore $I \cup J = \{1, 2, ..., m\}$.

Also $v^0 \geqq 0$, $g(x^0) \leqq 0$ and $v^{0^T} g(x^0) = 0 \Rightarrow v^0{}_j = 0$.

Now let $x^0 \epsilon X$. Then $g_I(x) \leqq 0 = g_I(x^0)$. Since g_I is quasi-convex at x^0, we have

$$\nabla g_I(x^0)(x - x^0) \leqq 0.$$

Therefore, from (2.6.41)

$$u^{0^T} \nabla f(x^0)(x - x^0) = -v^{0^T} \nabla g(x^0)(x - x^0) \geqq 0.$$

Using convexity of f, we get

$$u^{0^T} \{f(x) - f(x^0)\} \geqq 0$$

$$u^{0^T} f(x^0) \leq u^{0^T} f(x) \text{ for all } x \epsilon X$$

Hence by theorem 1 in Geoffrion [28], x^0 is a properly efficient solution of NMMP.

2.3.5 Fritz-John Type Sufficient Conditions for Efficiency

Theorem 2.6.5: Let f be convex and g_I quasi-convex at $x^0 \epsilon X$. If there exists $u^0 \epsilon R^k$ and $v^0 \epsilon R^m$ satisfying

$$u^{0^T} \nabla f(x^0) + v^{0^T} \nabla g(x^0) = 0, \quad ...(2.6.51)$$

$$v^{0^T} g(x^0) = 0, \quad ...(2.6.52)$$

$$u^0 \geqq 0, v^0 \geqq 0, (u_j{}^0, v_Q{}^0) \geq 0 \text{ for all } j \epsilon K \quad ... (2.6.53)$$

Where $Q = \{i \epsilon I$: g_i is strictly convex at $x^0\}$, then x^0 is an efficient solution of NMMP.

Proof: Suppose to the contrary that x^0 is not an efficient solution of NMMP. Then there exists $x^1 \in X$ and $r \in K$ such that

$$f_r\ (x^1) < f_r\ (x^0) \text{ and}$$

$$f_j\ (x^1) \leqq f_j\ (x^0) \text{ for } j \in K, j \neq r.$$

Since for each $j \in K$, f is convex at x^0, we get

$$\nabla f_r\ (x^0)\ (x^1 - x^0) < 0 \qquad \dots (2.6.54)$$

and

$$\nabla f_j\ (x^0)\ (x^1 - x^0) \leqq 0 \text{ for } j \in K, j \neq r \qquad \dots (2.6.55)$$

Let $Q' = I - Q = \{i : i \in I, \notin Q\}$. Since $x^0 \in X$,

$$g_Q\ (x^1) \leqq 0 = g_Q\ (x^0).$$

The strict convexity of g_Q at x^0 gives

$$\nabla g_Q\ (x^0)\ (x^1 - x^0) < 0. \qquad \dots (2.6.56)$$

Similarly the quasi-convexity of g_Q, at x^0 gives

$$\nabla g_{Q'}(x^0)\ (x^1 - x^0) \leq 0. \qquad \dots (2.6.57)$$

Also,

$$v_j^0 = 0$$

Where $J = \{i: g_i(x^0) < 0\}$, therefore

$$v^{0^T} \nabla g\,(x^0)\ (x^1 - x^0) = [v_Q{}^{0^T} \nabla g_Q\,(x^0) + v_{Q'}{}^{0^T} \nabla g_{Q'}\,(x^0)]\,(x^1 - x^0). \qquad \dots (2.6.58)$$

Now relations (2.6.53) to (2.6.58) imply that

$$[u^{0^T} \nabla f(x^0) + v^{0^T} \nabla g\,(x^0)]\,(x^1 - x^0) < 0$$

a contradiction to (2.6.51). Hence x^0 is an efficient solution of NMMP.

Remark: If $Q = \emptyset$, *i*.e., none of the components of g_I is strictly convex, then (2.6.53) implies that $u^0 > 0$. Thus in this case the assumptions of theorem (2.6.51) are reduced to that of theorem (2.6.4), which gives a stronger conclusion that x^0 is a properly efficient solution.

Theorem 2.6.6: Let $x^0 \in X$ be an efficient solution of NMMP. If the system (AB) satisfies the Kuhn-Tucker constraint qualification at x^0, f is convex and g_I be quasi-convex at x^0, then x^0 is a properly efficient solution of NMMP.

EXAMPLE WITH ILLUSTRATION

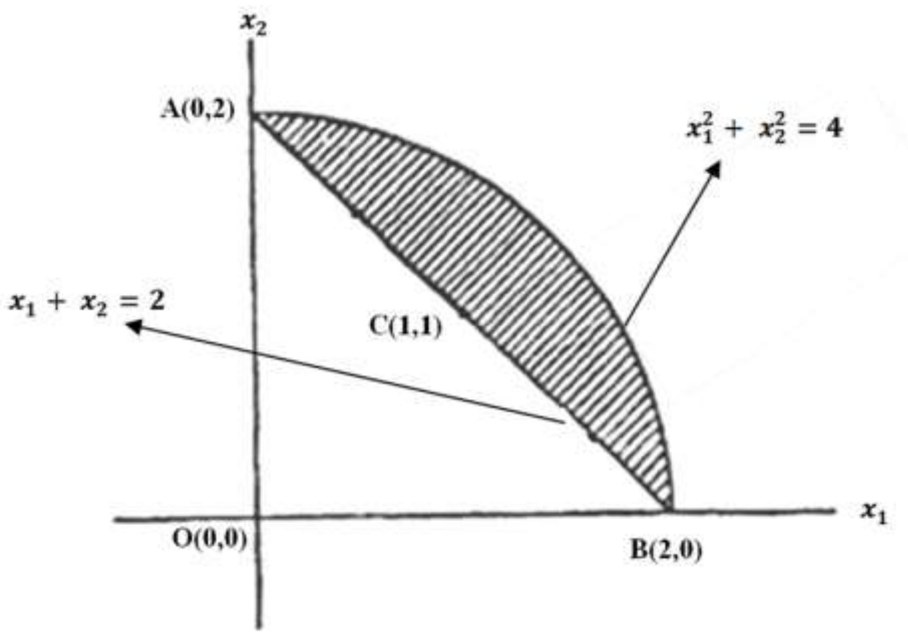
Example 2.3.6: consider the problem

(NMMP) Minimize $f(x) = [f_1(x) = {x_1}^2 + {x_2}^2, f_2(x) = x_1]^T$

Subject to

$$g_1(x) = {x_1}^2 + {x_2}^2 - 4 \leqq 0,$$

$$g_2(x) = 2 - x_1 - x_2 \leqq 0,$$



The feasible region X is the set of all points enclosed by the circle ${x_1}^2 + {x_2}^2 = 4$ and the line $x_1 + x_2 = 2$ (the shaded area in above

Figure). All the points on the line AC are efficient solutions. No other point is efficient. The constraint functions $g_i(x)$ are either linear or convex.

For an efficient point $x^0 = ({x_1}^0, {x_2}^0)$, the non-linear program (NLP) is

Minimize $Z = d_1 ({x_1}^2 + {x_2}^2) + d_2 x_1$

Subject to

$$
\begin{aligned}
&{x_1}^2 + {x_2}^2 - 4 \leqq 0,\\
&2 - x_1 - x_2 \leqq 0,\\
&{x_1}^2 + {x_2}^2 \le {x_1^0}^2 + {x_2^0}^2,\\
&x_1 \leqq {x_1}^0
\end{aligned}
$$

The above problem satisfies the Kuhn-Tucker constraints qualification at every point B (x_1, x_2) on the line AC except the point C. Hence by Theorem 2.6.6, all the points on the line AC except the point C are properly efficient. Let $x^0 = (1,1)$. Therefore $f_1(x^0) = 2$ and $f_2(x^0) = 1$. If x approaches to x^0 along the line AC, then $f_2(x) < f_2(x^0)$, $f_1(x^0) < f_1(x)$ and the ratio.

$$
\begin{aligned}
\frac{f_2(x^0) - f_2(x)}{f_1(x) - f_1(x^0)} &= \frac{1 - x_1}{{x_1}^2 + {x_2}^2 - 2}\\
&= \frac{1 - x_1}{{x_1}^2 + (2 - x_1)^2 - 2}\\
&= \frac{1 - x_1}{2({x_1}^2 - 2x_1 + 1)}\\
&= \frac{1}{2(1 - x_1)} \to \infty
\end{aligned}
$$

as $x_1 \to 1$ Hence C is not properly efficient

2.4 Second-Order Duality in Mathematical Programming

We consider the following nonlinear programming problem:

(NP): Minimum $f(x)$

Subject to

$$g(x) \leq 0$$

where $x \in R^n$, f and g are twice differentiable functions from R^n and R^m, respectively.

Mangasarian [41] formulated the Wolfe [56] type second-order dual of(NP).

(**ND-1):** Maximum $[f(u) + g^T g(u)] - \frac{1}{2} p^T \nabla^2 [f(u) + g^T g(u)]$

Subject to

$$\nabla [f(u) + g^T g(u)] + \nabla^2 [f(u) + g^T g(u)] p = 0,$$

$$y \geq 0$$

where $p \in R^n$ and for any function $\phi: R^n \to R$, the symbol $\nabla^2 \phi(x)$ designates n×n symmetric matrix of second-order partial derivatives. Mangasarian [41] established usual duality theorems between (NP) and (ND-1) under the assumptions that are involved and rather difficult to verify.

Chapter 3

Symmetric Duality In Mathematical Programming

3.1 Symmetric Duality In Mathematical Programming

Symmetric duality in mathematical programming in which the dual of the dual is the primal was first introduced by Dorn [23]. Subsequently, Dantzig, Eisenberg, and Cottle [17], Chandra and Husain [7], Mond and Weir [47] and others cited in these references developed significantly the notion of symmetric duality. Weir and Mond [54] discussed symmetric duality in multi-objective programming by using the concept of efficiency. Chandra and Prasad [8] presented a pair of multi-objective programming problems by associating a vector valued infinite game to this pair. Kumar and Bhatia [38] discussed multi-objective symmetric duality by using a nonlinear vector valued function of two variables corresponding to various objectives. Gulati, Husain and Ahmed [33] also established duality for multi-objective symmetric dual problems without non-negativity constraints.

A nonlinear programming problem and its dual are said to be symmetric if the dual is recast in the form of primal, its dual is the primal problem. First order symmetric and self-duality for differentiable mathematical programs have been studied by many authors. Dantzig. Eisenberg and Cottle [17] first formulated a pair of the symmetric dual nonlinear programs and established the weak and strong duality under convexity and concavity assumptions. Mond [43] presented a slightly different pair of symmetric dual nonlinear programs and obtained more generalized duality results than that of Dantzig, Eisenberg and Cottle [17]. Later Mond and Weir [47] gave another different pair of symmetric dual nonlinear programs in which the convexity and concavity assumptions were reduced to the pseudo-convexity and pseudo-concavity ones. Mond [44] was the first to study Wolfe type second-order symmetric duality bonvexity-boncavity. Subsequently Bector and Chandra [5] established second-order symmetric and self duality results for a pair of non-linear programs under pseudobonvexity-pseudoboncavity condition. Devi [22] formulated a pair of second-order symmetric dual programs and established corresponding duality results involving η-bonvex functions and Mishra [42] extended the results of to multiobjective nonlinear programming. Recently, Suneja et al [50] presented a pair of Mond-Weir type multiobjective second-order symmetric and self dual program without nonnegativity constraint and proved various duality results under bonvexity and pseudobonvexity.

3.2 Symmetric Duality in Differentiable Mathematical Programming

Consider a function $f(x,y)$ which is differentiable in $x \in R^m$ and y $\in R^m$. Dantzig et al [17] introduced the following pair of problems:

(SP): Minimize $f(x, y) - y^T \nabla_y f(x, y)$

Subject to

$$\nabla_y f(x, y) \leq 0$$

$$(x, y) \geq 0.$$

(SD): Maximize $f(x, y) - x^T \nabla_x f(x, y)$

Subject to

$$\nabla_x f(x, y) \geq 0$$

$$(x, y) \geq 0.$$

and proved the existence of a common optimal solution to the primal (SP) and (SD), when

(i) an optimal solution of (x_0, y_0) to the primal (SP) exists

(ii) f is convex in x for each y, concave in y for each x and

(iii) f, twice differentiable, has the property that at (x_0, y_0) its matrix of second partials with respect to y is negative definite.

Mond [43] further gave the following formulation of symmetric dual programming problems:

(MSP): Maximize $f(x, y) - y^T \nabla_y f(x, y)$

Subject to

$$\nabla_y f(x, y) \leq 0$$

$$x \geq 0.$$

(MSD): Maximize $f(x, y) - x^T \nabla_x f(x, y)$

Subject to

$$\nabla_x f(x, y) \geq 0$$

$$y \geq 0.$$

It may be remarked here that in [17], the constraints of both (SP) and (SD) include $x \geq 0, y \geq 0,$ but in only $x \geq 0$ is required in the primal and only $y \geq 0$ in the dual.

Later Mond and Weir [47] gave the following pair of symmetric dual nonlinear programming problems which allows the weakening of the convexity-concavity assumptions to pseudo-convexity–pseudo-concavity.

(M-WSP): Minimize $f(x, y)$

Subject to

$$\nabla_y f(x, y) \leq 0$$

$$y^T \nabla_y f(x, y) \geq 0,$$

$$x \geq 0.$$

(M-WSD): Maximize $f(x, y)$

Subject to

$$\nabla_x f(x, y) \leq 0$$

$$x^T \nabla_x f(x, y) \leq 0,$$

$$y \geq 0$$

3.3 Symmetric Duality in Non-differentiable Mathematical Programming

Let $f(x, y)$ be a real valued continuously differentiable in $x \in R^m$ and $y \in R^m$. Chandra and Husain [7] introduced pair of symmetric dual non-differentiable programs and proved duality results assuming convexity-concavity conditions on the kernel function $f(x,y)$:

(NP): Minimize $f(x, y) - y^T \nabla_y f(x, y) + (x^T B x)^{\frac{1}{2}}$

Subject to

$$-\nabla_y f(x, y) + Cw \geq 0,$$

$$w^T Cw \leq 1,$$

$$(x, y) \geq 0.$$

(ND): Maximize $f(x, y) - x^T \nabla_x f(x, y) - (y^T C y)^{\frac{1}{2}}$

Subject to

$$-\nabla_x f(x, y) - Bz \leq 0$$

$$z^T Cz \leq 1,$$

$$(x, y) \geq 0.$$

where B and C are $n \times m$ and $m \times m$ positive semi-definite matrices.

Further on the lines of Mond and Weir [46], Chandra, Craven and

Mond [6] presented another pair of symmetric dual non-differentiable programs by weakening the convexity conditions on the kernel function $f(x,y)$ to the pseudo-convexity and pseudo-concavity. The problems considered in [6] are:

(PS): Minimum $f(x, y) + (x^T Bx)^{\frac{1}{2}} - y^T Cz$

Subject to

$$\nabla_y f(x, y) - Cz \leq 0,$$

$$y^T [\nabla_y f(x, y) - Cz] \geq 0,$$

$$z^T Cz \leq 1,$$

$$x \geq 0.$$

(DS): Maximum $f(x, y) + (y^T Cy)^{\frac{1}{2}} - x^T Bw$

Subject to

$$\nabla_x f(x, y) + Bw \geq 0,$$

$$x^T [\nabla_x f(x, y) + Bw] \leq 0,$$

$$w^T Bw \leq 1,$$

$$y \geq 0.$$

Subsequently Mond and Schechter [45] introduced the following pair of symmetric dual programs one of which is Wolfe [56] type and another is Mond and Weir [47] type.

(P): Minimum $f(x, y) - y^T \nabla_2 f(x, y) + S(x| C_1)$

Subject to

$$\nabla_2 f(x, y) - z \leq 0,$$

$$z \in C_2, \quad x \geq 0.$$

(D): Maximum $f(u,v) - u^T \nabla_1 f(u,v) + S\,(v|C_2)$

Subject to

$$\nabla_1 f(u, v) + w \geq 0,$$

$$w \in C_1, \; v \geq 0. \qquad \text{and}$$

(P1): Minimum $f(x, y) - y^T z + S\,(x|\, C_1)$

Subject to

$$\nabla_2 f(x, y) - z \leq 0,$$

$$y^T (\nabla_2 f(x, y) - z) \geq 0,$$

$$z \in C_2, \quad x \geq 0.$$

(Dl): Maximum $f(u, v) - u^T w + S\,(v \mid C_2)$

Subject to

$$\nabla_1 f(u, v) + w \geq 0,$$

$$u^T (\nabla_1 f(u, v) + w) \leq 0,$$

$$w \in C_1, \; v \geq 0.$$

3.4 Symmetric Duality in Multi-objective Mathematical Programming

Weir and Mond [54] discussed symmetric duality in multi-objective programming by considering the following pair of programs

(PS): Minimum $f(x, y) - (y^T \nabla_2 \lambda^T f(x, y))\, e$

Subject to

$$\nabla_2 \lambda^T f(x, y) \leqq 0,$$

$$x \geq 0, \; \lambda \in \Lambda^+$$

(DS): Maximum $f(x, y) - (x^T \nabla_1 \lambda^T f(x, y))\, e$

Subject to

$$\nabla_1 \lambda^T f(x, y) \geqq 0,$$

$$y \geq 0, \; \lambda \in \Lambda^+$$

where $f: R^n \times R^m \to R^p$, $e=(1,1,\ldots,1) \epsilon R^p$ and $\wedge^+ = \{\lambda \in R^p : \lambda > 0, \lambda^T e = 1\}$ and proved the symmetric duality theorem under the convexity-concavity assumptions on $f(x,y)$. Here the minimization/ maximization is taken in the sense of proper efficiency as given by Geoffrion [28].

Further on the lines of scalar case (Mond and Weir [46]) also considered another pair of symmetric dual programs and proved symmetric duality results under weaker conditions of pseudo-convexity-pseudo-concavity:

(PS1): Minimum $f(x, y)$

Subject to

$$\nabla_2 \lambda^T f(x, y) \leqq 0,$$

$$y^T \nabla_2 \lambda^T f(x, y) \geqq 0,$$

$$x \geqq 0, \ \lambda \in \wedge^+$$

(DS1): Maximum $f(x, y) - (x^T \nabla_1 \lambda^T f(x, y))\, e$

Subject to

$$\nabla_1 \lambda^T f(x, y) \geqq 0,$$

$$x^T \nabla_1 \lambda^T f(x, y) \leqq 0,$$

$$y \geqq 0, \ \lambda \in \wedge^+$$

Later Chandra and D. Prasad [8] introduced following pair of multi-objective programs by associating a vector valued infinite game.

(PS*): Minimum $f(x, y) - (y^T \nabla_2 \mu^T f(x, y))\, e$

Subject to

$$\nabla_2 \mu^T f(x, y) \leqq 0,$$

$$x \geqq 0, \ \mu \in \wedge^+.$$

(DS*): Maximum $f(x, y) - (x^T \nabla_1 \mu^T f(x, y))\, e$

Subject to

$$\nabla_1 \lambda^T f(x, y) \leqq 0,$$

$$y \geqq 0, \;\; \lambda \in \wedge^+.$$

Here it may be noted that not the same λ is appearing in (PS*) and (DS*) and this creates certain difficulties which are also discussed in [8].

3.5 Mond-Weir type Second-Order Multi-Objective Symmetric Duality

We consider the following pair of Mond-Weir type second-order multi-objective symmetric dual nonlinear programming problems over arbitrary cones:

Primal (MP):

Minimize $F(x, y, p) = \{F_1(x, y, p),\; F_2(x, y, p),\, \dots,\, F_k(x, y, p)\}$

Subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)\, p_i] \in C_2^{\,*} \qquad \dots(3.51)$$

$$y^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)\, p_i] \geqq 0 \qquad \dots(3.52)$$

$$\lambda > 0,$$

Dual (MD):

Maximize $G(u, v, r) = \{G_1(u, v, r), G_2(u, v, r), \dots, G_k(u, v, r)\}$

Subject to

$$\sum_{i=1}^{k} \lambda_i [-\nabla_x f_i(u, v) - \nabla_{xx} f_i(u, v)\, r_i] \in C_1^{\,*}, \qquad \dots(3.53)$$

$$u^T \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)\, r_i] \leqq 0, \qquad \dots(3.54)$$

$$\lambda > 0,$$

where $F_i(x, y, p) = f_i(x, y) - \frac{1}{2}\ p_i^T\ \nabla_{yy} f_i(x, y) p_i$,

$G_i(u, v, r) = f_i(u, v) - \frac{1}{2}\ r_i^T\ \nabla_{xx} f_i(u, v) r_i$,

$\lambda_i \in R$, $p_i \in R^m$, $r_i \in R^n$, $i = 1, 2, \ldots, k$.

Also, $p = (p_1, p_2, \ldots, p_k)$, $q = (q_1, q_2, \ldots, q_k)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$. In the following theorems we take $\eta_1 : X \times X \to R^n$, $\eta_2 : Y \times Y \to R^m$, Where X and Y are open sets in R^n and R^m, respectively.

Theorem 3.5.1 (Weak Duality). Let (x, y, λ, p) and (u, v, λ, r) be feasible solutions of (MP) and (MD) respectively. Let either of the following conditions hold:

(i) For $i = 1, 2, \ldots, k$, f_i be η_1- bonvex in the first variable at u and $-f_i$ be η_2- bonvex in the second variable at y, or

(ii) $\sum_{i=1}^{k} \lambda_i\ f_i$ be η_1-pseudo-bonvex in the first variable at u and $-\sum_{i=1}^{k} \lambda_i f_i$ be η_2-pseudo-bonvex in the second variable at y.

Also, let

$$\eta_1(x, u) + u \in C_1, \qquad \ldots(3.55)$$

$$\eta_2(v, y) + y \in C_2. \qquad \ldots(3.56)$$

Then

$$F(x, y, p) \nleq G(u, v, r).$$

Proof: Since (u, v, λ, r) is feasible for (MD), from (3.53) and (3.55), it follows that $[\eta_1(x, u) + u]^T \sum_{i=1}^{k} \lambda_i\ [\nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)\ r_i] \geq 0$.

Using (3.54), we get

$$\eta_1^T(x, u) \sum_{i=1}^{k} \lambda_i [\ \nabla_x f_i(u, v) + \nabla_{xx} f_i(u, v)\ r_i] \geq 0 \qquad \ldots(3.57)$$

Since (x, y, λ, p) is feasible for (MP), from (3.51) and (3.56), it follows that $[\eta_2(v, y)\ +\ y]^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)\ p_i] \leq 0$.

Using (3.52), we get)

$$\eta_2{}^T(v, y) \ \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + \nabla_{yy} f_i(x, y)\, p_i] \le 0 \qquad \ldots(3.58)$$

(i) Since f_i be η_1- bonvex in the first variable at u, we have for $i=1, 2,\ldots, k,$

$$f_i(x,v) - f_i(u,v) + \frac{1}{2} r_i{}^T\ \nabla_{xx} f_i(u,v) r_i \ge \eta_1{}^T(x,u)[\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)\, r_i].$$

As $\lambda_i > 0$, $i = 1, 2,\ldots,k$, on using (3.57) we get

$$\sum_{i=1}^{k} \lambda_i\ [f_i(x, v) - f_i(u, v) + \frac{1}{2} r_i{}^T\ \nabla_{xx} f_i\,(u, v)\, r_i] \ge 0 \quad \ldots(3.59)$$

Since $-f_i$ be η_2-bonvex in the second variable at y, we have for $i=1,2,\ldots,k,$

$$-f_i(x, v) + f_i(x,y) - \frac{1}{2} p_i{}^T \nabla_{yy} f_i(x,y) p_i \ge -\eta_2{}^T(v,y)[\nabla_y f_i(x\ y) + \nabla_{yy} f_i(x,y) p_i].$$

As $\lambda_i > 0$, $i = 1, 2,\ldots,k$, on using (3.58) we get

$$-\sum_{i=1}^{k} \lambda_i\ [f_i(x,v) - f_i(x\ y) + \frac{1}{2} p_i{}^T \nabla_{yy}\ f_i(x,y)\, p_i]\ \ge 0\ \ldots(3.60)$$

Adding (3.59) and (3.60), we get

$$\sum_{i=1}^{k} \lambda_i\ [f_i(x,y) - \frac{1}{2} p_i{}^T \nabla_{yy}\ f_i(x,y)\, p_i] \ge \sum_{i=1}^{k} \lambda_i\ [f_i(u,v) - \ \frac{1}{2} r_i{}^T \nabla_{xx} f_i(u, v) r_i]$$

Hence $\qquad F(x,y,p) \not\le G(u, v, r).$

(ii) As $\sum_{i=1}^{k} \lambda_i\, f_i$ is η_1-pseudo-bonvex in the first variable, from (3.57), we get (3.59).

More over as $-\sum_{i=1}^{k} \lambda_i\, f_i$ is η_2-pseudo-bonvex in the second variable, from (3.58), we get (3.60).

On adding (3.59) and (3.60), we get the same results as in part (i).

Theorem 3.5.2 (Strong Duality). Let f: $R^n \times R^m \rightarrow R^k$ be thrice differentiable. Let $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ be a weakly efficient solution of (MP); *fix* $\lambda = \bar{\lambda}$ in (MD) and suppose that

(i) Either the Hessian matrix $\nabla_{yy} f_i$ is positive definite for each $i=1,2,\ldots,k$, and $\sum_{i=1}^{k} \bar{\lambda}_i \, \bar{p}_i^T \nabla_i \, f_i \geqq 0$ or the Hessian matrix $\nabla_{yy} f_i$ is negative definite for each $i=1,2,\ldots,k$, and $\sum_{i=1}^{k} \bar{\lambda}_i \, \bar{p}_i^T \nabla_i \, f_i \leqq 0$, and

(ii) The vectors $\{\nabla_y f_1 + \nabla_{yy} f_1 \; \bar{p}_1, \nabla_y f_2 + \nabla_{yy} f_2 \; \bar{p}_2, \ldots, \nabla_y f_k + \nabla_{yy} f_k \; \bar{p}_k\}$ are linearly independent,

where $f_i = f_i \, (\bar{x}, \bar{y})$, $i=1,2,\ldots,k$. then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r} = 0)$ is feasible for (MD) and objective function values of (MP) and (MD) are equal.

Furthermore, if the hypotheses of theorem (3.5.1) are satisfied for all feasible solutions of (MP) and (MD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r} = 0)$ is a properly efficient solution for (MD).

Proof: Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weak minimum of (MP), by Fritz John optimality conditions, there exist $\alpha \in R^k$, $\beta \in R^m$, $\gamma \in$ R, $\delta \in R^k$ such that

$$\sum_{i=1}^{k} \alpha_i \left[\nabla_x f_i - \frac{1}{2}(\nabla_{yy} f_i \bar{p}_i)_x \bar{p}_i\right] + \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{yx} f_i + (\nabla_{yy} f_i \bar{p}_i)_x](\beta - \gamma \bar{y}) = 0, \quad \ldots(3.61)$$

$$\sum_{i=1}^{k} \alpha_i \left[\nabla_y f_i - \frac{1}{2}(\nabla_{yy} f_i \bar{p}_i)_y \bar{p}_i\right] + \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_{yy} f_i + (\nabla_{yy} f_i \bar{p}_i)_y] (\beta - \gamma \bar{y}) - \gamma \sum_{i=1}^{k} \bar{\lambda}_i [\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i] = 0 \quad \ldots(3.62)$$

$$(\beta - \gamma \bar{y})^T [\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i] - \delta_i = 0, \; i=1,2,\ldots,k, \quad \ldots(3.63)$$

$$[(\beta - \gamma \bar{y}) \bar{\lambda}_i - \alpha_i \bar{p}_i]^T \nabla_{yy} f_i = 0, \; i=1,2,\ldots,k, \quad \ldots(3.64)$$

$$\beta^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{yy} f_i \bar{p}_i) = 0, \; i=1,2,\ldots,k, \quad \ldots(3.65)$$

$$\gamma\bar{y}\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i) = 0, \qquad ...(3.66)$$

$$\delta^T \bar{\lambda} = 0, \qquad ...(3.67)$$

$$(\alpha, \beta, \gamma, \delta) \geqq 0, \quad (\alpha, \beta, \gamma, \delta) \neq 0 \qquad ...(3.68)$$

As $\bar{\lambda} > 0$, it follows from (3.67), that $\delta = 0$. Therefore from (3.63), we get

$$(\beta - \gamma\bar{y})^T [\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i] = 0 \quad i = 1, 2, \ldots, k, \qquad ...(3.69)$$

As $\nabla_{yy} f_i$ is non-singular for $i=1,2,\ldots,k$, from (3.64), it follows that

$$(\beta - \gamma\bar{y})\, \bar{\lambda}_i = \alpha_i\bar{p}_i, \quad i=1,2,\ldots,k, \qquad ...(3.70)$$

From (3.62), we get

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)\nabla_y f_i + \sum_{i=1}^{k}\bar{\lambda}_i \nabla_{yy} f_i(\beta - \gamma\bar{y} - \gamma\bar{p}_i)$$

$$+\sum_{i=1}^{k}(\nabla_{yy} f_i\bar{p}_i)_y[(\beta - \gamma\bar{y})\bar{\lambda}_i - \frac{1}{2}\alpha_i\bar{p}_i] = 0$$

Using (3.70), it follows that

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)(\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i)$$

$$+\frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy} f_i\bar{p}_i)_y(\beta - \gamma\bar{y}) = 0 \qquad ...(3.71)$$

Pre-multiplying by $(\beta - \gamma\bar{y})^T$ and using (3.69), we get

$$(\beta - \gamma\bar{y})^T \sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy} f_i\bar{p}_i)_y (\beta - \gamma\bar{y}) = 0$$

Using the fact that $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy} f_i\bar{p}_i)_y$ is positive or negative definite, we get

$$\beta = \gamma\bar{y}. \qquad ...(3.72)$$

Using (3.72) in (3.71), we get

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)(\nabla_y f_i + \nabla_{yy} f_i\bar{p}_i) = 0$$

By condition (ii), we get

$$\alpha_i = \gamma\bar{\lambda}_i, \; i=1,2,\ldots,k, \qquad ...(3.73)$$

If $\gamma = 0$, from (3.72) and (3.73), it follows that $\beta = 0$, $\alpha = 0$ which contradicts (3.68). Hence $\gamma > 0$. Since $\bar{\lambda}_i > 0$, $i=1,2,\ldots,k$, from (3.73) we have $\alpha_i > 0$, $i=1,2,\ldots,k$, using (3.72) in (3.70), we have $\alpha_i \bar{p}_i = 0$, $i=1,2,\ldots,k$, and hence $\bar{p}_i = 0$, $i=1,2,\ldots,k$, in (3.61), it follows that

$$\sum_{i=1}^{k} \alpha_i \nabla_x f_i = 0,$$

Which by (3.73) gives

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i = 0,$$

And hence we have

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i = 0.$$

Thus we follows that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible solution of (MD) and

$$F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{r}). \qquad \ldots(3.74)$$

If $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$ is not efficient for (MD), then there exists a feasible solution $(u, v, \bar{\lambda}, r)$ of (MD) such that

$$G(\bar{x}, \bar{y}, \bar{p}) \leq G(u, v, r) \qquad \ldots(3.75)$$

Which by (3.74) gives

$$F(\bar{x}, \bar{y}, \bar{p}) \leq G(u, v, r)$$

Which is a contradiction to theorem (3.5.1).

If $(\bar{x},\bar{y},\bar{\lambda},\bar{r})$ is not efficient for (MD), then for some feasible $(u,v,\bar{\lambda},r)$ of (MD) and some i,

$$f_i(u, v) - \frac{1}{2} r_i^T \nabla_{xx} f_i(u, v)\, r_i > f_i(\bar{x}, \bar{y}) \text{ and}$$

$$f_i(u, v) - \frac{1}{2} r_i^T \nabla_{xx} f_i(u, v)\, r_i - f_i(\bar{x}, \bar{y})$$

$$> M\,[f_j(\bar{x}, \bar{y}) - f_j(u, v) + \frac{1}{2} r_i^T \nabla_{xx} f_j(u, v)\, r_j]$$

For all $M > 0$ and all j satisfying $f_j(\bar{x},\bar{y}) > f_j(u,v) - \frac{1}{2} r_j^T \nabla_{xx} f_j(u\ v)\, r_j$.

This means that $f_i(u,v) - \frac{1}{2} r_i^T \nabla_{xx} f_i(u,v)\, r_i - f_i\, (\bar{x},\bar{y})$ can be arbitrarily large. Thus for any $\bar{\lambda} > 0$,

$$\sum_{i=1}^{k} \overline{\lambda_i}\, [f_i(u, v) - \frac{1}{2} r_i^T \nabla_{xx} f_i(u, v)\, r_i]$$

$$> \sum_{i=1}^{k} \overline{\lambda_i}\, [f_j\, (\bar{x}, \bar{y}) - \frac{1}{2} \overline{r_j}^T \nabla_{xx} f_j(\bar{x}, \bar{y})\, r_j]$$

Which again contradictions to theorem (3.5.1).

Theorem 3.5.3 (Converse Duality). Let f:$R^n \times R^m \rightarrow R^k$ be thrice differentiable. Let ($\bar{u}$, $\bar{v}$, $\bar{\lambda}$, $\bar{r}$) be a weakly efficient solution of (MD). fix λ =$\bar{\lambda}$ in (MP) and suppose that

(i) Either the Hessian matrix $\nabla_{xx} f_i$ is positive definite for each $i=1,2,\ldots,k,$ and $\sum_{i=1}^{k} \overline{\lambda_i}\, \overline{r_j}^T\, \nabla_x f_i \geqq 0$ or the Hessian matrix $\nabla_{xx}\, f_i$ is negative definite for each $i=1,2,\ldots,k,$ and $\sum_{i=1}^{k} \overline{\lambda_i}\, \overline{r_i}^T \nabla_x\, f_i \leqq 0,$ and

(ii) The vectors $\{\nabla_x f_1 + \nabla_{xx} f_1\ \bar{r}_1, \nabla_x f_2 + \nabla_{xx} f_2\ \bar{r}_2, \ldots, \nabla_x f_k + \nabla_{xx} f_k\ \bar{r}_k\}$ are linearly independent,

where $f_i = f_i$ ($\bar{u}$, $\bar{v}$), $i=1,2,\ldots,k.$ then ($\bar{u}$, $\bar{v}$, $\bar{\lambda}$, $\bar{p}$ = 0) is feasible for (MP) and objective function values of (MP) and (MD) are equal. Furthermore, if the hypotheses of theorem (3.5.1) are satisfied for all feasible solutions of (MP) and (MD), then ($\bar{u}$, $\bar{v}$, $\bar{\lambda}$, $\bar{p}$ = 0) is a properly efficient solution for (MP).

Proof: Proof of which is analogous to that of the strong duality theorem.

Self duality: A mathematical programming problem is said to be self dual if it is formally identical with its dual, that is, the dual can be recast in the form of the primal. If we assume the functions f_i to be skew-symmetric, that is

$$f_i(x, y) = - f_i(y, x) \qquad \text{for each } i=1,2,\ldots,k,$$

Theorem 3.5.4 (Self Duality): Let f_i, $i=1,2,\ldots,k,$ be skew-symmetric. Then (MP) is self-dual. If (MP) and (MD) are dual problems and ($\bar{x}$, $\bar{y}$, $\bar{\lambda}$, $\bar{p}$) is a joint optimal solution, then so is ($\bar{y}$, $\bar{x}$, $\bar{\lambda}$, $\bar{p}$) and

$$F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{y}, \bar{x}, \bar{p}) = 0.$$

Proof: The dual problem (MD) can be restated as;

Minimize $\quad -G(u,v,r) = \{-G_1(u,v,r), -G_2(u,v,r), \ldots, -G_k(u,v,r)\}$

Subject to

$$\sum_{i=1}^{k} \lambda_i [-\nabla_x f_i(u,v) - \nabla_{xx} f_i(u,v)\, r_i] \leqq 0,$$

$$-u^T \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v)\, r_i] \geqq 0,$$

$$\lambda > 0,$$

Where

$$-G_i(u,v,r) = -f_i(u,v) + \frac{1}{2} r_i{}^T \nabla_{xx} f_i(u,v) r_i, \; i=1,2,\ldots,k,$$

Since f_i is skew symmetric,

$$f_i(u,v) = -f_i(v,u), \; \nabla_x f_i(u,v) = -\nabla_y f_i(v,u), \text{ and}$$

$$\nabla_{xx} f_i(u,v) = -\nabla_{yy} f_i(v,u).$$

Consequently, $\quad -G_i(u,v\, r) = f_i(v,u) - \frac{1}{2} r_i{}^T \nabla_{yy} f_i(v,u) r_i = G(v,u,r)$

Hence $\quad -G(u,v,r) = G(v,u,r)$

Thus the problem (MD) becomes

Minimize $\quad G(v,u,r) = \{G_1(v,u,r), G_2(v,u,r), \ldots, G_k(v,u,r)\}$

Subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(v,u) - \nabla_{yy} f_i(v,u)\, r_i] \leqq 0,$$

$$u^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(v,u) + \nabla_{yy} f_i(v,u) r_i] \geqq 0,$$

$$\lambda > 0,$$

Which is identical to (MP)

Thus $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is optimal for (MD), implies $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{p})$ is optimal for (MP), and by symmetric duality, also for (MD). Hence,

$$F(\bar{y}, \bar{x}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{p}) = -F(\bar{x}, \bar{y}, \bar{p}) = 0.$$

Chapter 4

Second-Order Symmetric Duality in Mathematical Programming

4.1 Second-Order Symmetric Duality in Mathematical Programming

Mangasarian [41] introduced the concept of second-order duality for nonlinear problems. Its study is significant due to computational advantage over first-order duality as it provides tighter bounds for the value of the objective function when approximations are used Mangasarian et al [41]. Later Mond [44] constructed the following pair of second-order symmetric dual problems:

(PP): Minimum $f(x,y) - y^T[\nabla_y f(x,y) + \nabla_y{}^2 f(x,y)\, p] - \frac{1}{2} p^T \nabla_y{}^2 f(x,y) p$

Subject to

$$\nabla_y f(x, y) + \nabla_y{}^2 f(x, y)\, p \leq 0,$$

$$x \geq 0.$$

(DD): Maximum $f(x,y) - x^T[\nabla_x f(x,y) + \nabla_x{}^2 f(x,y) q] - \frac{1}{2} q^T \nabla_x{}^2 f(x,y) q$

Subject to

$$\nabla_x f(x, y) + \nabla_x{}^2 f(x, y) q \geq 0,$$

$$y \geq 0.$$

In this chapter, we formulate Wolfe type second-order dual programs with cone constraints and prove weak, strong, converse and self duality theorems under bonvexity - boncavity condition.

***Proposition 4.11*:** Let X be a convex set with nonempty interior in R^n and C be a closed convex cone in R^m. Let F be real valued function and G be a vector valued function, both defined on X.

Consider the problem:

(P_0): Minimize $F(z)$

Subject to

$$G(z) \in C \text{ and } z \in X$$

If z solves the problem (P_0), then there exist $\alpha_0 \epsilon R$ and $\delta \epsilon C^*$ such that

$$[\alpha_0 \nabla F(z_0) + \nabla \delta^T G(z_0)]^T (z - z_0) \geq 0 \text{ for all } z \epsilon X,$$

$$\delta^T G(z_0) = 0,$$

$$(\alpha_0, \delta) \geq 0,$$

$$(\alpha_0, \delta) \neq 0.$$

4.2 Formulation of the Problems

In this section, we formulate a pair of second-order symmetric dual nonlinear programs with cone constraints and establish appropriate duality theorems.

Consider the following two programs:

Primal Problem

(SP): Minimize $F(x,y,p) = f(x,y) - y^T [\nabla_y f(x,y) + \nabla_y{}^2 f(x,y)p]$

$$-\frac{1}{2} p^T \nabla_y{}^2 f(x,y)p$$

Subject to

$$-\nabla_y f(x,y) - \nabla_y{}^2 f(x,y)\, p \,\epsilon C_2{}^* \quad \ldots(4.31)$$

$$(x,y) \epsilon C_1 \times C_2 \quad \ldots(4.32)$$

and

Dual problem

(SD): Maximum $H(x,y,q) = f(x,y) - x^T [\nabla_x f(x,y) + \nabla_x{}^2 f(x,y)q]$

$$-\frac{1}{2} q^T \nabla_x{}^2 f(x,y)q$$

Subject to

$$\nabla_x f(x,y) + \nabla_x{}^2 f(x,y)q \,\epsilon\, C_1{}^* \quad \ldots(4.33)$$

$$(x,y) \epsilon C_1 \times C_2 \quad \ldots(4.34)$$

where

(i) $f: C_1 \times C_2 \to R$ is a twice differentiable function,

(ii) C_1 and C_2 are closed convex cones with nonempty interior in R^n and R^m, respectively;

(iii) ${C_1}^*$ and ${C_2}^*$ are positive polar cones of C_1 and C_2 respectively.

4.2.1 Weak Duality

Theorem (4.2.1): Let *(x, y, p)* and *(u, v, q)* be feasible solutions of (SP) and (SD) respectively. Assume that *f(.,y)* is bonvex with respect to *x* for fixed *y* and *f(x,.)* is boncave with respect to *y* for fixed *x* for all feasible *(x,y,p,u,v,q).*

Then

$$\inf.(SP) \geq \sup.(SD).$$

Proof: By bonvexity of *f(., y)*, we have,

$$f(x,v)-f(u,v) \geq (x-u)^T[\nabla_x f(u,v)+{\nabla_x}^2 f(u,v)q]-\frac{1}{2}q^T{\nabla_x}^2 f(u,v)q \quad \ldots(4.35)$$

$$f(x,v)-f(x,y) \leq (v-y)^T[\nabla_y f(x,y)+{\nabla_y}^2 f(x,y\ p]-\frac{1}{2}p^T{\nabla_y}^2 f(x,y)p \quad \ldots(4.36)$$

multiplying (4.36) by (-1) and adding the resulting inequality to (4.35), we obtain,

$$\{f(x,y)-y^T[\nabla_y f(x,y)+{\nabla_y}^2 f(x,y)\,p]-\frac{1}{2}p^T{\nabla_y}^2 f(x,y)\,p\}$$

$$-\{f(u,v)+{\nabla_x}^2 f(u,v)q]-\frac{1}{2}q^T{\nabla_x}^2 f(u,v)q\}$$

$$\geq x^T[\nabla_x f(u,v)+{\nabla_x}^2 f(u,v)q]-v^T[\nabla_y f(x,y)+{\nabla_y}^2 f(x,y\ p] \quad \ldots(4.37)$$

Now since $x \epsilon C_1$ and $\nabla_x f(u,v)+{\nabla_x}^2 f(u,v)q \ \epsilon\ {C_1}^*$, we have

$$x^T [\nabla_x f(u, v) + \nabla_x{}^2 f(u, v) q] \geq 0. \qquad \ldots(4.38)$$

Now since v ϵ C_2 and $-[\nabla_y f(x, y) + \nabla_y{}^2 f(x, y)\, p] \,\epsilon\, C_2{}^*$, we have,

$$-v^T [\nabla_y f(x, y) + \nabla_y{}^2 f(x, y)\, p] \geq 0. \qquad \ldots(4.39)$$

The inequality (4.37) together with (4.38) and (4.39), yields,

$$f(x, y) - y^T [\nabla_y f(x, y) + \nabla_y{}^2 f(x, y)\, p] - \frac{1}{2}\, p^T \nabla_y{}^2 f(x, y)\, p$$

$$\geq f(u,v) - u^T [\nabla_x f(u,v) + \nabla_x{}^2 f(u,v) q] - \frac{1}{2} q^T \nabla_x{}^2 f(u,v) q$$

This implies,

$$\text{inf.(SP)} \geq \text{sup.(SD)}.$$

4.2.2 Strong Duality

Theorem 4.2.2: Let $(\bar{x},\bar{y},\bar{p})$ be an optimal solution of (SP). Also let

(i) the matrix $\nabla_y{}^2 f(\bar{x},\bar{y})$ is non singular, and

(ii) $\nabla_y(\nabla_y{}^2 f(\bar{x},\bar{y})\bar{p})$ be negative definite.

Then $(\bar{x}, \bar{y}, \bar{q}=0)$ is feasible for (SD) and the objective values of the programs (SP) and (SD) are equal. Moreover, if the requirements of Theorem (4.4) are fulfilled, then $(\bar{x}, \bar{y}, \bar{q})$ is an optimal solution of (SD).

Proof: We use Proposition (4.11) to prove this theorem.

Here $z = (x, y, p)$, $\bar{z} = (\bar{x},\bar{y},\bar{p})$, $x \epsilon C_1$, p ϵ R^m and y ϵ C_2

$$F(\bar{z}) = f(\bar{x}, \bar{y}) - \bar{y}^T [\nabla_y f(\bar{x}, \bar{y}) + \nabla_y{}^2 f(\bar{x}, \bar{y})\, \bar{p}\,] - \frac{1}{2}\, \bar{p}^T \nabla_y{}^2 f(\bar{x}, \bar{y})\bar{p}$$

$$G(\bar{z}) = -\nabla_y f(\bar{x}, \bar{y}) + \nabla_y{}^2 f(\bar{x}, \bar{y})\, \bar{p} \quad \text{and} \quad C = C^*$$

Since $(\bar{x},\bar{y},\bar{p})$ is an optimal solution of (SP), by proposition (4.11), there exist $\alpha \epsilon$R and β ϵ $C_2{}^*$ such that

$$[\alpha \nabla_x f(\bar{x},\bar{y}) - (\alpha\bar{y} + \beta)\nabla_x \nabla_y f(\bar{x},\bar{y}) - (\alpha\bar{y} + \frac{\alpha\bar{p}}{2} + \beta)\nabla_x \nabla_y{}^2 f(\bar{x}, \bar{y})\, \bar{p}]\, (x - \bar{x})$$

$$-[(\alpha\bar{y}+\beta+\alpha\bar{p})\nabla_y{}^2 f(\bar{x},\bar{y})+(\alpha\bar{y}+\frac{\alpha\bar{p}}{2}+\beta)\nabla_x\nabla_y{}^2 f(\bar{x},\bar{y})\,\bar{p}\,](y-\bar{y})\geq 0 \quad ...(4.40)$$

$$(\alpha y+\alpha p+\beta)\,\nabla_y{}^2 f(\bar{x},\bar{y}\,)=0\,. \quad ...(4.41)$$

$$\beta^T[\nabla_y f(\bar{x},\bar{y})+\nabla_y{}^2 f(\bar{x},\bar{y}\,)\,\bar{p}\,]=0 \quad ...(4.42)$$

$$(\alpha,\beta)\geq 0, \quad ...(4.43)$$

$$(\alpha,\beta)\neq 0. \quad ...(4.44)$$

the relation (4.41), in view of the hypothesis (i), gives,

$$\beta=-(\bar{y}+\bar{p}\,). \quad ...(4.45)$$

It follows that $\alpha\neq 0$, for if $\alpha=0$, (4.45) implies $\beta=0$. Hence $(\alpha,\beta)=0$ contradicts (4.44). Thus $\alpha>0$.

Now putting $\bar{x}=x$ and using (4.45) in (4.40), we obtain,

$$(\frac{\alpha\bar{p}}{2})^T[\nabla_y(\nabla_y{}^2 f(\bar{x},\bar{y}\,)\,\bar{p}\,)](y-\bar{y})\geq 0, \text{ for all } y\in C_2.$$

Putting $y=\bar{y}+\bar{p}$ and using $\alpha>0$, from the above inequality

$$p^T[\nabla_y(\nabla_y{}^2 f(\bar{x},\bar{y}\,)\,\bar{p}\,)]\,\bar{p}\geq 0$$

Which, because of (ii), yields,

$$\bar{p}=0 \quad ...(4.46)$$

Using (4.45) and (4.46) along with $\alpha>0$ in (4.40), we have,

$$\nabla_x f(\bar{x},\bar{y})\,(x-\bar{x})\geq 0, \text{ for all } x\in C_1 \quad ...(4.47)$$

Since C_1 is closed convex cone, therefore, for each $x\in C_1$ and $\bar{x}\in C_1$, it implies $(x+\bar{x})\in C_1$. Now, replacing x by $(x+\bar{x})$ in (4.47), we have,

$$x^T[\nabla_x f(\bar{x},\bar{y}\,)+\nabla_x{}^2 f(\bar{x},\bar{y}\,).0]\geq 0 \quad ...(4.48)$$

This implies,

$$x^T[\nabla_x f(\bar{x},\bar{y}\,)+\nabla_x{}^2 f(\bar{x},\bar{y}\,).0]\in C_1{}^*.$$

Thus $(\bar{x}, \bar{y}, \bar{q} = 0)$ is feasible for (SD).

Putting $x = 0$ in (4.47) and $x = \bar{x}$ in (4.48), we have respectively,

$$\bar{x}^T [\nabla_x f(\bar{x}, \bar{y}) + \nabla_x{}^2 f(\bar{x}, \bar{y}).0] \leq 0 \quad \text{and}$$

$$\bar{x}^T [\nabla_x f(\bar{x}, \bar{y}) + \nabla_x{}^2 f(\bar{x}, \bar{y}).0] \geq 0.$$

These together implies,

$$\bar{x}^T [\nabla_x f(\bar{x}, \bar{y}) + \nabla_x{}^2 f(\bar{x}, \bar{y}).0] = 0 \qquad \ldots(4.49)$$

Using $\beta = \alpha\bar{y}$ and $\bar{p} = 0$ along with $\alpha > 0$ in (4.42), we have,

$$\bar{y}^T [\nabla_y f(\bar{x}, \bar{y}) + \nabla_y{}^2 f(\bar{x}, \bar{y}).0] = 0 \qquad \ldots(4.50)$$

Consequently, we obviously have,

$$G(\bar{x},\bar{y},\bar{p}) = f(\bar{x}, \bar{y}) - \bar{y}^T [\nabla_y f(\bar{x}, \bar{y}) + \nabla_y{}^2 f(\bar{x}, \bar{y})\,\bar{p}\,] - \frac{1}{2}\,\bar{p}^T \nabla_y{}^2 f(\bar{x}, \bar{y})\,\bar{p}$$

$$= f(\bar{x},\bar{y}) - \bar{x}^T [\nabla_x f(\bar{x}, \bar{y}) + \nabla_x{}^2 f(\bar{x}, \bar{y})\,\bar{q}] - \frac{1}{2}\,\bar{q}^T \nabla_x{}^2 f(\bar{x}, \bar{y})\bar{q}$$

$$= H(\bar{x},\bar{y},\bar{q}).$$

That is, the objective values of (SP) and (SD) are equal. By Theorem (4.2.1), the optimality of $(\bar{x},\bar{y},\bar{z})$ for (SD) follows.

We will only state a converse duality theorem (Theorem 4.2.3) as the proof of this theorem would follow analogously to that of Theorem 4.2.2.

4.2.3 Converse Duality

Theorem 4.2.3: Let $(\bar{x}, \bar{y}, \bar{q})$ be an optimal solution of (SD). Also let

(i) the matrix $\nabla_x{}^2 f(\bar{x}, \bar{y})$ is nonsingular, and

(ii) $\nabla_x(\nabla_x{}^2 f(\bar{x}, \bar{y})\,\bar{q})$ be a positive definite.

Then $(\bar{x},\bar{y},\bar{p} = 0)$ is feasible for (SP) and the objective values of (SP) and (SD) are equal. Furthermore, if the hypothesis of Theorem (4.2.1) are met, then $(\bar{x},\bar{y},\bar{p})$ is an optimal solution of (SP).

4.2.4 Self Duality

Theorem 4.2.4: Let $f : R^n \times R^m \rightarrow R$ be skew symmetric and $C_1 = C_2$, then (SP) is self dual. Furthermore, if (SP) and (SD) are dual programs and $(\bar{x}, \bar{y}, \bar{s})$ is an optimal solution for (SP), then $(\bar{x},\bar{y},\bar{p} = 0)$ and $(\bar{y}, \bar{x}, \bar{q} = 0)$ are optimal solutions for (SP) and (SD), and

$$G(\bar{x},\bar{y},\bar{p}) = H(\bar{x},\bar{y},\bar{q}).$$

Proof: Recasting the problem (SD) as a minimization problem, we have

$(\mathbf{SD})_{\mathbf{1}}$:

Minimize $-\{f(x,y) - x^T [\nabla_x f(x,y) + \nabla_x{}^2 f(x,y)q] - \frac{1}{2} q^T \nabla_x{}^2 f(x,y)q\}$

Subject to

$$\nabla_x f(x, y) + \nabla_x{}^2 f(x, y)q \in C_1{}^*$$

$$(x, y) \in C_1 \times C_2$$

Since f is skew symmetric,

$$\nabla_x f(x,y) = -\nabla_y f(y,x) \text{ and } \nabla_x{}^2 f(x,y) = -\nabla_y{}^2 f(y,x);$$

and $C_1 = C_2$, the problem $(\mathbf{SD})_{\mathbf{1}}$ becomes,

Minimize $f(x, y) - x^T [\nabla_x f(x, y) + \nabla_x{}^2 f(x, y)q] - \frac{1}{2} q^T \nabla_x{}^2 f(x, y)q\}$

Subject to

$$-\nabla_y f(y, x) - \nabla_y{}^2 f(y, x) \in C_2{}^*$$

$$(x, y) \in C_1 \times C_2$$

which is just the primal problem (SP).Thus (SP) is self dual. Hence if $(\bar{x}, \bar{y}, \bar{q})$ is an optimal solution for (SP), then and conversely.

Also, $G(\bar{x},\bar{y},\bar{p}) = H(\bar{x},\bar{y},\bar{q})$.

Now we shall show that, $G(\bar{x},\bar{y},\bar{p}) = 0$.

$$G(\bar{x},\bar{y},\bar{p}) = f(\bar{x},\bar{y}) - \bar{y}^T[\nabla_y f(\bar{x},\bar{y})+\nabla_y{}^2 f(\bar{x},\bar{y})\,\bar{p}] - \frac{1}{2}\bar{p}^T\nabla_y{}^2 f(\bar{x},\bar{y})\bar{p} \quad ...(4.51)$$

Since $\bar{y} \epsilon C_2$ and $-\bar{y}^T\nabla_y f(\bar{x},\bar{y}) - \nabla_y{}^2 f(\bar{x},\bar{y})\,\bar{p} \epsilon C_2{}^* \geq 0$, therefore, we have

$$-[\bar{y}^T\nabla_y f(\bar{x},\bar{y}) + \nabla_y{}^2 f(\bar{x},\bar{y})\,\bar{p}] \geq 0. \quad ... (4.52)$$

Using (4.52) in (4.51), we have,

$$G(\bar{x},\bar{y},\bar{p}) \geq f(\bar{x},\bar{y}) - \frac{1}{2}\bar{p}^T\nabla_y{}^2 f(\bar{x},\bar{y})\bar{p}.$$

Using the conclusion $\bar{p} = 0$ of Theorem (4.2.2), we get

$$G(\bar{x},\bar{y},\bar{p}) \geq f(\bar{x},\bar{y}) \quad ...(4.53)$$

Similarly, in view of

$\bar{x} \epsilon C_1$ together with $\nabla_x f(\bar{x},\bar{y}) + \nabla_x{}^2 f(\bar{x},\bar{y})\,\bar{q} \epsilon C_1{}^*$, and $\bar{q} = 0$, we have,

$$H(\bar{x},\bar{y},\bar{q}) \leq f(\bar{x},\bar{y}) \quad ...(4.54)$$

By Theorem (4.2.2), we have,

$$f(\bar{x},\bar{y}) \leq G(\bar{x},\bar{y},\bar{p}) = H(\bar{x},\bar{y},\bar{q}) \leq f(\bar{x},\bar{y}).$$

This implies

$$G(\bar{x},\bar{y},\bar{p}) = H(\bar{x},\bar{y},\bar{q}) = f(\bar{x},\bar{y}) = f(y,x) = -f(x,y).$$

Consequently, we have,

$$G(\bar{x},\bar{y},\bar{p}) = 0.$$

BIBLIOGRAPHY

[1]. Abadie, J. nonlinear programming, North Holland Publishing Company, Amsterdam, (1967).

[2]. Avriel, M. Non linear programming. Analysis and Methods, Printice Hall, Englewood Cliffs, New Jersey, (1979).

[3]. Bazara, Sherali and Shetty; Nonlinear Programming-Theory and Algorithms (2^{nd} Ed.) John Wiley and Sons Inc.

[4]. Bector, C. R., and Chandra, S., "Generalized Bonvex Functions and Second Order Duality in Mathematical Programming," Research Report 85-2, Department of Actuarial and Management Sciences, The University of Manitoba, Winnipeg, January 1985.

[5]. Bector, C. R., and Chandra, S., "Second Order Symmetric and Self-Dual Programs," Opsearch 23(2), pp.89-95, 1986.

[6]. Chandra, S., Craven, B. D. and Mond, B., Generalized Concavity and Duality with a Square Root Term, Optimization, 16, (1985), 653-662.

[7]. Chandra, S., and Husain, I., Symmetric Dual Non-Differentiable Programs, Bull. Austral. Math. Soc., 24(1981), 259-307.

[8]. Chandra, S., and Prasad, D., "Symmetric Duality in Multiobjective Programming," Journal of Australian Mathematical Society, 35, pp. 198-206, 1993.

[9]. Charnes, A., and Cooper, W., "Programming with Linear Fractionals," Naval Research Logistics Quarterly, 9, pp. 181-186, 1962.

[10]. Chew, K. L., and Choo, E. U., "Pseudolinearity and Efficiency," Mathematical Programming, Vol.28, pp.226-239, 1984.

[11]. Choo, E. U., "Proper Efficiency and the Linear Fractional Vector Maximum Problem," Operations Research, Vol.32, pp.216-220, 1984.

[12]. Craven, B. D., "Lagrangian Conditions and Quasiduality," Bulletin of Australian Mathematical Society 16, pp.325-339, 1977.

[13]. Dantzig, G. B., "Maximization of Linear Function of Variables Subject to Linear Inequalities," Chapter XXI of Koopmans [31].

[14]. Dantzig, G. B., "Computational Algorithm of the Revised Simple Method," RAND Report RM-1266, The RAND Corporation, Santa Monica, Calif., October, 1953.

[15]. Dantzig, G. B., "Linear Programming and Extentions," Princeton University Press, Princeton, N.J., 1963.

[16]. Dantzig, G. B., "The Dual Simplex Algorithm, RAND Report RM-1276,The RAND Corporation, Santa Monica, Calif, October, 1954.

[17]. Dantzig, G. B., Eisenberg, E. and Cottle, R. W., "Symmetric Dual Nonlinear Programs," Pacific Journal of Mathematics 15, pp.809-812, 1965.

[18]. Dantzig, G. B., Ford, L. R. Jr. and Fulkerson, D. R., "A Primal-Dual Algorithm," RAND Report RM-1709, The RAND Corporation, Santa Monica, Calif., October, 1956.

[19]. Dantzig, G. B., Orden, A. and Wolfe. P., "Generalized Simplex Method for Minimizing a Linear Form under Linear Inequality Restraints," RAND Report RM-1264, The RAND Corporation, Santa Monica, Calif, October, 1954.

[20]. Dantzig, G. B. and Orden, A., "A Duality Theorem Based on The Simplex Method," pp. 51-55 of Directorate of Management Analysis.

[21]. Dantzig, G. B. and Orden, A., "Duality Theorems," RAND Report RM-1265, The RAND Corporation, Santa Monica, Calif, October, 1953.

[22]. Devi, G., Symmetric Duality for Nonlinear Programming Problem Involving η-Convex Functions, European Journal of Operational Research, 104, (1998), 615-621.

[23]. Dorn, W. S., "A Symmetric Dual Theorem for Quadratic Programs," Journal of the Operations Research Society of Japan 2, pp. 93-97, 1960.

[24]. Egudo, R. R., "Efficiency and Generalized Convex Duality for Multi-objective Program, J. M. A. A. 138, pp. 84-94, 1989.

[25]. Fritz John, Extremum problems with inequalities as subsidiary conditions. In “studies and Essays, Courant Anniversary Volume”. (K. O. Freidrichs, O. E. Nengebauer and J.J. Stoker. Eds.), Wiley (Interscience), New York,(1948),187-204.

[26]. Gale, D., Kuhn, H. W. and Tucker, A. W., "Linear Programming and The Theory of Games, Activity Analysis of Production and Allocation," John Wiley and Sons, New York, pp. 317-329, 1951.

[27]. Geoffiion, A. M., "Duality in Nonlinear Programming," A Simplified Applications Oriented Development SIAM Review, 13, pp. l-37, 1971 b.

[28]. Geoffrion, A. M., "Proper Efficiency and Theory of Vector

Maximization," Journal of Mathematical Analysis and Applications, 22, pp. 618-630, 1968.

[29]. Gulati, T. R. and Islam, M. A., "Efficiency and Proper Efficiency in Nonlinear Vector Maximum Problems," European Journal of Operations Research, Vol.44, pp. 378-382, 1990.

[30]. Gulati, T. R. and Islam, M. A., "Proper Efficiency in Linear Vector Maximum Problems with Nonlinear Constraints," Journal of Australian Mathematical Society, Series A, Vol.46, pp.229-235, 1989.

[31]. Gulati, T. R. and Nadia, Talaat, "Proper Efficiency in Convex Vector Minimum Problems, Journal of Information and Optimization Sciences, Vol.13. No.3, pp.437-444, 1992.

[32]. Gulati, T. R. and Talaat, N., "Duality in Nonconvex Multiobjective Programming," Asia-Pacific. J. Oper. Res.8, pp.62-69. 1991.

[33]. Gulati, T. R. Hussain, I. and Ahmed, A., "Multiobjective Symmetric Duality with Invexity," Bulletin of the Australian Mathematical Society 56, pp. 25-36, 1997.

[34]. Isermann, H., "Proper Efficiency and Linear Vector Maximum Problem," Operations Research 22, pp. 189-191, 1974.

[35]. Karush, W. “Minima of functions of Several Variables with Inequalities as Side Conditions”, M.S. Dissertation, Department of Mathematics, University of Chicago, (1939).

[36]. Koopmans, T. C. (Ed.), "Analysis of Production as an Efficient Combinations of Activities," in Analysis of Production and

Allocation," Cowels Commition Monograph 13, John Wiley and Sons, Inc, New York, pp. 33-97, 1951.

[37]. Kuhn, H. W. and Tucker, A. W., "Nonlinear Programming," in Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Edited by Neyman, University of California Press, Berkeley, California, pp. 481-493, 1951.

[38]. Kumar, P. and Bhatia, D., “A Note on Symmetric Duality for Multiobjective Nonlinear Programs, Opsearch 32(3), pp.172-183, 1995.

[39]. Mangasarian, O. L., "Nonlinear Programming" McGraw Hill, New York, 1969.

[40]. Mangasarian, O. L. and Formovitz, S., "The Fritz-John Necessary Optimality Conditions in the Presence of Equality and Inequality Constraints," J. Math. Anal. Appl., 17(1), pp. 37-47, 1967.

[41]. Mangasarian, O. L., "Second and Higher Order Duality in Nonlinear Programming," Journal of Mathematical Analysis and Applications 51, pp.607-620, 1975.

[42]. Mishra, S. K., Multi-objective Second-Order Symmetric Duality with Cone Constraints, European Journal of Operational Research, 126, (2000), 675-682.

[43]. Mond, B., "A Symmetric Dual Theorem for Nonlinear Programs, Quarterly Journal of Applied Mathematics 23, pp.265-269, 1965.

[44]. Mond, B., "Second Order Duality for Nonlinear Programs", Opsearch 11, pp. 90-99, 1974.

[45]. Mond, B., and Schechter, Non-differentiable Symmetric Duality, Bulletin of Australian Mathematical Society, 53, (1996) 177-188.

[46]. Mond, B., and Weir, T., "Generalized Concavity and Duality," in S. Schiable W. T. Ziemba (Eds), Generalized Concavity in Optimization and Economics, Academic Press, New York, 1981.

[47]. Mond, B., and Weir, T., Symmetric Duality for Nonlinear Programming, (Eds. Santosh Kumar, on the behalf of the Australian Society of Operations Research), Gorden and Breach Science Publisher, 137-153 1991.

[48]. Neumann, J. Von. On a Maximization Problem, Institute of Advance Study Princeton, New Jersey, (1947).

[49]. Pareto, V., "Coused Economics Politique Rouge, Lausanne, 1896.

[50]. Suneja, S. K., Lalitha, C. S., Seema Khurana, "Second Order Symmetric Duality in Multiobjective Programming," European Journal of Operations Research 144, pp. 492-500, 2003.

[51]. Taha, H. A., "Operations Research,"-An Introduction (VIth Ed.), Prentice Hall.

[52]. Weir, T., "Proper Efficiency and Duality for Vector Valued Optimization Problems," Department of Mathematics, Royal Military College, Duntroon (Australia), Research Report No. 12. 1985.

[53]. Weir, T. and Mond, B., "Generalized Convexity and Duality in Multiobjective Programming," Bulletin of Australian Mathematical Society, 39, pp. 287-299, 1989.

[54]. Weir, T. and Mond, B., "Symmetric and Self-Duality in Multiobjective Programming," Asia Pacific Journal of Operations Research 5 (2) pp.124-133, 1988.

[55]. Weir,T. and Mond, B., The Sufficient Fritz John Optimality Conditions and Duality for Non-linear Programming Problems, Opsearch, (1986), 23, No.3, 129-141.

[56]. Wolfe, P., "A Duality Theorem for Nonlinear Programming Problem," Quarterly Appl-Math 19(3), pp. 239-244, 1961.

[57]. Zangwill, W.I Non linear programming: a unified approach, Prentice Hall, Englewood cliffs, New Jersey, (1965).