SOME ASPECTS OF DUALITY IN VARIATIONAL PROBLEMS AND OPTIMAL CONTROL

THESIS SUBMITTED TO THE UNIVERSITY
OF KASHMIR FOR THE DEGREE OF

Doctor of Philosophy

IN

STATISTICS

BY

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POST GRADUATE DEPARTMENT OF STATISTICS
FACULTY OF PHYSICAL AND MATERIAL SCIENCES
UNIVERSITY OF KASHMIR
SRINAGAR-190006,J&K, India
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DEDICATION

To my (late) mother, Jana Begum, and father, Jalal-Ud-Din Malla,

To my wife Mahjabeena Parveen and daughter Fareeha Bilal

Bilal

Since the fabric of the universe is
Most perfect and the work of the
Most wise creator, nothing at all
Takes place in the Universe in
Which some rule of the maximum
Or minimum does not appear.

Leonhard Euler

1. ACKNOWLEDGEMENT

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ ○ إِنَّ الْفَضْلَ بِيَدِ اللَّهِ يُؤْتِيهِ مَن يَشَاءُ ۗ وَاللَّهُ وَاسِعٌ عَلِيمٌ ○

يَخْتَصُّ بِرَحْمَتِهِ مَن يَشَاءُ ۗ وَاللَّهُ ذُو الْفَضْلِ الْعَظِيمِ ○

(In the name of Allah Most Gracious, Most Merciful. All bounties Are in the hand of Allah: He grants them To whom He pleases And Allah cares for all, And He knows all things. For His Mercy He specially chooses Whom he pleases: For Allah is the Lord of bounties unbounded.)

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Bilal Ahmad

C O N T E N T S

ABSTRACT

This thesis is divided into six chapters. In the Ist chapter we present a brief survey of related work done in the area of multiobjective mathematical programming, optimal control and game theory.

Chapter Two: In this chapter sufficient optimality criteria are derived for a control problem under generalized invexity. A Mond-Weir type dual to the control problem is proposed and various duality theorems are validated under generalized invexity assumptions on functionals appearing in the problems. It is pointed out that these results can be applied to the control problem with free boundary conditions and have linkage with results for nonlinear programming problems in the presence of inequality and equality constraints already established in the literature.

Chapter Three: In this chapter a mixed type dual to the control problem in order to unify Wolfe and Mond-Weir type dual control problem is presented in various duality results are validated and the generalized invexity assumptions. It is pointed out that our results can be extended to the control problems with free boundary conditions. The duality results for nonlinear programming problems already existing in the literature are deduced as special cases of our results.

Chapter Four: In this chapter two types of duals are considered for a class of variational problems involving higher order derivative. The duality results are derived without any use of optimality conditions. One set of results is based on Mond-Weir type dual that has the same objective functional as the primal problem but different constraints. The second set of results is based on a dual of an auxiliary primal with single objective function. Under various convexity and generalized convexity assumptions, duality relationships between primal and its various duals are established. Problems with natural boundary values are considered and the analogues of our results in nonlinear programming are also indicated.

Chapter Five: In this chapter a certain constrained dynamic game is shown to be equivalent to a pair of symmetric dual variational problems which have more general formulation than those already existing in the literature. Various duality results are proved under convexity and generalized convexity assumptions on the appropriate functional. The dynamic game is also viewed as equivalent to a pair of dual variational problems without the condition of fixed points. It is also indicated that our equivalent formulation of a pair of symmetric dual variational problems as dynamic generalization of those already studied in the literature.

Chapter Six: In this chapter a mixed type second-order dual to a variational problem is formulated as a unification of Wolfe and Mond-Weir type dual problems already treated in the literature and various duality results are validated under generalized second order invexity. Problems with natural boundary values are formulated and it also is pointed out that our duality results can be regarded as dynamic generalizations of those of (static) nonlinear programming.

The subject matter of the present research thesis is fully published in the form of the following research papers written by the author:

(1) Sufficiency and Duality In Control Problems with Generalized Invexity, Journal of

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(2) Mixed Type Duality for Control Problems with Generalized Invexity, Journal of Applied

Mathematics and Informatics,Vol. 26(2008), No.5-6 , pp. 819-837.

(3) On Multiobjective Duality for Variational Problems, The Open Operational Research

Journal,2012, 6, 1-8.

(4) Constrained Dynamic Game and Symmetric Duality For Variational Problems, Journal of Mathematics and System Science 2(2012), 171-178.

(5) Mixed Type Second – Order Dulaity For Variational Problems, Journal of Informatics and Mathematical Sciences , Vol5,No.1, pp.1-13,(2013).

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Chapter 1

In this chapter we present a brief survey of related work done in the fields of multiobjective mathematical programming, optimal control and game theory followed by a precise summary of our own findings in the subsequent chapters of this thesis.

1.1.1 Mathematical Programming Problem

Many problems of practical importance can be transformed into different forms of minimization or maximization problems no matter whether such problems are from the field of engineering, science, business or finance. These problems share the characteristics of requirements of finding the most advantageous solution that offers certain optimal criteria under several limitations. Many of these problems concentrate primarily on optimizing the gain or the quality of performance: for instance the problem of optimal control (discrete or continuous), structural design, mechanical design, electrical network, water resource management, stochastic resource allocation, location facilities, etc., can be cast into optimization problems. Finally one can say that nothing at all takes place in the Universe in which some rule of the maximum or minimum does not appear.

Most of the optimization problems are concerned with more than a single objective function. Real life problems generally require the optimizing of multiple objectives at the same time. These objectives are often inter-conflicting. When objectives are conflicting, this implies that an objective cannot be improved without affecting the optimality of the other objectives. A possible solution to multiple criteria optimization should provide balance in objectives. These solutions may be suboptimal with respect to single objective programming problem. In fact, they are called trade-off solutions that are regarded as the best solution. Multiple criteria optimization is most often applied to deterministic problem in which the number of feasible alternatives is large.

Optimality criteria play a very significant role in determining the solution of the problem as the classical calculus suggests. Fritz-John [48] was the first to derive necessary optimality conditions for constrained single objective optimization problem using Lagrange multiplier rule. Later Kuhn and Tucker [52] established necessary optimality conditions for the existence of optimal solution under certain constraint qualification in 1951. It was revealed after wards that W.Karush [50] had presented way back in 1939 without imposing any constraint qualification; thus the Kuhn-tucker conditions are known as Karush-Kuhn-Tucker optimality conditions. Abadie [1] established a regularity condition that enabled him to derive Karush-Kuhn-Tucker conditions from Fritz John optimality conditions. Subsequently, Mangasarian and Fromovitz [55] generalized Fritz-John optimality conditions to treat equality and inequality constraints. Sufficiency of these conditions under convexity and generalized convexity were extensively treated by many authors notably, Mangasarian [53] and Martos [56].

1.1.2 Duality

Duality in nonlinear programming problems originated with duality results of quadratic programming, initially studied by Dannis [22]. Dual of convex primal program was given by Dorn [25] and Mangasarian [55].

Mond and Weir [67] modified the Wolfe dual moving a part of objective function of Wolfe dual to the constraints and thus introducing Mond-Weir dual programming problem. The resulting pair of dual programming was nonconvex program and was found that there was no involution between primal and dual that is, the dual of the dual was not primal in general. In the literature of mathematical programming, a primal-dual pair of problem is called symmetric if the dual of the dual is primal problem. In the sense, a linear problem and its dual is symmetric. However, the majority of the formulation, in nonlinear programming does not possess this property. The first symmetric dual formulation in nonlinear programming was proposed by Dantzig, Eisenberg and Cottle [23] which subsumed the duality formulation of linear programming and certain duality formations in quadratic programming. Making use of the Fritz John optimality conditions, they proved weak and strong duality theorems for their pair of symmetric dual programming problems under differentiability conditions. These ideas were further extended to single and multiple objective variational problems.

Kuhn and Tucker [52] were the first to incorporate some interesting results concerning multiobjective optimization in 1951 .Since then, research in this area has made remarkable progress both theoretically and practically. Some of the earliest attempts to obtain conditions for efficiency were carried out by Kuhn and Tucker [52], Arrow et al [3]. Their research has been inherited by Da Cunha and Polak [21], Neustadt [69], Ritter [70-72], Smale [76], Aubin [4], Husain et al. [36-40] and others. Duality, which plays an important role in traditional mathematical programming, has been extended to multiobjective optimization since the late 1970's. Isermann [44-47] developed multiobjective duality in linear case while results for nonlinear cases have been given by Schonfeld [74], Tanino and Sawaragi [78], Mazzoleni [57], Corley [16], Nakayama [68] and others.

Concept of mixed type multiobjective duality seems to be quite interesting and useful from practical as well as from algorithmic point of view. The computational advantage of mixed type dual formulations involves the flexibility of the choice of constraints to be put in the Lagrange function can be exploited to develop certain efficient solution procedures for solving mathematical programming problems.

The main contribution of this thesis is to study duality and mixed type duality for control problems, multiobjective duality and second order duality for variational problems and an equivalence of constrained dynamic games to a pair of symmetric dual variational problem which have more general formulations than those already existing in the literature. as mixed type duality in mathematical programming is interesting fro theoretical as well as computational view. Mixed type duality for control problems is also presented in this research. The linkage between control problems including variational problems and the corresponding nonlinear programming problems is incorporated in these problems.

1.2 PRE-REQUISITES

1.2.1 Notations

In this section, we shall incorporate major symbols which are used throughout the research work reported in this thesis.

R^n = n-dimensional Euclidean space,

R_+^n = The non-negative orthant in R^n,

A^T= Transpose of the matrix A,

Let f be a numerical function defined on an open set Γ in R^n, then $\nabla f(\bar{x})$ denotes the gradient of f at $\bar{x}$, that is, $\nabla f(\bar{x}) = \left[\frac{\partial f(\bar{x})}{\partial x^1}, \ldots, \frac{\partial f(\bar{x})}{\partial x^2}\right]^T$

Let ϕ be a real valued twice continuously differentiable function defined on an open set contained in $R^n \times R^m$. Then $\nabla_x \phi(x, y)$ and $\nabla_y \phi(x, y)$ denote the gradient (column) vector of ϕ with respect to x and y respectively i.e.,

$$\nabla_x \phi(\bar{x}, \bar{y}) = \left(\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \ldots, \frac{\partial \phi}{\partial x^n}\right)^T_{(\bar{x},\bar{y})}$$

$$\nabla_y \phi(\bar{x}, \bar{y}) = \left(\frac{\partial \phi}{\partial y^1}, \frac{\partial \phi}{\partial y^2}, \ldots, \frac{\partial \phi}{\partial y^m}\right)^T_{(\bar{x},\bar{y})}$$

Further $\nabla^2_{xx} \phi(\bar{x}, \bar{y})$ and $\nabla^2_{yy} \phi(\bar{x}, \bar{y})$ denote respectively the $(n \times n)$ and matrices of second order partial derivative i.e.,

$$\nabla^2_{xx} \phi(\bar{x}, \bar{y}) = \left(\frac{\partial^2 \phi}{\partial x^i x^j}\right)_{(\bar{x},\bar{y})}$$

$$\nabla^2_{xy}\phi(\bar{x},\bar{y}) = \left(\frac{\partial^2\phi}{\partial x^i x^j}\right)_{(\bar{x},\bar{y})}$$

The symbols $\nabla^2_{yy}\phi(\bar{x},\bar{y})$ and $\nabla^2_{xy}\phi(\bar{x},\bar{y})$ are similarly defined.

1.2.2 Definitions

Definition 1.1: Let $X \subseteq R^n$ be an open and convex set and $f : X \to R$ be differentiable. Then we define f to be

(i) **Convex**, if for all $x_1, x_2 \in X$,

$$f(x_1) - f(x_2) \geq (x_1 - x_2)\nabla f(x_2)$$

(ii) **Strictly convex**, if for all $x_1, x_2 \in X$ and $x_1 \neq x_2$

$$f(x_1) - f(x_2) > (x_1 - x_2)\nabla f(x_2)$$

(iii) **Quasi convex**, if for all $x_1, x_2 \in X$,

$$f(x_1) \leq f(x_2) \Rightarrow (x_1 - x_2)\nabla f(x_2) \leq 0$$

(iv) **Pseudo convex**, if for all $x_1, x_2 \in X$,

$$(x_1 - x_2)\nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2)$$

(v) **Strictly pseudoconvex**, if for all $x_1, x_2 \in X$, and $x_1 \neq x_2$

$$(x_1 - x_2)\nabla f(x_2) \geq 0 \Rightarrow f(x_1) > f(x_2)$$

(vi) **Invex,** if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$f(x_1) - f(x_2) \geq \eta(x_1, x_2)^T \nabla f(x_2)$$

(vii) **Pseudoinvex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$\eta^T(x_1, x_2)\nabla f(x_2) \geq 0 \Rightarrow f(x_1) \geq f(x_2)$$

(viii) **Quasiinvex**, if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that for all $x_1, x_2 \in X$,

$$f(x_1) \leq f(x_2) \Rightarrow \eta^T(x_1, x_2)\nabla f(x_2) \leq 0.$$

$$\left(x_1-x_2\right)^T \nabla f\left(x_2\right)+\left(x_1-x_2\right)^T \nabla^2 f\left(x_2\right) p \geq 0 \Rightarrow f\left(x_1\right) \geq f\left(x_2\right)-\frac{1}{2} p^T \nabla^2 f\left(x_2\right) p$$

Clearly, a differentiable convex, pseudoconvex, quasiconvex function is invex, pseudoinvex or quasi invex respectively with $\eta^T\left(x_1, x_2\right)=\left(x_1-x_2\right)$. Further we define f to be concave, strictly concave pseudoconcave, quasiconcave, strictly pseudo convex on X according as $-f$ is convex, strictly convex, quasi convex, pseudoconvex, strictly pseudoconvex. In the following definitions we shall use D and D^2 for customary symbols $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$.

Definition 1.2:

(i) **Invexity:** If there exists vector function $\eta(t,x,u) \in R^n$ with $\eta = 0$ and $x(t)=u(t)$, $t \in I$ = $[a,b]$, a real interval, such that for a scalar function $\phi(t,x,\dot{x})$, the functional $\Phi(x)=\int_I \phi(t,x,\dot{x})dt$ satisfies

$\Phi(u)$-$\Phi(x) \geqq \int_I \left\{\eta\phi_x(t,x,\dot{x})+(D\eta)^T \phi_{\dot{x}}(t,x,\dot{x})\right\}dt$, Φ is said to be invex in x and $\dot{x}$ on *I* with respect to η.

(ii) **Pseudoinvexity,** Φ is said to be pseudoinvex in x and $\dot{x}$ with respect to η if

$$\int_I \left\{\eta^T\phi_x(t,x,\dot{x})+(D\eta)^T \phi_{\dot{x}}(t,x,\dot{x})\right\}dt \geqq 0$$

implies $\Phi(x,\dot{u}) \geqq \Phi(x,\dot{x})$.

(iii) **Quasi-invex,** The functional Φ is said to quasi-invex in x and $\dot{x}$ with respect to η if

$\Phi(x,\dot{u}) \leqq \Phi(x,\dot{x})$ implies

$$\int_I \left\{\eta^T\phi_x(t,x,\dot{x})+(D\eta)^T \phi_{\dot{x}}(t,x,\dot{x})\right\}dt \leqq 0.$$

Consider the multiobjective variational problem (VP).

(VP): Minimize $\int_I \left(f^1(t,x,\dot{x}),\ldots,f^p(t,x,\dot{x})\right)dt$

Subject to

$$g(t,x,\dot{x}) \leq 0\ ,\ t \in I$$

Definition 1.3 **(*Efficient Solution*)*:*** A feasible solution $\bar{x}$ is efficient for (VP) if there exist no other feasible x for (VP) such that for some $i \in P = \{1, 2, ..., p\}$,

$$\int_I f^i(t, x, \dot{x})\,dt < \int_I f^i(t, \bar{x}, \dot{\bar{x}})\,dt$$

and

$$\int_I f^j(t, x, \dot{x})\,dt \leqq \int_I f^j(t, \bar{x}, \dot{\bar{x}})\,dt \text{ for all } j \in P,\ j \neq i.$$

Definition 1.4: Let $f : R^n \to R$ be a convex function, then a subgradient of f at a point $x \in R^n$ is a vector $\xi \in R^n$ satisfying

$$f(y) \geq f(x) + \xi^T(y - x), \text{ for all } y \in R^n$$

1.3 REVIEW OF THE RELATED WORK

1.3.1 Duality in Mathematical Programming

Nonlinear Programming

Consider the following nonlinear programming problem (P):

(P): Minimize $f(x)$

Subject to

$$h_j(x) \leq 0, \quad j = 1, 2, \ldots, m$$

where $f : R^n \to R$ and $h_j : R^n \to R$, $j = 1, 2, \ldots, m$ are continuously differentiable. The following problem (WD) is the Wolfe type dual to the problem (P):

(WD): Maximize $f(x) + y^T h(x)$

Subject to

$$\nabla\left(f(x) + y^T h(x)\right) = 0,$$

$$y \geq 0,\ y \in R^m$$

Mangasarian [53] explained by means of an example that certain duality theorems may not be valid if the objective or the constraint function is a generalized convex function. This motivated Mond and Weir [66] to introduce a different dual for (P) which is given below:

(MWD): Maximize $f(x)$

Subject to

$$\nabla f(x) + \nabla y^T h(x) = 0.$$

$$y^T h(x) \geq 0$$

$$y \in R_+^m$$

and they proved various duality theorems under pseudoconvexity of f and quasiconvexity of $y^T h(x)$ for all feasible solution of (P) and (MWD).

Later Weir and Mond [82] derived sufficiency of Fritz John optimality criteria under pseudoconvexity of the objective and quasiconvexity or semi-strict convexity of constraint functions. They formulated the following dual using Fritz John optimality conditions instead of Karush-Kuhn-Tucker optimality conditions and proved various duality theorems-thus the requirement of constraint qualification is eliminated.

(FrD): Maxmize $f(x)$

Subject to

$$\lambda_\circ \nabla f(x) + \nabla \lambda^T h(x) = 0.$$

$$\lambda^T h(x) \geq 0$$

$$(\lambda_\circ, \lambda) \geq 0, (\lambda_\circ, \lambda) \neq 0$$

Duality in Multiobjective Mathematical Programming

Whenever we shall study multiobjective programming problem we shall follow the following conventions for vectors in R^n

$$x < y, \quad \Leftrightarrow \quad x_i < y_i, \quad i = 1, 2, \ldots, n.$$

$$x \leqq y, \quad \Leftrightarrow \quad x_i \leqq y_i, \quad i = 1, 2, \ldots, n.$$

$$x \leq y, \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \ldots, n, \text{ but } x \neq y$$

$x \nleq y$, is the negation of $x \leq y$.

Consider the multiobjective programming problem:

(VP): V- Min $\quad F(x)=\left(f_1(x), f_2(x),..., f_p(x)\right)$

Subject to

$$h_j(\bar{x}) \leqq 0\ ,\ \left(j=1,2,...,m\right)$$

where $X \subseteq R^n$ is an open and convex set and f_i and h_j are differentiable functions where, $f_i : X \to R,\ i=1,2,\ldots, p$ and $h_j : X \to R,\ j=1,2,\ldots,m$. Here the symbol "V-Min" stands for vector minimization and minimality is taken in terms of either "*efficient points*" or "*properly efficient points*" given by Koopman [51] and Geoffrin [27] respectively.

***Definition 1.7* [27]:** A feasible point $\bar{x}$ for is said to be efficient solution of (VP), if there does

not exist any feasible x for (VP) such that

$$f_r(x) < f_r(\bar{x}) \text{ for some } r,$$

$$f_i(x) \leqq f_i(\bar{x}) \text{ for all } i=1,2,...,k\ ,\ i \neq r\ .$$

***Definition* 1.2 [27]:** A feasible point $\bar{x}$ is said to be properly efficient solution of (VP), if it is an efficient solution of (VP) and if there exists a scalar $M > 0$ such that for each i and $x \in X_0$ satisfying $f_i(x) < f_i(\bar{x})$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - \phi_j(\bar{x})} \leqq M\ ,$$

for some *j*, satisfying $f_j(x) < f_j(\bar{x})$.

Geoffrion [27] considered the following single objective minimization problems for fixed $\lambda \in R^p$:

(VP)$_\lambda$: Minimize $\sum_{i=1}^{p} \lambda_i f_i(x)$

Subject to

$$h_j(\bar{x}) \leqq 0\ ,\ \left(j=1,2,...,m\right)$$

and proved the following lemma connecting (VP) and (VP)$_\lambda$.

lemma 1.1:

(i) Let $\lambda_i > 0$, $(i = 1, 2, ..., p)$, $\sum_{i=1}^{p} \lambda_i = 1$ be fixed. If $\bar{x}$ is optimal for $(VP)_\lambda$, then $\bar{x}$ is properly efficient for (VP).

(ii) Let f_i and h_j be convex functions, Then $\bar{x}$ is properly efficient for (VP) iff $\bar{x}$ is optimal

for are differentiable functions $(VP)_\lambda$ for some $\lambda > 0$, $\sum_{i=1}^{p} \lambda_i = 1.$

If f_i and h_j are differentiable convex functions then $(VP)_\lambda$ is a convex programming problem. Therefore in relation to $(VP)_\lambda$ consider the scalar maximization problem:

(VD)$_\lambda$: Maximize $\lambda^T f(x) + y^T h(x) = \lambda^T \left(f(x) + y^T h(x) \right)$

Subject to

$$\nabla \left(\lambda^T f(x) + y^T h(x) \right) = 0$$

$$\lambda \in \Lambda^+ \ , \ y \geqq 0,$$

where $e = (1, 1, \ldots, 1) \in R^p$ and $\Lambda^+ = \left\{ \lambda \in R^P : \lambda > 0 \ , \lambda^T e = 1 \right\}$

Now as $(VD)_\lambda$ is a dual program of $(VP)_\lambda$, Weir [81] considered the following vector optimization problem in relation to (VP) as

(DV): Maximize $\lambda^T f(x) + y^T h(x) e$

Subject to

$$\nabla \left(w^T f(x) + y^T h(x) \right) = 0$$

$$w \in \Lambda^+ \ , \ y \geqq 0,$$

They termed (DV) as the dual of (VP) and proved various duality theorems between (VP) and (DV) under the assumption that f and h are convex functions.

Further, for the purpose of weakening the convexity requirements on objective and constraint functions, Weir [81] introduced another dual program (DV1).

(DV1): Maximize $f(x)$

Subject to

$$\nabla\left(\lambda^T f(x) + y^T h(x)\right) = 0$$

$$y^T h(x) \geq 0$$

$$\lambda \in \Lambda^+ \ , \ y \geqq 0,$$

For these problems, various duality theorems are proved by assuming the function f to be pseudo convex and $y^T h$ to be quasiconvex for their feasible solutions.

1.3.2 Symmetric Duality in Mathematical Programming

Symmetric Duality in Differentiable Mathematical Programming

Consider a function $f(x, y)$ which is differentiable in $x \in R^m$ and $y \in R^m$. Dantzig et al [23] introduced the following pair of problems:

(SP): Minimize $f(x, y) - y^T \nabla_y f(x, y)$

Subject to

$$\nabla_y f(x, y) \leq 0$$

$$(x, y) \geq 0.$$

(MSD): Maximize $f(x, y) - x^T \nabla_x f(x, y)$

Subject to

$$\nabla_x f(x, y) \leq 0$$

$$(x, y) \geq 0.$$

and proved the existence of a common optimal solution to the primal (SP) and (SD), when **(i)** an optimal solution of $(x_\circ, y_\circ)$ to the primal (SP) exists **(ii)** f is convex in x for each y, concave in

y for each x and

(iii) f, twice differentiable, has the property that at $(x_\circ, y_\circ)$ its matrix of second partials with respect to y is negative definite.

Mond [29] further gave the following formulation of symmetric dual programming problems:

(MSP): Maximize $f(x,y) - y^T \nabla_y f(x,y)$

Subject to

$$\nabla_y f(x,y) \leq 0$$

$$x \geq 0.$$

(MSD): Maximize $f(x,y) - x^T \nabla_x f(x,y)$

Subject to

$$\nabla_x f(x,y) \leq 0$$

$$y \geq 0.$$

It may be remarked here that in [23], the constraints of both (SP) and (SD) include $x \geq 0,\ y \geq 0$**,** but only $x \geq 0$ is required in the primal and only $y \geq 0$ in the dual.

Later Mond and Weir [67] gave the following pair of symmetric dual nonlinear programming problems which allows the weakening of the convexity-concavity assumptions to pseudoconvexity-pseudoconcavity.

(M-WSP): Minimize $f(x,y)$

Subject to

$$\nabla_y f(x,y) \leq 0$$

$$y^T \nabla_y f(x,y) \geq 0,$$

$x \geq 0.$

(M-WSD): Maximize $f(x,y)$

Subject to

$$\nabla_x f(x,y) \leq 0$$

$$x^T \nabla_y f(x,y) \geq 0,$$

$$y \geq 0.$$

Symmetric Duality in Multiobjective Programming

Mond and Weir [67] discussed symmetric duality in multiobjective programming by considering the following pair of programs:

(PS): Minimize $f(x,y) - \left(y^T \nabla_y \lambda^T f(x,y)\right)e$

Subject to

$$\nabla_y \lambda^T f(x,y) \leqq 0 \text{ ,}$$

$$x \geqq 0 \text{ , } \lambda \in \Lambda^+$$

$$where \;\; \Lambda^+ = \left\{ \lambda \in R^p I \;\; \lambda > 0, \; \sum_{i=1}^{p} \lambda_i = 1 \right\}$$

(DS): Maximize $f(x,y) - \left(x^T \nabla_x \lambda^T f(x,y)\right)e$

Subject to

$$\nabla_x \lambda^T f(x,y) \geqq 0 \text{ ,}$$

$$y \geqq 0 \text{ , } \lambda \in \Lambda^+$$

where $f : R^n \times R^m \to R^p$, and proved the symmetric duality theorem under the convexity – concavity assumptions on $f(x,y)$. Here the minimization is taken in the sense of proper efficiency as given by Geoffrion [27].

Further on the lines of scalar case Mond and Weir [66] also considered another pair of symmetric dual programs and proved symmetric duality results under pseudoconvexity-pseudoconcavity:

(PS1): Minimize $f(x,y)$

Subject to

$$\nabla_2 \lambda^T f(x,y) \leqq 0 \ ,$$

$$y^T \nabla_2 \lambda^T f(x,y) \geqq 0$$

$$x \geqq 0 \ , \ \lambda \in \Lambda^+$$

(DS1): Maximize $f(x,y) - \left(x^T \nabla_1 \lambda^T f(x,y)\right) e$

Subject to

$$\nabla_1 \lambda^T f(x,y) \geqq 0 \ ,$$

$$x^T \nabla_1 \lambda^T f(x,y) \leqq 0 \ ,$$

$$y \geqq 0 \ , \ \lambda \in \Lambda^+ .$$

Later Chandra and Durga Prasad [10] introduced the following pair of multiobjective programs by associating a vector valued infinite game:

(PS*): Minimize $f(x,y) - \left(y^T \nabla_y \mu^T f(x,y)\right) e$

Subject to

$$\nabla_y \mu^T f(x,y) \leqq 0 \ ,$$

$$x \geqq 0 \ , \ \mu \in \Lambda^+ .$$

(DS*): Maximize $f(x,y) - \left(x^T \nabla_x \lambda^T f(x,y)\right) e$

Subject to

$$\nabla_x \lambda^T f(x,y) \geqq 0 \ ,$$

$$y \geqq 0 \ , \ \lambda \in \Lambda^+$$

Here it may be noted that not the same λ is appearing in (PS*) and (DS*) and this creates certain difficulties which are also discussed in [10].

1.3.3 Variational Problems

Differentiable Variational Problems

A variational problem can be considered as a particular case of an optimal control problem in which the control function is the derivative of a state function.

In [15] Courant and Hilbert, quoting an earlier work of Friedrichs [26], gave a dual relationship for a simple type of unconstrained variational problem. Subsequently, Hanson [29] pointed out that some of the duality results of mathematical programming have analogues in variational calculus. Exploring this relationship between mathematical programming and the classical calculus of variations, Mond and Hanson [64] formulated a constrained variational problem as a mathematical programming problem and using Valentine's [79] optimality conditions for the same, presented its Wolfe type dual variational problem for validating various duality results under convexity.

Mathematically, a variational problem is of the form:

(VP): Minimize $\int_I f(t,x,\dot{x})\,dt$

Subject to

$$x(a)=\alpha\ ,\ x(b)=\beta$$

$$g(t,x,\dot{x})\le 0, \qquad t\in I,\ x\in C(I,R^n).$$

where $I=[a,b]$ is a real time interval, $\dot{x}$ denotes derivative of x with respect to t, $f: I\times R^n\times R^n \to R$ and $g: I\times R^n\times R^n \to R$ are continuously differentiable functions with respect to each of their arguments; $C(I,R^n)$ is the space of continuously differentiable functions $x: I\to R^n$, and is equipped with the norm $\|x\|=\|x\|_\infty+\|Dx\|_\infty$, where the differentiation operator D is given by $u=Dx \Leftrightarrow x(t)=\alpha+\int_a^t u(s)\,ds$ except at discontinuities.

The following necessary conditions for the existence for (VP) are derived by Valentine [79].

Theorem 1.1: For every minimizing arc $x=x^\circ(t)$ of the problem (VP), there exists a function of the form

$$H=\lambda_\circ f(t,x,\dot{x})-\lambda(t)^T g(t,x,\dot{x})$$

Such that

$$H_{\dot{x}}=\frac{d}{dt}H_x$$

$$\lambda(t)^T g(t,x,\dot{x})=0$$

$$\left(\lambda_\circ, \lambda(t)\right) \geq 0 \ , \ \left(\lambda_\circ, \lambda(t)\right) \neq 0 \ , \ t \in I$$

hold throughout *I* (except at corners of x° where $H_{\dot{x}} = \frac{d}{dt} H_x$, holds for unique right and left limits). Here $\lambda_\circ$ is constant and $\lambda(\cdot)$ is continuous except possibly for values of t corresponding to corners of x°. Following is the Wolfe type dual variational problem [64] for validating various duality results under convexity:

(WD): Maximize $\int_i \left(f(t,u,\dot{u}) + y(t)^T g(t,u,\dot{u}) \right) dt$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$\left(f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) \right) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) = 0$$

$$y(t) \geqq 0 \ , \ t \in I$$

Later Bector, Chandra and Husain [6] studied Mond-Weir type duality for the problem of [64] for weakening its convexity requirement.

(MWD): Maximize $\int_i f(t,u,\dot{u}) dt$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$\left(f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) \right) - D\left(f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right) = 0$$

$$\int_I y(t)^T g_{\dot{u}}(t,u,\dot{u}) dt > 0$$

$$y(t) \geqq 0 \ , \ t \in I$$

1.3.4 Multiobjective Variational Problems

Many authors have studied optimality and duality for multiobjective variational problems. Bector and Husain [7] were probably the first to introduce multiobjective programming in calculus of variation. They considered the following multiobjective variational problem (VP):

(VP): Minimize $\left(\int_I \left(f^1\left(t, x(t), \dot{x}(t)\right)\right) dt, \ldots, \int_I \left(f^p\left(t, x(t), \dot{x}(t)\right)\right) dt \right)$

Subject to

$$x(a) = \alpha \ , \ x(b) = \beta$$

$$g\left(t, x(t), \dot{x}(t)\right) \leqq 0 \ , \ t \in I,$$

$$x \in C\left(I, R^n\right)$$

where, $f^i : I \times R^n \times R^n \to R$, $i \in P = 1, 2, \ldots, p$, $g : I \times R^n \times R^n \to R^m$, are assumed to be continuously differentiable functions, for each $t \in I$, $i \in P$, $B^i(t)$ is an $n \times n$ positive semidefinite symmetric matrix with $B^i(\cdot)$ continuous on I .

Bector and Husain [7] constructed Wolfe type dual and Mond-Weir type dual and proved various duality theorems under convexity and generalized convexity of functionals.

(WD): Maximize $\left(\int_i \left(f^1(t, u, \dot{u}) + y(t)^T g(t, u, \dot{u}) \right) dt \right.$

$$\left. , \ldots, \int_i \left(f^p(t, u, \dot{u}) + y(t)^T g(t, u, \dot{u}) \right) dt \right)$$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta$$

$$\left(\lambda^T f_u + y(t)^T g_u\right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}}\right) = 0 \ , \ t \in I$$

$$y(t) \geqq 0 \ , \ t \in I$$

$\lambda > 0$, $\lambda^T e = 1$ where $e = (1, 1, \ldots, 1)^T$ and $\lambda \in R^k$.

The following Mond-Weir type dual to the problem (VP):

(M-WD): Maximize $\left(\int_I f^1(t, u, \dot{u}) dt, \ldots, \int_I f^p(t, u, \dot{u}) dt \right)$

Subject to

$$u(a) = \alpha \ , \ u(b) = \beta \ ,$$

$$\left(\lambda^T f_u + y(t)^T g_u\right) - D\left(\lambda^T f_{\dot{u}} + y(t)^T g_{\dot{u}}\right) = 0,\ t \in I$$

$$\int_1 y(t)^T g(t,u,\dot{u})\,dt \geqq 0\ ,$$

$$y(t) \geqq 0\ ,\ \ t \in I\,,$$

$$\lambda > 0\,.$$

1.3.5 Symmetric Duality for Variational Problems

Mond and Hanson [65] and Bector, Chandra and Husain [6] extended symmetric duality to Variational problems. In [65] they investigated Wolfe type duality symmetric duality for the variational problems (VP). Later [6] Bector, Chandra and Husain studied Mond-Weir type symmetric dual variational problems in order to weaken the convexity-concavity assumptions. Smart and Mond [61] applied invexity for Variational problems introduced by Mond, Chandra and Husain [63] to symmetric dual Variational problems without non-negativity constraints of Mond and Hanson [64], but subjecting invexity to an additional condition.

Mond and Hanson [65] studied symmetric duality for the following variational problem under convexity / concavity assumptions:

(Primal): Minimize $\int_a^b \Big\{ f(t,x,\dot{x},y,\dot{y}) - y(t)^T f_y(t,x,\dot{x},y,\dot{y})$

$$+ y(t)\frac{d}{dt} f_{\dot{y}}(t,x,\dot{x},y,\dot{y}) \Big\} dt$$

Subject to

$$x(a) = \alpha\ \ ,\quad x(b) = \beta$$
$$y(a) = \gamma\ \ ,\quad y(b) = \delta$$

$$\frac{d}{dt} f_{\dot{y}}(t,x,\dot{x},y,\dot{y}) \geq f_y(t,x,\dot{x},y,\dot{y}), t \in I$$

$$x(t) \geq 0, t \in I$$

(Dual): Maximize $\int_a^b \Big\{ f(t,u,\dot{u},v,\dot{v}) - u(t)^T f_x(t,u,\dot{u},v,\dot{v}) + u(t)\frac{d}{dt} f_{\dot{x}}(t,u,\dot{u},v,\dot{v}) \Big\} dt$

Subject to

$$u(a)=\alpha \quad , \quad u(b)=\beta$$
$$v(a)=\gamma \quad , \quad v(b)=\delta$$

$$\frac{d}{dt} f_{\dot{x}}(t,u,\dot{u},v,\dot{v}) \le f_x(t,u,\dot{u},v,\dot{v}), t \in I$$

$$v(t) \ge 0, t \in I$$

Let $I=[a,b]$ be the real interval, $x: I \to R^n$ and $y: I \to R^m$, x and $\dot{x}$ denote derivatives of x and y respectively with respect to t and $f(t,x,\dot{x},y,\dot{y})$ is a continuously differentiable scalar function. They needed f to be convex in x and $\dot{x}$ for each y and $\dot{y}$ and concave in y and $\dot{y}$ for each x and $\dot{x}$.

If the constraints $x(t) \ge 0$ and $y(t) \ge 0, t \in I$ are removed from the above problem primal and dual problems respectively, we get the pair considered by Smart and Mond [61], wherein weak duality theorem is proved assuming the functional $\int_a^b f\, dt$ to be invex in x and $\dot{x}$ and $-\int_a^b f\, dt$ to be invex in y and $\dot{y}$.

Subsequently, Bector, Chandra and Husain [6] presented a pair of Mond-Weir type symmetric dual variational problems in order to relax convexity-concavity to pseudoconvexity-pseudoconcavity.

The following are the primal and dual problems formulated in [6]:

Problem I (Primal) = P

Minimize $\int_a^b f(t,x,\dot{x},y,\dot{y})dt$

Subject to

$$x(a)=\alpha \quad , \quad x(b)=\beta$$
$$y(a)=\gamma \quad , \quad y(b)=\delta$$

$$f_y(t,x,\dot{x},y,\dot{y})-Df_{\dot{y}}(t,x,\dot{x},y,\dot{y}) \le 0$$

$$\int_a^b y(t)^T\left(f_y(t,x,\dot{x},y,\dot{y})-Df_{\dot{y}}(t,x,\dot{x},y,\dot{y})\right)dt \ge 0$$

$$x(t) \ge 0$$

Problem II (Dual) = D

Maximize $\int_a^b f(t,x,\dot{x},y,\dot{y})dt$

Subject to

$$x(a)=\alpha \quad , \quad x(b)=\beta$$
$$y(a)=\gamma \quad , \quad y(b)=\delta$$

$$f_x(t,x,\dot{x},y,\dot{y})-Df_{\dot{x}}(t,x,\dot{x},y,\dot{y})\geq 0$$

$$\int_a^b x(t)^T\left(f_x(t,x,\dot{x},y,\dot{y})-Df_{\dot{x}}(t,x,\dot{x},y,\dot{y})\right)dt\leq 0$$

$$y(t)\geq 0, t\in I .$$

The usual duality results are derived for above pair of Mond-Weir problems under pseudoconvexity and pseudoconcavity. The close relationship between the duality results for the pair in [6] and those of its counterpart is pointed out.

1.4 Control Problems

Optimal control models are very prominent amongst constrained optimization models because of their occurrences in a variety of popular contexts, notably, advertising investment, production and inventory, epidemic, control of a rocket etc. The planning of a river system, where it is required to make the best use of the water, can also be modelled as an optimal control problem. Optimal control models are also potentially applicable to economic planning, and to the world models of the 'Limits to Growth' kind.

1.4.1 Control Problem and Related Preliminaries

A control problem is to transfer the state vector from an initial state x(a)=α to a final state x(b) = β so as to minimize a functional, subject to constraints on the control and state variables.

A control problem can be stated formally as,

Problem (CP) (Primal): $\underset{x\in X, u\in U}{Minimize}\int_a^b f(t,x,u)dt,$

subject to

$$x(a) = \alpha,\ x(b) = \beta, \qquad (1.1)$$

$$h(t, x, u) = \dot{x},\ t \in I, \qquad (1.2)$$

$$g(t, x, u) \le 0,\ t \in I, \qquad (1.3)$$

f is as before, g: $I \times R^n \times R^m \to R^p$ and h: $I \times R^n \times R^m \to R^n$ are continuously differentiable functions with respect to each of its arguments.

The set X is the space of continuously differentiable state functions x: $I \to R^n$ such that x (a) =α, x (b) = β, equipped with the norm $||x|| = ||x||_\infty + ||Dx||_\infty$, and u is the space of piecewise continuous control functions $u: I \to R^m$ has the uniform norm $||.||_\infty$, and the differential equation (1.2) for x with the initial conditions expressed as $x(t) = x(a) + \int_a^t h(s, x(s), u(s))ds, t \in I,$ may be written as Dx =H(x, u), where the map : $X \times U \to C(I,R^n)$, $C(I,R^n)$ being the space of continuous functions from $I \to R^n$, defined by H(x, u)(t)=h(t, x(t),u(t)).

1.5 A **Brief Account of Games**

The theory of games started in 20th century but the mathematical treatment of games took fire in 1944 when John Von Neumann and Morgenstern [80] published their well known book, " The Theory of Games and economic behaviour the Neumann's approach uses the

minmax principle which involves the fundamental idea of the minimization of the maximum loss. Many of the competitive problems can be handled by this game theory. However, not all the competitive problems can be analyzed with the help of game theory.

A competitive situation is called a game if it has the following properties.

(i) There are finite numbers of competitors called players.

(ii) A list of finite or infinite number of possible courses of action is available to each

player. The list need not be the same for each player.

(iii) A play is played when each player choose one of his courses of action. The choices

are assumed to be made simultaneously so that no player knows his opponent's choice

until he has decided his own course of action.

(iii) Every play i.e., combination of courses of action is associated with an outcome

known as pay-off (generally many or some other quantitative measures for satisfactions)

this determines a set of games, one to each player. Here a loss is considered a negative

game. Thus after each play of the game, one player pays to others an amount determined

by the courses of action chosen.

The competition between firms, the conflict between management and labour, the fight to get bills through Congress, the power of judiciary, war and peace negotiations between countries, and so on, all provide examples of games in action. There are also psychological games played on a personal level, where the weapons are words, and the pay-offs are good or bad feelings.

There are biological games, the competition between is species, where natural selection can be modeled as a game played between genes. There is a connection between game

theory and mathematical areas of logic and computer science. One may view theoretical statistics as a two person game in which nature takes the roll of one of the players.

We denote the strategy set or action space of player i by A_i, i=1… n. Suppose the player I chooses $a_1 \in A_1$. Player two chooses $a_2 \in A_2$ etc. and player n chooses $a_n \in A_n$. Then we denote the payoff to the player j for $j = 1,2,...,n$ by $f_i\left(a_1, a_2, ..., a_n\right)$ and call it payoff function for the player j. The strategic form of a game is defined then by three objects.

i. The set, $N = \{1,2,...,n\}$, of players,

ii. The sequence, $A_1, A_2, ..., A_n$ of strategy sets of the players, and

iii. The sequence $f_1\left(a_1, a_2, ..., a_n\right), ..., f_n\left(a_1, a_2, ..., a_n\right)$ of real-valued payoff functions of the players.

A game in strategic form is said to be zero-sum if the sum of the payoffs to the players is zero no matter what actions are chosen by the players. That is, the game is zero-sum if

$$\sum_{j=1}^{n} f_1\left(a_1, a_2, ..., a_n\right) = 0, \text{ for all } a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n$$

A two-person game is a game in which the gain of one player after a play equals net loss of his opponent. The basic assumptions in a two-person zero-sum game are:

1. There are exactly two players with precisely opposite interests.

2. The number of strategies selected by a player is finite. The list may not be common.

3. For each specific strategy selected by a player, there results a payoff.

4. The amount won by one of the player is exactly equal to the amount lost by the other.

Dynamic game theory is related to the modeling of large scale systems which have individual decision makers. Application of these games lies in a variety of context such as environmental problems, resource problems, aerospace problems and energy managements.

Chapter -2

SUFFICIENCY AND DUALITY IN CONTROL PROBLEMS WITH GENERALIZED INVEXITY

2.1 Introduction

Optimal control models are very prominent amongst constrained optimization models because of their occurrences in a variety of popular contexts, notably, advertising investment, production and inventory, epidemic, control of a rocket etc. The planning of a river system, where it is required to make the best use of the water, can also be modelled as an optimal control problem. Optimal control models are also potentially applicable to economic planning, and to the world models of the ‘Limits to Growth’ kind.

Necessary optimality conditions for existence of extremal solution for a variational problem in the presence of inequality and equality constraints were obtained by Valentine [79]. Using Valentine’s results, Berkovitz [54] obtained corresponding Fritz John type necessary optimality conditions for a control problem. Mond and Hanson [9] pointed out that if the optimal solution for the problem is normal, then the Fritz John type optimality conditions reduce to Karush-Kuhn-Tucker conditions. Using these Karush- Kuhn – Tucker optimality conditions, Mond and Hanson [9] presented Wolfe type dual and established weak, strong and converse duality theorems under convexity conditions. Abraham and Buie [48] studied duality for continuous programming and optimal control from a unified point of view. Later Mond and Smart [10] proved that for invex functions, the necessary conditions of Berkovitz [54] together with normality conditions are sufficient for optimality and also derived some duality results under invexity.

In this chapter, it is shown that for generalized invexity assumptions on functionals, the necessary conditions [54] in the control problems are also sufficient. As an application of Berkovitz’s [54] optimality conditions with normality, a Mond-Weir [65] type dual to the control problem is constructed and under generalized invexity of functionals, various duality results are derived. It is indicated that these duality results are applicable to the control problem with free boundary conditions and also related to those for nonlinear programming problems already existing in the literature.

2.2 Control Problem and Related Preliminaries

Let R^n denotes an n-dimensional Euclidean space, I = [a, b] be a real interval and $f:I\times R^n \times R^m \rightarrow R$ be a continuously differentiable with respect to each of its arguments. For the function f (t, x, u), where $x:I\rightarrow R^n$ is differentiable with its derivative $\dot{x}$ and u: $I\rightarrow R^m$ is the smooth function, denote the partial derivatives of f by f_t, f_x and f_u, where

$$f_t : \frac{\partial f}{\partial t},\ f_x = \left(\frac{\partial f}{\partial x^1},\ldots,\frac{\partial f}{\partial x^n}\right)^T,\ f_u = \left(\frac{\partial f}{\partial u^1},\ldots,\frac{\partial f}{\partial u^m}\right)^T,\ x = (x^1,\ldots,x^n)^T \ and\ u = (u_1,\ldots,u_m)^T.$$

For an m-dimensional vector function g (t, x, u), the gradient with respect to x is

$$g_x = \begin{pmatrix} \frac{\partial g^1}{\partial x^1},\ldots,\frac{\partial g^p}{\partial x^n} \\ ------- \\ ------- \\ \frac{\partial g^1}{\partial x^n},\ldots,\frac{\partial g^p}{\partial x^n} \end{pmatrix},$$ an n×p matrix of first order derivatives.

Here u(t) is the control variable and x(t) is the state variable, u is related to x via the state equation $\dot{x}$ = h(t,x,u). Gradients with respect to u are defined analogously.

A control problem is to transfer the state vector from an initial state x(a)=α to a final state x(b) = β so as to minimize a functional, subject to constraints on the control and state variables.

A control problem can be stated formally as,

Problem (CP) (Primal): $\underset{x\in X, u\in U}{Minimize} \int_a^b f(t,x,u)dt,$

subject to

$$x(a) = \alpha,\ x(b) = \beta, \tag{2.1}$$

$$h(t,x,u) = \dot{x},\ t \in I, \tag{2.2}$$

$$g(t,x,u) \leq 0,\ t \in I, \tag{2.3}$$

(i) f is as before, $g: I\times R^n \times R^m \rightarrow R^p$ and $h: I\times R^n \times R^m \rightarrow R^n$ are continuously differentiable functions with respect to each of its arguments.

(ii) X is the space of continuously differentiable state functions $x: I \rightarrow R^n$ such that x(a) = α, x (b) = β, equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, and u is the space of piecewise continuous control functions $u: I\rightarrow R^m$ has the uniform norm $\|.\|_\infty$, and

The differential equation (2.2) for x with the initial conditions expressed as $x(t) = x(a) + \int_a^t h(s, x(s), u(s))ds, t \in I,$ may be written as Dx =H(x,u), where the map : X×U → $C(I,R^n)$, $C(I,R^n)$ being the space of continuous functions from $I\rightarrow R^n$, defined by H(x,u)(t)=h(t,x(t),u(t)).

Following Craven [7], the control problem can be expressed as,

(ECP): $\underset{x\in X, u\in U}{\text{Minimize}} \quad F(x, u)$

subject to Dx = H(x,u),–G(x,u) ∈ S,

Where G is function from X×U into $C(I, R^p)$ given by G(x,u) (t) = g(t,x(t), u(t)) from x∈X, u∈U, and t∈I; S is the convex cone of functions in $C(I,R^p)$ whose components are non-negative; thus S has interior points.

Necessary optimality conditions for existence of extermal solution for a variational problem subject to both equality and inequality constraints were given by valentine [26]. Invoking Valentine's [26] results, Berkovitz [54] obtained corresponding necessary optimality conditions for the above control problem (CP). Here we mention the Fritz John optimality conditions derived by Craven [7] in the form of the following proposition which will be required in the sequel.

Proposition 2.1 (Necessary optimality conditions). If $(\bar{x},\bar{u}) \in X\times U$ an optimal solution of (CP) and the Fre/che/t derivatives $Q' = \left(D - H_x(x,u), -H_u(x,u)\right)$ is surjective, then there exist Lagrange multipliers $\lambda_0 \in R$, and piecewise smooth functions $\lambda: I \to R^p$ and $\mu: I \to R^n$ satisfying, for all $t \in I$,

$$\lambda_0 f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t) = 0,$$

$$\lambda_0 f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u}) = 0,$$

$$\lambda(t)^T g(t,\bar{x},\bar{u}) = 0,$$

$$\left(\lambda_0, \lambda_{(t)}\right) \geq 0,$$

$$\left(\lambda_0, \lambda_{(t)}, \mu_{(t)}\right) \neq 0.$$

The above conditions will become Karush-Kuhn-Tucker conditions if $\lambda_0 > 0$. Therefore, if we assume that the optimal solutions $(\bar{x},\bar{u})$ is normal, then without any loss of generality, we can set $\lambda_0 = 1$. Thus from the above we have the Karush-Kuhn- Tucker type optimality conditions

$$f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t) = 0, t \in I, \qquad (2.4)$$

$$f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u}) = 0, t \in I, \qquad (2.5)$$

$$\lambda(t)^T g(t,\bar{x},\bar{u}) = 0, t \in I, \qquad (2.6)$$

$$\lambda(t) \geq 0, t \in I. \qquad (2.7)$$

Using these optimality conditions, Mond and Hanson [63] constructed following Wolfe type dual. Problem (CD) (Dual):

$$\text{Maximize } \int_a^b \left\{ f(t,x,u) + \lambda(t)^T g(t,x,u) + \mu(t)^T \left(h(t,x,u) - \dot{x}\right)\right\} dt$$

subject to

$$f_x(t,x,u) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,x,u) + \dot{\mu}(t) = 0, t \in I,$$

$$f_u(t,x,u) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,x,u) = 0, t \in I,$$

$$\lambda(t) \geq 0, t \in I.$$

In [5], [CP] and (CD) are shown to be a dual pair if *f*, g and *h* are all convex in x and u. Subsequently, Mond and Smart [10] extended this duality by introducing the following invexity requirement.

Definition 2.1*(Invex)* [10]: If there exists vector function $\eta(t,x,\bar{x}) \in R^n$ with $\eta = 0$ at t if x(t)= $\bar{x}$(t), and there exists vector function $\xi(t,u,\bar{u}) \in R^m$ such that for scalar function $\Phi(t,x,\dot{x},u)$, the functional

$$\Phi(x,\dot{x},u)=\int_a^b \phi(t,x,\dot{x},u)dt$$

satisfies

$$\Phi(x,\dot{x},u)-\phi(\bar{x},\dot{\bar{x}},\bar{u})\geq\int_a^b\left[\eta^T\phi_x(t,\dot{\bar{x}},\bar{x},\bar{u})+\left(\frac{dn}{dt}\right)^T\phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u})+\xi^T\phi_u(t,\bar{x},\dot{\bar{x}},\bar{u})\right]dt$$

then ϕ is said to invex at $x, \dot{x}$ and u on I with respect to η and ξ.

In [10] Mond and Smart proved weak, strong and converse duality theorems under the invexity of $\int_a^b f\ dt$, $\int_a^b \lambda^T g\ dt$, for λ (t) $\in R^p$ with $\lambda(t) \geq 0$, t $\in$I and $\int_a^b \mu^T hdt$ for any $\mu(t) \in R^n$, t$\in$I.

2.3 Generalized Invexity

In this section, we extend the notion of invexity for a functional given in [10] to a large class of functionals, as these will be required for subsequent analysis.

Definition 2.2 For a scalar function $\phi(t,x,\dot{x},u)$ the functional $\Phi(x,\dot{x},u)=\int_a^b \phi(t,x,\dot{x},u)dt$ is said to be pseudoinvex at x, $\dot{x}$ and u if there exist vector function $\eta(t,x,\bar{x})\in R^n$ with η=0 at t if $x(t)=\bar{x}(t)$ and $\xi(t,u,\bar{u})\in R^m$ such that for all $(x,\dot{x},u)\neq(\bar{x},\dot{\bar{x}},\bar{u})$.

$$\int_a^b\left(\eta^T\phi_x\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)+\left(\frac{d\eta}{dt}\right)^T\phi_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)+\xi^T\phi_u\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)\right)dt\geq 0\Rightarrow$$

$$\Phi(x,\dot{x},u)\geq\Phi(\bar{x},\dot{\bar{x}},\bar{u}).$$

Definition 2.3 (Strictly Pseudoinvex): The functional Φ is said to be strictly pseudoinvex, if there exist vector functions $\eta(t,x,\bar{x})\in R^n$ with η = 0 at t if x(t) = $\bar{x}$ (t) and $\xi(t,u,\bar{u})\in R^m$ such that

$$\int_a^b\left(\eta^T\phi_x\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)+\left(\frac{d\eta}{dt}\right)^T\phi_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)+\xi^T\phi_u\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)\right)dt\geq 0\Rightarrow$$

$$\Phi(x,\dot{x},u)>\Phi(\bar{x},\dot{\bar{x}},\bar{u})..$$

Definition 2.4 (Quasi-invex): The functional Φ is said to be quasi-invex, if there exist vector functions $\eta(t,x,\bar{x})\in R^n$ with η = 0 at t if x (t) = $\bar{x}$ (t) and $\xi(t,u,\bar{u})\in R^m$ such that

$$\Phi(x,\dot{x},u)\leq\Phi(\bar{x},\dot{\bar{x}},\bar{u})\Rightarrow\int_a^b\left((\eta^T\phi_x\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)+\left(\frac{d\eta}{dt}\right)^T\phi_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)\right.$$

$$\left.+\xi^T\phi_u\left(t,\bar{x},\dot{\bar{x}},\bar{u}\right)\right)dt\leq 0.$$

2.4 Sufficiency of Optimality Conditions

It can be proved that for generalized invex functionals, the Karush-Kuhn-Tucker optimality conditions given in Section 2.2 are sufficient for optimality.

Theorem 2.1 If there exists $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ such that the conditions (2.4) – (2.7) hold with $(\bar{x},\bar{u})$ feasible for (CP) and $\int_a^b f dt$ is pseudoinvex and $\int_a^b \left(\bar{\lambda}^T g+\bar{\mu}^T (h-\dot{x})\right)dt$ `is quasi-invex with respect to the same η and ξ, then $(\bar{x},\bar{u})$ is an optimal solution of (CP).

Proof: Assume that $(\bar{x},\bar{u})$ is not optimal for (CP).Then there exists $(x,u)\neq(\bar{x},\bar{u})$, i.e., (*x, u*) feasible for (CP), such that $\int_a^b f(t,x,u)\,dt< \int_a^b f(t,\bar{x},\bar{u})dt$.

This, because of pseudoinvexity of $\int_a^b f dt$ with respect to the same η and ξ, it follows that

$$\int_a^b \left(\eta^T f_x(t,\bar{x},\bar{u})+\xi^T f_u(t,\bar{x},\bar{u}\right)dt<0$$

Using (2.4) and (2.5), this yields

$$0<\int_a^b \Big[\eta^T\left(\bar{\lambda}^T(t)g_x(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_x(t,\bar{x},\bar{u})+\dot{\bar{\mu}}(t)\right)$$

$$+\xi^T\left(\bar{\lambda}^T(t)^T g_u(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_u(t,\bar{x},\bar{u}\right)\Big]dt$$

$$=\int_a^b\Bigg[\eta^T\left(\bar{\lambda}(t)^T g_x(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_x(t,\bar{x},\bar{u})dt\right)+\int_a^b \eta^T\dot{\bar{\mu}}(t)dt$$

$$+\int_a^b \xi^T\left(\bar{\lambda}(t)^T g_u(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_u(t,\bar{x},\bar{u}\right)dt$$

$$=\int_a^b\Bigg[\eta^T\left(\bar{\lambda}^T(t)g_x(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_x(t,\bar{x},\bar{u})dt\right)-\left(\frac{d\eta}{dt}\right)^T\bar{\mu}(t)$$

$$+\xi^T\left(\bar{\lambda}^T(t)g_u(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_u(t,\bar{x},\bar{u}\right)\Big\}dt+\eta\mu(t)\Big|_{t=b}^{t=a}$$

(by integrating by parts)

$$=\int_a^b\Bigg[\eta^T\left(\bar{\lambda}^T(t)g_x(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_x(t,\bar{x},\bar{u})dt\right)-\left(\frac{d\eta}{dt}\right)^T\bar{\mu}(t)$$

$$+\xi^T\left(\bar{\lambda}^T(t)g_u(t,\bar{x},\bar{u})+\bar{\mu}(t)^T h_u(t,\bar{x},\bar{u})\right)\Big\}\,dt$$

(using η =0 at t if $x(t)=\bar{x}$ (t))

By quasi-invexty of $\int_a^b\left(\bar{\lambda}g+\bar{\mu}(h-\dot{x})\right)dt$, this implies

$$\int_a^b\left\{\bar{\lambda}(t)^T g(t,x,u)+\bar{\mu}(t)^T\left(h(t,x,u)-\dot{x}\right)\right\}dt$$

$$>\int_a^b\left\{\bar{\lambda}(t)^T g(t,\bar{x},\bar{u})+\bar{\mu}(t)^T\left(h(t,\bar{x},\bar{u})-\dot{\bar{x}}\right)\right\}dt$$

Using (2.6) and also $\bar{\mu}(t)^T\left(h(t,\bar{x},\bar{u})-\dot{\bar{x}}\right)=0$, the above inequality gives

$$\int_a^b\left\{\bar{\lambda}(t)^T g(t,x,u)+\bar{\mu}(t)^T\left(h(t,x,u)-\dot{x}\right)\right\}dt>0. \tag{2.8}$$

Since (x,u) is feasible for (CP), g(t,x,u) ≤ 0, t∈I and $h(t,x,u)-\dot{x}=0$. Hence for $\bar{\lambda}(t)\geq 0$, t ∈ T and $\bar{\mu}$ (t) ∈ R^n, we have

$$\int_a^b\left\{\bar{\lambda}(t)^T g(t,x,u)+\bar{\mu}(t)^T\left(h(t,x,u)-\dot{x}\right)\right\}dt\leq 0. \tag{2.9}$$

Consequently (2.8) contradicts (2.9). Thus $(\bar{x},\bar{u})$ is, indeed, an optimal solution of the control problem (CP).

2.5 Duality

We formulate the following dual (CD) to the primal problem (CP) in the spirit of Mond and Weir [65].

Problem (CD) (Dual): Maximize $\int_a^b f(t,x,u)dt$

Subject to $x(a)=\alpha,\ x(b)=\beta,$ (2.10)

$$f_x(t,x,u)+\lambda(t)^T g_x(t,x,u)+\mu(t)^T h_x(t,x,u)+\dot{\mu}(t)=0,\quad t\in I, \tag{2.11}$$

$$f_u(t,x,u)+\lambda(t)^T g_u(t,x,u)+\mu(t)^T h_u(t,x,u)=0,\quad t\in I, \tag{2.12}$$

$$\int_a^b\left(\lambda(t)^T g(t,x,u)+\mu(t)^T\left(h(t,x,u)-\dot{x}\right)\right)dt\geq 0, \tag{2.13}$$

$$\lambda(t)\geq 0,\qquad t\in I. \tag{2.14}$$

Theorem 2.2(Weak Duality): Let $(\bar{x}, \bar{u})$ and (x,u,λ,μ) be feasible solution for (CP) and (CD) respectively. If for all feasible $(\bar{x}, \bar{u}, x, u, \lambda, \mu)$, $\int_a^b$ f dt is pseudoinvex and $\int (\lambda^T g + \mu^T (h - \dot{x}))$ dt for $\lambda(t) \in R^n, \lambda(t) \geq 0, t \in I$ and $\mu(t) \in R^n$ is quasi-invex with respect to the same η and ξ, then

max (CP) ≥ min (CD).

Proof: Since $(\bar{x}, \bar{u})$ is feasible for the problem (CP) and (x, u,λ,μ) feasible for the problem (CD), it implies that

$$\int_a^b \left(\lambda(t)^T g(t, \bar{x}, \bar{u}) + \mu(t)^T \left(h(t, \bar{x}, \bar{u}) - \dot{\bar{x}}\right)\right) dt \leq \int_a^b \left(\lambda(t)^T g(t, x, u) + \mu(t)^T \left(h(t, x, u) - \dot{x}\right)\right) dt$$

This, because of quasi-invexity of $\int_a^b (\lambda^T g + \mu^T \left(h - \dot{x} \right)) dt$, implies

$$0 \geq \int_a^b \left\{\eta^T \left(\lambda\ (t)^T g_x(t, x, u) + \mu(t)^T h_x(t, x, u)\right) + \xi^T \left(\lambda\ (t)^T g_u(t, x, u) + \mu(t)^T h_u(t, x, u)\right)\right\} dt - \int_a^b \left(\frac{d\eta}{dt}\right)^T \mu(t) dt$$

$$= \int_a^b \left\{\eta^T \left(\lambda(t)^T g_x(t, x, u) + \mu(t)^T h_x(t, x, u)\right) + \xi^T \left(\lambda(t)^T g_u(t, x, u) + \mu(t)^T h_u(t, x, u)\right)\right\} dt$$

$$- \mu(t)\eta\Big|_{t=a}^{t=b} + \int_a^b \dot{\mu}(t)^T \eta dt,$$

(By integration by parts)

$$= \int_a^b \left\{\eta^T \left(\lambda(t)^T g_x(t, x, u) + \mu(t)^T h_x(t, x, \dot{u}) + \dot{\mu}(t)\right) + \xi^T \left(\lambda(t)^T g_u(t, x, u) + \mu(t)^T h_u(t, x, u)\right)\right\} dt$$

(as fixed boundary conditions give η = 0 at t = a and t =b)

Using (2.11) and (2.12), we have

$$\int_a^b \left\{\eta^T f_x(t, x, u) + \xi^T f_x(t, x, u)\right\} dt \geq 0.$$

By psudoinvexity $\int_a^b \mathrm{fdt}$, this gives

$$\int_a^b f(t, \bar{x}, \bar{u}) dt \geq \int_a^b f(t, x, u) dt.$$

That is,

infimum (CP) ≥ supremum (CD).

Theorem 2.3 (Strong Duality): Under generalized invexity conditions of Theorem 2.2, if $(\bar{x}, \bar{u})$ is an optimal solution of the problem (CP) and is also normal, then there exist piecewise smooth functions

$\overline{\lambda}: I \to R^p$ and $\overline{\mu}: I \to R^n$ such that $(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\mu})$ is an optimal solution of (CP) and the corresponding objective values are equal.

Proof: Since $(\overline{x}, \overline{u})$ is optimal solution for (CP) and is normal, by Proposition 2.1, there exist piecewise smooth functions $\overline{\lambda}: I \to R^p$ and $\overline{\mu}: I \to R^n$ such that the condition (2.4) – (2.7) are satisfied. Since $\overline{\lambda}(t)^T g(t, \overline{x}, \overline{u}) = 0$ and $\overline{\mu}(t)^T (h(t, \overline{x}, \overline{u}) - \dot{\overline{x}}) = 0$, $\int_a^b \left(\overline{\lambda}(t)^T g(t, \overline{x}, \overline{u}) + \overline{\mu}(t)^T g(t, \overline{x}, \overline{u})\right) dt = 0.$ Thus, this together with (2.4), (2.5) and (2.7) implies that $(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\mu})$ is feasible for (CD) and the corresponding objective values are the same as it is evident from the formulation of the primal and dual problems. So by Theorem 2.2, $(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\mu})$ is an optimal solution for (CD).

Theorem 2.4 (Strict Converse Duality): Let $(\overline{x}, \overline{u})$ be an optimal solution of (CP) and also normal. If $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$ is an optimal solution; and $\int_a^b f\, dt$ is strictly pseudoinvex and $\int_a^b \left(\hat{\lambda}^T g + \hat{\mu}^T (h - \dot{x})\right) dt$ is quasi-invex at $(\hat{x}, \hat{u})$ with respect to the same η and ξ, then $(\overline{x}, \overline{u}) = (\hat{x}, \hat{u})$, i.e., $(\hat{x}, \hat{u})$ is an optimal solution of (CP).

Proof: Assume that $(\overline{x}, \overline{u}) \neq (\hat{x}, \hat{u})$.

Since $(\overline{x}, \overline{u})$ is an optimal of (CP) at which normality condition is met, and since conditions of Theorem 2.1 are satisfied, then, by Theorem 2.3, there exist piecewise smooth $\overline{\lambda}: I \to R^p$ and $\overline{\mu}: I \to R^n$ such that $(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\mu})$ is and optimal solution of (CD) and

$$\int_a^b f(t, \overline{x}, \overline{u}) dt = \int_a^b f(t, \hat{x}, \hat{u}) dt. \qquad (2.15)$$

By the feasibility of $(\overline{x}, \overline{u})$ for (CP) and $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$ for (CD), it implies,

$$\int_a^b \left(\hat{\lambda}(t)^T g(t, \overline{x}, \overline{u}) + \hat{\mu}(t)^T \left(h(t, \overline{x}, \overline{u}) - \dot{\overline{x}}\right)\right) dt \leq 0,$$

and

$$\int_a^b \left(\hat{\lambda}(t)^T g(t, \hat{x}, \hat{u}) + \hat{\mu}(t)^T \left(h(t, \hat{x}, \hat{u}) - \dot{\hat{x}}\right)\right) dt \geq 0.$$

Combining these inequalities we have

$$\int_a^b \left(\hat{\lambda}(t)^T g(t, \overline{x}, \overline{u}) + \hat{\mu}(t)^T (h(t, \overline{x}, \overline{u}))\right) dt \leq \int_a^b \left(\hat{\lambda}(t)^T g(t, \hat{x}, \hat{u}) + \hat{\mu}(t)^T (h(t, \hat{x}, \hat{u}))\right) dt$$

Because of the quasi-invexity of $\int_a^b \left(\hat{\lambda}^T g + \hat{\mu}^T h(x - \dot{x})\right) dt$? at ($\hat{x}, \hat{u}$), this yields

$$0 \geq \int_a^b \left\{ \eta^T \left(\hat{\lambda}(t)^T g_x(t,\hat{x},\hat{u}) + \hat{\mu}(t)^T h_x(t,\hat{x},\hat{u}) \right) - \left(\frac{d\eta}{dt} \right)^T \hat{\mu}(t) \right.$$

$$\left. + \xi^T \left(\hat{\lambda}(t)^T g_u(t,\hat{x},\hat{u}) + \hat{\mu}(t)^T h_u(t,\hat{x},\hat{u}) \right) \right\} dt$$

$$= \int_a^b \left\{ \eta^T \left(\hat{\lambda}(t)^T g_x(t,\hat{x},\hat{u}) + \hat{\mu}(t)^T h_x(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}(t) \right) \right.$$

$$\left. + \xi^T \left(\hat{\lambda}(t)^T g_u(t,\hat{x},\hat{u}) + \hat{\mu}(t)^T h_u(t,\hat{x},\hat{u}) \right) \right\} dt - \eta^T \hat{\mu}(t) \Big|_{t=a}^{t=0}$$

(by integration by parts)

$$0 \geq \int_a^b \left\{ \eta^T \left(\hat{\lambda}(t)^T g_x(t,\hat{x},\hat{u}) + \hat{\mu}(t)^T h_x(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}(t) \right) \right.$$

$$\left. + \xi^T \left(\hat{\lambda}(t)^T g_u(t,\hat{x},\hat{u}) + \hat{\mu}(t)^T h_u(t,\hat{x},\hat{u}) \right) \right\} dt \qquad (2.16)$$

Because ($\hat{x},\hat{u}$) is feasible for (CD), we have that

$$f_x(t,\hat{x},\hat{u}) + \hat{\lambda}(t)^T g_x(t,\hat{x},\hat{u}) + \hat{\mu}(t) h_x(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}(t) = 0, \quad t \in I,$$

$$f_u(t,\hat{x},\hat{u}) + \hat{\lambda}(t) g_u(t,\hat{x},\hat{u}) + \hat{\mu}(t) h_u(t,\hat{x},\hat{u}) = 0, \qquad t \in I$$

Using these equations in (2.16), we have

$$\int_a^b \left(\eta^T f_x(t,\hat{x},\hat{u}) + \xi^T f_u(t,\hat{x},\hat{u}) \right) dt \geq 0$$

Thus, by strict pseudoinvexity of $\int_a^b f dt$ yield,

$$\int_a^b f(t,\bar{x},\bar{u})\, dt > \int_a^b f(t,\hat{x},\hat{u})\, dt.$$

This contradicts (2.15). Hence $(\hat{x},\hat{u}) = (\bar{x},\bar{u})$, i.e., $(\hat{x},\hat{u})$ is an optimal solution of (CP).

Now, we shall prove converse duality under the assumption that f, g and h are twice continuously differentiable. The problem (CD) may be written in minimization form as follows:

Minimize $-\psi(x,u,\lambda,\mu)$

Subject to

$x(a) = \alpha, \quad x(b) = \beta$

$$\theta_1(t, x(t), u(t), \lambda(t), \mu(t), \dot{\mu}(t)) = f_x + \lambda(t)^T g_x + \mu(t)^T h_x + \dot{\mu}(t) = 0, t \in I,$$

$\theta_2(t, x(t), u(t), \lambda(t), \mu(t)) = f_u + \lambda(t)^T g_u + \mu(t)^T h_u = 0, t \in I,$

with $f_x \equiv f_x(t, x(t), u(t))$, $g_x \equiv g_x(t, x(t), u(t))$, $h_x \equiv h_x(t, x(t), u(t))$,etc.

Consider $\theta_1(., x(.), u(.), \lambda(.), \mu(.), \dot{\mu}(.))$ as defining a mapping $Q^1 : X \times U \times V \times \Lambda \to B_1$, where V is the space of piecewise smooth functions λ, Λ is the space of differentiable functions μ and B_1, is a Banach Space; and also consider $\theta_2(., x(.), u(.), \lambda(.), \mu(.))$ as defining a mapping $Q^2 : X \times U \times V \times \Lambda \to B_2$,where B_2 is another Banach Space. In order to apply Proposition 2.1 or results of Valentine [79], some restrictions are needed on the equality constraints

$$\theta_1(.) = 0 \text{ and } \theta_2(.) = 0$$

It suffices if Freche/t derivatives

$$Q^{1'} = [Q_x^1, Q_u^1, Q_\lambda^1, Q_\mu^1] \text{ and } Q^{2'} = [Q_x^2, Q_u^2, Q_\lambda^2, Q_\mu^2]$$

have weak *closed range. Denote $\bar{f} \equiv f(t, \bar{x}(t), \bar{u}(t))$, $\bar{f}_x \equiv f_x(t, \bar{x}(t), \bar{u}(t))$,etc.

Theorem 2.5 (Converse Duality): Let $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})$ be an optimal solution of (CD). Assume that

(i) the Freche/t derivative $Q^1$and Q^2 have weak closed range,

(ii) Corresponding to (2.5), there exists a piecewise smooth Lagrange multiplier β:I$\to R^n$ with its derivative $\dot{\beta}(t) \geq 0, t \in I$ and β(a)=0= β(b).

(iii) $\int_a^b \sigma(t)^T M(t) \sigma(t) dt = 0 \Rightarrow \sigma(t) = 0,$ where σ(t) $\in R^{n+m}$ and

$$M(t) = \begin{pmatrix} \bar{f}_{xx} + \bar{\lambda}(t)^T \bar{g}_{xx} + \bar{\mu}(t)^T \bar{h}_{xx}, \bar{f}_{ux} + \bar{\lambda}(t)^T g_{ux} + \bar{\mu}(t)^T \bar{h}_{ux} \\ \bar{f}_{xu} + \bar{\lambda}(t)^T g_{xu} + \bar{\mu}(t)^T h_{xu}, f_{xu} + \bar{\lambda}(t)^T g_{uu} + \bar{\mu}(t)^T h_{uu} \end{pmatrix}$$,is a positive definite and

(iv) $\bar{f}_x + \bar{\lambda}(t)^T g_x + \bar{\mu}(t)^T \bar{h}_x + \dot{\bar{\mu}}(t) \neq 0, \bar{f}_u + \bar{\lambda}(t)^T g_u + \bar{\mu}(t)^T h_u \neq 0, t \in I.$

If the hypotheses of Theorem 2.2 are satisfied, $(\bar{x}, \bar{u})$ is an optimal solution of (CP) and the objective values of (CP) and (CD) are equal.

Proof: Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ is an optimal solution of (CD), an application of Proposition 2.1 shows that there exist Lagrange multipliers $\alpha \in R$, piecewise smooth functions β: I $\to R^n$ with β(a) = 0 = β (b) and its derivative $\dot{\beta}(t) \geq 0$, t$\in$I,θ: I$\to R^m$, and ζ: I$\to R^m$ and $\gamma \in R$ such that

$$\alpha f_x + \beta(t)^T (f_{xx} + \bar{\lambda}(t)^T g_{xx} + \bar{\mu}(t)^T h_{xx}) + \theta(t)^T (\bar{f}_{ux} + \bar{\lambda}(t)^T g_{ux} + \bar{\mu}(t)^T h_{ux})$$
$$+ \gamma(\bar{\lambda}(t)^T g_x + \bar{\mu}(t)^T h_x) = 0, t \in I \quad (2.17)$$

$$\alpha f_u + \beta(t)^T (f_{xu} + \bar{\lambda}(t)^T g_{xu} + \bar{\mu}(t)^T h_{xu}) + \theta(t)^T (\bar{f}_{uu} + \bar{\lambda}(t)^T g_{uu} + \bar{\mu}(t)^T h_{uu})$$

$$+\gamma\left(\bar{\lambda}(t)^T g_u + \bar{\mu}(t)^T h_u + \dot{\bar{\mu}}(t)\right) = 0, t \in I \qquad (2.18)$$

$$\beta(t)^T g_x + \theta(t)^T g_u + \gamma g + \xi(t) = 0, t \in I\ , \qquad (2.19)$$

$$\beta(t)^T h_x - \dot{\beta}(t) + \theta(t)^T h_u + \gamma(h - \dot{x}) = 0, t \in I\ , \qquad (2.20)$$

$$\gamma \int_a^b \left(\bar{\lambda}(t)^T g + \bar{\mu}(t)^T (\bar{h} - \dot{x})\right) dt = 0 \qquad (2.21)$$

$$\bar{\lambda}(t)^T \zeta(t) = 0, t \in I, \qquad (2.22)$$

$$(\alpha, \zeta(t), \gamma) \geq 0,\ t \in I, \qquad (2.23)$$

$$(\alpha,\ \beta(t),\ \theta(t),\ \zeta\ (t),\ \gamma) \neq 0,\ t \in I. \qquad (2.24)$$

Using (2.11) and (2.12) in (2.17) and (2.18) respectively, we have

$$(\gamma - \alpha)\left(\bar{\lambda}(t)^T g_x + \bar{\mu}(t)^T h_x + \dot{\mu}(t)\right) + \beta(t)^T \left(\bar{f}_{xx} + \bar{\lambda}(t) g_{xx} + \bar{\mu}(t)^T h_{xx}\right)$$

$$+\theta(t)^T \left(f_{xx} + \bar{\lambda}(t)^T g_{xu} + \bar{\mu}(t)^T \bar{h}_{xu}\right) = 0,\ \ t \in I \qquad (2.25)$$

$$(\gamma - \alpha)\left(\bar{\lambda}(t)^T g_u + \bar{\mu}(t)^T \bar{h}_u\right) + \beta^T(t)^T \left(\bar{f}_{xx} + \bar{\lambda}(t)\bar{g}_{xx} + \bar{\mu}(t)^T h_{xx}\right)$$

$$+\theta(t)^T \left(\bar{f}_{uu} + \bar{\lambda}(t)^T g_{uu} + \bar{\mu}(t)^T h_{uu}\right) = 0, t \in I \qquad (2.26)$$

Multiplying (2.19) and (2.20) by $\bar{\lambda}(t)^T$ respectively and then adding the resulting equations, we have

$$\int_a^b \left\{\beta(t)^T \left(\lambda(t)^T g_x + \bar{\mu}(t)^T h_x\right) + \theta(t)^T \left(\lambda(t)^T g_u + \mu(t)^T h_u\right)\right\} dt$$

$$+\gamma \int_a^b \left(\lambda(t)^T g + \mu(t)^T h\right) dt + \int_a^b \lambda(t)^T \zeta(t)\, dt$$

$$= \int_a^b \beta(t)\mu(t) dt = \mu(t)^T \beta(t)\Big|_{t=a}^{t=b} - \int_a^b \beta(t)^T \dot{\mu}(t) dt$$

(By integration by parts)

Using β(a) = 0 = β(b), (2.21) and (2.22), this implies,

$$\int_a^b \left\{\beta(t)^T \left(\bar{\lambda}(t)^T g_x + \bar{\mu}(t)^T h_x + \dot{\bar{\mu}}(t)\right) + \theta(t)^T \left(\bar{\lambda}(t)^T g_u + \bar{\mu}(t)^T h_u\right)\right\} dt = 0$$

Equivalently, this can be written as,

$$\int_a^b \left(\beta(t), \theta(t)\ \right)^T \begin{pmatrix} \bar{\lambda}(t)^T \bar{g}_x + \bar{\mu}(t)^T h_x + \dot{\mu}(t) \\ \bar{\lambda}(t)^T g_u + \bar{\mu}(t)^T h_u \end{pmatrix} dt = 0. \qquad (2.27)$$

The equation (2.25) and (2.26) can be combined to be written in the following matrix form,

$$(\gamma-\alpha)\begin{pmatrix}\bar{\lambda}(t)^T+\bar{\mu}(t)^T h_x+\dot{\mu}(t)\\ \bar{\lambda}(t)^T g_u+\bar{\mu}(t)^T h_u\end{pmatrix}$$

$$+\begin{pmatrix}f_{xx}+\bar{\lambda}(t)^T+g_{xx}\bar{\mu}(t)^T h_{xx} & \mathrm{f}_{\mathrm{xx}}+\bar{\lambda}(t)g_{xu}+\bar{\mu}(t)^T h_{xu}\\ f_{xu}+\bar{\lambda}(t)^T g_{xu}+\bar{\mu}(t)^T h_{xu}, & \mathrm{f}_{\mathrm{uu}}+\bar{\lambda}(t)^T g_{uu}+\bar{\mu}(t)^T h_{uu}\end{pmatrix}\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}=0, t\in I \qquad (2.28)$$

Multiplying this by $(\beta(t),\theta(t))^T$, and then integrating we obtain

$$(\gamma-\alpha)\int_a^b(\beta(t),\theta(t))^T\begin{pmatrix}\bar{\lambda}(t)^T+\bar{g}_x+\bar{\mu}(t)^T h_x+\dot{\mu}(t)\\ \bar{\lambda}(t)^T \bar{g}_u+\bar{\mu}(t)^T h_u\end{pmatrix}dt$$

$$+\int_a^b(\beta(t),\theta(t))^T\begin{pmatrix}f_{xx}+\bar{\lambda}(t)^T+g_{xx}\bar{\mu}(t)^T h_{xx}, & \mathrm{f}_{\mathrm{xx}}+\bar{\lambda}(t)g_{xu}+\bar{\mu}(t)^T h_{xu}\\ f_{xu}+\bar{\lambda}(t)^T gx_u+\bar{\mu}(t)^T h_{xu}, & \mathrm{f}_{\mathrm{uu}}+\bar{\lambda}(t)^T g_{uu}+\bar{\mu}(t)^T h_{uu}\end{pmatrix}\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}\mathrm{dt} \qquad (2.29)$$

Using (2.27) in (2.29), we have

$$\int_a^b(\beta(t),\theta(t))^T\begin{pmatrix}\bar{f}_{xx}+\bar{\lambda}(t)^T g_{xx}+\bar{\mu}(t)^T h_{xx}, & \mathrm{f}_{\mathrm{xu}}+\bar{\lambda}(t)^T g_{xu}+\bar{\mu}(t)^T h_{xu}\\ f_{xu}+\bar{\lambda}(t)^T g_{xu}+\bar{\mu}(t)^T h_{xu}, & \mathrm{f}_{\mathrm{uu}}+\bar{\lambda}(t)^T g_{uu}+\bar{\mu}(t)^T h_{uu}\end{pmatrix}\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}\mathrm{dt}=0$$

In view of the hypothesis (iii), this implies

$$\sigma(t)=(\beta(t),\theta(t))=0\Rightarrow\ \beta(\mathrm{t})=\theta(\mathrm{t})=0,\ \mathrm{t}\in\mathrm{I} \qquad (2.30)$$

The relation (2.28) together with (2.27) yields

$$(\gamma-\alpha)\begin{pmatrix}\bar{\lambda}(t)^T g_x+\bar{\mu}(t)^T h_x+\dot{\bar{\mu}}(t)\\ \bar{\lambda}(t)g_u+\bar{\mu}(t)^T h_u\end{pmatrix}=0,\ \mathrm{t}\in\mathrm{I}$$

Because of the hypothesis (iv), this gives

$$\alpha=\gamma \qquad (2.31)$$

If $\alpha = 0$, then $\gamma = 0$. Consequently using (2.30), (2.19) implies that $\zeta(t) = 0$, $t \in I$

Thus $(\alpha, \beta(t), \theta(t), \zeta(t), \gamma) = 0$, $t\in I$. This contradicts the Fritz John condition (2.24). Hence $\gamma = \alpha > 0$.

Using (2.30) and $\gamma > 0$ in (2.19), we have $g(t,\bar{x},\bar{u})=-\dfrac{\zeta(t)}{\gamma}\leq 0$, $t \in I$. Also from (2.20), we have $h(t,\bar{x},\bar{u})-\dot{\bar{x}}=\dfrac{\dot{\beta}(t)}{\gamma}\geq 0.$

Thus, it shows that $(\bar{x},\bar{u})$ is feasible for (CP) and the objective values of (CP) and (CD) are equal. In view of the hypotheses of Theorem (2.1), the optimality of $(\bar{x},\bar{u})$ for (CP) follows.

2.6 Control Problem with Free Boundary Conditions

The results validated in the preceding sections may be applied to the control problems with free boundary conditions. If the 'targets' x(a) and x(b) are not restricted, we have

Problem (CPF) (Primal): Minimize $\int_a^b f(t,x,u)dt$

subject to

$$h(t,x,u)=\dot{x},\ t\in I,$$

$$g(t,x,u)\leq 0,\ t\in I.$$

The dual control problem now includes the transversality conditions $\mu(t) = 0$, t = a and t = b as the new constraints. This yields

Problem CDF (Dual): Maximize $\int_a^b f(t,x,u)dt$

subject to

$$\mu(a) = 0,\ \mu(b) = 0$$

$$f_x(t,\bar{x},\bar{u})+\lambda(t)^T g_x(t,\bar{x},\bar{u})+\mu(t)^T h_x(t,\bar{x},\bar{u})+\dot{\mu}(t)=0, t\in I,$$

$$f_u(t,\bar{x},\bar{u})+\lambda(t)^T g_u(t,\bar{x},\bar{u})+\mu(t)^T h_u(t,\bar{x},\bar{u})= 0, t\in I,$$

$$\int_a^b \left(\lambda(t)^T g(t,x,u)+\mu(t)^T (h(t,x,u)-\dot{x}(t)\right)dt \geq 0$$

$$\lambda(t)^T \geq 0, t\in I,$$

In order the prove to results, corresponding to Theorem 2.1 to Theorem 2.4, we will have the term $\eta^T \mu(t)|_{t=a}^{t=b}$ vanished by using $\mu(a) = 0$ and $\mu(b) = 0$ instead of having $x(a) = \alpha$ and $x(b) = \beta$ so that $\eta = 0$ at t = a and t = b.

2.7 Mathematical Programming Problems

If f, g and h are independent of t (without any loss of generality b–a = 1) then the problems (CP) and (CD) reduce to the static primal and dual of mathematical programming problems treated by Mond and Weir [65] under generalized convexity and also under invexity by Craven and Glover [6].

Put $z=\begin{pmatrix} x \\ u \end{pmatrix}$, we have

Problem (PS): Minimize f (z)

subject to h (z) = 0, g (z) ≤ 0.

Problem (DS): Maximize f (z)

subject to

$$f_z(z) + \lambda^T g_z(z) + \mu^T h_z(z) = 0,$$

$$\lambda^T g(z) + \mu^T h(z) \geq 0,$$

$$\lambda \geq 0.$$

Chapter 3

MIXED TYPE DUALITY FOR CONTROL PROBLEMS WITH GENERALIZED INVEXITY

3.1. Introduction

Recently Husain et al [41] for relaxing invexity requirements in [10] for duality to hold constructed the following dual in the spirit of Mond and Weir [65]:

Problem (CD) (Dual): Maximize $\int_a^b f(t,x,u)dt$

subject to

$$x(a) = \alpha,\ x(b) = \beta, \tag{3.1}$$

$$f_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \dot{\mu}(t) = 0,\quad t \in I, \tag{3.2}$$

$$f_u(t,x,u) + \lambda(t)^T g_u(t,x,u) + \mu(t)^T h_u(t,x,u) = 0,\quad t \in I, \tag{3.3}$$

$$\int_a^b \left(\lambda(t)^T g(t,x,u) + \mu(t)^T \left(h(t,x,u) - \dot{x}\right)\right)dt \geq 0, \tag{3.4}$$

$$\lambda(t) \geq 0,\qquad t \in I. \tag{3.5}$$

3.2. Mixed Type Duality

We propose the following mixed type dual (Mix CD) to the control problem (CP) and establish usual duality results:

(Mix CD): Maximize $\int_a^b \left[f(t,x,u) + \sum_{i \in I_0} \mu^i(t)\left(h^i(t,x,u) - \dot{x}^i\right) + \sum_{j \in J_0} \lambda^j(t) g^j(t,x,u) \right] dt$

subject to

$$x(a) = \alpha,\quad x(b) = \beta \tag{3.6}$$

$$f_x(t,x,u) + \mu(t)^T h_x(t.x,u) + \lambda(t)^T g_x(t,x,u) + \dot{\mu}(t) = 0,\quad t \in I \tag{3.7}$$

$$f_u(t,x,u) + \mu(t)^T h_u(t,x,u) + \lambda(t)^T g_u(t,x,u) = 0,\quad t \in I \tag{3.8}$$

$$\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t)\left(h^i(t,x,u) - \dot{x}^i\right) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,x,u) \right) dt \geq 0,\ \alpha = 1,2,\ldots r \tag{3.9}$$

$$\lambda(t) \geq 0,\qquad t \in I \tag{3.10}$$

where for N = {1,2,…,n} and K = (1,2,…,k),

(i) $I_\alpha \subseteq M$, $\alpha = 0,1,2,\ldots r$

and $I_\alpha \cap I_\beta = \phi$, $\alpha \neq \beta$ and $\bigcup_{\alpha=0}^{r} I_\alpha = N$.

(ii) $J_\alpha \subseteq k$, $\alpha = 0, 1, 2,\ldots r$ with $J_\alpha \cap J_\beta = \phi$, $\alpha \neq \beta$ and $\bigcup_{\alpha=0} J_\alpha = K$, and

(iii) $r = \max(r_1, r_2)$, where r_1 is the number of disjoint subsets of M and r_2 is the number of disjoint subsets of K. Then I_α or J_α is empty for

$\alpha > \min(r_1, r_2)$.

Theorem 3.1 (Weak duality): Let $(\bar{x}, \bar{u})$ be feasible for (CP) and (x, u, λ, μ) be feasible for (Mix CD). If for all feasible $(\bar{x}, \bar{u}, x, u, \lambda, \mu)$, $\int_a^b \left(f + \sum_{i \in I_0} \mu^i \left(h^i - \dot{x}^i \right) + \sum_{j \in J_0} \lambda^j g^j \right) dt$

is pseudoinvex and $\int_a^b \left(\sum_{i \in I_\alpha} \mu^i \left(h^i - \dot{x}^i \right) + \sum_{j \in J_\alpha} \lambda^j g^j \right) dt$

is quasi-invex with respect to the same η and ξ, then

inf (CP) $\geq$ Sup (Mix CD).

Proof: Since $(\bar{x}, \bar{u})$ be feasible for (CP) and (x, u, μ, λ) be feasible for

(Mix CD), we have

$$\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t) \left(h^i(t,x,u) - \dot{x}^i \right) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,x,u) \right) dt$$

$$\leq \int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t) \left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i \right) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right) dt, \quad \alpha = 1,2,\ldots, r$$

By quasi-invexity of $\int_a^b \left(\sum_{i \in I_\alpha} \mu^i \left(h^i - \dot{x}^i \right) + \sum_{j \in J_\alpha} \lambda^j g^j \right) dt$, $\alpha = 1, 2,\ldots, r$ this inequality yields,

$$0 \geq \int_a^b \Bigg[\eta^T \left(\sum_{i \in I_\alpha} \mu^i(t) h_x^i(t,x,u) + \sum_{j \in J_\alpha} \lambda^j(t) g_x^j(t,x,u) \right) - \left(\frac{d\eta}{dt} \right) \sum_{i \in I_\alpha} \dot{\mu}^i$$

$$+ \xi^T \left(\sum_{i \in I_\alpha} \mu^i(t) h_u^i(t,x,u) + \sum_{j \in J_\alpha} \lambda^j(t) g_u^j(t,x,u) \right) \Bigg] dt$$

$$= \int_a^b \Bigg[\eta^T \left(\sum_{i \in I_\alpha} \left(\mu^i(t) h_x^i(t,x,u) + \dot{\mu}^i(t) \right) + \sum_{j \in J_\alpha} \lambda^j(t) g_x^j(t,x,u) \right)$$

$$+\xi^T\left(\sum_{i\in I_\alpha}\mu^i(t)h_u^i(t,x,u)+\sum_{j\in J_\alpha}\lambda^j(t)g^j(t,x,u)\right)\Bigg]\text{ dt-}\eta\sum_{i\in I_\alpha}\dot{\mu}^i\Bigg|_{t=a}^{t=b}$$

(By integration by parts)

$$=\int_a^b\Bigg[\eta^T\left(\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i(t,x,u)+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j(t,x,u)\right)$$

$$+\xi^T\left(\sum_{i\in I_\alpha}\mu^i(t)h^i(t,x,u)+\sum_{j\in J_\alpha}\lambda^j(t)g^j(t,x,u)\right)\Bigg]\text{ dt}$$

(using η = 0, at t = a and t = b)

$$=\int_a^b\Bigg[\eta^T\left(\sum_{i\in N\setminus I_0}\left(\mu^i(t)h_x^i(t,x,u)+\dot{\mu}^i(t)\right)+\sum_{j\in K\setminus J_0}\lambda^j(t)g^j(t,x,u)\right)$$

$$+\xi^T\left(\sum_{i\in N\setminus I_0}\mu^i(t)h^i(t,x,u)+\sum_{j\in K\setminus J_0}\lambda^j(t)g^j(t,x,u)\right)\Bigg]\text{ dt}$$

Using (2.5) and (2.6), this implies

$$\int_a^b\Bigg[\eta^T\left(\sum_{i\in I_0}\left(\mu^i(t)h_x^i(t,x,u)+\dot{\mu}^i(t)\right)+\sum_{j\in J_0}\lambda^j(t)g^j(t,x,u)\right)$$

$$+\xi^T\left(\sum_{i\in I_0}\mu^i(t)h^i(t,x,u)+\sum_{j\in J_0}\lambda^j(t)g^j(t,x,u)\right)\Bigg]\text{ dt}\geq 0$$

This, because of pseudo-invexity of $\int_a^b\left(f+\sum_{i\in I_0}\mu^i(t)\left(h^i-\dot{x}^i\right)+\sum_{j\in J_0}\lambda^j(t)g^j\right)\text{dt}$ yields,

$$\int_a^b\left\{f(t,\bar{x},\bar{u})+\sum_{i\in I_0}\mu^i(t)\left(h^i(t,\bar{x},\bar{u})-\dot{\bar{x}}^i\right)+\sum_{j\in J_0}\lambda^j(t)g^j(t,\bar{x},\bar{u})\right\}\text{ dt}$$

$$\geq\int_a^b\left\{f(t,x,u)+\sum_{i\in I_0}\mu^i(t)\left(h^i(t,x,u)-\dot{x}^i\right)+\sum_{j\in J_0}\lambda^j(t)g^j(t,x,u)\right\}\text{ dt}\qquad(3.11)$$

Since $\mu(t)^T\left(h(t,\bar{x},\bar{u})-\dot{\bar{x}}\right)=0,$ and $\lambda(t)^T g(t,\bar{x},\bar{u})\leq 0,$ these respectively imply

$$\sum_{i\in I_0}\mu^i(t)\left(h^i(t,\bar{x},\bar{u})-\dot{\bar{x}}^i\right)=0\text{ and }\sum_{j\in J_0}\lambda^j(t)g^j(t,\bar{x},\bar{u})\leq 0,\ t\in I$$

Consequently (3.11) gives

$$\int_a^b f(t,\bar{x},\bar{u})dt \geq \int_a^b \left\{ f(t,x,u) + \sum_{i\in I_0} \mu^i(t)\left(h^i(t,x,u) - \dot{x}^i\right) + \sum_{j\in J_0} \lambda^j(t) g^j(t,x,u) \right\} \text{dt}$$

That is,

infimum (CP) ≥ Supremum (Mix CD).

Theorem 3.2 (Strong Duality): If $(\bar{x},\bar{u})$ is an optimal solution of (CP) and is normal, then there exist piecewise smooth $\bar{\mu}: I \to R^n$ and $\bar{\lambda}: \mathrm{I} \to \mathrm{R}^p$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ be feasible and the corresponding values of (CP) and (Mix CD) are equal.

If, also $\int_a^b \left\{ f + \sum_{i\in I_0} \mu^i (h^i - \dot{x}^i) + \sum_{j\in J_0} \lambda^j g^j \right\} \text{dt}$

is pseudoinvex and $\int_a^b \left\{ f + \sum_{i\in I_\alpha} \mu^i (h^i - \dot{x}^i) + \sum_{j\in J_\alpha} \lambda^j g^j \right\}$ dt is quasi-invex with respect to the same η and ξ, then $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ is an optimal solution of (Mix CD).

Proof: Since $(\bar{x},\bar{u})$ is an optimal solution to (CP) and is normal then from Proposition 2.1, there exist piecewise smooth $\bar{\mu}: \mathrm{I} \to \mathrm{R}^n$ and $\bar{\lambda}: \mathrm{I} \to \mathrm{R}^p$ such that

$$f_x(t,\bar{x},\bar{u}) + \bar{\mu}(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t) + \bar{\lambda}(t)^T g(t,\bar{x},\bar{u}) = 0, \quad t \in \mathrm{I} \tag{3.12}$$

$$f_u(t,\bar{x},\bar{u}) + \bar{\mu}(t)^T h_u(t,\bar{x},\bar{u}) + \bar{\lambda}(t)^T g(t,\bar{x},\bar{u}) = 0, \quad t \in \mathrm{I} \tag{3.13}$$

$$\bar{\lambda}(t)^T g(t,\bar{x},\bar{u}) = 0, \quad t \in \mathrm{I} \tag{3.14}$$

$$\bar{\lambda}(t) \geq 0, \quad t \in \mathrm{I} \tag{3.15}$$

The relation (3.13) implies $\sum_{j\in J_0} \lambda^j(t) g^j(t,\bar{x},\bar{u}) = 0$ and $\sum_{j\in J_\alpha} \bar{\lambda}^j(t) g^j(t,\bar{x},\bar{u}) = 0$,

α =1,2,…, r.

Also $\bar{\mu}(t)^T\left(h(t,\bar{x},\bar{u}) - \dot{\bar{x}}\right) = 0,$ implies $\sum_{i\in I_0} \mu^i(t)\left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i\right) = 0,\ t \in \mathrm{I}$ and $\sum_{i\in I_\alpha} \mu^i(t)\left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i\right) = 0,\ t \in \mathrm{I}$.

Consequently,

$$\sum_{i\in I_\alpha} \bar{\mu}^i(t)\left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i\right) = 0,\ t \in \mathrm{I} \text{ and } \sum_{j\in J_\alpha} \bar{\lambda}^j(t) g^j(t,\bar{x},\bar{u}) = 0,\ t \in \mathrm{I}$$

imply

$$\int_a^b \left[\sum_{i \in I_\alpha} \bar{\mu}^i(t)\left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i\right) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right] dt = 0 \tag{3.15}$$

From the relations (3.11), (3.12), (3.14) and (3.15), it implies that $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ is feasible for (Mix CD) and the corresponding objective values of (CP) and (Mix CD) are equal in view of $\sum_{i \in I_0} \bar{\mu}^i(t)\left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i\right) = 0$ and $\sum_{j \in J_\alpha} \bar{\lambda}^j(t) g^j(t,\bar{x},\bar{u}) = 0, \quad t \in I.$

If $\int_a^b \left(f + \sum_{i \in I_0} \mu^i(h^i - \dot{x}^i) + \sum_{j \in J_0} \lambda^j g^j \right)$ dt is pseudoinvex and $\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(h^i - \dot{x}^i) + \sum_{j \in J_\alpha} \lambda^j g^j \right)$ dt α = 1, 2, … r is quasi-invex with respect to the same η and ξ, then from Theorem 3.1, $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ must be an optimal solution of (Mix CD).

Theorem 3.3 (Strict Converse duality): Let $(\bar{x},\bar{u})$ be an optimal solution of (CP) and normality condition be satisfied at $(\bar{x},\bar{u})$. Let $(\hat{x},\hat{u},\hat{\lambda},\hat{\mu})$ be an optimal solution of (Mix CD). If for all feasible $(\hat{x},\hat{u},\hat{\lambda},\hat{\mu})$, $\int_a^b \left(\sum_{i \in I_\alpha} \hat{\mu}^i(h^i - \hat{\dot{x}}^i) + \sum_{j \in J_\alpha} \hat{\lambda}^j(t) g^j \right) dt$ α=1,2…r is quasi-invex and $\int_a^b \left(f + \sum_{i \in I_0} \hat{\mu}^i(t)(h^i - \hat{\dot{x}}^i) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j \right)$ dt is strictly pseudoinvex with respect to the same η and ξ, then $(\hat{x},\hat{u})$ = $(\bar{x},\bar{u})$, i.e., $(\hat{x},\hat{u})$ is an optimal solution of (CP).

Proof: We assume that $(\hat{x},\hat{u}) \neq (\bar{x},\bar{u})$ and show that this assumption leads to a contradiction. Since $(\bar{x},\bar{u})$ is an optimal solution of (CP) and is normal, it follows by strong duality (Theorem 3.2) that there exist piecewise smooth $\mu : I \to R^n$ and $\lambda : I \to R^k$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ is an optimal solution of (Mix CD) and

$$\int_a^b f(t,\bar{x},\bar{u})dt = \int_a^b \left[f(t,\bar{x},\bar{u}) + \sum_{i \in I_0} \hat{\mu}^i(t)\left(h^i(t,\hat{x},\hat{u}) - \dot{\hat{x}}^i\right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u}) \right] dt \tag{3.17}$$

Also since $(\bar{x},\bar{u})$ and $(\hat{x},\hat{u},\hat{\lambda},\hat{\mu})$ are feasible for (CP) and (Mix CD), therefore,

for α = 1, 2, …, r

$$\int_a^b \left(\sum_{i \in I_\alpha} \hat{\mu}^i(t)\left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i\right) + \sum_{j \in J_\alpha} \hat{\lambda}^j(t) g^i(t,\bar{x},\bar{u}) \right) dt$$

$$\leq \int_a^b \left(\sum_{i \in I_\alpha} \hat{\mu}^i(t)\left(h^i(t,\hat{x},\hat{u}) - \dot{\hat{x}}^i\right) + \sum_{j \in J_\alpha} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u}) \right) dt \tag{3.18}$$

This, because of quasi-invexity of $\int_a^b \left(\sum_{i \in I_\alpha} \hat{\mu}^i(t)\left(h^i - \hat{\dot{x}}^i\right) + \sum_{j \in J_\alpha} \hat{\lambda}^j g^j \right) dt$, α = 1, 2, …, r is quasi-invex for all feasible $(\bar{x},\bar{u},\hat{x},\hat{u},\hat{\lambda},\hat{\mu})$ with respect to η and ξ, therefore, (3.18) implies that for α = 1, 2, …, r,

$$\int_a^b \left[\eta^T \left(\sum_{i\in I_\alpha} \hat{\mu}^i(t) h_x^{\,i}(t,\hat{x},\hat{u}) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g_x^{\,j}(t,\hat{x},\hat{u}) \right) - \sum_{i\in I_\alpha} \left(\frac{d\eta}{dt} \right)^T \hat{\mu}^i \right.$$

$$\left. + \xi^T \left(\sum_{i\in I_\alpha} \hat{\mu}^i(t) \left(h^i(t,\hat{x},\hat{u}) - \dot{\hat{x}}^i \right) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u}) \right) \right] dt \le 0$$

$$\int_a^b \left[\eta^T \left(\sum_{i\in I_\alpha} \left(\hat{\mu}^i(t) h^i(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}^i(t) \right) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) \left(g_x^j(t,\hat{x},\hat{u}) \right) \right) \right.$$

$$\left. + \xi^T \left(\sum_{i\in I_\alpha} \hat{\mu}^i(t) h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g_u^j(t,\hat{x},\hat{u}) \right) \right] dt - \eta \sum_{i\in I_\alpha} \dot{\hat{\mu}}^i(t) \Bigg|_{t=a}^{t=b}$$

(By integration by parts)

$$\int_a^b \left[\eta^T \left(\sum_{i\in I_\alpha} \left(\hat{\mu}^i(t) h_x^i(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}^i(t) \right) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u}) \right) \right.$$

$$\left. + \xi^T \left(\sum_{i\in I_\alpha} \hat{\mu}^i(t) h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u}) \right) \right] dt \le 0$$

(Using $\eta = 0$ at $t = a$, $t = b$)

Or

$$\int_a^b \left[\eta^T \left(\sum_{i\in N\backslash I_0} \left(\hat{\mu}^i(t) h_x^i(t,\hat{x},\hat{u}) + \hat{\mu}^i(t) \right) + \sum_{j\in K\backslash I_0} \hat{\lambda}^j(t) g_x^j(t,\hat{x},\hat{u}) \right) \right.$$

$$\left. + \xi^T \left(\sum_{i\in N\backslash I_0} \hat{\mu}^i(t) h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in K\backslash I_0} \hat{\lambda}^j(t) g_u^j(t,\hat{x},\hat{u}) \right) \right] dt \le 0$$

Since $(\hat{x},\hat{u},\hat{\lambda},\hat{\mu})$ is feasible for (Mix CD), therefore, by using (3.7) and (3.8) in the above inequality, we have

$$\int_a^b \left[\eta^T \left(f_x(t,\hat{x},\hat{u}) + \sum_{i\in I_0} \left(\hat{\mu}^i(t) h_x^i(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}^i(t) \right) + \sum_{j\in J_0} \hat{\lambda}^j(t) g_x^j(t,\hat{x},\hat{u}) \right) \right.$$

$$\left. + \xi^T \left(f_u(t,\hat{x},\hat{u}) + \sum_{i\in I_0} \hat{\mu}^i(t) h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in J_0} \hat{\lambda}^j(t) g_u^j(t,\hat{x},\hat{u}) \right) \right] dt \ge 0$$

This, because of strict pseudo-invexity of $\int_a^b \left(f + \sum_{i \in J_0} \hat{\mu}^i(t)\left(h^i - \bar{x}^i\right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j \right) dt$ with respect to η andξ, yields

$$\int_a^b \left(f(t,\bar{x},\bar{u}) + \sum_{i \in I_0} \hat{\mu}^i(t)\left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}\right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j(t,\bar{x},\bar{u}) \right) \mathrm{dt}$$

$$> \int_a^b \left[\left(f(t,\hat{x},\hat{u}) + \sum_{i \in I_0} \left(\hat{\mu}^i(t)(h^i(t,\hat{x},\hat{u}) - \dot{\hat{x}}) \right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u}) \right) dt \right.$$

Since $\sum_{i \in I_0} \hat{\mu}^i \left(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i \right) = 0$ and $\sum_{j \in J_0} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u})$, which are consequence of feasibility of $(\bar{x},\bar{u})$ for (CP) and $(\hat{x},\hat{u},\hat{\lambda},\hat{\mu})$ for (Mix CD), we have

$$\int_a^b f(t,\bar{x},\bar{u})dt > \int_a^b \left(f(t,\hat{x},\hat{u}) + \sum_{i \in I_0} \hat{\mu}^i(t)\left(h^i(t,\hat{x},\hat{u}) - \dot{\hat{x}}^i\right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j(t,\hat{x},\hat{u}) \right) \mathrm{dt}$$

This is a contradiction to (3.10). Hence $(\hat{x},\hat{u}) = (\bar{x},\bar{u})$, i.e., $(\hat{x},\hat{u})$ must be an optimal solution of (CP).

We now write

$$\psi_1 \equiv \psi_1(t,x,u,\lambda,\mu,\dot{\mu})$$

$$= f_x + \mu(t)^T (h_x(t,x,u) + \dot{\mu}) + \lambda^T(t) g(t,x,u)$$

$$\psi_2 \equiv \psi_2(t,x,u,\lambda,\mu)$$

$= f_x + \mu(t)^T h_x(t,x,u) + \dot{\mu} + \lambda^T(t) g(t,x,u)$ where $f_x = f_x(t,x,u)$, $f_u = f_u(t,x,u)$, $g_x = g_x(t,x,u)$ and $h_x = h_x(t,x,u)$.

Consider $\psi_1(t,x(t),u(t),\lambda(\cdot),\mu(\cdot),\dot{\mu}(\cdot))$ as defining a mapping

Q_1: X×U×Y×Z → B, where Y is the space of piecewise smooth functionsλ: I → R^k, Z is the space of differentiable function μ: I → R^n and B is a banach space; X and U are already defined. Also consider $\psi_2(t,x(\cdot),u(\cdot),\lambda(\cdot),\mu(\cdot))$ as defining a mapping Q_2: X×U×Y × Z → C where C is another banach space. In order to apply Proposition 2.1 to the problem (CD), some assumptions on $\psi_1(\cdot) = 0$ and $\psi_2(\cdot) = 0$ are in order. For this it suffices to assume that Frechet derivatives.

$$Q_1' = \left(Q_{1x}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}), Q_{1u}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}), Q_{1\lambda}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}), Q_{1\mu}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}) \right)$$

$$Q_2' = \left(Q_{2x}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}), Q_{2u}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}), Q_{2\lambda}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}), Q_{2\mu}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}) \right)$$

have weak *closed range. For notational convenience, we shall write in the sequel $\bar{f} = f(t,\bar{x},\bar{u}), \bar{g} = g(t,\bar{x},\bar{u}), \bar{h} = h(t,\bar{x},\bar{u}), \bar{f}_x = f_x(t,\bar{x},\bar{u})$, $\bar{g}_x = g_x(t,\bar{x},\bar{u}), \bar{h}_x = h_x(t,\bar{x},\bar{u})$, etc.

Theorem 3.4 (Converse duality): Let f, g, and h be twice continuously differentiable and $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})$ be an optimal solution of (Mix CD). Let the Frechet derivatives Q_1' and Q_2' have weak closed range. Assume that

(H$_1$): $$\int_a^b \sigma(t)^T (M(t)\sigma(t)dt = 0 \Rightarrow \sigma(t) = 0, t \in I$$

where $\sigma(t) \in R^{n+m}$ and

$$M(t) = \begin{bmatrix} \bar{f}_{xx} + \bar{\mu}(t)^T h_{xx} + \bar{\lambda}(t)^T g_{xx}, & \bar{f}_{ux} + \bar{\mu}(t)^T h_{ux} + \bar{\lambda}(t)^T g_{ux} \\ \bar{f}_{xu} + \bar{\mu}(t)^T h_{xu} + \bar{\lambda}(t)^T g_{xu}, & \bar{f}_{uu} + \bar{\mu}(t)^T h_{uu} + \bar{\lambda}(t)^T g_{uu} \end{bmatrix}$$

(H$_2$): $$\left\{ \sum_{i \in I_\alpha} \left(\mu^i(t) h_x^i(t, \bar{x}, \bar{u}) + \dot{\mu}^i(t) \right) + \sum_{j \in J_\alpha} \lambda^j(t) g_x^j(t, \bar{x}, \bar{u}), \alpha = 1,2,\ldots,r \right\}$$

and

$$\left\{ \sum_{i \in I_\alpha} \mu^i(t) h_u^i(t, \bar{x}, \bar{u}) + \dot{\mu}^i(t) + \sum_{j \in J_\alpha} \lambda^j(t) g(t, \bar{x}, \bar{u}), \alpha = 1,2,\ldots,r \right\}$$ are linearly independent, there exist corresponding to (3.8) a piecewise smooth Lagrange multiplier $\beta: I \to R^n$ with $\beta(t) \geq 0$, $t \in I$ with $\beta(a) = 0 = \beta(b)$.

If, for all feasible $(\bar{x}, \bar{u}, x, u, \lambda, \mu)$, $\int_a^b \left(f + \sum_{i \in I_0} \mu^i(t) \left(h^i - \dot{x}^i \right) + \sum_{j \in J_0} \lambda^j(t) g^j \right) dt$ is pseudoinvex and $\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t) \left(h^i - \dot{x}^i \right) + \sum_{j \in J_\alpha} \lambda^j(t) g^j \right) dt$ is quasi-invex with respect to the same η and ξ, then $(\bar{x}, \bar{u})$ is an optimal solution of (CP).

Proof: Since $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})$ is an optimal solution to (CP), therefore, by Proposition 2.1, there exist $\tau \in R$, $\gamma_\alpha \in R$, $\alpha = 1,2,, \ldots, r$, and piecewise smooth $\beta : I \to R^n$ and $\theta : I \to R^m$ such that

$$\tau \left(f_x + \sum_{i \in I_0} \left(\mu^i(t) h_x^i + \dot{\mu}^i(t) \right) + \sum_{j \in J_0} \lambda^j(t) g_x^j \right) + \beta(t)^T \left(f_{xx} + \lambda(t)^T g_{xx} + \mu(t)^T h_{xx} \right)$$

$$+ \theta(t)^T \left(f_{ux} + \lambda(t)^T g_{ux} + \mu(t)^T h_{ux} \right)$$

$$+ \sum_{\alpha=1}^r \gamma_\alpha \left\{ \sum_{i \in I_\alpha} \left(\mu^i(t) h_x^i + \dot{\mu}^i(t) \right) + \sum_{i \in J_\alpha} \lambda^j(t) g_x^i \right\} = 0, \quad t \in I \qquad (3.19)$$

$$\tau \left(f_u + \sum_{i \in I_0} \mu^i(t) h_u^i + \sum_{j \in J_0} \lambda^j(t) g_u^j \right) + \beta(t)^T \left(f_{xu} + \lambda(t)^T g_{xu} + \mu(t)^T h_{xu} \right)$$

$$+ \theta(t)^T \left(f_{uu} + \lambda(t)^T g_{uu} + \mu(t)^T h_{uu} \right)$$

$$+\sum_{\alpha=1}^{r}\gamma_\alpha\left\{\sum_{i\in I_\alpha}\mu^i(t)h_u^i+\sum_{i\in J_\alpha}\lambda^j(t)g_u^i\right\}=0,\quad \mathrm{t}\in \mathrm{I} \tag{3.20}$$

$$\tau(h^i-\dot{x}^i)+\beta(t)^T h_x^i-\dot{\beta}^i(t)+\theta^T(t)h_u^i=0,\ \ \mathrm{i}\in \mathrm{I}_0 \tag{3.21}$$

$$\beta(t)^T h_x^i-\beta^i(t)+\theta(t)^T h_x^i+\gamma_\alpha(h^i-\dot{x}^i)=0,\quad \mathrm{i}\in \mathrm{I}_\alpha,\ \ \alpha=1,2,\ldots,\mathrm{r} \tag{3.22}$$

$$\tau g^i+\beta(t)g_x^i+\theta(t)g_u^i+\eta^i(t)=0, i\in I_o \tag{3.23}$$

$$\beta(t)^T\ g_x^i+\theta(t)^T\ g_u^i+\gamma_\alpha g^i+\eta^i(t)=0, i\in J_\alpha, \alpha=1,2,..,r \tag{3.24}$$

$$\gamma_\alpha\int_a^b\left(\sum_{i\in I_\alpha}\mu^i(t)\left(h^i-\dot{x}^i\right)+\sum_{j\in J_\alpha}\lambda^j(t)g^j\right)dt=0,\quad \alpha=1,2,\ldots,\mathrm{r} \tag{3.25}$$

$$\eta(t)^T\lambda(t)=0,\quad \mathrm{t}\in I, \tag{3.26}$$

$$\left(\tau,\gamma_1,\ldots,\gamma_r\eta(t)\right)\geq 0,\ \mathrm{t}\in I \tag{3.27}$$

$$\left(\tau,\beta(t),\theta(t),\gamma_r\ldots\gamma_r,\eta(t)\right)\neq 0,\ \mathrm{t}\in I \tag{3.28}$$

Multiplying (3.21) by $\mu^{\mathrm{i}}(\mathrm{t}),\ \mathrm{i}\in \mathrm{I}_0$ and $\mathrm{t}\in \mathrm{I},$ and summing over i $\in$ I_0 and then integrating, we have

$$\tau\int_a^b\sum_{i\in I_0}\mu^i(t)\left(h^i-\dot{\bar{x}}^i\right)dt+\int_a^b\left\{\beta(t)^T\sum_{i\in I_0}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)+\theta(t)^T\left(\sum_{i\in I_0}\mu^i(t)h_u^i\right)\right\}dt$$
$$-\sum\mu^i(t)\beta^i(t)\Big|_{t=a}^{t=b}=0$$

Using β(a) = 0 = β(b), we have

$$\tau\int_a^b\sum_{i\in I_0}\mu^i(t)\left(h^i-\dot{x}^i\right)dt$$
$$+\int_a^b\left\{\beta(t)^T\sum_{i\in I_0}\left(\mu^i(t)h_{x+}^i\dot{\mu}^i(t)\right)+\theta(t)^T\left(\sum_{i\in I_0}\mu^i(t)h_u^i\right)\right\}\quad dt=0 \tag{3.29}$$

Multiplying (3.22) by $\mu^i(t),\ \mathrm{i}\in \mathrm{I}_0$ and $\mathrm{t}\in \mathrm{I},$ and summing over I $\in$ I_α and then integrating, we have

$$\int_a^b\left\{\beta(t)^T\left(\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)\right)+\theta(t)^T\left(\sum_{i\in I_\alpha}\mu^i(t)h_u^i\right)\right\}\mathrm{dt}$$
$$+\gamma_\alpha\int_a^b\left(\sum_{i\in I_\alpha}\mu^i(t)(h^i-\dot{x}^i)\right)\mathrm{dt}=0,\quad \alpha=1,2,\ldots,\mathrm{r} \tag{3.30}$$

Similarly from (3.23) and (3.24) together with (3.26), it implies respectively

$$\tau\int_a^b \sum_{j\in J_0}\lambda^j(t)g^j dt+\int_a^b\left\{\beta(t)^T\left(\sum_{j\in J_0}\lambda^j(t)g_x^j\right)+\theta(t)^T\left(\sum_{j\in J_0}\lambda^j(t)g_u^j\right)\right\}dt=0, \tag{3.31}$$

and

$$\int_a^b\left\{\beta(t)^T\left(\sum_{j\in J_\alpha}\lambda^j(t)g_x^j\right)+\theta(t)^T\left(\sum_{j\in J_\alpha}\lambda^j(t)g_u^j\right)\right\}dt+\gamma_\alpha\int_a^b\left(\sum_{j\in J_\alpha}\lambda^j(t)g^j\right)\mathrm{dt}=0,$$

$$\alpha=1,2,\ldots,\mathrm{r} \tag{3.32}$$

Adding (3.29) to (3.31) and (3.30) to (3.32), we have

$$\tau\int_a^b\left(\sum_{i\in I_0}\mu^i(t)\left(h^i+\dot{x}^i\right)+\sum_{j\in J_0}\lambda^j(t)g^j\right)dt+\int_a^b\left(\beta(t)^T\left(\sum_{i\in I_0}\mu^i(t)h_x^i+\dot{\mu}^i(t)+\sum_{j\in J_0}\lambda^j(t)g_x^j\right)\right.$$

$$\left.+\theta(t)^T\left(\sum_{i\in I_0}\mu^i(t)h_u^i+\sum_{j\in J_0}\lambda^j(t)g_u^j\right)\right)dt=0 \tag{3.33}$$

and

$$\gamma_\alpha\int_a^b\left(\sum_{i\in I_\alpha}\mu^i(t)\left(h^i-\dot{x}^i\right)+\sum_{j\in J_\alpha}\lambda^j(t)g^j\right)dt+\int_a^b\left(\beta(t)^T\left(\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j\right)\right.$$

$$\left.+\theta(t)^T\left(\sum_{i\in I_\alpha}\mu^i(t)h_u^i+\sum_{j\in J_\alpha}\lambda^j(t)g_u^j\right)\right)dt=0,\ \alpha=1,2,\ldots,\mathrm{r} \tag{3.34}$$

Using (3.25) in (3.34), we have

$$\int_a^b\left\{\beta(t)^T\left(\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^i\right)\right.$$

$$\left.+\theta(t)^T\left(\sum_{i\in I_\alpha}\mu^i(t)h_u^i+\sum_{j\in J_\alpha}\lambda^j(t)g_u^j\right)\right\}dt=0,\ \alpha=1,2,\ldots\mathrm{r}$$

This can be written as

$$\int_a^b\left(\beta(t),\theta(t)\right)^T\begin{pmatrix}\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^i\\ \sum_{i\in I_\alpha}\mu^i(t)h_u^i+\sum_{j\in J_\alpha}\lambda^j(t)g_u^i\end{pmatrix}\mathrm{dt}=0,\alpha=1,2,\ldots,\mathrm{r} \tag{3.35}$$

Using (3.7) and (3.8) in (3.19) and (3.20) respectively

$$\sum_{\alpha=1}^{r}(\gamma_\alpha-\tau)\left(\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j\right)+\beta(t)^T\left(f_{xx}+\lambda(t)^Tg_{xx}+\mu(t)^Th_{xx}\right)$$
$$+\theta(t)^T\left(f_{ux}+\lambda(t)^Tg_{ux}+\mu(t)^Th_{ux}\right)=0,\ \text{t}\in\text{I}$$

and

$$\sum_{\alpha=1}^{r}(\gamma_\alpha-\tau)\left(\sum_{i\in I_\alpha}\bar{\mu}^i(t)h_u^i+\sum_{j\in J_\alpha}\lambda^j(t)g_u^j\right)+\beta(t)^T\left(f_{xu}+\bar{\lambda}(t)^Tg_{xu}+\bar{\mu}(t)^Th_{xu}\right)$$
$$+\theta(t)^T\left(f_{uu}+\lambda(t)^Tg_{uu}+\mu(t)^Th_{uu}\right)=0,\ \text{t}\in\text{I}$$

Combining these relations, we have

$$\sum_{\alpha=1}^{r}(\gamma_\alpha-\tau)\begin{pmatrix}\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^i\\ \sum_{i\in I_\alpha}\mu^i(t)h_u^i+\sum_{j\in J_\alpha}\lambda^j(t)g_u^i\end{pmatrix}$$

$$+\begin{pmatrix}f_{xx}+\lambda(t)^Tg_{xx}+\mu(t)^Th_{xx},\ f_{ux}+\lambda(t)^Tg_{ux}+\mu(t)^Th_{ux}\\ f_{xu}+\lambda(t)^Tg_{xu}+\mu(t)^Th_{xu},\ f_{ux}+\lambda(t)^Tg_{ux}+\mu(t)^Th_{ux}\end{pmatrix}$$

$$\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}=0,\quad \text{t}\in\text{I} \tag{3.36}$$

Pre-multiplying (3.36) by $(\beta(t),\theta(t))^T$ and then using (3.35), we have

$$\int_a^b(\beta(t),\theta(t))^TM(t)\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}\text{dt}=0,$$

where $\text{M(t)}=\begin{pmatrix}f_{xx}+\lambda(t)^Tg_{xx}+\mu(t)^Th_{xx},\ f_{ux}+\lambda(t)^Tg_{ux}+\mu(t)^Th_{ux}\\ f_{xu}+\lambda(t)^Tg_{xu}+\mu(t)^Th_{xu},\ f_{ux}+\lambda(t)^Tg_{ux}+\mu(t)^Th_{ux}\end{pmatrix}$

This, in view of (H_1), yields

$$\sigma(t)=\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}=0,\qquad t\in I.$$

That is,

$$\beta(t)=0=\theta(t),\quad t\in I \tag{3.37}$$

Using (3.21) in (3.22), we have

$$\sum_{\alpha=1}^{r}(\gamma_\alpha-\tau)\begin{pmatrix}\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j\\ \sum_{i\in I_\alpha}\mu^i(t)h_u^i+\sum_{j\in J_\alpha}\lambda^j(t)g_u^j\end{pmatrix}=0.$$

This, because of the hypothesis (H_2), gives

$$\gamma_\alpha = \tau, \quad \alpha = 1, 2, \ldots, r \tag{3.38}$$

If $\tau = 0$, then $\gamma_\alpha = 0$, $\alpha = 1, 2, \ldots, r$ from (3.38), $\eta = 0$ from (3.23) and (3.24), consequently, $(\tau, \gamma, \ldots, \gamma_r, \beta(t), \theta(t), \eta(t)) = 0$, $t \in I$ but this contradicts (3.28). Hence $\tau = \gamma_\alpha > 0$, $\alpha = 1, 2, \ldots, r$.

Using (3.37) in (3.21) and (3.22) along with $\tau > 0$, γ_α (α, 1, 2, …, r) and $\dot{\beta}^i(t) > 0$, $t \in I$ we have

$$h^i - \dot{x}^i \geq 0, \quad i \in I_0 \text{ and } h^i - \dot{x}^i \geq 0, \quad i \in I_\alpha \;, \; \alpha = 1,2,...,\mathrm{r}$$

This implies $\quad h(t,\bar{x},\bar{u}) - \dot{\bar{x}}(t) \geq 0, \; t \in I \qquad (3.39)$

Using (3.37) in (3.23) and (3.24) together with $\tau > 0$, $\gamma_\alpha > 0$, $\alpha = 1, 2, \ldots r$, we have

$$g(t,\bar{x},\bar{u}) \leq 0, \quad t \in I \tag{3.40}$$

The relation (3.39) and (3.40) implies that, $(\bar{x},\bar{u})$ is feasible for (CP).

Using (3.37) with $\tau > 0$ in (3.33), we have

$$\int_a^b \left(\sum_{i \in I_o} \mu^i(t)\left(h^i - \bar{x}^i\right) + \sum_{j \in J_o} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right) \mathrm{dt} = 0$$

This accomplishes the equality of objective values of (CP) and (Mix CD), i.e.,

$$\int_a^b f(t,\bar{x},\bar{u})dt = \int_a^b \left(f(t,\bar{x},\bar{u}) + \sum_{i \in I_o} \mu^i(t)\left(h^i - \bar{x}^i\right) + \sum_{j \in J_o} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right) \mathrm{dt}$$

If, all feasible (x, u, λ, μ), $\int_a^b \left(\mathrm{f} + \sum_{i \in I_0} \mu^i (h^i - \dot{x}^i) + \sum_{i \in J_0} \lambda^j g^j \right) \mathrm{dt}$ is pseudoinvex and $\int_a^b \left(\sum_{i \in I_\alpha} \mu^i h^i + \sum_{j \in J_\alpha} \lambda^j g^j \right) \mathrm{dt}$ is quasi-invex with respect to the same η and ξ, then from Theorem 3.1, $(\bar{x},\bar{u})$ is an optimal solution of (CP).

3.4. Control Problem with Free Boundary Conditions

The duality results established in the preceding section can be applied to the control problem with free boundary conditions. If the “targets” x(a) and x(b) are not restricted, we have

Problem PF (Primal): Maximize $\int_a^b \mathrm{f}(\mathrm{t},\mathrm{x},\mathrm{u})\mathrm{dt}$

subject to

$$\mathrm{h}(\mathrm{t}, \mathrm{x}, \mathrm{u}) = \dot{\mathrm{x}}, \; \mathrm{t} \in I$$

$$\mathrm{g}(\mathrm{t}, \mathrm{x}, \mathrm{u}) \leq 0, \; \mathrm{t} \in I$$

This duality now includes the transversality $\mu(t) = 0$, at $t = a$ and $t = b$ as new constraints. This implies

Problem DF (Dual):

Maximize $\int_a^b \left[f(t,x,u) + \sum_{i \in I_0} \mu^i(t)\left(h^i(t,x,u) - \dot{x}^i\right) + \sum_{j \in J_0} \lambda^j(t) g^j(t,x,u) \right] dt$

Subject to

$\mu(a) = 0, \ \mu(b) = 0$

$$f_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \dot{\mu}(t) = 0, \ t \in I$$

$$f_u(t,x,u) + \mu(t)^T h_u(t,x,u) + \lambda(t)^T g_u(t,x,u) = 0, \ t \in I$$

$$\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t)\left(h^i(t,x,u) - \dot{x}\right) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,x,u) \right) dt \geq 0$$

$$\lambda(t) \geq 0, \ t \in I$$

3.5 Related Control Problems and Mathematical Programming

We now consider some special cases of (Mix CD). If $I_0 = N$ and $J_0 = K$, then (Mix CD) becomes the following Wolfe type dual, considered by Mond and Smart [10] under invexity of

$\int_a^b f dt, \int_a^b \mu^T \ (h - \dot{x}) dt$ and $\int_a^b \lambda^T g \, dt$

(WCD): Maximize $\int_a^b \left(f(t,x,u) + \mu(t)^T \left(h(t,x,u) - \dot{x}\right) + \lambda(t)^T g(t,x,u) \right) dt$

subject to

$x(a) = \alpha, \ x(b) = \beta$

$$f_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \dot{\mu}(t) = 0, \ t \in I$$

$$f_u(t,x,u) + \mu(t)^T h_u(t,x,u) + \lambda(t)^T g_u(t,x,u) = 0, \ t \in I$$

$$\lambda(t) \geq 0, \ t \in I$$

If $I_0 = \phi$ and $J_0 = \phi$, then (Mix CD) becomes following Mond – Weir type dual recently considered by Husain et al $[41]$ in order to relax invexity requirement on suitable forms of functionals involved in the formulation of the dual:

(M-WCD): Maximize $\int_a^b f(t,x,u) dt$

subject to

$$x(a) = \alpha, x(b) = \beta$$

$$f_x(t,x,u)+\lambda(t)^T g_x(t,x,u)+\mu(t)^T h_x+\dot{\mu}(t)=0,\ \text{t} \in \text{I}$$

$$f_u(t,x,u)+\lambda(t)^T g_u(t,x,u)+\mu(t)^T h_u=0,\ \text{t} \in \text{I}$$

$$\int_a^b \left(\sum_{i\in I_\alpha} \mu(t)^T (h-\dot{x}) + \sum_{j\in J_\alpha} \left(\lambda\ (t)^T g(t,x,u)\right) \right) dt \geq 0,$$

$$\lambda(t) \geq 0,\ t \in I$$

If f, g and h are independent of t (without any loss of generality, assume b –a = 1), then the control problems (CP) and Mix (CD) reduce to a pair of static primal and dual of mathematical programming, consider by Mond and Weir [65] the duality results of this

Putting $z = \begin{pmatrix} x \\ u \end{pmatrix}$, we have

Problem (PS): Minimize f(z)

Subject to

h(z) = 0

g(z) ≤ 0

Problem (Mix DS): Maximize $f(z) + \sum_{i\in I_0} \mu^i h^i(z) + \sum_{j\in J_0} \lambda^j g^j(z)$

Subject to

$$f(z) + \mu^T h_z(z) + \lambda^T g_z(z) = 0$$

$$\sum_{i\in I_\alpha} \mu^j h^j(z) + \sum_{j\in J_\alpha} \lambda^j g^j(z) \geq 0,\ \alpha = 1,2,...,\text{r}.$$

$\lambda \geq 0$, where $\lambda \in R^k$ and $\mu \in R^n$.

Chapter 4

ON MULTIOBJECTIVE DUALITY FOR VARIATIONAL PROBLEMS

4.1 INTRODUCTION

Calculus of Variations is a powerful technique for the solution of various important problems appearing in dynamics of rigid bodies, optimization of orbits, theory of vibrations and many areas of science and engineering. The subject of calculus of variation primarily concerns with finding optimal value of a definite integral involving a certain function subject to fixed point boundary conditions. Mond and Hanson [58] were the first to represent the problem of calculus of variation as a mathematical programming in infinite dimensional space. Since that time many researches contributed to this subject extensively. For somewhat comprehensive list of references, one may consult Husain and Jabeen [38] and Husain and Rumana [39]. The treatment in [38] has been for the real valued objective function while in [39] for vector valued function.

In this chapter, we consider a vector valued function for the primal problem and its minimality in the Pareto sense. Both equality and inequality constraints are considered in the formulations. In establishing duality results we consider two types of dual problems to the primal problem. The first one has vector valued objective where as the second set of results are based on the duality relations between an auxiliary problems and its associated dual as defined in Mond and Hanson [58]. Duality theorems, unlike in case of classical mathematical programming, are not based on optimality criteria but on certain types of convexity and generalized convexity requirements. Finally multiobjective variational problems with natural boundary values rather than fixed end points are mentioned and the analogues of our results in nonlinear programming are pointed out.

4.2. PRE-REQUISITES

In the treatment of the following problem (VP), by minimality we mean Pareto minimality. Now consider the following multiobjective variational problem involving higher order derivatives.

(VP) Minimize $\left(\int_I f^1\left(t,x,\dot{x},\ddot{x}\right)dt, \ldots, \int_I f^p\left(t,x,\dot{x},\ddot{x}\right)dt \right)$

Subject to

$$x(a)=0=x(b) \tag{4.1}$$

$$\dot{x}(a)=0=\dot{x}(b) \tag{4.2}$$

$$g(t,x,\dot{x},\ddot{x}) \leqq 0 \quad , \quad t\in I \tag{4.3}$$

$$h(t,x,\dot{x},\ddot{x})=0 \quad , \quad t\in I \tag{4.4}$$

Where **(i)** for $I=[a,b]\subseteq R$, $f:I\times R^n\times R^n\times R^n\to R$, $g:I\times R^n\times R^n\times R^n\to R^m$ and $h:I\times R^n\times R^n\times R^n\to R^k$ are continuously differentiable functions, and

(ii) *X* designates the space of piecewise smooth function $x:I\to R^n$ having its first and second order derivatives $\dot{x}$ and $\ddot{x}$ respectively equipped with the norm.

$$\|x\|=\|x\|_\infty+\|Dx\|_\infty+\|D^2x\|_\infty ,$$

where the differentiation operator D is given by

$$\omega=Dx \Leftrightarrow x(t)=\int_a^t \omega(s)\,ds$$

Thus $D\equiv\frac{d}{dt}$ except at discontinuities.

We denote the set of feasible solutions of the problem (VP) by K_P , i.e.,

$$K_P=\left\{x\in X \left| \begin{array}{llll} x(a)=0=x(b) & , & g(t,x,\dot{x},\ddot{x})\leqq 0 & , \; t\in I \\ \dot{x}(a)=0=\dot{x}(b) & , & h(t,x,\dot{x},\ddot{x})=0 & , \; t\in I \end{array}\right.\right\}$$

It is pointed out that the conventions for equalities and inequalities for vectors in R^n given in Mangasarian [51] will be used throughout the development of the theory.

Definition 4.1: A feasible solution of the problem (VP) i.e., $\bar{x}\in K_P$ is said to be Pareto minimum if there exists no $\hat{x}$ such that

$$\left(\int_I f^1(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})dt,\ldots,\int_I f^p(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})dt\right)$$
$$\leq\left(\int_I f^1(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt,\ldots,\int_I f^p(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})dt\right)$$

Pareto maximality can be defined in the same way except that the inequality in the above definition is reversed.

In the subsequent analysis the following result plays a significant role.

PROPOSITION 4.1: Suppose there exists a $\lambda > 0$, $\lambda \in R^p$ such that Let $\bar{x}(t) \in K_P$ is an optimal solution of the problem,

$$(P_\lambda): \qquad \underset{x(t)\in K_P}{Min} \int_I \lambda^T f(t, x, \dot{x}, \ddot{x})\, dt$$

Then $\bar{x}(t)$ is an optimal solution of (MP) in the Pareto sense.

Proof: Assume $\bar{x}(t)$ is not a Pareto optimal of (MP). Then there exists an $\hat{x}(t) \in K_P$ such that

$$\int_I f^i(t, \hat{x}, \dot{\hat{x}}, \ddot{\hat{x}})\, dt \leqq \int_I f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})\, dt \ , \ i = 1, 2, ..., p.$$

$$\int_I f^j(t, \hat{x}, \dot{\hat{x}}, \ddot{\hat{x}})\, dt < \int_I f^j(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})\, dt \ , \ i \neq j.$$

Hence

$$\int_I \lambda^T f(t, \hat{x}, \dot{\hat{x}}, \ddot{\hat{x}})\, dt < \int_I \lambda^T f(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})\, dt \ .$$

This contradicts the assumption that $\bar{x}$ minimizes $\int_I \lambda^T f(t, x, \dot{x}, \ddot{x})\, dt$ over K_P .

In the subsequent sections some duality results by introducing two types of duals to (VP) will be established.

4.3 MOND-WEIR TYPE MULTIOBJECTIVE DUALITY

Consider the following Mond-Weir dual to (VP)

(M-WD): Maximize $\left(\int_I f^1(t, u, \dot{u}, \ddot{u})\, dt, \ldots, \int_I f^p(t, u, \dot{u}, \ddot{u})\, dt \right)$

Subject to

$$u(a) = 0 = u(b) \tag{4.5}$$

$$\dot{u}(a) = 0 = \dot{u}(b), \tag{4.6}$$

$$\begin{aligned} &\lambda^T f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \\ &-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) \\ &+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) = 0 \ , \ t \in I \end{aligned} \tag{4.7}$$

$$\int_1 \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt \geqq 0 \ , \tag{4.8}$$

$$\lambda > 0 \ , \ \lambda \in R^p \ , \ y(t) \geqq 0 \ , \ t \in I \tag{4.9}$$

Let K_D be the set of the feasible solutions of (M-WD).

Theorem 4.1: Suppose

(A_1): $\bar{x}(t) \in K_P$

(A_2): $(\lambda, u, y(t), z(t)) \in K_D$

(A_3): $\int_I \lambda^T f(t,.,.,.)dt$ is pseudo-convex

(A_4): $\int_1 \left\{ y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \right\} dt$ is quasiconvex.

Then $\int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt \geqq \int_I \lambda^T f(t,u,\dot{u},\ddot{u})dt$.

Proof: Since $y(t) \geqq 0$, $t \in I$, $g(t,x,\dot{x},\ddot{x}) \leqq 0$, $t \in I$ and $h(t,x,\dot{x},\ddot{x}) = 0$, $t \in I$, we have

$$\int_1 \left(y(t)^T g(t,x,\dot{x},\ddot{x}) + z(t)^T h(t,x,\dot{x},\ddot{x}) \right) dt \leqq 0 \qquad (4.10)$$

Combining this inequality with (4.8)

We have,

$$\int_I y(t)^T g(t,x,\dot{x},\ddot{x}) + z(t)^T h(t,x,\dot{x},\ddot{x}) dt$$
$$\leqq \int_1 \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt$$

By the hypothesis (A_4), this yields

$$0 \geqq \int_I \Big[(x-u)^T \left(y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right)$$
$$+ (\dot{x}-\dot{u})^T \left(y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+ (\ddot{x}-\ddot{u})^T \left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt$$

$$= \int_I (x-u)^T \Big(y(t)^T g_u(t,u,\dot u,\ddot u) + z(t)^T h_u(t,u,\dot u,\ddot u) \Big) dt$$

$$+ (x-u)^T \Big(y(t)^T g_{\dot u}(t,u,\dot u,\ddot u) + z(t)^T h_{\dot u}(t,u,\dot u,\ddot u) \Big) \Big|_{t=a}^{t=b}$$

$$- \int_I (x-u)^T D\Big(y(t)^T g_{\dot u}(t,u,\dot u,\ddot u) + z(t)^T h_{\dot u}(t,u,\dot u,\ddot u) \Big) dt$$

$$+ (\dot x-\dot u)^T \Big(y(t)^T g_{\dot u}(t,u,\dot u,\ddot u) + z(t)^T h_{\dot u}(t,u,\dot u,\ddot u) \Big) \Big|_{t=a}^{t=b}$$

$$- \int_1 (\dot x-\dot u)^T D\Big(y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u) + z(t)^T h_{\ddot u}(t,u,\dot u,\ddot u) \Big) dt$$

(By integration by parts)

$$= \int_I (x-u)^T \Big[\Big(y(t)^T g_u(t,u,\dot u,\ddot u) + z(t)^T h_u(t,u,\dot u,\ddot u) \Big)$$

$$- D\Big(y(t)^T g_{\dot u}(t,u,\dot u,\ddot u) + z(t)^T h_{\dot u}(t,u,\dot u,\ddot u) \Big) \Big] dt$$

$$- (x-u)^T D\Big(y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u) + z(t)^T h_{\ddot u}(t,u,\dot u,\ddot u) \Big) \Big|_{t=a}^{t=b}$$

$$+ \int_1 (x-u)^T D^2 \Big(y(t)^T g_{\ddot u}(t,u,\dot u,\ddot u) + z(t)^T h_{\ddot u}(t,u,\dot u,\ddot u) \Big) dt$$

(By integration by parts)

`Using (4.7), we have,

$$0 \leq \int_I (x-u)^T \Big\{ \lambda^T f_u(t,u,\dot u,\ddot u) - D\lambda^T f_{\dot u}(t,u,\dot u,\ddot u)$$

$$+ D^2 \lambda^T f_{\ddot u}(t,u,\dot u,\ddot u) \Big\} dt$$

$$0 \leq \int_I \Big\{ (x-u)^T \lambda^T f_u(t,u,\dot u,\ddot u) + (\dot x-\dot u)^T \lambda^T f_{\dot u}(t,u,\dot u,\ddot u) \Big\} dt$$

$$- (x-u)^T \lambda^T f_u(t,u,\dot u,\ddot u) \Big|_{t=a}^{t=b}$$

$$- (\dot x-\dot u)^T D\lambda^T f_{\ddot u}(t,u,\dot u,\ddot u) \Big] dt + (\dot x-\dot u)^T D\lambda^T f_{\ddot u}(t,u,\dot u,\ddot u) \Big|_{t=a}^{t=b}$$

$$\int_I \Big\{ (x-u)^T \lambda^T f_u(t,u,\dot u,\ddot u) + (\dot x-\dot u)^T \lambda^T f_{\dot u}(t,u,\dot u,\ddot u)$$

$$+ (\ddot x-\ddot u)^T D\lambda^T f_{\ddot u}(t,u,\dot u,\ddot u) \Big\} dt \geqq 0$$

This by integration by parts using the boundary conditions, we have,

$$\int_I \Big\{ (x-u)^T \lambda^T f_u(t,u,\dot{u},\ddot{u}) + (\dot{x}-\dot{u})^T \lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u})$$
$$+ (\ddot{x}-\ddot{u})^T \lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Big\} dt \geqq 0$$

This, because of the hypothesis (A_3) implies,

$$\int_I \lambda^T f(t,x,\dot{x},\ddot{x})\,dt \geqq \int_I \lambda^T f(t,u,\dot{u},\ddot{u})\,dt \;.$$

Theorem 4.2: Assume

(B_1): $\bar{x}(t) \in K_P$

(B_2): $(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t)) \in K_D$

(B_3): $\int_1 \Big(y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \Big) dt$ is quasi-convex

(B_4): $\int_I \lambda^T f(t,.,.,.)\,dt$ is pseudoconvex

(B_5): $\int_I \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt = \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt$

Then $\bar{x}(t)$ is an optimal solution of (VP) and $(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t))$ is an optimal solution of the problem (M-WD).

Proof: Assume that $\bar{x}$ is not Pareto-optimal of (VP). Then there exists an $\bar{x}(t) \in K_P$ such that

$$\int_I f^i(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})\,dt < \int_I f^i(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt \;, \quad \text{for all } i$$

And $\int_I f^j(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})\,dt < \int_I f^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt$ for some j, $1 \le j \le p$

Since $\bar{\lambda} > 0$, this implies,

$$\int_I \bar{\lambda}^T f(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})\,dt < \int_I \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt$$

By the hypothesis (B_5), this inequality implies,

$$\int_I \bar{\lambda}^T f(t,\hat{x},\dot{\hat{x}},\ddot{\hat{x}})\,dt < \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt \;.$$

This contradicts the conclusion of Theorem 4.1 thus establishing the Pareto optimality of $\bar{x}(t)$ for (VP). Similarly we can show that $(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t))$ is Pareto optimal for (M-WD).

We state the following theorem without proof as it is similar to Theorem 3.4 of [69].

Theorem 4.3: Assume,

(C1): $\bar{x}(t) \in K_P$; $(\lambda, \bar{u}(t), \bar{y}(t), \bar{z}(t)) \in K_D$;

(C2): $\int_I \bar{\lambda}^T f(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) dt = \int_I \bar{\lambda}^T f(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}) dt$;

(C3): $\int_1 \left(y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \right) dt$ is convex;

(C4): $\int_I \lambda^T f(t,.,.,.) dt$ is quasiconvex.

Then $\bar{x}(t) = \bar{u}(t),\ t \in I.$

4.4. WOLFE TYPE MULTIOBJECTIVE DUALITY

To establish duality results similar to the preceding ones but under

different convexity and generalized convexity assumptions, we formulate the following Wolfe type dual to the problem (P_λ) stated in the Preposition 4.1.

We assume that $\bar{\lambda}$ is known and $\bar{\lambda} > 0$.

(WCD_λ): Maximize: $\int_I \left(\bar{\lambda}^T f(t, x, \dot{x}, \ddot{x}) + y(t)^T g(t, x, \dot{x}, \ddot{x}) + z(t)^T h(t, x, \dot{x}, \ddot{x}) \right) dt$

Subject to:

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b) \tag{4.11}$$

$$\begin{aligned} &\left(\lambda^T f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) \right) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \\ &- D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \Big) \\ &+ D^2 \left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Big) = 0 \ , \ t \in I \end{aligned} \tag{4.12}$$

$$y(t) \geqq 0,\ t \in I \tag{4.13}$$

In the following L_P represents the set of feasible solutions of (P_λ) and L_D the set of feasible solutions of (WCD_λ).

Theorem 4.4: Assume

(H1): $\bar{x}(t) \in L_P$; $(\bar{u}(t), \bar{y}(t), \bar{z}(t)) \in L_D$

(H$_2$): $\int_I \bar{\lambda}^T f(t,.,.,.)\,dt$ and $\int_1 \left(y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.) \right) dt$ are convex.

Then,

$$\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x})\,dt \geqq \int_I \left(\bar{\lambda}^T f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt.$$

Proof: By the convexity of $\int_I \bar{\lambda}^T f(t,.,.,.)\,dt$, we have

$$\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x})\,dt \geqq \int_I \bar{\lambda}^T f(t,u,\dot{u},\ddot{u})\,dt + \int_I \Big[(x-u)^T \bar{\lambda}^T f_u(t,u,\dot{u},\ddot{u})$$
$$+ (\dot{x}-\dot{u})^T \bar{\lambda}^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + (\ddot{x}-\ddot{u})^T \bar{\lambda}^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Big] dt \qquad (4.14)$$

From the dual constraint (4.12), we have,

$$\int_I (x-u)^T \Big[\left(\lambda^T f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right)$$
$$-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt = 0$$

This, by integrating by parts and using the boundary conditions as earlier, implies

$$\int_I (x-u)^T \Big[\lambda^T \left(f_u(t,u,\dot{u},\ddot{u}) + y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right)$$
$$+(\dot{x}-\dot{u})^T \left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+(\ddot{x}-\ddot{u})^T \left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt = 0$$

Using this, in (4.14) we have

$$\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x})\,dt \geq \int_I \bar{\lambda}^T f(t,u,\dot{u},\ddot{u})\,dt$$
$$-\int_I \Big[(x-u)^T \left(y(t)^T g_u(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u}) \right)$$
$$+(\dot{x}-\dot{u})^T \left(y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) \right)$$
$$+(\ddot{x}-\ddot{u})^T \left(y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \right) \Big] dt$$

By the hypothesis (H$_2$), this implies

$$\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x})\,dt \geq \int_I \bar{\lambda}^T f(t,u,\dot{u},\ddot{u})\,dt + \int_I \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt$$
$$-\int_I \left(y(t)^T g(t,x,\dot{x},\ddot{x}) + z(t)^T h(t,x,\dot{x},\ddot{x}) \right) dt,$$

Since $x \in L_P$, this implies

$$\int_I \lambda^T f(t,x,\dot{x},\ddot{x})\,dt \geqq \int_I \left(\bar{\lambda}^T f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u})\right)dt.$$

This proves the theorem.

The following theorem gives a situation in which a Pareto optimal solution of (VP) exists.

Theorem 4.5: Suppose

(F_1): $\bar{x}(t) \in L_P$, $\left(\bar{u}(t), \bar{y}(t), \bar{z}(t)\right) \in L_D$;

(F_2): $\int_I \bar{\lambda}^T f(t,.,.,.)\,dt$; and $\int_I y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.)\,dt$ are convex,

(F_3): $\int_I \bar{\lambda}^T f(t,x,\dot{x},\ddot{x})\,dt = \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + y(t)^T g(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + z(t)^T h(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt$

Then $\bar{x}(t)$ and $\left(\bar{y}(t), \bar{z}(t), \bar{u}(t)\right)$ are optimal solutions of (P_λ) and (WCD_λ). Hence $\bar{x}(t)$ is a Pareto optimal solution of (VP).

The last part of the conclusion follows from Proposition 4.1.

Proof: Suppose $\bar{x}(t)$ does not minimize (P) then there exist, $x^*(t) \in L_P$ such that

$$\int_I \bar{\lambda}^T f\left(t, x^*(t), \dot{x}^*(t), \ddot{x}^*(t)\right)dt < \int_I \bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt$$

$$= \int_I \bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + y(t)^T g(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + z(t)^T h(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\,dt$$

This contradicts the conclusion of Theorem 4.1. Hence $\bar{x}(t)$ minimizes (P_λ).

We can similarly prove that $\left(\bar{y}(t), \bar{z}(t), \bar{u}(t)\right)$ maximizes (WCD_λ).

Theorem 4.6: Assume

(G_1): $\bar{x}(t) \in L_P$, $\left(\bar{y}(t), \bar{z}(t), \bar{u}(t)\right) \in L_D$;

(G_2): $\int_I \bar{\lambda}^T f(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt = \int_I \left(\bar{\lambda}^T f(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + y(t)^T g(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}) + z(t)^T h(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}})\right)dt$

(G_3): $\int_1 \left(\bar{\lambda}^T f(t,.,.,.) + y(t)^T g(t,.,.,.) + z(t)^T h(t,.,.,.)\right)dt$ is convex

Then

$$\int_I y(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})\,dt = 0 \ , \ t \in I$$

Proof: By hypotheses (G_2) and (G_3), we have

$$\int_I \bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt=\int_I\left(\bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+y(t)^T g\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+z(t)^T h\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)dt$$

$$\leq\int_I\left(\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt$$

$$-\int_I\Big[\left(\bar{x}-\bar{u}\right)^T\left(\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+\left(\ddot{\bar{x}}-\ddot{\bar{u}}\right)^T\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

$$=\int_I\left(\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt$$

$$-\int_I\Big[\left(\bar{x}-\bar{u}\right)^T\left(\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$-\left(\bar{x}-\bar{u}\right)^T\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big|_{t=a}^{t=b}$$

$$-\int_I\left(x-u\right)^T D\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

$$+\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big|_{t=a}^{t=b}$$

$$-\int_I\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T D\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)dt$$

$$+\left(\ddot{\bar{x}}-\ddot{\bar{u}}\right)^T\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

(Integrating by parts)

$$=\int_I\left(\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt$$

$$+\int_I\Big[\left(\bar{x}-\bar{u}\right)^T\left(\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$-\left(\bar{x}-\bar{u}\right)^T D\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h\dot{u}_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

$$-\left(\bar{x}-\bar{u}\right)^T D\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big|_{t=a}^{t=b}$$

$$+\int_I.\left(\bar{x}-\bar{u}\right)^T D^2\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)dt$$

(Using boundary conditions and Integrating by parts)

$$=\int_I \left(\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt$$

$$+\int_I \Big[\left(\bar{x}-\bar{u}\right)^T \Big\{\left(\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$-D\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h\dot{u}_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+D^2\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{y}(t)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}(t)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big\}\Big]dt$$

$$=\int_I \left(\bar{\lambda}^T f\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt$$

(Using (4.13))

This implies

$$\int_I \left(\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt \geqq 0 \qquad (4.15)$$

But since $\bar{y}(t)\geq 0$, $g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\leq 0$ and $h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0$, $t\in I$ yield,

$$\int_I \left(\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt \leq 0 \quad \ldots\ldots (4.16)$$

Combining (4.15) and (4.16), we have

$$\int_I \left(\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)+\bar{z}(t)^T h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\right)dt = 0$$

This, because of $h\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0$, $t\in I$, gives

$$\int_I \bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)dt = 0.$$

This, together with $\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)\leqq 0,\ t\in I,$ implies $\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}\right)=0,\ t\in I$

Theorem 4.7: Suppose

(R_1): $\left(\bar{y}(t),\bar{z}(t),u(t)\right)\in L_D$ and $\bar{u}(t)\in L_P$;

(R_2): $\bar{y}(t)^T g\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)=0$, $t\in I$;

(R_3): $\int_I \bar{\lambda}^T f(t,.,.,.)dt$ and $\int_I \left(\bar{y}(t)^T g(t,.,.,.)+\bar{z}(t)^T h(t,.,.,.)\right)dt$ are convex;

Then $\bar{u}(t)$ is an optimal solution of (P_λ) and hence of (VP).

Proof: If $\bar{u}(t)$ is the only feasible solution of (P_λ), the conclusion is self evident. So, assume that $x(t)$ is another feasible solution of (P_λ). Then by the hypotheses (R_1) and (R_3), we have

$$\int_I \bar{\lambda}^T f\left(t,\bar{x},\dot{x},\ddot{x}\right)dt \geqq \int_I \bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)dt$$

$$\int_I \Big[\left(\bar{x}-\bar{u}\right)^T\left(\bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)+\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+\left(\ddot{\bar{x}}-\ddot{\bar{u}}\right)^T\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

Now integrating by parts, we have,

$$=\int_I \bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)dt-\int_I \Big[\left(\bar{x}-\bar{u}\right)^T \bar{\lambda}^T f_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)dt$$

$$-\left(\bar{x}-\bar{u}\right)^T\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big|_{t=a}^{t=b}-\int_I \left(x-u\right)^T D\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

$$+\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big|_{t=a}^{t=b}-\int_I \left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T D\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)dt$$

$$=\int_I \bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)dt+\int_I \left(\bar{x}-\bar{u}\right)^T D\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)dt$$

$$+\int_I \left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T D\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)dt$$

(Using boundary conditions (4.11)

$$=\int_I \bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)dt+\int_I \left(\bar{x}-\bar{u}\right)^T\Big[\bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)-D\left(\bar{\lambda}^T f_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+D^2\left(\bar{\lambda}^T f_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

$$=\int_I \left(\bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)dt$$

$$-\int_I \left(\bar{x}-\bar{u}\right)^T\Big[\bar{y}\left(t\right)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}\left(t\right)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)$$

$$-D\left(\bar{y}\left(t\right)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}\left(t\right)^T h\dot{u}_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+D^2\left(\bar{y}\left(t\right)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}\left(t\right)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

This, by integrating by parts and using boundary conditions, as earlier, we get

$$=\int_I \bar{\lambda}^T f\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)dt$$

$$-\int_I \Big[\left(\bar{x}-\bar{u}\right)^T\left(\bar{y}\left(t\right)^T g_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}\left(t\right)^T h_u\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+\left(\dot{\bar{x}}-\dot{\bar{u}}\right)^T\left(\bar{y}\left(t\right)^T g_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}\left(t\right)^T h_{\dot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)$$

$$+\left(\ddot{\bar{x}}-\ddot{\bar{u}}\right)^T\left(\bar{y}\left(t\right)^T g_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)+\bar{z}\left(t\right)^T h_{\ddot{u}}\left(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}}\right)\right)\Big]dt$$

$$\geqq \int_I \left(\bar{\lambda}^T f\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right)\right) dt$$
$$+ \int_I \left[\bar{y}(t)^T g\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right) + \bar{z}(t)^T h\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right) \right] dt$$
$$- \int_I \left[\bar{y}(t)^T g\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) + \bar{z}(t)^T h\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) \right] dt$$

$$\geqq \int_I \bar{\lambda}^T f\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right) dt$$

(Using hypothesis (A_1), (A_2) and $x \in L_p$).

Thus implies that $\bar{u}$ minimizes $\int_I \bar{\lambda}^T f\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt$ over L_p .

Remark: In Theorem 4.7, we assume that a part of feasible solution of (WCD_λ)

is a feasible solution of (P_λ). It is a natural question if there is any set of some appropriate conditions under which this assumption is true. The following theorem gives one such set of conditions.

Theorem 4.8: Assume

(Q_1): $x(t) \in L_p$ and $\left(\bar{y}(t), \bar{z}(t), u(t)\right) \in L_D$;

(Q_2): $g(t,.,.,.)$ and $h(t,.,.,.)$ are differentiable convex functions;

(Q_3): $\int_I \left(g\left(t, u(t), \dot{u}(t), \ddot{u}(t)\right) + h\left(t, u(t), \dot{u}(t), \ddot{u}(t)\right)\right) dt = 0$

(Q_4): $(x-u)^T g_u\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right) + (\dot{x}-\dot{u})^T \left(g_{\dot{u}}\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right)\right)$

$+ (\ddot{x}-\ddot{u})^T g_{\ddot{u}}\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right) \geqq 0$, $t \in I$

(Q_5): $(x-u)^T h_u\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right) + (\dot{x}-\dot{u})^T \left(h_{\dot{u}}\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right)\right)$

$+ (\ddot{x}-\ddot{u})^T h_{\ddot{u}}\left(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}\right) \geqq 0$, $t \in I$

Then $\bar{u} \in L_p$.

Proof: By the convexity of $g(t,.,.,.)$ and $h(t,.,.,.)$, we have

$$\begin{aligned} g(t, x, \dot{x}, \ddot{x}) &\geqq g(t, u, \dot{u}, \ddot{u}) + (x-u)^T g_u(t, u, \dot{u}, \ddot{u}) + (\dot{x}-\dot{u})^T g_{\dot{u}}(t, u, \dot{u}, \ddot{u}) \\ &+ (\ddot{x}-\ddot{u})^T g_{\ddot{u}}(t, u, \dot{u}, \ddot{u}) \geqq 0 \end{aligned} \tag{4.17}$$

$$\begin{aligned} h(t, x, \dot{x}, \ddot{x}) &\geqq h(t, u, \dot{u}, \ddot{u}) + (x-u)^T h_u(t, u, \dot{u}, \ddot{u}) + (\dot{x}-\dot{u})^T h_{\dot{u}}(t, u, \dot{u}, \ddot{u}) \\ &+ (\ddot{x}-\ddot{u})^T h_{\ddot{u}}(t, u, \dot{u}, \ddot{u}) \geqq 0 \end{aligned} \tag{4.18}$$

Using (4.13) and (4.14) together with the hypotheses (Q_1), (Q_2) and (Q_3), we have

$$g\left(t,u,\dot{u},\ddot{u}\right) \leqq 0\ ,\ t \in I \tag{4.19}$$

and

$$h\left(t,u,\dot{u},\ddot{u}\right) \leqq 0\ ,\ t \in I \tag{4.20}$$

By (4.15) and (4.16), we have

$$g\left(t,u\left(t\right),\dot{u}\left(t\right),\ddot{u}\left(t\right)\right)+h\left(t,u\left(t\right),\dot{u}\left(t\right),\ddot{u}\left(t\right)\right) \leqq 0\ ,\ t \in I \tag{4.21}$$

The hypothesis (Q_2) with (4.17) implies

$$g\left(t,u\left(t\right),\dot{u}\left(t\right),\ddot{u}\left(t\right)\right)+h\left(t,u\left(t\right),\dot{u}\left(t\right),\ddot{u}\left(t\right)\right)=0\ ,\ t \in I \tag{4.22}$$

But $g\left(t,u,\dot{u},\ddot{u}\right) \leqq 0\ ,\ t \in I$. Hence by (4.22) we have

$$h\left(t,u,\dot{u},\ddot{u}\right) \geqq 0\ ,\ t \in I\ . \tag{4.23}$$

The inequalities (4.20) and (4.21) imply

$$h\left(t,u,\dot{u},\ddot{u}\right)=0\ ,\ t \in I\ . \tag{4.24}$$

The relations (4.19) and (4.24) imply that $\bar{u}\left(t\right) \in L_p$

4.5. Variational problems with natural boundary values

It is possible to construct variational problems with natural boundary values rather than the problem with fixed end point considered in the preceding sections. The problems of Section4.2 can be formulated as follows:

$$(VP_N):Maximize\left(\int_I f^{'}(t,x,\dot{x},\ddot{x})dt,....,\int_I f^{'}(t,x,\dot{x},\ddot{x})dt\right)$$

Subject to

$$g(t,x,\dot{x},\ddot{x}) \leq 0,\ t \in I$$
$$h(t,x,\dot{x},\ddot{x}) = 0,\ t \in I$$

$$\left(M-WD\right)_N:\ Maximize\left(\int_I f'(t,u,\dot{u},\ddot{u})\,dt,...........,\int f'(t,u,\dot{u},\ddot{u})dt\right)$$

$$
\begin{aligned}
&\left(\lambda^T f_u(t,u,\dot{u},\ddot{u})\right) + y(t) gu(t,u,\dot{u},\ddot{u}) + z(t)^T h_u(t,u,\dot{u},\ddot{u})) \\
&-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})) + \\
&D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u})) = 0,\ t \in I, \\
&\int_1 \left(y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt \geqq 0 \\
&\lambda > 0,\ \ y(t) \geqq 0,\ \ t \in I. \\
&\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) = 0,\ at\ t = a, t = b \\
&\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) = 0,\ at\, t = a, t = b
\end{aligned}
$$

The proof of the theorems of Section 4.3 for (VP_N) and $(M - WD_N)$ can easily be recoursed for their proofs with slight modification. The problems of the Section 4.4 can be written with natural boundary values as follows:

For given $0 < \lambda \in R^p$.

$(P_\lambda)_N$: $\quad$ $Minimize \int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt$

Subject to $\quad$ $g(t,x,\dot{x},\ddot{x}) \geqq 0,\ t \in I.$

$\quad$ $h(t,x,\dot{x},\ddot{x}) = 0,\ t \in I.$

$(WCD_\lambda)_N$: $\quad$ $Maximize\ \int_I \left(\lambda^T f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u}) + z(t)^T h(t,u,\dot{u},\ddot{u}) \right) dt$

$$
\begin{aligned}
&\left(\lambda^T f_u(t,u,\dot{u},\ddot{u})\right) + y(t) gu(t,u,u,u) + z(t)^T h_u(t,u,\dot{u},\ddot{u})) \\
&-D\left(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u})\right) + y(t)^T g_{\dot{u}}(t,u,u,u) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})) + \\
&D^2\left(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u})\right) + y(t)^T g_{\ddot{u}}(t,u,u,u) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u})) = 0, t \in I, \\
&\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\dot{u}}(t,u,u,u) + z(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u}) = 0, at\, t = a, t = b \\
&\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + y(t)^T g_{\ddot{u}}(t,u,u,u) + z(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}) = 0, at\, t = a, t = b \\
&\lambda > 0, y(t) \geqq 0, t \in I.
\end{aligned}
$$

4.6. Multiobjective nonlinear programming pair of Mond-Weir type Multiobjective.

When all the functions in the problems $\left(VP\right)_N$, $(WCD_\lambda)_N$, $(P_\lambda)_N$ are independent of t. For simplicity b-a =1 and the pairs of dual problems reduce to the following problems:

(VP_0): $\quad$ $Minimize\ (f^1(u), f^2(u), \ldots\ldots f^p(u))$

$Subject\, to$

$g(x) \leq 0,\ \ h(x) = 0$.

$(M-WP_0)$: $Maximize\ (f^1(u), f^2(u), \ldots\ldots f^p(u))$

$$\begin{aligned}
&Subject\ to\\
&\lambda^T f_u(u) + y^T g_u(u) + z^T h_u(u) = 0,\\
&y^T g(u) + z^T h(u) \geqq 0.\\
&For\ given\ \lambda > 0, \lambda \in R^p, y \geqq 0.
\end{aligned}$$

$(VP_\lambda)_0$: $Maximize\ \lambda^T f(u)$

$$\begin{aligned}
&Subject\ to\\
&g(u) \geqq 0, \quad h(u) = 0\ .
\end{aligned}$$

$(WCD_\lambda)_0$: $Maximize\ \lambda^T f(u) + y^T g(u) + z^T h(u)$

$$\begin{aligned}
&Subject\ to\\
&\lambda^T f_u(u) + y^T g_u(u) + z^T h_u(u) = 0,\\
&y^T g(u) + z^T h(u) \geqq 0.\ y \geqq 0.
\end{aligned}$$

Theorems 4.1-4.3 for the pair of Mond-Weir type dual problems $(VP_0)\ and\ (M-WP_0)$ and Theorems 4.4-4.8 for the pair of Wolf type dual problems $(VP_\lambda)_0$ and $(WCD_\lambda)_0$ are simple to be validated, albeit validations of these theorems are not explicitly mentioned in the literature.

Chapter 5

CONSTRAINED DYNAMIC GAME AND SYMMETRIC DUALITY FOR VARIATIONAL PROBLEMS

5.1 Introduction

The applications of game –theoretic ideas are quite extensive and lie at the root of almost every human activity. So the search for elegant methods for solving a general strategic game is very natural. Dantzig [23] studied equivalence of the programming problem and the game problem, and Charnes [11] established that every matrix game is equivalent to linear programming. The results in [23] and [11] yield that every two- person, zero-sum can be solved by simplex method of linear programming. The measure advantage of linear programming techniques is that it provides solution to a mixed strategy game of any size. Motivated with this observation, in the recent past many researchers were interested in studying equivalence between a scalar valued game and a certain mathematical programming problem. Cottle [18] was the first to establish the equivalence between an unconstrained game having a non-linear Convex-Concave payoff function and the corresponding symmetric dual programming problems. Since then several authors notably, Chandra et al [14], Corley [16], and Prasad and Sreenivas [25].Later Mond et al [61] extended the result of Kawaguchi and Maruyama [50] to the nonlinear setting and proved that a constrained game is proved that a Constrained game is equivalent to a pair of Symmetric dual nonlinear programming problems, which appearing similar those of Mond-Weir [65].

The dynamic games are basically concerned with the modeling of large scale systems which have independent decision makers with individual payoff (or reward) functions. Applications of dynamic games can be experienced in solving some important problems relating to environment resources, aerospace and energy management. So their domain of applications is naturally wider than those of static games. The purpose of this research is to extend the results of Mond et al [61] to the dynamic setting involving variational problems and show that such a constrained game is equal to pair of symmetric dual variational problems which are similar to that of Bector et al [4] but contain certain additional constraints. It is pointed out that results of this research can be considered as the dynamic generalization of those of [61].

5.2 Problem Formulation and Motivation

Consider the time independent nonlinear game G (X, Y, F), where,

$X = \{x: I \to R^n | x(a) = 0 = x(b),\ p_i(t, x, \dot{x}) \geq 0, t \in I,\ \ i = 1,2, \ldots\ldots, k\}$

$Y = \{y: I \to R^m |\ y(a) = 0 = y(b), q_j(t, y, \dot{y}) \geq 0, t \in I, j = 1,2, \ldots\ldots, l\}$

and F: X × Y → R,

where,

(i) $p_i : I \times R^n \times R^n \to R, (i = 1,2, \ldots, k)$, and

$q_j : I \times R^m \times R^m \to R, (j = 1,2, \ldots, l)$, with $I = [a, b] \subseteq R$

are twice continuously differentiable with respect to each of their arguments, $x: I \rightarrow R^n$ and $y: I \rightarrow R^m$. The functions x and y are differentiable with their derivatives $\dot{x}$ and $\dot{y}$. Each of these spaces X and Y is equipped with the following norm:

$$\|v\| = \|v\|_\infty + D\|v\|_\infty,$$

where differentiation operator D is given by

$$w = Dv \Rightarrow v(t) = v(a) + \int_a^t w(s)ds$$

in which $D = {}^{d}/_{dt}$ except at discontinuities.

(ii) X and Y represent the strategy spaces of players A and B respectively and

(iii) $F: (x, y)$ represents the pay-off to the player B from the player A when player A selects strategy x and the player B selects strategy Y. In analogy with Cottle [2], it is assumed that the player A is the minimizing player and the player B is the maximizing player. The player A wishes to solve $\min_{x \in X} \max_{y \in Y} F(x, y)$ and the player B wishes to solve $\max_{x \in X} \min_{y \in Y} F(x, y)$. In the spirit of Cottle [18], both $\min_{x \in X} \max_{y \in Y} F(x, y)$ will be reduced to certain nonlinear variational problems and the symmetric duality in the subsequent section. These symmetric dual nonlinear variational problems will also be related to the non-linear time dependent constrained game in the spirit of Shreevivas [25].

Fix $\bar{x} \in X$ and consider $\max_{y \in Y} F(\bar{x}, y)$ i.e.

(CP$_1$): Maximize $F(\bar{x}, y) = \int_a^b f(t, \bar{x}, \dot{\bar{x}}, y, \dot{y})dt$

Subject to

$$x(a) = 0 = x(b)$$
$$y(a) = 0 = y(b)$$
$$q_j(t, y, \dot{y}) \leq 0, j = 1,2, \dots, r$$
$$y(t) \geq 0, t \in I$$

where $f : R^n \times R^m \times R^m \rightarrow R$ is continuously differentiable and possesses fourth order derivatives with respect to each of its arguments.

It can be shown on the lines of Mangasarian [55] involving the analysis in [64] that $\bar{y}$ is optimal to (CP$_1$) if and only if

$$f_y(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}) - \mu(t)^T q_y(t, \bar{y}, \dot{\bar{y}})$$
$$-D\left(f_{\dot{y}}(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}) - \mu(t)^T q_{\dot{y}}(t, \bar{y}, \dot{\bar{y}})\right) \leq 0, t \in I$$
$$y(t)^T \left[\left(f_y(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}) - \mu(t)^T q_y(t, \bar{y}, \dot{\bar{y}})\right)\right.$$
$$\left.-D\left(f_{\dot{y}}(t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_{\dot{y}}(t, \bar{y}, \dot{\bar{y}})\right)\right] = 0, t \in I$$

$$\mu(t)^T q(t, \bar{y}, \dot{\bar{y}}) = 0,$$
$$q(t, \bar{y}, \dot{\bar{y}}) \leq 0, \ t \in I$$

$$\mu(t) \geq 0, \bar{y}(t) \geq 0$$

where f_y and $f_{\dot{y}}$ denote the gradients of f with respect to y and $\dot{y}$; and q_y and $q_{\dot{y}}$ denote the gradient of q with respect to y and $\dot{y}$.

Using the above conditions, $\min_{x\in X}\max_{y\in Y} F(x,y)$ is equivalent to the following variational problems:

(CP$_2$) Maximize $\int_a^b f(t,x,\dot{x},y,\dot{y})dt$

Subject to

$$x(a) = 0 = x(b)$$

$$y(a) = 0 = y(b)$$

$$f_y(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_y(t,y,\dot{y})$$
$$-D\big(f_{\dot{y}}(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_{\dot{y}}(t,y,\dot{y})\big) \le 0, t \in I$$

$$y(t)^T\Big[\big(f_y(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_y(t,y,\dot{y})\big)$$
$$-D\big(f_{\dot{y}}(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_{\dot{y}}(t,y,\dot{y})\big)\Big] = 0,$$

$$\mu(t)^T q(t,y,\dot{y}) = 0, \ t \in I,$$

$$p(t,x,\dot{x}) \ge 0, \ q(t,y,\dot{y}) \le 0, t \in I.$$

$$x(t) \ge 0, y(t) \ge 0, \mu(t) \ge 0, t \in I.$$

Similarly $\max_{x\in X}\min_{y\in Y} F(x,y)$ will be reduced to the following variational problem:

(CD$_2$): Maximize $\int_a^b f(t,u,\dot{u},v,\dot{v})dt$

Subject to

$$u(a) = 0 = u(b)$$

$$v(a) = 0 = v(b)$$

$$f_u(t,u,\dot{u},v,\dot{v}) - \lambda(t)^T p_u(t,u,\dot{u})$$
$$-D\Big(f_{\dot{u}}(t,u,\dot{u},v,\dot{v}) - \lambda(t)^T p_{\dot{u}}(t,u,\dot{u})\Big) \ge 0, t \in I$$

$$u(t)^T\big[\big(f_u(t,u,\dot{u},v,\dot{v}) - \lambda(t)^T p_u(t,u,\dot{u})\big)$$
$$-D\Big(f_{\dot{u}}(t,u,\dot{u},v,\dot{v}) - \mu(t)^T p_{\dot{u}}(t,u,\dot{u})\Big) = 0, t \in I$$

$$\lambda(t)^T p(t,u,\dot{u})dt = 0, t\epsilon I$$

$$p(t,u,\dot{u}) \ge 0, \ q(t,v,\dot{v}) \le 0, t \in I,$$

$$u(t) \ge 0, v(t) \ge 0, \lambda(t) \ge 0, t \in I$$

where f_u and $f_{\dot{u}}$ denote the gradients of f with respect to u and $\dot{u}$, and p_u and $p_{\dot{u}}$ denote the gradient of p with respect to u and $\dot{u}$ respectively.

In view of the formulation of symmetric dual variational problems by Mond and Hanson [61] and Bector, et al [4], we shall drop the constraints $q(t,y,\dot{y}) \le 0$, $y(t) \ge 0, t \in I$ from (CP$_2$) and $p(t,u,\dot{u}) \ge 0, u(t) \ge 0, t \in I$ from (CD$_2$), as these will automatically be satisfied. Thus the two problems (CP_2) and (CD_2) corresponding to $\min_{x\in X}\max_{y\in Y} F(x,y)$ and $\max_{x\in X}\min_{y\in Y} F(x,y)$ can be constructed in the following forms.

(NVP): Minimize $\int_a^b f(t,x,\dot{x},y,\dot{y})dt$

Subject to

$$x(a) = 0 = x(b), y(a) = 0 = y(b) \quad (5.1)$$

$$f_y(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_y(t,y,\dot{y}) - D(f_{\dot{y}}(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_{\dot{y}}(t,y,\dot{y})) \leq 0, \quad (5.2)$$

$$\int_a^b y(t)^T \left[\left(f_y(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_y(t,y,\dot{y})\right) - D(f_{\dot{y}}(t,x,\dot{x},y,\dot{y}) - \mu(t)^T q_{\dot{y}}(t,y,\dot{y}))\right]dt \geq 0, \quad (5.3)$$

$$\int_a^b \mu(t)^T q(t,y,\dot{y})\, dt \geq 0, \quad (5.4)$$

$$x(t) \geq 0, \mu(t) \geq 0, p(t,x,x) \geq 0, t \in I$$
(5.5)

(NVD): Maximize $\int_a^b f(t,u,\dot{u},v,\dot{v})dt$

Subject to

$$u(a) = 0 = u(b), v(a) = 0 = v(b) \quad (5.6)$$

$$f_u(t,u,\dot{u},v,\dot{v}) - \lambda(t)^T p_u(t,u,\dot{u}) - D\left(f_{\dot{u}}(t,u,\dot{u},v,\dot{v}) - \lambda(t)^T p_{\dot{u}}(t,u,\dot{u})\right) \geq 0, t \in I \quad (5.7)$$

$$\int_a^b u(t)^T [(f_u(t,u,\dot{u},v,\dot{v}) - \lambda(t)^T p_u(t,u,\dot{u})) - D\left(f_{\dot{u}}(t,u,\dot{u},v,\dot{v}) - \lambda(t)^T p_{\dot{u}}(t,u,\dot{u})\right) dt \leq 0, \quad (5.8)$$

$$\int_a^b \lambda(t)^T p(t,u,\dot{u})dt \leq 0, \quad (5.9)$$

$$u(t) \geq 0, v(t) \geq 0, \lambda(t) \geq 0, t \in I \quad (5.10)$$

In the subsequent section, it will be shown with above problems constitute a pair of symmetric dual variational problems.

5.3 Symmetric Duality

It is easy to see that if the dual (NVD) is recast in the form of the problem (NVP), its dual is primal (NVP).We shall prove the following duality theorems.

Theorem 5.1 (Weak Duality). Let $(\bar{x},\bar{y},\bar{\mu})$ be feasible to the problem (NVP) and $(\bar{u},\bar{v},\bar{\lambda})$ be feasible for (NVD). For all feasible solutions (x,y,μ,u,v,λ), let $\int_a^b \left\{ f(t,.,.,y,\dot{y}) - \lambda^T(t)\, p(t,.,.)\right\} dt$ be pseudoconvex for each $(y, \dot{y})$ and $\int_a^b \left\{ f\left(t, x, \dot{x}, .,.\right) - \mu^T\left(t\right) q\left(t, .,.\right) \right\} dt$ be pseudoconcave for each $(x, \dot{x})$. Then

$$\int_a^b f(t, x, \dot{x}, y, \dot{y}) dt \geq \int_a^b f(t, u, \dot{u}, v, \dot{v}) dt$$

If, in the above, the equality holds, then$(\bar{x}, \bar{y}, \bar{\mu})$ is optimal to the problem (NVP) and$(\bar{u}, \bar{v}, \bar{\lambda})$ is optimal to the problem (NVD).

Proof: From (5.2) and (5.3) we obtain

$$\left(v(t) - y(t)\right)^T \left[f_y(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q_y}(\mathrm{t}, y, \dot{y}) \right.$$

$$\left. - \mathrm{D}\left(f_{\dot{y}}\,(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q}_{\dot{y}}(\mathrm{t}, y, \dot{y})\right) \right] \leq 0, \mathrm{t} \in \mathrm{I}$$

This implies,

$$\int_a^b \left(v(t) - y(t)\right)^T \left[f_y(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q_y}(\mathrm{t}, y, \dot{y}) \right.$$

$$\left. - \mathrm{D}\left(f_{\dot{y}}\,(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q}_{\dot{y}}(\mathrm{t}, y, \dot{y})\right) \right] \mathrm{dt} \leq 0, \quad .$$

This, by integration by parts, yields,

$$\left[\int_a^b \left(v(t) - y(t)\right)^T \left(f_y(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q_y}(\mathrm{t}, y, \dot{y}) \right) \right.$$

$$\left. + \left(\dot{v}(t) - \dot{y}(t)\right)^T \left(f_{\dot{y}}\,(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q}_{\dot{y}}(\mathrm{t}, y, \dot{y})\right) \right] dt$$

$$+ \left(v(t) - y(t)\right)^T f_y\,(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q}_y(\mathrm{t}, y, \dot{y}) \Big|_{t=b}^{t=b} \leq 0$$

This, by using fixed point conditions (5.1) and (5.6), we have

$$\int_a^b \left[\, \left(v(t) - y(t)\right)^T \left(f_y(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q_y}(\mathrm{t}, y, \dot{y}) \right) \right.$$

$$\left. + \left(\dot{v}(t) - \dot{y}(t)\right)^T \left(f_{\dot{y}}\,(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q}_{\dot{y}}(\mathrm{t}, y, \dot{y})\right) \right] dt \leq 0$$

By pseudo-concavity of $\int_a^b \left(f(\mathrm{t}, x, \dot{x}, ., .) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q_y}(\mathrm{t}, ., .) \right) dt$ in y and $\dot{y}$ for each x and $\dot{x}$, this implies,

$$\int_a^b \left(f(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q_y}(\mathrm{t}, v, \dot{v}) \right) dt$$

$$\leq \int_a^b \left(f(\mathrm{t}, x, \dot{x}, y, \dot{y}) - \mu\,(\mathrm{t})^{\mathrm{T}}\, \mathrm{q_y}(\mathrm{t}, y, \dot{y}) \right) dt$$

Since$\int_a^b \mu(t)^T (q(t, v, \dot{v})) dt \leq 0$, $\int_a^b \mu(t)^T\, (q(t, y, \dot{y})) \geq 0$, this implies

$$\int_a^b f(t, x, \dot{x}, v, \dot{v}) dt \;\leq\; \int_a^b f(t, x, \dot{x}, y, \dot{y}) dt \tag{5.11}$$

The relations (5.7) and (5.8) imply,

$$\int_a^b (x(t)-u(t))^T[f_u(t,u,\dot u,v,\dot v)-\lambda(t)^T p_u(t,u,\dot u)$$

$$-D(f_{\dot u}(t,u,\dot u,v,\dot v)-\lambda(t)^T p_{\dot u}(t,u,\dot u))]dt \geq 0\ ,t\in I$$

As earlier, this becomes,

$$\int_a^b (x(t)-u(t))^T[f_u(t,u,\dot u,v,\dot v)-\lambda(t)^T p_u(t,u,\dot u)$$

$$-(x(t)-u(t))^T(f_{\dot u}(t,u,\dot u,v,\dot v)-\lambda(t)^T p_{\dot u}(t,u,\dot u))]dt \geq 0\ ,t\in I$$

This, because of pseudo-convexity of $\int_a^b\{(f(t,\ldots.y,y)-\lambda(t)^T\,p(t,\ldots.))\}\,dt$ for each y and $\dot y$, implies,

$$\int_a^b\{(f(t,x,\dot x,y,\dot y)-\lambda(t)^T\,p(t,x,\dot x)))\}\,dt\ \geq \int_a^b\{(f(t,u,\dot u,v,\dot v)-\lambda(t)^T\,p(t,u,\dot u))\}\,dt$$

This, in view of $\int_a^b \lambda(t)^T\,p(t,x,x)dt \geq 0,\ and \int_a^b \lambda(t)^T\,p(t,u,\dot u)dt \leq 0\ gives,$

$$\int_a^b f(t,x,\dot x,v,\dot v)\,dt\ \geq \int_a^b f(t,u,\dot u,v,\dot v)\,dt \qquad (5.12)$$

Combining (5.11) and (5.12), we have

$$\int_a^b f(t,x,\dot x,y,\dot y)\,dt\ \geq \int_a^b f(t,u,\dot u,v,\dot v)\,dt\ .$$

For derivation of optimality conditions required in Theorem 5.2, the following result will be used:

$$\frac{d}{dt}f_{\dot y}=f_{\dot y t}+f_{\dot y y}\dot y+f_{\dot y\dot y}\ddot y+f_{\dot y x}\dot x+f_{\dot y\dot x}\ddot x$$

Implying

$$\frac{\partial}{\partial y}\frac{d}{dt}f_{\dot y}=\frac{d}{dt}f_{\dot y y},\quad \frac{\partial}{\partial \dot y}\frac{d}{dt}f_{\dot y}=\frac{d}{dt}f_{\dot y\dot y}+f_{\dot y y},\quad \frac{\partial}{\partial \ddot y}\frac{d}{dt}f_{\dot y}=f_{\dot y\dot y},\quad \frac{\partial}{\partial x}\frac{d}{dt}f_{\dot y}=\frac{d}{dt}f_{\dot y x},$$
$$\frac{\partial}{\partial \dot x}\frac{d}{dt}f_{\dot y}=\frac{d}{dt}f_{\dot y\dot x}+f_{\dot y x},\qquad \frac{\partial}{\partial \ddot x}\frac{d}{dt}f_{\dot y}=f_{\dot y\dot x}.$$

Theorem 5.2: (Strong duality) Assume that

(A$_1$): $(\bar x,\bar y,\bar\mu)$ is an optimal solution of (NP).

(A$_2$): $\left(f_y-Df_{\dot y}\right)$ and $\left(\mu(t)^T q_y-D\mu(t)^T q_{\dot y}\right)$ are linearly independent.

(A$_3$): $\mu(a)=0=\mu(b)$

(A$_4$): $\int_a^b[\theta(t)^T\{\left(f_{yy}-\mu(t)^T q_{yy}\right)-\ D\left(f_{\dot y y}-\mu(t)^T q_{\dot y y}\right)\}$

$$-\ \mathrm{D}\left\{\theta(t)^T D\left(f_{\dot y\dot y}-\mu(t)^T q_{\dot y\dot y}\right)\right\}$$

$$+D^2\{\theta(t)^T D\left(f_{\dot y\dot y}-\mu(t)^T q_{\dot y\dot y}\right)\}]\theta(t)dt=0,\quad \Rightarrow \theta(t)=0,\ \ t\in I.$$

and

(A$_5$): $\rho(t)^T p_x-D(\rho(t)^T p_{\dot x})\leq 0\ and\ \rho(t)\geq 0,\ \ t\in I,\Rightarrow\ \rho(t)=0,\ \ t\in I.$

Then there exists $(t) \in R^k$, $t \in I$ such $(\bar{x}, \bar{y}, \bar{\mu})$ is feasible for (ND). If, in addition, the Theorem 5.1 holds, then $(\bar{x}, \bar{y}, \bar{\mu})$ is an optimal solution of (ND).

Proof: If $(\bar{x}, \bar{y}, \bar{\mu})$ minimize (NVP), there exists a function of the form

$$H(x, y, \mu) = r_0 f + \left(\eta(t) - wy(t)\right)^T \left\{\left(f_y - \mu(t) q_y\right) - D\left(f_{\dot{y}} - \mu(t) q_{\dot{y}}\right)\right\}$$

$$-z\mu(t)q - \delta(t)^T p - \alpha(t)^T x(t) - \beta(t)^T \mu(t)$$

Where $r_0 \in R_+$, $\omega \in R_+, z \in R_+$ *and* piecewise smooth $\alpha : I \to R_+^n$ *and* $\eta : I \to R_+^n$,

$\delta : I \to R_+^s$ *and* $w : I \to R_+^t$, such that

$$H_x - DH_{\dot{x}} + D^2 H_{\ddot{x}} = 0 , t \in I \quad (5.13)$$

$$H_y - DH_{\dot{y}} + D^2 H_{\ddot{y}} = 0 , t \in I \quad (5.14)$$

$$H_\mu - DH_{\dot{\mu}} + D^2 H_{\ddot{y}} = 0 , t \in I \quad (5.15)$$

$$\eta(t)^T \left\{\left(f_y - \mu(t)^T q_y\right) - D\left(f_{\dot{y}} - \mu(t)^T q_{\dot{y}}\right)\right\} = 0, t \in I \quad (5.16)$$

$$\omega \int_a^b y(t)^T \left\{\left(f_y - \mu(t)^T q_y\right) - D\left(f_{\dot{y}} - \mu(t)^T q_{\dot{y}}\right)\right\} dt = 0 \quad (5.17)$$

$$z \int_a^b \mu(t)^T q \; dt = 0 , \quad (5.18)$$

$$\delta(t)^T p = 0 , t \in I \quad (5.19)$$

$$\alpha(t)^T x(t) = 0 , t \in I. \quad (5.20)$$

$$\beta(t)^T \mu(t) = 0 , t \in I. \quad (5.21)$$

$$\left(r_{0,} \eta\,(t), \omega\,, \alpha(t), \beta(t), \delta(t)\right) \neq 0 \quad (5.22)$$

From (5.13), we have,

$$r_0(f_x - Df_{\dot{x}}) + \left(\eta(t) - wy(t)\right)^T \left(f_{yx} - Df_{\dot{y}x}\right)$$

$$-D\left(\eta(t) - wy(t)\right)^T \left(f_{y\dot{x}} - Df_{\dot{y}\dot{x}} - f_{\dot{y}x}\right)$$

$$-(\delta(t)^T p_x - D\delta(t)^T p_{\dot{x}}) - D^2(\eta(t)^T - r\, y(t)^T) f_{\dot{y}\dot{x}} - \alpha(t) = 0, \; t \in I \quad (5.23)$$

From (5.14) we have,

$$(r_0 - \omega)\left(f_y - Df_{\dot{y}}\right) + (w - z)\left(\mu(t)^T q_y - D\mu(t)^T q_{\dot{y}}\right)$$

$$+\left(\eta(t) - wy(t)\right)^T \left[\left(f_{yy} - \mu(t)^T q_{yy}\right) - D\left((f_{\dot{y}y} - \mu(t)^T q_{\dot{y}y}\right)\right]$$

$$+ \text{D} \left[\left(\eta(t) - wy(t)\right)^T D\left((f_{\dot{y}\dot{y}} - \mu(t)^T q_{\dot{y}\dot{y}}\right)\right]$$

$$+D^2 \left[\left(\eta(t) - wy(t)\right)^T D\left((f_{\dot{y}\dot{y}} - \mu(t)^T q_{\dot{y}\dot{y}}\right)\right] = 0, t \in I \quad (5.24)$$

From (5.15), we have

$$-\left(\eta(t) - \omega y(t)\right)^T q_y + \left(\eta(t) - \omega y(t)\right)^T Dq_{\dot{y}}$$

$$-D\left\{\left(\eta(t)-\omega y(t)\right)^T q_{\dot{y}}\right\}-\beta(t)-zq=0, t\in I \tag{5.25}$$

Multiplying this by $\mu(t)$, and using (5.18) and (5.21), we have

$$0=\int_a^b\Big[-\left(\eta(t)-\omega y(t)\right)^T\mu(t)^T q_y+\left(\eta(t)-\omega y(t)\right)^T\mu(t)\,Dq_{\dot{y}}$$

$$-\mu(t)^T D\left(\eta(t)-\omega y(t)\right)^T q_{\dot{y}}\Big]dt$$

$$=\int_a^b\Big[-\left(\eta(t)-\omega y(t)\right)^T\left(\mu(t)^T q_y\right)+\left(\eta(t)-\omega y(t)\right)^T\left(D\mu(t)q_{\dot{y}}-\dot{\mu}(t)q_{\dot{y}}\right)$$

$$-\mu(t)^T D(\eta(t)-\omega y(t))^T q_{\dot{y}}\Big]dt$$

$$=\int_a^b\Big[-\left(\eta(t)-\omega y(t)\right)^T\left(\mu(t)^T q_y-D\mu(t)^T q_{\dot{y}}\right)$$

$$-\left(\eta(t)-\omega y(t)\right)^T\dot{\mu}(t)^T q_{\dot{y}}-\mu(t)^T D(\eta(t)-\omega y(t))^T q_{\dot{y}}\Big]dt,$$

$$=\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\left(\mu(t)^T q_y-D\mu(t)^T q_{\dot{y}}\right)$$

$$+\left(\eta(t)-\omega y(t)\right)^T\dot{\mu}(t)^T q_{\dot{y}}+\mu(t)^T D(\eta(t)-\omega y(t))^T q_{\dot{y}}\Big]dt\ ,$$

$$=\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\left(\mu(t)^T q_y-D\mu(t)^T q_{\dot{y}}\right)\Big]dt$$

$$+\int_a^b D\left[\mu(t)\left(\eta(t)-\omega y(t)\right)q_{\dot{y}}\ \right]dt$$

$$=\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\left(\mu(t)^T q_y-D\mu(t)^T q_{\dot{y}}\right)\Big]dt$$

$$+\left(\eta(b)-\omega y(b)\right)\mu(b)^T q_{\dot{y}}(b)$$

$$-\left(\eta(a)-\omega y(a)\right)\mu(a)^T q_{\dot{y}}(a)$$

This, in view of the hypothesis (A_5), yields,

$$\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\left(\mu(t)^T q_y-D\mu(t)^T q_{\dot{y}}\right)\Big]dt=0 \tag{5.26}$$

From (5.16) and (5.17) we have

$$\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\left(f_y-Df_{\dot{y}}\right)\Big]dt=\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\left(\mu(t)^T q_y-D\mu(t)^T q_{\dot{y}}\right)\Big]dt$$

By (5.26), this reduces to

$$\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\left(f_y-Df_{\dot{y}}\right)\Big]dt=0 \tag{5.27}$$

Multiplying (5.24) by$(\eta(t)-\omega y(t))$, and then using (5.26) and (5.27), we have,

$$\int_a^b\Big[\left(\eta(t)-\omega y(t)\right)^T\{\left(f_{yy}-\mu(t)^T q_{yy}\right)-D\left(f_{\dot{y}y}-\mu(t)^T q_{\dot{y}y}\right)\}\left(\eta(t)-\omega y(t)\right)$$

$- \ \mathrm{D}\left\{\left(\eta(t)-\omega y(t)\right)^T D\left(f_{\dot{y}\dot{y}}-\mu(t)^T q_{\dot{y}\dot{y}}\right)\right\}\left(\eta(t)-\omega y(t)\right)$

$+ D^2\left\{\left(\eta(t)-\omega y(t)\right)^T D\left(f_{\dot{y}\dot{y}}-\mu(t)^T q_{\dot{y}\dot{y}}\right)\right\}\left(\eta(t)-\omega y(t)\right)\Big] dt = 0,$

This in view of hypothesis (A_4) implies,

$$\left(\eta(t)-\omega y(t)\right) = 0,\ t \in I \qquad (5.28)$$

Using (5.28) in (5.24), we have

$(r_0 - w)\left(f_y - Df_{\dot{y}}\right) + (z-\omega)\left(\mu(t)^T q_y - D\mu(t)^T)\ q_{\dot{y}}\right) = 0, t \in I$

which because of linear independence condition (A_2) gives,

$$r_0 = \omega = z \qquad (5.29)$$

Using $\left(\eta(t)-wy(t)\right) = 0,\ t \in I,$ (5.24) implies

$$r_0\left(f_x - Df_{\dot{x}}\right) - \left(\delta(t)^T p_x - D\delta(t)^T p_{\dot{x}}\right) - \alpha(t) = 0, t \in I \qquad (5.30)$$

Let $r_0 = 0, then$ (5.29) imples $\omega = z = 0$. Consequently (5.25) and (5.28) respectively yield $\beta(t) = 0$ and $\eta(t) = 0, t \in I$.

If $r_0 = 0,$ (5.30) implies

$$\delta(t)^T p_x - D\delta(t)^T p_{\dot{x}} \leq 0, t \in I\ ,\ \delta(t) \geq 0, t \in I.$$

By the hypothesis (A_5), this implies $\delta(t) = 0\ , t \in I$. From (5.30) we obtain

Thus we get, $\left(r_0, \omega, z, \beta(t), \delta(t), \eta(t)\right) = 0$ contradicting (5.22)

Hence $r_0 > 0$

$$\left(f_x - \frac{\delta(t)^T}{r_0}\ p_x\right) - D\left(f_{\dot{x}} - \frac{\delta(t)^T}{r_0}\ p_{\dot{x}}\right) = 0, t \in I$$

Multiplying (5.30) by $\bar{x}$(t) and using (5.20) we have

$$x(t)\left(f_x - \frac{\delta(t)^T}{r_0}\ p_x\right) - D\left(f_{\dot{x}} - \frac{\delta(t)^T}{r_0}\ p_{\dot{x}}\right) = 0, t \in I$$

From the above analysis, it readily follows that $\left(x, y, \frac{\delta(t)}{r_0}\right)$ is feasible for (NCD). An application of Theorem 5.1 completes the validation of Theorem 5.2.

The following is the converse duality (Theorem 5.3) whose proof follows by virtue of symmetry of the formulation of the primal and dual variational problems:

Theorem 5.3: (Converse duality) Assume that

(H_1): $(\bar{x}, \bar{y}, \bar{\lambda})$ is an optimal solution of (NP).

(H_2): $(f_x - Df_{\dot{x}})$ and $(\lambda(t)^T p_x - D\lambda(t)^T p_{\dot{x}})$ are linearly independent.

(H_3): $\lambda(a) = 0 = \lambda(b)$

(H_4): $\int_a^b [\phi(t)^T\{(f_{xx} - \lambda(t)^T p_{xx}) - D(f_{\dot{x}x} - \lambda(t)^T p_{\dot{x}x})\}$

$$- \ \mathrm{D}\,\{\phi(t)^T D(f_{\dot{x}\dot{x}} - \lambda(t)^T p_{\dot{x}\dot{x}})\}$$

$$+ D^2\{\phi(t)^T D(f_{\dot{x}\dot{x}} - \lambda(t)^T p_{\dot{x}\dot{x}})\}]\phi(t)dt = 0, \ \Rightarrow \phi(t) = 0, \ t \in I.$$

and

(H_5): $\psi(t)^T q_y - D\big(\psi(t)^T q_{\dot{y}}\big) \leq 0 \ and \ \psi(t) \geq 0, \ t \in I \ \ \Rightarrow \ \psi(t) = 0, \ t \in I.$

Then there exists $(t) \in R^l$, $t \in I$ such that $(\bar{x}, \bar{y}, \bar{\mu})$ is feasible for (ND). If, in addition, the Theorem 5.1 holds, then $(\bar{x}, \bar{y}, \bar{\mu})$ is an optimal solution of (ND).

Remarks 5.1: In view of the duality relationship between the variational problems (NVP) and (NVD) and method of their formulation we obtain the following results whose proof is simple:

Theorem 5.4: For the constrained games $G(X, Y, F)$, $\min_{x\in X} \max_{y\in Y} F(x, y)$ exists if and only if $\max_{y\in Y} \min_{x\in X} F(x, y)$, and when this happens,

$$\min_{x\in X} \max_{y\in Y} F(x, y) = \max_{y\in Y} \min_{x\in X} F(x, y)$$

5.4 Dynamic Game Equivalent Variational Problems with Natural Boundary

It is possible to formulate symmetric dual variational problems with natural boundary values rather than fixed end points.

(P): Minimize $\int_a^b f(t, x, \dot{x}, y, \dot{y})dt$

Subject to

$$f_y(\,(t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y(t, y, \dot{y})$$
$$-D\big(f_{\dot{y}}\,(t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y(t, y, \dot{y})\big) \ \leq \ 0, t \in \ I \ )$$
$$\int_a^b y(t)^T \Big[\Big(f_y(\,(t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y(t, y, \dot{y})\Big)$$
$$-D\big(f_{\dot{y}}\,(t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_y(t, y, \dot{y})\big)\Big]dt \ \geq \ 0,$$

$$\int_a^b \mu(t)^T q_y\,(t, y, \dot{y})\,dt \geq 0$$
$$x(t) \geq 0, \ \mu(t) \geq 0, \ p(\,(t, x, \dot{x}) \geq 0, \ \ t \in I$$
$$f_{\dot{y}}\,(t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_{\dot{y}}(t, y, \dot{y})\big|_{t=a} = 0$$
$$f_{\dot{y}}\,(t, x, \dot{x}, y, \dot{y}) - \mu(t)^T q_{\dot{y}}(t, y, \dot{y})\big|_{t=b} = 0$$

(D) Maximize $\int_a^b f(t, u, \dot{u}, v, \dot{v})dt$

Subject to

$$f_u(t, u, \dot{u}, v, \dot{v}) - \lambda(t)^T p_u(t, u, \dot{u})$$
$$-D\,\Big(f_{\dot{u}}\,(t, u, \dot{u}, v, \dot{v}) - \lambda(t)^T p_{\dot{u}}(t, u, \dot{u})\Big) \ \geq \ 0, t \in \ I \ )$$
$$\int_a^b u(t)^T \Big[\Big(f_u(t, u, \dot{u}, v, \dot{v}\,) - \lambda(t)^T p_y(t, u, \dot{u})\Big)$$

$$-\mathrm{D}\left(f_{\dot{y}}\,(t,u,\dot{u},v,\dot{v}\,) - \lambda\,(\mathrm{t})^{\mathrm{T}}\,\mathrm{p}_{\dot{u}}(t,u,\dot{u})\right)\Big]\,\mathrm{dt}\ \le\ 0,$$

$$\int_a^b \lambda(t)^T p\,(t,u,\dot{u})\,dt \le 0$$

$$v(t) \ge 0,\ \lambda(t) \ge 0,\ q(t,u,\dot{u}) \ge 0,\ \ t \in I$$

$$f_{\dot{u}}\,(t,u,\dot{u},v,\dot{v}\,) - \lambda\,(\mathrm{t})^{\mathrm{T}}\,p_{\dot{u}}(t,u,\dot{u})\Big|_{t=a} = 0$$

$$f_{\dot{u}}\,(t,u,\dot{u},v,\dot{v}\,) - \lambda\,(\mathrm{t})^{\mathrm{T}}\,p_{\dot{u}}(t,u,\dot{u})\Big|_{t=a} = 0$$

If only one end point in fixed, say $x(a) = 0$ and $y(a) = 0$ in (VP) and (VD), then the corresponding boundary value condition (5.1) and (5.6) are deleted. It can be easily seen that (VP) and (VD) are still symmetric and the Theorems 5.1-5.2 remain valid.

5.5 Static Game Equivalent Nonlinear Programming Problems

If all functions in the problems (P) and (D) are independent of t, the problem reduces to the following problems considered by Mond et al [61] as static game equivalent nonlinear programming problems:

(PS): Minimize $f(x,y)$,

Subject to $\left[f_y(x,y) - \mu^T q_y(y)\right] \le 0,$

$y^T\left[f_y(x,y) - \mu^T q_y(y)\right] \ge 0,$

$\mu^T q(y) \ge 0,$

$x \ge 0,\ \mu \ge 0,\ p_k(x) \ge 0, (k = 1,2,\dots,s)$

(ND): Maximize $f(u,v)$,

Subject to $[f_u(u,v) - \lambda^T p_u(u)] \ge 0,,$

$u^T[f_u(u,v) - \lambda^T p_u(u)] \le 0,$

$\lambda^T p(u) \le 0$

$v \ge 0,\ \lambda \ge 0,\ q_r(v) \le 0, (r = 1,2,\dots,t)$

5.6 Conclusions

In this exposition, the authors have discussed the equivalence between certain constrained dynamic game and a pair of symmetric dual variational problems which have more general formulations than those formulated by Mond and Hanson [63]. Usual duality results for the pair of variational problem are validated under appropriate generalized convexity assumptions. It is briefly indicated that dynamic game formulated in this research is equivalent to a pair of dual variational problems without the conditions of fixed points. When the functions occurring in the formulations of the problems without fixed point do not depend explicitly on *t*, the authors' result reduces to those of Chandra and Durga Prasad [13]. Further, the formulations of variational problems of this research can be revisited in setting of multiobjective dynamic games.

Chapter 6

MIXED TYPE SECOND-ORDER DUALITY FOR VARIATIONAL PROBLEMS

6.1 INTRODUCTION

Duality in continuous programming problem has been investigated by many authors. Hanson [30] pointed out that some of the duality results in nonlinear programming have the analogues in calculus of variations. Exploring this relationship of mathematical programming and classical calculus of variation, Mond and Hanson [63] formulated a constrained variational problem in abstract space and using Valentine [26] optimality conditions for the same, constructed its Wolfe type dual variational problem for proving duality results under usual convexity conditions. Later Bector, Chandra and Husain [4] studied Mond-Weir type duality for the problem of Mond and Hanson [63] for relaxing its convexity requirement for duality to hold.

In view of Mond's [8] remarks that the second-order dual for a nonlinear programming problem gives a tighter bound and has computational advantage over a first order dual, it is natural to find its analogue in continuous programming. Motivated with this observation ,Chen [84] formulated Wolfe type second order dual problem to the classical variational problem and studied usual duality results under invexity-like conditions on the function that appear in the formulation of the problem along with some strange and hard relations. Recently Husain et al [42] presented Mond-Weir type second-order dual to the variational problems considered in [84] and establish various duality theorems under second-order generalized invexity conditions. In [42], the relationship between second-order duality results in calculus of variation and their counterparts in nonlinear programming is also pointed out.

The concept of mixed type duality seems to be interesting and useful both from theoretical and algorithmic point of view. In this research, in sprit of Xu [85], a mixed second-order dual to the variational problem [84] to combine Wolfe type dual and Mond –Weir type dual problems is presented. A pair of mixed type dual variational problem with natural boundary values is formulated and the validation of its duality results in indicated. The formulation of natural boundary value problems is essential for seeing our results as having analogues in nonlinear programming and hence it is pointed out that our duality results can be viewed as dynamic generalizations of nonlinear programming already existing in the literature.

6.2 DEFINITIONS AND RELATED PRE-REQUISITES

Let $I=[a,b]$ be a real interval, $f: I\times R^n\times R^n \to R$ *and* $g: I\times R^n\times R^n \to R^m$ be twice continuously differentiable functions. In order to consider $f\left(t,x(t),\dot{x}(t)\right)$, where $x: I\to R^n$ is differentiable with derivative $\dot{x}$, denoted by f_x *and* $f_{\dot{x}}$ the partial derivative of f with respect to $x(t)$ *and* $\dot{x}(t)$, respectively, that is,

$$f_x=\begin{pmatrix}\dfrac{\partial f}{\partial x^1}\\ \dfrac{\partial f}{\partial x^2}\\ \vdots\\ \dfrac{\partial f}{\partial x^n}\end{pmatrix},\qquad f_{\dot{x}}=\begin{pmatrix}\dfrac{\partial f}{\partial \dot{x}^1}\\ \dfrac{\partial f}{\partial \dot{x}^2}\\ \vdots\\ \dfrac{\partial f}{\partial \dot{x}^n}\end{pmatrix};$$

denote by f_{xx} the Hessian matrix of f with respect to x, that is,

$$f_{xx} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} & \frac{\partial^2 f}{\partial x^1 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^1 \partial x^n} \\ \frac{\partial^2 f}{\partial x^2 \partial x^1} & \frac{\partial^2 f}{\partial x^2 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^2 \partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x^n \partial x^1} & \frac{\partial^2 f}{\partial x^n \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^n} \end{pmatrix}_{n\times n}$$

It is obvious that f_{xx} is a symmetric $n\times n$ matrix. Denote by g_x the $m\times n$ Jacobian matrix with respect to x, that is,

$$g_x = \begin{pmatrix} \frac{\partial g_1}{\partial x^1} & \frac{\partial g_1}{\partial x^2} & \cdots & \frac{\partial g_1}{\partial x^n} \\ \frac{\partial g_2}{\partial x^1} & \frac{\partial g_2}{\partial x^2} & \cdots & \frac{\partial g_2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x^1} & \frac{\partial g_m}{\partial x^2} & \cdots & \frac{\partial g_m}{\partial x^n} \end{pmatrix}_{m\times n}$$

Similarly $f_{\dot{x}}$, $f_{\dot{x}x}$, $f_{x\dot{x}}$ *and* $g_{\dot{x}}$ can be defined.

Denote by X, the space of piecewise smooth functions $x: I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\,ds ,$$

where α is given boundary value; thus $\frac{d}{dt} = D$ except at discontinuities.

We introduce the following definitions which are needed for duality results to hold.

DEFINITION 6.1 ***(Second-order Invexity):*** If there exists a vector function $\eta(t, x, \dot{x}) \in R^n$ where $\eta: I \times R^n \times R^n \to R^n$ and with $\eta = 0$ at t = *a* and t = *b*, such that for the functional $\int_I \phi(t, x, \dot{x})\,dt$ where $\phi: I \times R^n \times R^n \to R$ satisfies

$$\int_I \phi(t, x, \dot{x})\,dt - \int_I \left\{ \phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2} p(t)^T Gp(t) \right\} dt$$
$$\geq \int_I \left\{ \eta^T \phi_x(t, \bar{x}, \dot{\bar{x}}) + (D\eta)^T \phi_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \eta^T Gp(t) \right\} dt,$$

where $G = \phi_{xx} - D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}}$ and $p \in C(I, R^n)$, the space of continuous n-dimensional vector function.

DEFINITION 6.2 ***(Second-order Pseudoinvex):*** If the functional $\int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\int_I \left\{ \eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G p(t) \right\} dt \geq 0 \;\Rightarrow$$

$$\int_I \phi(t,x,\dot{x})\,dt \geq \int_I \left\{ \phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2} p(t)^T G p(t) \right\} dt,$$

then $\int_I \phi(t,x,\dot{x})\,dt$ is said to be second-order pseudoinvex with respect to η.

DEFINITION 6.3: (*Strictly Second- order Pseudoinvex:* If the functional $\int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\int_I \left\{ \eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G p(t) \right\} dt \geq 0,$$

$$\Rightarrow \int_I \phi(t,x,\dot{x})\,dt > \int_I \left\{ \phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2} p(t)^T G p(t) \right\} dt$$

DEFINITION 6.4*: (Second- order Quasi-invex):*If the functional $\int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\int_I \phi(t,x,\dot{x})\,dt \leq \int_I \left\{ \phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2} p(t)^T G p(t) \right\} dt \Rightarrow$$

$$\int_I \left\{ \eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G p(t) \right\} dt \leq 0,$$

then the functional $\int_I \phi(t,x,\dot{x})\,dt$ is said to be second-order quasi-invex with respect to η.

If ϕ is independent of t, then the above definition reduces to those given in [86].

Consider the following constrained variational problem

(VP): Minimize $\int_I f(t,x,\dot{x})\,dt$

subject to

$$x(a) = 0 = x(b) \tag{6.1}$$

$$g(t,x,\dot{x}) \leq 0\,,\; t \in I \tag{6.2}$$

where $f : I \times R^n \times R^n \to R$ and $g : I \times R^n \times R^n \to R^m$ are continuously differentiable.

The Fritz-John optimality conditions for the problem (VP) derived in [74] are given in the proposition below.

PROPOSITION 6.1 ([2] *Fritz-John Conditions*): If (VP) attains a local (or) global minimum at $x = \bar{x} \in X$ then there exist Lagrange multiplier $\tau \in R$ and piecewise smooth $y : I \to R^m$ such that

$$\tau f_x\left(t,\bar{x},\dot{\bar{x}}\right)+y(t)^T g_x\left(t,\bar{x},\dot{\bar{x}}\right)-D\left[f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+y(t)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right]=0, \quad t \in I,$$

$$y(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)=0, \quad t \in I,$$

$$\left(\tau, y(t)\right) \geq 0, \quad t \in I,$$

$$\left(\tau, y(t)\right) \neq 0, \quad t \in I.$$

The Fritz John necessary conditions for (VP), become the Karush-Kuhn-Tucker conditions if $\tau = 1$. If $\tau = 1$, the solution $\bar{x}$ is said to be normal.

Chen [84] presented the following Wolfe type dual to (VP) in the spirit of Mangasarian [56] and proved various duality results under somewhat strange invexity-like condition.

(WVD): Maximize $\int_I \left\{ f\left(t,u,\dot{u}\right)+\alpha(t)^T g\left(t,u,\dot{u}\right)-\frac{1}{2}p(t)^T H\left(t,u,\dot{u},\alpha(t)\right)p(t)\right\}dt$

Subject to

$$u(a)=0=u(b)$$

$$f_u\left(t,u,\dot{u}\right)+\alpha(t)^T g_u\left(t,u,\dot{u}\right)-D\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\alpha(t)^T g_{\dot{u}}\left(t,u,\dot{u}\right)\right)$$
$$+H(t)\,p(t)=0\,,\; t \in I$$

$$\alpha(t) \in R_+^m,\; p(t) \in R^n$$

where $$H = f_{uu}\left(t,u(t),\dot{u}(t)\right)+\left(y(t)^T g_u\left(t,u(t),\dot{u}(t)\right)\right)_u$$
$$-2D\left(f_{u\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(y(t)^T g_u\left(t,u(t),\dot{u}(t)\right)\right)_{\dot{u}}\right)$$
$$+D^2\left(f_{\dot{u}\dot{u}}\left(t,u(t),\dot{u}(t)\right)+\left(y(t)^T g_{\dot{u}}\left(t,u(t),\dot{u}(t)\right)\right)_{\dot{u}}\right).$$

It is remarked here that f and g are independent of t then (WVD) becomes second-order dual problem studied by Mond [8]. Recently Husain et al [42] presented the following Mond-Weir dual with the view to weaken the second order invexity requirements and proved duality theorems connecting the problems (CP) and (CD) under generalized second order invexity hypothesis.

(CD): Maximize $\int_I \left\{ f\left(t,u,\dot{u}\right)-\frac{1}{2}p(t)^T F(t)\,p(t)\right\}dt$

Subject to

$$u(a)=0=u(b)$$

$$f_u + y(t)^T g_u - D\left(f_{\dot{u}} + y(t)^T g_{\dot{u}} \right) + \left(F(t) + G(t)\right) p(t) = 0 \text{ , } t \in I$$

$$\int_I \left\{ y(t)^T g(t,u,\dot{u}) - \frac{1}{2} p(t)^T G(t)\, p(t) \right\} dt \geq 0 \text{ ,}$$

$$y(t) \geq 0.$$

where $F(t) = f_{uu} - Df_{u\dot{u}} + D^2 f_{\dot{u}\dot{u}}$ and $G(t) = \left(y(t)^T g_u \right)_u + D\left(y(t)^T g_u \right)_{\dot{u}} + D^2 \left(y(t)^T g_{\dot{u}} \right)_{\dot{u}}$

where D is defined as earlier.

6.3. Mixed Type second order Duality

In this section we construct a mixed type second-order dual model for the variational problem (VP):

(Mix VD): Maximize $\int_I \left\{ f(t,u,\dot{u}) + \sum_{i \in I_\circ} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T H^\circ(t,u,\dot{u},y)\, p(t) \right\} dt$

subject to

$$u(a) = 0 = u(b) \qquad (6.3)$$

$$f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) - D\left(f_{\dot{u}} + y(t)^T g_{\dot{u}} \right)$$
$$+ H(t)\, p(t) = 0 \text{ , } t \in I \qquad (6.4)$$

$$\int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T G(\alpha,t)\, p(t) \right\} dt \geq 0 \text{ , } \quad \alpha = 1,2,\ldots,r \qquad (6.5)$$

$$y(t) \geq 0, \quad t \in I \quad , \quad p(t) \in R^n \qquad (6.6)$$

where

(i) $$H^\circ(t) = f_{uu} + \sum_{i \in I_0} \left(y^i(t) g_u^i(t,u,\dot{u}) \right)_u - D\left(f_{u\dot{u}} + \sum_{i \in I_0} \left(y^i(t) g_u^i(t,u,\dot{u}) \right)_{\dot{u}} \right)$$
$$+ D^2 \left(f_{\dot{u}\dot{u}} + \sum_{i \in I_0} \left(y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) \right)_{\dot{u}} \right)$$

(ii) $$G(\alpha,t) = \sum_{i \in I_\alpha} \left(y^i(t) g_u^i(t,u,\dot{u}) \right)_u - D \sum_{i \in I_\alpha} \left(y^i(t) g_u^i(t,u,\dot{u}) \right)_{\dot{u}} + D^2 \sum_{i \in I_\alpha} \left(y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) \right)_{\dot{u}}$$

and

(iii) $I_\alpha \subset M = \{1,2,3,\ldots m\}, \alpha = 0,1,2,\ldots,r$ with $M = \bigcup_{\alpha=0} I_\alpha$ and $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$.

We present the following duality theorems for the pair of dual problems (VP) and (Mix VD).

THEOREM 6.1. (Weak duality): Let $x(t) \in X$ be a feasible solution of (VP) and $(u(t), y(t), p(t))$ be feasible solution of (MixVD).If for all feasible $(x(t), u(t), y(t), p(t))$, $\int_I \left(f(t,.,.) + \sum_{i \in I_0} \left(y^i(t) g^i(t,.,.) \right) \right) dt$ be second-order pseudoinvex and $\sum_{i \in I_\alpha} \int_I \left(y^i(t) g^i(t,.,.) \right) dt$ be second-order quasi-invex with respect to the same $\eta : I \times R^n \times R^n \to R^n$ satisfying η =0 at $t = a$ and $t = b$, then

$$\int_I f(t,x,\dot{x}) dt \geq \int_I \left\{ f(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T H \ (t,u,\dot{u},y) p(t) \right\} dt$$

PROOF: The relations $g(t,x,\dot{x}) \leq 0$, $t \in I$ and $y(t) \geq 0$, yield

$$\int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,u,\dot{u}) \right\} dt \leq 0, \qquad \alpha = 1,2,...r$$

This together with (6.5) implies

$$\int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,x,\dot{x}) \right\} dt \leq \int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T G(\alpha,t) p(t) \right\} dt \quad , \ \alpha = 1,2,...r$$

This, because of second-order quasi-invexity of $\int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,\cdot,\cdot) \right\} dt$, $\alpha = 1,2,...r$, gives

$$0 \geq \int_I \left\{ \eta^T \left(\sum_{i \in I_\alpha} y^i(t) g_u^i \right) + (D\eta)^T \left(\sum_{i \in I_\alpha} y^i(t) g_{\dot{u}}^i \right) + \eta^T G(\alpha,t) p(t) \right\} dt$$

$$= \int_I \eta^T \left\{ \sum_{i \in I_\alpha} y^i(t) g_u^i - D \sum_{i \in I_\alpha} y^i(t) g_{\dot{u}}^i + G(\alpha,t) p(t) \right\} dt$$

$$+ \eta \sum_{i \in I_\alpha} y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) \Big|_{t=a}^{t=b}$$

(by integration by parts)

Using $\eta(t,u,\dot{u}) \ \Big|_{t=a}^{t=b} = 0$,we have,

$$\int_I \eta^T \left\{ \sum_{i \in I_\alpha} y^i(t) g_u^i(t,u,\dot{u}) - D \sum_{i \in I_\alpha} y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) + G(\alpha,t) p(t) \right\} dt \leq 0, \ \alpha = 1,2,...r$$

Hence,

$$\int_I \eta^T \left\{ \sum_{M-I_0} y^i(t) g_u^i(t,u,\dot{u}) - D \sum_{M-I_0} y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) + G(\alpha,t) p(t) \right\} dt \leq 0$$

By (6.4), this yields

$$\int_I \eta^T \left\{\left(f_u\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right)g_u^i\left(t,u,\dot{u}\right)\right)-D\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right)g_u^i\left(t,u,\dot{u}\right)\right)\right.$$
$$\left. +H^{\circ}\left(t\right)p\left(t\right)\right\}dt\geq 0$$

Integrating by parts, this gives,

$$0\leq \int_I \left\{\eta^T\left(f_u\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right)g_u^i\left(t,u,\dot{u}\right)\right)+\left(D\eta\right)^T\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right)g_u^i\left(t,u,\dot{u}\right)\right)\right.$$
$$\left. +\eta^T H^0\left(t\right)p\left(t\right)\right\}dt-\eta^T\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right)g_u^i\left(t,u,\dot{u}\right)\right)\Bigg|_{t=a}^{t=b}$$

Using $\eta=0$ at $t=a$ and $t=b$ in the above inequality, we obtain,

$$\int_I \eta^T \left\{\left(f_u\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right)g_u^i\left(t,u,\dot{u}\right)\right)+\left(D\eta\right)^T\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right)g_u^i\left(t,u,\dot{u}\right)\right)\right.$$
$$\left. +\eta^T H^0\left(t\right)p\left(t\right)\right\}dt\geq 0$$

This, in view of second order invexity of $\int_I\left\{ f\left(t,.,.\right)+\sum_{i\in I_0} y^i(t)g^i\left(t,.,.\right)\right\}dt$ with respect to η gives

$$\int_I\left\{ f\left(t,x,\dot{x}\right)+\sum_{i\in I_0} y^i(t)g^i\left(t,x,\dot{x}\right)\right\}dt\geq \int_I\left\{ f\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i(t)g^i\left(t,u,\dot{u}\right)-\frac{1}{2}p\left(t\right)^T H^0\left(t\right)p\left(t\right)\right\}dt \quad (6.7)$$

Since $y\left(t\right)\geq 0, t\in I$ and $g\left(t,x,\dot{x}\right)\leq 0$, $t\in I$ yielding $\sum_{i\in I_0} y^i(t)g^i\left(t,x,\dot{x}\right)\leq 0, t\in I$, (6.7) gives

$$\int_I f\left(t,x,\dot{x}\right)dt\geq \int_I\left\{ f\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i(t)g^i\left(t,u,\dot{u}\right)-\frac{1}{2}p\left(t\right)^T H^0\left(t\right)p\left(t\right)\right\}dt .$$

THEOREM 6.2. (Strong Duality): If $\bar{x}\in X$ is an optimal solution of (VP) and meets the normality condition, then there exists a piece wise smooth factor $\bar{y}:I\rightarrow R^m$ such that $\left(\bar{x}\left(t\right),\bar{y}\left(t\right),\bar{p}\left(t\right)=0\right)$, $t\in I$ is a feasible for (MixVD) and the two objective values are equal. Furthermore, if the hypothesis of Theorem 6.1 holds, then $\left(\bar{x}\left(t\right),\bar{y}\left(t\right),\bar{p}\left(t\right)\right)$ is optimal for

(Mix VD).

PROOF: From Proposition 1 [74], there exists a piecewise smooth function $\bar{y}:I\rightarrow R^m$ satisfying the following conditions:

$$\left(f_x\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{y}\left(t\right)^T g_x\left(t,\bar{x},\dot{\bar{x}}\right)\right)-D\left(f_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)+\bar{y}\left(t\right)^T g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\right)$$

$$+H\left(t\right)\bar{p}\left(t\right)=0, t\in I \text{ with } \bar{p}\left(t\right)=0$$

$$\bar{y}(t)^T g\left(t,\bar{x},\dot{\bar{x}}\right)=0, \ \bar{y}(t)\geq 0, \quad t\in I$$

The last relation implies,

$$\sum_{i\in I_\circ}\bar{y}^i\left(t\right)g^i\left(t,\bar{x},\dot{\bar{x}}\right)=0=\sum_{i\in I_\alpha}\bar{y}^i\left(t\right)g^i\left(t,\bar{x},\dot{\bar{x}}\right), \ \alpha=1,2,...,r$$

$$\int_I\left\{\sum_{i\in I_\alpha}\bar{y}^i\left(t\right)^T g^i\left(t,\bar{x},\dot{\bar{x}}\right)-\frac{1}{2}p\left(t\right)^T G\left(\alpha,t\right)p\left(t\right)\right\}dt=0, \ \alpha=1,2,...r \text{ with } p\left(t\right)=0$$

From the above relation it implies that $\left(\bar{x}\left(t\right),\bar{y}\left(t\right),p\left(t\right)=0\right)$ is feasible for (MixVD).

In view of $\sum_{i\in I_0}\bar{y}^i\left(t\right)^T g^i\left(t,\bar{x},\dot{\bar{x}}\right)=0, \ t\in I,$ and $p\left(t\right)=0, t\in I$, we have,

$$\int_I f\left(t,x,\dot{x}\right)dt=\int_I\left\{f\left(t,x,\dot{x}\right)+\sum_{i\in I_0}y^i(t)g^i\left(t,x,\dot{x}\right)-\frac{1}{2}p\left(t\right)^T H^\circ\left(t\right)p\left(t\right)\right\}dt$$

Furthermore, for every feasible solution $\left(u\left(t\right),y\left(t\right),p\left(t\right)\right)$, from the condition we have,

$$\int_I\left\{f\left(t,\bar{x},\dot{\bar{x}}\right)+\sum_{i\in I_\circ}y^i(t)g^i\left(t,\bar{x},\dot{\bar{x}}\right)-\frac{1}{2}p\left(t\right)^T H^\circ\left(t\right)p\left(t\right)\right\}dt$$
$$=\int_I f\left(t,\bar{x},\dot{\bar{x}}\right)dt$$
$$\geq\int_I\left\{f\left(t,u\left(t\right),\dot{u}\left(t\right)\right)+\sum_{i\in I_\circ}y^i(t)g^i\left(t,u\left(t\right),\dot{u}\left(t\right)\right)-\frac{1}{2}p\left(t\right)^T H^\circ\left(t\right)p\left(t\right)\right\}dt$$

So, $\left(\bar{x}\left(t\right),\bar{y}\left(t\right),\bar{p}\left(t\right)\right)$ is also an optimal solution of (Mix VD).

The theorem given below is the Mangasarian [56] type converse duality theorem:

THEOREM 6.3 (Strict Converse duality): Let $\bar{x}$ be optimal solution of (VP) and normal. If $\left(\hat{u},\hat{y},\hat{p}\right)$ is an optimal solution to (Mix VD) and if $\int_I\left\{f\left(t,.,.\right)+\sum_{i\in I_\circ}y^i(t)g^i\left(t,.,.\right)\right\}dt$ is second order strict pseudoinvex and $\sum_{i\in I_\alpha}\int_I\hat{y}^i(t)g^i\left(t,.,.\right)dt$, $\left(\alpha=1,2,...,r\right)$ is a second-order quasi-invex with respect to $\eta=\eta\left(t,\bar{x},\hat{u}\right)$, then $\bar{x}=\hat{u}$ i.e., $\hat{u}$ is an optimal solution of (VP).

Proof: We assume that $x(t) \neq \hat{u}(t)$, $t \in I$ and show that the contradiction occurs. Since $\bar{x}$ is an optimal solution of (VP) and normal, it follows from Theorem 6.2 that there exists piecewise smooth functions $\bar{y}: R \to R^m$ with $\bar{y}(t) = \left(\bar{y}^1(t)\bar{y}^2(t),...,\bar{y}^m(t)\right)^T$ such that $\left(\bar{x}(t), \bar{y}(t), \bar{p}(t)\right)$ is optimal for

(MixVD) and $\int_I \left\{ f\left(t,\bar{x},\dot{\bar{x}}\right)\right\} dt = \int_I \left\{ f\left(t,\bar{x},\dot{\bar{x}}\right) + \sum_{i \in I_\circ} \bar{y}^i(t) g^i\left(t,\bar{x},\dot{\bar{x}}\right) - \frac{1}{2}\bar{p}(t)^T H^\circ(t)\bar{p}(t) \right\} dt$

$$= \int_I \left\{ f\left(t,\hat{u},\dot{\hat{u}}\right) + \sum_{i \in I_\circ} \hat{y}^i(t) g^i\left(t,\hat{u},\dot{\hat{u}}\right) - \frac{1}{2}\bar{p}(t)^T H^\circ(t)\bar{p}(t) \right\} dt \tag{6.8}$$

Since $\bar{x}(t)$ is feasible for (VD) and $\left(\hat{u}(t), \hat{y}(t), \hat{p}(t)\right)$ is feasible for (Mix VD), we have

$$\sum_{i \in I_\alpha} \int_I \hat{y}^i(t) g^i\left(t,\bar{x},\dot{\bar{x}}\right) dt \leq 0, \ \ \alpha = 1,2,...,r$$

This, together with the feasibility $\left(\hat{u}(t), \hat{y}(t), \hat{p}(t)\right)$ for the dual problem (Mix VD)

$$\sum_{i \in I_\alpha} \int_I \hat{y}^i(t) g^i\left(t,\bar{x},\dot{\bar{x}}\right) dt \leq \int_I \left\{ \sum_{i \in I_\alpha} \hat{y}^i(t) g^i\left(t,\hat{u},\dot{\hat{u}}\right) - \frac{1}{2}\hat{p}(t)^T G(\alpha,t)\hat{p}(t) \right\} dt \ , \ (\alpha = 1,2,...r)$$

This, in view of second-order quasi-invexity of $\sum_{i \in I_\alpha} \int_I y^i(t) g^i(t,.,.) dt, \ \ (\alpha = 1,2,...r)$ gives

$$\int_I \left\{ \eta^T \sum_{i \in I_\alpha} \hat{y}^i(t) g^i_{\hat{u}}\left(t,\hat{u},\dot{\hat{u}}\right) + (D\eta)^T \left(\sum_{i \in I_\alpha} \hat{y}^i(t) g^i_{\dot{\hat{u}}}\left(t,\hat{u},\dot{\hat{u}}\right) \right) + \eta^T G(\alpha,t)\hat{p}(t) \right\} dt \leq 0.$$

This, by integration by parts, gives,

$$0 \geq \int_I \left\{ \eta^T \sum_{i \in I_\alpha} \hat{y}^i(t) g^i_{\hat{u}}\left(t,\hat{u},\dot{\hat{u}}\right) + (D\eta)^T \left(\sum_{i \in I_\alpha} \hat{y}^i(t) g^i_{\dot{\hat{u}}}\left(t,\hat{u},\dot{\hat{u}}\right) \right) + \eta^T G(\alpha,t)\hat{p}(t) \right\} dt$$
$$+ \eta^T \sum_{i \in I_\alpha} \hat{y}^i(t) g^i_{\hat{u}}\left(t,\hat{u},\dot{\hat{u}}\right) \Bigg|_{t=a}^{t=b}$$

Using $\eta\Big|_{t=a}^{t=b} = 0$, this gives,

$$\int_I \eta^T \left\{ \sum_{i \in I_\alpha} \hat{y}^i(t) g^i_{\hat{u}}\left(t,\hat{u},\dot{\hat{u}}\right) + D\left(\sum_{i \in I_\alpha} \hat{y}^i(t) g^i_{\dot{\hat{u}}}\left(t,\hat{u},\dot{\hat{u}}\right) \right) + G(\alpha,t)\hat{p}(t) \right\} dt \leq 0$$

From (6.4) we have,

$$\int_I \eta^T \left\{ f_{\hat{u}} + \sum_{i \in I_\circ} \hat{y}^i(t) g^i_{\hat{u}}\left(t,\hat{u},\dot{\hat{u}}\right) + D\left(f_{\dot{\hat{u}}} + \sum_{i \in I_\circ} \hat{y}^i(t) g^i_{\dot{\hat{u}}}\left(t,\hat{u},\dot{\hat{u}}\right) \right) + H^\circ(t)\hat{p}(t) \right\} dt \geq 0$$

This inequality, by integration by parts, gives,

$$\int_I \left\{ \eta^T \left(f_{\hat{u}} + \sum_{i\in I_\circ} \hat{y}^i(t) g_{\hat{u}}^i\left(t,\hat{u},\dot{\hat{u}}\right) \right) + \left(D\eta\right)^T \left(f_{\dot{\hat{u}}} + \sum_{i\in I_\circ} \hat{y}^i(t) g_{\dot{\hat{u}}}^i\left(t,\hat{u},\dot{\hat{u}}\right) \right) + \eta^T H^\circ\left(t\right)\hat{p}\left(t\right) \right\} dt \geq 0$$

which in view of second-order strict pseudoinvexity of $\int_I \left\{ f\left(t,.,.\right) + \sum_{i\in I_0} \hat{y}^i(t) g^i\left(t,.,.\right) \right\} dt$ gives

$$\int_I \left\{ f\left(t,\bar{x},\dot{\bar{x}}\right) + \sum_{i\in I_0} \hat{y}^i(t) g^i\left(t,\bar{x},\dot{\bar{x}}\right) \right\} dt > \int_I \left\{ f\left(t,\hat{u},\dot{\hat{u}}\right) + \sum_{i\in I_0} \hat{y}^i(t) g^i\left(t,\hat{u},\dot{\hat{u}}\right) - \frac{1}{2}\hat{p}\left(t\right)^T H^\circ\left(t\right)\hat{p}\left(t\right) \right\} dt$$

Using $\sum_{i\in I_0} \hat{y}^i(t) g^i\left(t,\bar{x},\dot{\bar{x}}\right) \leq 0$, $t \in I$ this yields

$$\int_I \left\{ f\left(t,\bar{x},\dot{\bar{x}}\right) \right\} dt > \int_I \left\{ f\left(t,\hat{u},\dot{\hat{u}}\right) + \sum_{i\in I_0} \hat{y}^i(t) g^i\left(t,\hat{u},\dot{\hat{u}}\right) - \frac{1}{2}\hat{p}\left(t\right)^T H^0\left(t\right)\hat{p}\left(t\right) \right\} dt$$

This contradicts the relation (6.7).Hence $\bar{x}\left(t\right) = \hat{u}\left(t\right), t \in I$ i.e $\hat{u}\left(t\right)$is optimal solution of (VP).

6.4. SPECIAL CASES

If I_α is empty for each $\alpha \in 1,2,...r$, then $H^\circ\left(t\right) = H\left(t\right)$(MixVD) reduces to the following Wolfe type second-order dual variational problem treated by Chen[84].

If $I_\circ$ is empty, then (MixVD) reduces to the following Mond-weir type second-order dual variational problem recently treated by Husain et al [42]

6.5. NATURAL BOUNDARY VALUES

In this section, we present dual variational problem with natural boundary values rather than fixed end points.

(VP$_0$): Minimize $\int_I f\left(t,x,\dot{x}\right) dt$

subject to

$$g\left(t,x,\dot{x}\right) \leq 0, \qquad t \in I$$

(MixVD$_0$): Maximize $\int_I \left\{ f\left(t,u,\dot{u}\right) + \sum_{i\in I_0} y^i(t) g^i\left(t,u,\dot{u}\right) - \frac{1}{2} p\left(t\right)^T H^0\left(t\right) p\left(t\right) \right\} dt$

subject to

$$f_u\left(t,u,\dot{u}\right) + y\left(t\right)^T g_u\left(t,u,\dot{u}\right) - D\left(f_{\dot{u}} + y\left(t\right)^T g_{\dot{u}} \right)$$
$$+ H\left(t\right) p\left(t\right) = 0 \text{ , } t \in I$$

$$\int_I \left\{ \sum_{i\in I_\alpha} y^i(t) g^i\left(t,u,\dot{u}\right) - \frac{1}{2} p\left(t\right)^T G\left(\alpha,t\right) p\left(t\right) \right\} dt \geq 0$$

$$\left(f_{\dot{u}}\left(t,u,\dot{u}\right)+\sum_{i\in I_0} y^i\left(t\right) g_u^i\left(t,u,\dot{u}\right)\right) \Big|_{t=a}^{t=b} = 0$$

$$\sum_{i\in I_\alpha} y^i\left(t\right) g_u^i\left(t,u,\dot{u}\right)\Bigg|_{t=a}^{t=b} = 0 \quad , \qquad \alpha = 1,2,\ldots$$

6.6 MIXED TYPE NONLINEAR PROGRAMMING PROBLEM

If all the functions are independent of t, then we have following pair of problems treated in Zhang and Mond [86] except that square root of a quadratic form is to be omitted from the expression of the problems.

(VPo): Minimize $f\left(x\right)$

subject to

$$g\left(x\right) \le 0$$

(Mix CDo): Maximize $f\left(u\right)+\sum_{i\in I_\circ} y^i g^i\left(u\right) - \frac{1}{2} p\left(t\right)^T \left[\nabla^2 f\left(u\right)+\sum_{i\in I_\circ} y^i g^i\left(u\right)\right] p$

subject to

$$\nabla\left(f\left(u\right)+\sum_{i\in I_\circ} y^i g^i\left(u\right)\right)+\nabla^2\left(f\left(u\right)+\sum_{i\in I_\circ} y^i g^i\left(u\right)\right) p = 0$$

$$\sum_{i\in I_\alpha} y^i g^i\left(u\right) - \frac{1}{2} p^T \left[\sum_{i\in I_\alpha} y^i\left(u\right) g^i\left(u\right)\right] p \ge 0$$

$$y \ge 0$$

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