RELIABILITY ANALYSIS OF DISCRETE LIFETIME DISTRIBUTIONS

DISSERTATION

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CERTIFICATE

This is to certify that the scholar **Mr. Adil Hamid Khan,** has carried out the present dissertation entitled **"Reliability Analysis of Discrete Lifetime Distributions"** under my supervision and the work is suitable for submission for the award of degree of Master of Philosophy in Statistics. It is further certified that the work has not been submitted in part or full for the award of M.Phil or any other degree.

Dr. Tariq Rashid Jan
Supervisor

Dedicated to…

My

Parents

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Preface

Deliberating on mathematical reliability theory, it is generally taken as set of ideas, mathematical models, and methods directed towards the solution of problems in predicting, estimating, or optimizing the probability of survival, mean life, or, more generally, life distribution of components or systems; other problems considered in reliability theory are those involving the probability of proper functioning of the system at either a specified or an arbitrary time, or the proportion of time the system is functioning properly. In a large class of reliability situations, maintenance, such as replacement, repair, or inspection, may be performed, so that the solution of the reliability problem may influence decisions concerning maintenance policies to be followed. The present dissertation is divided into five chapters and a comprehensive bibliography at the end.

Chapter I presents comprehensive survey and some basics of discrete lifetime probability distributions used in reliability for modeling discrete distributions. Distributions are classified into two families, the first class is constituted with discrete distributions derived from usual continuous lifetime distributions and the second class contains distributions based on a Pòlya urn scheme.

Chapter II reveals characterizations of some discrete distributions; using properties of the reversed hazard rate and reversed mean residual life are established. It is known that in the case of absolutely continuous distributions on the positive real line, no model with constant reversed hazard rate is in vogue. In this chapter discrete models are identified for which the reversed hazard rate is constant; the product of the reversed hazard rate and the reversed mean residual life is constant.

Chapter III deals with discrete versions of the additive Weibull distribution. The distribution has the twin virtues of mathematical tractability and the ability to produce bathtub-shaped hazard rate functions. Conditions on the parameters for the hazard rate function to be increasing, decreasing or bathtub shaped are derived. Results are illustrated using several real-life data sets.

Chapter IV presents Bayesian estimators for the reliability measures (the failure rate, reliability function and the mean time to failure) of the individual components in multi component systems. The life time of each component using masked system life test data is assumed to be geometric distribution. The problem is illustrated on a series system consisting of two components. At the end of this chapter numerical simulation study is given in order to explain masking level on the accuracy of point estimates.

Chapter V presents Bayesian estimation of reliability functions of discrete distributions like Consul, Geeta and Size-biased Geeta distributions. The prior distribution of parameter is considered as two parameter Beta distribution. Also reliability functions of Geometric, Negative-binomial and Haight distributions are also obtained as special cases.

CONTENTS

CHAPTER 1

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Basics

of

Discrete Reliability

1.1 Introduction

The mathematical theory of reliability has grown out of the demands of modern technology and particularly out of the experiences in World War II. With the increased complexity of component structure and the continuous requirements of the high quality reliable products, the importance of reliability has increased greatly. The high reliability is strictly required for the functionality of the system and safety of people using the products. It is believed that unreliable components and systems will cause inconvenience to the productivity in our daily lives. In even worse situations, any unstable component of a product can cause huge economic loss and serious damage to customers, producers, government and the society. The increased emphasis on reliability is also due to other factors worth considering, including the awareness of stability of high quality products, complexity and sophistication of systems, new industrial regulations concerning product liability, and product cost for testing, repairing and warranty. The theories and tools of reliability is applied in to widespread fields such as electronic and manufacturing products, aerospace equipment, earthquake and volcano forecasting, communication system, navigations and transportation control, medical treatments to the survival analysis of the human being or biological species and so on. Reliability has always been considered as one of the most important characteristics for industrial products and systems. Reliability engineering studies the life data analysis which deals with the specific issues to study and predict the lifetime of the products using statistical parametric distributions or non-parametric methods, and subsequently these methodologies and results can be applied to product testing and prediction, optimization of warranty policy and quality and reliability enhancement. The introduction of every new device must be accompanied by provision for maintenance, repair parts, and protection against failures. This is certainly apparent to the military, where the life-cycle maintenance costs of systems far exceeds the original purchase costs. Maintenance of units after failure may be costly, and sometimes requires a long time corrective maintenance of the failed units. The most important problem is to determine the maintenance of system before failure. However, it is not wise to maintain units with unnecessary frequency. From this viewpoint, the commonly considered maintenance policies are preventive replacement for units without repair and preventive maintenance for units with repair on a specific schedule. Consequently, the object of maintenance optimization problems is to determine the frequency and timing of corrective maintenance, preventive replacement or preventive maintenance according to cost and effect. Units under age replacement and preventive maintenance are replaced or repaired at failure, or at a planned time after installation. Units under periodic and block replacements are replaced at periodic times, and undergo repair or replacement between planned replacements.

Lots of research and applications have been carried out in order to explore and understand the methodologies and applications of reliability analysis for the product enhancements and many researchers have investigated statistically and stochastically complex phenomena of real systems to improve their reliability. In the early 1950's certain areas of reliability, especially life testing and electronic and missile reliability problems, started to receive a great deal of attention both from mathematical statisticians and from the engineers in the military-industrial complex. Among the first groups to face up seriously to the problem of tube reliability were the commercial airlines. Accordingly, the airlines set up an organization called Aeronautical Radio, Inc. (ARINC) which, among other functions, collected and analyzed defective tubes and returned them to the tube manufacturer. In its years of operation with the airlines, ARINC achieved notable success in improving the reliability of a number of tube types. The ARINC program since 1950 has been focused on military reliability problems. In December 1950 the Air Force formed an adhoc Group on Reliability of Electronic Equipment to study the whole reliability situation and recommend measures that would increase the reliability of equipment and reduce maintenance.

In 1951 Epstein and Sobel initiated this work in the field of life testing which was to result in a long stream of important and extremely influential papers. This work marked the beginning of the widespread assumption of the exponential distribution in life-testing research. At that time in the missile industry Robert Lusser, Richard R. Carhart, and others were also active in promoting interest in reliability and stating the problems of most interest to their technology. According to one source, the basic definition of reliability (as used by engineers) was first presented by Robert Lusser.

Davis [1952] published a paper presenting failure data and the results of several goodness-of-fit tests for various competing failure distributions. This data seemed to give a distinct edge to the exponential distribution, and for this reason the Davis paper has been widely referred to in support of the assumption of an exponential failure distribution. With the publication of this paper and the Epstein-Sobel paper [1953], the exponential distribution acquired a unique position in life testing. This position became even more secure in 1957 with the AGREE report. A fundamental reason for the popularity of the exponential distribution and its widespread exploitation in reliability work is that it leads to simple addition of failure rates and makes possible the compilation of design data in a simple form. However, in 1955 serious consideration began to be given to other life distributions. Kao [1956, 1958], among others, was influential in bringing attention to the Weibull distribution. This interest in the Weibull was to grow ever stronger until it gained major importance with the publication of the Zelen Dannemiller [1961] paper pointing out that many life test procedures based on the exponential are not robust.

If one takes a look at the first works on reliability of the end of 50s and of the beginning of 60s, he could see pure pragmatic nature of those works. Even “pure mathematicians” wrote for users rather than for themselves: their results were transparent and their applicability was evident. However, in the middle of 70s there appeared papers considering unrealistic models, math results began to be non-understandable with no commonsense interpretation. That situation led to definite discredit of reliability theory as a whole. This situation was expressed by one of leading specialist in reliability engineering:"The reliability theory is for those who understand nothing in reliability. Those who understand reliability, they design and produce reliable equipment!"

The somewhat classical statistical treatment of reliability based only on the reliability function is not, at all, sufficient to handle the reliability assessment processes of complex and multiphase 21st century problems. These complexities rarely lead to the rise of a static coherent system. Therefore, one must develop a methodology that can integrate other related information and be able to propagate information up and down throughout the system representation.

1.2 Some Definitions

Reliability is the **probability** of performing without **failure**, a specific **function** under given **condition** for a specified period of **time**.

This definition includes five elements:

1) **Probability**: Reliability is a probability, a probability of performing without failure; thus, reliability is a number between zero and one.

2) **Failure**: What constitutes a failure must be agreed upon in advance of the testing and use of the component or system under study. For example if the function of a pump is to deliver at least 200 gallons of fluid per minute and it is now delivering 150 gallons/per minute, the pump has failed, by this definition.

3) **Function**: The device whose reliability is in question must perform a specific function. For example, if I use my gasoline-powered lawn mower to trim my hedges and a blade breaks, this should not be charged as a failure.

4) **Conditions**: The device must perform its function under given conditions. For example, if my company builds and sells small gasoline-powered electrical generators intended for use in ambient temperatures of 0-120 degrees Fahrenheit and

several are brought to Nome, Alaska and fail to operate in the winter, we should not charge failures to these units.

5) **Time**: The device must perform for a period of time. One should never cite a reliability figure without specifying the time in question. The exception to this rule is for one-shot devices such as munitions, rockets, automobile air-bags, and the like. In this case one think of the reliability as the probability that the device will operate properly when deployed or used. Or equivalently one-shot reliability may be thought of as the proportion of all identical devices which will operate properly (once) when deployed or used. In reliability, unless otherwise specified, time begins at zero.

The elements 2, 3 and 4 are important to the reliability of a device, but they differ in different situations; elements 1 and 5 are more basic. Since reliability is a probability, thus the probability element of reliability allows one to calculate reliabilities in a quantitative way, that is, the assessment of reliability can be done probabilistically so that the quantity given to the reliability has the meaning and structure of probability for its manipulation and interpretation.

The time element is also basic in reliability. In fact, the basic distinction between reliability and quality control is related to this element. In this way of comparing reliability and quality control, quality control studies failure at a given time whereas reliability studies failure over time.

In a sense, this comparison introduces a new definition of reliability, that is, a study of failure over time. Also the term failure is introduced and to be consistent, it is important to define failure. Thus, a failure is defined as any functioning of the device or component which is not considered within the prescribed limits of satisfactory functioning.

Since the element time is so basic to reliability, it is quite natural then, that the primary random variable in reliability studies is time and that the purpose of such studies is often life length. When this emphasis on life length is the focus of a reliability study, the study is often referred to as a life test and this terminology is often used to describe the reliability study.

1.3 Reliability Function and Failure Rate

Let variable T be the lifetime or time to failure of a component having probability density function (p.d.f) $f(t)$ and distribution function $F(t)$. The probability that the component survives beyond sometime t is called the reliability $R(t)$ of the component. Thus,

$$R(t) = 1 - F(t) = P(T > t), \qquad t > 0 \qquad (1.3.1)$$

Although the term "reliability" has many technical meanings, the above used is becoming more commonly accepted. The definition given here simply says that the reliability of a component equals the probability that the component does not fail during the interval [0,t] (or equivalently, reliability equals the probability that the component is still functioning at time t).

In other words, the component is assumed to be working properly at time $t = 0$ means $R(0) = 1$ and no component can work forever without failure means $R(\infty) = \lim_{t\to\infty} R(t) = 0$. $R(t)$ is a monotone non-increasing function of t and has no meaning for $t < 0$. $F(t)$ is called unreliability.

The probability that a component will fail in the interval $(t, t + \Delta t)$given that the component is working at time t is:

$$P(t < T \leq t + \Delta t / T > t) = \frac{P(t < T \leq t + \Delta t)}{P(T > t)}$$

$$= \frac{F(t + \Delta t) - F(t)}{R(t)} \qquad (1.3.2)$$

By dividing this probability by the length of the length of the time interval Δt and letting $\Delta t \to 0$, the failure rate (hazard) function $h(t)$ at time t is

$$h(t) = \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} \cdot \frac{1}{R(t)} \qquad (1.3.3)$$

A failure rate can be classified as increasing failure rate (IFR) or decreasing failure rate (DFR).

Since,

$$f(t) = \frac{d}{dt}F(t) = -R'(t) \quad (1.3.4)$$

then

$$h(t) = -\frac{R'(t)}{R(t)} = -\frac{d}{dt}\log R(t) \quad (1.3.5)$$

using $R(0) = 1$, we have,

$$R(t) = \exp\left(-\int_0^t h(x)dx\right) \quad (1.3.6)$$

For some purposes it is also useful to define the cumulative hazard function

$$H(t) = \int_0^t h(x)dx$$

which by (1.3.6), is related to the survivor function by

$$R(t) = \exp[-H(t)]$$

It can be observed that since $R(\infty) = 0$, then

$$H(\infty) = \lim_{t\to\infty} H(t) = \infty$$

Finally, in addition to (1.3.6), it follows immediately from (1.3.4) and (1.3.5) that

$$f(t) = h(t)\exp\left(-\int_0^t h(x)dx\right) \quad (1.3.7)$$

From the above concepts and formulae, the reliability function $R(t)$ and the distribution function $F(t) = 1 - R(t)$ are uniquely determined by the failure rate function $h(t)$. Also the relationship between the functions $F(t)$, $f(t)$, $R(t)$ and $h(t)$ as given below:

Expressed by	$F(t)$	$f(t)$	$R(t)$	$h(t)$
$F(t) =$	$-$	$\int_0^t f(u)du$	$1 - R(t)$	$1 - \exp\left(-\int_0^t h(u)du\right)$
$f(t) =$	$\frac{d}{dt}F(t)$	$-$	$-\frac{d}{dt}R(t)$	$h(t)\exp\left(-\int_0^t h(u)du\right)$
$R(t) =$	$1 - F(t)$	$\int_t^\infty f(u)du$	$-$	$\exp\left(-\int_0^t h(u)du\right)$
$h(t) =$	$\frac{dF(t)/dt}{1 - F(t)}$	$\frac{f(t)}{\int_t^\infty f(u)du}$	$-\frac{d}{dt}\log R(t)$	$-$

1.4 Shapes of Hazard Functions

The pdf (or pf), the distribution and survivor functions are common representations of a probability distribution, but hazard functions function is particularly useful with lifetime distributions, since it describes the way in which the instantaneous probability of death for an individual changes with time. Often, in applications, there may be qualitative information about the hazard function, which can help in selecting a life distribution model. For example, there may be reasons to restrict consideration to models with non-decreasing hazard functions or with hazard functions having some other well-defined characteristic.

The failure rate function is an important concept in reliability. Failure rate functions often falling into one of three categories are considered: (a) monotonic failure rates, where the failure rate curve is either increasing or decreasing; (b) bathtub failure rates, where the curve has a bathtub or a U shape; and (c) generalized bathtub failure rates, where the failure rate curve is a polynomial, or has roller-coaster shape or some other generalization. Many lifetime distributions may be categorized with respect to the shape of their failure rate functions:

Let $r(t)$ be the failure rate function of a lifetime distribution. It is

(i) an IFR (increasing failure rate) distribution if $r(t)$ is nondecreasing in t;

(ii) a DFR (decreasing failure rate) distribution if $r(t)$ is nonincreasing in t;

(iii) a BT (bathtub-shaped) distribution if there exists a $t_0 > 0$ such that $r(t)$ is non-increasing for $0 \leq t \leq t_0$ and non-decreasing for $t \geq t_0$;

(iv) an UBT (upside-down bathtub-shaped) distribution if there exists a $t_0 > 0$ such that $r(t)$ is non-decreasing for $0 \leq t \leq t_0$ and non-increasing for $t \geq t_0$.

Fig. 1.1 shows hazards functions and pdf's for three continuous distributions. The shapes of the hazard functions are qualitatively quite different; distribution (a) has a monotone increasing hazard function, distribution (b) has a monotone decreasing function, and (c) has a so-called "bathtub –shaped", or U-shaped hazard function. Model with these and other shaped of hazard function are all useful in practice. If, for example, individuals in a population are followed right from actual birth to death, a bathtub shaped hazard function is often appropriate. We are, for example, familiar with this pattern in human populations: after an initial period in which deaths result primarily from birth defects or infant diseases, the death rate drops and is relatively constant until the age of 30 or so, after which is increased with age. This pattern also manifests itself in many other populations, including ones consisting of manufactured items.

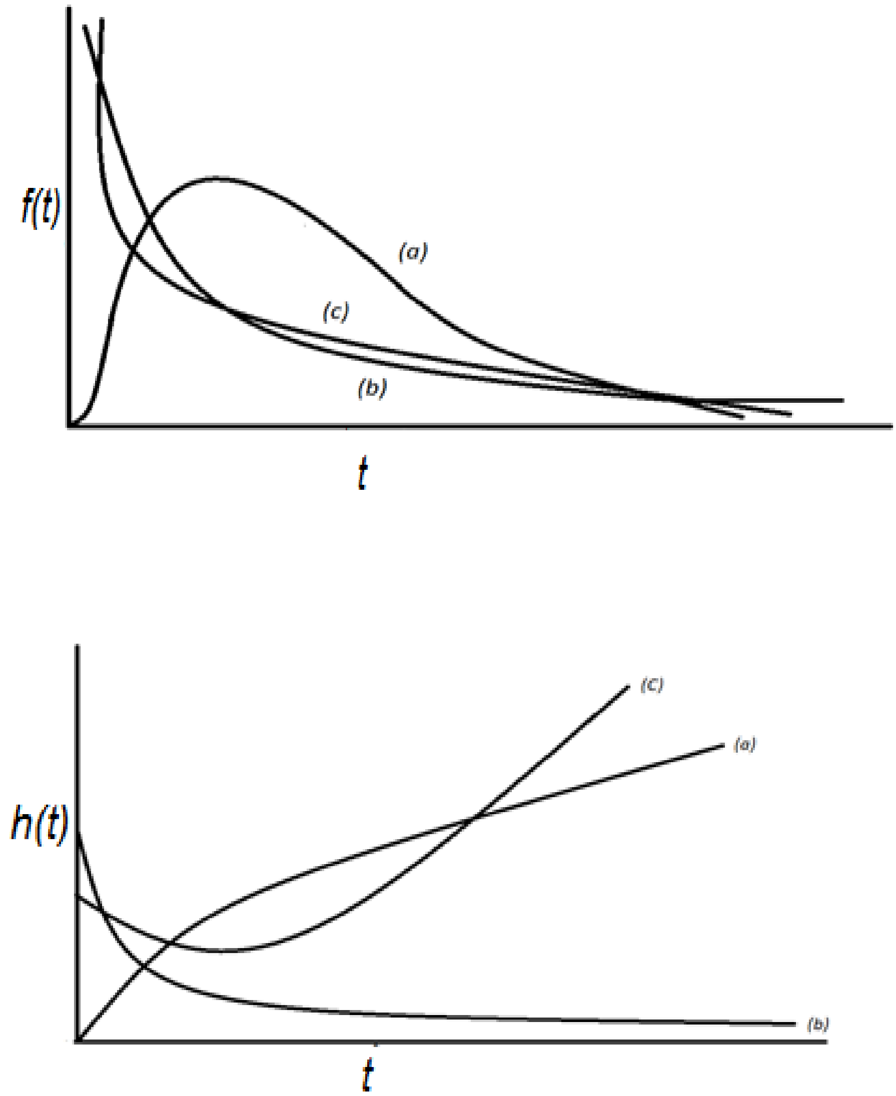

Fig.1.1 Some Hazard and probability functions

Models with increasing hazard functions are used the most. One reason for this is that interest often centers on a period in the life of an individual over which some kind of gradual ageing takes place, yielding an increasing hazard functions. Also, populations that display a bath tub-shaped hazard function are sometimes purged of weak individuals, leaving a reduced population with an increasing hazard function. For example, manufacturers often use “burn-in” process in which items are subjected to a brief period of operation before being sent to customers. In this way defective items that would fail very early are removed from the population; this frequently leaves a residual population in which individuals exhibit gradual ageing, with an increasing hazard function.

Models with a constant hazard function are important and have particularly simple structure and models with decreasing hazard functions are less common, but are sometimes used. For example, certain types of electronic devices appear to have decreasing failure rate, at least over some fairly long initial period of use. Non-monotone hazard functions other than bathtub-shaped once are even less common, but possible. All in all, the main point to be remembered is that the hazard function represents an aspect of a distribution that has direct physical meaning and that information about the nature of the hazard function is helpful in selecting a model.

1.5 Life Testing

Life testing is concerned with measuring the pertinent characteristics of the life of the unit under study. Often this is accomplished by making statistical inferences about probability distributions or their parameters.

In general, units are put on test, observed and the times of failure recorded as they occur. For example, a group of similar components are placed on test and the failure times observed. Obviously, the times at which individual units fail will vary. Sometimes, assignable causes can be found that contribute to that variation. Suppose some components have been subjected to testing at a high temperature environment and it is possible that such components will fail sooner than those tested at an ambient temperature environment. However, the components at the high temperature will still have different failure times; and, if there are no assignable causes in operation, these components will still have different failure times, that is, it is always assumed that the failure times of the components have some random elements and will be assumed to be a random variable with a probability distribution.

To make statistical inferences about the probability distribution of the failure time random variable, one uses the failure times that have been observed from a life test, ideally a test that has been statistically designed for the purpose of the study. If the failure times of a particular component under a given set of conditions, can be adequately described by a probability distribution, there are considerable practical benefits. The failure times can then be used to estimate the parameters of the distribution and to perhaps study the relationship of these parameters to associated explanatory variables. The estimates can be used to make predictions, determine component configurations in systems, determine replacement procedures, specify guarantee periods and make other decisions about the use of the component.

1.6 Failure Times

Before a study of the effects of a group of failure times is begun, it must be determined precisely what these data values involve. There must be agreement among participating parties about certain characteristics of the failure data. That is, the start of the time measurement, the scale of the time measurement and the definition of a failure are not always consistent in life test situations and must be precisely specified in a given study.

The time origin in some studies is obvious. In some other studies, however, there is enough confusion about the origin of time measurements that some agreement as to the origin must be reached before the study begins. For example, in some studies the unit under test may have under gone earlier testing in development studies and some agreement must be reached as to whether to include the earlier times on test as running times for the present study. The same is true of the time scale. Usually the scale is clock time but other measures may also be used, such as the number of cycles, the mileage to the first puncture of a tire, etc.

There may also be differing definitions of what constitutes a failure. It is important that one definition be specified or that different modes of failure be recognized and allowed as failures. It is usually informative in the data analysis if the differing modes of failure are distinguished and recorded in the test results. For many components, failure is catastrophic and the definition of a failure is obvious. But for some components, the performance slowly degrades and the amount of degradation to be judged a failure must be defined.

1.7 Censoring of Data

One of the circumstances that has traditionally caused concern and some difficulty in statistical studies has been the occurrence of missing observations.

Although techniques have been proposed for accommodating missing observations in most types of statistical analyses, the problem of missing or incomplete observations in general does not seem to occur as often as in modern reliability studies. With highly reliable components, it is unusual if all the components have failed by the end of the time allotted for the test. In human survival studies and in some engineering studies, some of the units on test may be withdrawn from the test for various reasons. Such incomplete data observations in reliability studies are called censored items. Although the failure time information on such an item is incomplete, there is usually still some information in the time data that is available in the item and so the censoring time should always be recorded in a study.

Censoring is often distinguished according to type and order. The type of censoring reflects the rule for censoring and influences which variables in the study that are random. A consideration of which variables are random affects the distributional assumptions of estimates and will be discussed later.

Type I censoring is the rule that specifies that the testing is terminated at a specific, fixed time t_c. In this case, the time t_c is a fixed value and the number of units which are censored in a study is a random variable. Type I censoring is the most common type of censoring used in practice because it is the easiest to implement since the duration of the study is determined and fixed beforehand. However, it is not the most convenient in terms of the distributional considerations.

Type II censoring is the rule that specifies that the testing is terminated when a preset number of units, say r, have failed. In the case of Type II censoring the time at which the test is stopped is a random variable, that is, the time at which the rth failure occurred. This type of censoring is less practical because it does not allow an upper bound on the total time duration. It does, however, result in a more convenient theory.

The order of censoring indicates whether there is a single or there are multiple rules for censoring in a test. Multiply censored data are made up of failure times and a mixture of censored times.

More generally, for the ith unit from a sample of n on life test, one could record the observation (x_i, d_i), where x_i is the failure time if the indicator variable

$d_i = 1$ and x_i is the censored time if $d_i = 0$. In Type I censoring, all the x_i values are equal to t_c when $d_i = 0$ and when $d_i = 1$, the x_i values have the values t_i which are observations of the random failure variable T. in type II censoring, the censoring time is a random variable, the rth order statistics $T_{(r)}$, if the test is stopped at the time of rth failure. The (x_i, d_i) notation can handle multiple censoring also, and will be particularly useful in maximum likelihood derivations of estimators.

It is important that the censoring mechanism remains independent of the failure mechanism. It would be impossible to obtain meaningful data if units were censored when they appeared to have high probability of failure at the time of censoring. Any unit censored at the time t_c should be representative of all the units under the same test conditions at time t_c.

1.8 A Survey on Discrete Lifetime Distributions

The literature on the reliability theory mainly deals with the non-negative absolutely continuous random variables. However, quite often we come across with situations where product life can be described through non-negative integer valued random variable. In these situations, system lifetime is discrete random variable. Therefore one needs to develop tools, analogous to the continuous case, for studying the discrete failure data. In particular, discrete analogous of usual distributions for continuous lifetimes, such as, the exponential or Weibull distributions, have to be defined. It is well known that the geometric distribution is the discrete counterpart of the exponential distribution, but it is not so easy for Weibull, since at least three distributions are known as “discrete Weibull distributions”. Moreover, discrete lifetime distributions can be defined without any continuous counterpart.

So far little work has been done in discrete reliability. Several discrete lifetime distributions have been proposed, but the links between them have not been studied. In this chapter a brief survey of discrete lifetime distributions given by Cyril Bracquenmond and Olivier Gaudion [2003] is presented, which can also be understood as discrete non repairable system reliability models.

1.9 Basic Discrete Reliability Concepts

It is assumed that a discrete lifetime is the number K of system demands until the first failure. Then, K is a random variable defined over the set $N^*$of positive integers.

The probability function and cumulative distribution function (CDF) of K are respectively defined as $p(k) = P(K = k)$ and $F(k) = P(K \leq k) \sum_{i=1}^{k} p(i), \forall k \in N^*$. The basic discrete reliability concepts are defined hereafter.

(i) The reliability is:

$\forall k \in N^*, \quad R(k) = P(K > k) = 1 - F(k) = 1 - \sum_{i=1}^{k} p(i)$

(ii) The mean residual life is:

$\forall k \in N^*, \quad m(k) = E(K - k/K > k)$

(iii) The Mean Time To Failure is(if the series converges):

$$MTTF = E(K) = m(0) = \sum_{i=1}^{\infty} ip(i)$$

(iv) The failure rate is:

$$\forall k \in N^*, \lambda(k) = P(K = k)/K \geq k) = \frac{P(K = k)}{P(K \geq k)} = \frac{p(k)}{R(k-1)}$$

Barlow, Marshall and Proschan [1963] defined failure rate (or hazard rate) as the conditional probability of failure of the system at times k, given that it did not fail before. In discrete time, $\lambda(k) \leq 1$, while the usual failure rate in continuous time is not bounded. Since a failure rate determines completely a lifetime distribution, Shaked, Shanthikumar and Valdez-Torres [1995] gave necessary and sufficient conditions for a sequence $\{\lambda(k)\}_{k \geq 1}$ to be a failure rate:

(i) $\exists m \in N^*, \forall i < m, \lambda(i) < 1 \ and \ \lambda(m) = 1$

or

(ii) $\forall k \in N^*, \lambda(k) \in [0,1[$ and $\sum_{k=1}^{\infty} \lambda(k) = +\infty$

The distribution is defined over $\{1, \dots, m\}$ in case (i), and over N^* in case (ii).

The sense of variation of the failure rate is of major concern since it indicates system wear-out (IFR: Increasing Failure Rate) or burn-in (DFR: Decreasing Failure Rate). According to Barlow and Proschan [1975], determining the failure rate monotonicity is often easy when its expression is given and when the expression is not given in that case it is usual in continuous time to look at the log-concavity or log-convexity of the distribution. Analogous statements for discrete distributions with unbounded support $(\forall k \in N^*, p(k) \neq 0)$ were proposed by Gupta, Gupta and Tripathi [1997]:

(i) The distribution is log-concave if and only if $\left\{\frac{p(k+1)}{p(k)}\right\}_{k\geq 1}$ is decreasing. Then the failure rate is increasing (IFR).

(ii) The distribution is log-concave if and only if $\left\{\frac{p(k+1)}{p(k)}\right\}_{k\geq 1}$ is increasing. Then the failure rate is decreasing (DFR).

(iii) If the sequence $\left\{\frac{p(k+1)}{p(k)}\right\}_{k\geq 1}$ is constant, the failure rate is constant and the distribution is geometric.

Distributions can be classified into two families. The first class is constituted with discrete distributions derived from usual continuous lifetime distributions and the second class contains distributions based on a Pòlya urn scheme.

1.10 Discrete Lifetime Distribution Derived from Continuous ones

There are several ways to derive discrete lifetime distributions from continuous ones. The first possibility is to consider characteristic property of a continuous distribution and to build the similar property in discrete time. The second one is to consider discrete lifetime as the integer part of continuous lifetime.

1.10.1 *Geometric Distribution*

The geometric distribution is the analogous in discrete time of exponential distribution, since it has the lack of memory property (no ageing, no burn-in): the system failure probabilities on each demand are independent and all equal to $p \in]0,1[$. Equivalently, the failure rate is constant. This property can also be reformulated as:

$$\forall(i,j) \in N^{*2}, P\,(K > i + k)/K > i) = P(K > k)$$

The geometric distribution $\mathcal{g}(p)$ is defined by:

(i) $P(k) = p(1-p)^{k-1}$

(ii) $R(k) = (1-p)^k$

(iii) $\lambda(k) = p$

The Mean Time To Failure is MTTF= $\frac{1}{p}$.

1.10.2 *Shifted Negative Binomial Distribution*

Assume that $K_1, K_2, \dots, K_r$ are independent random variables from a geometric distribution with parameter p. Then $\sum_{i=1}^{r} K_i$ has a negative binomial distribution with parameters r and p, $BN(r,p)$. So the negative binomial distribution is the analogous in discrete time of the Gamma distribution. In order to obtain a random variable defined over N^*, we have to shift the $BN(r,p)$ distribution by setting $K = X - r + 1$ where, X has the $\mathcal{BN}(r,p)$ distribution.

The shifted negative binomial distribution $BN(r,p)$ is defined by:

(i) $P(k) = \binom{r-1}{k+r-2} p^r (1-p)^{k-1}$

(ii) $R(k) = 1 - \sum_{i=1}^{k-1} \binom{r-1}{k+r-2} p^r (1-p)^{k-1}$

(iii) $\lambda(k) = \frac{\binom{r-1}{k+r-2} p^r (1-p)^{k-1}}{R(k-1)}$

$\frac{p(k+1)}{p(k)} = \left(1 + \frac{r-1}{k}\right)(1-p)$ is a decreasing function of k, so that the distribution is log-concave and the failure rate is increasing. For $r = 1$, the distribution reduces to geometric distribution.

The Mean Time To Failure is $\frac{r(1-p)}{p} + 1$. Due to the shifting, parameter r and p have no practical interpretation.

1.10.3 *Type I Discrete Weibull Distribution*

Nakagawa and Osaki [1975] were the first to propose a specific discrete lifetime distribution which is defined to correspond with the Weibull distribution in continuous time. It was the first time a probability distribution was specifically defined to be a discrete lifetime distribution. The model is based on the similarity of expression of the reliability between discrete and continuous time. It T has a continuous Weibull distribution $\mathcal{W}(\eta, \beta)$, then $R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}$. A similar expression for the reliability in discrete time is $R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}$ or equivalently $R(t) = q^{k^{\beta}}$, where $\beta \in R^{+*}$ and $q \in]0,1[$.

Thus, the type first Weibull distribution $\mathcal{W}_1(q, \beta)$ is defined by:

(i) $P(k) = q^{(k-1)^{\beta}} - q^{k^{\beta}}$

(ii) $R(k) = q^{k^{\beta}}$

(iii) $\lambda(k) = 1 - q^{(k-1)^{\beta}}$

q is the probability of surviving the first demand. As for the continuous distribution, β is shape parameter: the distribution is IFR for $\beta > 1$, DFR for $0 < \beta < 1$, and for $\beta = 1$, it reduce to geometric distribution.

1.10.4 *Type II Discrete Weibull Distribution*

Stein and Dattero [1984] introduced another Weibull (II) distribution in which they showed a connection to the famous birthday problem and to the lifetime of a series system of components. It T has a continuous Weibull distribution $\mathcal{W}(\eta, \beta)$, then $\lambda(t) = \left(\frac{t}{\eta}\right)^{\beta-1}$. A similar expression for discrete time is $\lambda(k) = \left(\frac{k}{\eta}\right)^{\beta-1}$ with $\eta \in R^{+*}$ and $\beta \in R^{+*}$.

But in discrete time $\lambda(k) \leq 1$, then k has to be less than η. So this distribution has the bounded support. In order to check the conditions present in section 2 for λ to be failure rate, η has to be an integer. It is more usually denoted by m.

Then, the type II discrete Weibull distribution $\mathcal{W}_2(\mathrm{m},\beta)$ with support in $\{1,2,\dots,\mathrm{m}\}$ can be redefined as:

(i) $\mathrm{P(k)} = \left(\frac{k}{m}\right)^{\beta-1} \prod_{i=1}^{k-1}\left[1-\left(\frac{i}{m}\right)^{\beta-1}\right]$ for $k \in \{1,2,\dots,m\}$

(ii) $\mathrm{R(k)} = \prod_{i=1}^{\inf(k,m)}\left[1-\left(\frac{i}{m}\right)^{\beta-1}\right]$

(iii) $\lambda(k) = \left(\frac{k}{m}\right)^{\beta-1}$ for $k \in \{1,2,\dots,m\}$

m is the maximum lifetime of the system and β is shape parameter.

1.10.5 *Type III Discrete Weibull Distribution*

Padgett and Spurrier [1985] provided three families of discrete parametric distributions which are versatile in fitting increasing, decreasing and constant failure rate models to either uncensored or right-censored discrete life-test data with respect to the choice of a shape parameter, analogous to the Weibull distribution in the continuous case. The type III discrete Weibull distribution $\mathcal{W}_3(\mathrm{c},\beta)$ is defined for $c \in R^{+*}$ and $\beta \in R$ by:

(i) $\mathrm{P(k)} = (1-e^{-ck^{\beta}})e^{-c\sum_{i=1}^{k-1} i^{\beta}}$

(ii) $\mathrm{R(k)} = e^{-c\sum_{j=1}^{k} j^{\beta}}$

(iii) $\lambda(k) = 1-e^{-ck^{\beta}}$

The monotonicity of the failure rate depends on the value of the shape parameter β:

(i) For $\beta = 0$, the distribution reduces to geometric distribution.

(ii) For $\beta > 0$, the distribution is IFR.

(iii) For $\beta < 0$, the distribution is DFR.

c is linked with the probability of failure at the first demand since $\mathrm{P}(1) = 1-e^{-c}$.

1.10.6 "*s*" *Distribution*

"*s*" distribution was introduced by Soler [1996] to describe the continuous lifetimes of systems subjected to random stress. Cyril Bracquenmond and Olivier

Gaudion propose the analogous in discrete time. Consider a system such that, on each demand, a shock can occur with probability p and not occur with probability $1 - \mathrm{p}$. It is natural to assume that the failure rate at the $\mathrm{k^{th}}$ demand, conditionally to the shock sequence, is an increasing function of the number $\mathrm{N_k}$ of shocks occurred at that time. One way of taking this assumption into account is to set:

$$\forall \mathrm{k} \in \mathrm{N}^*, \lambda_{\mathrm{N}}(\mathrm{k}) = \mathrm{P}\left(\mathrm{K} = \mathrm{k}\right) / \mathrm{K} \geq \mathrm{k}, \left\{\mathrm{N_j}\right\}_{\mathrm{j} \geq 1}) = 1 - \pi^{\mathrm{N_k}}, \text{with } \pi \in]0,1[.$$

Conditionally to the shock sequence, the reliability is:

$$\begin{aligned}
\mathrm{R}(\mathrm{k}) = \mathrm{P}(\mathrm{K} > k) = \mathrm{E}\left[\mathrm{P}(\mathrm{K} > k) / \left\{\mathrm{N_j}\right\}_{\mathrm{j} \geq 1}\right] \\
= \mathrm{E}\left[\prod_{\mathrm{i}=1}^{\mathrm{k}} \left(1 - P\left[K = i / \mathrm{K} \geq \mathrm{i}, \left\{\mathrm{N_j}\right\}_{\mathrm{j} \geq 1}\right]\right)\right] \\
= \mathrm{E}\left[\prod_{\mathrm{i}=1}^{\mathrm{k}} \pi^{\mathrm{N_i}}\right] \\
= \mathrm{E}\left[\prod_{\mathrm{i}=1}^{\mathrm{k}} \pi^{\sum_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{N_i}}\right]
\end{aligned}$$

The random variables $\mathrm{N_i}$ are not independent. $\forall \mathrm{i} \geq 1$, let $\mathrm{U_i} = \mathrm{N_i} - \mathrm{N_{i-1}}$. The $\mathrm{U_i}$'s are independent and have the Bernoulli distribution $B(\mathrm{p})$, describing the occurrence of the shock at each demand. Then,

$$\sum_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{N_i} = \sum_{\mathrm{i}=1}^{\mathrm{k}} (\mathrm{k} - \mathrm{i} + 1)\mathrm{U_i}$$

So, the reliability becomes:

$$\mathrm{R}(\mathrm{k}) = \mathrm{E}\left[\prod_{\mathrm{i}=1}^{\mathrm{k}} \pi^{\sum_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{N_i}}\right] = \mathrm{E}\left[\prod_{\mathrm{i}=1}^{\mathrm{k}} \pi^{\sum_{\mathrm{i}=1}^{\mathrm{k}} (\mathrm{k}-\mathrm{i}+1)\mathrm{U_i}}\right]$$

$$= \prod_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{E}\left[\prod_{\mathrm{i}=1}^{\mathrm{k}} \pi^{(\mathrm{k}-\mathrm{i}+1)\mathrm{U_i}}\right] = \prod_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{G}\left(\pi^{\mathrm{k}-\mathrm{i}+1}\right)$$

where is the probability generating function of the $B(\mathrm{p})$ distribution: $\mathrm{G}(\mathrm{u}) = \mathrm{pu} + 1 - \mathrm{p}$.

Finally, the $S(\mathrm{p}, \pi)$ distribution is defined by:

(i) $\mathrm{P(k)} = \mathrm{p}(1-\pi^{\mathrm{k}}) \prod_{\mathrm{i}=1}^{\mathrm{k}} (1-\mathrm{p}+\mathrm{p}\pi^{\mathrm{i}})$

(ii) $\mathrm{R(k)} = \prod_{\mathrm{i}=1}^{\mathrm{k}} (1-\mathrm{p}+\mathrm{p}\pi^{\mathrm{i}})$

(iii) $\lambda(\mathrm{k}) = \mathrm{p}(1-\pi^{\mathrm{k}})$

p is the probability that the shock occurs on demand and π is the probability of surviving the first demand given that a shock has occurred.

If the shock occurs at each demand, then $\mathrm{p} = 1$ and a very simple expression of the failure rate is obtained

$$\lambda(\mathrm{k}) = 1 - \pi^{\mathrm{k}}$$

This is a particular case of type III discrete Weibull distribution with $\beta = 1$ and $\mathrm{c} = -\ln\pi$.

1.11 Discrete Distributions Derived from Continuous ones by Time Discretization

Let T be a real positive random variable describing a system lifetime in continuous time. Let $\mathrm{K} = \lfloor \mathrm{T} \rfloor + 1$ (where, $\lfloor\ \rfloor$ is the integer part). K is the random variable defined over N^*. Let λ_{K}, $\mathrm{F_K}$, $\mathrm{R_K}$ and λ_{T}, $\mathrm{F_T}$, $\mathrm{R_T}$ denote the failure rate, CDF and reliability related respectively to the random variables K and T.

The relation between the probability function of K and the CDF of T:

$$\forall \mathrm{k} \in \mathrm{N}^*, \mathrm{p(k)} = \mathrm{P(K = k)} = \mathrm{P(k-1 \leq T} < k) = \mathrm{F_T(k)} - \mathrm{F_T(k-1)}$$

Furthermore:

$$\mathrm{F_K(k)} = \mathrm{P(K \leq k)} = \mathrm{P}(\lfloor \mathrm{T} \rfloor + 1 \leq \mathrm{k}) = \mathrm{P(T} < k) = \mathrm{F_T(k)}$$

Hence:

$$\mathrm{R_T(k)} = \mathrm{R_T(k)}$$

The failure rate of K can be written as:

$$\forall k \in N^*, \lambda_K(k) = 1 - \frac{R_T(k)}{R_T(k-1)} = 1 - e^{-\int_{k-1}^{k} \lambda_T(u)du}$$

λ_T and λ_K have the same monotonicity property.

1.11.1 *Exponential Distribution*

Let T have the exponential distribution, with CDF $F_T(t) = 1 - e^{-\lambda t}$. Using the above equations, we obtain:

$$R_K(k) = e^{-\lambda k} = (1 - (1 - e^{-\lambda})^k$$

This is the reliability function of the geometric distribution with parameter $1 - e^{-\lambda}$, whose (constant) failure rate is equal to $1 - e^{-\lambda}$. Consequently, the failure rate of geometric distribution is not equal to geometric rate of the corresponding exponential distribution.

1.11.2 *Weibull Distribution*

Let T have the Weibull distribution $\mathcal{W}(\eta, \beta)$, then

$$R_K(k) = e^{-\left(\frac{k}{\eta}\right)} = q^{k^\beta}$$

with $q = e^{-\frac{1}{\eta^\beta}}$. This is the type I discrete Weibull distribution.

1.11.3 *Truncated Logistic Distribution*

Let T have the logistic distribution $\mathcal{L}og(c, d)$, truncated on R^+ ($d \in R^{+*}$ and $c \in R$, with CDF:

$$F_T(t) = \frac{1 - e^{-\frac{t}{d}}}{1 + e^{-\frac{t-c}{d}}}$$

Then, the discrete truncated logistic distribution is defined by:

(i) $$P(k) = \frac{e^{-\frac{k-1}{d}}\left(1-e^{-\frac{1}{d}}\right)\left(1+e^{\frac{c}{d}}\right)}{\left(1+e^{-\frac{k-c}{d}}\right)\left(1+e^{-\frac{k-1-c}{d}}\right)}$$

(ii) $R(k) = \frac{e^{-\frac{k-c}{d}} + e^{-\frac{k}{d}}}{1+e^{-\frac{k-c}{d}}}$

(iii) $\lambda(k) = \frac{1-e^{-\frac{1}{d}}}{1+e^{-\frac{k-c}{d}}}$

The failure rate is increasing. Parameters c and d have no practical interpretation.

1.11.4 *Geometric-Weibull Distribution*

Experts believe that systems have three steps in their life. In the first step, the system has decreasing failure rate called early life, in the second step, the system has a constant failure rate until time τ, then, in the third step, the failure rate is larger than in step I and is increasing. The step II is the stable phase of the system and is called useful life while the step III is its wear-out phase, and τ is the change point.

In continuous time, if we consider that after time τ, the failure rate is increasing like one of a Weibull distribution, one obtain the exponential-Weibull distribution introduced by Zacks [1984], with CDF given by:

$$F_T(t) = 1 - e^{-\lambda t-[\lambda(t-\tau)^+]^\beta}$$

where, $Y^+ = \max(0, Y)$, $\lambda \in R^{+*}$ is a scale parameter, $\beta \in R^+$ is a shape parameter and $\tau \in R^{+*}$ is the change point.

For the construction of the analogous of this distribution in discrete time, which will of course be called the geometric-Weibull distribution, τ takes values in N^*.

Then, geometric-Weibull distribution is defined by:

(i) $P(k) = e^{-\lambda(k-1)-[\lambda(k-\tau-1)^+]^\beta} - e^{-\lambda k-[\lambda(k-\tau)^+]^\beta}$

(ii) $R(k) = e^{-\lambda k-[\lambda(k-\tau)^+]^\beta}$

(iii) $\lambda(k) = 1 - e^{-\lambda+\lambda^\beta[[(k-\tau-1)^+]^\beta-[(k-\tau)^+]^\beta]}$

1.12 Pòlya Urn Distribution

Numerous discrete distributions can be built from urn representations, Jhonson, Kotz and Kemp [1992]. The urn scheme considered by Eggenberger and Pòlya [1923] is the following. An urn contains “W” white balls “R” red balls. After each drawing of a ball, a replacement policy is chosen. Pòlya distributions are the distributions of the number of times a red ball is drawn in N drawings. Inverse Pòlya distributions are the distributions of the number of drawings to obtain a specified number r of red balls. There are as many different distributions as possible replacement policies. For example, if after each drawing, only the chosen ball is returned in the urn, the Pòlya distribution reduces to binomial distribution, and the corresponding inverse Pòlya distribution is the negative binomial distribution (geometric distribution for r=1).

In the discrete reliability context, the drawing of a white ball corresponds to a demand successfully completed and drawing of a red ball to a failure on demand. Thus, system lifetime is described by an inverse Pòlya distribution with r=1.

In this case, the failure rate at k^{th} drawing can be understood as the probability of drawing for the first time a red ball at k^{th} drawing, given that no red ball has been drawn during the first $\mathrm{k}-1$ drawings.

If the replacement policy consists in returning the draw (white) ball, together with red balls, this scheme increases failure probability. So the corresponding lifetime will have an increasing failure rate. Conversely, if the draw (white) ball is returned together with other white balls, the lifetime distribution has a decreasing failure rate.

Inverse Pòlya distributions can be very useful in reliability studies since they have practical interpretations and a simple expression of the failure rate. Moreover, most of these distributions have closed form expressions for the MTTF and variance, which provide a simple way to estimate parameters.

1.12.1 *IFR Inverse Pòlya Distribution*

If the drawn white ball is returned in the urn with Δ red balls, the corresponding failure rate is increasing and given by:

$$\lambda(k) = \frac{R + (k-1)\Delta}{R + W + (k-1)\Delta}$$

Let $\theta = \frac{R}{R+W} \in]0,1[$ and $\delta = \frac{\Delta}{R+W} \in R^+$. From the lifetime point of view, θ is the probability of failure on the first demand and δ qualifies the importance of ageing. For $\delta = 0$, the distribution is geometric.

The IFR inverse Pòlya distribution is defined by:

(i) $P(k) = \frac{(1-\theta)^k[\theta+(k-1)\delta]}{\prod_{i=1}^{k}[1+(i-1)\delta]} = \frac{(1-\theta)^k[\theta+(k-1)\delta]}{\delta^k\left(\frac{1}{\delta}\right)_{(k)}}$

where, $(a)_k$ is the Pochhammer symbol:

$$(a)_k = \begin{cases} \prod_{i=1}^{k}(a+i-1) & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \end{cases}$$

(ii) $R(k) = \frac{(1-\theta)^k}{\delta^k\left(\frac{1}{\delta}\right)_{(k)}}$

(iii) $\lambda(k) = \frac{\theta+(k-1)\delta}{1+(k-1)\delta} = 1 - \frac{1-\theta}{\theta+(k-1)\delta}$

The Mean Time To Failure is $MTTF = \frac{(1-\delta)\delta^{\frac{1}{\delta}-2}}{\delta^{\frac{1-\delta}{\delta}}} e^{\frac{1-\theta}{\delta}} \gamma\left(\frac{1-\delta}{\delta}, \frac{1-\theta}{\delta}\right)$,

where, $\gamma(b,x) = \int_0^x t^{b-1}e^{-t}\,dt$ is the incomplete Gamma function.

1.12.2 *DFR Inverse Pòlya Distribution*

If the white ball drawn is replaced in the urn together with Δ white balls,

the failure rate is decreasing and given by:

$$\lambda(k) = \frac{R}{R + W + (k-1)\Delta}$$

With the same notations as before, the DFR inverse Pòlya distribution is defined by:

(i) $P(k) = \frac{\theta\prod_{i=1}^{k-1}[1-\theta+(i-1)\delta]}{\prod_{i=1}^{k}[1+(i-1)\delta]} = \frac{\theta}{1-\theta-\delta}\frac{\left(\frac{1-\theta-\delta}{\delta}\right)_{(k)}}{\left(\frac{1}{\delta}\right)_{(k)}}$

(ii) $R(k) = \frac{\prod_{i=1}^{k}[1-\theta+(i-1)\delta]}{\prod_{i=1}^{k}[1+(i-1)\delta]} = \frac{\left(\frac{1-\theta}{\delta}\right)_{(k)}}{\left(\frac{1}{\delta}\right)_{(k)}}$

(iii) $\lambda(k) = \frac{\theta}{1+(k-1)\delta}$

The MTTF is only defined for $\theta > \delta$ and is equal to:

$$\text{MTTF} = \sum_{k=0}^{+\infty} \frac{\left(\frac{1-\theta}{\delta}\right)_{(k)}}{\left(\frac{1}{\delta}\right)_{(k)}}$$

$$= {}^{2}F_{1}\left(\frac{1-\theta}{\delta}, 1, \frac{1}{\delta}, 1\right) = \frac{1-\theta}{\theta-\delta}$$

where, $${}^{2}F_{1} = (a, b, c, z) = \sum_{k=0}^{+\infty} \frac{(a)_{(k)}(b)_{(k)}}{(c)_{(k)}} \frac{z^k}{k!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{+\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}$$

is a Gaussian hyper geometric series.

The distribution variance, only defined for $\theta > 2\delta$, is:

$$\text{Var}(k) = \frac{(1-\delta)\theta(1-\theta)}{(\theta-\delta)^2(\theta-2\delta)}$$

The DFR inverse Pòlya distribution is also called the Waring distribution, Irwin [1975]. Its failure rate is inversely proportional to a linear functional of time: $\lambda(k) = \frac{1}{a+bk}$. The studied case here corresponds to $b > 0$. Xekalaki [1983] also considered the case $b < 0$. In this case the random variable is defined on $\{1,2,\ldots,m\}$ and its failure rate is:

$$\lambda(k) = \frac{1}{1+c(m-k)}, c > 0$$

For the particular case where $\frac{1}{c}$ is an integer, the probability function is given by:

$$p(k) = \frac{\binom{\frac{1}{c}-1}{m+\frac{1}{c}-k-1}}{\binom{\frac{1}{c}}{m+\frac{1}{c}-1}}$$

and K has shifted hyper geometric distribution.

For DFR inverse Pòlya distribution, if $\theta > \delta$

$$m(k) = \frac{\delta(k-1)+1}{\theta-\delta}$$

As a particular case, MTTF $= m(0) = \frac{1-\delta}{\theta-\delta}$.

Gupta, Gupta and Tripathi [1997] showed that the Waring distribution is a particular case of extended Katz family, defined by the ratio of two consecutive probabilities given by:

$$\frac{p(k+1)}{p(k)} = \frac{\alpha+\beta k}{\gamma+k}, \text{for } \alpha = \frac{1-\theta-\delta}{\delta}, \beta = 1 \text{ and } \gamma = \frac{1}{\delta}$$

In the same way, the IFR inverse Pòlya distribution can be defined by:

$$\frac{p(k+1)}{p(k)} = \frac{1-\theta}{\delta} \frac{\frac{\theta}{\delta}+k}{\left(\frac{\theta}{\delta}-1+k\right)\left(\frac{1}{\delta}+k\right)}$$

which is a particular case of Kemp family.

1.13 Salvia and Bollinger Distributions

Salvia and Bollinger [1982] introduced two distributions with only one parameter $c \in (0,1)$ which are based on very simple expression of the hazard rate. For $\delta = 1$, in fact, they are inverse Pòlya distributions.

1.13.1 *IFR SB Distribution*

The IFR SB distribution is such that

(i) $p(k) = (k-c)\frac{c^{k-1}}{k!}$

(ii) $R(k) = \frac{c^k}{k!}$

(iii) $\lambda(k) = 1 - \frac{c}{k}$

$MTTF = e^c$. The distribution variance is $Var(k) = 2ce^c - e^c - 1$.This distribution has few particular interests because its mean is between 1 and e, so the values taken by K are mainly equal to 1.

1.13.2 *DFR SB Distribution*

The DFR SB distribution is such that:

(i) $p(k) = \frac{c}{k!}(1 - c)_{(k-1)}$

(ii) $R(k) = \frac{(1-c)_{(k)}}{k!}$

(iii) $\lambda(k) = \frac{c}{k}$

The MTTF does not exist because the assumption $\theta > \delta$, here $c > 1$, is not verified.

1.14 Generalized Saliva and Bollinger Distributions

Saliva and Bollinger distributions are not flexible enough to fit a wide variety of situations. Padgett and Spurrier [1985] proposed to generalize these distributions by adding a second parameter $\alpha \in R^{+}$. When, $\alpha = 1$ the distribution reduces to Salvia and Bollinger distribution. When, $\alpha = 0$ the distribution is geometric.

The IFR generalized SB distribution is defined by its failure rate:

$$\lambda(k) = 1 - \frac{c}{(k-1)\alpha + 1}$$

And the DFR generalized SB distribution by:

$$\lambda(k) = \frac{c}{(k-1)\alpha + 1}$$

It appears that these distributions are exactly inverse Pòlya distributions.

1.15 Eggenberger - Pòlya Distribution

Eggenberger-Pòlya distribution can be understood as a limit point of Pòlya distribution when the number of drawings goes to infinity.

The probability of system failure at demand k is:

$$p(k) = \frac{1}{(1+d)^{h/d}} \frac{\left(h/d\right)_{(k-1)}}{(k-1)!} \left(\frac{d}{d+1}\right)^{k-1}$$

The failure rate is given by:

$$\frac{1}{\lambda(k)} = \sum_{j=0}^{+\infty} \frac{\left(h/d + k - 1\right)_{(j)}}{(k)_{(j)}} \left(\frac{d}{d+1}\right)^{j}$$

$$= {}^{2}F_{1}\left(\frac{h}{d} + k - 1, 1, k, \frac{d}{1+d}\right)$$

The expression of the failure rate is very complex and the result on the ratios of the successive probabilities is required to study its monotonicity:

$$\frac{p(k+1)}{p(k)} = \frac{d}{1+d} + \frac{h-d}{1+d}\frac{1}{k}$$

(i) If $h = d$, then failure rate is constant and the model reduces to geometric distribution with parameter $\frac{1}{1+d}$,
(ii) If $h < d$, the distribution is log-convex and then DFR ,
(iii) If $h > d$, the distribution is log-concave and then IFR.

In the three cases,

$$\lim_{k\to\infty} \lambda(k) = \frac{1}{1+d}$$

Using the probability generating function of this distribution, the mean and variance can be derived:

(i) $\mathrm{MTTF} = h + 1$
(ii) $\mathrm{Var}(k) = h(d+1)$

h represents the mean number of demands until the first failure minus one. There is no interpretation for parameter d.

CHAPTER 2

Characterizations of Discrete Distributions using Reliability Concepts in Reversed Time

2.1 Introduction

In lifetime data analysis, the concepts of reversed hazard rate has potential application when the time elapsed since failure is a quantity of interest in order to predict the actual time of failure. The reversed hazard rate is more useful in estimating reliability function when the data are left censored or right truncated. The reversed hazard rate is defined as the ratio of the density to the distribution function, had attracted the attention of researchers. Being in a certain sense a dual function to an ordinary hazard rate, it still bears some interesting features useful in reliability analysis. Ordinary hazard rate functions are most useful for lifetimes, and reverse hazard rates are natural if the time scale is reversed. Mixing up these concepts can often, although not always, lead to anomalies. For example, one result gives that if the reversed hazard rate function is increasing, its interval of support must be $(-\infty, b)$ where b is finite. Consequently non negative random variables cannot have increasing reversed hazard rates. Reversed hazard rates are also important in the study of systems. Hazard rates have an affinity to series systems; reversed hazard rates seem more appropriate for studying parallel systems. Several results are given that demonstrate this. In studying systems, one problem is to relate derivatives of hazard rate functions and reversed hazard rate functions of systems to similar quantities for components; one can also find applications of these concepts in different topics of investigation. Some analogous results and characterizations in the case of the reversed hazard rate and mean residual life for discrete distributions are discussed by Gupta, Nair and Asha [2006] and Goliforushani and Asadi [2008].

There has been growing interest in recent times in the study of reliability functions in reversed time and their applications. The functions of primary interest discussed in the continuous case in the literature are the reversed hazard rate, Block, Savits and Singh [1998] and Finkelstein [2002]. The reversed mean (variance) residual life or mean (variance) inactivity time (Nanda et al., [2003]; Li and Lu, [2003]; Kundu and Nanda, [2010]), and the reversed percentile residual life (Nair and Vineshkumar, [2010]). In this chapter characterizations of some discrete distributions established by Unnikrishnan Nair and Sankaran [2013], using properties of the reversed hazard rate and reversed mean residual life are discussed and discrete distributions having a constant reversed hazard rate, the reversed lack of memory property, and the product of the reversed hazard rate and the mean residual life a constant are identified.

2.2 Reversed Hazard Rate

Let X be a discrete random variable defined on the set $S = \{0, 1, 2, \dots, b\}$, where b is a positive integer and can be ∞. If the probability mass function and distribution function of X is denoted by f (x) and F (x) respectively. Then the reversed hazard rate of X is defined as

$$\lambda(x) = P(X = x | X \leq x) = \frac{f(x)}{F(x)} \tag{2.2.1}$$

The distribution of X is determined uniquely by $\lambda(x)$ through the formula

$$F(x) = \prod_{t=x+1}^{b} \left(1 - \lambda(t)\right) \tag{2.2.2}$$

Further, the random variable X is said to satisfy the reversed lack of memory property if and only if

$$P(X \leq t | X \leq t + s) = P(X \leq 0 | X \leq s) \qquad (2.2.3)$$

for all t, s in S. The property (2.2.3) can be interpreted in the following way in the context of maintenance problems. When X represents the lifetime of a device, its inactivity time (time since failure) is independent of the age of the device.

With these definitions, we have the following characterizations, in which we assume that $b < \infty$.

Theorem 2.2.1 **:** The random variable X is distributed as

$$F(x) = \begin{cases} (1 + c)^{x-b} & , x = 0, 1, 2, \ldots, b;\ c > 0, b < \infty \\ 1 & , \quad x \geq b \end{cases} \qquad (2.2.4)$$

if and only if any one of the following conditions is satisfied:

(a) $\lambda(x) = k$, a constant for all $x > 0$, where $0 < k < 1$;

(b) X has the reversed lack of memory property.

Proof**:** Let

$$f(x) = \begin{cases} (1 + c)^{-b} & , x = 0 \\ c(1 + c)^{x-b-1} & , x = 1, 2, \ldots, b \end{cases}$$

The terms in the above probability mass function for $x = 1, 2, 3, \ldots$ are in increasing geometric progression with common ratio $(1 + c)$, as opposed to the usual geometric distribution where the terms are decreasing, therefore;

$$F(x) = P(X \leq x) = \begin{cases} (1 + c)^{x-b} & , x = 0, 1, 2, \ldots, b;\ c > 0, b < \infty \\ 1 & , \quad x \geq b \end{cases}$$

In view of the reversal of the monotonicity of successive probabilities $F(x)$ is called reversed geometric distribution.

also, from equation (2.2.1), we get

$$\lambda(x+1) = \frac{c}{1+c}, x = 0, 1, 2, \ldots, b - 1$$

and $\lambda(0) = 1$ for all discrete distributions with zero as the left end point for their supports. Thus, $\lambda(x)$ is constant.

Conversely, suppose

$$\lambda(x+1) = \frac{c}{1+c}, x = 0, 1, 2, \ldots, b - 1$$

then by using equation (2.2.2), we get

$$F(x) = \begin{cases} (1+c)^{x-b} & , x = 0, 1, 2, \ldots, b;\ c > 0, b < \infty \\ 1 & , \quad x \geq b \end{cases}$$

thus, $\lambda(x)$ is constant if and only if X has a reversed geometric law.

Also, the distribution (2.2.4) satisfies (b) can be easily verified from (2.2.3).

Conversely, when the reversed lack of memory property holds, we have

$$F(t+s)F(0) = F(t)F(s)$$

$$F(t+s) = \frac{F(t)}{F(0)} . F(s)$$

$$\frac{F(t+s)}{F(0)} = \frac{F(t)}{F(0)} . \frac{F(s)}{F(0)}$$

$$a(t+s) = a(t)a(s), \qquad \text{where } a(t) = \frac{F(t)}{F(0)}$$

To solve the functional equation, set $s = 1$, so we have

$$a(t+1) = a(t)a(1)$$

$$= \frac{(1+c)^{t-b}}{(1+c)^{-b}} \times \frac{(1+c)^{1-b}}{(1+c)^{-b}}$$

$$= (1+c)^{t+1}$$

$$= [a(1)]^{t+1}$$

for all $t = 1, 2, \ldots, b - 1$.

and,

$$\frac{[F(1)]^{x+1}}{[F(0)]^{x}} = (1 + c)^{x+1-b} = F(x + 1)$$

$$F(x + 1) = \frac{[F(1)]^{x+1}}{[F(0)]^{x}} \tag{2.2.5}$$

with

$$F(b) = \frac{p^{b}}{[F(0)]^{b-1}} = 1$$

$$F(0) = p^{\frac{b}{b-1}}, \qquad p = F(1)$$

substituting for $F(0)$ in (2.2.5) and setting $p = (1 + c)^{-1}$, we have (2.2.4).

Thus X has a reversed geometric law if and only if X has the reverse lack of memory property.

2.3 Reversed Mean Residual Life

The reversed mean residual lifetime is defined as

$$r(x) = E(x - X|X < x) = \frac{1}{F(x-1)} \sum_{t=1}^{x} F(t + 1) \tag{2.3.1}$$

with $r(0)$ defined as zero. Goliforushani and Asadi [2008] have shown that

$$\lambda(x) = \frac{1 - r(x + 1) + r(x)}{r(x)}, x = 1, 2, \ldots, b \tag{2.3.2}$$

and

$$F(x) = \left[\prod_{t=1}^{x} \frac{r(t)}{r(t+1)-1}\right] \Big/ \left[\prod_{t=1}^{b} \frac{r(t)}{r(t+1)-1}\right], b < \infty \qquad (2.3.3)$$

In the case of the usual reliability functions, X having constant hazard rate, having constant mean residual life and having a geometric distribution are all equivalent. Further the geometric law is characterized by the property that the product of the hazard rate and the mean residual life is unity. A different scenario arises when the functions in reversed time are considered.

While $\lambda(x)$ is constant for all x for the reversed geometric law, its reversed mean residual life is

$$r(x) = \frac{1+c}{c}\left[1 - \frac{1}{(1+c)^x}\right]$$

a non constant function.

***Theorem 2.3.1* :** The random variable X has reversed mean residual life function

$$r(x) = \frac{c+1}{c}, \qquad c > 0, x = 2, 3, \ldots, b, b < \infty$$

if and only if

$$F(x) = \begin{cases} c^{-1}(1+c)^{1-b}, & x = 0 \\ (1+c)^{x-b}, & x = 1, 2, \ldots, b \end{cases} \qquad (2.3.4)$$

***Proof*:** The probability mass function corresponding to (2.3.4) is

$$f(x) = \begin{cases} c^{-1}(1+c)^{1-b}, & x = 0 \\ \dfrac{c-1}{c}(1+c)^{1-b}, & x = 1 \\ c(1+c)^{x-b-1}, & x = 2, 3, \ldots, b \end{cases} \quad : c > 1$$

and from equation (2.2.1), we get

$\lambda(x) = \frac{c}{1+c}$ for the assumed values of x.

Since, the property $\lambda(x)r(x) = 1$ for all $x = 2, \ldots, b$ characterizes the model (2.3.4), thus

$$r(x) = \frac{c + 1}{c}$$

Conversely from (2.3.2), if $\lambda(x)r(x) = 1$, then

$$r(x + 1) - r(x) = 0$$

so $r(x)$ is a constant.

Thus, X has reversed mean residual life function $r(x) = \frac{c+1}{c}$ if and only if $F(x)$ is given by (2.3.4).

2.4 Reversed Variance Residual Life

The reversed variance residual life is defined as

$$v(x) = E((x - X)^2 | X < x) - r^2(x) \tag{2.4.1}$$

It can also be computed as

$$v(x) = E(X^2 | X < x) - E^2(X | X < x)$$

Another useful representation of $v(x)$ is given below; using (2.4.1), we have

$$E((x - X)^2 | X < x) = \frac{1}{F(x - 1)} \sum_{t=0}^{x-1} (x - t)^2 f(t)$$

$$= \frac{1}{F(x - 1)} \sum_{t=1}^{x} (x - t)^2 [F(t) - F(t - 1)]$$

$$= \frac{2}{F(x - 1)} \sum_{t=1}^{x} (x - t) F(t - 1) - r(x)$$

$$= \frac{2}{F(x - 1)} \sum_{t=1}^{x} \sum_{u=1}^{t} F(u - 1) - r(x)$$

giving

$$v(x) = \frac{2}{F\ (x - 1)} \sum_{t=1}^{x} \sum_{u=1}^{t} F\ (u - 1) - r(x)(r(x) + 1) \qquad (2.4.2)$$

***Example 2.4.1*:** Let X follows the uniform distribution

$$F\ (x) = \frac{x}{b}, x = 1, 2, \ldots, b$$

Now,

$$r(x) = \frac{x}{2}$$

$$\lambda(x) = \frac{1 - r(x + 1) + r(x)}{r(x)}$$

$$= \frac{1 - \frac{x+1}{2} + \frac{x}{2}}{\frac{x}{2}}$$

$$= \frac{2 - x - 1 + x}{x} = \frac{1}{x}$$

and

$$v(x) = \frac{2}{F\ (x - 1)} \sum_{t=1}^{x} \sum_{u=1}^{t} F\ (u - 1) - r(x)(r(x) + 1)$$

$$= \frac{2}{(x - 1)} \sum_{t=1}^{x} \sum_{u=1}^{t} (u - 1) - \frac{x}{2}\left(\frac{x}{2} + 1\right)$$

$$= \frac{2}{(x - 1)} \sum_{t=1}^{x} \left[\frac{t(t + 1)}{2}\right] - \frac{x(x + 2)}{4}$$

$$= \frac{1}{(x - 1)} \sum_{t=1}^{x} [t^2 - t] - \frac{x(x + 2)}{4}$$

$$= \frac{1}{(x-1)}\left[\frac{x(x+1)(2x+1)}{2} - \frac{x(x+1)}{2}\right] - \frac{x(x+2)}{4}$$

$$= \frac{x(x+1)}{3} - \frac{x(x+2)}{4} = \frac{x(x-2)}{12}$$

Thus, the functions calculated are

$$\lambda(x) = \frac{1}{x}, \qquad r(x) = \frac{x}{2} \ \ and \ \ v(x) = \frac{x(x-2)}{12}, \ \ x > 2$$

Example 2.4.2**:** The arithmetic distribution is defined by the distribution function

$$F\,(x) = \frac{x(x\ +\ 1)}{b(b\ +\ 1)}\ ,\quad x\ =\ 1, 2, \ldots, b$$

Its probability mass function

$$f\,(x) = \frac{2x}{b(b\ +\ 1)}$$

is the length-biased version of the uniform distribution in Example 2.4.1.

Now,

$$r(x) = \frac{x\ +\ 1}{3}$$

$$\lambda(x) = \frac{1 - \frac{x+2}{3} + \frac{x\ +\ 1}{3}}{\frac{x\ +\ 1}{3}}$$

$$= \frac{3 - x - 2 + x + 1}{x+1} = \frac{2}{x+1}$$

also,

$$v(x) = \frac{2}{F\,(x-1)} \sum_{t=1}^{x} \sum_{u=1}^{t} F\,(u - 1) - r(x)(r(x) + 1)$$

$$= \frac{2}{x\,(x-1)}\sum_{t=1}^{x}\sum_{u=1}^{t}\,(u^2 - u) - \frac{x+1}{3}\left(\frac{x+1}{3} + 1\right)$$

$$= \frac{2}{x\,(x-1)}\sum_{t=1}^{x}\left[\frac{t(t+1)(2t+1)}{6} - \frac{t(t+1)}{2}\right] - \frac{(x+1)(x+4)}{9}$$

$$= \frac{2}{x\,(x-1)}\sum_{t=1}^{x}\frac{t^3 - t}{3} - \frac{(x+1)(x+4)}{9}$$

$$= \frac{2}{3x\,(x-1)}\left[\left(\frac{x(x+1)}{2}\right)^2 - \frac{x(x+1)}{2}\right] - \frac{(x+1)(x+4)}{9}$$

$$= \frac{(x+1)(x+2)}{6} - \frac{(x+1)(x+4)}{9}$$

$$= \frac{(x+1)(x-2)}{18}$$

Thus,

$$\lambda(x) = \frac{2}{x+1}\ , \quad r(x) = \frac{x+1}{3} \quad and \quad v(x) = \frac{(x+1)(x-2)}{18}\ , \quad x > 2.$$

***Theorem 2.4.1*:** For a random variable X in the support of $(1, 2, \ldots, b), b < \infty$, the relationship

$$\lambda(x) r(x) = k, 0 < k < 1 \qquad (2.4.3)$$

holds for $x = 2, 3, \ldots, b$ if and only if the distribution of X is specified by

$$F\,(x) = \frac{(b-1)!}{(x-1)!}\frac{(\theta)_x}{(\theta)_b}\ , \quad x = 1, 2, \ldots, b \qquad (2.4.4)$$

where $\theta = \frac{k}{1-k}$ is a positive integer and

$$(\theta)_x = \theta(\theta + 1)\cdots(\theta + x - 1)$$

is the Pochhammer symbol.

***Proof*:** Let $\lambda(x)r(x) = k, 0 < k < 1$, then using (2.3.2) to obtain the first-order difference equation

$$r(x + 1) - r(x) = 1 - k$$

To solve for $r(x)$, we iterate for x values and use the boundary condition $r(x) = 1$, to obtain

$$r(x + 1) = k + (1 - k)x$$

and hence

$$\frac{r(x + 1) - 1}{r(x)} = \frac{k + (1 - k)x - 1}{k + (1 - k)(x - 1)}$$

$$= \frac{x - 1}{\frac{k}{1 - k} + x - 1}$$

$$= \frac{x - 1}{\theta + x - 1}$$

where, $\theta = \frac{k}{1-k}$

using the value of $r(x)$ in (2.3.3), we get

$$F(x) = \frac{(b - 1)!}{(x - 1)!}\frac{(\theta)_x}{(\theta)_b}$$

Conversely, suppose X is specified by (2.4.4), then the probability mass function

is

$$f(x) = \frac{(b - 1)!}{(x - 1)!}\frac{\theta(\theta)_{x-1}}{(\theta)_b}$$

and by using (2.2.1), we get

$$\lambda(x) = \frac{\theta}{\theta + x - 1} = \frac{k}{k + (1 - k)(x - 1)}$$

Thus, $r(x)\lambda(x) = k.$

Particular cases: $\theta = 1$ and $\theta = 2$ correspond to the uniform and arithmetic distributions respectively.

There are some identities satisfied by $\lambda(x), r(x)$ and $v(x)$ that could be of use in characterization problems for determining one from another.

From (2.4.2),

$$[v(x) + r(x)(r(x) + 1)]F(x - 1) = 2\sum_{t=1}^{x}\sum_{u=1}^{t} F(u - 1) \qquad (2.4.5)$$

changing x to $x + 1$ and then subtracting (2.4.5), we obtain

$$[v(x+1) + r(x+1)(r(x+1)+1)]F(x) - [v(x) + r(x)(r(x)+1)]F(x-1)$$

$$= 2r(x+1)F(x)$$

dividing by $F(x)$ and noting that

$$\frac{F(x - 1)}{F(x)} = 1 - \lambda(x) = \frac{r(x + 1) - 1}{r(x)}$$

we have,

$$v(x+1) + r^2(x+1) + r(x+1) - [v(x) + r(x)(r(x)+1)][1 - \lambda(x)]$$

$$= 2r(x+1)$$

$$v(x+1) - v(x) + r^2(x+1) + r(x+1) - r(x)(r(x)+1)\left(\frac{r(x + 1) - 1}{r(x)}\right)$$

$$= 2r(x+1) - v(x)\lambda(x)$$

$$v(x+1) - v(x) + r^2(x+1) - 2r(x+1) + 1 - r(x)(r(x)+1)$$

$$= -v(x)\lambda(x)$$

$$v(x+1) - v(x) + (r(x+1) - 1)^2 - r(x)(r(x)+1)$$

$$= -v(x)\lambda(x)$$

$$v(x+1) - v(x) = (r(x) - r(x+1) + 1)(r(x)+1) - v(x)\lambda(x)$$

$$= r(x)\left(1 - \frac{r(x+1) - 1}{r(x)}\right)(r(x) + 1) - v(x)\lambda(x)$$

$$= r(x)\lambda(x)[r(x) + 1] - v(x)\lambda(x)$$

$$= \lambda(x)\big(r(x)(r(x+1) - 1) - v(x)\big) \tag{2.4.6}$$

$$v(x+1) = \lambda(x)r(x)(r(x+1) - 1) - \lambda(x)\, v(x) + v(x)$$

$$v(x+1) = \left(1 - \frac{r(x+1) - 1}{r(x)}\right) r(x)(r(x+1) - 1) + v(x)\big(1 - \lambda(x)\big)$$

$$v(x+1) =$$

$$r(x)(r(x+1) - 1) - (r(x+1) - 1)^2 + v(x)\left(\frac{r(x+1) - 1}{r(x)}\right)$$

$$v(x+1) + (r(x+1) - 1)^2$$

$$= r(x)(r(x+1) - 1) + \frac{v(x)}{r(x)}(r(x+1) - 1)$$

$$\frac{v(x+1)}{r(x+1) - 1} + r(x+1) - 1$$

$$= \frac{v(x)}{r(x)} + r(x) \tag{2.4.7}$$

Kundu and Nanda [2010] have identified the class of distributions in the continuous case for which the square of the coefficient of variation of the residual life is a constant. In the discrete case, the result is slightly modified.

***Theorem 2.4.2*:** A random variable X in the support of $\{(1, 2, \ldots, b)\}$ satisfies the property

$$a = \frac{v(x)}{r(x)(r(x) - 1)} \tag{2.4.8}$$

a constant if and only if its distribution is

$$F(x) = \frac{(b-1)!}{(x-1)!}\frac{(\theta)_x}{(\theta)_b}, \quad x = 1, 2, \ldots, b \tag{2.4.9}$$

for $0 < a < 1$

and

$$F(x) = \begin{cases} c^{-1}(1 + c)^{1-b}, & x = 0 \\ (1 + c)^{x-b}, & x = 1,2,\dots,b \end{cases} \quad (2.4.10)$$

for $a = 1$.

***Proof*:** We know that

$$r_2(x) = [E(x - X)^2 | X < x] = \frac{1}{F(x - 1)} \sum_{t=1}^{x-1} (x - t)^2 f(t)$$

$$= \frac{2}{F(x - 1)} \sum_{t=1}^{x-1} (x - t) F(t) - r(x)$$

$$[r_2(x) + r(x)] F(x - 1) = 2 \sum_{t=1}^{x-1} F(t)$$

Working as in the case of (2.4.6) and (2.4.7), we obtain

$$r_2(x + 1) r(x) - r_2(x)(r(x + 1) - 1) - r(x)(r(x + 1) - 1)$$

$$= r(x) r(x + 1) \quad (2.4.11)$$

Now, assume that $a = 1$. Then

$$v(x) = r(x)(r(x) - 1) = v(x) + r^2(x)$$

giving

$$r_2(x) = r(x)(r(x) - 1) + r^2(x)$$

using this in (2.4.9) and simplifying, we obtain

$r(x + 1) - r(x) = 0 \quad , x = 2, 3, \dots, b$

and hence $r(x)$ is a constant, so by Theorem 2.3.1 the distribution is

$$F(x) = \begin{cases} c^{-1}(1 + c)^{1-b}, & x = 0 \\ (1 + c)^{x-b}, & x = 1,2, \dots, b \end{cases}$$

For this distribution $r(x) = \frac{1+c}{c}$

and so,

$$r(x)(r(x) - 1) = \frac{c + 1}{c^2}$$

also,

$$v(x) = \frac{2(c + 1)}{c(1 + c)^{x-1}} \left[\sum_{t=2}^{x} (1 + c)^{t-1-b} + \frac{(1 + c)^{t-b}}{c} \right] - \frac{(1 + c)^2}{c^2} - \frac{1 + c}{c}$$

$$= \frac{c + 1}{c^2}$$

For any distribution, $v(1) = 0$ and $r(1) = 1$, therefore;

$$v(x) = r(x)(r(x) - 1)$$

is trivially satisfied for $x = 1$ and this completes the proof for $a = 1$.When $0 < a < 1$ and the distribution is

$$F(x) = \frac{(b - 1)!}{(x - 1)!} \frac{(\theta)_x}{(\theta)_b}, \quad x = 1, 2, \dots, b$$

$$r(x) = k + (1 - k)(x - 1) = \frac{\theta + x - 1}{\theta + 1}$$

$$\frac{2}{F(x - 1)} \sum_{t=1}^{x} \sum_{u=1}^{t} F(i - 1) = \frac{2}{F(x - 1)} \sum_{t=2}^{x} r(t) F(t - 1)$$

$$= \frac{2(b - 1)!}{(\theta + 1)(\theta)_b F(x - 1)} \sum_{t=2}^{x} \frac{(\theta)_t}{(t - 2)!} \qquad (2.4.12)$$

from the identity

$$\sum_{j=0}^{r} \binom{j + a - 1}{a - 1} = \binom{r + a}{a}$$

$$\sum_{t=2}^{x} \frac{(\theta)_t}{(t-2)!} = \frac{(x+\theta)!}{(\theta+2)!\,(x-2)!}$$

substituting for $F\,(x-1)$ from (2.4.9) and simplifying (2.4.12), we obtain

$$\frac{2}{F\,(x-1)} \sum_{t=1}^{x} \sum_{u=1}^{t} F\,(u-1) = \frac{2(\theta+x-1)(\theta+x)}{(\theta+1)(\theta+2)}$$

the last expression together with

$$r(x)(r(x)+1) = \frac{(2\theta+x)(\theta+x-1)}{(\theta+1)^2}$$

yields

$$v(x) = \frac{\theta(\theta+x-1)(x-2)}{(\theta+1)^2(\theta+2)}$$

and

$$\frac{v(x)}{r(x)(r(x)-1)} = \frac{\theta}{\theta+2} = a\ , \qquad 0 < a < 1$$

Conversely, suppose

$$\frac{v(x)}{r(x)(r(x)-1)} = a\ , \qquad 0 < a < 1$$

and by using identity (2.4.7), we get the difference equation

$$r(x+1) - r(x) = \frac{1-a}{1+a}$$

which on solution using the boundary condition $r(2) = 1$ leads to

$$r(x) = 1 + \frac{1-a}{1+a}(x-2)$$

$$= k + (1-k)(x-1) \quad , \quad 0 < k < 1$$

where,

$$a = \frac{k}{2 - k}$$

therefore by theorem 2.2.1, the distribution is (2.4.9).

CHAPTER 3

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The Discrete Additive Weibull Distribution

3.1 Introduction

An important concept in reliability is the bathtub shaped hazard rate function which consists of an initially decreasing hazard (the 'wear-in' phase) followed by an approximately constant hazard (the 'useful' phase) and finally an increasing hazard (the 'wear-out' phase). The point of minimum hazard is termed the 'turning point'. This obviously provides an opportunity to improve the expected operating time by means of 'burn-in (using up the initial wear-in phase). Bebbington, Lia and Zitikis [2007a] proposes computationally tractable formal mathematical definitions for the 'useful period' of lifetime distributions with bathtub shaped hazard rate functions and detailed analysis of the reduced additive Weibull hazard rate function. Bebbington et al. [2007b] also focused on bathtub-type distributions and provide a view of certain problems, methods and solutions that can be encountered in reliability engineering, survival analysis, demography and actuarial science. Again, Bebbington et al. [2007c] estimated the optimal burn-in time, for bathtub shaped failure-rate lifetime distributions the optimal burn-in time is frequently defined as the point where the corresponding mean residual life function achieves its maximum. For this point, they construct an empirical estimator and develop the corresponding statistical inferential theory for the difference between the minimum point of the corresponding failure rate function and the aforementioned maximum point of the mean residual life function. The difference measures the length of the time interval after the optimal burn-in time during which the failure rate function continues to decrease and thus the burn-in process can be stopped.

The turning point of a hazard rate function is useful in assessing the hazard in the useful life phase and helps to determine and plan appropriate burn-in, maintenance, and repair policies and strategies. For many bathtub-shaped distributions, the turning point is unique, and the hazard varies little in the useful life phase. Bebbington et al. [2008] investigate the performance of an empirical estimator for the turning point in the case of the modified Weibull distribution, a bathtub-shaped generalization of the Weibull distribution, that has been found to be useful in reliability engineering and other areas concerned with life-time data.

Several discrete lifetime distributions have been introduced in the literature, many derived by discretizing their continuous counterparts. For example, Nakagawa and Osaki [1975] defined the discrete Weibull distribution to correspond with the Weibull distribution in continuous time. Roy [2003] proposes a discrete version of the continuous normal distribution and ensured Increasing Failure Rate property in the discrete setup. Roy [2004] used a general approach for discretization of continuous life distributions in the univariate & bivariate situations and proposed a discrete Rayleigh distribution; he examined this distribution in detail with respect to two measures of failure rate. He also characterization results to establish a direct link between the discrete distributions (Normal and Rayleigh) and their continuous counterpart. This discretization approach not only expands the scope of reliability modeling, but also provides a method for approximating probability integrals arising out of a continuous setting. As an example, he approximated the reliability value of a complex system. Krishna and Pundir [2009] obtained discrete Burr and Pareto distributions using the general approach of discretizing a continuous distribution and proposed them as suitable lifetime models. A discrete analogue of the standard continuous Weibull distribution was proposed in the literature to meet the need of fitting discrete-time reliability and survival data sets. Its properties were studied and the methods of estimation of its parameters were also investigated by various authors.

Analogous to its continuous counterpart, the discrete Weibull does not provide a good fit to data sets that exhibit non-monotonic hazard rate shapes. Aghababaei, Lai and Alamatsaz [2010] proposed a discrete inverse Weibull distribution, which is a discrete version of the continuous inverse Weibull variable, defined as X^{-1} where X denotes the continuous Weibull random variable.

For discrete distribution with reliability function $R(k), k = 1, 2, \ldots, [R(k-1) - R(k)]/R(k-1)$ has been used as the definition of the failure rate function in the literature. However, this is different from that of the continuous case. This discrete version has the interpretation of a probability while it is known that a failure rate is not a probability in the continuous case. This discrete failure rate is bounded, and hence cannot be convex, e.g., it cannot grow linearly. It is not an additive for series system while the additivity for series system is a common understanding in practice. Xie, Gaudoin and Bracquemond [2002] introduced another definition of discrete failure rate function as In[R(k - 1)/R(k)] and showed that the two failure rate definitions have the same monotonicity property. That is, if one is increasing / decreasing, the other is also increasing / decreasing. Gupta et. al in 1997 developed techniques for the determination of increasing failure rate (IFR) and decreasing failure rate (DFR) property for a wide class of discrete distributions. Instead of using the failure rate, they make use of the ratio of two consecutive probabilities. The method developed is applied to various well known families of discrete distributions which include the binomial, negative binomial and Poisson distributions as special cases. These formulas are explicit but complicated and cannot normally be used to determine the monotonicity of the failure rates. In this chapter discrete versions of the additive Weibull distribution is presented, explored by Bebbington , Lai , Wellington and Zitikis [2012]. The distribution has the twin virtues of mathematical tractability and the ability to produce bathtub-shaped hazard rate functions. Conditions are defined on the parameters for the hazard rate function to be increasing, decreasing, or bathtub shaped.

3.2 Continuous Additive Weibull Distribution

The continuous additive Weibull distribution is defined by its reliability function

$$R(t) = \exp\left(-\lambda_1 t^{\alpha} - \lambda_2 t^{\beta}\right) \text{ for all } t \in [0, \infty), \tag{3.2.1}$$

where $\lambda_1, \lambda_2, \alpha, \beta \in (0,1)$ are parameters. Intuitively, $R(t)$ is the reliability function of $X = \min\{W_1, W_2\}$ where W_1 and W_2 are independent Weibull random variables with parameters (λ_1, α) and (λ_2, β) respectively.

The hazard rate function

$$h(t) = \frac{-R'(t)}{R(t)} = \alpha\lambda_1 t^{\alpha-1} + \beta\lambda_2 t^{\beta-1} \tag{3.2.2}$$

of the additive Weibull distribution has a bathtub shape, meaning that it is initially decreasing but ultimately increasing. Specifically, it has been shown by Xie and Lai [1995] that $h(t)$ is bathtub shaped when $\alpha < 1 < \beta$ or, by symmetry, when $\beta < 1 < \alpha$. The 'scale' parameters λ_1 and λ_2 have no influence on the shape. The turning point of the hazard rate function $h(t)$, that is, the point t where the bathtub-shaped hazard rate function achieves its minimum, is given by

$$t_{\alpha,\beta} = \left(\frac{\alpha(1-\alpha)\lambda_1}{\beta(1-\beta)\lambda_2}\right)^{\frac{1}{(\beta-\alpha)}} \tag{3.2.3}$$

when, $\alpha < 1 < \beta$, as well as by symmetry when $\beta < 1 < \alpha$.

Throughout this chapter, whenever a function, say g, is defined on $N = \{1,2, \dots\}$, its argument is denote by i, thus writing g(i). If, however, the function g is defined on a continuous scale such as $[0, \infty)$ or $[1, \infty)$, then its argument is denote by t, thus writing $g(t)$.

3.3 Definitions, Shapes and Turning Points

The discrete version of the continuous hazard rate function $h(t)$ at the points t = i, called the 'naive' discrete hazard rate function is defined as;

$$h_0(i) = \alpha\lambda_1 i^{\alpha-1} + \beta\lambda_2 i^{\beta-1} \tag{3.3.1}$$

And the 'classical' discrete hazard rate function $h : N \to [0, \infty)$ is defined by

$$h(i) = \frac{p(i)}{R(i-1)} \tag{3.3.2}$$

where, $p(i) = R(i-1) - R(i)$ is a probability mass function. However, this definition implies that the function h maps the natural numbers N into the interval [0, 1]. This means that the hazard is not scalable relative to any time unit, making comparisons between situations difficult. This shortcoming of the hazard rate function $h(i)$, when it comes to cost analysis has inspired the search for other definitions. A method for converting the naive discrete hazard function into a classical formulation has been proposed by Stein and Dattero [1984] for the discrete Weibull, but as it requires truncating the support of the distribution. Nevertheless, continuing with the 'classical' discrete hazard rate function $h(i)$ defined by (3.3.2) in the case of the additive Weibull model (3.2.1), we have that

$$h(i) = 1 - q_1^{i^\alpha - (i-1)^\alpha} q_2^{i^\beta - (i-1)^\beta} \qquad (3.3.3)$$

where, $q_1 = \exp\{-\lambda_1\}$ and $q_2 = \exp\{-\lambda_2\}$.

For, $\alpha = \beta = 1$, this is the geometric distribution. We see that the discrete hazard function $h(i)$ is quite different from $h_0(i)$. The degree of similarity between the shape and other properties of the two depends heavily on the (discrete) time scaling. For example, if the domain of (time) interest is large, so that the increments between possible discrete failures are a relatively insignificant part of the lifetimes, then the discrete failures will occur in a process resembling the continuous model. However, if the increments are a noticeable fraction of the lifetimes, this may not be so, particularly for more complicated properties. For this reason, there is no general theory relating the properties of the discrete and continuous hazard functions, and hence the shape of the former needs to be established in a rigorous fashion despite the fact that the shape of the continuous version might already be well known. In the following theorem the possible shapes of the hazard rate function $h(i)$ are discussed.

***Theorem 3.3.1*:**

i) If either (i) $\alpha \geq 1$ and $\beta > 1$ *or* (ii) $\alpha > 1$ *and* $\beta \geq 1$, then $h: N \to [0, 1]$ is strictly increasing on $[0, \infty)$.
ii) If $\alpha = 1$ and $\beta = 1$ then $h: N \to [0, 1]$ is constant on $[0, \infty)$.

iii) If either (i) $\alpha \leq 1$ and $\beta < 1$ *or* (ii) $\alpha < 1$ *and* $\beta \leq 1$, then $h: N \to [0,1]$ is strictly decreasing on $[0, \infty)$.

iv) If $\alpha < 1 < \beta$, then $h: N \to [0,1]$ is bathtub shaped with minimum achieved at one of the three points $\lfloor t_{\alpha,\beta} \rfloor, 1 + \lfloor t_{\alpha,\beta} \rfloor$ and $2 + \lfloor t_{\alpha,\beta} \rfloor$, where, $t_{\alpha,\beta} = \left(\frac{\alpha(1-\alpha)\lambda_1}{\beta(1-\beta)\lambda_2}\right)^{\frac{1}{(\beta-\alpha)}}$ and $\lfloor . \rfloor$ is the floor function, that is, $\lfloor x \rfloor$ is the largest integer k such that $k \leq x$. By symmetry, the same conclusion holds in the case $\beta < 1 < \alpha$.

***Proof*:** In the continuous case we need to investigate the sign of the derivative of the function

$$h(t) = \alpha\lambda_1 t^{\alpha-1} + \beta\lambda_2 t^{\beta-1} \tag{3.3.4}$$

where as in the discrete case we need to investigate the sign of the derivative of the function

$$h(t) = 1 - q_1^{t^\alpha-(t-1)^\alpha} q_2^{t^\beta-(t-1)^\beta} \tag{3.3.5}$$

this function is a continuous version of the hazard rate function

$$h(i) = 1 - q_1^{i^\alpha-(i-1)^\alpha} q_2^{i^\beta-(i-1)^\beta} \tag{3.3.6}$$

we explore the shape of the hazard rate function $h: N \to [0,1]$ by exploring the shape of its continuous version $h(t)$ defined by (3.3.5). For this, we investigate the sign of the derivative $h'(t)$, which is same as the sign of the function $h'(t)/[1 - h(t)]$.

Now,

$$\frac{h'(t)}{1-h(t)} = -\frac{d}{dt}\log\big(1-h(t)\big)$$

$$= -\frac{d}{dt}\log\left(q_1^{t^\alpha-(t-1)^\alpha} q_2^{t^\beta-(t-1)^\beta}\right)$$

$$= \frac{d}{dt}\Big(\lambda_1(t^\alpha - (t-1)^\alpha) + \lambda_2\big(t^\beta - (t-1)^\beta\big)\Big) \tag{3.3.7}$$

where, $\lambda_1 = \log\left(\frac{1}{q_1}\right)$ and $\lambda_2 = \log\left(\frac{1}{q_2}\right)$.

Thus, the sign of derivative $h'(t)$, is same as the sign of the function

$$\Theta(t) = \alpha\lambda_1(t^{\alpha-1} - (t-1)^{\alpha-1}) + \beta\lambda_2\left(t^{\beta-1} - (t-1)^{\beta-1}\right)$$

$$= \alpha(\alpha-1)\lambda_1 \int_{t-1}^{t} s^{\alpha-2} ds + \beta(\beta-1)\lambda_2 \int_{t-1}^{t} s^{\beta-2} ds \qquad (3.3.8)$$

Clearly.

1. If either (i) $\alpha \geq 1$ and $\beta > 1$ or (ii) $\alpha > 1$ and $\beta \geq 1$, then $\Theta(t)$ is strictly positive for all $t > 1$, and thus $h(t)$ is strictly increasing on $(0, \infty)$.
2. If $\alpha = 1$ and $\beta = 1$, then $\Theta(t) \equiv 0$, and $h(t)$ is constant on $(0, \infty)$.
3. If either (i) $\alpha \leq 1$ and $\beta < 1$ or (ii) $\alpha < 1$ and $\beta \leq 1$, then $\Theta(t)$ is strictly positive for all $t > 1$, and thus $h(t)$ is strictly decreasing on $(0, \infty)$.

and, the only two cases that remain to look at are, first,

$\alpha < 1 < \beta$ and $\beta < 1 < \alpha$

By symmetry we analyze one of the two cases, and hence we concentrate on $\alpha < 1 < \beta$. We next determine the number of zeros of the function $\Theta(t)$ which are the solution of the equations

$$\alpha(\alpha-1)\lambda_1 \int_{t-1}^{t} s^{\alpha-2} ds = \beta(\beta-1)\lambda_2 \int_{t-1}^{t} s^{\beta-2} ds \qquad (3.3.9)$$

Now, $t > 1$. Let

$$\Theta_1(t) = \alpha(\alpha-1)\lambda_1 \int_{t-1}^{t} s^{\alpha-2} ds \qquad (3.3.10)$$

$$\Theta_2(t) = \beta(\beta-1)\lambda_2 \int_{t-1}^{t} s^{\beta-2} ds \qquad (3.3.11)$$

Hence, $\Theta(t) = \Theta_2(t) - \Theta_1(t)$.

Clearly $\Theta_2(t), \Theta_1(t)$ are both positive for all $t > 1$ and since $\alpha < 1$. Therefore the function $\Theta_1(t)$ is strictly decreasing on $(1, \infty)$ with the limiting value $+\infty$

when $t \to 1$ and 0 when $t \to \infty$ and the function $\Theta_2(t)$ is strictly increasing on $(1, \infty)$ when $\beta > 2$, is equal to the constant $\beta\lambda_2$ when $\beta = 2$, and is strictly decreasing when $\beta \in (1, 2)$.

Now to show minimum is achieved at one of the three points $\lfloor t_{\alpha,\beta} \rfloor, 1 + \lfloor t_{\alpha,\beta} \rfloor$ or $2 + \lfloor t_{\alpha,\beta} \rfloor$, we first define bonds

$$(t-1)^{\beta-\alpha} \int_{t-1}^{t} s^{\alpha-2} ds < \int_{t-1}^{t} s^{\beta-2} ds < t^{\beta-\alpha} \int_{t-1}^{t} s^{\alpha-2} ds$$

which imply the existence of $\xi \in (t-1, t)$ such that

$$\int_{t-1}^{t} s^{\beta-2} ds = \xi^{\beta-\alpha} \int_{t-1}^{t} s^{\alpha-2} ds$$

applying the later equation on the right-hand side of (3.3.8), we have that

$$\Theta(t) = \left(\frac{\beta(\beta-1)\lambda_2}{\alpha(1-\alpha)\lambda_1} \xi^{\beta-\alpha} - 1 \right) \alpha(-\alpha)\lambda_1 \int_{t-1}^{t} s^{\alpha-2} ds \tag{3.3.12}$$

Hence, for all $t > 1$ such that

$$\frac{\beta(\beta-1)\lambda_2}{\alpha(1-\alpha)\lambda_1} t^{\beta-\alpha} \leq 1 \tag{3.3.13}$$

The function $\Theta(t)$ is negative and thus the function $h(t)$ is decreasing. The condition (3.3.13) is equivalent to $t \leq t_{\alpha,\beta}$. Furthermore, for all $t > 1$ such that

$$\frac{\beta(\beta-1)\lambda_2}{\alpha(1-\alpha)\lambda_1} (t-1)^{\beta-\alpha} \geq 1 \tag{3.3.14}$$

The function $\Theta(t)$ is positive and thus the function $h(t)$ is increasing. Condition (3.3.14) is equivalent to $t \geq 1 + t_{\alpha,\beta}$. Therefore, the points $t > 1$ such that $\Theta(t) = 0$ must be in the interval $(t_{\alpha,\beta}, 1 + t_{\alpha,\beta})$, and so the minimum of the function $h(t)$ must be in the same interval. This in turn implies that the minimum of the discrete hazard rate function $h(i)$ must be at one of the three integral points $\lfloor t_{\alpha,\beta} \rfloor, 1 + \lfloor t_{\alpha,\beta} \rfloor$ and $2 + \lfloor t_{\alpha,\beta} \rfloor$.

It has been argued by Xie et al. [2002] that a different definition of the discrete hazard rate function would be more appropriate for a number of reasons and the ‘non-classical’ hazard rate function $h_1: N \to [0, \infty)$ is given by the formula

$$h_1(i) = \log\left(\frac{R(i-1)}{R(i)}\right) \qquad (3.3.15)$$

According to Lai and Xie [2006], unlike the classical hazard rate function (3.2.2), this has the intuitively attractive property of being additive for a system consisting of a series of independent components. The function $h_1: N \to [0, \infty)$ has also been termed the second-rate of failure by Roy and Gupta [1992], which was actually defined, given a ‘classical’ discrete hazard rate function $h: N \to [0, 1]$, by the equation

$$h_1(i) = \log\left(\frac{1}{1-h(i)}\right) \qquad (3.3.16)$$

Clearly, (3.3.16) is an obvious consequence of (3.3.2) and (3.3.15). In the case of the additive Weibull survival function, h_1 has the expression

$$h_1(i) = \lambda_1(i^{\alpha} - (i-1)^{\alpha}) + \lambda_2(i^{\beta} - (i-1)^{\beta}) \qquad (3.3.17)$$

***Theorem 3.3.2*:** The statements of Theorem 1 hold with $h: N \to [0, 1]$, replaced by $h_1: N \to [0, \infty)$.

***Proof*:** Since, the notation $h_1(t)$, (3.3.7) becomes $h'(t)/[1 - h(t)] = h_1'(t)$. Thus the sign of the derivatives $h'(t)$ and $h_1'(t)$ are same. And in the proof of theorem 1 the sign of $h'(t)$ has already been determined, this establishes the sign of $h_1'(t)$ and also the turning point of $h_1(t)$.

The reason for the shapes–but not the functions themselves– being identical in both formulations of the hazard rate function can be explained by first extending (3.3.16) to the continuous scale (with t), then differentiating the result with respect to t, and in this way arriving at the equation $h_1'(t) = h'(t)/[1 - h(t)]$. Obviously, the sign of derivatives $h_1'(t)$ and $h'(t)$ coincides, and hence so do the monotonicity property of $h_1(t)$ and $h(t)$ including their turning points. However, the values of the two functions $h_1(t)$ and $h(t)$ differ since the magnitudes of the derivatives $h_1'(t)$

and $h'(t)$ differ, unless of course the hazard rate function $h(t)$ is identically equal to zero which is the case of no interest do the monotonicity property of $h_1(t)$ and $h(t)$ including their turning points. However, the values of the two functions $h_1(t)$ and $h(t)$ differ since the magnitudes of the derivatives $h_1'(t)$ and $h'(t)$ differ, unless of course the hazard rate function $h(t)$ is identically equal to zero which is the case of no interest.

The factors $i^{\alpha} - (i-1)^{\alpha}$ and $i^{\beta} - (i-1)^{\beta}$ on the right hand side of (3.3.17) are discrete analogous of the derivatives $(d/dt)t^{\alpha}$ and $(d/dt)t^{\beta}$, which are equal to $\alpha t^{\alpha-1}$ and $\beta t^{\beta-1}$, respectively. In turn, discrete analogues of the later functions are $\alpha i^{\alpha-1}$ and $\beta i^{\beta-1}$, respectively. Substituting these on the right hand side of (3.3.17) in place of $i^{\alpha} - (i-1)^{\alpha}$ and $i^{\beta} - (i-1)^{\beta}$ respectively. We arrive at 'naive' discrete version $h_0(i)$ given by (3.2.1). Thus via the classical and 'non-classical' discrete hazard rate functions $h(i)$ and $h_1(i)$ we have arrived at the discretized version of the original continuous additive Weibull hazard rate function $h(t)$.

Also, when applied on the right-hand side of (3.3.3), lead to the following definition of a discrete hazard rate function:

$$h_2(i) = 1 - q_1^{\alpha i^{\alpha-1}} q_2^{\beta i^{\beta-1}} \qquad (3.3.18)$$

Furthermore, the interpretation of the continuous additive Weibull model in terms of $X = \min\{W_1, W_2\}$ leads to yet another natural definition of the discrete hazard rate function:

$$h_3(i) = 2 - q_1^{i^{\alpha}-(i-1)^{\alpha}} q_2^{i^{\beta}-(i-1)^{\beta}} \qquad (3.3.19)$$

which is the sum of two discrete Weibull hazard rate functions. These two definitions alongside the earlier introduced ones has been explored on several data sets later in this chapter.

Conclude by highlighting an issue with the 'hazard rate function' (3.3.1). Namely, since the set $h_0(k), k \in N$ has not been defined in terms of the set $\{R(k), k \in N\}$, restoring the latter set from the former is problematic. In the case of the 'classical' and ‘non-classical’ discrete hazard rate functions this problem does not manifest itself. Indeed, in both cases

$$R(k) = \prod_{i=1}^{k}\big(1 - h(i)\big) = \prod_{i=1}^{k} e^{-h_1(i)} \qquad (3.3.20)$$

for all $k \in N$. Adopting the usual convention $\prod_{i=1}^{0}(\dots) = 1$, we have $R(0) = 1$. One may use the discrete hazard rate function of (3.2.1) and define the following discrete version of additive Weibull survival function:

$$R_0(k) = \prod_{i=1}^{k} e^{-h_0(i)} \qquad (3.3.21)$$

which differs from (3.3.20). Naturally, the set $\{R_0(k), k \in N\}$ differs from the set $\{R(k), k \in N\}$, where, $R(k) = \exp\big(-\lambda_1 k^{\alpha} - \lambda_2 k^{\beta}\big)$ since the two sets have been generated by two different discrete hazard rate functions have using the same generating procedure.

3.4 Empirical Illustrations

Authors (Mark Bebbington , Chin-Diew Lai , Morgan Wellington and Ricardas Zitikis) [2012] have examined how the various discrete versions of the additive Weibull distribution differ in their fit to data. In all cases they estimate the parameters $\lambda_1, \lambda_2, \alpha$ and β by maximizing the log-likelihood

$$\sum_{k=1:n_u} \log h(i_k) + \sum_{k=1:n_u} \log R(i_k - 1) + \sum_{l=1:n_c} \log R(j_l) \qquad (3.4.1)$$

where, $i_1, \dots, i_{n_u}$ denote the observed (uncensored) failure times, and $j_1, \dots, j_{n_c}$ the observed censored failure times. In the case of the continuous model, $h(i)$ and $R(i)$ are given by (3.2.2) and (3.2.1), respectively. As the classical model (3.3.3) and non classical model (3.3.17) are tied together by (3.3.16), they have the same parameters, which are obtained by maximizing (3.4.1) with $h(i)$ and $R(i)$ given by (3.3.3) and (3.3.20), respectively. Substituting the hazard rate (3.3.18) and reliability function (3.3.21) into (3.4.1) leads to another set of estimates, which can be interpreted as either the hazard rate (3.3.18), or as the non-classical version (3.3.1). Finally, the competing risks model (3.3.19) can be fitted using the reliability function $R(i) = R_A(i)R_B(i) = q_A^{\alpha^i} q_B^{\beta^i}$, the product of two discrete Weibull distributions (3.3.3), and hazard rate $h(i) = [R(i-1) - R(i)]/R(i-1)$ in (3.4.1).

In all the figures, the discrete survival and hazard functions are drawn as continuous curves rather than step functions to improve readability. In the figures, showing the fitted survival function, the Kaplan–Meier (''Empirical'') PL estimate will be shown by a grey step function.

The data sets are:

a) Time to failure of 18 electronic devices.
b) Failure times of 50 devices. The first two failure times are rounded up to 1
c) Lifetimes of 20 batteries.
d) Unit testing failure data. (311 observations).
e) BT-Serum reversal times (148 observations, 64 of them censored).

The estimated parameters and turning points are given in table 1

The case for data set A is shown in Fig 3.1, the model fits are almost indistinguishable, and all produce a bathtub shape, as is suggested by the parameter estimates in Table 1 and the theory developed above. The turning points in Table 3.1 are almost indistinguishable given the resolution of the data, although even here the various discrete versions differ slightly. In short a 'nice' example.

The slightly more complicated case of data set B is shown in Fig. 3.2. Here there are multiple failures at many integers, and hence a higher degree of resolution in the data. This is picked up by the models, with noticeable differences in the fit. Although all the fitted models are of bathtub shape, there are considerably differences in the estimated turning points, with important consequences for any burn-in-plan.

Case C, as shown in Fig 3.3 is a case where the data do not appear to support a bathtub shape. Instead, with its sigmoid curvature, the hazard rate functions appear to be reminiscent of that produced by the flexible Weibull, Bebbington, Lai and Zitikis [2007a]. The shapes of the various hazard rate functions are quite variable

In case D, as shown in Fig. 3.4, things become even more interesting. Notably, using a Weibull plot, as producing a bathtub shape, the MLE estimates are not compatible with a bathtub shape. Instead, with its sigmoid curvature, the continuous hazard rate function again appears to be reminiscent of that produced by the flexible Weibull, Bebbington et al. [2007a]. Most interestingly, this property is shared by the competing risks version of the discrete hazard rate $h_3(i)$. However, the remaining discrete versions are bathtub shaped.

Table 3.1:

Parameter estimation and turning points t_p				
Data set	$h(t)$ 'Continuous'	$h(i)$ and $h_1(i)$ 'Classical'	$h_0(i)$ and $h_2(i)$ 'Native'	$h_3(i)$'Competing Risk'
A	$\hat{\lambda}_1 = 9.2 \times 10^{-3}$ $\hat{\lambda}_2 = 6.3 \times 10^{-17}$ $\hat{\alpha} = 0.86$ $\hat{\beta} = 6.33$ $t_p = 140.1$	$\hat{\lambda}_1 = 9.9 \times 10^{-3}$ $\hat{\lambda}_2 = 1.1 \times 10^{-16}$ $\hat{\alpha} = 0.85$ $\hat{\beta} = 6.24$ $t_p = 140$	$\hat{\lambda}_1 = 1.1 \times 10^{-2}$ $\hat{\lambda}_2 = 3.1 \times 10^{-16}$ $\hat{\alpha} = 0.82$ $\hat{\beta} = 6.06$ $t_p = 139$	$\hat{\lambda}_1 = 8.6 \times 10^{-3}$ $\hat{\lambda}_2 = 3.9 \times 10^{-17}$ $\hat{\alpha} = 0.87$ $\hat{\beta} = 6.40$ $t_p = 139$
B	$\hat{\lambda}_1 = 0.060$ $\hat{\lambda}_2 = 5.9 \times 10^{-24}$ $\hat{\alpha} = 0.63$ $\hat{\beta} = 12.1$ $t_p = 48.2$	$\hat{\lambda}_1 = 0.120$ $\hat{\lambda}_2 = 4.7 \times 10^{-19}$ $\hat{\alpha} = 0.44$ $\hat{\beta} = 9.62$ $t_p = 42$	$\hat{\lambda}_1 = 2.6 \times 10^{4}$ $\hat{\lambda}_2 = 1.9 \times 10^{-16}$ $\hat{\alpha} = 5.3 \times 10^{-6}$ $\hat{\beta} = 8.28$ $t_p = 38$	$\hat{\lambda}_1 = 0.067$ $\hat{\lambda}_2 = 7.8 \times 10^{-11}$ $\hat{\alpha} = 0.56$ $\hat{\beta} = 5.33$ $t_p = 29$
C	$\hat{\lambda}_1 = 3.7 \times 10^{-4}$ $\hat{\lambda}_2 = 6.9 \times 10^{-81}$ $\hat{\alpha} = 1.12$ $\hat{\beta} = 23.0$ $t_p = NA$	$\hat{\lambda}_1 = 1.9 \times 10^{-4}$ $\hat{\lambda}_2 = 4.8 \times 10^{-60}$ $\hat{\alpha} = 1.21$ $\hat{\beta} = 17.0$ $t_p = NA$	$\hat{\lambda}_1 = 3.1 \times 10^{-4}$ $\hat{\lambda}_2 = 1.1 \times 10^{-74}$ $\hat{\alpha} = 1.15$ $\hat{\beta} = 21.2$ $t_p = NA$	$\hat{\lambda}_1 = 3.9 \times 10^{-4}$ $\hat{\lambda}_2 = 9.5 \times 10^{-40}$ $\hat{\alpha} = 1.11$ $\hat{\beta} = 11.2$ $t_p = NA$
D	$\hat{\lambda}_1 = 0.085$ $\hat{\lambda}_2 = 1.2 \times 10^{-8}$ $\hat{\alpha} = 1.27$ $\hat{\beta} = 6.76$ $t_p = NA$	$\hat{\lambda}_1 = 0.177$ $\hat{\lambda}_2 = 1.3 \times 10^{-5}$ $\hat{\alpha} = 0.88$ $\hat{\beta} = 4.43$ $t_p = 4$	$\hat{\lambda}_1 = 0.209$ $\hat{\lambda}_2 = 9.4 \times 10^{-6}$ $\hat{\alpha} = 0.82$ $\hat{\beta} = 4.51$ $t_p = 4$	$\hat{\lambda}_1 = 0.053$ $\hat{\lambda}_2 = 3.3 \times 10^{-10}$ $\hat{\alpha} = 7.90$ $\hat{\beta} = 5.18$ $t_p = NA$
E	$\hat{\lambda}_1 = 0.044$ $\hat{\lambda}_2 = 5.0 \times 10^{-85}$ $\hat{\alpha} = 0.52$ $\hat{\beta} = 32.9$ $t_p = 284.9$	$\hat{\lambda}_1 = 0.053$ $\hat{\lambda}_2 = 4.8 \times 10^{-78}$ $\hat{\alpha} = 0.48$ $\hat{\beta} = 30.1$ $t_p = 280$	$\hat{\lambda}_1 = 0.116$ $\hat{\lambda}_2 = 2.3 \times 10^{-68}$ $\hat{\alpha} = 0.36$ $\hat{\beta} = 26.3$ $t_p = 272$	$\hat{\lambda}_1 = 0.041$ $\hat{\lambda}_2 = 8.5 \times 10^{-63}$ $\hat{\alpha} = 0.53$ $\hat{\beta} = 24.1$ $t_p = 268$

The final example of data set E is illustrated in Fig. 3.5. The distinctive aspect of this data set is the large number of censored observations (64 of 148, predominantly the large values). All the fitted models are of bathtub shape, and very similar in shape, although the estimated turning points differ considerably.

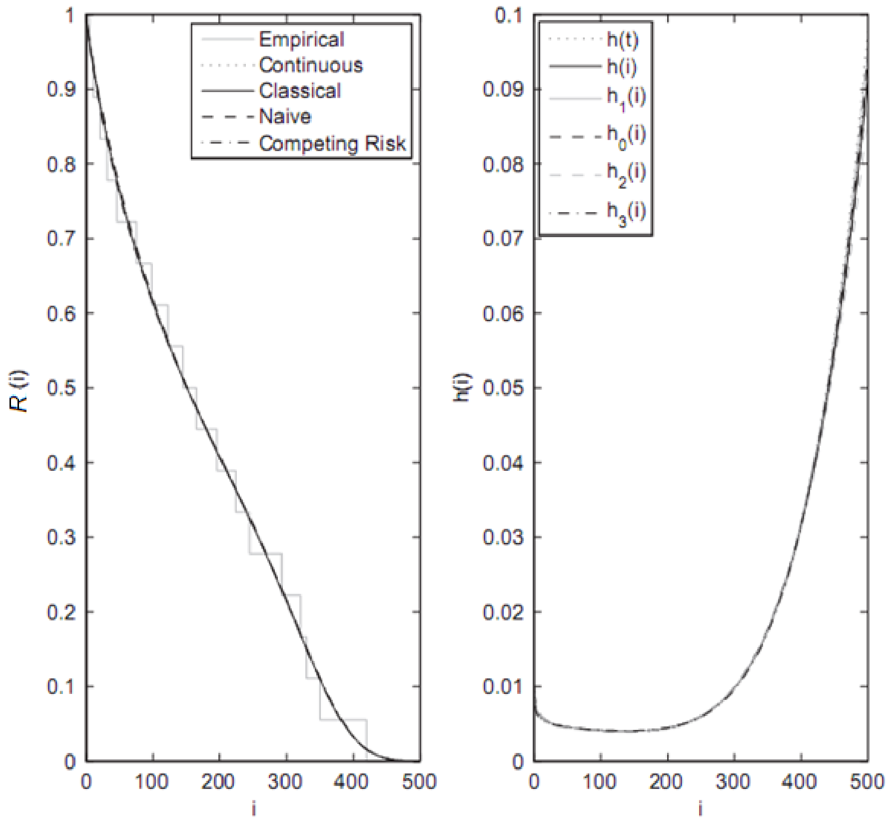

Fig 3.1: Estimated survival (left) and hazard (right) functions for data set A.

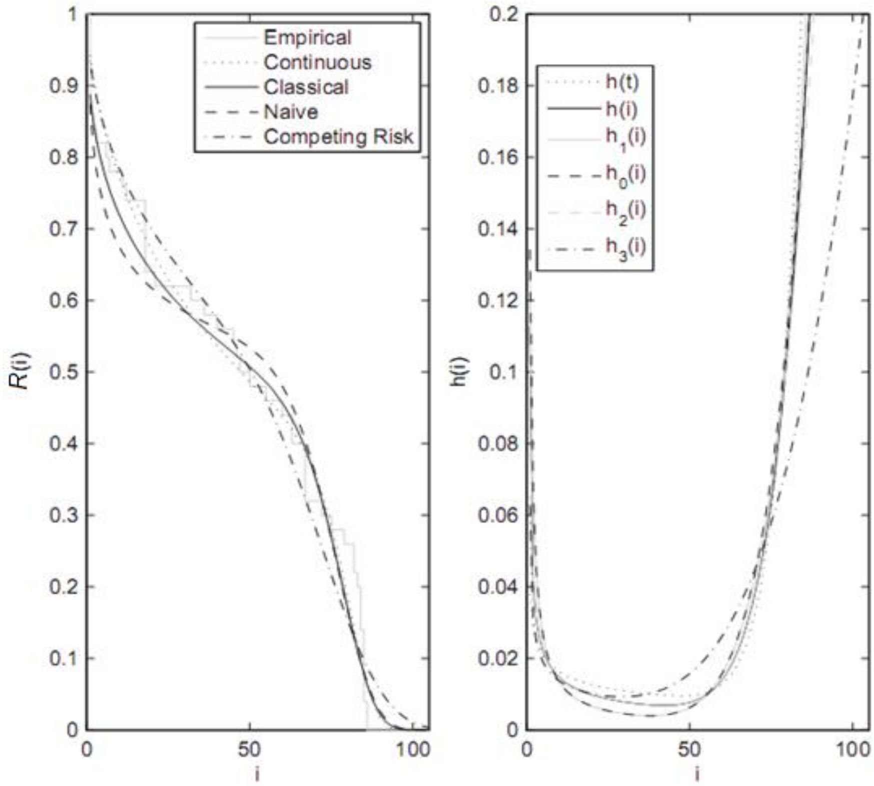

Fig 3.2: Estimated survival (left) and hazard (right) functions for data set B.

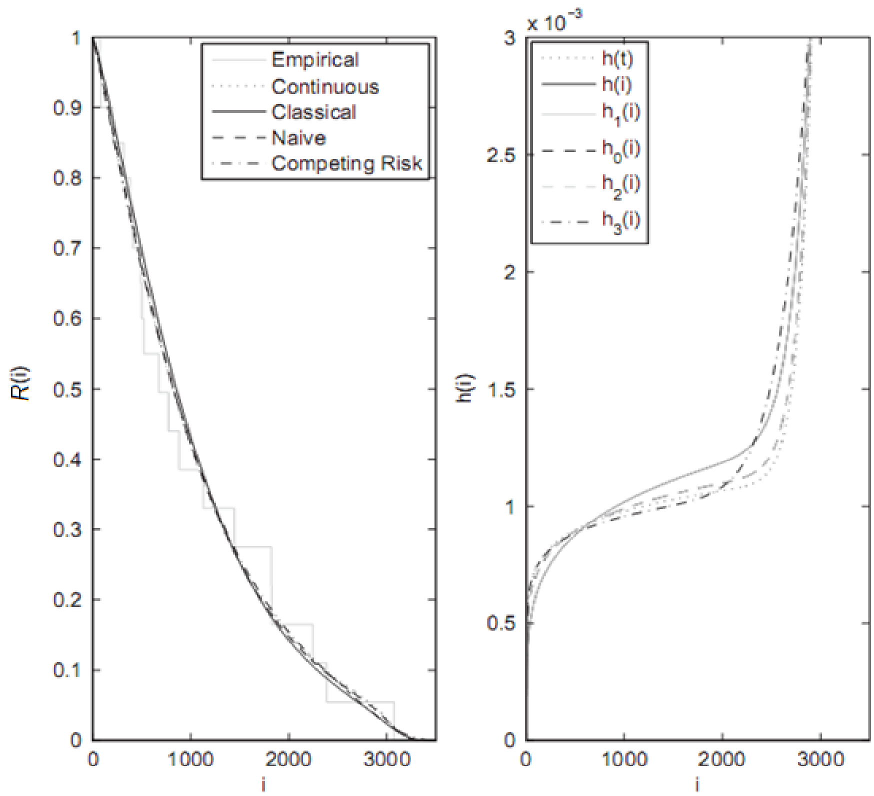

Fig 3.3: Estimated survival (left) and hazard (right) functions for data set C.

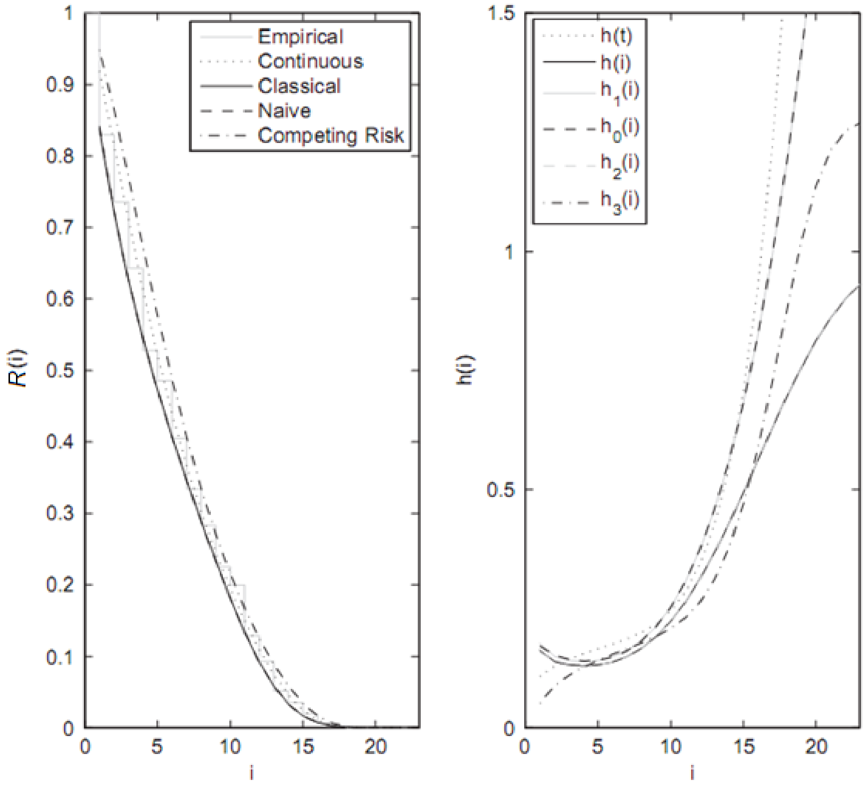

Fig 3.4: Estimated survival (left) and hazard (right) functions for data set D.

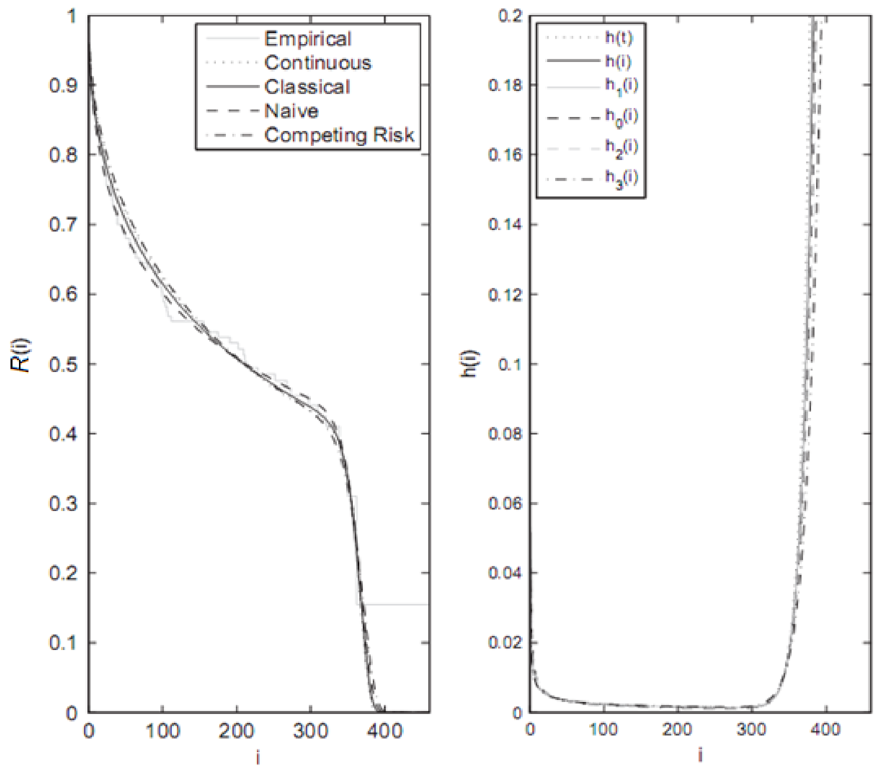

Fig 3.5: Estimated survival (left) and hazard (right) functions for data set E.

CHAPTER 4

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Bayes Estimators for Reliability Measures in Geometric Distribution

4.1 Introduction

Analysis of system life data is often used to derive estimation of the reliability of individual components in multi-component systems. Under ideal circumstances, this system life data contains the system time to failure along with information on the exact component that causes the system to fail. In many cases the exact component that causes the system to fail is unknown. Instead, it may only be known that the failing component is one of a subset of components that are considered potentially responsible for the failure. When this occurs, the cause of the system fail is masked, Usher and Hodgson [1988]. Masking may occur due to variety of reasons. Some of these reasons, for example, are: the cost, time constraints associated with failure analysis, recording errors, lack of proper diagnostic components, and the destructive nature of certain component failures that makes exact diagnosis impossible. However, in many situations the exact cause of the system failure, while still unknown, can be isolated to some subset of system components. That is, many components might be isolated as potential causes based on such things as the mode of system failure or some brief diagnostic routine. For example, consider that large computer system has failed. A repairman may immediately be able to find the cause of system failure down to a single circuit card (that may contain many components). In order to minimize the system downtime he may replace the entire circuit card with a new one and hence he never knew exactly which component caused the system failure. While the exact cause of the system failure is still unknown, the isolation to a smaller subset of possible causes provides additional information that can help in the estimation process. This motivates the need for using masked system life data in estimation process. Under a simplifying assumption that components have constant failure rates, Miyakawa [1984] considered a 2-component series system and he derived closed form expressions for the maximum likelihood estimate when some of the sample observations are masked. Under the same exponential assumption, Usher and Hodgson [1988] extended Miyakawas's results to a three component system.

These previously developed models are based upon the assumption that masking occurs independently of the true cause of system failure. That is, the probability of observing a particular masked set does not depend upon which component failed in the system. However, in some cases, this assumption may not hold. For instance, consider the case of a circuit card with two components where, under certain environmental conditions, the failure of either component can result in fire and complete destruction of circuit card. If the card is destroyed, then cause of failure cannot be identified. Dependence occurs when the probability of card's destruction differs based upon which component fails. Moreover this probability of destruction given that a particular component failed may depend on time but it seems reasonable to assume that the ratio of these masking probabilities will not be a function of time. The relevance of the independence assumption and its effect on the development of the likelihood function were first described by Guess, Usher and Hodgson [1991]. The effect of the dependency between the masking and the true cause of the system failure was investigated by Lin, Usher, and Guess [1993]. They suggested a simple means of checking for the independence via sub-sampling for the case when the system's components have constant failure rates (exponentially distributed).

Tan [2007] estimated reliability of components in series and parallel systems from masking system testing data. He takes into account a second type of uncertainty: censored lifetime, when system components have constant failure rates. To efficiently estimate failure rates of system components in presence of combined uncertainty, he proposes a useful concept for components: equivalent failure and equivalent lifetime. For a component in a system with known status and lifetime, its equivalent failure is defined as its conditional failure probability and its equivalent lifetime is its expectation of lifetime. For various uncertainty scenarios, he derives equivalent failures and test times for individual components in both series and parallel systems

In this chapter, the problem on a series system consisting of two components, illustrate by Ammar M. Sarhana and Debasis Kundu [2008] is presented. Bayesian estimators of the reliability measures (the failure rate, reliability function and the mean time to failure) of the individual components in a multi-component series system are obtained when the life time of each component has a geometric distribution, using masked system life test data.

4.2 The Model Assumptions

Throughout this chapter the following assumptions are considered.

Assumptions

1.1. The system is series with $J, J \geq 2$, independent components.

1.2. N identical systems are put on the life test. The test is terminated when n systems

failed. That is the data are censored.

1.3. The random variables $X_{ij}, j = 1, 2, \dots, J;\ i = 1, 2, \dots, N$ are independent with

$X_{1j}, X_{2j}, \dots, X_{Nj}$ being identical and having geometric distribution with parameter p_j.

1.4. The observable quantities for the system i, which failed, on the test are: (i) the random variable T_i, represents the number of success trials of using system i to get its first failure, and (ii) a set S_i of system's components that may cause the system I failed. But for the censored observation, we only observe $X_i, i = n + 1, \cdots, N$. The data collected from this process are

$(X_1, S_1), (X_2, S_2), \ldots, (X_n, S_n), (X_{n+1},*), \cdots, (X_N,*)$. Here $(X,*)$ means the observation is censored.

1.5. Masking is s-independent of the true cause of system failure. That is, for all $\ell, j \in S_i$, $P(S_i = s_i | T_i = t_i, K_i = j) = P(S_i = s_i | T_i = t_i, K_i = \ell)$, where K_i denotes the index of the component causes the system i to fail.

1.6. The system may fail due to component 1 or component 2 or both components 1 and 2.

Based on the assumption (1.3), for $j = 1, 2, \ldots, J$, the random variables $X_{1j}, X_{2j}, \ldots, X_{Nj}$ can be written as a random variable X_j, $j = 1, 2, \ldots, J$, having geometric distribution with probability of success q_j. That is, $p_j = 1 - q_j$ denotes to the failure probability of component j 'per time-unit' or 'at each time' $0 < p_j < 1$, $j = 1, 2, \ldots, J$. That is, the probability mass function of X_j, $j = 1, 2, \ldots, J$, is given by

$$f_j(x) = p_j q^{j-1}, x = 1,2, \ldots \tag{4.2.1}$$

The reliability function of X_j, $j = 1, 2, \ldots, J$, is

$$F_j(x) = q^j, x = 1,2, \ldots \tag{4.2.2}$$

The mathematical expectation of X_j, say M_j, $j = 1, 2, \ldots, J$, is

$$M_j = E[X_j] = \frac{1}{p_j} \tag{4.2.3}$$

There are two different definitions of the failure rate function of the discrete distributions. In that follows we present the two different forms of the failure rate functions denoted respectively by $\lambda_j(x)$ and $r_j(x)$, $j = 1, 2, \ldots, J$.

1. According to Barlow, Marshall and Proschan [1963], the hazard rate function of X_j, $j = 1, 2, \ldots, J$ is

$$\lambda_j(x) = P(X_j = x / X_j \geq x) = 1 - \frac{\bar{F}_j(x)}{\bar{F}_j(x-1)} = p_j \tag{4.2.4}$$

2. According to Roy and Gupta [1999] and Xie, Gaudoin and Bracquemond [2002], the hazard rate function of X_j, $j = 1, 2, \ldots, J$ is

$$r_j(x) = \ln\frac{\bar{F}_j(x-1)}{\bar{F}_j(x)} = -\ln(1-p_j) \qquad (4.2.5)$$

There is a simple relation between $\lambda_j(x)$ and $r_j(x)$, Xie et al. [2002],

$$\lambda_j(x) = 1 - e^{-r_j(x)}$$

In this chapter the aim is to estimate the reliability measures of the individual components $\lambda_j(x) = p_j, r_j(x), \bar{F}_j(x)$ and M_j based on masked system life test data.

4.3 The Likelihood Function

Based on the random sample $(X_1, S_1), (X_2, S_2), \dots, (X_n, S_n), (X_{n+1}, *), \cdots, (X_N, *)$, the likelihood function is Gauss et al. [1991]

$$L(data; p_1, \dots, p_m) = \prod_{i=1}^{n}\left[\sum_{j\in s_i}\left(f_i(x_i)\prod_{\ell=i,l\neq j}^{J}\bar{F}_l(x_i)\right)\right]\prod_{k=n+1}^{N}\bar{F}(x_k) \qquad (4.3.1)$$

where, $\bar{F}(x) = \prod_{j=1}^{J}\bar{F}_k(X)$ is the survival function of the system. Then for the geometric distribution model, we have

$$L(data; p_1, \dots, p_m) = \prod_{i=1}^{n}\left[\sum_{j\in s_i}\left(p_j(1-p_j)^{x_i-1}\prod_{\ell=i,l\neq j}^{J}(1-p_j)^{x_i}\right)\right]\prod_{k=n+1}^{N}\prod_{j=1}^{J}(1-p_j)^{x_i} \qquad (4.3.2)$$

From now and henceforth, assume that $J = 2$. That is the system consists of two components. In this case we need the following notations. Let n_1 be number observations when the component 1 causes the system failure. That is n_1 is the number of the observation when $S_i = \{1\}$. Let t_i be the observed value of X when $S_i = \{1\}, i = 1,2,\dots,n_1$. Let n_2 be number observations when the component 2 causes the system failure. That is n_2 is the number of the observation when $S_i = \{2\}$. Let y_i be the observed value of X when $S_i = \{2\}, i = 1,2,\dots,n_2$. Let n_0 be number observations when both components1 and 2 cause the system failure. That is n_0 denotes the number of the observation when $S_i = \{0\}$. Here we mean by $S_0 = \{0\}$ that the cause of system i failure is due to both components1 and 2. Let z_i be the ob served value of X when $S_i =$

$\{0\}, i = 1,2,\dots,n_0$. Also, let n_{12} be the number of observation when the cause of system failure is masked (either component 1 or component 2 or both components 1 and 2). That is n_{12} is the number of observation when $S_i = \{1, 2\}$. Let z_i be the observed value of X when $S_i = \{1,2\}, i = 1,2,\dots,n_{12}$. Thus, the likelihood function (2.7), in this case, reduces to

$$L(data; p_1, p_2)$$
$$= \left(\prod_{i=1}^{n_1} p_1 q_1^{t_i-1} q_2^{t_i} \prod_{j=1}^{n_2} p_2 q_2^{y_j-1} q_2^{y_j} \prod_{j=1}^{n_0} p_1 p_2 (q_1 q_2)^{z_j-1}\right) \prod_{k=n+1}^{N} (q_1 q_2)^{x_k}$$
$$\times \prod_{\ell=1}^{n_{12}} \left(p_1 q_1^{z_\ell-1} q_2^{z_\ell} + p_2 q_2^{z_\ell-1} q_2^{z_\ell} + p_1 p_2 (q_1 q_2)^{z_\ell-1}\right)$$

$$\text{where}, q_j = 1 - p_j \ , \qquad j = 1,2,3,\dots$$

$$L(data; p_1, p_2) = p_1^{n_1} (1-p_1)^{\sum_{i=1}^{n_i}(t_i-1)} (1-p_2)^{\sum_{i=1}^{n_2} t_i}$$
$$\times\ p_2^{n_2} (1-p_2)^{\sum_{j=1}^{n_2}(y_j-1)} (1-p_1)^{\sum_{j=1}^{n_2} y_j}$$
$$\times\, p_1^{n_0} p_2^{n_0} (1-p_1)^{\sum_{j=1}^{n_0}(z_j-1)}\ (1-p_2)^{\sum_{j=1}^{n_0}(z_j-1)}$$
$$\times\, (1-p_1)^{\sum_{k=n+1}^{N} x_k} (1-p_2)^{\sum_{k=n+1}^{N} x_k}$$
$$\times \prod_{\ell}^{n_{12}} \left[q_1^{z_\ell-1} q_2^{z_\ell-1} (p_1 q_2 + p_2 q_1 + p_1 p_2)\right]$$

$$L(data; p_1, p_2) = p_1^{n_1+n_0} p_2^{n_2+n_0}$$
$$\times\, (1-p_1)^{\sum_{i=1}^{n_i} t_i + \sum_{j=1}^{n_2} y_j + \sum_{j=1}^{n_0} z_j + \sum_{k=n+1}^{N} x_k - n_1 - n_0}$$
$$\times\, (1-p_2)^{\sum_{i=1}^{n_i} t_i + \sum_{j=1}^{n_2} y_j + \sum_{j=1}^{n_0} z_j + \sum_{k=n+1}^{N} x_k - n_2 - n_0}$$
$$\times\, (1-p_1)^{\sum_{\ell}^{n_{12}} z_\ell - n_{12}} (1-p_2)^{\sum_{\ell}^{n_{12}} z_\ell - n_{12}}$$
$$\times\, (p_1 - p_1 p_2 + p_2 - p_1 p_2 + p_1 p_2)^{n_{12}}$$

$$L(data; p_1, p_2) = p_1^{n_1+n_0} p_2^{n_2+n_0} (1-p_1)^{T-n_1-n_{12}} (1-p_2)^{T-n_2-n_{12}}$$

$$\times (p_1 + p_2 - p_1p_2)^{n_{12}}$$

where, $$T = \sum_{i=1}^{N} x_i \qquad (4.3.3)$$

By using binomial expansion of $(p_1 + p_2 - p_1p_2)^{n_{12}}$,the likelihood function (4.3.3) can be written as

$$L(data; p_1, p_2) = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} {p_1}^{n_{12}+n_1+n_0+j-i} (1 - p_1)^{T-n_1-n_{12}}$$

$$\times {p_2}^{n_{12}+n_2+n_0+j} (1 - p_2)^{T-n_2-n_{12}} \qquad (4.3.4)$$

4.4 Bayesian Analysis

Bayesian estimators for the reliability measures of the individual components in the system have been obtained, for that the following additional assumptions are assumed:

Assumptions

2.1. The parameters p_1 and p_2 behave as independent random variables.

2.2. The random variable p_j has Beta prior distribution with known shape and scale parameters α_j and $\beta_j, j = 1, 2$. That is, the prior probability density function (pdf) of $p_j, j = 1, 2$, takes the following form

$$g_j(p_j) = \frac{1}{B(\alpha_j , \beta_j)} p_j^{\alpha_j - 1} (1 - \beta_j)^{\beta_j - 1} \quad , \qquad 0 < p_j < 1 \qquad (4.4.1)$$

2.3. The loss incurred when p_1 and p_2 are estimated, respectively, by $\hat{p}_1$ and $\hat{p}_2$ is a quadratic

$$\mathcal{L}\big((p_1, p_2), (\hat{p}_1, \hat{p}_2)\big) = k_1(\hat{p}_1 - p_1)^2 + k_2(\hat{p}_2 - p_2)^2 , k_1, k_2 > 0 \qquad (4.4.2)$$

The Beta prior distribution is assumed not only to give nicely results but also permits closed forms of the required estimators in terms of Beta functions.

***Theorem 4.4.1*:** Based on the assumptions 2.1 to 2.3, the joint posterior pdf of p_1, p_2 is

$$g(p_1,p_2)=\frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{l=1}^{2}p_\ell^{a_\ell-1}(1-p_\ell)^{b_\ell-1}$$

where, $0<p_1,p_2<1$

$a_\ell=n_{12}+n_0+n_\ell-(-1)^\ell j-(2-l)i$

$b_\ell=T+\beta_\ell-n_\ell-n_{12}$, $\ell=1,2$ and

$$I_0=\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{\ell=1}^{2}B(a_\ell,b_\ell)$$

Proof: Based on the assumptions 2.1 and 2.2, the joint prior pdf of p_1 and p_2 be come

$$g(p_1,p_2)=\frac{p_1^{\alpha_1-1}(1-p_1)^{\beta_1-1}p_2^{\alpha_2-1}(1-p_2)^{\beta_2-1}}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)}\ ,0<p_1,p_2<1 \qquad (4.4.3)$$

But the joint posterior pdf of p_1,p_2 is related with their joint prior pdf and the likelihood function according to the following relation, see Martz and Waller [1982],

$$g(p_1,p_2/\text{data})=\frac{g(p_1,p_2)L(\text{data};p_1,p_2)}{\int_0^1\int_0^1 g(p_1,p_2)L(\text{data};p_1,p_2)dp_1dp_2} \qquad (4.4.4)$$

Substituting from equations (2.9) and (3.3) into (4.4.4) we get

$$g(p_1,p_2/\text{data})=\frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{=1}^{2}p_\ell^{a_\ell-1}(1-p_\ell)^{b_\ell-1}$$

where, $0<p_1,p_2<1$

$$I_0=\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{\ell=1}^{2}\int_0^1 p_\ell^{a_\ell-1}(1-p_\ell)^{b_\ell-1}\ dp_\ell \qquad (4.4.5)$$

but

$$\int_0^1 p_\ell^{a_\ell-1}(1-p_\ell)^{b_\ell-1}\ dp_\ell=\ B(a_\ell,b_\ell) \qquad (4.4.6)$$

using (4.4.6) in (4.4.5), one get I_0 as given by

$$I_0 = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{\ell=1}^{2} B(a_\ell, b_\ell)$$

Corollary 1**:** The marginal posterior pdf's of p_1 and p_2 are give, respectively, by

$$g_1(p_1/\text{data}) = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i}\, B(a_2, b_2)\, {p_1}^{a_1-1}(1-p_1)^{b_1-1}$$

$$,0 < p_1 < 1 \qquad (4.4.7)$$

and

$$g_2(p_2/\text{data}) = \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i}\, B(a_1, b_1)\, {p_2}^{a_2-1}(1-p_2)^{b_2-1}$$

$$,0 < p_2 < 1 \qquad (4.4.8)$$

Proof**:** The relation between the joint and marginal pdf's given by

$$g_\ell(p_\ell/\text{data}) = \int_0^1 g(p_1, p_2/\text{data}) \left[\delta_{\ell 1} d_{p_2} + \delta_{\ell 2} d_{p_1}\right]$$

where, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, One can deduce $g_1(p_1/\text{data})$ and $g_2(p_2/\text{data})$.

Now,

$$g_1(p_1/\text{data}) = \int_0^1 g(p_1, p_2/\text{data}) \left[\delta_{11} d_{p_2} + \delta_{12} d_{p_1}\right]$$

$$g_1(p_1/\text{data}) = \int_0^1 g(p_1, p_2/\text{data})\ d_{p_2}$$

$$g_1(p_1/\text{data}) = \int_0^1 \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{\ell=1}^{2} {p_\ell}^{a_\ell-1}(1-p_\ell)^{b_\ell-1}\ d_{p_2}$$

$$g_1(p_1/\text{data}) = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\left(\int_0^1 {p_2}^{a_2-1}(1-p_2)^{b_2-1}\, d_{p_2}\right)$$

$$\times\ {p_1}^{a_1-1}(1-p_1)^{b_1-1}$$

$$g_1(p_1/\text{data}) = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_2,b_2)\, {p_1}^{a_1-1}(1-p_1)^{b_1-1}$$

Similarly,

$$g_2(p_2/\text{data}) = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_1,b_1)\, {p_2}^{a_2-1}(1-p_2)^{b_2-1}$$

***Corollary 2*:** The following statements are fulfilled for all $m = 1, 2, ..$.

$$\mu_\ell^{(m)} = \frac{J_\ell^{(m)}}{I_0}\,, \tag{4.4.9}$$

where,

$$J_\ell^{(m)} = \sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2} B(a_k + m\delta_{k\ell}, b_k) \tag{4.4.10}$$

***Proof*:** The marginal posterior m^{th} moment of p_ℓ is related with the marginal posterior pdf of p_ℓ, $\ell = 1,2$, according to the following relation

$$\mu_\ell^{(m)} = \int_0^1 p_\ell^m g_\ell(p_\ell/\text{data})\, dp_\ell \tag{4.4.11}$$

for $\ell = 1$, substituting from (4.4.7) into (4.4.11) we have

$$\mu_1^{(m)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_2,b_2)\int_0^1 {p_1}^{m+a_1-1}(1-p_1)^{b_1-1}\, dp_1$$

$$= \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_2,b_2)\, B(a_1+m, b_1)$$

$$= \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2} B(a_k + m\delta_{k1}, b_k)$$

Similarly for $\ell = 2$, we can derive

$$\mu_2^{(m)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2}B(a_k + m\delta_{k2}, b_k)$$

***Theorem 4.4.2*:** Under the group of assumptions 1 and 2:

1. The Bayesian estimator for p_ℓ, $\ell = 1,2$, is

$$\hat{p}_\ell = \frac{J_\ell^{(1)}}{I_0} \tag{4.4.12}$$

2. The minimum posterior risk associated with $\hat{p}_\ell$, say $R_{\hat{p}_\ell}$, is

$$R_{\hat{p}_\ell} = \frac{J_\ell^{(2)}}{I_0} - \left(\frac{J_\ell^{(1)}}{I_0}\right)^2 \tag{4.4.13}$$

***Proof*:** Under the squared error loss, the Bayesian estimators for an unknown parameter are defined as its posterior expectation and the associated minimum posterior risk is the posterior variance, Martz and Waller [1982]). That is, the Bayesian estimator for p_ℓ is

$$\hat{p}_\ell = E[p_\ell/\text{data}] = \mu_\ell^{(1)}, \ell = 1,2. \tag{4.4.14}$$

for $\ell = 1$and $m = 1$

$$\hat{p}_1 = \mu_1^{(1)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2}B(a_k + \delta_{k1}, b_k)$$

$$= \frac{J_1^{(1)}}{I_0}$$

for $\ell = 2$and $m = 1$

$$\hat{p}_2 = \mu_2^{(1)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2}B(a_k + \delta_{k2}, b_k)$$

$$= \frac{J_2^{(1)}}{I_0}$$

and the minimum posterior risk associated with $\hat{p}_\ell$ is

$$R_{\hat{p}_\ell} = Var[p_\ell/data] = \mu_\ell^{(2)} - \left[\mu_\ell^{(1)}\right]^2, \ell = 1,2. \qquad (4.4.15)$$

for $\ell = 1$

$$R_{\hat{p}_1} = \mu_1^{(2)} - \left[\mu_1^{(1)}\right]^2$$

$$= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k + 2\delta_{k1}, b_k)$$

$$- \left(\frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k + \delta_{k1}, b_k) \right)^2$$

$$= \frac{J_1^{(2)}}{I_0} - \left(\frac{J_1^{(1)}}{I_0} \right)^2$$

Similarly, for $\ell = 2$

$$R_{\hat{p}_1} = \mu_2^{(2)} - \left[\mu_2^{(1)}\right]^2$$

$$= \frac{J_2^{(2)}}{I_0} - \left(\frac{J_2^{(1)}}{I_0} \right)^2$$

***Theorem 4.4.3*:** Under the group of assumptions 1 and 2:

1. The Bayes estimator for the reliability function $\bar{F}_j(x_0), \ell = 1,2$ is

$$\hat{\bar{F}}_\ell(x_0) = \frac{k_\ell^{(1)}}{I_0} \qquad (4.4.16)$$

2. The minimum posterior risk associated with $\hat{\bar{F}}_\ell(x_0)$, say $R_{\hat{\bar{F}}_\ell}$, is

$$R_{\hat{\hat{F}}_\ell} = \frac{k_\ell^{(2)}}{I_0} - \left(\frac{k_\ell^{(1)}}{I_0}\right)^2 \qquad (4.4.17)$$

where for $\ell = 1,2,$ and $m = 1,2,\dots$

$$K_\ell^{(m)} = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} \prod_{k=1}^{2} B(a_k, b_k + m x_0 \delta_{k\ell}) \qquad (4.4.18)$$

***Proof*:** The Bayesian estimator for a function of p_ℓ and the associated minimum posterior risk are defined respectively as the posterior expectation and variance of that function, Martz and Waller [1982]. That is, the Bayesian estimator for $\bar{F}_j(x_0), \ell = 1,2$ is

$$\hat{\hat{F}}_\ell(x_0) = E[\bar{F}_j(x_0)/\text{data}] = \int_0^1 (1 - p_\ell)^{x_0} g_\ell(p_\ell/\text{data}) dp_\ell \qquad (4.4.19)$$

Put $\ell = 1$, we get

$$\hat{\hat{F}}_1(x_0) = \int_0^1 (1 - p_1)^{x_0} g_1(p_1/\text{data}) dp_1$$

$$= \int_0^1 (1 - p_1)^{x_0} \left(\frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} B(a_2, b_2) {p_1}^{a_1-1} (1 - p_1)^{b_1-1} \right) dp_1$$

$$= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} B(a_2, b_2) \int_0^1 {p_1}^{a_1-1} (1 - p_1)^{b_1+x_0-1} dp_1$$

$$= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i} B(a_2, b_2) B(a_1, b_1 + x_0) = \frac{k_1^{(1)}}{I_0}$$

where,

$$k_1^{(1)} = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i}\ B(a_1, b_1 + x_0)\ B(a_2, b_2)$$

Similarly, put $\ell = 2$, we get

$$\hat{\bar{F}}_2(x_0) = \frac{k_2^{(1)}}{I_0}$$

where,

$$k_2^{(1)} = \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i}\ B(a_1, b_1)\ B(a_2, b_2 + x_0)$$

and the associated minimum posterior risk is

$$R_{\hat{\bar{F}}_\ell} = V\left[\bar{F}_j(x_0)/\text{data}\right]$$

$$= E\left[\left\{\bar{F}_j(x_0)\right\}^2/\text{data}\right] - \left\{E\left[\bar{F}_j(x_0)/\text{data}\right]\right\}^2$$

$$= \int_0^1 (1 - p_\ell)^{2x_0}\, g_\ell(p_\ell/\text{data})dp_\ell$$

$$- \left\{\int_0^1 (1 - p_\ell)^{x_0}\, g_\ell(p_\ell/\text{data})dp_\ell\right\}^2 \qquad (4.4.20)$$

put $\ell = 1$, we get

$$R_{\hat{\bar{F}}_1} = \int_0^1 (1 - p_1)^{2x_0}\, g_1(p_1/\text{data})dp_1 - \left\{\int_0^1 (1 - p_1)^{x_0}\, g_1(p_1/\text{data})dp_1\right\}^2$$

$$= \frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i}\, B(a_2, b_2)\, B(a_1, b_1 + 2x_0)$$

$$- \left(\frac{1}{I_0} \sum_{i=0}^{n_{12}} \sum_{j=0}^{i} \binom{n_{12}}{i} \binom{i}{j} (-1)^{n_{12}-i}\, B(a_2, b_2)\, B(a_1, b_1 + x_0)\right)^2$$

$$R_{\hat{\hat{F}}_1} = \frac{k_1^{(2)}}{I_0} - \left(\frac{k_1^{(1)}}{I_0}\right)^2$$

where,

$$k_1^{(1)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_1, b_1 + x_0)B(a_2, b_2)$$

and,

$$k_1^{(2)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_1, b_1 + 2x_0)\, B(a_2, b_2)$$

Similarly, put $\ell = 2$, we get

$$R_{\hat{\hat{F}}_2} = \frac{k_2^{(2)}}{I_0} - \left(\frac{k_2^{(1)}}{I_0}\right)^2$$

where,

$$k_2^{(1)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_1, b_1)B(a_2, b_2 + x_0)$$

and,

$$k_2^{(2)} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_1, b_1)\, B(a_2, b_2 + 2x_0)$$

Theorem 4.4.4**:** Under the previous assumptions:

1. The Bayesian estimator for $r_\ell(x_0), \ell = 1,2$ is

$$\hat{r}_\ell(x_0) = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}[\psi(a_\ell + b_\ell) - \psi(b_\ell)]\prod_{k=1}^{2} B(a_k, b_k)$$

(4.4.21)

2. The minimum posterior risk associated with $\hat{r}_\ell(x_0)$ is

$$R_{\hat{r}_\ell} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2}B(a_k, b_k)$$

$$\times \{[\psi(b_\ell) - \psi(a_\ell + b_\ell)]^2 + \psi'(b_\ell) - \psi'(a_\ell + b_\ell)\} \qquad (4.4.22)$$

$$-\left\{\frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}[\psi(b_\ell) - \psi(a_\ell + b_\ell)]\prod_{k=1}^{2}B(a_k, b_k)\right\}^2,$$

where, $\psi(z)$ and $\psi'(z)$ are digamma and poly-gamma functions defined, respectively, as

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \psi'(z) = \frac{d\psi(z)}{dz}.$$

***Proof*:** As before, the Bayes estimator for $r_\ell(x_0)$ is given by

$$\hat{r}_\ell(x_0) = \int_0^1 r_\ell(x_0)\, g_\ell(\, p_\ell/\text{data})dp_\ell$$

substituting from (4.4.5) into the above relation, we get

$$\hat{r}_\ell(x_0) = \int_0^1 -\ln[1 - p_\ell]\, g_\ell(\, p_\ell/\text{data})dp_\ell$$

$$= -E[\ln(1 - p_\ell)/\text{data}] \qquad (4.4.23)$$

where,

$$E[\ln(1 - p_\ell)/\text{data}] = \int_0^1 \ln[1 - p_\ell]\, g_\ell(\, p_\ell/\text{data})dp_\ell \qquad (4.4.24)$$

for $\ell = 1$, using the form of the function $g_1(\, p_1/\text{data})$ together with (4.4.24), one can derive that

$$E[\ln(1 - p_1)/\text{data}] = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}$$

$$\times\ B(a_2, b_2)\, \mathcal{K}(a_1, b_1) \qquad (4.4.25)$$

where,

$$\mathcal{K}(a_1, b_1) = \int_0^1 p_1{}^{a_1-1}(1-p_1)^{b_1-1} \ln[1- p_1]dp_1$$

let $x = 1 - p_1$, then

$$\mathcal{K}(a_1, b_1) = \int_0^1 x^{b_1-1}(1-x)^{a_1-1} \ln x \, dp_1$$

also,

$$\mathcal{K}(a_1, b_1) = B(a_1, b_1)[\psi(b_1) - \psi(a_1 + b_1)] \qquad (4.4.26)$$

substituting from (4.4.26) into (4.4.25), one can get

$$E[\ln(1- p_1)/\text{data})] = \frac{-1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}$$

$$\times [\psi(b_1) - \psi(a_1 + b_1)]\prod_{k=1}^{2} B(a_k, b_k) \qquad (4.4.27)$$

substituting from (4.4.27) into (4.4.23) we get $r_1(x_0)$ as given by (4.4.21) when $\ell = 1$. Similarly we can prove that the relation (4.4.21) is correct for $\ell = 2$. Let us now prove that the relation (4.4.22) is fulfilled. As it was stated before, the minimum posterior risk associated with $r_\ell(x_0)$ is the posterior variance of $r_\ell(x_0)$ that is

$$R_{\hat{r}_\ell} = E\left[r_\ell^2(x_0)/\text{data}\right] - \{E[r_\ell(x_0)/\text{data}]\}^2$$

$$= E\left[\left(-\ln(1- p_\ell)\right)^2/\text{data}\right] - \{E[-\ln(1- p_\ell)/\text{data}]\}^2$$

$$= E[\{\ln(1- p_\ell)\}^2/\text{data}] - \{E[\ln(1- p_\ell)/\text{data}]\}^2 \qquad (4.4.28)$$

for $\ell = 1$, $E[\ln(1- p_1)/\text{data}]$ is given by (4.4.27) and $E[\{\ln(1- p_1)\}^2/\text{data}]$ can be derived as follows

$$E[\{\ln(1- p_1)\}^2/\text{data}] = \int_0^1 \{\ln(1- p_1)\}^2 g_1(p_1/\text{data})dp_1 \qquad (4.4.29)$$

substituting from (4.4.7) into (4.4.28), one can get

$$E[\{\ln(1-\ p_1)\}^2/\text{data}] = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_2,b_2)\,\mu(a_1,b_1)$$

(4.4.30)

where,

$$\mu(a_1,b_1) = \int_0^1 {p_1}^{a_1-1}(1-p_1)^{b_1-1}\ln^2(1-\ p_1)dp_1$$

let $x = 1-\ p_1$, then

$$\mu(a_1,b_1) = \int_0^1 x^{b_1-1}(1-x)^{a_1-1}\ln^2 x\, dp_1$$

also,

$$\mu(a_1,b_1) = B(a_1,b_1)[\{\psi(b_1)-\psi(a_1+b_1)\}^2+\psi'(b_1)-\psi'(a_1+b_1)]$$

(4.4.31)

substituting from (4.4.31) into (4.4.30)

$$E[\{\ln(1-\ p_1)\}^2/\text{data}] = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\, B(a_2,b_2)$$

$$\times\ B(a_1,b_1)[\{\psi(b_1)-\psi(a_1+b_1)\}^2+\psi'(b_1)-\psi'(a_1+b_1)]$$

therefore,

$$R_{\hat{r}_1} =\ E[\{\ln(1-\ p_1)\}^2/\text{data}] - \{E[\ln(1-\ p_1)/\text{data}]\}^2$$

$$= \quad \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\ \prod_{k=1}^{2} B(a_k,b_k)$$

$$\times\ [\{\psi(b_1)-\psi(a_1+b_1)\}^2+\psi'(b_1)-\psi'(a_1+b_1)]$$

$$-\left\{\frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\ [\psi(b_1)-\psi(a_1+b_1)]\prod_{k=1}^{2}B(a_k,b_k)\right\}^2$$

similarly for $\ell = 2$

$$R_{\hat{r}_2} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\ \prod_{k=1}^{2}B(a_k,b_k)$$

$$\times\ [\{\psi(b_2)-\psi(a_2+b_2)\}^2+\psi'(b_2)-\psi'(a_2+b_2)]$$

$$-\left\{\frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\ [\psi(b_2)-\psi(a_2+b_2)]\prod_{k=1}^{2}B(a_k,b_k)\right\}^2$$

Also, under the groups of assumptions 1, 2:

1. The Bayes estimator for $M_\ell,\ \ell = 1,2$, is

$$\widehat{M}_\ell = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2}B(a_k-\delta_{\ell k},b_k)\ ,$$

2. The minimum posterior risk associated with $\widehat{M}_\ell, \ell = 1,2,$ is

$$R_{\widehat{M}_\ell} = \frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2}B(a_k-2\delta_{\ell k},b_k)$$

$$-\left\{\frac{1}{I_0}\sum_{i=0}^{n_{12}}\sum_{j=0}^{i}\binom{n_{12}}{i}\binom{i}{j}(-1)^{n_{12}-i}\prod_{k=1}^{2}B(a_k-\delta_{\ell k},b_k)\right\}^2$$

4.5 Simulation Study

Authors illustrated the result with simulation study. It is assumed in this example that there exists a series system with two independent components, where $f_j(x) = p_j(1-p_j)^{x-1}, j = 1,2$ and $p_1 = 0.1$ and $p_2 = 0.12$. For each system a pair of random variables X_1 and X_2 from f_1 and f_2, respectively is generate. Then $X = \min(X_1, X_2)$ is calculate and record the index of the minimum if it is available.

This step is repeated n times to simulate a random sample $(X_1, S_1), (X_2, S_2), \ldots, (X_n, S_n)$ with size n from the underlying system. Note that X_i and S_i represent random successive trails of system i before the first failure and the set contains the index of component causes its failure, respectively. In the case of masking data about 50% of the observations are randomly mask. The parameters p_1 and p_2 behave as random variables with beta prior distributions with parameters (7.22, 48.33) and (2.85, 25.89), respectively. The values of the parameters of the prior distributions are determined by following the technique given in Martz and Waller [1982].

***Example*:** In this example, a random sample has been generated with size $N = n = 15$ from the underlying model. Table 4.1 shows the data generated. Then this data is used to calculate:

(i) the point estimates of p_1 and p_2, (ii) the percentage errors associated with the point estimates obtained, (iii) 95% TBPI for each parameter, when the parameters have beta and non-informative prior distributions. Also, the prior and posterior probability density functions of the parameters p_1 and p_2 are plotted when the prior density functions are beta. Figures 4.1 and 4.2 show these functions for p_1 and p_2, respectively.

The percentage error associated with the point estimate of ω, say $PE_{\hat{\omega}}$, is given by the following formula:

$$PE_{\hat{\omega}} = \frac{|\text{exact value of } \omega - \text{estimated value of } \omega\,|}{\text{exact value of } \omega} \times 100\% \qquad (4.5.1)$$

Table 4.2 gives the point estimates of p_1, p_2, PE_{p_1}, PE_{p_2} and 95%TBPI for p_1 and p_2

Table 4.1: The simulated data							
I	t_i	No masking	General masking	I	t_i	No masking	General Masking
		S_i	S_i			S_i	S_i
1	5	{1}	{1,2}	9	1	{1}	{1,2}
2	7	{1}	{1}	10	2	{1}	{1,2}
3	2	{2}	{2}	11	1	{1}	{1,2}
4	7	{2}	{2}	12	1	{0}	{1,2}
5	1	{2}	{2}	13	8	{2}	{2}
6	2	{1}	{1}	14	12	{2}	{2}
7	4	{2}	{2}	15	12	{1}	{1,2}
8	2	{2}	{2}				

In table 4.1 that i denotes to the system number, t_i denotes to the number of successive trails of system i before its first failure, and S_i denotes the set of components that may cause the system i to fail. Further, $S_i = \{0\}$ means that the system i fails due to both components 1 and 2.

Based on the results shown in table 2, one can conclude that follows, for the current example

(i) The percentage error when the there is no masking in the observations is smaller than its value when there is no masking.

(ii) The percentage error associated with the point estimate when the prior distribution is beta is smaller than one associated with the point estimate when the prior distribution is non-informative.

Table 4.2: Point estimates, PE and 95% TBPI for p_1 and p_2						
Parameter	Beta			Non informative		
	Estimate	PE	95% TBPI	Estimate	PE	95% TBPI
	No masking					
p_1	0.112	12.16	(0.101,0.221)	0.118	17.65	(0.101,0.22
p_2	0.123	2.66	(0.101,0.221)	0.118	1.96	(0.091,0.22
	General masking					
p_1	0.074	26.27	(0.07,0.231)	0.053	47.52	(0.06,0.271)
p_2	0.147	22.48	(0.131,0.291)	0.171	42.83	(0.131,0.36

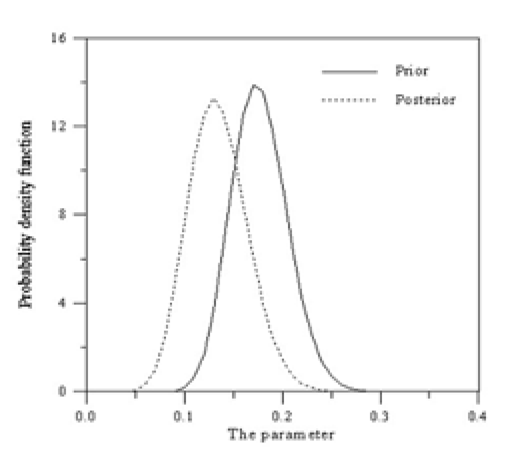

Fig 4.1 The prior and posterior pdf's of p_1

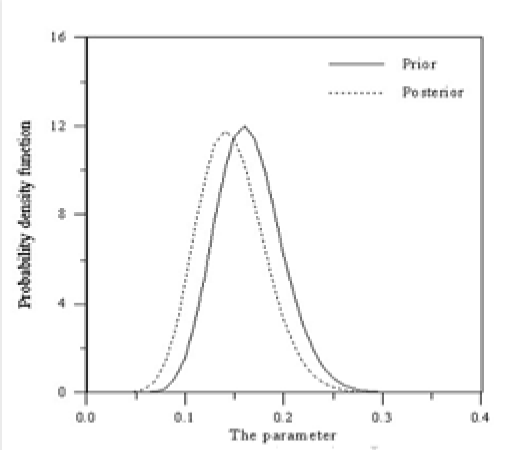

Fig 4.2 The prior and posterior pdf's of p_2

Therefore, one may say that: (i) beta prior distributions performs better than the non-informative ones in the sense of giving point estimates with smaller percentage errors, (ii) a sample with known cause of system failure gives better estimates, in the same sense, than that obtained when the masking takes place.

CHAPTER 5

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Estimation of Reliability Function of Consul and Geeta Distributions

5.1 Introduction

Since the beginning of seventies, attention of research workers in discrete distribution appears to be shifted towards 'Lagrangian probability distributions'. The Lagrangian probability distributions provide generalization of classic discrete distributions that have been found more general in nature and wide in scope. The discovery of Lagrangian distribution has relaxed the research workers who were previously vexed in a large number of generalized mixture and compound discrete distributions as they have been found to have tremendous capability to fit well into observed distributions of any type. Consul and Shenton [1972] gave a method for generating new families of generalized discrete distributions with the help of Lagrangian expansion. Gupta [1974] defined Modified Power Series Distributions (MPSD) which is a sub class of Lagrangian probability distribution. The discrete Lagrangian probability distribution have been systematically studied in a number of papers by Consul and Shenton [1972, 1973a, 1973b, 1975].

In the present chapter we have obtained Bayesian Reliability of parametric function of θ, of Consul, Geeta and Size-biased Geeta distributions assuming β as known, and the prior distribution of parameter θ, is considered as two parameter Beta distribution. In addition reliability functions of Geometric, Negative-binomial and Haight distributions are also obtained. This research paper is to appear in the Journal of JRSS and is presently under minor revision

5.2 Consul Distribution

Famoye in 1997 introduced the Consul distribution with parameters θ and m, where the probability mass function is given by

$$p(X = x) = \frac{1}{x}\binom{mx}{x-1}\theta^{x-1}(1-\theta)^{mx-x+1} \;;\quad x = 1,2,3,\ldots \qquad (5.2.1)$$

Where, $\quad 0 < \theta < 1 \quad and \quad 1 \le m \le \theta^{-1}$

the mean and variance of the model exists when $m < \theta^{-1}$. The mean and variance of (5.2.1) are given by the expressions

$$\mu = (1-\theta m)^{-1} \text{ and } \sigma^2 = m\theta(1-\theta)(1-\theta m)^{-3}$$

using these values the Consul distribution can be expressed as a location-parameter discrete probability distribution in the form

$$p(X = x) = \begin{cases} \frac{m}{m+\beta x}\binom{m+\beta x}{x}\left(\frac{\mu-1}{m\mu}\right)^{x-1}\left(1-\frac{\mu-1}{m\mu}\right)^{m+\beta x-x} & ; x = 1,2,\ldots \\ 0 & ; \text{otherwise} \end{cases} \qquad (5.2.2)$$

all the moments of the Consul distribution exist for $0 < \theta < 1 \quad and \quad 1 \le m < \theta^{-1}$.

Famoye [1997] showed that the Consul distribution is the limit of zero-truncated Generalized Negative Binomial Distribution (GNBD)

$$P_x(\theta,\beta,m) = \frac{m}{m+\beta x}\binom{m+\beta x}{x}\theta^x(1-\theta)^{m+\beta x-x}/[1-(1-\theta)^m]\,;\; x = 1,2,3,\ldots \qquad (5.2.3)$$

as the parameter $\beta \to 1$ and is unimodal but not strongly unimodal for all values of $m \ge 1$ and $0 < \theta < 1$ and the mode is at a point $x = 1$. He also obtained moment estimates, the estimates based upon the sample mean and first frequency, and the maximum likelihood estimates. The model (5.2.1) is a member Lagrangian probability distribution. We now obtain the Bayesian estimators of a number of parameters functions of the parameter θ and the Bayesian Reliability function. Since, $0 < \theta < 1$, it is assumed that the prior information on θ may be summarized by a beta distribution, $\beta(a, b)$ where the parameters ‘a’ and ‘b’ are not known.

5.2.1 *Bayesian Estimator (β Known) of Reliability Function*

The likelihood function of Consul distribution is given by

$$L_1\left(\frac{x}{\theta}, \beta\right) = k_1 \theta^{z-n}(1-\theta)^{mz-z+n}$$

$$\text{where,} \quad k_1 = \prod_{i=1}^{n}\left(\frac{1}{x}\binom{mx}{x-1}\right) \quad \text{and} \quad z = \sum_{i=1}^{n} x_i$$

since $0 < \theta < 1$, it is assumed that prior information on θ is given by a beta distribution with density function

$$\lambda(\theta, a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a, b)} \qquad ; a > 0, b > 0, 0 < \theta < 1$$

thus, the joint p.d.f of (X_1, $X_2 \ldots X_n$, θ) is given by

$$H\left(\frac{X_1, X_2, \ldots, X_n}{\theta}\right) = L_1(x/\theta, \beta).\lambda(\theta, a, b)$$

then using Bayes theorem, the posterior distribution of θ becomes

$$\phi_1\left({}^{\theta}/_{t}\right) = \frac{\theta^{z+a-n-1}(1-\theta)^{(m-1)z+n+b-1}}{B(z+a-n, (m-1)z+n+b)}$$

Bayesian estimator of any function $\varphi(\theta) = \theta^{x-1}(1-\theta)^{mx-x+1}$ with respect to squard error loss function is

$$\hat{\varphi}(\theta) = \frac{\int_0^1 \varphi(\theta)\theta^{z+a-n-1}(1-\theta)^{(m-1)z+n+b-1}\, d\theta}{B(z+a-n, (m-1)z+n+b)} \qquad (5.2.2.1)$$

$$= \frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)}$$

Bayesian estimator of Reliability function $R_c(t_0)$ for Consul distribution at a sepecified value $t_0(\geq 0)$ is

$$\hat{R}_c(t_0) = \sum_{x=t_0}^{\infty} \frac{1}{x}\binom{mx}{x-1} \frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)}$$

Let $x_1, x_2, x_3, \ldots, x_n$ are i.i.d Consul random variables as defined in (5.2.1), then the sample sum Y= $\sum x_i$ has Delta-binomial distribution given by

$$p_{db}(Y=y) = \begin{cases} \frac{n}{y}\binom{my}{y-n}\theta^{y-n}(1-\theta)^{my-y+n} & ; y = n, n+1, n+2, \ldots \\ 0 & ;\ \text{elsewhere} \end{cases} \qquad (5.2.2.2)$$

using (5.2.2.1) we have Bayesian estimate for $\theta^{y-n}(1-\theta)^{\beta y-1}$ is

$$\frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)}$$

Bayesian estimator of Reliability, $R_{db}(t_0)$ at a sepecified value $t_0(\geq 0)$ is

$$\hat{R}_{db}(t_0) = \sum_{x=t_0}^{\infty} \frac{n}{y}\binom{my}{y-n} \frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)}$$

Particular cases

1. When $m=1$ (5.2.1) reduces to Geometric distribution with probability of success$1-\theta$. Then, the reliability function $\hat{R}(t_0)$ for Geometric distribution at time $t_0(\geq 0)$ is

$$\hat{R}_g(t_0) = \sum_{x=t_0}^{\infty} \frac{B(z+a+x-n-1,n+b+1)}{B(z+a-n,n+b)}$$

2. When $m = 1$ (5.2.2.2) reduces to negative binomial distribution with probability of success $1-\theta$. Then, the reliability function $R_{nb}(t_0)$ for negative binomial distribution at time $t_0(\geq 0)$ is

$$\widehat{R}_{nb}(t_0) = \sum_{x=t_0}^{\infty} \binom{y-1}{n-1} \frac{B(z+a+x-n-1, n+b+1)}{B(z+a-n, n+b)}$$

5.3 Geeta Distribution

Consul [1990a] introduced the Geeta distribution, with parameters θ and β, where the probability mass function is defined

$$p(X=x) = \begin{cases} \frac{1}{\beta x - 1}\binom{\beta x - 1}{x} \theta^{x-1}(1-\theta)^{\beta x - 1}\,; & x = 1,2,3 \ldots \\ 0 & ;\ \text{elsewhere} \end{cases} \qquad (5.3.1)$$

$$\text{where,} \qquad 0 < \theta < 1\,, 1 < \beta < \frac{1}{\theta}$$

the upper limit on β has been imposed for the existence of the mean of the distribution. When $\beta \to 1$, the Geeta distribution degenerates and its probability mass gets concentrates at point $x = 1$. Consul [1990a] studied the estimation of the model (5.3.1), using moments, sample mean and frequency, maximum likelihood estimation (MLE) and minimum variance unbiased estimation (MVUE) methods, and gave the MVUE estimates for some functions of parameter θ, which are similar to the results obtained by Gupta [1974] for modified power series distribution. Consul [1990b] gave two stochastic models for Geeta distribution and showed that the distribution can be obtained as an urn model and that it is also generated as a model based on a difference differential equation. The model (5.3.1) is a member of Consul and Shentons's [1972], Lagrangian probability distribution and also Gupta's [1974] modified power series distribution.

the mean and variance of (5.3.1) are given by the expressions

$$\mu = \frac{(1-\theta)}{(1-\theta\beta)^{-1}} \text{ and } \sigma^2 = (\beta-1)(1-\theta)(1-\theta\beta)^{-3}$$

The family of Geeta probability models belongs to the classes of the modified power series distribution and the Lagrangian series distribution. Consul [1990b] also expressed it as a location–parameter probability distribution given below:

$$p(X = x) = \begin{cases} \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}\left(\frac{\mu - 1}{\beta\mu - 1}\right)^{x-1}\left(\frac{\mu(\beta - 1)}{\beta\mu - 1}\right)^{\beta x - x} & ; x = 1,2,3,\dots \\ 0 & ; \text{otherewise} \end{cases} \quad (5.3.2)$$

a numerical approach indicates that Geeta distribution has maximum at x = 1, and can have a either short or long or heavy tail, depending upon the values of β and θ. Estimation using (1) moments (2) sample mean and first frequency, (3) M.L and (4) M.V.U.E are studied by Consul [1990a], two models of genesis (a two-urn model and a regenerative stochastic process) are given in Consul [1990b]. Generating function and recurrence relations for central moments are given by Consul [1990a]. We now obtain the Bayesian estimators of a number of parameters functions of the parameter θ and the Bayesian Reliability function. Since, $0 < \theta < 1$, it is assumed that the prior information on θ may be summarized by a beta distribution, $\beta(a, b)$ where the parameter 'a' and 'b' are not known.

5.3.1 *Bayesian Estimator (β Known) f Reliability Function*

The likelihood function of Geeta distribution is given by

$$L_2\,(x/\theta, \beta) = k_2\theta^{t-1}(1-\theta)^{\beta t - 1}$$

where,

$$k_2 = \prod_{i=1}^{n} \frac{1}{\beta x_i - 1}\binom{\beta x_i - 1}{x_i} \quad \text{and} \quad t = \sum_{i=1}^{n} x_i$$

the part of the likelihood function which is relevant to Bayesian inference on unknown parameter θ is

$$\theta^{t-1}(1-\theta)^{\beta t - 1}$$

we assume that before the observations were made, our knowledge about θ was only vague, since $0 < \theta < 1$, it is assumed that prior information on θ is given by a beta distribution with density function

$$\lambda(\theta, a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} \quad ; a > 0, b > 0, 0 < \theta < 1$$

thus the joint p.d.f of $(X_1, X_2, \dots, X_n, \theta)$ is given by

$$H\left(\frac{X_1, X_2, \dots, X_n}{\theta}\right) = L_2(x/\theta, \beta).\lambda(\theta, a, b)$$

and the posterior distribution of θ becomes

$$\phi_2(\theta/_t) = \frac{\theta^{t+a-n-1}(1-\theta)^{(\beta-1)t+b-1}}{B(t+a-n, \beta(t-1)+b)}$$

Bayesian estimator of any function $\varphi(\theta) = \theta^{x-1}(1-\theta)^{\beta x-1}$ with respect to squard error loss function is

$$\widehat{\varphi}(\theta) = \frac{\int_0^1 \varphi(\theta)\theta^{t+a-n-1}(1-\theta)^{(\beta-1)t+b-1}\, d\theta}{B(t+a-n, \beta(t-1)+b)} \qquad (5.3.1.1)$$

$$= \frac{B(t+a+x-n-1, (\beta-1)(x+t)+b)}{B(t+a-n, \beta(t-1)+b)}$$

Table: 5.1 Bayesian estimators of some parametric functions in Geeta distribution

$\varphi(\theta)$	$\widehat{\varphi}(\theta)$
$\theta^l(1-\theta)^k$, l & k are non-negative integers	$\frac{B(n\bar{x}+a-n+1, (\beta-1)n\bar{x}+b+k)}{B(n\bar{x}+a-n, (\beta-1)n\bar{x}+b)}$
$[\theta(1-\theta^{\beta-1})]^k$, k is positive integer	$\frac{B(n\bar{x}+a-n+k, \beta n\bar{x}-n\bar{x}+b+k\beta-k)}{B(n\bar{x}+a-n, (\beta-1)n\bar{x}+b)}$
$p(X=k)$ $k = 0,1,2, \dots.$	$\frac{1}{\beta k-1}\binom{\beta k-1}{x}\frac{B(n\bar{x}+a-n+k, \beta n\bar{x}-n\bar{x}+b+k\beta-k)}{B(n\bar{x}+a-n, (\beta-1)n\bar{x}+b)}$

Bayesian estimator of Reliability, $R_{ge}(t_0)$ at a sepecified value $t_0(\geq 0)$ is

$$\hat{R}_{ge}(t_0) = \sum_{x=t_0}^{\infty} \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}\frac{B(t+a+x-n-1,(\beta-1)(x+t)+b)}{B(t+a-n,\beta(t-1)+b)}$$

Let $x_1, x_2, x_3, \ldots, x_n$ are i.i.d Geeta random variables as defined in (5.3.1), then the sample sum Y= $\sum x_i$ has Geeta-n distribution given by

$$p(Y=y) = \frac{n}{y}\binom{\beta y - n - 1}{y-n}\theta^{y-n}(1-\theta)^{\beta y - y} \quad ; y = n, n+1, n+2, \ldots \qquad (5.3.1.2)$$

using (5.3.1.1) we have Bayesian estimate for $\theta^{y-n}(1-\theta)^{\beta y - 1}$ is

$$\frac{B(t+a+y-n-1,(\beta-1)(t+y)+b)}{B(t+a-n,(\beta-1)t+b}$$

therefore, Bayesian estimate of Reliability, $R_{gn}(t_0)$ at time $t_0(\geq 0)$ is

$$\hat{R}_{gn}(t_0) = \sum_{y=t_0}^{\infty} \frac{n}{y}\binom{\beta y - n - 1}{y-n}\frac{B(t+a+y-n-1,(\beta-1)(t+y)+b)}{B(t+a-n,(\beta-1)t+b)}$$

Particular case

When $\beta = 2$, (5.3.1.2) reduces to Haight distribution, therefore reliability function $\hat{R}_h(t_0)$ for Haight distribution at time $t_0(\geq 0)$ is

$$\hat{R}_h(t_0) = \sum_{y=t_0}^{\infty} \frac{n}{y}\binom{2y-n-1}{y-n}\frac{B(t+a+y-n-1,(t+y)+b)}{B(t+a-n,t+b)}$$

5.4 Size-Biased Geeta Distribution (SBGET)

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weight function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. Size-biased distributions are a special case of the more general form known as weighted distributions. The concept of weighted distribution can be traced to Fisher [1934] in his paper “The effects of methods of ascertainment upon the estimation of frequencies”; while this of length-biased sampling was introduced by Cox 1962 [Patill 2002]

If $f(x;\theta)$ is the distribution of a random variable X with unknown parameter θ, then the corresponding size-biased distribution is of the form

$$f^*(x;\theta) = \frac{x^\alpha f(x;\theta)}{\mu'_\alpha} \qquad \text{where,} \quad \mu'_\alpha = \sum x^\alpha f(x;\theta)$$

for $\alpha = 1$ and 2, we get the simple size-biased and area-biased distributions respectively. A Size-biased Geeta Distribution (SBGET) is obtained by applying the weight x^α, where $\alpha = 1$ to the Geeta distribution (5.3.1). This gives the size-biased Geeta distribution as

$$p_{sb}(X = x) = (1-\theta\beta)\binom{\beta x - 2}{x-1}\theta^{x-1}(1-\theta)^{(\beta-1)x-1} \quad ; x = 1,2,3,\ldots \quad (5.4.1)$$

$$\text{where,} \qquad 1 < \beta < \frac{1}{\theta} \quad \text{and} \quad 0 < \theta < 1$$

5.4.1 *Bayesian Estimator (β Known) of Reliability Function*

The likelihood function of SBGET (5.4.1) is

$$L_3(x/\theta,\beta) = \left(\prod_{i=1}^{n}\binom{\beta x_i - 2}{x_i - 1}\right)(1-\theta\beta)^n\theta^{\sum x - n}(1-\theta)^{(\beta-1)\sum x - n}$$

$$L_3(x/\theta,\beta) = k_3(1-\theta\beta)^n\theta^{z-n}(1-\theta)^{(\beta-1)z-n}$$

where,

$$k_3 = \prod_{i=n}^{n}\binom{\beta x_i - 2}{x_i - 1} \qquad \text{and} \qquad z = \sum_{i=1}^{n} x_i$$

since, $0 < \theta < 1$, therefore we assume that the prior information about θ when β is know from Beta distribution, $\beta(a,b)$ where the parameter a and b are not known.

$$\lambda(\theta,a,b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} \quad ; \quad a > 0, b > 0, 0 < \theta < 1$$

and the posterior distribution of θ from becomes

$$\phi_3\left(\theta/_{t}\right) = \frac{(1-\theta\beta)^{n}\theta^{z+a-n-1}(1-\theta)^{(\beta-1)z+b-n-1}}{\int_0^1 (1-\theta\beta)^{n}\theta^{z+a-n-1}(1-\theta)^{(\beta-1)z+b-n-1}\, d\theta}$$

$$= \frac{(1-\theta\beta)^{n}\theta^{z+a-n-1}(1-\theta)^{(\beta-1)z+b-n-1}}{B(z+a-n,(\beta-1)z+b-n)\ {}^{2}F_1[-n, z+a-n; \beta z+a+b-2n; \beta]}$$

Bayesian estimator of any function $\varphi(\theta) = (1-\beta\theta)\theta^{x-1}(1-\theta)^{(\beta-1)x-1}$ with respect to squard error loss function is

$$\hat{\varphi}(\theta) = \frac{\int_0^1 (1-\theta\beta)^{n+1}\,\theta^{z+a+x-n-2}(1-\theta)^{(\beta-1)(x+z)+b-n-2} d\theta}{B(z+a-n,(\beta-1)z+b-n)\ {}^{2}F_1[-n, z+a-n; \beta z+a+b-2n; \beta]}$$

$$\hat{\varphi}(\theta) = \frac{B(z+a+x-n-1,(\beta-1)(x+z)+b-n-1)}{B(z+a-n,(\beta-1)z+b-n)}$$

$$\times \frac{{}^{2}F_1[-(n+1), z+a+x-n-1; \beta(x+z)+a+b-2(n+1); \beta]}{{}^{2}F_1[-n, z+a-n; \beta z+a+b-2n; \beta]}$$

Therefore, Bayesian estimator of Reliability function $R_{sb}(t_0)$ for Size-biased Geeta distribution at a specified value t_0 (≥ 0) is

$$\hat{R}_{sb}(t_0) = \sum_{x=t_0}^{\infty} \binom{\beta x-2}{x-1} \frac{B(z+a+x-n-1,(\beta-1)(x+z)+b-n-1)}{B(z+a-n,(\beta-1)z+b-n)}$$

$$\times \frac{{}^{2}F_1[-(n+1), z+a+x-n-1; \beta(x+z)+a+b-2(n+1); \beta]}{{}^{2}F_1[-n, z+a-n; \beta z+a+b-2n; \beta]}$$

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