Quantum Chaos And Transition Densities In Atomic Nuclei

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Asloob Ahmad Rather

Dedicated

to

My Parents

and

Brothers

Khalid Mehmood Rather

and

Aasif Mehmood Rather

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Chapter 1

General Introduction

Chaotic behaviour is ubiquitous in nature and plays an important role in most fields of science. In classical physics, chaos is characterized by hypersensitivity of the time evolution of a system from initial conditions. Quantum mechanics does not permit a similar definition owing in part to the uncertainty principle and in part to the Schrodinger equation, which preserves the overlap between quantum states. This fundamental disconnect poses a serious challenge to quantum-classical correspondence and has motivated a long-standing search for quantum signatures of classical chaos.

In classical mechanics, the state of a physical system is specified by a set of dynamical variables, for example, the position and momentum of a point particle, whose values define a point in phase space. Regular motion is associated with periodic orbits in phase space, whereas chaos is characterized by complex, aperiodic trajectories that diverge exponentially as a function of initial separation. This description of states and time evolution is fundamentally incompatible with quantum mechanics, where conjugate observables such as position and momentum cannot take on well-defined values at the same time. However, it is still possible to represent a quantum state in phase space in the form of a delocalized quasi-probability distribution whose evolution is governed by the Schrodinger equation. This suggests an experiment in which one prepares an initial minimum uncertainty state centred on a point in phase space, subjects it to a desired evolution, measures the quantum state at successive points in time and observes the degree to which the dynamically evolving quantum phase space distribution reflects the classical phase space structures. Experiments of this type can be simulated with classical waves,

but are very challenging for true quantum systems because of the overhead involved in state preparation, control and reconstruction.

The point of concern here in this dissertation, is how to study quantum chaos in quantum many-body systems like atomic nuclei. The appropriate model to study chaos in these systems is the random matrix theory and the reason for modelling the systems in this manner is that as excitation energy increases, the many-body level density grows exponentially by pure combinatorial reasons and this global behaviour is not changed qualitatively by the interaction between the particles. In other-words, the manifestation of chaos in nuclei are described in terms of Wigner-Dyson random matrix theory for level and strength fluctuations, i.e., in terms of Gaussian orthogonal ensemble of random matrices and its extensions [1, 2, 3]. The assumption that is made here as per Wigner, " the Hamiltonian which governs the behaviour of a complicated system is a random symmetric matrix, with no special properties except for its symmetric nature". Going beyond this as established by Bohigas and Berry [4, 5, 6] and summarized by Altshuler in the abstract of the colloquium he gave in memory of French at the university of Rochester in 2004; " Classical dynamical systems can be separated into two classes- integrable and chaotic. For quantum systems this distinction manifests itself, e.g. in spectral statistics. Roughly speaking integrability leads to Poisson distribution for the energies while chaos implies Wigner-Dyson statistics of levels, which are characteristic for the ensemble of random matrices. The onset of chaotic behaviour for a rather broad class of systems can be understood as a delocalization of quantum numbers that characterize the original integrable system.....".

It is now a well established fact that neutron resonance spacings in heavy and medium heavy nuclei follow GOE, regular rotational levels are Poisson, excited 2^+ levels in even-even nuclei obey intermediate statistics. However, a more general random matrix theory that describes not only the fluctuations, but also spectral averages or global (smoothed with respect to energy) quantities such as level densities, single-particle orbit occupation probabilities, Gamow-Teller matrix elements etc. is based on embedded Gaussian orthogonal ensemble of one plus two-body interactions [EGOE(1+2)] where the two-body interaction is treated as random, subjected to some symmetries. By treating the two-body interaction to be random in nuclear shell model spaces, the embedded ensembles provide the basis for statistical nuclear spectroscopy [7] with applications in nuclear as-

trophysics. Statistical nuclear spectroscopy theory has been applied sucessfully to level densities [8, 9, 10, 11] and occupancies [12, 13]. In addition, supplementing statistical spectroscopy theory with empirical data for low-lying levels, gives a good method for calculating nuclear structure inputs for nuclear astrophysical processes [14, 15].

Quantum mechanical study of classically chaotic systems is the subject matter of quantum chaos [16, 17]. A major challenge of quantum chaos is to identify quantum signatures of classical chaos. Various signatures have been identified, such as the spectral properties of the generating Hamiltonian [4], phase space scarring [18], hypersensitivity to perturbation [19], and fidelity decay [20], which indicate presence of chaos in underlying classical system. Recent studies have shown that entanglement in chaotic systems can also be a good indicator of the regular to chaotic transition in its classical counterpart [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] . A study of the connections between chaos and entanglement is interesting because the two phenomena are prima facie uniquely classical and quantum, respectively. This is definitely an important reason to study entanglement in chaotic systems. Moreover, presence of chaos has also been identified in some realistic model of quantum computers [34, 35].

The transition strength $R(E_i, E_f)$ from the initial state $|E_i>$ at energy $|E_i>$ to a final state $|E_f>$ is defined as $R(E_i, E_f) = |<E_f|\hat{O}|E_i>|^2 = <E_i|\hat{O}^\dagger|E_f><E_f|\hat{O}|E_i>$ where $\hat{O}$ is the transition or excitation operator. Depending upon the nature of $\hat{O}$, the two states involved may be in the same space, as e.g, for the electromagnetic transitions like E_2, M_1 etc. between states of the same (J^π, T); or they may be in two spaces with different particle number, as for a one-nuclear transfer reaction in which the initial state is in a nucleus with A nucleons and the final state in one with $(A\pm1)$ nucleons, for example the beta decay operator. In either case, the strength function is a function of both the initial and final state. The transition strength sum is defined as follows. Given $K = O^\dagger O$, the transition strength sum is given by the expectation value $<K>^E$, and can be written in terms of the expectation value density

$$<K>^E = <\hat{O}^\dagger\hat{O}> = [d\rho(E)]^{-1}\left[\sum_{\alpha\in E}<E\alpha|K|E\alpha>\right] = \frac{I_K(E)}{I(E)} = \frac{\rho_K(E)}{\rho(E)}$$

The chaos and complexity measures like number of principal components and localization length in wave-functions and transition strength distributions are used to study chaos

in atomic nuclei. However, these measures are of considerable importance for the reason that transition strengths are observables while wavefunctions are not. The nuclear shell model has proven to be a very valuable tool and is a testing laboratory for understanding various aspects of chaos in atomic nuclei [2, 36, 37, 38, 39, 40]. It has been established from the study of embedded Gaussian orthogonal ensemble of random matrices, the strength sum generated by a transition operator acting on an eigenstate vary with the excitation energy as the ratio of two Gaussians and this general result when compared to the exact-shell model calculations of Gamow-Teller strength sums in nuclei, a good agreement is is obtained in the chaotic domain of the spectrum and strong deviations are observed as nuclear motion approaches a regular regime [41]. Further, from the study of the shell model results, the electric quadrupole (E_2), magnetic dipole(M_1) and occupation numbers when calculated using different valence spaces and compared to the EGOE predictions, the transition strength sums emerge as a new kind of statistics capable of distinguishing between regular and chaotic motion [42]. In addition to this, established is the fact that the EGOE and not GOE provides the reasonable description of the shell-model strength sums in the chaotic domain and in order to arrive at this result the study of behaviour of strength sums had been studied in order to chaos transitions generated by means of a family of Hamiltonians $H(\lambda) = h(1) + \lambda V(2)$, built from the realistic one- and two-body interactions [42]. Comparison of the predictions of EGOE of one-plus two body interactions, in the Gaussian domain for the complexity and chaos measures number of principal components and localization length in transition strengths from an eigenstate with energy E with the shell model calculations had been found to be quite consistent with the E_2 and M_1 transition strengths from the shell model example of 2p-1f shell nucleus ^{46}V [43].

This dissertation is organized as follows:
In chapter 2, we describe the random matrix theory, a very valuable tool to study quantum chaos in atomic nuclei. The motivation behind the introduction of RMT in nuclear structure can be thought of as two-fold. Firstly, as is well established from the empirical evidences that the nuclear models are inadequate at higher excitation energies for probing the individual nuclear energy levels because of the reason that the many-body level density increases exponentially with the increase in excitation energy and it becomes next to impossible on part of the nuclear models to provide the individual description of nuclear energy levels. So, the choice left is to resort to some statistical approach, that is RMT.

Second reason that can be cited about the introduction of RMT is due to the overwhelming success of Bohr's compound nucleus around 1940's and the quest to derive information about level and strength fluctuations about compound nuclear resonances. The chaos and complexity measures in wave-function and transition strength distributions like number of principal components and localization length along-with the transition strength sums are described in chapter 3. The re-derivation of the formulas for number of principal components and information entropy is also described in the chapter 3, supplemented by the already obtained results and comparison of random matrix results with the shell model results is also described. In chapter 4, laws of statistical nuclear spectroscopy along with the moments of distribution, the level density formula is described. The different polynomial density expansions like Edgeworth, Gram-Charlier and Cornish-Fischer expansion shall also be discussed along with their domain of validity in nuclear statistical spectroscopy in chapter 4. In this chapter 4 the distribution of eigen-values and transition strengths shall be also covered. Finally, the last chapter 5 gives the summary of the titled work.

Chapter 2

Random Matrix Theory

2.1 Introduction

The subject of Random Matrix Theory (RMT) has matured into an independent field with far-reaching applications in many branches of Physics and Mathematics. A large number of Physicists and Mathematicians have been fascinated by it that has led to major advances in this area. Paraphrasing, J. Freeman Dyson, RMT is a new kind of statistical mechanics where the realisation of the system is not relevant. Instead of having an ensemble of states of a system, we have in RMT an ensemble of Hamiltonians and ergodicity is the equivalence of spectral averaging and the averaging over this ensemble. RMT has been applied to a huge number of fields like multivariate statistics, combinatorics, graph theory, number theory, biology, genomics, wireless communications [44] and of course physics [45]. Figure below shows the diverse applications of random matrix theory. The

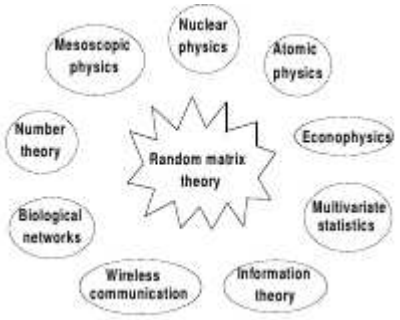

Figure 2.1: Representation of wide range applicability of RMT.

basic idea of RMT is to presume that the unobtainable Hamiltonian matrix of a system

under inspection is one of some ensemble of possible Hamiltonian matrices. Now, instead of trying to find the specific Hamiltonian in question, one studies the properties of this ensemble of matrices, hoping that these properties will be the same as, or at least close to, the properties of the specific but unknown Hamiltonian matrix of the system in question. It is clear that the success of the RMT approach lies in the right choice of ensemble of matrices-right meaning that the ensemble be chosen in such a way that the Hamiltonian matrix of the system in question, and those of similar systems, be in a sense the most probable ones in the ensemble. The first to approach a problem in this way was Wigner, and it is his Gaussian ensembles that form the entry point to RMT. This tool has been particularly successsful in three areas: first in describing level correlations on the scale of average level spacing; second in providing the generating functions for the combinatorial factorials of planar diagrams; and third as an exactly solvable model with intimate relationship to the theory of integrable systems. One of the several reasons for the success of RMT is its universality, i.e., eigenvalue correlations on the scale of average level spacing do not depend on the probability distribution, a property which is at the very foundation of RMT. Thus, it suggests that in RMT eigenvalue correlations should be a rule rather than the suggestion. Hence, the most important reason for studying RMT is that the predictions made by it do occur in systems like nuclear energy levels, zeros of Riemann Zeta function (ζ) and the sound waves in quartz crystals. Another important role played by the RMT is that the large N limit of its partition function is a generating function for planar diagrams which have played an important role in quantum field theory. For example, they are the leading contributions to Quantum Chromodynamics with a large number of colors, and they are dual to triangulation of a random surface and thus decribes two dimensional quantum gravity.

In addition, random matrix theory has attracted a great deal of attention because of the mathematical challenges it poses. The subject matter of random matrix theory is highly non-trivial, but with sufficient effort, most of the problems that arise in this field can be answered in detail. Nowadays, RMT is considered by the physics community as some sort of new statistical mechanics that can be successfully applied to describe generic statistical properties of very different systems, like atomic nuclei, complex atoms and molecules, disordered systems, one-dimensional interacting fermion systems, QCD and quantum gravity. A comprehensive review of the most important concepts and developments of RMT in quantum physics was given recently by Guhr et. al [45].

The random matrix approach does not aim at calculating individual spectra and at comparing them with data. Rather, one determines the combined probability distribution of the eigenvalues and from here calculates certain spectral fluctuation measures such as nearest-neighbour spacing distribution as averages over the ensemble. The random matrix approach to spectral fluctuations and to other properties of complex systems has some similarity to classical thermodynamics. There in classical thermodynamics one is interested in the generic description of systems in terms of few parameters like specific heat, magnetic susceptibility etc., but all these parameters are system-specific and in classical thermodynamics they need not be determined from the system's Hamiltonian and in this respect RMT and classical thermodynamics are phenomenological theories that do not refer to an underlying system-specific Hamiltonian. Remarkably, RMT appeared only a few years after the introduction of the nuclear shell model [46, 47]. In its simplest form the shell model neglects completely the interaction between nucleons, which are treated as independent particles moving in an average potential. This model yields a reasonable evaluation of the nuclear level density (the effect of the residual interaction turns out to be relatively small), but it is unable to explain many other statistical properties of nuclear spectra and transitions.

The subject of RMT has fascinated both Physicists and Mathematicians since it was first introduced in mathematical statistics by Wishart in 1928 [48]. After a bit of a slow start,the subject got a big boost when Wigner [49] introduced the concept of statistical distribution of nuclear energy levels in 1950. However, it was in 1955 that Wigner [50] introduced the ensembles of random matrices. In that very paper he also introduced the large-N expansion and came to realise that the leading order contribution to the expectation values of the moments of the random Hamiltonian is given by Planar diagrams. In 1956, Wigner [51] derived the Wigner Surmise from the level spacing distribution of an ensemble of 2×2 matrices after level repulsion was predicted by Landau and Smorodinsky [52] and observed by Gurevich and Pevsner [53].

The idea of invariant random matrix ensembles was introduced in Physics by Porter and Rosenzweig [54] after it had appeared earlier in the mathematical literature. Rigorous analysis of spacing distributions was first given by Gaudin [55]. For the analysis of the eigenvalue density Mehta [56] invented the orthogonal polynomial method. The mathematical foundations of random matrix theory was established in a series of beauti-

ful papers by Dyson [57, 58, 59, 60, 61]. Dyson introduced the classification of random matrix ensembles according to their invariance properties under time reversal [57, 61]. As we know that for a system there are only three possibilities: a system is not time reversal invariant, or a system is time reversal invariant with the square of the time reversal invariance operator either equal to 1 or -1. The matrix elements of the corresponding random matrix elements are complex, real and self-dual quaternion, respectively which from a mathematical point of view exhaust the distinct real commutative normed division algebras, or in effect number systems. The corresponding invariant Guassain ensembles of Hermitian random matrices, are known as the Guassian unitary ensemble (GUE), the Guassian orthogonal ensemble (GOE) and the Guassain symplectic ensemble, in that order.

The philosophical foundations of RMT has also been laid down by Dyson [57]. In the words of Dyson, ”What is here required is a new kind of statistical mechanics, in which we renounce the exact knowledge not of the state of the sysytem but of the system itself. We picture a complex nucleus a “black box” in which a large number of particles are interacting according to unknown laws. The problem then is to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally possible”. This was made more precise by Balian [62] who obtained the Guassian random matrix ensembles from minimising the information entropy. The second important result deducted from the Dyson’s papers [61, 62] was the establishment of relation between random matrix theory and the theory of exactly integrable systems: the partition functions of a random matrix ensemble and of a log-potential coulomb gas in one dimension at three special temperatures are equivalent, each with solvability properties not shared for general temperature. In addition to this Fokker-Plank operator, which also specifies the Brownian evolution of the coulomb gas, was shown to have control over the evolution of eigenvalues of parameter-dependent extensions of the Guassian ensembles. These results were further confirmed by Sutherland [63] when he came to realise that Calogero-Sutherland quantum many body system, the Hamiltonian of which is constructed from N independent commuting operators, and so is integrable, is mathematically equivalent to the Dyson’s Brownian motion model. Detailed account of the realtionship between random matrix theory and integrable systems is discussed in the monograph by Forrester [64]. A review of one-dimensional integrable systems that touches on many ideas and also which form the subject matter of random matrix theory is given in the book by Korepin et. al [65]. A

third idea that made its appearance in Dyson's paper [57] is the application of Shannon's information entropy to random matrix spectra.

The early development in the random matrix theory are well summarised in the first edition of the monograph by Mehta [66] which has proven to be a very influential book containing many mathematical details and was quite useful for several years. A second significant book in the field of random matrix theory is by Porter [1] which contains the reprints of the important papers on the subject of random matrix theory that were written before 1965.

The field of disordered systems was born from the work of Anderson [67] on the localisation of wavefunctions in one-dimensional disordered systems at the same time when random matrix theory was in its infancy in nuclear physics. What he did is that he considered a one-dimensional lattice with random potential at each lattice point and concluded that eigenfunctions of this system are exponentially localised. His work had a stong impact on both theoretical and experimental solid state physics. Another exciting application of random matrix theory is the theory of small metallic grains by Gorkov and Eliasberg [68] which comes within the domain of mesoscopic physics. After a rapid growth period, RMT became a minor field until the early 1980. Nevertheless, the basic ideas and concepts as well as its mathematical formulation were developed in the period 1950-1963. Most of the references of the first historical period can be found in [1]. Later, the theory was consolidated as many experimental data were gathered, like Ericsons cross-section fluctuations [69] or the nuclear data ensemble [70]. Around 1984 two developments took place which lead to an exponential development of the theory: the adoption of Efetov s supersymmetry method and the ensuing coalescence of RMT and localization theory [71], and the link between RMT and the spectral fluctuation properties of quantum systems with a chaotic classical analog provided by the Bohigas-Giannoni-Schmidt conjecture (BGS) [4].

2.1.1 Random Matrix Theory and Mathematics

The random matrix theory which was first formulated in mathematical statistics, continued to develop in mathematics independently of the developments in physics. The important results regarding the integration measure of invariant random matrix ensembles

were obtained by Hua [72] and his results of more than one decade of work are summarised in his book that appeared in 1959 but which remained largely unknown. Very few mathematicians worked on the integrals of random matrix theory. By evaluating a unitary matrix integral, that is now known as the Harish-Chandra-Itzykson-Zuber integral [73, 74], an important result was obtained by Harisch-Chandra [73]. Zinn-Justin and Zuber [75] reviewed this topic in the present special issue. Also the work of Seelberg [76] is well known, not in the least because Mehta devoted a chapter of the second edition of his book [77] to this subject. Another noteworthy contribution is the introduction of zonal polynomials by James [78]. The book by Muirhead [79] in 1982 ties together the matrix integrals and zonal polynomials as they are relevant in mathematical statistics. Girkov has written a number of mathematical books (see, e.g.,[80]) relating to the the analytic properties of the eigenvalue distribution of large random matrices. Voiculescu [81] used random matrices as a primary example of free non-commutative random variables in operator algebras.

2.1.2 Random Matrix Theory and Quantum Field Theory

Few years before the discovery of universal conductance fluctuations, random matrix theory was applied to quantum field theory. From the work of 't Hooft [82] it is clear that in the limit of a large number of colors, the QCD partition function is dominated by planar diagrams and is also the case for the large N limit of [83] the combinatorial factors were calculated that enter in the large N_c limit of QCD by means of random matrix theory. A second innovative idea that appeared in this paper is the formulation of the calculation of the resolvent in random matrix theories as a Riemann-Hilbert problem and this approach is being focussed much in mathematical literature [84].

RMT has made impact on several areas of quantum field theory: lattice QCD, two-dimensional gravity, the Euclidean Dirac spectrum and the Seiberg-Witten [85] solution of two dimensional supersymmetric gauge theories. An important result is the Eguchi-Kawai [86] reduction that showed that in the limit of a large number of colors, certain gluonic correlation functions of pure Yang-Mills theory can be reduced to an integral over 4 unitary matrices. In two spatial dimensions this reduction results in an integral over a single unitary matrix which can be evaluated in the large-N limit. A unitary matrix integral also occurs in the low-energy limit of QCD. Because of the spontaneous breaking

of chiral symmetry, its low-energy degrees of freedom are the Goldstone modes which are parameterized by a unitary matrix valued field [87]. Below the Thouless energy for this system the kinetic term of the effective Lagrangian can be neglected and the low-energy limit of the QCD partition function is given by the unitary matrix integral [88]. In this domain the eigenvalues of the Dirac operator are correlated according to a random matrix theory with the additional involutive (chiral) symmetry of the QCD Dirac operator [89, 90]. The same symmetry is also found in two-sublattice disordered systems where hopping only occurs in between the sub-lattices [91]. The eigenvalue spectrum around zero of these chiral ensembles was first derived in reference [92]. An important difference between two-sublattice systems and QCD is the topology of the random matrix (i.e. the number of exact zeros) and the fermion determinant. In two-sublattice systems one is only interested in quenched results at zero topology whereas in QCD the fermion determinant and its zero modes are essential. Also, in the case of the chiral ensembles we have three different symmetry classes depending on the reality content of the matrix elements. Most of the work on chiral random matrix theory and its applications to the Dirac spectrum in QCD was done in the second half of the nineties [93]. There have been other attempts to derive QCD from a matrix model. Perhaps the best known is the induced QCD partition function of Kazakov and Migdal [94] where the lattice guage field is coupled to an adjoint scalar field. The guage field can be integrated out by means of the Harish-Chandra-Itzykson-Zuber integral resulting in a partition function for the adjoint scalar field. This partition function can be evaluated by the saddle point methods in the large N-limit. It has been shown that the so-called prepotential of N =2 supersymmetric theory can be derived from the large-N limit of a random matrix theory [95].

2.1.3 Random Matrix Theory In Nuclear Physics

The subject of statistical nuclear physics, as stressed by French (1984), evolved around the ideas introduced by Bohr's (1936) compound nucleus, Bethe's (1936) level density and Wigner's (1955) treatment of spectral fluctuations. Random matrices were introduced to nuclear physics in 1960s by Wigner and the reason for introducing the random matrix theory in nuclear physics was due to his quest to derive information about level and strength fluctuations in compound nucleus resonances. Intoduction of RMT in nuclear structure was motivated by the fact that if we consider low-energy region of the excitation spectrum of a nucleus the level density is small and description of the most of the states can be pro-

vided by nuclear models. However, due to rapid increase of level density with excitation energy,

$$\rho(E) \simeq \frac{C}{(E-\triangle)}^{\frac{5}{4}} \times exp(a\sqrt{(E-\triangle)}) \tag{2.1}$$

the number of levels is so high by the time one reaches the region, for example, the neutron threshold E$\sim 6MeV$, the microscopic description of individual states by nuclear models becomes meaningless. Instead at such energies, nuclear models provide adequate description of special states like giant resonances, analogue states which have peculiar structure. In the absence of a dynamical nuclear theory (the nuclear shell model had only just been discovered, and had not yet found a universal acceptance), Wigner focussed emphasis on the statistical aspects of nuclear spectra as revealed in neutron scattering data. At first sight, such a statistical approach to nuclear spectroscopy may seem bewildering. Indeed, the spectrum of any nucleus (and, for that matter, of any conservative dynamical system) is determined unambiguously by the underlying Hamiltonian, leaving seemingly no room for statistical concepts. Nonetheless, such concepts may be a useful and perhaps even the only tool available to deal with spectral properties of systems for which the spectrum is sufficiently complex. The approach introduced by Wigner differs in a fundamental way for that in standard statistical mechanics, one considers an ensemble of identical physical systems, all governed by the same Hamiltonian but differing in initial conditions, and calculates thermodynamic functions by averaging over this ensemble. Wigner proceeded differently: he considered ensembles of dynamical systems governed by different Hamiltonians with some common symmetry property. This novel statistical approach focusses attention on the generic properties which are common to (almost) all members of the ensemble and which are determined by the underlying fundamental symmetries. The application of the results obtained within this approach to individual physical systems is justified provided there exists a suitable ergodic theorem. Actually, the approach taken by Wigner was not quite as general as discussed above. The ensembles of Hamiltonian matrices considered by Wigner are defined in terms of invariance requirements: With every Hamiltonian matrix belonging to the ensemble, all matrices generated by suitable unitary transformations of Hilbert space are likewise members of the ensemble. This postulate guarantees that there is no preferred basis in Hilbert space. Many recent applications of RMT use extensions of Wigners original approach and violate this invariance principle. It is always assumed in the sequel that all conserved quantum numbers like spin or parity are utilized in such a way that the Hamiltonian matrix becomes block-diagonal, each block being characterized by a fixed set of such quantum numbers. We deal with only one

such block in many cases, and this block has dimension N . The basis states in Hilbert space relating to this block are labelled by Greek indices like μ and ν which run from 1 to N . Since Hilbert space is infinite-dimensional, the limit $N \to \infty$ is taken at some later stage. Taking this limit, signals that we do not address quantum systems having a complete set of commuting observables. Taking this limit also emphasises the generic aspects of the random matrix approach. In as much as RMT as a "new kind of statistical mechanics" bears some analogy to standard statistical mechanics, the limit $N \to \infty$ is akin to the thermodynamic limit. Before the introduction of random matrices by Wigner, Bohr argued that nuclei are systems of great complexity. What led Bohr to argue this, is the experiments conducted by Fermi and his group in Rome on neutron scattering by light nuclei which had revealed the existence of numerous narrow resonances. A similar type of data was taken by Rainwater and his group at Coulumbia University which used time of flight spectroscopy of slow neutrons to measure total neutron cross-section on a number of heavy even-even nuclei. The cross-section versus neutron energy E_n shown in figure for the target nucleus displays narrow resonances with width < 1 eV and spacing of about 20 eV. As the target nucleus ^{232}Th has spin 0 and positive parity and the incident slow neutrons carry zero angular momentum and has spin $\frac{1}{2}$, so the all resonances have spin/parity $\frac{1}{2}^+$, which correspond to excited states of the compound nucleus ^{232}Th with an excitation energy slightly above the neutron separation energies of 4.786 (the neutron threshold). The number of resonances observed in each compound nucleus was limited by the resolution of the spectrometer and was never much greater than 200. Similarly the data on proton resonances at the coulomb barrier in lighter nuclei were later taken up by the Triangle University group [96]. Together these data form what has been called the nuclear data ensemble (NDE) by [97] and [98]. This discovery led to the compound nucleus hypothesis by Bohr, that basically stated that the existence of these sequences of narrow resonances is incompatible with a pure independent particle picture and there must exist strong interactions between the nucleons inside the nucleus. Indeed, by assuming an indeependent particle model with a nuclear radius of about 5 fm and a potential well depth of several 10 MeV, one comes to the idea that single particle states have a typical spacing of several keV and widths of the order of 10 keV or larger, in complete disagreement with the data. What Bohr proposed, in compound-nucleus model (fig. 2.3) is that incident nucleon carries kinetic energy (as indicated by the billiard cue), collides with the nucleons in the target, and shares its energy with many nucleons. In units of the time for passage of nucleon through the nuclear interior, it takes the system a long time until one of its constituents acquires sufficient energy to be reemitted from the system.

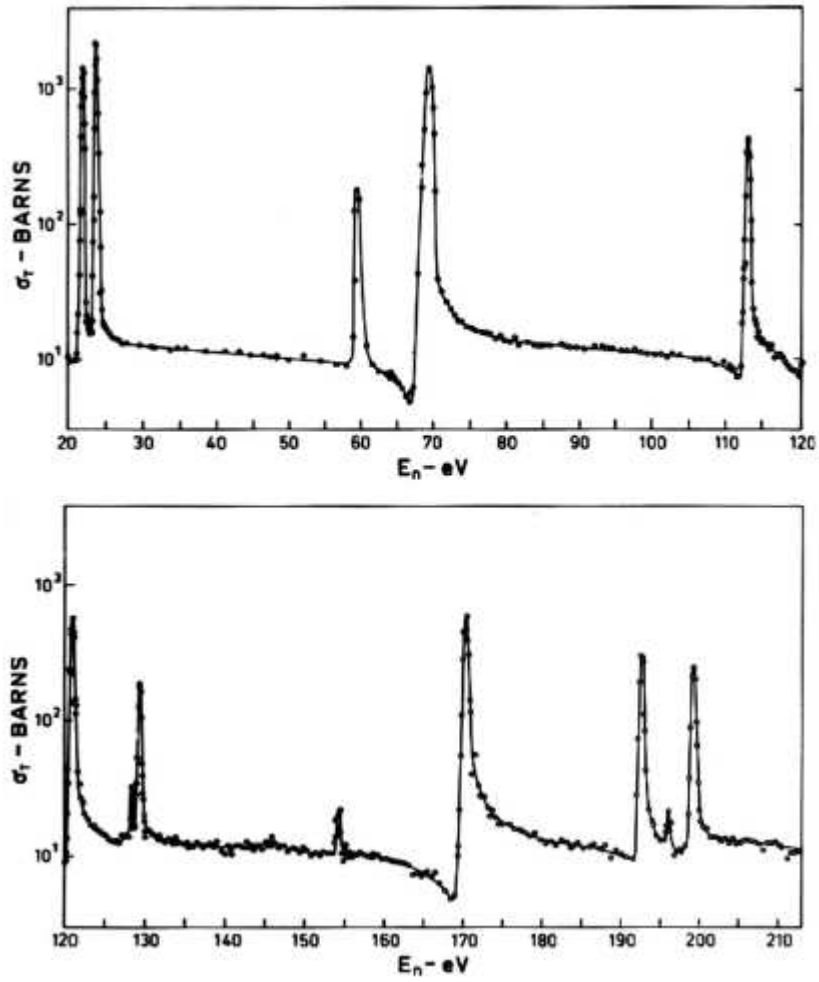

Figure 2.2: The total neutron cross-section on ^{232}Th vs neutron energy E_n in eV. From neutron cross-section, 1964, as reproduced in Bohr and Mottelson, 1969, Vol. 1, p. 178.

Bohr's concept of nucleus as a complex, strongly interacting system was adopted by the scientific community and survived until the discovery of nuclear shell model in 1949. The introduction of random matrices by Wigner was certainly motivated by the Bohr's idea. In order to explain the spirit of this approach, we focus attention on nuclear levels with the same quantum numbers (total spin J, parity π, and, at least, in light nuclei, total isospin) and ask the following question: can we identify the generic spectral properties of a system with strong interactions? Figure 2.4 shows six spectra, all having the same total number of levels. and spanning the same total energy interval, and therefore having the same average level spacing. The spectra of all the six systems differ in the way the spac-

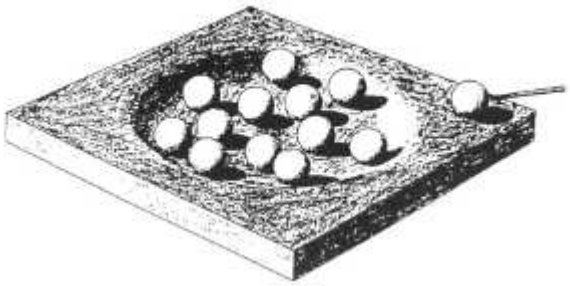

Figure 2.3: Bohr's wooden toy model of the compound nucleus. From Niels Bohr, Nature 137, 344 1936.

ings between the neighbouring levels are distributed. For the case of harmonic oscillator potential shown on right side of the figure, the spacings between the levels is identical. The spacing distribution differ more and more from delta function as we move towards the left.

The random matrix approach characterizes spectra by their fluctuation properties. The distribution of spacings of nearest neighbors is the first and obvious measure for spectral fluctuations. This is called as nearest-neighbour (NNS) distribution. the other fluctuation measures such as the correlation between nearest spacings, between next-nearest spacings, etc. Remarkably, RMT appeared only a few years after the introduction of the nuclear shell model. In its simplest form the shell model neglects completely the interaction between nucleons, which are treated as independent particles moving in an average potential. This model yields a reasonable clear level density (the effect of the residual interaction turns out to be relatively small), but it is unable to explain many other statistical properties of nuclear spectra and transitions. Concerning slow neutron resonances, independent-particle calculations give s-wave level spacings of about 1 MeV, and widths of about 0.1 MeV. Clearly the non-interacting shell model is not appropriate to describe these states, since the residual interaction plays an essential role. Nowadays, it is widely accepted that Bohr's hypothesis is to a large extent the physical basis of RMT.

The knowledge of the nuclear interaction was rather limited at that time and, therefore, Wigner was compelled to use a stochastic approach. According to Wigner, the Hamiltonian which governs the behavior of a complicated system can be represented by a random matrix with no particular properties, except for the symmetry properties of the system.

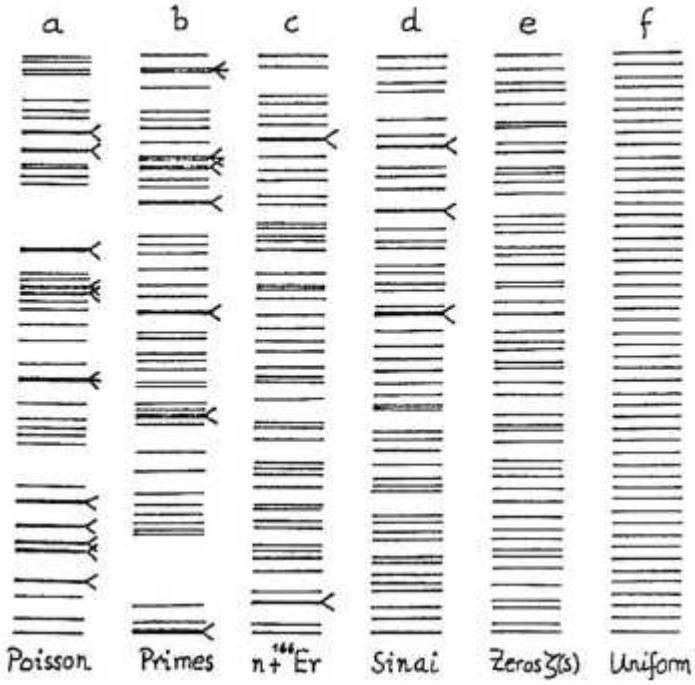

Figure 2.4: Six spectra with 50 levels each and the same mean level spacing. Fom right to left: The one-dimensional harmonic oscillator, a sequence of zeros of the Riemann zeta function, a sequence of eigenvalues of the Sinai billiard,a sequence of resonances seen in neutron scattering on ^{166}Er, a sequence of prime numbers, and a set of eigenvalues obeying Poisson statistics. This figure is taken from from Bohigas and Giannoni 1984.

In fact, he went one step further, substituting the random matrix representing the Hamiltonian by a whole ensemble of random matrices, all with the same symmetry properties, and applied ensemble averages to explain the statistical properties of individual nuclei.

2.2 Ensembles Of Random Matrices

There are only relatively few simple theoretical problems that physicists can solve exactly. As the complexity of systems under investigation grow, one soon has to resort to approximation and even these may not be able to deliver the results. As a last resort, one then has to turn to a statistical approach to the problem at hand. There are in general two ways of doing this. Firstly, there is the more conventional and normally intuitively more acceptable bottom-up approach, whereby one constructs a statistical theory of a system

taking into account all of it's detailed microscopic dynamics. Then there is the other more, ad-hoc top-down approach, where one ignores the small scale detailed dynamics, and builds the theoretical model from only a few broad physical considerations. One can then, after comparison between theoretical and experimental results, try and infer details of the unknown microscopic structure. The RMT approach can, for most applications, be summed up as follows:

- first, define an ensemble of matrices;
- secondly, try to find, analytically or numerically, some characteristics of this theoretical ensemble;
- and finally, compare the obtained characteristics of the theoretical ensemble with the experimental data.

A natural question to ask is: What are we going to learn by comparing the charac+-teristics of some theoretical ensemble of matrices and measurements from the real world? This is unfortunately not a simple question to answer. The large and diverse spectrum of physical systems to which the level sequence predictions of random matrix theory is applicable, is remarkable. But it is this uncanny success that poses the largest, and as of yet, unsolved mystery in RMT. Why does it work? There is as of yet no system that has, to our knowledge, been approached from a fundamental first approach that has led to a RMT. The gap between the bottom-up approach and the top-down approach is still largely a mystery. As we have just stated, the first step in the random matrix approach to a problem is by defining an ensemble of matrices, and it is to this that we now turn to.

Following the idea of a top-down approach, one builds in the broad physical considerations of the systems that one wishes to investigate by taking them into account when constructing the ensemble of matrices. It should be constructed in such a manner that the Hamiltonian matrix of the system under consideration, and of physically similar systems, should be in a sense more probable. Although there are many different sets of physical considerations that over the years have led to many different ensembles of random matrices, the first and probably most famous ensembles were constructed mainly by Wigner himself. These are known as the Gaussian orthogonal, the Gaussian unitary

and the Gaussian symplectic ensembles, GOE, GUE and GSE for short. Wigners three famous ensembles were built with three distinct physical situations in mind: the GOE for systems with time-reversal invariance, the GUE for systems without time reversal invariance and the GSE for systems with time-reversal symmetry, but specifically where there is no rotational symmetry. In the next two sections we will introduce and discuss these three classical ensembles from where RMT for all intents and purposes got started.

To get an initial feel for the basic ideas behind RMT, we will devote the whole of the next section to the Gaussian orthogonal ensemble, as it is in some respects the simplest, as well as the most physically relevant, of the three ensembles. The introduction of the GOE and the discussion of the considerations that went into its construction will also serve as an introduction to the basic ideas behind RMT. With the basic ideas under the belt, the Gaussian Unitary and Symplectic ensembles will then be introduced in a more compact manner in the section. The systems with which we deal in RMT are characterised by their Hamiltonians which can be represented by Hermitian matrices. When there are some exact quantum numbers corresponding to exact integrals of motion, like angular momentum and parity (J^π), and if the basis states are labelled by these exact quantum numbers, the Hamiltonian matrix splits into two blocks, and matrix elements connecting two blocks vanish. The underlying space-time symmetries obeyed by the system put important restrictions on the admissible matrix elements. If the Hamiltonian is time-reversal invariant and invariant under rotations, the Hamiltonian matrices can be chosen real and symmetric. If the Hamiltonian is not time-reversal invariant then, irrespective of its behaviour under rotations, the Hamiltonian matrices are complex Hermitian. Finally, if the system is time-reversal invariant but not invariant under rotations, and if it has half-odd integer total angular momentum, the matrices are quarternion real. The three classical ensembles constructed by Wigner: the GOE, the GUE and the GSE were built on the following considerations :

2.2.1 Independently Distributed Matrix Elements

The matrices in the ensemble are made up of matrix elements that are independently distributed of one another. This requirement was imposed solely for the purpose of making, the ensembles easier to handle analytically.

2.2.2 Invariance Under Transformation Of Basis

In an ensemble, each element in it has associated with it a probability. For ensembles of matrices, this is called the joint probability density function, or j.p.d.f. for short. It is required that the j.p.d.f. for matrices in the ensemble that are within a basis transformation of each other be the same, in other words, invariant under transformation of basis. This requirement was made due to the physical consideration that the dynamics of a system are not dependent on the choice of basis used to describe it.

2.2.3 Symmetry

Symmetry is the distinguishing point between the ensembles. They were constructed with three distinct groups of systems in mind, the systems being grouped by the same broad physical symmetries. The three groups, and the ensembles that were built for them, are

- **The GOE -** Systems with time reversal invariance as well as integer spin, with or without rotational symmetry, or half integer spin with rotational symmetry.
- **The GSE -** Systems with time reversal invariance as well as half integer spin with broken rotational symmetry.
- **The GUE -** Systems with broken time reversal invariance.

It is important to first of all distinguish between systems with time reversal symmetry, and systems without it. For systems without it, the GUE is the applicable ensemble. Systems with time reversal symmetry are, however, split into two subgroups, corresponding to the GOE and the GSE. The GOE covers virtually all systems with time reversal invariance, the exception being systems that also have half integer spin and broken rotational symmetry. This subgroup of systems is then covered by the GSE.

2.3 Gaussian Orthogonal Ensemble

In the literature there are many different ways that GOE is introduced. The usual way is to write down the joint probability density function for the matrices in the ensemble. In

GOE, we consider the real and symmetric Hamiltonian matrices H in a Hilbert space of dimension N,with μ, ν = 1,2,.....N, the matrix elements obey $H_{\mu\nu} = H_{\nu\mu} = H^*_{\mu\nu}$. For realistic systems Hilbert space is infinite dimensional, so we consider the limit $N \longrightarrow \infty$ in what follows. The ensemble is defined in terms of an integration over matrix elements. The volume element in matrix space

$$d[H] = \prod_{\mu \leq \nu} d[H]_{\mu\nu} \tag{2.2}$$

is the product of the differentials $dH_{\mu\nu}$ of the independent matrix elements (i.e. of the matrix elements not connected by symmetry). The ensemble is defined by the probability density P(H) of the matrices H,

$$P(H)d[H] = N_0 e^{\frac{-N}{4\lambda^2} Tr(H^2)} d[H] \tag{2.3}$$

Here N_0 is a normalisation factor and λ is a parameter which defines the average level density.

$$\rho(E) = \frac{N}{\Pi\lambda}\sqrt{1 - \frac{E^2}{4\lambda^2}} \tag{2.4}$$

When GOE is applied to data, λ is determined by the empirical average level density and the spectral fluctuation properties are then predicted in a parameter-free fashion. The Gaussian weight factor in eq.(2.3) ensures the convergence of the ensemble averages for large values of the integration variables. Using the symmetry properties of the matrices , we write the trace in the exponent as $\sum_{\mu<\nu} 2H^2_{\mu\nu} + \sum_{\mu} H^2_{\mu\mu}$. The probability density in eq.(2.3) takes the form

$$P(H)d[H] = N_0 \prod_{\mu} exp\left\{\frac{-N}{4\lambda^2} H^2_{\mu\mu}\right\} dH_{\mu\mu} \times \prod_{\rho<\sigma} exp\left\{\frac{-N}{2\lambda^2 H^2_{\rho\sigma}}\right\} dH_{\rho\sigma} \tag{2.5}$$

The above equation is a product of terms each of which depends only a single matrix element. Thus GOE has the properties of having uncorrelated Guassian distributed random variables with a zero mean value and a second moment given by

$$\overline{H_{\mu\nu} H_{\rho\sigma}} = \frac{\lambda^2}{N}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) \tag{2.6}$$

In eq.(2.6) the overbar denotes the ensemble average. Defining the GOE by these properties is equivalent to the definition (2.3). As far as the form of the probability measure is

concerned, it is fixed by symmetry while as the Guassian cut-off in that equation seems to be completely arbitrary. Rosenzweig and Porter (1960) have shown that the distribution (2.3) is obtained when one assumes that the ensemble is orthogonality invariant and the matrix elements not connected by symmetry are statistically independent. R. Balian (1968) has derived the distribution (2.3) from a maximum entropy principle. In GOE, every state is connected to itself and to every other state by a matrix element of H as all the non-diagonal matrix elements have the same first and second moments, every state is coupled to all other states with equal average strength, which results in level repulsion between any pair of levels and in a complete mixing of states in Hilbert space. The importance of such coupling becomes more evident when we consider a more general ensemble with probability density

$$P_\alpha(H)d[H] = \tilde{N}_0 \prod_\mu exp\left\{\frac{-N}{4\lambda^2}H_{\mu\mu}^2\right\}dH_{\mu\mu} \times \prod_{\rho\leq\sigma} exp\left\{\frac{-N}{2\alpha\lambda^2}H_{\rho\sigma}^2\right\} \tag{2.7}$$

where the positive integer α ranges from 0 to 1. For α=0, all non-diagonal elements vanish, and the ensemble (2.7) consists of diagonal matrices with independent Guassian-distributed diagonal elements. The shape of the average spectrum is Guassian, there is no level repulsion, and the spectral fluctuations are Poissonian. For $\alpha = 1$, the ensemble coincides with the GOE. For values of α between these two limits the shape of the spectrum and the spectral fluctuations interpolate between those two limiting cases. Significant mixing occurs between levels when the mean square mixing matrix element $\overline{H_{\mu\nu}^2}$ with $\mu \neq \nu$ is roughly equal to the square of the mean-level spacing. For the case of GOE, the mean level spacing is $d = \frac{\pi\lambda}{N}$ at the centre of the semicircle which follows from

$$\rho(E) = \frac{N}{\pi\lambda}\sqrt{1 - \left(\frac{E}{2\lambda}\right)^2} \tag{2.8}$$

showing that significant mixing occurs when α is of the order $\frac{1}{\sqrt{N}}$. Mixing sets in as soon as α differs from zero. The ensemble defined by eq.(2.3) is chosen in such a way that it is invariant under orthogonal transformations,under which reality and symmetry of the matrices $H_{\mu\nu}$ is preserved. The matrices obtained from the orthogonal transformation of a given matrix H also belong to the ensemble as a result of which there does not exist any preferred direction in Hilbert space and the ensemble is termed as generic. Because of the invariance under orthogonal transformations and Guassian-cut-off, the ensemble is referred to as Guassian orthogonal ensemble of random matrices. Instead of $\frac{N(N+1)}{2}$

integration variables $H_{\mu\nu}$ with $\mu \leq \nu$ in eq. (2.3) we can use the N eigenvalues E_μ of the matrices H and the $\frac{N(N-1)}{2}$ generators of the orthogonal transformations O, which diagonalises H. Then the volume element dH takes the form

$$dH = dO \prod_{\mu<\nu} |E_\mu - E_\nu| \prod_\rho dE_\rho \tag{2.9}$$

The factor dO represents the Haar measure of the orthogonal group in N dimensions. The Haar measure is a unique invariant measure that can be assigned to every compact group and that is used to define integrals over that group [99]. One of the factors depends only on the eigenvalues, and the second depends only on the diagonalising matrices which ensures that the eigenvalues and the eigenvectors of the matrices H are uncorrelated random variables. The factor $\prod_{\mu<\nu} \mid E_\mu - E_\nu \mid$ originates from the volume element in matrix space and reflects the orthogonal invariance of the ensemble. This factor causes the probability density to go to zero as the two eigenvalues approach each other which is a manifestation of level repulsion, a basic feature of quantum mechanics.

2.3.1 Derivation of Probability distributions for a simple 2 × 2 GOE matrix

The construction of Gaussian ensembles will be illustrated here by considering real and symmetric 2 × 2 matrices with O(2) as their group of canonical transformations. What we shall be seeking here is a probability density P(H) for the three independent matrix elements H_{11}, H_{22} and H_{12} normalised as

$$\int_{-\infty}^{\infty} dH_{11} dH_{22} dH_{12} P(H) = 1. \tag{2.10}$$

The two requirements suffice to determine P(H). Firstly, P(H) must be invariant under any canonical, i.e. orthogonal transformation of the two-dimensional basis:

$$P(H) = P(H'), H' = OHO^T, O^T = O^{-1}. \tag{2.11}$$

Secondly, the three independent matrix elements must be uncorrelated. The function P(H) must, therefore, be the product of three densities, one for each element

$$P(H) = P_{11}(H_{11}) P_{22}(H_{22}) P_{12}(H_{12}). \tag{2.12}$$

In order to exploit second and third equations, it suffices to consider an infinitisemal change of basis

$$O = \begin{pmatrix} 1 & -\Theta \\ \Theta & 1 \end{pmatrix} \tag{2.13}$$

. For which

$$H' = OHO^T \tag{2.14}$$

gives

$$\begin{pmatrix} H'_{11} & H'_{12} \\ H'_{12} & H'_{22} \end{pmatrix} = \begin{pmatrix} 1 & -\Theta \\ \Theta & 1 \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} \begin{pmatrix} 1 & \Theta \\ -\Theta & 1 \end{pmatrix}$$

$$\begin{pmatrix} H'_{11} & H'_{12} \\ H'_{12} & H'_{22} \end{pmatrix} = \begin{pmatrix} 1 & -\Theta \\ \Theta & 1 \end{pmatrix} \begin{pmatrix} H_{11} - H_{12}\Theta & H_{11}\Theta + H_{12} \\ H_{12} - H_{22}\Theta & H_{12}\Theta + H_{22} \end{pmatrix}$$

$$\begin{pmatrix} H'_{11} & H'_{12} \\ H'_{12} & H'_{22} \end{pmatrix} = \begin{pmatrix} H_{11} - H_{12}\Theta - H_{12}\Theta + H_{22}\Theta^2 & H_{11}\Theta + H_{12} - H_{12}\Theta^2 - H_{22}\Theta \\ H_{11}\Theta + H_{12} - H_{12}\Theta^2 - H_{22}\Theta & H_{11}\Theta^2 + H_{12}\Theta + H_{12}\Theta + H_{22} \end{pmatrix}$$

Neglecting Θ^2 terms, we get

$$H'_{11} = H_{11} - 2H_{12}\Theta \tag{2.15}$$

$$H'_{12} = H_{12} + (H_{11} - H_{22})\Theta \tag{2.16}$$

$$H'_{22} = H_{22} + 2H_{12}\Theta \tag{2.17}$$

Factorisation and invariance of P(H) yield

$$P(H) = P(H') = P(H)\left\{1 - \Theta\left[2\frac{H_{12}}{P_{11}}\frac{dP_{11}}{dH_{11}} - 2\frac{H_{12}}{P_{22}}\frac{dP_{22}}{dH_{22}} - (H_{11} - H_{22})\frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}}\right]\right\}$$

Since infinitisemal angle Θ is arbitrary, its coefficients in above equation must vanish.

$$\begin{aligned} \left(2\frac{H_{12}}{P_{11}}\frac{dP_{11}}{dH_{11}} - 2\frac{H_{12}}{P_{22}}\frac{dP_{22}}{dH_{22}} - \big(H_{11} - H_{22}\big)\frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}}\right) &= 0 \\ \big(H_{11} - H_{22}\big)\frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}} - 2\frac{H_{12}}{P_{11}}\frac{dP_{11}}{dH_{11}} + 2\frac{H_{12}}{P_{22}}\frac{dP_{22}}{dH_{22}} &= 0 \\ \frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}} - 2\frac{H_{12}}{(H_{11} - H_{22})}\frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}} + 2\frac{H_{12}}{(H_{11} - H_{22})P_{22}}\frac{dP_{22}}{dH_{22}} &= 0 \end{aligned}$$

$$\frac{1}{H_{12}}\frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}} - \frac{2}{(H_{11}-H_{22})}\left(\frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}} - \frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}}\right) = 0$$

This gives three independent differential equations, one for each of the three independent functions $P_{ij}(H_{ij})$ since each P_{ij} has its own exclusive argument H_{ij}. From above equation, we have

$$\begin{aligned}
\frac{1}{H_{12}}\frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}} &= \frac{2}{(H_{11}-H_{22})}\left(\frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}} - \frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}}\right) \\
\frac{1}{H_{12}}\frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}} &= -\frac{2}{(H_{22}-H_{11})}\left(\frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}} - \frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}}\right) \\
&= -A'
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{1}{H_{12}}\frac{dP_{12}}{dH_{12}}\frac{1}{P_{12}} &= -A' \\
\frac{dP_{12}}{dH_{12}} &= -A'H_{12}dH_{12} \qquad (2.18)
\end{aligned}$$

Integrating the above equation, we get

$$\begin{aligned}
\int \frac{dP_{12}}{P_{12}} &= -A'\int H_{12}dH_{12} \\
logP_{12} &= -A'\frac{H_{12}^2}{2} + logA'' \\
P_{12} &= A''e^{-A'\frac{H_{12}^2}{2}} \qquad (2.19)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
-\frac{2}{(H_{22}-H_{11})}\left(\frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}} - \frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}}\right) &= -A' \\
\frac{2}{(H_{22}-H_{11})}\left(\frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}} - \frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}}\right) &= -A' \\
\left(\frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}} - \frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}}\right) &= -\frac{A'}{2}(H_{22}-H_{11})
\end{aligned}$$

$$\begin{aligned}\frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}}+\frac{A'}{2}H_{22} &= \frac{1}{P_{11}}\frac{dP_{11}}{dH_{11}}+\frac{A'}{2}H_{11}\\ &= -B\end{aligned}$$

The above equation implies that

$$\begin{aligned}\frac{1}{P_{22}}\frac{dP_{22}}{dH_{22}}+\frac{A'}{2}H_{22} &= -B\\ \frac{dP_{22}}{dH_{22}} &= -BdH_{22}-\frac{A'}{2}H_{22}dH_{22}\end{aligned} \tag{2.20}$$

Integrating the above equation, we get

$$\begin{aligned}\int\frac{dP_{22}}{dH_{22}} &= -B\int dH_{22}-\frac{A'}{2}\int H_{22}dH_{22}\\ logP_{22} &= -BH_{22}-\frac{A'}{4}{H_{22}}^2+logA'''\\ P_{22} &= A'''e^{-BH_{22}-\frac{A'}{4}{H_{22}}^2}\end{aligned} \tag{2.21}$$

Working along the similar lines, we get

$$P_{11}=A''''e^{-BH_{11}-\frac{A'}{4}{H_{11}}^2} \tag{2.22}$$

Hence the probability distribution is given by, for a 2 $\times$ 2 GOE matrix,

$$\begin{aligned}P(H) &= A''A'''A''''e^{\frac{A'}{2}H_{12}^2-BH_{22}-\frac{A'}{4}H_{22}^2-BH_{11}-\frac{A'}{4}H_{11}^2}\\ P(H) &= Ce^{-A'\left(\frac{H_{12}^2}{2}+\frac{H_{22}^2}{4}+\frac{H_{11}^2}{4}\right)-B(H_{11}+H_{22})}\\ P(H) &= Ce^{-\frac{A'}{4}\left(2H_{12}^2+H_{22}^2+H_{11}^2\right)-B(H_{11}+H_{22})}\\ P(H) &= Ce^{-A(H_{11}^2+H_{22}^2+2H_{12}^2)-B(H_{11}+H_{22})}\end{aligned} \tag{2.23}$$

Of the three integration constants, B can be made to vanish by appropriately choosing the zero of energy. The constant A fixes the unit of energy and C is determined by the normalisation. Hence the above equation reduces to

$$P(H)=Ce^{-ATr.(H^2)} \tag{2.24}$$

Where Tr. stands for the trace of the matrix. Generalisation of the above equations to higher dimensions is discussed in the section 4.

2.3.2 GOE Fluctuation Measures

Porter-Thomas Distribution

We know that in GOE eigenvalues and eigenfunctions are uncorrelated random variables. For N $\rightarrow \infty$, the projections of the eigenfunctions onto an arbitrary vector in Hilbert space have a Guassian distribution centered at zero. Therefore, the squares ψ^2 of such projection have a χ^2 with one degree of freedom. Let us introduce the variable

$$y = \frac{\psi^2}{\overline{\psi^2}} \tag{2.25}$$

The resulting distribution is also known as Porter-Thomas distribution and has the form

$$P(y) = \frac{1}{\sqrt{2\pi y}} exp(-\frac{y}{2}) \tag{2.26}$$

The function is given in terms of the mean value $\Gamma = \overline{\psi^2}$. That parameter is an input parameter and is not predicted by random matrix theory. This distribution can be checked experimentally. Transition prrobabilities of nuclear levels to a fixed final state and decay widths to a fixed channel are proportional to squares of matrix elements containing the nuclear wavefunctions. These matrix elements can be read as projections of the wavefunctions onto a particular vector in Hilbert space. However, it may happen that the mean value Γ undergoes a secular variation and this is the case with, for instance, for doorway states. Then it is necessary to unfold the fluctuations by scaling the intensities properly.

Nearest-neighbour Spacing distribution and Dyson-Mehta or Δ_3 statistic

The two fluctuation measures most frequently employed in analyzing the experimental data are the nearest-neighbour spacing (NNS) distribution and Dyson-Mehta or Δ_3 statistic. These are obtained in the limit N $\rightarrow \infty$. Prior to using these measures for data analysis, it is necessary to unfold the experimental data. Let us try have a look at why need of unfolding arises i.e what is the the origin of unfolding. As is clear from the average level density $\rho(E)$ of GOE that it is constant in every energy interval containing finite number

of levels and same is true for average level spacing, "d" in the limit N $\rightarrow \infty$. However, in nuclei the situation is quite different as the level density grows nearly exponentially with energy and in many cases even a fairly short stretch of levels displays this fact. The spacing of the lowest-lying levels are consistently larger than those of the highest lying levels. So, this fact distorts the spectral fluctuation measures and must be taken into account prior to comparing data with GOE predictions. This is done by unfolding tha spectra. What is done in unfolding is that actual spectrum is modified such that average level spacing is constant. GOE predictions relate to spectra consisting of levels with identical quantum numbers. Spectra obtained experimentally may be incomplete [i.e. miss levels (especially those with small spacing or very large widths)], or not be pure(i.e., may contain levels with uncertain or incorrect quantum number assignments). It is important to know that how lack of completeness affects the compaison of data with GOE predictions.

Unfolding require knowledge of the average level density $\rho(E)$ for the data at hand. The situation is easy if a theoretical prediction for the average level density is available. This is the case, for instance, in billiards (where a point particle moving in two dimensions is scattered elastically on some surface). Here the Weyl formula gives the average level density in closed form in terms of the area enclosed by the surface and length of the boundaries of that surface. Given $\rho(E)$, the spectrum (or the spectra) is subsequently unfolded by mapping the eigenvalues E_μ onto new eigenvalues ϵ_μ by the prescription

$$\epsilon_\mu = \int_{-\infty}^{E_\mu} dE \rho(E) \tag{2.27}$$

By construction, the new eigenvalues are dimensionless and have an average level spacing equal to unity. The ϵ_μ can be used to construct the NNS distribution and the Δ_3. We observe the right hand side of above equation is the average of the staircase function defined by

$$N(E) = \int_{-\infty}^{E} dE' \sum_\mu \delta(E' - E_\mu) \tag{2.28}$$

The unfolded eigenvalues ϵ_μ are the values of that function taken at E_μ. Usually, however, the exact form of the average level density is not known. If, however, the data is obtained by numerical simulation of an ensemble (diagonalisation of matrices), the average level density is best found by numerically averaging over the ensemble. If, we, have

to deal with an empirical spectrum of say, several tens of levels, it is advantageous to use the data to construct the staircase function rather than the level density (the representation of the latter in the form of a histogram depends on the bin width chosen), and to fit a low-order polynomial to that function. The unfolded eigenvalues are again given by the values of the staircase function taken at the original eigenvalues E_μ.

The nearest-neighbour spacing distribution (NNS), P(s) depends on s, which is the ratio of actual level spacing and the mean level spacing d. However, to write it in a closed form is not possible. However, an excellent approximation due to Wigner is known as the Wigner's surmise

$$P(s) = \frac{\pi}{2} exp(-\pi s^2/4) \tag{2.29}$$

The linear increase with s for small s is due to GOE level repulsion. Universality shows that the Guassian cutoff factor defining the GOE and simply accounts for the fact that very large spacings are unlikely to occur. The exact expression for$P(s)$ was first derived by Gaudin in 1961. P(s) is displayed in figure below. The NNS distribution describes the dis-

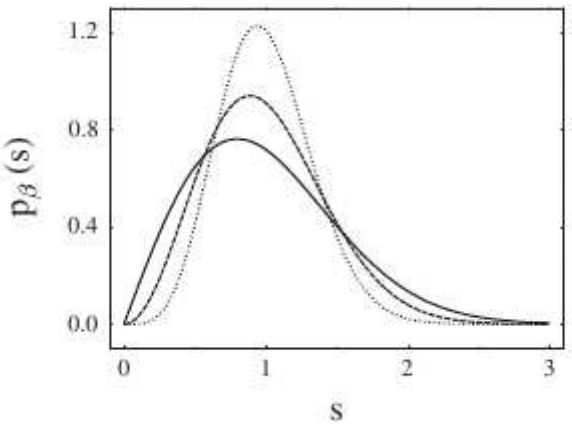

Figure 2.5: The nearest-neighbor spacing (NNS) distribution of the GOE (solid line)vs s, the ratio of the actual level spacing and mean level spacing. For comparison, we also show the NNS distributions for GUE (dashed line) and the GSE (dotted line). The parameter β is the Dyson index with β=1,2, and 4 for GUE, GOE and GSE, respectively.

tribution of level spacings but does not contain information about their correlations. Such information is provided by another fluctuation measure, the Δ_3 statistics. The number staircase function

$$N(E) = \int_{-\infty}^{E} dE' \sum_{\mu} \delta(E' - E_\mu) \tag{2.30}$$

counts the number of eigenvalues below energy E. With increasing energy, it increases by unity as E passes a (nondegenerate) eigenvalue and is otherwise constant. The number of eigenvalues in the energy interval $[E_0, E_0+L]$ is given by $n(E_0, L) = N(E_0+L)-N(E_0)$. By the definition of mean level spacing $d(E)$, We have $\overline{n(E_0, L)} = \frac{L}{d(E_0)}$. We use the fact that for $N \to \infty$, d(E) is constant (independent of E) in any energy interval containing a finite number of levels. The number variance $\Sigma^2_\beta(L) = \overline{n^2(E_0, L)} - \overline{(n(E_0, L))^2}$ is a fluctuation measure that contains information about correlations between level spacings. Suppose, for example, that actual GOE spectra can be constructed by drawing spacings at random from the NNS distribution. In this case, $\Sigma^2_\beta(L)$ is, for large L, proportional to (ln L). The slow growth indicates that large spacings and small spacings do not follow each other at random but almost alternate, and reflects the stiffness of GOE spectra. For the three canonical ensembles, the number variance is shown in figure below. The number

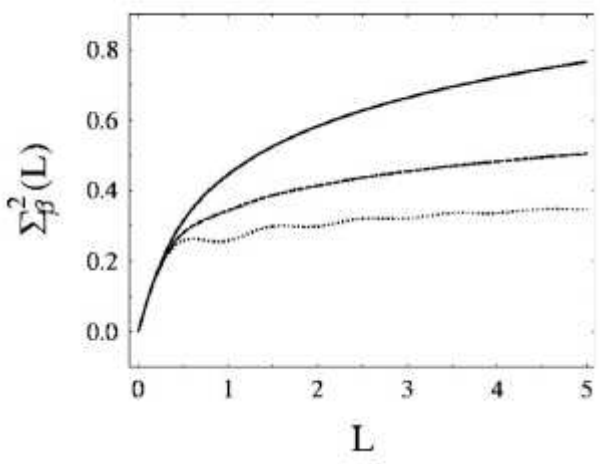

Figure 2.6: The number variance vs the length L of the interval (L is in units of the mean level spacing), for the three canonical ensembles. Top curve, GOE; middle curve GUE, GUE; bottom curve, GSE. The parameter β is the Dyson index.

variance is seldom used in nuclear physics because it fluctuates too strongly, so that is why the Δ_3 statistic by Dyson and Mehta is used. The latter is defined by

$$\Delta_3(L) = min_{a,b}\frac{1}{L} < \int_{E_0}^{E_0+L} dE' \overline{[N(E') - a - bE']^2} >_{E_0} \tag{2.31}$$

We integrate the ensemble average of the square of the difference between the number staircase function and the straight line $(a + bE')$ over an energy interval, divide by the length of the interval, and minimize the result with respect to the parameters a and b of the

straight line. The angular brackets denote an average over the initial point E_0. It can be shown that Δ_3 can be written as an integral over the number variance $\Sigma^2_\beta(L)$. Therefore, Δ_3 is much smoother than $\Sigma^2_\beta(L)$ and is better suited for data analysis. Similar to $\Sigma^2_\beta(L)$, $\Delta_3(L)$ grows logarithmically with L. For large L,

$$\Delta_3(L) \approx \frac{1}{\pi^2}\{lnL - 0.0687\}. \tag{2.32}$$

Similar to Σ^2, the Δ_3 statistic reflects the stiffness of GOE spectra. Figure below shows Δ_3 vs L for the GOE.

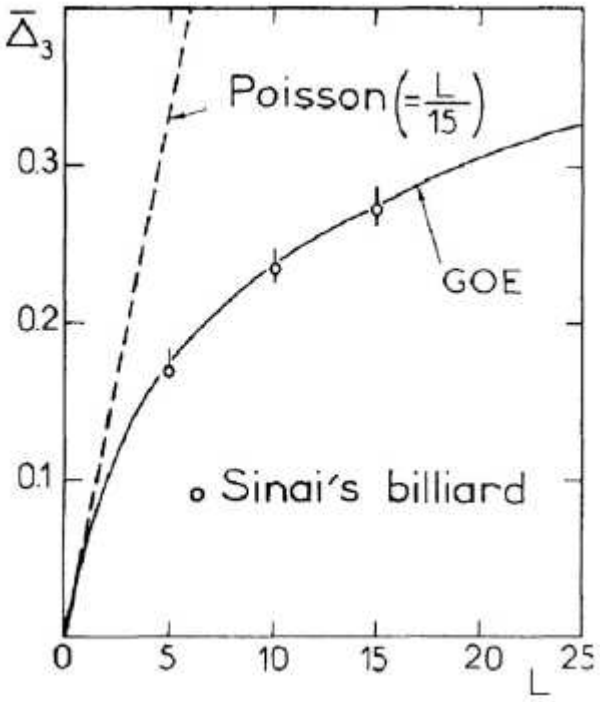

Figure 2.7: The Δ_3 statistic for the Sinai billiard (open circles), the GOE prediction(solid line), and the Poisson result (dashed line). From Bohigas et. al., 1984.

2.3.3 Properties of GOE

Universality

The form of GOE spectrum is because of the Guassian cut-off factor. However, this form is quite unrealistic as hardly any real physical system possess such a spectrum. While

as reality and symmetry of the matrices $H_{\mu\nu}$ reflect time reversal invariance and are thus a consequence of quantum theory, the Guassian cut-off is not, although the arguments of Rosenzweig and Porter (1960) and of Balian (1968) lend some plausibility to its use. The Guassian cut-off is preferred from a practical point of view because of ease with which the Guassian integrals can be handled. But GOE is interesting from physics point of view only if it furnishes information that is independent of the form of cut-off factor, which is guaranteed by the universality of GOE. In the usage of GOE emphasis is not on the overall shape of the spectrum but interest is rather on local spectral fluctuation measures nearest-neighbour spacing distribution or correlation between level spacings. These measures are predicted in a parameter-free fashion which means that all the local spectral fluctuation properties are functions of a dimensionless parameter s, which is the ratio of actual level spacing and the mean level spacing. The energy scale on which the local spectral fluctuations properties chatracterise properties of the spectrum, is negligibly small as compared to the length 4λ of the spectrum, in the limit N $\rightarrow \infty$. On that very energy scale, the spectral fluctuations are universal in the sense that they are functions of s and have same form for both the GOE and all non-Guassian cutoff factors, as long as the latter are orthogonally invariant and confine the spectrum to a finite singly piece of the energy axis (Hackenbroich and Weidenmuller, 1995). Non-Guassian cutoffs modify the overall shape of the spectrum but in principle, it is possible to find a cutoff factor for any given form of the spectrum such that the resulting random-matrix ensemble has an average spectrum of that form, which leaves local spectral fluctuation properties unaffected. In fact, the local spectral fluctuations, in the limit N $\rightarrow \infty$ separate from the global properties and become universal.

Ergodicity

The question that arises, in case of GOE, is that how can we compare theoretical predictions obtained from an ensemble of Hamiltonians, in a meaningful way, with the data taken from a physical system with a single Hamiltonian and not from an ensemble of Hamiltonians. The question is answered by the ergodicity property of GOE. Spectral fluctuation measures such as the mean level spacing or the NNS distribution as running averages, can be calculated from the spectral data of a given system and such running average is denoted by angular brackets. We would like to ascertain that $\overline{O} =< O >$ holds true for all the members of the ensemble and for all observables O that describe local spectral properties. The above equation cannot be proved in general because there is no way

to evaluate $< O >$ in the framework of GOE. However,a slightly weaker proof (Brody et al., 1981) is possible for $\overline{(\overline{O} - < O >)^2}$. The proof is possible for the reason that all the terms are ensemble averages. The above statement implies that for almost all members of the ensemble [with the exception of a set of measure zero and the measure defined in the equation $d[H] = \prod_{\mu \leq \nu} d[H]_{\mu\nu}$] the running average of an observable O (calculated for a single member of the ensemble) is equal to ensemble average of the observable. This property is referred to as ergodicity and the name derives from the similarity of the statement with the ergodicity in the classical statistical mechanics which states that the phase-space average and time average along a single trajectory are equal.

Information content of GOE Spectra

The eq.(2.5) shows that in GOE every state in Hilbert space is coupled to every other one by a Gaussian-distributed random-matrix element and hence in GOE all states in Hilbert space are completely mixed with each other. By choosing the parameters N and λ and drawing all independent matrix elements from the resulting Gaussian distribution generates a random GOE matrix. Diagonalisation of that matrix yields a GOE spectrum and by construction that spectrum contains no information beyond the input parameters N and λ. In particular, the spectral fluctuations are void of physical information and if the spectral fluctuation of an experiment agree with the GOE predictions and if there is no further information on that system, then the spectral data alone cannot be used to extract any physical information on the system beyond the mean level density. One arrives at the same conclusion while asking the question that how many pieces of spectral data are needed to determine the underlying Hamiltonian. Counting in GOE shows that all N eigenvalues and all N orthonormal eigenfunctions are needed to determine the $\frac{N(N+1)}{2}$ independent matrix elements of H. By comparing this with the usual dynamical approach to physical systems where the Hamiltonian is given in terms of a few (say n) parameters. Then n pieces of data suffice to determine the Hamiltonian and further data can be used to check the consistency of the underlying theory.

Average level density

A central property of the GOE is the mean level density $\rho(E)$, a function of energy E. It is defined as

$$\rho(E) = \overline{\sum_n \delta(E - E_n)} \tag{2.33}$$

The delta function in the above equation can be written as the limit of a Lorentzian curve with vanishing width

$$\delta(E) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \sum_n \frac{1}{E^2 + \epsilon^2} \tag{2.34}$$

The above equation indeed is a representation of Dirac-Delta function as shown below. In order to qualify for a representation of Dirac-Delta function, it must satisfy the two defining properties of Dirac-Delta function. In order to show this, let us define a function as $F(\epsilon, E) = \frac{1}{\pi}\frac{\epsilon}{E^2+\epsilon^2}$. Hence, the above equation becomes $\delta(E) = \lim_{\epsilon\to 0} F(\epsilon, E)$ We have $\lim_{\epsilon\to 0} F(\epsilon, E)$ = 0, if E $\neq$ 0
and,

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} F(\epsilon, E)dE &= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\epsilon}{E^2+\epsilon^2}dE \\
&= \lim_{\epsilon\to 0}\frac{2\epsilon}{\pi}\int_{0}^{+\infty}\frac{1}{E^2+\epsilon^2}dE \\
&= \frac{2\epsilon}{\pi}\frac{1}{\epsilon}\frac{\pi}{2} \\
&= 1
\end{aligned}$$

In the last but one step, use has been made of the integral, $\int_0^{+\infty}\frac{dx}{x^2+a^2} = \frac{1}{a}\arctan(\frac{x}{a})$. Hence, it is a representation of Dirac-Delta function so that equation (2.33) can be written as

$$\begin{aligned}
\rho(E) &= \lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\sum_n \frac{1}{(E-E_n)^2+\epsilon^2} \\
\rho(E) &= -\lim_{\epsilon\to 0}\frac{1}{\pi} Im\left(\sum_n \frac{1}{(E-E_n)} + i\epsilon\right)
\end{aligned} \tag{2.35}$$

Now using

$$\sum_n \frac{1}{E - E_n} = Tr\left(\frac{1}{E-H}\right) \tag{2.36}$$

Hence, the density of states becomes

$$\rho(E) = -\frac{1}{\pi} Im\left(Tr\left(\frac{1}{E-H}\right)\right) \tag{2.37}$$

In the above eqn.(2.37) the limit $\epsilon \rightarrow 0$ has been omitted for simplification purpose. Wherever we meet an expression of this kind, it is to be understood that there is infinitisemal small positive imaginary part of E.
The density of states can alternatively be expressed in terms of the quantum mechanical Green's function

$$\begin{aligned} G(q_A, q_B, E) &= \sum_n \frac{\psi_n^*(q_A)\psi_n(q_B)}{E - E_n} \\ G(q_A, q_B, E) &= \sum_n \psi_n^*(q_A)\frac{1}{E-H}\psi_n(q_B) \end{aligned} \tag{2.38}$$

Where $\psi_n(q)$ are the eigenfunctions of H to the eigenvalue E_n. Now, we have

$$\sum_n \psi_n^*(q_A)\psi_n(q_B) = \delta(q_A - q_B) = \int \delta(q_A - q)\delta(q_B - q)dq \tag{2.39}$$

Hence the above equation can be written as

$$\begin{aligned} G(q_A, q_B, E) &= \int \delta(q_A - q)\frac{1}{E-H}\delta(q_B - q)dq \\ G(q_A, q_B, E) &= < q_A|\frac{1}{E-H}|q_B > \end{aligned} \tag{2.40}$$

where in the above equation the delta function $\delta(q_A - q) = |q_A >$ as the eigenfunction of the position operator $q|q_A >= q_A|q_A >$. The Green's function can thus be interpreted as the matrix element of the operator $(E - H)^{-1}$ with the eigenfunctions of the position operator as the basis functions. Thus, eqn. (2.37) for the density of states may be alternatively be expressed as

$$\rho(E) = -\frac{1}{\pi} Im\Big(Tr(G)\Big) \tag{2.41}$$

In eqn. (2.37) the eigenvalues no longer enter explicitly. Instead the average of the trace of an operator inverse must be calculated

$$S = \quad < Tr(\frac{1}{E-H}) > \tag{2.42}$$

This is done by expanding $(E-H)^{-1}$ into its Taylor series

$$S = \sum_{n=0}^{\infty} \frac{1}{E^{n+1}} < TrH^n > \tag{2.43}$$

The series converges only if $|E|$ exceeds all eigenvalues of H in magnitude. For the calculation of the average density of states, on the other hand, we need S in the range of the eigenvalues, where the expansion diverges. The problem is now reduced to the calculation of the ensemble average of the trace of H^n. Here, we shall perform the average for GUE. We know that the probability distribution for GUE is given by

$$P(H_{11}, ..., H_{NN}) = \left(\frac{A}{\pi}\right)^{\frac{N}{2}} \left(\frac{2A}{\pi}\right)^{N(N-1)} exp\left\{ - A \sum_{n,m} [(H_R)^2_{nm} + (H_I)^2_{mm}] \right\} \tag{2.44}$$

Where $(H_R)_{nm}$ and $(H_I)_{nm}$ are the real and imaginary parts of H_{nm}, respectively, we obtain

$$< H_{\alpha\beta} H_{\beta\alpha} >= \int H_{\alpha\beta} H_{\beta\alpha} P(H_{11}, ..., H_{NN}) dH_{11} ..... dH_{NN} = \frac{1}{2A} \tag{2.45}$$

holding both for $\alpha = \beta$ and $\alpha \neq \beta$. The ensemble average of all other products of two matrix elements vanish,$< H_{\alpha\beta} H_{\gamma\delta} >= 0, (\alpha, \beta) \neq (\delta, \gamma)$. From eqn. (2.45), we get for the average of the trace of H^2

$$< TrH^2 > \quad = \quad \sum_{\alpha\beta} < |H_{\alpha\beta}|^2 > \quad = \frac{N^2}{2A} \tag{2.46}$$

The averages of the traces of all odd powers of H vanish, $TrH^{2n+1} = 0$ as is clear from symmetry considerations. First non-trivial case is n=4,

$$< TrH^4 > \quad = \quad < \sum_{\alpha,\beta,\gamma,\delta} H_{\alpha\beta} H_{\beta\alpha} H_{\gamma\delta} H_{\delta\alpha} > \tag{2.47}$$

In the ensemble average only terms survive where matrix elements $H_{\alpha\beta}$ and $H_{\beta\alpha}$ occur pairwise. To this end we introduce the bracket notation $\overbrace{H_{\alpha\beta}\ H_{\gamma\delta}}$ denoting that only terms with $(\alpha,\beta)=(\delta,\gamma)$ are taken in the sums. Then there are four surviving terms, the first one given by

$$\Big\langle \sum_{\alpha,\beta,\gamma,\delta} \overbrace{H_{\alpha\beta}\ H_{\beta\gamma}}\overbrace{H_{\gamma\delta}\ H_{\delta\alpha}}\Big\rangle \quad = \quad \Big\langle \sum_{\alpha,\beta,\delta} H_{\alpha\beta}H_{\beta\alpha}H_{\alpha\delta}H_{\delta\alpha}\Big\rangle \quad = O(N^3) \tag{2.48}$$

where $O(N^3)$ denotes the number of terms in the sum is N^3. Two further terms are given by

$$\Big\langle \sum_{\alpha,\beta,\gamma,\delta} \overbrace{H_{\alpha\beta}\underbrace{H_{\beta\gamma}H_{\gamma\delta}}H_{\delta\alpha}}\Big\rangle \quad = \quad \Big\langle \sum_{\alpha,\beta,\gamma} H_{\alpha\beta}H_{\beta\gamma}H_{\gamma\beta}H_{\beta\alpha}\Big\rangle \quad = O(N^3) \tag{2.49}$$

and

$$\begin{aligned}\Big\langle \sum_{\alpha,\beta,\gamma,\delta} \overbrace{H_{\alpha\beta}\underbrace{H_{\beta\gamma}H_{\gamma\delta}}H_{\delta\alpha}}\Big\rangle \quad &= \quad \Big\langle \sum_{\alpha,\beta} H_{\alpha\beta}H_{\beta\beta}H_{\beta\alpha}H_{\alpha\alpha}\Big\rangle \\ &= \quad \Big\langle \sum_{\alpha} H_{\alpha\alpha}^4\Big\rangle \\ &= \quad O(N)\end{aligned} \tag{2.50}$$

The case is left where $H_{\alpha\beta}$ and $H_{\beta\alpha}$ occur twice within one term

$$\Big\langle \sum_{\alpha,\beta,\gamma,\delta} \overbrace{H_{\alpha\beta}H_{\beta\gamma}H_{\gamma\delta}H_{\delta\alpha}}\Big\rangle = \Big\langle \sum_{\alpha,\beta} |H_{\alpha\beta}|^4\Big\rangle = O(N^2) \tag{2.51}$$

This shows that sums with interlacing brackets as well as sums containing the same matrix element repeatedly are of lower order in N than the sums with non-interlacing brackets. In the limit of large N therefore only the latter terms have to be considered. This facilitates the calculation considerably. Introducing the abbreviation

$$M_n = \quad < TrH^{2n} > \tag{2.52}$$

Equation (2.43) can be written as

$$S = \sum_{n=0}^{\infty} \frac{1}{E_{2n+1}} M_n \tag{2.53}$$

To derive a recursion formula for the M_n, we write

$$\begin{aligned} M_n = & \quad < \sum_{\alpha} (H^{2n})_{\alpha\alpha} > \\ M_n = & \quad < \sum_{\alpha} H_{\alpha\beta} (H^{2n-1})_{\beta\alpha} > \end{aligned} \tag{2.54}$$

In order that a given term survives the averaging, one of the factors of H^{2n-1} must be identical with $H_{\beta\alpha}$. As each of the (2n-1) factors can assume this role, we get

$$\begin{aligned} M_n &= \sum_{k=0}^{2n-2} \sum_{\alpha,\beta,\gamma,\delta} < H_{\alpha\beta} (H^k)_{\beta\gamma} H_{\gamma\delta} (H^{2n-k-2})_{\delta\alpha} > \\ M_n &= \sum_{k=0}^{2n-2} \sum_{\alpha,\beta} < H_{\alpha\beta} (H^k)_{\beta\beta} H_{\beta\alpha} (H^{2n-k-2})_{\alpha\alpha} > \end{aligned} \tag{2.55}$$

Now, we use the fact that contributions of the types (2.49) and (2.50) are negligible in the limit of large N. Then the ensemble average in eqn.(2.55) factorises

$$M_n = \sum_{k=0}^{2n-2} \sum_{\alpha,\beta} < |H_{\alpha\beta}|^2 >< (H^k)_{\beta\beta} >< (H^{2n-k-2})_{\alpha\alpha} >< (H^{2n-k-2})_{\alpha\alpha} > \tag{2.56}$$

Using equation (2.45) this may be written as

$$\begin{aligned} M_n &= \frac{1}{2A} \sum_{k=0}^{2n-2} \sum_{\alpha,\beta} < (H^k)_{\beta\beta} >< (H^{2n-k-2})_{\alpha\alpha} > \\ M_n &= \sum_{k=0}^{2n-2} < TrH^k >< (H^{2n-k-2} > \\ M_n &= \frac{1}{2A} \sum_{k=0}^{n-1} M_k M_{n-k-1} \end{aligned} \tag{2.57}$$

Where in the last step we took into account that only the traces of the even powers of H survive the ensemble average. By means of the initial conditions

$$M_0 = Tr(1) = N \tag{2.58}$$

The eqn. (2.57) may be used to calculate M_n recursively. But an explicit knowledge of the M_n is not even needed here. We may instead directly enter the recursion relation (2.57)

into expression (2.53) for S

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{1}{E^{2n+1}} M_n \\ &= \frac{1}{E}\Big(N + \sum_{n=1}^{\infty} \frac{1}{E^{2n+1}} M_n\Big) \\ &= \frac{1}{E}\Big(N + \sum_{n=1}^{\infty} \frac{1}{E^{2n}} \frac{1}{2A} \sum_{K=0}^{\infty} M_k M_{n-k-1}\Big) \end{aligned} \tag{2.59}$$

By changing the order of summation and subsequently shifting the summation index, we get

$$\begin{aligned} S &= \frac{1}{E}\Big(N + \frac{1}{2A} \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{E^{2n}} M_k M_{n-k-1}\Big) \\ S &= \frac{1}{E}\Big(N + \frac{1}{2A} \sum_{K=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{E^{2(n+k+1)}} M_k M_n\Big) \\ S &= \frac{1}{E}\Big(N + \frac{1}{2A} \sum_{k=0}^{\infty} \frac{1}{E^{2k+1}} M_k \sum_{n=0}^{\infty} \frac{1}{E^{2n}} M_n\Big) \\ S &= \frac{1}{E}\Big(N + \frac{1}{2A} S^2\Big) \end{aligned} \tag{2.60}$$

We have now end up with an equation which is quadratic and can be easily solved.

$$\begin{aligned} S &= \frac{N}{E} + \frac{1}{2AE} S^2 \\ SE &= N + \frac{1}{2A} S^2 \\ S^2 - 2ASE + 2AN &= 0 \\ S &= \frac{2AE \pm \sqrt{4A^2E^2 - 8AN}}{2} \\ S &= AE\Big(1 \pm \sqrt{1 - \frac{2AN}{A^2E^2}}\Big) \end{aligned} \tag{2.61}$$

However, we take only the term with the negative sign for the reason that S $\rightarrow$ 0 as E $\rightarrow \infty$ (see equation(2.53)). Using equation (2.37) the average density of states is now

immediately obtained as

$$\rho(E) = \left\{ \begin{array}{rl} \frac{A}{\pi}\sqrt{\frac{2N}{A} - E^2} & \text{if} |E| < \frac{\sqrt{2N}}{A}, \\ 0 & \text{if} |E| > \frac{\sqrt{2N}}{A} \end{array} \right\} \tag{2.62}$$

In the limit E $\rightarrow$ 0, the ensemble averaged density of states becomes constant. i.e. $\rho(E) = \frac{\sqrt{2NA}}{\pi}$. It is a common practice to normalise this quantity to one by taking $A = \frac{\pi^2}{2N}$. Then the average density of states becomes

$$\rho(E) = \left\{ \begin{array}{rl} \sqrt{1 - \left\{\frac{\pi E}{2N}\right\}^2} & \text{if} |E| < \frac{2N}{\pi}, \\ 0 & \text{if} |E| > \frac{2N}{\pi} \end{array} \right\} \tag{2.63}$$

This is the famous Wigner's semicircle law.

2.3.4 Physical Considerations Built Into The GOE

As said before, the broad physical properties of the system to which a RMT approach is to be applied, is built into the ensemble of matrices. Now that we have defined the GOE, it is perhaps a good time to take a look at what physical considerations went into its construction in the first place. As briefly discussed in the introduction, the physical system that Wigner was investigating when he first introduced the GOE was that of energy levels of heavy nuclei [100]. With this system in mind, let us now take a look at what went into the construction of the GOE.

Symmetry

The Hamiltonian operator in the Schroedinger equation that characterizes a quantum mechanical system is required to be Hermitian. For Hamiltonians in matrix form, this implies that the Hamiltonian matrix of the system has to be such that $H = H^\dagger$, where the $\dagger$ operator denotes the conjugate transpose of a matrix, i.e., $A^\dagger = (A^*)^T$ with the * operator denoting the complex conjugate. A matrix that is its own conjugate transpose is called a Hermitian matrix. As is clear from the section (2) that some further restriction has been made while defining GOE, as the matrices in the ensemble are not only Hermitian, but also symmetrical. This restriction on the possible Hamiltonians allowed in the GOE stems from a restriction on the physical systems under inspection, namely that these systems all

exhibit time-reversal symmetry. To get an idea of why time reversal symmetry restricts the Hamiltonian matrix of a system to being symmetrical, it may be instructive to take a brief look at the time reversal operator.

When the time reversal operator acts on a system, it, by definition, reverses linear and angular momentum, but leaves position unchanged. From this, it can be deduced [101] that the time reversal operator be anti-unitary. Now, an anti-unitary operator can always be written as the product of a unitary operator and the complex conjugation operator. In other words, for the anti-unitary time reversal operator T, we can write

$$T = YK_0 \tag{2.64}$$

with Y being a unitary operator, and K_0 denoting the complex conjugation operator. The explicit form of the time-reversal operator depends on the basis that is chosen to describe the system at hand. Without going into much detail how the properties of the time reversal operator constrains the Hamiltonian of a time reversal invariant system to being symmetrical, let us consider as an example the coordinate representation specifically. In this basis the time dependent Schredinger equation can be written as follows.

$$\left[\frac{-\hbar^2}{2m}\bigtriangledown^2 + V(x)\right]\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t) \tag{2.65}$$

with V(x) denoting the potential. The bracket denoted by H is the Hamiltonian of the system. If one now takes the complex conjugate of both sides of eqn. (2.65), one obtains,

$$\left[\frac{-\hbar^2}{2m}\bigtriangledown^2 + V^*\right]\psi^*(x,t) = -i\hbar\frac{\partial}{\partial t}\psi^*(x,t) \tag{2.66}$$

If we now replace the dummy variable t with t , it is apparent that both $\psi(x,t)$ and $\psi^*(x,t)$ will be solutions of the original eqn.(2.65) if we require that the Hamiltonian in eqn.(2.66) be the same as the Hamiltonian in eqn.(2.65), in other words, by requiring that

$$V(x) = V^*(x). \tag{2.67}$$

For this to hold, it is clear that V(x) has to be real, and by implication, so too the Hamiltonian H. A unitary matrix that is also real, is by implication symmetrical. In coordinate

representation the form of the time reversal operator T, from what we have seen, is simply the complex conjugation operator:

$$T = K_0 \tag{2.68}$$

with the unitary operator Y, from eqn.(2.64), in this case being equal to the identity operator. In general however, Y is not equal to the identity operator, and in fact the requirement that the Hamiltonian of a system is invariant under time reversal, is given by

$$THT^{-1} = H \tag{2.69}$$

For more detail in this regard, see [77] and [102].That said, by far most quantum mechanical systems that normally occur in nature, exhibit time reversal symmetry, making the GOE, at least from a quantum-mechanical point of view, the most applicable of the three ensembles introduced by Wigner.

Invariance under basis transformation

To write down the Hamiltonian of a physical system in a matrix form, it is necessary first to choose an orthonormal basis in which you are going to do so. There are many different ways of doing this, each leading to a seemingly different Hamiltonian matrix. In general, one can transform the Hamiltonian matrix H of a system resulting from one choice of basis to a Hamiltonian 'H' for a different choice of basis by the linear tansformation

$$H^{'} = T^{-1}HT \tag{2.70}$$

the only requirement on the transformation matrix T being that its inverse exists. In quantum mechanics, however, Hamiltonian matrices are always required to be Hermitian. The GOE does not hold for systems exhibiting time reversal symmetry that have broken spin-reversal symmetry. An ensemble was however constructed for this special case, the GSE, which we shall briefly discuss ahead. A quantum mechanical system lives in a Hilbert space and when choosing a basis for this Hilbert space, it is usually done so that this basis is orthonormal, in other words the basis vectors are so chosen that they are not only orthogonal to each other, but also all have a norm of 1 and $H^{'}$ will only be guarantied of being so if we further restrain the transformation matrix to being unitary. For a unitary matrix U, with the property

$$UU^{\dagger} = U^{\dagger}U = I \tag{2.71}$$

the transformation of basis is now

$$H^{'} = U^{\dagger}HU \tag{2.72}$$

Taking the conjugate transpose on both sides of equation (3.18) we then have

$$(H^{'})^{\dagger} = (U^{\dagger}HU)^{\dagger} = (U)^{\dagger}(H)^{\dagger}(U^{\dagger})^{\dagger} = (U^{\dagger}HU)^{\dagger} = H^{'} \tag{2.73}$$

by using the property of conjugate transposition that for two matrices A and B

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \tag{2.74}$$

as well as the known fact that H is Hermitian to begin with. Eqn. (2.73) shows that $H^{'}$, the result of a unitary transformation of H, is equal to the conjugate transpose of itself, or is in other words, Hermitian. For Hamiltonian matrices describing systems with time reversal symmetry, we have to restrict the form of the transformation matrix in eq.(2.70) even further. As discussed in the previous section, Hamiltonians of such systems all have the property of being symmetrical. If H therefore describes a system that is invariant under time reversal, the matrix $H^{'}$ also has to be symmetrical as it too describes a system where time reversal symmetry holds. This can only be guaranteed if the transformation matrix T of eq.(2.70) is even further restricted to being orthogonal. An orthogonal transformation preserves symmetry in the same way that a unitary transformation preserves Hermiticity. This can be shown in much the same as in 2.73, using the fact that for an orthogonal matrix O we have

$$O^{T}O = OO^{T} = I \tag{2.75}$$

Here, I represents the identity matrix. Even though the form of a Hamiltonian matrix that describes a system is dependent on choice of basis, the actual mechanics of the physical system are not. Hamiltonian matrices that are within a unitary transformation of another should lead to the same, basis independent solutions of the Schroedinger equation. This brings us to an important feature of the GOE. Since matrices that are within an orthogonal transformation of one another describe the same physical system, it stands to reason that these related matrices should carry the same statistical weight in ones ensemble. The GOE was constructed that this is indeed so. To verify this, let us take a look at the j.p.d.f.

of the matrix ‘H’ given by equation

$$P(H^{'}) = N_0 e^{\frac{-N}{4\lambda^2}Tr\left((O^T HO)^2\right)} \tag{2.76}$$

Furthermore,

$$\begin{aligned} Tr\big((O^T HO)^2\big) &= Tr\big(O^T HOO^T HO\big) \\ &= Tr\big(O^T HHO\big) = Tr\big(O^T H^2 O\big) \\ &= Tr\big(OO^T H^2\big) = Tr\big(H^2\big) \end{aligned} \tag{2.77}$$

by using eqn. 2.75, and the characteristic of the trace function that

$$Tr(AB) = Tr(BA) \tag{2.78}$$

for any two square matrices A and B of equal dimension. By inserting equation 2,77 into equation 2.76, we then obtain

$$\begin{aligned} P(H^{'}) &= N_0 e^{\frac{-N}{4\lambda^2}Tr\left((O^T HO)^2\right)} \\ &= N_0 e^{\frac{-N}{4\lambda^2 Tr(H^2)}} \\ &= P(H) \end{aligned} \tag{2.79}$$

Thus the j.p.d.f. for the matrix $H^{'}$ is the same as the j.p.d.f. for the matrix H as we expected (hoped), as they are merely an orthogonal transformation away from another.

2.3.5 Size Of The Matrices And Block Diagonal Form

The choice of basis is an important issue, as a good choice of basis may simplify the problem at hand greatly. Ideally, for example, one could choose the basis of the Hilbert space in which the system lives in such a manner that the Hamiltonian matrix of the system would simplify to a diagonal matrix, which is as simple as it gets. To do this, however, one would have to solve the Schroedinger equation, as the set of basis vectors that results in a diagonal Hamiltonian matrix, is in fact the set of allowed states of the system, i.e., the eigenstates of the Schredinger equation in the first place. The states of the quantum mechanical system are labelled by what are called quantum numbers, each state corresponding uniquely to a unique set of quantum numbers. What these quantum

numbers represent, and the way that they label the states, differs from system to system. The states of the hydrogen atom, for example, can be labelled by a set of three numbers, (n,l,m), n representing the so called principle quantum number, l representing total angular momentum, and m the projection of angular momentum onto a certain fixed direction. For much more complicated systems, such as that of a heavy nucleus, labelling of individual states in such a manner is very difficult. In principle it would be possible to label all the states exactly, but that would require an exact solution of the systems Schredinger equation that which we cannot do in the first place. In attempting an approximate solution - it turns out that at higher excitation levels some of the quantum numbers very quickly get washed out, as the levels get close to one another and start to mix. There are, however, quantum numbers that are exactly conserved throughout the spectrum - the so called good quantum numbers. For a heavy nucleus, for example, these good quantum numbers are total spin, and parity. Even though labelling individual states is not practical, it is possible to group states with the same good quantum numbers together when choosing a basis for ones system in such a way that the Hamiltonian matrix of the system reduces to a block-diagonal form such as in fig.(2.8). Each of these blocks can then be seen as a Hamiltonian matrix of a sub-system, and each of these smaller sub-system problems can be tackled individually. Unfortunately, these sub-problems can not be solved exactly either. It is in fact these sub-problems that RMT was applied to in the first place, the matrices of the GOE representing such a sub-block of a possible Hamiltonian of the entire system.

2.4 Circular Ensembles

Not only the lack of physical motivation for the independence of matrix elements posed a problem but also it was against the basic premise of random matrix theory. What Dyson [57] argued is that an ensemble of matrices should be constructed in such a way that all interactions are equally probable and that was impossible to do if matrix elements were required to be independent of one another, which then lead to construction of Dyson's famous ensembles with same physical considerations as Wigner's ensembles, but without the added requirement of independent matrix elements. Dyson did it in such a way that the ensemble remained analytically tractable. The question that naturally arises is that why to bother about the Wigner's ensembles if those of the Dyson are physically more justifiable. The remarkable fact is that the analytical results obtained from Wigner's Guassian ensembles and those from Dyson's circular ensembles are the same [57]. This

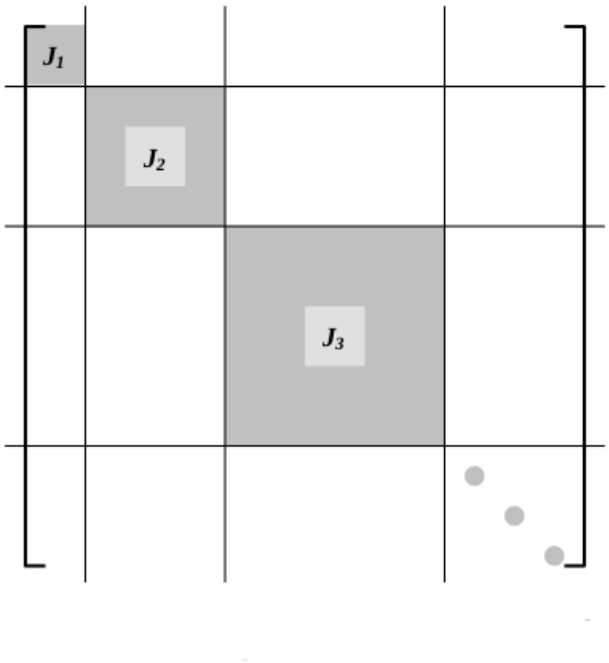

Figure 2.8: A Hamiltonian matrix in block-diagonal form. In this case a basis has been chosen is such a way that each of the blocks correspond to a sub-system of states each with a fixed total angular momentum J.

was the very first indication of a very important concept in random matrix theory called universality that is still today not understood fully.

2.5 Gaussian Unitary Ensemble

Whereas the GOE was constructed for the systems with time-reversal invariance, the GUE was was constructed for the systems with that do not have this property. It is, in principle, easy to create a quantum mechanical system without time reversal invariance by just putting a quantum mechanical system that has a time reversal symmetry in a strong external magnetic field. However, it was not possible at the time ensembles were first constructed because the magnetic fields required to sufficiently break the time-reversal symmetry of atomic nuclei systems were not experimentally possible. Dyson [57] however, mentioned the possibility of future application to atomic and molecular systems. The GUE has proven to be a very valuable ensemble, with applications far away from the nuclear systems it was originally intended for. The difference between between GOE and

GUE lies in the requirement of time reversal symmetry. As is discussed in the section of GOE that systems with time-reversal symmetry have symmetric Hamiltonian matrices describing them and it is therefore the symmetric matrices that the GOE is built of. So, without the requirement of time-reversal symmetry all one can say about the Hamiltonian matrices of such systems is that they are Hermitian, and GUE is therefore built simply out of Hermitian matrices. The other requirements on matrices in GUE are the same as those on matrices in the GOE. Just as in the case of GOE the matrices in the GUE are such that their individual matrix elements are independently distributed. As the matrices in GOE were required to be symmetric, the individual matrix elements were restricted to being real. With the requirement on the matrices in the GUE being weakened to Hermiticity, the individual matrix elements, except for those on the diagonal, can now in general have complex values. Let us suppose that the matrix elements of a Hermitian matrix A, of size N $\times$ N, is given by $a_{ij} = x_{ij} + iy_{ij}$. The requirement of Hermiticity does restrict the possible values of the matrix elements. As, mentioned above the matrix elements on the diagonal of a Hermitian matrix cannot have a complex value. From the definition of Hermiticity one can first of all deduce that the matrix elements on the diagonal of a Hermitian matrix are restricted to having real values, or in other words $y_{ii} = 0, i = 1...N$. Secondly the matrix elements opposite of the diagonal from each other are related by $a_{ij} = a^*_{ji}$. These restrictions imply that, just as with the symmetric matrices in the GOE, not all the matrix can be freely choosen. Let us suppose for the arguments sake, that the matrix elements we are free choose are those lying in the upper triangular part of the matrix, the above restrictions pinning rest of them down, so to speak. As in the case of GOE the freely choosen elements are required to be independently distributed, but in this case with the added meaning that the real and imaginary parts also be independent of each other. Considering the above, an N $\times$ N thus has N^2 elements that are free to be chosen, N of them lying on the diagonal, and $2 \times \frac{N(N-1)}{2} = N(N-1)$ lying above the diagonal. The joint probability distribution function thus gives the probability of finding matrix H in the differential volume element or the invariant measure has the form

$$d[H^{GUE}] = \prod_{\mu<\nu} d[ReH^{GUE}_{\mu\nu}] d[ImH^{GUE}_{\mu\nu}] \prod_{\sigma} d[H^{GUE}_{\sigma\sigma}] \tag{2.80}$$

With this definition, the equation for the probability density of the GUE is similar to the expression of GOE and is given by

$$P(H^{GUE})d[H^{GUE}] = N_0 exp\left\{ -\frac{N}{2\lambda^2} Tr(H^{GUE})^2 \right\} d[H^{GUE}] \tag{2.81}$$

The GUE is invariant under unitary transformations of Hilbert space. The real and imaginary parts of the matrix elements are uncorrelated random variables with equal Gaussian probability distributions centered at zero. The factors in the exponent are chosen in such a way that the second moments have the values

$$\overline{H^{GUE}_{\mu\nu} H^{GUE}_{\rho\sigma}} = \frac{\lambda^2}{N} \delta_{\mu\sigma} \delta_{\nu\rho} \tag{2.82}$$

In the GUE, the transformation to eigenvectors and eigenvalues as new integration variables involves a unitary transformation u and yields

$$\begin{aligned} P(H^{GUE})d[H^{GUE}] &= N_0 \quad du \quad exp\left\{ \frac{-N}{2\lambda^2} \sum_{\mu} E_\mu^2 \right\} \\ &\quad \times \prod_{\rho<\sigma} (E_\rho - E_\sigma)^2 \prod_{\nu} dE_\nu \end{aligned} \tag{2.83}$$

Here du denotes the Haar measure of the unitary group in N dimensions. Instead of the factor $|E_\rho - E_\sigma|$ occuring in the expression $P(H)d[H] = N_0 \quad dO \quad exp\left\{ \frac{-N}{4\lambda^2} \sum_\mu E_\mu^2 \right\} \prod_{\rho<\sigma} |E_\rho - E_\sigma| \prod_\nu dE_\nu$ for GOE, the above equation for GUE contains the factor $(E_\rho - E_\sigma)^2$. Hence, the level repulsion for the GUE is quadratic. This difference between GOE and GUE can be easily understood: In GOE, the coupling of any pair of levels is described by a single parameter, namely, the real coupling matrix element. For the two levels to have small spacing, the value of real coupling matrix element must be small. However, in case of GUE the coupling is described by two parameters, namely, the real and imaginary parts of the real coupling matrix element. In order to have small spacing, both parameters must be small, and the probability of small spacings is reduced accordingly.

2.6 Gaussian Symplectic Ensemble

As discussed in the subsection of symmetry, that there is a special class of quantum mechanical systems that exhibit time-reversal symmetry to which GOE does not apply. These are systems (for example, systems with strong spin-orbit coupling) with half-integer total angular momentum that are not symmetrical under rotation. For such kind of systems a specific kind of ensemble was created, known as Gaussian symplectic ensemble. Quantum mechanics demands that all Hamiltonian matrices to be Hermitian, whereas in case of GOE these matrices are further constrained to be symmetrical, the constituents of the GSE are constrained to being real. Any 2×2 matrix with complex valued entries can be expressed as a linear combination of the following four matrices:

$$\left.\begin{array}{ll} \mathbf{E}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \mathbf{E}_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ \mathbf{E}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \mathbf{E}_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \end{array}\right\}. \tag{2.84}$$

In other words, one can write any complex valued 2×2 matrix Q as

$$Q = \sum_{n=0}^{3} c_n E_n, \tag{2.85}$$

the coefficients c_n in general being complex numbers. If however they are real, the matrix Q is said to be a real Quaternion. It is important to note that, even though called real quaternion, such a matrix does not in general have only real valued entries. A N $\times$ N real quaternion matrix H is constructed out of N $\times$ N real quaternion 2×2 matrices such as depicted in equation above. Counting the individual matrix elements, it is evident that the dimension of H is in fact 2N $\times$ 2N. For the matrix H to also be Hermitian, one has to be able to write it as follows:

$$H = H_0 \otimes E_0 + H_1 \otimes E_1 + H_2 \otimes E_2 + H_3 \otimes E_3 \tag{2.86}$$

with H_0 a N $\times$ N real symmetric matrix, and H_1, H_2 and H_3 real antisymmetric matrices. Here the $\otimes$ operator denotes the direct product. To get a feeling of all this, let us, as an

example construct 2 $\times$ 2 Hermitian real quaternion matrix. For the matrices

$$\left.\begin{array}{ll} \mathbf{H}_0 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, & \mathbf{H}_1 = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \\ \mathbf{H}_2 = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}, & \mathbf{H}_3 = \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}, \end{array}\right\}. \tag{2.87}$$

We have

$$\mathbf{H} = \begin{pmatrix} a & 0 & b+ig & -f+id \\ 0 & a & f+id & b-ig \\ b-ig & f-id & b & 0 \\ -f-id & b+ig & 0 & b \end{pmatrix} \tag{2.88}$$

with a, b, c,d, f and g all real numbers.

As H_0 is a real and symmetric matrix so it has $\frac{N(N+1)}{2}$ free parameters, and as H_1, H_2 and H_3 are real antisymmetric matrices, each of them has $\frac{N(N-1)}{2}$ free parameters. Adding this all up, a real N $\times$ N quaternion matrix has therefore $2N^2 - N$ free parameters, and for matrices in the GSE, these are once again required to be independently distributed. If the distribution of the matrix elements are Gaussian, the resultant j.p.d.f. for matrices in the GSE, just as for the GOE and GUE, has the general form given by $P(H) = \frac{1}{C}e^{-aTr.(H^2)+bTr(H)+c}$ where a,b, and c real numbers, with a required to be positive and C is a normalisation constant. Wheras the j.p.d.f. of the GOE is invariant under orthogonal transformations, the j.p.d.f. of the GSE is invariant under symplectic transformations, brought about by the transformation matrices from the symplectic unitary group. A matrix 'B' is a member of this group if it satisfies the identity

$$Z = BZB^T \tag{2.89}$$

where the matrix Z is defined by

$$Z = I \otimes E_2 \tag{2.90}$$

with I the N $\times$ N identity matrix. For a detailed discussion in this regard, see section 2.4 in [77].

2.7 Embedded Ensembles

2.7.1 Introduction

Gaussian (and circular) ensembles were introduced in statistical nuclear spectroscopy because of the fact that these ensembles possess invariant properties under the adequate symmetry transformations [103]. However, the matrices of these kind of ensembles has the limitation that they are always filled and therefore the systems modelled represented by these kind of ensemble of matrices possess d-body interactions. As is well confirmed from the the wealth of the experimental data that the real systems like atomic nuclei or atoms are well described by the real or effective two-body interaction in the mean-field basis. Let $|k>$ denotes the mean-field single-particle states with k = 1, 2, 3,. . . . . , the Hamiltonian for such kind of systems can be written as

$$H = \sum_{k<l,p<q} < pq|H|kl > \hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_l \hat{a}_k \tag{2.91}$$

where $\hat{a}^\dagger(\hat{a})$ creates (destroys) a fermion in the kth single-particle state, and the two body matrix elements $< pq|Hkl >$ are properly antisymmetrized.
The main motivation behind the introduction of embedded ensembles in nuclear statistical spectroscopy was to tackle the problem of interaction rank, which is actually two body dominant as compared to the multi-body interactions predicted by Gaussian orthogonal ensemble. Hence such kind of ensembles provide present a more realistic picture of many-body quantum systems particularly because of the fact that it takes account of the number of particles, the rank of the interaction or the size of the Hilbert space which otherwise are not present in Gaussian ensembles. Embedded ensembles, in particular, the embedded Gaussian orthogonal ensemble of random matrices with k-body interactions (EGOE(K)), were introduced by French and Wong [104, 105] and Bohigas and Flores [106, 107]. The earlier studies used for analysing the EGOE(K) were the nuclear shell model and Monte-carlo methods. A good physical insight into EGOE(K) can be obtained by using the binary correlation approximation [108, 109, 36]. The EGOE(K) for many fermion (boson) systems assumes that many particle spaces are direct product spaces of single particle states, as in the nuclear shell model. Now let us define EGOE(k)for m$(m > k)$ particle sysytems (bosons or fermions) with the particles say distributed in N single-particle states. The EGOE(k)is generated by defining the Hamiltonian, which is k-body, to be GOE in the

k-particle spaces and then propagating it to the m-particle spaces by using the geometry (direct product structure) of the m-particle spaces. To make the definition more obvious, let us consider one of the simplest ensembles, EGOE(2) for fermions which is appropriate for atomic nuclei when studied using the shell model. Given the single particle states $|\nu>_i$, i=1,2,. . . . . ,N, the two-body Hamiltonian is defined by

$$H(2) = \sum_{\nu_i<\nu_j,\nu_k<\nu_l} < \nu_k\nu_l|H|\nu_i\nu_j >_a \hat{a}^\dagger_{\nu_l}\hat{a}^\dagger_{\nu_k}\hat{a}_{\nu_i}\hat{a}_{\nu_j} \tag{2.92}$$

where $\hat{a}^\dagger_{\nu_l}$ creates a fermion in the state in $|\nu_l>$ and similarly a_{ν_l} destroys a fermion in the state $|\nu_l>$. The symmetries for the antisymmetrised two-body matrix elements (TBME) $< \nu_k\nu_l|H|\nu_i\nu_j >_a$ being,

$$< \nu_k\nu_l|H|\nu_j\nu_i >_a = \quad - < \nu_k\nu_l|H|\nu_i\nu_j >_a \tag{2.93a}$$

$$< \nu_k\nu_l|H|\nu_i\nu_j >_a = \quad < \nu_i\nu_j|H|\nu_k\nu_l >_a \tag{2.93b}$$

The Hamiltonian H(m) in m-particle spaces is defined of the TBME via the direct product structure. The non-zero matrix elements of H(m) are of three types,

$$< \nu_1\nu_2 . . . . . \nu_m|H|\nu_1\nu_2 . . . . . \nu_m >_a = \sum_{\nu_i<\nu_j\leq\nu_m} < \nu_i|H|\nu_j >_a \tag{2.94a}$$

$$< \nu_p\nu_2 . . . . . \nu_m|H|\nu_1\nu_2 . . . . . \nu_m >_a = \sum_{\nu_i=\nu_2}^{\nu_m} < \nu_p\nu_q|H|\nu_1\nu_i >_a \tag{2.94b}$$

$$< \nu_p\nu_q\nu_3 . . . . . \nu_m|H|\nu_1\nu_2\nu_3 . . . . . \nu_m >_a = < \nu_p\nu_q|H|\nu_1\nu_2 >_a \tag{2.94c}$$

and all other $< . . . . . |H| . . . . . >_a = 0$ due to two-body selection rules. Thus EGOE(2) is defined by above equations with GOE representation for H in the two-particle spaces, i.e., $< \nu_k\nu_l|H|\nu_i\nu_j >_a$ are independent Gaussian random variables.

$$\overline{< \nu_k\nu_l|H|\nu_i\nu_j >_a} = 0 \tag{2.95a}$$

$$\overline{| < \nu_k\nu_l|H|\nu_i\nu_j >_a |^2} = \nu^2(1 + \delta_{(ij),(kl)}) \tag{2.95b}$$

Here bar denotes ensemble average and ν is a constant. The dimensions of matrix H(m), d is d(N,m) = $\binom{N}{m}$ and the number of independent matrix elements are ime(N) = $\frac{d_2(d_2+1)}{2}$ where the two-particle space dimension, $d_2 = \frac{N(N-1)}{2}$. e.g; d(11,4) = 330, d(12,5) = 792, d(12,6) = 924, d(14,6) = 3003, d(14,7) = 3432, d(40,6) = 3838380, d(80,4) = 1581580 etc

and Ime(11) = 1540, Ime(12) = 2211, Ime(14) = 4186. The EGOE(2) is also called as two-body random matrix ensemble(TBRE). Extensions of above equations for boson systems is straightforward. But as far as Hamiltonians of many-body interacting particle systems are concerned, they contain a mean-field part (one-body part h)and two-body residual interaction V mixing the configurations built out of the distribution of the particles in the mean-field single-particle orbitals; h is defined by single-particle energies (SPE), $\epsilon_i i = 1 \ldots\ldots N$ and V is defined by the TBME. Thus,the EE$(1+2)$, which is the embedded ensemble of (1+2)-body Hamiltonians, $EE(1+2)$: $\{H\} = [h(1)] + \lambda\{V(2)\}$,gives a more realistic picture of quantum many-body systems. Here $\{\mathrm{V}\}$ is EE(2), i.e., it is EGOE(2) with $\nu = 1$ in above equation or an ensemble with TBME being independent variables with a distribution different from Gaussian (for example uniform distribution). Similarly [h] is a fixed hamiltonian or an ensemble with single particle energies (SPE) chosen random but following some distribution. Finally, [h] and $\{\mathrm{V}\}$ are independent. It is to be expected that the generic features of EE(1+2) approach those of EGOE(k) for sufficiently large values of λ and significant results emerge as λ is varied starting from λ = 0.

Many different types of embedded Gaussian ensembles have been introduced in the literature as shown in the fig.(2.9). They are generated by incorporating symmetries and

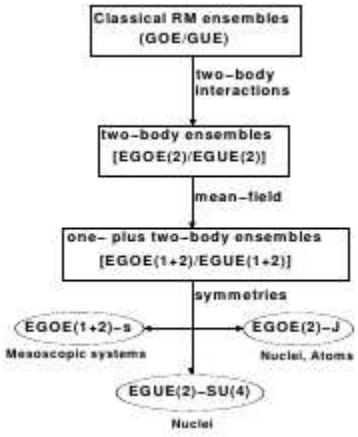

Figure 2.9: The information content of various random matrix ensembles. Also shown are the areas in which the embedded ensembles with various symmetries are relevant.

other information in the interactions. Besides EGOE(k) and EGOE(1+2), the other embedded ensembles are EGUE(k), TBRIM, RIMM, EGUE(2)-s, EGOE(2)-s, EGOE(1+2)-s [this ensemble is sometimes called RIMM [110] and TBRE-s [111]], BEGUE(k), BE-

GOE(k) and BEGOE(1+2). The ensembles generated by three-body, four-body interactions, etc., are also called 2-BRE, 3-BRE, N-BRE [112]. Going beyond these, with JT-symmetry for a two-body Hamiltonian we have EGOE(2)-J, which is nothing but the TBRE mentioned in the beginning. Adding a spherical one-body part will make the TBRE more realistic and the resulting ensemble is EGOE(1+2)-J (for nuclei EGOE(1+2)-JT) [7]. It is also called RTBRE [113]. Similarly the TBRE for a single j shell is called TBRE-j in [111]. Also studied in literature are displaced TBRE (called DTBRE) [114] where a constant is added to all the two-body matrix elements, a fixed Hamiltonian plus EGOE called K+EGOE [7], EGOE with particle-hole symmetry called RQE [115, 116], induced TBRE [110], EGOE(2) with good parity [117], EGOE with a partitioned GOE in 2-particle space called p-EGOE [7, 118], and finally EGUE(2)-SU (4) [119] with good spin-isospin SU(4) symmetry. For bosons there are studies of BEGOE(1+2)-L (also called BTBRE-L) with bosons in sp orbits [120] and sd orbits [121] with the Hamiltonian preserving the many-boson orbital angular momentum L, and also BEGOE(2) with SO(N1)$\oplus$ SO(N2) symmetry in the interacting boson model [122]. Although the GOE and GUE versions of embedded Gaussian ensembles have received attention, there are no studies yet of the GSE versions of these ensembles.

2.7.2 Definition Of The TBRE

The TBRE is defined within the framework of spherical nuclear shell model, which assumes that, nucleons move independently in a central potential with a strong spin-orbit force. Let us consider one of the major shells of that model. Numerical examples are calculated for the sd shell with the single particle-states labelled by $s_{1/2}$, $d_{3/2}$ and $d_{5/2}$ and single-particle energies $\epsilon_{1/2}$, $\epsilon_{3/2}$ and $\epsilon_{5/2}$. This can be generalised to other major shells of the heavier nuclei since number of many-particle states becomes forbiddingly large for numerical work. Sometimes, also a single j shell with half-integer single particle total spin j is considered to yield useful insights, although not realistic for nuclei. By putting several nucleons into a major shell, a basis of orthonormal antisymmetrised many-body states of fixed total spin J, parity P and isospin T. These states are labelled as $|\mathbf{J}\mu>$ with **J** standing for the quantum numbers J, P and T and with $\mu = 1, \ldots\ldots$, d(J) a running index with range given by the dimension d(J) of the Hilbert H(J). We will focus attention on a fixed but arbitrary z-projection of M of total spin J so that d(**J**) is the actual number of states not counting their degeneracy regarding M. In the middle of

the sd shell and for low values J, d(**J**) is typically of the order 10^3 and as much larger for heavier nuclei (other major shells). The actual construction used for the basis of states $|\mathbf{J}\mu>$ because basis resulting from from different modes of construction are connected by a unitary transformation. In the sd shell nuclei, all single-particle states have positive parity and therefore, quantum number P is omitted and more oftenly, we consider only sd shell states with isospin T =0, so that it suffices to label the many-body states by the total spin J only. The number of nucleons in the major shell is denoted by m and sometimes, we consider several nuclei simultaneously with different values of m. For this case, we denote the dimension of Hilbert space by d(J,m) and similarly for other quantities. The many-body states $|\mathbf{J}\mu>$ are eigenstates of the single-particle shell model Hamiltonian with a very high degree of degeneracy. The degeneracy is lifted when we take account of the residual interaction of the shell model. Also, it is assumed that the residual interaction mixes states only within the same major shell. However, this assumption is unrealistic in the sense that intruder states from higher shells occur even at low excitation energies, and mixing with higher shells is bound to play a major role at the upper end of the spectrum. The residual interaction is assumed to be two body although there are evidences for three body forces and also coulomb interaction betweeen the protons is neglected. In order to elucidate role of the residual interaction, it is assumed that the single-particle energies within a major shell are all degenerate. For the case of sd shell, it means that we put $\epsilon_{1/2} = \epsilon_{3/2} = \epsilon_{5/2} = 0$. Then the problem reduces to finding the residual interaction which entirely determines the full shell model Hamiltonian H. Let us consider the matrix elements $H_{\mu\nu}(\mathbf{J})$ with respect to the basis of states $|\mathbf{J}\mu>$.

The residual two-body interaction V_2 possess a finite number of two-body matrix elements within a major shell and we can arrive at the form of these in the following manner. Let $\mathbf{j_i}$ where i=1, 2, 3, 4 designate the four (equal or different) values of total single-particle spin, parity and isospin$1/2$. Coupling $\mathbf{j_1}$ and $\mathbf{j_2}$, ($\mathbf{j_3}$ and $\mathbf{j_4}$) to total two-body spin s_1 (s_2 respectively), denoting the parity of the resulting wavefunctions by π_1 (π_2 respectively), and introducing the notation s for the quantum numbers s, π, the reduced two-body matrix elements of V_2 within a major shell have the form $< \mathbf{j_3 j_4 s}||V_2||\mathbf{j_1 j_2 s} >$, where we have put $s_1 = s_2 = s$ because V_2 conserves spin, parity and isospin. The number 'a' of such two-body matrix elements within a major shell is limited as for example for the sd shell, $a = 63$ whereas for the case of a single j shell and identical nucleons, $a = j + 1/2$. The matrix elements are denoted by v_α with $\alpha = 1, 2, ...a$ and also the two-body specific operator whose matrix elements are designated by v_α. V_2 is completely characterised by the a matrix elements v_α within a major shell and is immaterial of the actual form ofV_2.

The Hamiltonian of the shell model is linear in the matrix element v_α and is of the form

$$H_{\mu\nu}(\mathbf{J}) = \sum_\alpha v_\alpha C_{\mu\nu}(\mathbf{J}, \alpha) \tag{2.96}$$

The matrices $C_{\mu\nu}(\mathbf{J}, \alpha)$ carry the two-body interaction into the Hilbert space $H(\mathbf{J})$ and depends on the quantum numbers **J** and on the particular states μ and ν and on the particular two-body operator α under consideration. The values of the $C_{\mu\nu}(\mathbf{J}, \alpha)$ s are completely specified by the underlying shell model, i.e., the single-particle states that occur within a given major shell, the coupling scheme used to construct the many-body states $|\mathbf{J}\mu >$, and the exclusion principle. The $C_{\mu\nu}(\mathbf{J}\alpha)$ depends upon the values of the matrix elements v_α and is independent of the choice of two-body interaction. The equation (105) gives the decomposition of H into parts that are determined by the symmetries of the shell model [the matrices $C_{\mu\nu}(\mathbf{J}\alpha)$] and matrix elements v_α that carry the information on specific details of two-body interaction. The aim is to give generic statements about spectral properties of H that apply (almost) to all two body interactions for which TBRE is employed. The matrix elements v_α are assumed to be uncorrelated Gaussian-distributed random varaibles with mean value zero and a common second moment v^2. Without loss of generality, we can put $v^2 = 1$ as all single particle energies are equal so that the scale of the spectrum is determined by v^2. Now the mean value of the observables is worked out by integrating the random variables v_α, the measure being given by the product of differentials of the v_α s and a Gaussian factor $\exp(-\sum_\alpha v^2\alpha/2)$. After calculating the mean values and square root of the variance of the observables, we are sure (within the error given by the latter) that the mean values applies to all memebers of the ensemble i.e., to all two-body interactions, with the exception of a set of measure 0. With the v_α s Gaussian random variables, the Hamiltonian $H_{\mu\nu}(\mathbf{J})$ represents an ensemble of Gaussian-distributed random matrices, the TBRE. Numerical studies have shown that the spectral fluctuations of the TBRE generally coincide with those of the GOE, that is because of the complete mixing of basis states $|\mathbf{J}\mu >$ by H and is independent of the specific choice of the v_α s and thus reflects a property of the matrices $C_{\mu\nu}(\mathbf{J}, \alpha)$. In order to acheive such mixing, almost every linear combination of these matrices must be sufficiently dense matrix in Hilbert space, with sufficiently complex matrix elements. This is rather a remarkable statement as the matrices are defined entirely in terms of an independent particle model (which is integrable). In principle, the $C_{\mu\nu}(\mathbf{J}, \alpha)$ can be worked out using group-theoretical methods and using the fact that the same possibility exists for the embedded Gaussian ensembles and has been used [123, 124]. Intuitively, that mixing property of the $C_{\mu\nu}(\mathbf{J}, \alpha)$ can be

understood by observing that each of the matrix element of $C_{\mu\nu}(\mathbf{J},\alpha)$ contains sums of products of Clebsch-Gordon and Racah coefficients and coefficients of fractional parentage. The combination of these three coeeficients become highly complex for more than three particles in a major shell, irrespective of the well defined nature and simplicity of the three coefficients. A more detailed discussion of of the properties of the matrices was given in [125].

2.8 Comparison of GOE and TBRE

As far as GOE is concerned it has three important properties. Firstly, it is invariant under orthogonal transformations (and hence is mathematically manageable). Secondly, it is universal and thirdly, it is ergodic. TBRE does not have any such such properties in common with GOE. In case of TBRE the set of matrices $C_{\mu\nu}(\mathbf{J},\alpha)$ is fixed and a unitary transformation of all matrices generates another representation of the ensemble and does not lead to the another memeber of the ensemble, which makes, therefore, TBRE non-unitarily invariant and is not orthogonally invariant. Till now, it is not clear whether TBRE is universal i.e., yields results that do not depend on the assumed Gaussian distribution of the matrix elements v_α or in otherwords it is not clear how a non-Gaussian distribution of v_α would affect the spectral fluctuations of the TBRE. Whereas in case of GOE, local spectral fluctuation properties and global spectral properties become separated in the limit N $\rightarrow \infty$ and this separation lies at the root of universality i.e., local fluctuation properties donot depend on the form of the distribution of the matrix elements. Also TBRE is not ergodic because the limit of infinite matrix dimension cannot be taken in a meaningful way except for the case of a single j shell, where j $\rightarrow \infty$ is a meaningful limit that has not been explored yet. Contrary to this fact GOE is ergodic which is proved by showing that correlation functions vanish with increasing distance of their energy arguments and also in the proof, N $\rightarrow \infty$ is made use of. Inspite of all these shortcommings of TBRE, it has certain attractive features as well. The TBRE produces spectra with Wigner-Dyson level statistics and at the same time TBRE does carry information content because the number of random variables is small compared to typical matrix dimensions. Ideally, it takes 'a' data points to completely determine the values of random variables in the TBRE and the number is typically small compared to the number of eigenvalues pertaining to fixed values of J, T, and π and this shows the important role played by the matrices $C_{\mu\nu}(\mathbf{J},\alpha)$ in the TBRE. These matrices are fixed by the geometry

of the shell model itself and these are responsible for the strong mixing of the shell-model configurations and the choice of residual interaction only determines the particular linear combination of the C's that forms the shell-model Hamiltonian $H_{\mu\nu}(\mathbf{J})$. In the GOE, the analogs of the matrices $C_{\mu\nu}(\mathbf{J},\alpha)$ exist and these are the $N(N+1)/2$ matrices G_μ which either have a unit element somewhere in the main diagonal and zeros everywhere else or have a unit element somewhere above the main diagonal , its mirror image below, and zeros everywhere else. The set $\{G_\mu\}$ forms a complete basis for real and symmetric matrices. In contradistinction, the matrices $C_{\mu\nu}(\mathbf{J},\alpha)$ do not form such a complete set. To be sure, every matrix $C_{\mu\nu}(\mathbf{J},\alpha)$ may be thought of as a linear combination of the G_μ. But the number of matrices $C_{\mu\nu}(\mathbf{J},\alpha)$ is typically much smaller than the number of independent random variables $N(N+1)/2$ in GOE. Therefore, many other linear combinations of the G_μ which are linearly independent of $C_{\mu\nu}(\mathbf{J},\alpha)$ and which do not occur in the TBRE and TBRE may be negatively defined by constraining all such linear combinations to be zero.

2.9 Conclusions

Random matrix theory has been applied to a huge number of fields with considerable success as described in this chapter by means of a figure 1. As far as physics is concerned, and in particular to nuclear physics, it has produced results and inferences that are quite consistent with the predictions from shell model calculations. The main motivation behind the introduction of random matrix theory in nuclear physics by Wigner in 1955 was to get an understanding about level and strength fluctuations. Another apparent reason for the use of RMT in nuclear physics one can cite, is that at higher excitation energies the level density becomes very high as is clear from equation (1) so that by the time one reaches, for example, at neutron threshold, E $\sim$ 6 MeV the nuclear models fail to provide finer details about the individual states of a quantum many-body system like atomic nuclei. Paraphrasing Wigner, the assumption made while applying random matrix theory to nuclear physics is that Hamiltonians which govern the behaviour of a complicated system is a random symmetric matrix with no particular properties except for its symmetric nature. The significant results that follow while applying random matrix theory to nuclear physics i.e., the inferences drawn from the random matrix ensembles are: (i) the nearest-neighbour spacing (S) distribution (NNSD) P(S) dS (of unfolded spectra)is well represented by Wigner's surmise $P(S)d(S) = Se^{-S^2}dS$ showing level repulsion ass discussed in section. (ii) the Dyson-Mehta Δ_3 statistic showing spectral rigidity as described

in section. (iii) the locally renormalised transition strengths (x) obey the Porter- Thomas law $P(x)dx \sim x^{-1/2}e^{x}$ as elaborated in section. The classical random matrix ensembles had been quite successful, specifically GOE in modelling the physical realities. As far as GOE is concerned, the spectral fluctuation properties of complex nuclear spectra often agree with the predictions of random matrix theory or to be more precise with those of GOE, the truth of which is established for resonances observed at neutron threshold and the coulomb barrier for protons [126] and also in a number of cases likewise to the levels at lower excitation energies [127]. The basic tenet of GOE is not in keeping with the shell model, which is basically a single-particle model with a residual interaction, the interaction of which is dominated by two-body forces and is the fundamental dynamical model of the nuclear physics [128]. In a representation where the many-body basis states are Slater determinants of single-particle states, a two-body interaction will have non-zero matrix elements only between those Slater determinants that differ by at most two units in occupation number of single-particle states. Of the total number of such determinants, this is a small fraction. In otherwords, in an arbitrary basis for the many-body states, the number of independent matrix elements of the two-body interaction is very much smaller than that of GOE and this fact is changed only quantitatively but not qualitatively when we allow for the three-body residual interaction. Around 1970s this fact led to the following question: Are the predictions of GOE for standard spectral fluctuation measures (nearest-neighbor spacing distribution and Dyson-Mehta statistics) consistent with the results of the shell model calculations with a residual two-body interaction.The answer, based on numerical calculations, has been affirmative [129, 106], and numerous more recent calculations have confirmed it [39] . The calculations were based on a random-matrix ensemble [the two-body random ensemble (TBRE)] that differs from the GOE and accounts for the specific properties of the nuclear shell model: The existence of a residual two-body interaction that conserves total spin, parity, and isospin. Unfortunately, this realistic feature of the TBRE poses a severe challenge for an analytical understanding precisely because the many-body states carrying fixed total, spin and isospin are very complex. As a consequence, very little is known analytically about the TBRE. There are several open questions and directions for future research as far as TBRE is concerned.(i) We are still lacking a deeper analytical understanding of the TBRE and its fluctuation properties. An analytical approach must be based on properties of the matrices denoted here by $C_{\mu}(J, \alpha)$. While a theoretical description for shells with several subshells is probably difficult, focusing on a single j shell might simplify the problem. (ii) The TBRE predicts correlations between spectra with different quantum numbers e.g., different masses, spins, or isospins for nu-

clei within a major shell. Experimental verification is difficult due to limitations in the length and completeness of observed nuclear spectra, but other Fermi systems might be more accessible. (iii) The correlations between spectra with different quantum numbers might also affect the scattering matrix, more precisely, such correlations might induced correlations among S-matrix elements carrying different total spin quantum numbers. The present analysis of fluctuating cross sections in compound nuclei neglects any such correlations. A better understanding of this problem would be highly desirable.

Chapter 3

Chaos Measures in Wave-functions and Transition Strength Distributions

3.1 Introduction and Review of Literature

There has been an unprecedented growth in the use of random matrix theory to quantum systems particularly in the context of quantum chaos. There are number of chaos and complexity measures for quantifying the quantum chaos. Among them are number of principal components, information entropy in wave-functions and transition strength distributions, transition strength and transition strength sums. For example, the statistical properties of total Gamow-Teller strength as a function of excitation energy is related to regular or chaotic features of nuclear dynamics and this strength has astrophysical importance in pre-supernova evolution and stellar collapse. In fact, the smoothed behaviour of the total Gamow-Teller strength versus excitation energy will be adequate for many astrophysical purposes and it will give information about order-chaos transitions, just as energies, wave-function amplitudes, and transition strengths.

In this unit we are going to discuss about the chaos measures in wave-functions and transition strength distributions. In the section 1 introduction an overview of the literature regarding the chaotic measures in wave-functions and transition strength distributions shall be covered. This will be followed by the basic results for (1+2)-body random matrix ensembles. In the next section we are going to re-derive the expresions for chaotic measures (number of principal components and information entropy) in wave-functions and

transition strength distributions. Also in this section a brief discussion on the comparison of exact shell model calculations and GOE and EGOE predictions shall be discussed, supplemented by some results. In the last section we shall introduce transition strength sums as a measure of chaos.

The study of quantum chaos in finite-interacting many particle quantum systems has underwent a change from the study of spectral statistics to the study of wave-functions and transition strength distributions (for example, electromagnetic and Gamow-Teller transition strengths in atomic nuclei, dipole strengths in atoms etc.). There has been a great deal of spectroscopic activity using the measures of chaos and complexity like the number of principal components and localisation length in wavefunctions and transition strength distributions. The results obtained from such studies is being then tested against the predictions from realistic EGOE(1+2) ensembles. For example, the number of principal components and localization length in E_2 and M_1 transitions strengths in ^{46}V measuring complexity and chaos in transition strength from an eigenstate with energy E_i has been studied and results when compared with the predictions from EGOE(1+2) in the Gaussian domain, a good agreement has been found [43]. The study of eigenvalue amplitudes of many fermionic systems and construction of information entropy, number of principal components and similar other measures of complexity and chaos in the system is of great current interest. Firstly, the investigations of Izrailev [130] and detailed study of nuclear shell model studies by Zelevinsky and collaborators [131, 39] established the importance of these measures. Further, these studies confirmed that Gaussian orthogonal ensemble (GOE) of random matrices is totally inadequate to explain the strong energy dependence of these quantities. The measures of chaos and complexity, that is, the localisation length related to the information entropy had also been calculated with the nuclear shell model wave-functions in large shell model basis states for several Ca, Sc and Ti isotopes, and compared to the predictions of embedded Gaussian orthogonal ensemble. The dimensionalities involved in the calculations are so large, upto many thousands, ensures good statistics and there is a good agreement in the chaotic region (central region) of the energy spectrum, while some deviations are observed at ground-state region. Also, from these studies, it has been established that localisation length of shell-model wave-functions in Ca isotopes is much smaller than in Sc, showing a strong dependence of nuclear chaos, in good agreement with previous results based on energy level fluctuation properties [132]. The formulas for the complexity and chaos measures like information

entropy and number of principal components has been derived by using the results from statistical spectroscopy or to be more accurate from the EGOE, of the bivariate Gaussian forms for smoothed strength densities in transition strength distributions. These measures describe the shell model results in terms of the bivariate correlation coefficient ζ and which reduce to GOE results for $\zeta = 0$ [133]. The chaos measures, number of principal components(NPC) and information entropy (S^{inf}) are normally defined for the eigenfunctions expanded in terms of a given set of basis states. However, imagining that a given basis state is a compound state generated by the action of a transition operator on an eigenstate with energy E, it is possible to extend the measures NPC and S^{inf} for transition strength distributions [133]. The inverse participation ratio (or NPC) of an eigenstate is the effective number of basis functions contributing to it. It provides a measure for the presence of chaos in the system. For example, smallness of NPC signifies presence of collective states [134], and also NPC can be used as a measure for defining the region of onset of chaos in the spectrum [135]. Interestingly, it was also employed, without actually realisingly the connection to the work of [133], in the study of rotational damping using the particle-rotor model [136]. NPC and S^{info} in transition strength are significant because transition strength are observables, while wave-functions in general are not observables. Here, we will discuss these measures for the EGOE(1+2) ensemble operating in the Gaussian domain. Working along these lines i.e., studying quantum chaos in finite-interacting many-body systems, using transition strengths and wave-functions, several research groups have recognised the fact that the two-body random matrix ensembles and their various extensions form good models for understanding various aspects of chaos in interacting particle systems [7]. In particular, using the so called EGOE(1+2), embedded Gaussian orthogonal ensemble of (1+2)-body interactions defined by a mean-field one-body plus a chaos generating random two-body interaction, several studies has been made on the nature of occupancies of single particle states, strength functions (or local density of states), information entropy, transition strength sums and transition matrix elements of one-body transition operators, Fock-space localisation by etc. in the chaotic domain of interacting particle systems such as atoms [137], nuclei [7, 39], quantum dots [138, 139, 140], quantum computers [34, 141] and so on. Reference [7] gives a overview of the subject. The common feature shared by the Hamiltonians for all these kinds of systems consists of mean-field one-body part plus a complexity generating two body interaction. With one plus two-body interactions [$H = h(1) + \lambda V(2), \lambda$ is a parameter] one has EGOE(1+2) (the embedded Gaussian orthogonal ensemble of one plus two-body interactions). For EGOE(1+2), h(1) is fixed (or an ensemble) with some average single-

particle level spacing with unit variance for the matrix elements, so that λ is the interaction strength in units of average single-particle level spacing. With m fermions distributed over N single particle states, firstly it is a well known fact that the EGOE(1+2) state density is Gaussian for all λ values and in the strict sense of the word, it is Gaussian in the dilute limit defined by $m \to \infty$, N $\to \infty$, and $\frac{m}{N} \to 0$. Two important chaos markers λ_c and λ_f are also known for EGOE(1+2) [139, 142, 143, 144, 145]. For $\lambda > \lambda_c$ there is chaos in the sense that the level fluctuations start coming close to GOE fluctuations; λ_c marks the transition from Poisson to GOE. Similarly as λ is increasing from λ_c, the strength functions change from Breit-Wigner (BW) to Gaussian form (the BW to Guassian transition was discussed first by Lewenkopf and Zelevinsky [146]). The $\lambda = \lambda_F$ (it is to be noted that $\lambda_F > \lambda_C$) marks the onset of Gaussain and the $\lambda > \lambda_F$ region is called the Gaussian domain and here not only the state densities and strength functions are Gaussian and level (strength) fluctuations follow GOE but also the bivariate transition strength densities take bivariate Guassian form [7]. The region $\lambda_C \leq \lambda \leq \lambda_F$ region is called Breit-Wigner (BW) domain. Shell model with realistic interactions has established the operation of quantum chaos, and EGOE(1+2) in Gaussian domain, for 2s1d shell nuclei [7, 2, 39]. In this unit the two measure of chaos (in wave functions and transition strength distributions): (i) number of principal components NPC (or the inverse participation ratio); (ii) localisation length l_H as defined by the information entropy (S^{info}) will be discussed. It is well established that the NPC in wavefunctions characterises various layers of chaos in interacting particle systems [147]. NPC for transition strengths is a measure of fluctuations in transition strength sums. Similarly the role of l_H in quantum chaos studies is well emphasized by Izrailev [130] and more significantly, using nuclear physics examples [148]. It is well demonstrated that the wave-function entropy S^{info} coincides with the thermodynamic entropy for many particle systems with two-body interactions of sufficient strength but only in the presence of mean-field, i.e., in the chaotic domain but with mean-field - therefore the significance of EGOE(1+2). Clearly deriving the predictions of EGOE(1+2) for NPC and l_H are of considerable importance. The problem was addressed in [149, 150]. In [149] results for NPC in wave-functions, in the so called Breit-Wigner (BW) domain, are derived. On the other hand in [150] results in so called Gaussian domain are derived for NPC and $\mathbf{l_H}$ in transition strength distributions with only the final results mentioned for wave-functions.

3.2 Basic Results For (1+2)-Body Random Matrix Ensembles

By distributing m fermions in N single particle states, assuming at the very outset that many-particle spaces are direct-product spaces of the single-particle states , two-body random matrix ensembles (usually called TBRE) are generated by defining the Hamiltonian H, which is 2-body, to be a random matrix in the 2-particle spaces and then propagating it to the $\binom{N}{m}$ dimensional m-particle spaces by using their geometry (direct product structure); often one considers a GOE representation in the 2-particle spaces and then the TBRE is called EGOE(2). More details regarding it are given in reference [7]. For EGOE(2), with $N >> m >> 2$, the normalized state density $\rho(E) = \ < \delta(H-E) >$ take Gaussain form and is defined by the centroid $\epsilon = \ < H >$ and variance $\sigma^2 = \ < (H-\epsilon)^2 >$. In order to explicitly state that the state density is generated by the Hamiltonian H, sometimes $\rho(E)$ is denoted as $\rho^H(E)$ and similarly ϵ as ϵ_H and σ as σ_H. The averages $<>$ are over the m-particle spaces and in case of nuclear physics examples, they are usually over the m-particle spaces with fixed angular momentum (J) and isospin (T) which are good quantum numbers. Just as with the state density, given a transition operator O, the normalized bivariate strength densities (matrix elements of O weighted by the state densities at the initial and final energies) $\rho(E_i, E_f) = \left[< O^\dagger O > \right]^{-1} < O^\dagger \delta(H-E_f) O \delta(H-E_i) >$ take bivariate Gaussian form EGOE(2) and it is defined by the centroids (ϵ_i, ϵ_f) and widths (σ_i, σ_f) of its two marginals and the bivariate correlation coefficient is given by $< O^\dagger[(H-\epsilon_f)/\sigma_f] O[(H-\epsilon_i)/\sigma_i] > / < O^\dagger O >$. Thirdly, the level and strength fluctuations follow GOE. Also with the Gaussian forms for the state densities and bivariate Gaussian forms for the strength densities, the strength sums $< E|O^\dagger O|E > \ = \sum_{E'} | < E'|O|E > |^2$ take the form of ratio of two Gaussians, $< E|O^\dagger O|E > \ = \rho_{O^\dagger O:G}(E)/\rho_G(E)$ where $\rho_{O^\dagger O:G}(E) = \ < O^\dagger O \delta(H-E) >$ is defined by its centroid $\epsilon_{O^\dagger O} = \ < O^\dagger O H > / < O^\dagger O >$ and variance $\sigma^2_{O^\dagger O} = \ < O^\dagger O H^2 > / < O^\dagger O > - \epsilon^2_{O^\dagger O}$ where G stands for Gaussian.

However, for realistic interacting particle systems we have a mean-field part [one body part h(1)] and a two-body residual interaction, which mixes the configurations built out of the the distribution of particles in the field single-particle states where h(1) is defined by the single-particle energies ϵ_i, i= 1,2,.....,N and V(2) is defined by its two-body matrix elements. It is to be noted that all the EGOE results mentioned above are indeed applicable to EGOE(1+2), but only in the domain of chaos. Given (m, N) and the average

spacing Δ [generated by h(1)] of the single-particle states (without loss of generality one can put($\Delta = 1$), it is possible to find the critical λ value λ_c such that for $\lambda \geq \lambda_c$, there is the onset of chaos (GOE fluctuations) in many ($m >> 1$) particle spaces. In fact, λ_c is of the order of spacing between m-particle mean-field basis states that are directly coupled by the two-body interaction. For details in this regard see [139, 140]. For $\lambda > \lambda_c$, for instance it has been well established that the transition strength sums in EGOE(1+2) follow the EGOE forms as shown in fig. (3.1). The most useful quantity for deriving the formulas for NPC and l_H in wave-functions is the strength function or local density of states $F_k(E)$. Given the mean-field basis states $|k>$ with energies $E_k = \ < k|H|k >$, the eigenstates $|E >$ can be expanded as $|E > \ = \sum_k C_k^E |k >$. Then the strength function $F_k(E) = \ < \delta(H-E)^k > \ = \sum_{E'} |C_k^{E'}|^2 \delta(E-E')$ and therefore it gives information about the structure of wave-functions. In order to proceed further, let us say that the E_k energies are generated by a Hamiltonian H_k (the structure of H_k is discussed ahead). With this, it is easy to identify $F_k(E)$ as a conditional density of of the bivariate $\rho_{biv}(E, E_k) = \ < \delta(H-E)\delta(H_k - E_k) >$. Taking degeneracies of E and E_k into account, we have

$$\begin{aligned} \rho_{biv}(E, E_k) &= < \delta(H-E) >< \delta(H_k - E_k) > \\ &= \Big(\frac{1}{d}\Big) \sum_{\alpha \in k, \beta \in E} |C_{k,\alpha}^{E,\beta}|^2 \\ &= \Big(\frac{1}{d}\Big) \overline{|C_k^E|^2} [d\rho^H(E)][d\rho^{H_k}(E_k)] \end{aligned} \tag{3.1}$$

$$\begin{aligned} F_k(E) &= \rho_{biv}(E, E_k)/\rho^{H_k}(E_k) \\ \overline{|C_k^E|^2} &= \rho_{biv}(E, E_k)/[d\rho^H(E)][d\rho^{H_k}(E_k)] \end{aligned} \tag{3.2}$$

In the above equations, d stands for the dimensionality of the m-particle spaces and $\overline{|C_k^E|^2}$ is the average of$|C_k^E|^2$ over all the degenerate states. Now let us try to have look over the structure of H_k and $\rho(E, E_k)$. It should be noted that the two-body interaction V(2) can be decomposed into two parts $V(2) = V^{[0]} + V$ so that $h(1) + V^{[0]}$ generates the E_k energies (diagonal matrix elements of H in the m-particle mean-field basis states). By distributing the m particles in N single particle states, there is an underlying U(N) group and with respect to this group $V^{[0]}$ contains a scalar part $V^{[0],0}$ (a function of m), an effective (m- dependent) one-body (Hartree-Fock-like) part $V^{[0],1}$ and an irreducible two-body part$V^{[0],2}$. The $V^{[0],0} + V^{[0],1}$ will add to h(1) giving an effective one-body part

of H; h(1)$\Rightarrow h(1) + V^{[0],0} + V^{[0],1} = h$. The important point to be worth noticing is that, with respect to a U(N) norm, the size of $V^{[0],2}$ is usually very small compared to the size of h in the m-particle spaces.

3.3 Chaos Markers λ_c, λ_f and λ_t

As is well known that the realistic systems such as nuclei contain a mean-field one-body interaction, which is defined by a set of single-particle states, plus a complexity generating two-body interaction, so the appropriate random matrix ensemble for their description is EGOE(1+2), first studied by Flambaum et al. [151] is defined by

$$\{H\} = h(1) + \lambda\{V(2)\}$$

where $\{\}$ denotes an ensemble. The mean-field one-body Hamiltonian $h(1) = \sum_i \epsilon_i n_i$ is a fixed one-body operator defined by the single particle energies ϵ_i with average spacing Δ, where n_i is the number operator for the single-particle state $|\nu_i >$ and in general one can choose the ϵ's to form an ensemble. The $V(2)$ is the EGOE(2) with the unit variance for two-body matrix elements, which form the GOE and λ is the strength two-body interaction in units of Δ. Hence, EGOE(1+2) is defined by the four parameters m,n, Δ and λ and Δ =1 without loss of generality. The construction of EGOE(1+2) is discussed in chapter 2. Before proceeding further let us not forget to mention that EGOE(1+2) with h(1) a fixed Hamiltonian, usually generating a single-particle spectrum is called the two-body random interaction model (TBRIM) by Flambaum and Izrailev [151]. If h(1) is defined with single-particle energies drawn from the eigenvalues around the centre of the semicircle density of a GOE (or a GUE), it is called random interaction matrix model (RIMM) by Alhassid [152]. Alternatively jacquod et al. [139, 140, 153] considered RIMM with single-particle energies random, such that ϵ_i = Δ + δ_i, where δ_i are uniform random variables.

The important aspect about EGOE(1+2) is that as λ changes, in terms of state density, level fluctuations, strength functions and entropy, the ensemble is described by three chaos markers. Firstly, the state densities $\rho^{H,m} = < \delta(H - E) >^m$ take Gaussian form, for large enough m, for all λ values. This follows from the fact that EGOE(2) gives Gaussian state densities and also in general the h(1)'s produce Gaussian densities. From

now on the superscripts H or m or both in $\rho^{H,m}$ will be dropped as long as there is no confusion. With the increase in λ, there is a chaos marker λ_c such that for $\lambda \geq \lambda_c$ the level fluctuations follow GOE, i.e.,λ_c marks the transition in nearest-neighbour spacing distribution from Poisson to Wigner form. This transition occurs when the interaction strength λ is of the order of the spacing Δ_c between the states that are directly coupled by the two-body interaction. This definition has the origin from nuclear structure calculations by Aberg [154]. Thus, for the Poisson to Wigner transition chaos marker, $\lambda_c \propto 1/m^2N$ [139]. Given mean-field h(1) basis states $|k> \; = \sum_E C_k^E|E>$, the strength functions are defined by $F_k(E) = \sum_{\beta \in E} |C_k^{E,\beta}|^2 = \overline{|C_k^E|^2} d\rho^H(E)$ where d is the dimension of m-particle space. As λ increases further from λ_c, the strength functions change from Breit-Wigner (BW) [149] to Gaussian form and the transition point is denoted by λ_F. The Breit-Wigner to Gaussian chaos marker λ_F can be understood as follows. Firstly, there are two scales in EGOE(1+2) with the first one being Δ_c, the other one being the m-particle spacing Δ_m. The estimation using the h(1) spectrum, for the spectrum spanned by m-particle spectrum is $B_m = m(N-m) \simeq mN$, we have $\Delta_m = mN/d(N,m)$. The Fermi golden rule gives the spreading width to be $\Gamma \propto \lambda^2/\Delta_c \sim mN\lambda^2$ [149]. Thus, participation ratio is $\zeta \propto \Gamma/\Delta_m = \lambda^2 md(N,m)$. For the BW domain $\Gamma < B_m/f_0$ where $f_0 > 1$ and $\zeta >> 1$. This gives $\frac{1}{\sqrt{md(N,m)}} << \lambda < \frac{1}{f_0 m}$ [155, 156]. As d(N,m) is usually large, the BW form sets in fast and $\lambda_F \propto 1/\sqrt{m}$. The $\lambda_c \leq \lambda \leq \lambda_F$ region is called the BW domain, with the strength functions close to Gaussian form. In principle, the BW form starts in a region below λ_c. There is a λ_0 such that below λ_0, the strength functions are close to a δ-function form and for $\lambda > \lambda_0$ there is onset of the BW form, but fluctuations here will be close to Poisson for $\lambda < \lambda_0$. This transition from BW to Gaussian was first recognized by Zelevinsky et al. ^{24}Mg shell-model results [157] and it has been shown to be a feature of EGOE(1+2) by Kota and Sahu [158]. Fig. 3.2 shows the BW to Gaussian transition in atoms. An important question concerning isolated finite interacting particle systems is [159] that in the chaotic domain will there be a point or a region where thermalization occurs i.e., there will be a region where different definitions of entropy, temperature, specific heat and other thermodynamic variables give the same results, as for infinite systems. For obvious reasons, this has to happen beyond λ_f and this gives the third chaos marker λ_t. To understand this marker, in the Gaussian domain of EGOE(1+2), three different entropies are considered: thermodynamic (S^{therm}), information (S^{inf}) and single-particle (S^{sp}) entropies. The definitions of thermodynamic and single-particle entropy is given as $(S^{therm})_E = ln\rho^{H,m}(E)$ and $(S^{sp})_E = -\sum\{< n_i >^E ln(< n_i >^E$

$+(1-(< n_i >^E) \quad ln(1-(< n_i >^E)\}; < n_i >^E$ is the occupancy of the ith single-particle state energy E. The EGOE(1+2) formulas, for the three entropies are [158, 160]

$$\left.\begin{array}{l} exp[(S^{ther})_E - (S^{ther})_{max}] \rightarrow exp\Big(-1/2\hat{E}^2\Big) \\ \hat{E} = (E - \epsilon_H(m))/\sigma_{\sigma_H}(m) \\ exp[(S^{inf})_E - (S^{info})_{GOE}] \rightarrow \sqrt{1-\zeta^2} exp\Big(1/2\zeta^2\Big) exp\zeta^2\hat{E}^2/2 \\ exp[(S^{inf})_E - (S^{info})_{GOE}] \rightarrow exp\Big(-\frac{1}{2}\zeta^2\hat{E}^2\Big) \\ \zeta^2 \sim \sigma_h^2/\sigma_h^2(H) = \sigma_h^2/(\sigma_h^2 + \lambda^2\sigma_V^2) \end{array}\right\}. \qquad (3.3)$$

Here $\epsilon_H(m)$ is the spectrum centroid and $\sigma_H^2(m)$ is the spectral variances. Results stated above are compared with numerical EGOE(1+2) calculations in fig.(3.3) and can be understood as follows [161]. For H = h(1) + V(2) with h(1) defined by single-particle level spacing Δ and V(2) with matrix elements variance λ^2, there are two natural basis defined by h and V respectively. Then for the thermodynamic considerations to apply, the entropy measures should be independent of the chosen basis. Firstly, in the dilute limit h and V will be orthogonal. The variance of h in m-particle space is $\sigma_H^2(m) = [(mN^2)/12]\Delta^2 = f^2\Delta^2$. Similarly, the variance of V is $\sigma_V^2(m) \sim [(m^2N^2)/4]\lambda^2 = g^2\lambda^2$. The S^{info} and S^{sp} are determined by ζ and for, for strength functions expanded in h(1) basis, $\zeta_0(\lambda) = \sqrt{(g^2\lambda^2)/(f^2\Delta^2 + g^2)\lambda^2}$. Now the obvious thing is that as $\lambda \rightarrow \infty$, ζ_0 goes close to zero. Similarly when $\Delta \rightarrow \infty$, ζ_∞ gets close to zero. In both of these situations S^{info} takes GOE values and S^{sp} approaches its maximum value. The condition $\zeta_0(\lambda_t) = \zeta_\infty(\lambda_t)$ gives $\lambda_t = |\Delta/g|$ and here $\zeta^2 = 0.5$. Also note that $\lambda_t \sim \Delta/(3m)^{1/2}$. With λ_t defined, it is easily seen that $\zeta_\infty(\lambda) = \zeta_0(\lambda_t^2/\lambda)$, thus there is a duality in EGOE(1+2) and at the duality point$\lambda = \lambda_t$ the entropies are basis independent. Moreover at this point $\zeta^2 = 0.5$; i.e. the spreadings produced by h and V are equal. Using eqn. (3.3), it is easily verified that at and around $\zeta^2 = 0.5$, all the three entropies will be close to each other (see fig 3). Thus $\lambda \sim \lambda_t$ with $\zeta^2 \sim 0.5$ defines the thermodynamic region for interacting particle systems. Comparison of the figure with shell model calculations by Horoi et al. [148] for ^{28}Si and by Kota and Sahu [160] for ^{28}Mg, it is seen that the nuclei are in general in the thermodynamic regime (i.e., $\lambda \sim \lambda_t$).

The three chaos markers of EGOE(1+2) are summarized as shown in fig.(3.4) and for more details see [161, 162]. The important point that needs to be mentioned here that the broad structure shown in figure is a general feature of EGOE(1+2)'s with additional

good quantum numbers [163] and also for BEGOE [164, 165]. In the study of multi-partite entanglement and fidelity decay in the context of quantum computers and quantum information theory the importance of embedded random matrix ensembles and the BW and Gaussian domains defined by the chaos markers have been recognized [166, 167, 168, 169]. This defines a new roadmap for the future developments in embedded ensembles.

3.4 EGOE(1+2) Results for NPC and l_H in Wave-functions

For EGOE(1+2), in the chaotic domain with $\lambda > \lambda_{F_k}$ from the previous section we have the results: (i) E_k are generated by $H_k = h(1)$, therefore the variance of $\rho^{H_k}(E_k)$ is $\sigma^2_{\mathbf{h}}$; (ii) widths of the strength functions are constant and are generated by V(2), the average variance $\overline{\sigma^2_k} = \sigma^2_{\mathbf{V}}$; (iii) $F_k(E)'s$ are in Guassian form; (iv) $F_k(E)$ is a conditional density of the bivariate Gaussian $\rho_{biv:G}(E, E_k)$. The correlation coefficient ζ of $\rho_{biv:G}(E, E_k)$ is given by

$$\zeta = \frac{< (H - \epsilon_H)(H_k - \epsilon_H) >}{\sqrt{< (H - \epsilon_H)^2 >< (H_k - \epsilon_H)^2 >}} = \sqrt{\Big(1 - \frac{\overline{\sigma^2_k}}{\sigma^2_H}\Big)} \tag{3.4}$$

The centroids of the E and E_k energies are both given by $\epsilon_H = \ < H >$. In the the above equation, the second equality is obtained by using the orthogonality between h(1) and V(2) operators. It can be immediately seen that ζ^2 is nothing but the variance of E_k's [the centroids of $F_k(E)$] normalised by the state density-variance. The $\rho_{biv:G}(E, E_k)$, which takes into account the fluctuations in the centroids of $F_k(E)$ and assumes that variances are constant, is used to derive formula for NPC and l_H in the wavefunctions (methods of taking into account variance fluctuations will be discussed ahead) $\psi_E = |E >$ expanded in the mean-field basis defined by the states ϕ_k. Before proceeding further, let us define NPC and l_H.

$$|E >= \sum_k C^E_k |k >$$

$$\Longrightarrow (NPC)_E = \Big[\sum_k |C^E_k|^4\Big]^{-1} \tag{3.5}$$

$$l_H(E) = exp[(S^{info})_E]/(0.48d) \tag{3.6}$$

$$(S^{info})_E = -\sum_k |C^E_k|^2 ln|C^E_k|^2 \tag{3.7}$$

In the eq.(3.6) 0.48d is the GOE value for S^{info}, thus, $l_H = 1$ for GOE. Similarly, NPC is $d/3$ for GOE. In terms of the locally renormalized amplitudes $\mathcal{C}_k^E = C_k^E/\sqrt{\overline{|C_k^E|^2}}$ where the bar denotes the ensemble average with respect to EGOE(1+2), $\sum_k |C_k^E|^4 = \sum_k |\mathcal{C}_k^E|^4 \overline{(|C_k^E|^2)^2}$. Then the ensemble averaged $(NPC)_E$ is obtained as follows.

$$\overline{|C_k^E|^4} \stackrel{EGOE(1+2)}{\longrightarrow} \sum_k \overline{|\mathcal{C}_k^E|^4}\,\overline{(|C_k^E|^2)^2} \tag{3.8}$$

In the above step use has been made of the fact that EGOE exhibits average fluctuations separation (with little communication between the two). For example, in the normal mode decomposition of the EGOE state density, it is seen that the long wavelength parts generate the smooth Gaussian density (with corrections) and the short-wavelength parts the GOE fluctuations with the damping of the intermediate ones (see [2, 108, 170, 171] for a detailed discussions on this important result). This allows to carry out the $|\mathcal{C}_k^E|^4$ ensemble average independent of the other smoothed (average) term. Now using the fact that the local fluctuations follow Porter Thomas and thus $\overline{|\mathcal{C}_k^E|^4} = 3$, a GOE result. Hence the above equation becomes

$$\overline{|C_k^E|^4} = 3\sum_k \overline{(|C_k^E|^2)^2} \tag{3.9}$$

Finally, using the result from previous section that Gaussian form, valid in the chaotic domain ($\lambda > \lambda_{F_k}$), of all densities for EGOE(1+2) gives the final formula.

$$\begin{aligned} \overline{|C_k^E|^4} &= \frac{(3/d)}{[\rho_G^H(E)]^2}\int dE_k \frac{[\rho_{biv:G}(E,E_k)]^2}{\rho_G^{H_k}(E_k)} \\ &= \frac{(3/d)}{[\rho_G^H(E)]^2}\int dE_k \rho_G^{H_k}(E_k)[F_{k:G}(E)]^2 \end{aligned} \tag{3.10}$$

Hence the final form of NPC is

$$(NPC)_E = (d/3)\sqrt{1-\zeta^4} exp - \left\{\frac{\zeta^2 \hat{E}^2}{1+\zeta^2}\right\} \tag{3.11}$$

This result was quoted first in [133] without details. Before turning to the formula for localization length l_H, let us briefly discuss about the corrections to eq.(3.11) due to the fluctuations in the variances of $F_k(E)$; the form with $F_k(E)$ the form with $F_k(E)$ shown explicitly, is written in eq.(3.11) for this purpose and this form also allows one to understand the results equation [172] as discussed ahead. The correction to NPC due to $\delta\sigma_k^2 = \sigma_k^2 - \overline{\sigma_k^2}$ is obtained by using, for small $|\sigma_k^2|$, the Hermite polynomial expansion

which gives [173], $F_{k:G}(E) \rightarrow F_{k:G}(E)\{1+c_2(\mathcal{E}_k^2-1)\}$ where $c_2 = \frac{\delta\sigma_k^2}{2\overline{\sigma_k^2}}$ and $\mathcal{E}_k = \frac{(E-E_k)}{\sqrt{\overline{\sigma_k^2}}}$. This corrected $F_k(E)$ is used in the integral form with $F_k(E)$ in eq.(3.11). As NPC involves sum over all the $|k>$ states, it is a valid assumption to treat $\delta\sigma_k^2$'s as a random in $[F_k(E)]^2$ only the terms that are quadratic in (σ_k^2) will contribute (see [172]). Replacing $[\frac{\sigma_k^2}{\overline{\sigma_k^2}}]$ by $\frac{\sigma^2}{\overline{\sigma_k^2}} = [(d)^{-1}\{\sum_k(\delta\sigma_k^2)^2]^{1/2}/\overline{\sigma_k^2}$ and substituting the corrected $F_k(E)$ for $F_{k:G}(E)$ in eq.(3.11), we get

$$\begin{aligned}(NPC)_E &= \frac{(3/d)}{[\rho_G^H(E)]^2}\int_{-\infty}^{+\infty} dE_k \frac{[\rho_{biv:G}(E,E_k)]^2}{\rho_G^{H_k}(E_k)} \times \Big(1+\frac{(\delta\sigma^2)}{2\overline{\sigma_k^2}}(\mathcal{E}_k^2-1)\Big)^2 \\ &= (d/3)\sqrt{1-\zeta^4}exp-\Big\{\frac{\zeta^2\hat{E}^2}{1+\zeta^2}\Big\} \times \Big\{1+\frac{1}{4}\Big[\frac{(\delta\sigma^2)}{\sigma_H^2}\Big]^2 X(E)\Big\}^{-1} \quad (3.12)\end{aligned}$$

where

$$X(E) = \frac{1}{(1+\zeta^2)^4}\left[\hat{E}^4 - 2\frac{(1+\zeta^2)(1-2\zeta^2)}{1-\zeta^2}\hat{E}^2 + \left(\frac{1+\zeta^2}{1-\zeta^2}(1+2\zeta^4)\right]\right. \quad (3.13)$$

The $\delta\sigma^2$ correction term in the above eq.(3.12) is valid only when the fluctuations in the variances of $F_k(E)$'s are small and this is in general true. For small ζ values, this formula for NPC in the above eq.(3.12) reduces to the expression given by Kaplan and Papenbrock [172] for EGOE(2) where they used the idea related to the scar theory. For EGOE(1+2) Hamiltonian $H = h(1) + \lambda V(2)$, with $\lambda \rightarrow \infty$ one obtains EGOE(2) and then it is clear from the definition given in that in this limit $\zeta \sim 0$. To be more precise, with $N >> m >> 1$, $\zeta^2 \sim (\binom{N}{2})^{-1}$ and $[(\delta\sigma^2)/\sigma_H^2)]^2 \sim [\binom{m}{2}][\binom{N}{2}]^{-1}$ for $H = V(2)$. Therefore, for finite N, the correlation coefficient and the variance corrections are small but nonzero and in the large N limit, they are zero giving the GOE result as pointed out in [150]. As we add the mean-field part to the EGOE(2), ζ increases and at the same time the variance correction decreases. Thus, the formula with $(\delta\sigma^2)$ term is important only for small ζ. Eq. (3.11) is accurate for reasonably large ζ (say for $\zeta \geq 0.3$) as in the examples discussed in [150]. All these results are well tested by the numerically. Proceeding exactly as in equation of NPC, formula for the localization length l_H as a function of excitation energy is derived. Firstly, using the definition of l_H and writing $|C_k^E|^2$ in terms of $|\mathcal{C}_k^E|^2$ and $|C_k^E|^2$ as there occurs separation of averages and fluctuations. Then using the GOE results $|C_k^E|^2 = 1$ and $\overline{|C_k^E|^2 ln(|C_k^E|^2)} = 1 = -ln0.48$. Finally applying the eq.(3.2) and replacing all the densities by their Gaussian forms and converting the sum in eq.(3.5) into an integral and finally carrying over the integration, the expression for l_H in wave-

functions is obtained,

$$l_H \overset{EGOE(1+2)}{\longrightarrow} -\int dE_k \frac{\rho_{biv:G}(E,E_k)}{\rho^H_G(E)} ln\left\{\frac{\rho_{biv:G}(E,E_k)}{\rho^{H_k}_G(E_k)\rho^H_G(E)}\right\}$$
$$= \sqrt{1-\zeta^2} exp(\frac{\zeta^2}{2}) exp(\frac{\zeta^2\hat{E}^2}{2}) \tag{3.14}$$

the result in above equation was reported in [150] without details. By rewriting the integral in the above equation in terms of $F_k(E)$ and making small $(\delta\sigma^2)$ expansion just in the case of NPC, the formula incorporating corrections due to fluctuations (with respect to k) in the variances of $F_k(E)$ is derived following the arguments that led to eq.(3.12). Neglecting the higher order terms in $[(\delta\sigma^2)/\sigma^2_H)]$, the final result is

$$l_H(E) = \sqrt{1-\zeta^2} exp(\frac{\zeta^2}{2} exp\Big[-\Big(\frac{\zeta^2\hat{E}^2}{2}\Big] \times \Big(1-\frac{1}{8}\Big[\frac{(\delta\sigma^2)}{\sigma^2_H}\Big]Y(E)\Big); \tag{3.15}$$

where

$$Y(E) = \frac{1}{(1-\zeta^2)^2}\Big\{(1-\zeta^2)^2(\hat{E}^2-1)^2 + 4\zeta^2(1-\zeta^2)\hat{E}^2 + 2\zeta^4\Big\} \tag{3.16}$$

3.4.1 Derivation Of Number Of Principal Components For Transition Strength Distributions

The two important results of statistical spectroscopy are that in strongly interacting shell model spaces (essentially in $0\hbar\omega$ spaces). (i) the state densities take Gaussian form and (ii) the bivariate strength densities take bivariate Gaussian form. These results have their basis in the EGOE representation of the Hamiltonian H (which is in general one plus two-body in nuclear case).

$$I(E) = \quad << \delta(H-E) >> \quad = d \times < \delta(H-E) > \quad = d \times \rho(E) \tag{3.17}$$

$$\rho(E) \overset{EGOE}{\longrightarrow} \overline{\rho(E)} = \rho_G(E) = \frac{1}{\sqrt{2\pi}\sigma} \exp \frac{-1}{2}\left(\frac{E-\epsilon}{\sigma}\right)^2 \tag{3.18}$$

In the above eq.(3.17) $<< ... >>$ denotes trace (similarly $< ... >$ denotes average), the ϵ, σ and d are centroid, width (σ^2 is variance) and dimensionality respectively. Note that $\epsilon = \quad < H >, \quad \sigma^2 = \quad < (H-\epsilon^2) >$, G stands for Gaussian and the bar over $\rho(E)$ indicates ensemble average (smoothening) with respect to EGOE. The strength $R(E_i, E_f)$

generated by a transition operator O in the H-diagonal basis is $R(E_i, E_f) = \ |< E_f \mid O \mid E_i >|^2$. Correspondingly the bivariate strength density $I_{biv;o}(E_i, E_f)$ or $\rho_{biv;o}(E_i, E_f)$ which is positive definite and normalized to unity is defined by

$$\begin{aligned}
I_{biv;o}(E_i, E_f) &= << O^\dagger \delta(H - E_f) O \delta(H - E_i) >> \\
&= \sum_E < E \mid O^\dagger \delta(H - E_f) O \delta(H - E_i) \mid E > \\
&= I(E) < E \mid O^\dagger \delta(H - E_f) O \delta(H - E_i) \\
&= I(E) \sum_{E\prime} < E \mid O^\dagger \delta(H - E_f) \mid E' >< E' \mid O \delta(H - E_i) \mid E > \\
&= I(E_i) \times I(E_f) < E \mid O^\dagger \delta(H - E_f) \mid E' < E' \mid O \delta(H - E_i) \mid E > \\
&= I(E_i) \times I(E_f) < E \mid O^\dagger \delta(E' - E_f) \mid E' >< E' \mid O \delta(E - E_i) \mid E > \\
&= I(E_i) \times I(E_f) < E_i \mid O^\dagger \mid E_f >< E_f \mid O \mid E_i > \\
&= I(E_i) \times I(E_f) < E_f \mid O \mid E_i >^* < E_f \mid O \mid E_i > \\
&= I(E_i) \times I(E_f) \ |< E_f \mid O \mid E_i \ |^2 \\
&= << O^\dagger O >> \rho_{biv;o}(E_i, E_f) \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
\rho_{biv;o}(E_i, E_f) \stackrel{EGOE}{\longrightarrow} \overline{\rho_{biv;o}(E_i, E_f)} &= \rho_{biv-G;o}(E_i, E_f) \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\zeta^2}} \\
&\times \ exp\Bigg\{ \frac{-1}{2(1-\zeta^2)} \Bigg[\left(\frac{E-\epsilon_1}{\sigma_1}\right)^2 - 2\zeta \left(\frac{E-\epsilon_1}{\sigma_1}\right)\left(\frac{E_f-\epsilon_2}{\sigma_2}\right) \\
&\times \ \left(\frac{E_f-\epsilon_2}{\sigma_2}\right)^2 \Bigg] \Bigg\} \qquad (3.20)
\end{aligned}$$

In the eq.(3.20), ϵ_1 and ϵ_2 are the centroids and σ_1^2 and σ_2^2 are the variances of the marginal densities $\rho_{1;O}(E_i)$ and $\rho_{2;O}(E_f)$ respectively. The bivariate reduced central moments of $\rho_{biv;O})$ are $\mu_{pq} = \left\langle O^\dagger \left(\frac{H-\epsilon_2}{\sigma_2}\right)^q O \frac{H-\epsilon_1}{\sigma_1})^p \right\rangle \Big/ < O^\dagger O >$ and $\zeta = \mu_{11}$ is the bivariate correlation coefficient. Although the EGOE forms in eq.(3.18) and (3.20) are derived by evaluating the averages over fixed m-spaces, however in a large number of shell model examples, it is verified that [39, 2, 108, 174, 175, 176, 177, 178] they are equally applicable in fixed-m, mT and mJT spaces. In practice, just as in the case of state densities, bivariate Edgeworth corrections are added to the bivariate Gaussian form in (3.20). The point worth mentioning here is that, in general, the (mJT) values for the E_i and E_f need

not be same. The Gaussian forms in (3.18) and (3.20) give compact formulas for NPC and S in transition strength distributions incorporating the information that the hamiltonian and transition operators are of lower particle rank (i.e.$k, t << m$, where k and t are maximum particle ranks of H and O respectively). Firstly, we will define mathematically the NPC and information entropy S for transition strengths and then the ensemble average with respect to EGOE is carried out. Let us introduce the statistical quantities normalised strength $\mathcal{R}$, average (smoothed) normalised strength $\overline{\mathcal{R}}$ and locally renormalized strength $\hat{R}$ where

$$\begin{aligned}
\mathcal{R}(E_i, E_f) &= \left\{< E_i|O^\dagger O|E_i >\right\}^{-1} | < E_f|O|E_i > |^2 && (3.21)\\
\overline{\mathcal{R}(E_i, E_f)} &= \overline{\left\{< E_i|O^\dagger O|E_i >\right\}^{-1}} \quad \overline{| < E_f|O|E_i > |^2} && (3.22)\\
\hat{R} &= \overline{\{| < E_f|O|E_i > |^2\}^{-1}} \quad | < E_f|O|E_i > |^2 && (3.23)
\end{aligned}$$

The eq.(3.21) can be shown normalized as follows

$$\begin{aligned}
\frac{\sum_{E_f} | < E_f|O|E_i > |^2}{< E_i|O^\dagger O|E_i >} &= \frac{\sum_{E_f} | < E_f|O|E_i > |^2}{\sum_{E_f} < E_i|O^\dagger|E_f >< E_f|O|E_i >}\\
&= \frac{\sum_{E_f} | < E_f|O|E_i > |^2}{\sum_{E_f} < E_f|O|E_i >^* < E_f|O|E_i >}\\
&= \frac{\sum_{E_f} | < E_f|O|E_i > |^2}{\sum_{E_f} | < E_f|O|E_i > |^2}\\
&= 1 && (3.24)
\end{aligned}$$

Then the measures NPC and entropy S for strength distributions are

$$(NPC)_{E_i} = \Big\{ \sum_{E_f} \{\mathcal{R}(E_i, E_f)\}^2 \Big\}^{-1}, (S)_{E_f} = -\sum_{E_f} \mathcal{R}(E_i, E_f) ln \mathcal{R}(E_i, E_f) \quad (3.25)$$

In the first step for the derivation of $NPC(E)_i$, it is written in terms of $\overline{(\hat{R}^2)}$ and $\overline{\mathcal{R}}$

$$\begin{aligned}(NPC)_{E_i} &= \left\{\overline{\sum_{E_f}\left\{\frac{|<E_f|O|E_i>|^2}{\overline{|<E_f|O|E_i>|^2}}\right\}^2\frac{\left\{\overline{|<E_f|O|E_i>|^2}\right\}^2}{\left\{\overline{<O^\dagger O>}\right\}^2}}\right\}^{-1} \\ &= \left\{\sum_{E_f}\frac{\overline{\mathcal{R}^2(E_i,E_f)}}{\left\{\overline{\mathcal{R}(E_i,E_f)}\right\}^2}\times\left\{\overline{\mathcal{R}(E_i,E_f)}\right\}^2\right\}^{-1} \\ &= \left\{\sum_{E_f}\overline{\left\{\hat{R}(E_i,E_f)\right\}^2}\left\{\overline{\mathcal{R}(E_i,E_f)}\right\}^2\right\}^{-1} \end{aligned} \tag{3.26}$$

In the second step of the derivation use is made of the fact that there is a separation of average and fluctuation in transition strengths, so that we can evaluate $\overline{\left\{\hat{R}(E_i,E_f)\right\}^2}$ separately. Also, the numerically observed result that the EGOE fluctuations follow GOE is used. i.e.,$\hat{R}(E_i,E_f)$ distribution is Porter-Thomas [2, 179]. In otherwords, it also implies that the locally renormalized amplitudes $\frac{|<E_f|O|E_i>|}{\left\{<E_f|O|E_i>|^2\right\}^{1/2}}$ are Guassian distributed with zero center and unit variance. In the study of strength fluctuations [180] and in many other similar investigations [181] the local averages $\overline{|<E_f|O|E_i>|^2}$ are obtained via a numerical smoothening procedure while in [2, 179], SS forms are used; in [2] double polynomial expansion given in [177] is used and in [179] bivariate Gaussian is employed. The P-T law for $\hat{R}$ gives [182]

$$\overline{(\hat{R})}=1,\overline{(\hat{R}^2)}=3,\overline{(\hat{R}ln\hat{R})}=ln(0.48) \tag{3.27}$$

$$\begin{aligned}NPC(E)_i &= \left\{3\sum_{E_f}\left\{\overline{\mathcal{R}(E_i,E_f)}\right\}^2\right\}^{-1} \\ &= \frac{d_{eff}(E_i)}{3}\end{aligned} \tag{3.28}$$

where $d_{eff}(E_i)$ stands for effective dimension which depends on the energy E_i. Hence above equation becomes

$$NPC(E)_i=\left\{3\frac{\sum_{E_f}\left[\overline{|<E_f|O|E_i>|^2}\right]^2}{\left[\overline{<E_i|O^\dagger O|E_i>}\right]^2}\right\}^{-1} \tag{3.29}$$

To proceed further the bivariate strength density $I_{biv;o}(E_i, E_f)$ or $\rho_{biv;o}(E_i, E_f)$ which is positive definite and normalized to unity is defined by

$$\begin{aligned} I_{biv;o}(E_i, E_f) &= I(E_i) \times I(E_f) \mid< E_f \mid O \mid E_i \mid^2 \\ &= << O^\dagger O >> \rho_{biv;o}(E_i, E_f) \end{aligned} \tag{3.30}$$

For EGOE(1+2) in the Gaussian domain they take bivariate Gaussian form with normalization $<< O^\dagger O >>$. Now writing the numerator and denominator in eq. (3.29) in terms of $\rho' s$ and replacing the sum over E_f by the integral $\int(-----)\rho(E_f)dE_f$ will lead to the equation

$$\begin{aligned} NPC(E)_i &= \left[3\int_{-\infty}^{+\infty} d_f \times \rho(E_f)dE_f \left\{\left\{\overline{R(E_i, E_f)}\right\}^2\right\}\right]^{-1} \\ &= \left[3\int_{-\infty}^{+\infty} d_f \times \rho(E_f)dE_f \left\{\frac{\overline{\mid < E_f|O|E_i > \mid}}{\overline{< E_i|O^\dagger O|E_i >}}\right\}^2\right]^{-1} \\ &= \left[3\int_{-\infty}^{+\infty} d_f\rho(E_f)dE_f \frac{\left(\overline{<<O^\dagger O>>}\right)^2 \left(\overline{\rho_{G;O^\dagger O}(E_i, E_f)}\right)^2}{d_i^2 \times \left(\overline{\rho_{i;G}(E_i)}\right)^2 \quad d_f^2 \left(\overline{\rho_{f:G}(E_f)}\right)^2 \left(\overline{< E_i|O^\dagger O|E_i > \mid}\right)^2}\right]^{-1} \\ NPC(E)_i &= \left[3\int_{-\infty}^{+\infty} d_f \times \rho(E_f)dE_f \frac{d_i^2 \times \left(\overline{<O^\dagger O>}\right)^2 \left(\overline{\rho_{biv;O}(E_i, E_f)}\right)^2}{d_i^2 \left(\overline{\rho(E_i)}\right)^2 d_f^2 \left(\overline{\rho(E_f)}\right)^2 \times \left(\overline{<O^\dagger O>}\right)^2 \frac{\left(\overline{\rho_{1:O}(E_i)}\right)^2}{\left(\overline{\rho(E_i)}\right)^2}}\right]^{-1} \\ &= \frac{d_f}{3} \times \left(\overline{\rho_{1:O}(E_i)}\right)^2 \left[\int_{-\infty}^{+\infty} dE_f \frac{\left(\overline{\rho_{biv;O}(E_i, E_f)}\right)^2}{\rho'_{E_f}(E_f)}\right]^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} exp - \left(\frac{E-\epsilon_1}{\sigma_1}\right)^2 \\
&\times \left[\int_{-\infty}^{+\infty} dE_f \frac{\frac{1}{4\pi^2\sigma_1^2\sigma_2^2(1-\zeta^2)} e^{-\frac{1}{1-\zeta^2}\left\{\left(\frac{E-\epsilon_1}{\sigma_1}\right)^2 - 2\zeta\left(\frac{E_i-\epsilon_1}{\sigma_1}\right)\left(\frac{E_f-\epsilon_2}{\sigma_2}\right)+\left(\frac{E_f-\epsilon_2}{\sigma_2}\right)^2\right\}}}{\frac{1}{\sqrt{2\pi}\sigma_f} e^{-\frac{1}{2}\left(\frac{E_f-\epsilon_f}{\sigma_f}\right)^2}}\right]^{-1} \\
&= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\left(\frac{E-\epsilon_1}{\sigma_1}\right)^2} \left[\int_{-\infty}^{+\infty} dE_f \frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\sigma_2^2(1-\zeta^2)}\right. \\
&\times \left. e^{-\frac{1}{2(1-\zeta^2)}\left\{2\left(\frac{E-\epsilon_1}{\sigma_1}\right)^2 - 4\zeta\left(\frac{E_i-\epsilon_1}{\sigma_1}\right)\left(\frac{E_f-\epsilon_2}{\sigma_2}\right)+2\left(\frac{E_f-\epsilon_2}{\sigma_2}\right)^2-\left(1-\zeta^2\right)\left(\frac{E_f-\epsilon_f}{\sigma_f}\right)^2\right\}}\right]^{-1}
\end{aligned} \tag{3.31}
$$

In order to simplify the above, let us make the following substitutions.

$$
\frac{E-\epsilon_1}{\sigma_1} = \hat{E}; \hat{\Delta} = \frac{\epsilon_2-\epsilon_f}{\sigma_f}; \frac{\epsilon_f-\epsilon_2}{\sigma_2} = y \tag{3.32}
$$

$$
\begin{aligned}
(NPC)_{E_i} &= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\int_{-\infty}^{+\infty} dE_f \frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\sigma_2^2(1-\zeta^2)}\right. \\
&\times \left. e^{-\frac{1}{2(1-\zeta^2)}\left\{2\hat{E}^2 - 4\zeta\hat{E}y + 2y^2 - \left(1-\zeta^2\right)\left(y\hat{\sigma_2}+\hat{\Delta}_2\right)^2\right\}} \sigma_2 dy\right]^{-1} \\
(NPC)_{E_i} &= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\hat{\sigma_2}(1-\zeta^2)} \int_{-\infty}^{+\infty} dE_f\right. \\
&\times \left. e^{-\frac{1}{2(1-\zeta^2)}\left\{2\hat{E}^2 - 4\zeta\hat{E}y + 2y^2 - \left(1-\zeta^2\right)\left(y^2\hat{\sigma_2}+\hat{\Delta}^2+2y\hat{\sigma_2}\hat{\Delta_2}\right)\right\}} \sigma_2 dy\right]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
(NPC)_{E_i} &= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\hat{\sigma_2}(1-\zeta^2)} \int_{-\infty}^{+\infty} dE_f \right. \\
&\times \left. e^{-\frac{1}{2(1-\zeta^2)}\left\{ \left(2-\hat{\sigma_2}^2(1-\zeta^2)\right)y^2 + 2y\left(-2\zeta\hat{E} - (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right) + \left(2\hat{E}^2 - (1-\zeta^2)\hat{\Delta_2}^2\right) \right\}} dy \right]^{-1} \\
&= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\hat{\sigma_2}(1-\zeta^2)} \int_{-\infty}^{+\infty} dE_f \right. \\
&\times \left. e^{-\frac{1}{2(1-\zeta^2)}\left\{ y^2X^2 + 2\hat{E}^2 - (1-\zeta^2)\hat{\Delta_2}^2 - 2y\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right) \right\}} dy \right]^{-1} \\
&= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\hat{\sigma_2}(1-\zeta^2)} \int_{-\infty}^{+\infty} dE_f \right. \\
&\times \left. e^{-\frac{X^2}{2(1-\zeta^2)}\left\{ y^2 - \frac{2y}{X^2}\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right) \right\}} \times e^{-\frac{1}{2(1-\zeta^2)}\left\{ 2\hat{E}^2 - (1-\zeta^2)\hat{\Delta_2}^2 \right\}} dy \right]^{-1} \\
&= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\hat{\sigma_2}(1-\zeta^2)} \int_{-\infty}^{+\infty} dE_f \right. \\
&\times e^{-\frac{X^2}{2(1-\zeta^2)}\left\{ y^2 - \frac{2y}{X^2}\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right) \right\} + \frac{1}{X^4}\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right)^2 - \frac{1}{X^4}\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right)^2} \\
&\times \left. e^{-\frac{1}{2(1-\zeta^2)}\left\{ 2\hat{E}^2 - (1-\zeta^2)\hat{\Delta_2}^2 \right\}} dy \right]^{-1} \\
&= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\hat{\sigma_2}(1-\zeta^2)} e^{-\frac{1}{2(1-\zeta^2)}\left\{ 2\hat{E}^2 - (1-\zeta^2)\hat{\Delta_2}^2 \right\}} \right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right)^2} \int_{-\infty}^{+\infty} e^{-\frac{X^2}{2(1-\zeta^2)}\left[\left\{ y - \frac{1}{x^2}\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right) \right\}^2 \right]} dy \right]^{-1} \\
&= \frac{d_f}{3} \times \frac{1}{2\pi\sigma_1^2} e^{-\hat{E}^2} \left[\frac{\sqrt{2\pi}\sigma_f}{4\pi^2\sigma_1^2\hat{\sigma_2}(1-\zeta^2)} e^{-\frac{1}{2(1-\zeta^2)}\left\{ 2\hat{E}^2 - (1-\zeta^2)\hat{\Delta_2}^2 \right\}} \right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left(2\zeta\hat{E} + (1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right)^2} \times \sqrt{2\pi} \times \frac{\sqrt{1-\zeta^2}}{X} \right]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d_f}{3} \times e^{-\hat{E}^2}\left[\frac{1}{\sqrt{1-\zeta^2}X} e^{\left\{\frac{1}{2X^2(1-\zeta^2)}\left(2\zeta\hat{E}+(1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\right)^2 - X^2\left(2\hat{E}^2-(1-\zeta^2)\hat{\Delta_2}^2\right)\right\}}\right]^{-1} \\
&= \frac{d_f}{3}\left[\frac{1}{\sqrt{1-\zeta^2}X}\right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left\{2X^2(1-\zeta^2)\hat{E}^2+4\zeta^2\hat{E}^2-2X^2\hat{E}^2+4\hat{E}\zeta(1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}+X^2(1-\zeta^2)\hat{\Delta_2}^2(1-\zeta^2)^2\hat{\sigma_2}^2\hat{\Delta_2}^2\right\}}\right]^{-1} \\
&= \frac{d_f}{3}\left[\frac{1}{\sqrt{1-\zeta^2}X}\right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left\{(2X^2-2X^2\zeta^2+4\zeta^2-2X^2)\hat{E}^2+4\hat{E}\zeta(1-\zeta)^2\hat{\sigma_2}\hat{\Delta_2}+X^2(1-\zeta^2)\hat{\Delta}^2+(1-\zeta^2)^2\hat{\sigma_2}^2\hat{\Delta_2}^2\right\}}\right]^{-1} \\
&= \frac{d_f}{3}\left[\frac{1}{\sqrt{1-\zeta^2}X\hat{\sigma_2}}\right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left\{2\zeta^2(2-X^2)\hat{E}^2+4\zeta(1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}+(1-\zeta^2)\hat{\Delta_2}+(1-\zeta^2)^2\hat{\Delta_2}^2\hat{\sigma_2}^2+X^2(1-\zeta^2)\hat{\Delta_2}^2\right\}}\right]^{-1} \\
&= \frac{d_f}{3}\left[\frac{1}{\sqrt{1-\zeta^2}X\hat{\sigma_2}}\right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left\{2\zeta^2\left(2-2+\hat{\sigma_2}^2(1-\zeta^2)\right)+4\zeta(1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}+(1-\zeta^2)\hat{\Delta}^2\hat{\sigma_2}^2+X^2(1-\zeta^2)\hat{\Delta}^2\right\}}\right]^{-1} \\
&= \frac{d_f}{3}\left[\frac{1}{\sqrt{1-\zeta^2}X\hat{\sigma_2}}\right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left\{\left(2\zeta^2\hat{\sigma_2}^2(1-\zeta^2)\right)\hat{E}^2+4\zeta(1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\hat{E}+(1-\zeta^2)\hat{\Delta}^2\left(X^2+(1-\zeta^2\hat{\sigma_2}\right)\right\}}\right]^{-1} \\
&= \frac{d_f}{3}\left[\frac{1}{\sqrt{1-\zeta^2}X\hat{\sigma_2}}\right. \\
&\times \left. e^{\frac{1}{2X^2(1-\zeta^2)}\left\{\left(2\zeta^2\hat{\sigma_2}^2(1-\zeta^2)\right)\hat{E}^2+4\zeta(1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\hat{E}+(1-\zeta^2)\hat{\Delta}^2\left(2-\hat{\sigma_2}^2(1-\zeta^2)+(1-\zeta^2)\hat{\sigma_2}^2\right)\right\}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d_f}{3}\Bigg[\frac{1}{\sqrt{1-\zeta^2}X\hat{\sigma_2}} \\
&\times \; e^{\frac{1}{2X^2(1-\zeta^2)}\Big\{\left(2\zeta^2\hat{\sigma_2}^2(1-\zeta^2)\right)\hat{E}^2+4\zeta(1-\zeta^2)\hat{\sigma_2}\hat{\Delta_2}\hat{E}+2(1-\zeta^2)\hat{\Delta_2}^2\Big\}}\Bigg]^{-1} \\
&= \frac{d_f}{3}\Bigg[\frac{1}{\sqrt{1-\zeta^2}X\hat{\sigma_2}} \\
&\times \; e^{\frac{1}{2X^2(1-\zeta^2)}\times 2(1-\zeta^2)\Big\{\zeta^2\hat{\sigma_2}^2\hat{E}^2+2\zeta\hat{\sigma_2}\hat{\Delta_2}\hat{E}+\hat{\Delta_2}^2\Big\}}\Bigg]^{-1} \\
&= \frac{d_f}{3}\times\sqrt{1-\zeta^2}X\hat{\sigma_2}e^{\frac{-1}{X^2}\Big\{\zeta^2\hat{\sigma_2}^2\hat{E}^2+2\zeta\hat{\sigma_2}\hat{\Delta_2}\hat{E}+\hat{\Delta_2}^2\Big\}} \\
&= \frac{d_f}{3}\sqrt{1-\zeta^2}X\hat{\sigma_2}e^{-\left(\frac{\hat{\sigma_2}\zeta\hat{E}+\hat{\Delta_2}}{X}\right)^2} \qquad (3.33)
\end{aligned}
$$

Working along the similar lines the EGOE expression for the information entropy can be derived.

$$
\begin{aligned}
(S)_{E_i} \;\xrightarrow{EGOE}\; & \sum_{E_f}\overline{\mathcal{R}(E_i,E_f)}\overline{\Big\{\hat{R}(E_i,E_f)ln\hat{R}(E_i,E_f)\Big\}} \\
\times \quad & -\sum_{E_f}\overline{\left(\hat{R}(E_i,E_f)\right)}\overline{\mathcal{R}(E_i,E_f)}ln\overline{\mathcal{R}(E_i,E_f)} \\
= \quad & ln(0.48d_f)-\int dE_f\overline{\rho(E_f|E_i)}\left[ln\frac{\overline{\rho_{biv;O}(E_i,E_f)}}{\overline{\rho_{1;O}(E_i)}\;\;\overline{\rho'(E_f)}}\right] \\
= \quad & ln\Bigg\{0.48d_f\Big[\hat{\sigma_2}\sqrt{1-\zeta^2}exp\frac{1-\hat{\sigma_2^2}(1-\zeta^2)}{2}exp-\frac{(\hat{\sigma_2}\zeta\hat{E}+\Delta_2)^2}{2}\Big]\Bigg\} \\
\Longrightarrow exp\big((S)_{E_i}\big) \quad = \quad & 0.48d_f\Big[\hat{\sigma_2}\sqrt{1-\zeta^2}exp\frac{1-\hat{\sigma_2^2}(1-\zeta^2)}{2}exp-\frac{(\hat{\sigma_2}\zeta\hat{E}+\Delta_2)^2}{2}\Big] \qquad (3.34)
\end{aligned}
$$

In the derivation of information entropy from first step to last step results in (3.18), (3.20) and (3.27) are used. It is to be noted that $\rho(E_f,E_i)=\frac{\rho_{biv;O}(E_i,E_f)}{\rho_{1;0}(E)}$ is a conditional density and it takes a Gaussian form with $\rho_{biv;O}$ and $\rho_{1;O}$ taking Gaussian forms. The third equality in equation (3.34) is obtained by substituting the Gaussian forms in (3.18) and (3.20)for the densities in second equality and carrying out the integrations. In the dilute limit, with EGOE(k) for H and an independent EGOE(t) for O in m-particle space (i.e.,

in the situation, as it is the case with the numerical examples discussed ahead, that the initial and final spaces connected by the transition operators O are same and H and O are representable by EGOE,) it is seen that [160, 161, 162] $d_i = d'$, $\sigma_1 = \sigma_2 = \sigma = \sigma'$, $\epsilon_1 = \epsilon_2 = \epsilon = \epsilon'$ and

$$\zeta = \binom{m}{k}^{-1} \binom{m-t}{k} \tag{3.35}$$

Then the formulas for $(NPC)_{E_i}$ and $(S)_{E_i}$ get reduced to the forms determined only by the correlation coefficient ζ

$$(NPC)_{E_i} = \frac{d_i}{3}\sqrt{1-\zeta^4} exp - \frac{\zeta^2 \hat{E}^2}{1+\zeta^2}; \hat{E} = \frac{E_i - \epsilon}{\sigma} \tag{3.36}$$

$$exp\big((S)_{E_i}\big) = 0.48 d_i \sqrt{1-\zeta^2} exp\left(\frac{\zeta^2}{2}\right) exp\left(-\frac{-\zeta^2 \hat{E}}{2}\right) \tag{3.37}$$

For GOE obviously $\zeta = 0$ [162] and then the above equations reduce to the well known GOE results i.e., $(NPC)_{E_i} = \frac{d_i}{3}$ and $(S)_{E_i} = ln(0.48d)$. Shell model calculations had been performed using Rochester-OakRidge shell model code in 307 dimensional space $(2s1d)^{m=6,J=2,T=0}$ for testing the EGOE results given by equations (3.33) and (3.34). The operator 'O' choosen is two-body in nature and is basically same as in [183], the two-body part of H without the configuration-isospin centroid producing part. The Hamiltonian $H = h(1) + V(2)$ is defined by Kuo's [184] two-body matrix elements (V(2)) and ^{17}O single-particle energies (h(1) $\Leftrightarrow$ $\epsilon_{d_{5/2}} = -4.15MeV, \epsilon_{d_{3/2}} = 0.93MeV, \epsilon_{s_{1/2}} = -3.28MeV$). The diagonal matrix element $< E|O|E >$ of O in H diagonal representation are put equal to zero for the reasons explained in detail in [182, 183]. With these choices it is seen that $\epsilon = \epsilon^{'} = -32.78$, $\sigma = \sigma^{'} = 10.24$ MeV, $\epsilon_1 = \epsilon_2 = -29.88$, $\sigma_1 = \sigma_2 = 10.67$ MeV, $\hat{\Delta} = 0.28$, $\hat{\sigma_2} = 1.04$ and $X = 1.25$. $\zeta = 0.55$, while the EGOE estimate given by eq. (3.35) is 0.67 for one-body H and 0.4 for two-body H as m = 6 and the rank '$t' = 2$. Using these parameters in eqns. (3.33) and (3.34) the EGOE curves for NPC and exp(S) are constructed and compared in the fig.(3.6) with the exact shell model results and the theoretical predictions given by EGOE is in excellent agreement with the shell model results. Further, these results show clear departures from GOE results just as seen in the 3276 dimensional space $(2s1d)^{m=12,J=2,T=0}$ shell model results in [131, 39]. The EGOE results give also a formula for the ratio $exp(S)/NPC$,

$$exp(S)/NPC = (1.44)\frac{exp\frac{\zeta^2}{2}}{\sqrt{1+\zeta^2}} exp\frac{\zeta^2(1-\zeta^2)\hat{E}^2}{2(1+\zeta^2)} \tag{3.38}$$

The above equation shows that $exp(S)/NPC$ increases as the energy is away from the centre and this behaviour is clearly seen in the fig.(3.6).

3.5 Transition Strength Sums

EGOE(k) is constructed in m-particle spaces [i.e., in the $\binom{N}{m}$ dimensional space generating by distributing the m fermions over N single-particle states] by defining it to be a GOE in k-particle space for k-body operators (usually$k << m$) and then using the direct-product structure of m-particle spaces. The two important results given by EGOE, are that in strongly interacting shell model spaces (essentially in $0\hbar\omega$ spaces), (i) the state densities $I(E) = \ << \delta(H-E) >>$ take Gaussian form [108, 185, 9] and (ii) with the strength $R(E_i, E_f) = | < E_f|O|E_i > |^2$ generated by a transition operator O in the H-diagonal basis, the bivaraiate strength densities $I_{biv;O}(E_i, E_f) = \ << O^\dagger \delta(H-E_f) O \delta(H-E_i) >> = I'_{E_f} | < E_f|O|E > |^2 I(E_i)$ take bivariate Gaussian form [186, 36]. Here $< ... >$ stands for average and$<< \ .... \ >>$ stands for trace. Although EGOE forms in (i) and (ii) are derived by evaluating the averages over fixed m-spaces, they hold equally well in fixed m, mT and mJT [2, 150, 108, 183, 185, 187, 188, 189] spaces in a large number of numerical shell model calculations. Edgeworth corrections are added to the Gaussain forms in practice. One of the important byproducts of (ii) is that the transition strength sum density $<< O^\dagger O \delta(H-E) >>$, which is a marginal density of the bivariate strength density, takes a Gaussian form, since the marginal of a bivariate Gaussian is a Gaussian. Therefore, it immediately follows from (i) and (ii) that the transition strength sums generated by a transition operator acting on an eigenstate vary with the excitation energy as the ratio of two Gaussians. Given $K = O^\dagger O$, the transition strength sum is given by the expectation

value $< K >^E$

$$\begin{aligned} < K >^E &= < \hat{O}^\dagger \hat{O} >^E \\ &= < E_i | \hat{O}^\dagger \hat{O} | E_i > \\ &= \sum_{E_f} < E_i | \hat{O}^\dagger | E_f > < E_f | \hat{O} | E_i > \\ &= \sum_{E_f} < E_i | \hat{O}^\dagger | E_f > < E_f | \hat{O} | E_i > \\ &= \sum_{E_f} < E_f | \hat{O} | E_i > {}^* < E_f | \hat{O} | E_i > \\ &= \sum_{E_f} | < E_f | \hat{O} | E_i > |^2 \end{aligned} \tag{3.39}$$

also it can be written as the expectation value density as $\rho_K(E)$ [183, 188, 189] as

$$\left.\begin{aligned} < K >^E &= [d\rho(E)]^{-1} \left[\textstyle\sum_{\alpha \in E} < E\alpha | K | E\alpha > \right] \\ &= I_K(E)/I(E) \\ &= \rho_K(E)/\rho(E) \overset{EGOE}{\longrightarrow} \overline{\rho_K(E)}/\overline{\rho(E)} \\ &= \rho_{K:G}(E)/\rho_G(E) \end{aligned}\right\}. \tag{3.40}$$

where

$$\rho(E) =< \delta(H - E) >= d^{-1} I(E) = d^{-1} << \delta(H - E) >> \tag{3.41}$$

and,

$$\rho_K(E) =< K\delta(H - E) >= d^{-1} I_k(E) = d^{-1} << K\delta(H - E) >>; K = O^\dagger O \tag{3.42}$$

In eq. (3.40) d is the dimensionality and G stands for Gaussian, and the bars over $\rho(E)$ and $\rho_k(E)$ indicates the ensemble average (smoothed) with respect to EGOE. While deriving eq.(3.40) it is assumed that the smoothed forms of $\rho_k(E)/\rho(E)$ reduces to the ratio of smoothed form of $\rho_k(E)$ and $\rho(E)$. This result ignores the fluctuation in both $\rho_k(E)$ and $\rho(E)$ and the rms error due to neglect of fluctuation is given in terms of the number of principal components or the inverse participation ratio for the transition operator O [2, 150]. The smoothed EGOE form for $< K >^E$ takes into account (K,H) and (K, H^2) correlations, which define the centroid ϵ_k and width σ_k of $\rho_K(E)$; $\epsilon_K = \ < KH > / < K >$ and $\sigma_k^2 = \ < KH > / < K > -\epsilon_k^2$. The results in the eq.(3.40) are quite general

and in order to study its domain of validity detailed shell model calculations has been done by using the operator that generates GT strength sums, which is defined by

$$O^{(\pm)}_{GT;\mu} = \sum_{i=nucleons} \sigma_\mu(i) t_\pm(i) \tag{3.43}$$

where $t_\pm$ converts a neutron into proton and vice-versa. The total Gamow-Teller strength originating from an initial state at energy E to all final states is given by the expectation value

$$K^{(\pm)}(GT) = \sum_\mu O^{(\pm)\dagger}_{GT;\mu} O^{(\pm)}_{GT;\mu} \tag{3.44}$$

Exact shell-model calculations for the total GT strength have been carried out for all the J=0 states of ^{46}V in the 814 dimensional $(1f2p)^{m=6,J=0,T=0}$ space. The calculations were performed with the NATHAN code of the Strasbourg-Madrid, using the effective interaction KB3, which successfully reproduces the experimental binding energies, excitation spectra, and transition strengths for nuclei in this region [190, 191]. On the other hand, the expectation value density $\rho_{K(GT):G}$ for the K(GT) operator is constructed in terms of its centroid and width and, similarly, the state density Gaussian. Then, using eq. (3.40), the smoothed form of the GT strength sum as a function of excitation energy is constructed and compared with the exact shell model results. In fig. (3.7), it is very clear that the smoothed EGOE curve describes very well the shell model results, except at the edges of the spectra. Thus, it seems that the agreement is good in chaotic region and that the deviations are just in the ground-state region, where the states are not sufficiently complex (chaotic). Similar kind of deviations are observed at the upper end due to the finite shell-model space.

In embedded Gaussian orthogonal ensemble (EGOE) of random matrices, the transition strength sums generated by a transition operator acting on an eigenstate vary with the excitation energy as the ratio of two Gaussians. This general result when compared to the exact shell model calculations of Gamow-Teller Strength sums in nuclei, shows good agreement in the chaotic domain of the spectrum and strong deviations are observed as nuclear motion approaches a regular regime.Thus transition strength sums seem to be a new statistic sensitive to the chaoticity of the system.

3.6 Conclusions

The study of quantum chaos in atomic nuclei using the chaos measures like number of principal components and information entropy in shell-model wave-functions and transition strength distributions has became a much debated subject. Calculations performed by the French's group with the, then Rochester-Oak Ridge shell model code had established this fact that the smoothed (with respect to energy E) level densities (fixed-J or JT density of eigenvalues) I(E) take Gaussian form while as for classical ensembles, it takes semi-circular form, as discussed in detail in chapter 2. Further, for the smoothed transition strengths, it was found in 80's that, they follow bivariate (in the two energies involved) Gaussian while classical ensembles give constant values. The chaos measures like the number of principal components and information entropy in shell model wave-functions have quite different behaviour compared to GOE results while as the level and strength fluctuations after unfolding individual spectra, are seen to follow GOE. From the shell model studies it is also established that, it generates separation in averages (smooth forms) and fluctuations and cross-correlations (absent in GOE) in spectra with different quantum numbers. Thus, both one and two-point functions are different for shell model and the validity of these results have been extensively established both in 2s1d and 2p1f shell examples. All these differences show that we need to take into account the two-body nature of nucleon-nucleon interaction in RMT. Shell model with ensembles of random two-body interactions is seen to produce the forms for various quantities seen in the shell model calculations with realistic interactions and hence the random matrix ensembles generated by random interactions are called embedded ensembles. The important insights, drawn from the study of embedded ensembles in this chapter are: by starting from the EGOE(1+2) Hamiltonian defined by $H(1+2) = h(1) + \lambda V(2)$ and increasing the λ value from zero, the following results are observed: (i) Eigenvalue density will be essentially of Guassian form for all λ values. (ii) As λ increases, there is transition from Poisson to GOE fluctuations with the onset of GOE fluctuations at $\lambda = \lambda_c$. (iii) For $\lambda \sim 0$ strength functions will be delta functions and then quickly turn into Breit-Wigner (BW) form at $\lambda = \lambda_0$ with $\lambda_0 << \lambda_c$. As λ increases beyond λ_c there will be a transition from BW form to Gaussian with the onset of this transition at $\lambda = \lambda_F > \lambda_c$ (iv). As we increase λ further, there will be a region around $\lambda \sim \lambda_t \sim \lambda_F$ where different definitions of entropy, temperature etc. will coincide defining 'thermodynamic region'. The existence of three chaos markers or transition markers λ_c, λ_F and λ_t has been established numerically for both fermion and boson systems by analyzing spin-less and spin embedded ensembles.

Same structure is also seen in shell model calculations with random two-body interactions having J or JT symmetry [EGOE(1+2)-J or EGOE(1+2)-JT] and more importantly, also with realistic interactions in presence of a mean-field by changing all the two-body matrix elements by a factor. In this chapter, compact formulas for NPC and l_H are re-derived and the derivation is based on the results: (i) The Gaussian form for strength functions $F_k(E)'s$ and the bivariate Gaussian form for $\rho_{biv}(E, E_k)$ [with $F_k(E)$ being a conditional density of $\rho_{biv}(E, E_k)$] which are valid in the chaotic domain defined by $\lambda > \lambda_{F_k}$; (ii) there is average fluctuation separation (with little communication between the two) in energy levels and strengths with local fluctuations following the Porter-Thomas law, and (iii) there is a significant unitary group decomposition of the Hamiltonian. For the case of EGOE(1+2), the NPC and l_H take Gaussian forms as a function of excitation energy and they are defined by the bivariate correlation coefficient ζ which measures the variance of the distribution of $F_k(E)$'s relative to the state-density variance. Also in this unit, theory for incorporating corrections due to fluctuations in the variances (with k) of $F_k(E)$ is also given. For the small value of ζ, the present formulation gives back the results for pure EGOE(2) [i.e., in the limit $\lambda \to \infty$] as derived in [172]. By re-deriving the statistical spectroscopy (EGOE) expressions for the measures NPC and exp(S) in transition strength distributions the important inference drawn is that the bivariate correlation coefficient ζ that characterizes the strength distributions determines the energy variation of the measures as seen in shell model results and the agreement as shown in fig(3.6) between the exact shell model results and the EGOE forms makes it obvious the fact that the hamiltonian and transition operator in numerical example are well represented by EGOE. Thus, EGOE (and SS) considerations are essential for dealing with questions related to chaos and complexity in finite interacting many-particle quantum systems, like atomic nuclei. For example, to study the region of onset of chaos [151, 139, 155], chaos and thermalization [148, 151, 139, 155], nature of chaos near yrast line at high spins [192] etc. it is necessary to go beyond the simple EGOE (and SS) and consider interpolating [151, 139, 155] and partitioned [185] EGOE's just as it is done before for the Gaussian ensembles [183, 193, 194, 195]. Some of these more general EGOE ensembles are being investigated by using the large body of results available in statistical spectroscopy [2, 108, 174, 175, 176, 177, 178, 183, 185, 187, 189, 14, 196] and by further extending them. The formulas derived for NPC and l_H are subjected to numerical EGOE(1+2) tests with ζ ranging from 0.1 to 0.8. These and the analysis of the results for a EGOE(2)-S example and some nuclear shell model examples, clearly point out that isolated finite realistic interacting particle systems, in the chaotic domain $(\lambda \geq \lambda_{F_k})$, will have the wave-

function structure as given by EGOE(1+2). Finally, the fomulas for NPC and l_H depend on just one parameter and this appears to be an aspect of geometric chaos.

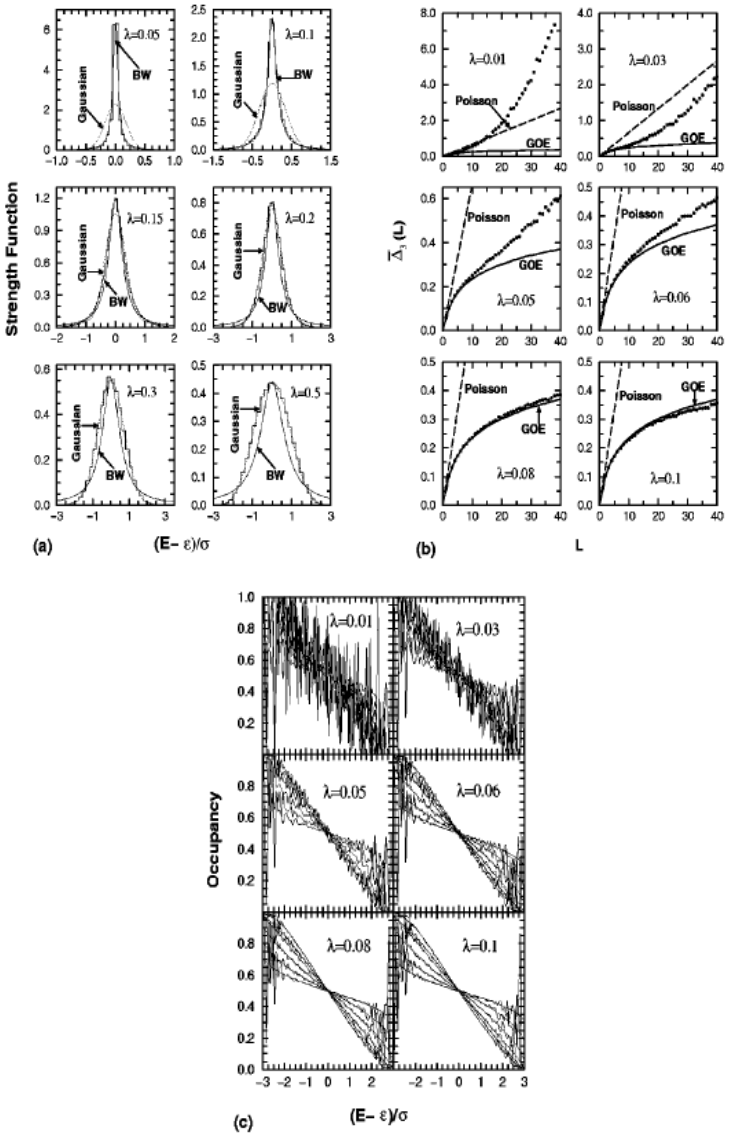

Figure 3.1: Strength functions $F_k(E)$, Dyson-Mehta $\overline{\Delta_3}$ statistic for level fluctuations and occupancies $< E|n_i|E >$ for EGOE(1+2) for various values of interaction strength $\lambda\{H\} = h(1) + \lambda\{V(2)\}$ for a system of 7 fermions (only one member is considered here because of computational contraints just as in [146]); the matrix dimension is 3432. The single particle energies used in the calculations are $\epsilon_i = (i + 1/i)$, i=1,2,....14 just as in [146]. (a) The histograms are EGOE(1+2) results for strength functions, continuous curves are BW fit and the dotted curves are Gaussian for$\lambda \leq 0.1$ and the Edgeworth corrected Gaussian for $\lambda > 0.1$. In constructing the strength functions,$|C_k^E|^2$ are summed over the basis states $|k >$ in the energy window $\hat{E_k} \pm \Delta$ and then the ensemble averaged$F_{\hat{E_k}}(\hat{E})$ vs $\hat{E}$ is constructed as a histogram; the value of Δ is chosen to be 0.025 for $\lambda \leq 0.1$ and beyond this $\Delta = 0.1$. Here, $\hat{E_k} = \frac{(E_k - \epsilon_H)}{\sigma_H}$ and in the figure $\hat{E_k} = 0$. Note that for $\lambda_{F_k} \sim 0.2$, there is BW to Gaussian transition.(b)The $\overline{\Delta_3}$ statistic for overlapping intervals of length L ≤ 40 are compared with poisson and GOE values. For $\lambda \sim 0.06$, there is a Poisson to GOE transition in the Δ_3 statistic.(c) The wavy curves are numerical EGOE(1+2) results for occupancies and the smoothed curves with $\lambda \geq 0.06$ correspond to the results of EGOE(2) theory (ratio of Gaussians). Note that for $\lambda < 0.06$, there are wide fluctuations in occupancies and the smoothed forms here are meaningless. All the results are shown for lowest six single-particle states. Results similar to those in the figure, for the N = 12, m = 6 case, are reported in [7]

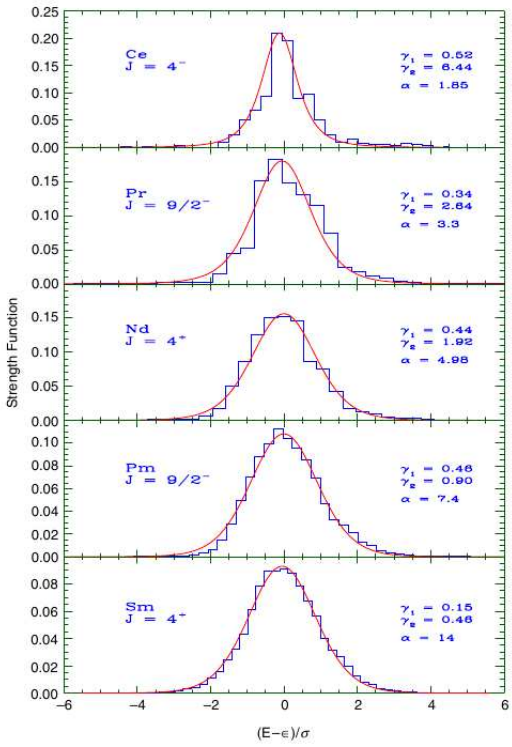

Figure 3.2: Strength functions $F_k(E)$ for CeI to SmI. Histograms are calculated strength functions and the smooth curves are the best fit $F_{k:BW-G}(E)$ with $E_k = 0$. Also given in the figure are the calculated γ_1 (skewness) and γ_2 (excess) values and the deduced values, from the best fits, of α characterizing $F_{k:BW-G}(E)$ with $E_k = 0$. In the figure, ϵ and σ are the spectral centroids and widths. Note that the BW to Gaussian interpolating function is the t-distribution well known in statistics and its explicit form is $F_{k:BW-G}(E : \alpha, \beta)dE = \frac{\alpha\beta)^{\alpha-\frac{1}{2}}\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha-\frac{1}{2})} \frac{dE}{[(E-E_k)^2+\alpha\beta]^{\alpha}}$. Here β is a scale parameter [fixed by the width of $F_k(E)$]. More important is thatα =1 gives BW and $\alpha \to \infty$ gives Gaussian. As we go from CeI to SmI the α parameter changes from 1.85 to 14 showing BW to Gaussian transition. See [149] for further details.

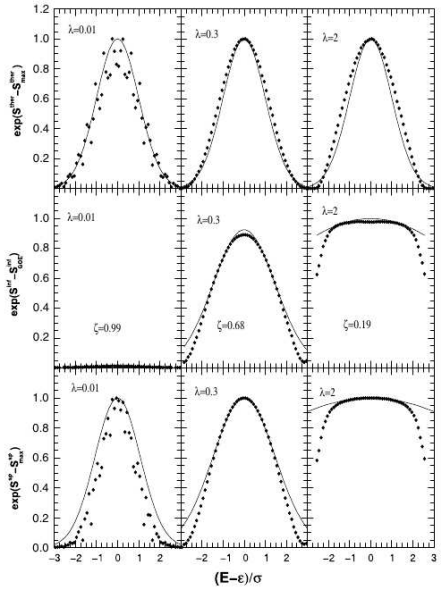

Figure 3.3: Thermodynamic,information and single-particle entropies, in terms of the values of interaction strength $\lambda(\{H\} = h(1) + \lambda\{V(2)\})$, for a 10 member EGOE(1+2) with 7 fermions in 14 single-particle states; the matrix dimension is 3432. The single-particle entropies used are $\epsilon_i = (i + 1/i)$, i= 1,2, ...,14. The numerical EGOE(1+2) results are obtained by averaging over a bin size of 0.1 and they are shown in the figure as filled circles. The continuous curves are the theoretical EGOE(1+2) predictions as given by eqns.(3.3). Results similar to those in the figure were reported earlier [154] for a six fermion system. note that for $\lambda = 0.01$, $exp(S^{info} - S^{info}_{GOE})$ is almost zero for all E.

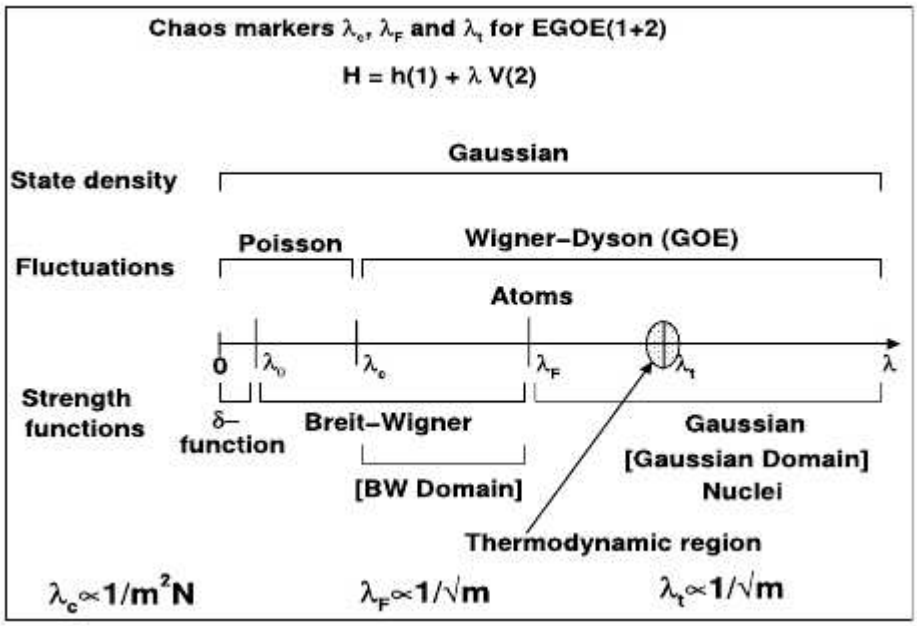

Figure 3.4: Chaos markers for EGOE(1+2). In the figure, m is the number of fermions and N is the number of single-particle states. Behaviour of the chaos markers as a function (m,N) is also indicated in the figure.

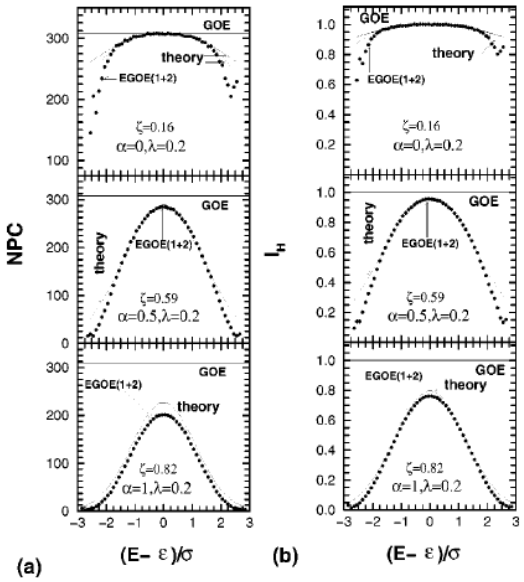

Figure 3.5: (a)Number of principal components NPC and (b) the localizationl_H in wave-functions for a system of six interacting particles in 12 single-particle states (matrix dimension is 924). For convenience, the EGOE(1+2) Hamiltonian is changed to $\{H_{(\alpha,\lambda)}\} = \alpha h(1) + \lambda\{V(2)\}$. Numerical EGOE(1+2) results correspond to filled circles. The continuous curves correspond to the theory (3.11) for NPC and (3.14) forl_H. For the case with $\alpha = 0$, the dashed curves correspond to the theory (3.12) for NPC and equation (3.14) for l_H. For the other cases, the correction due to variance fluctuations is negligible, and hence only the results of eqns. (3.11) and (3.14) are shown in the figure. Note that$NPC = d/3$ and $l_H = 1$ for GOE.

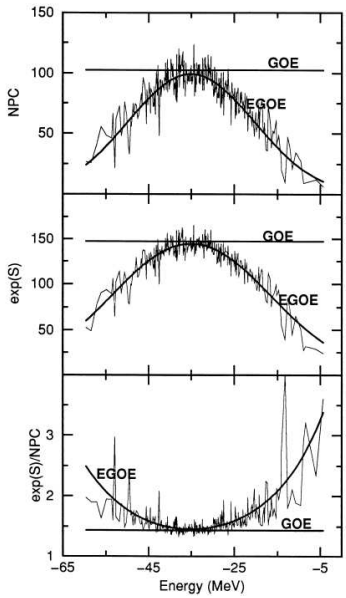

Figure 3.6: Number of principal components (NPC) and information entropy (S) versus energy (E) for a strength distribution in six particle (2s1d) shell space with J = 2, T = 0. The hamiltonian and the transition operator are defined in the text. Shown in the figure is also the ratio exp(S)/NPC versus. The exact shell model results are compared with the GOE and EGOE predictions; the EGOE predictions are given by equations (3.33) and (3.34).

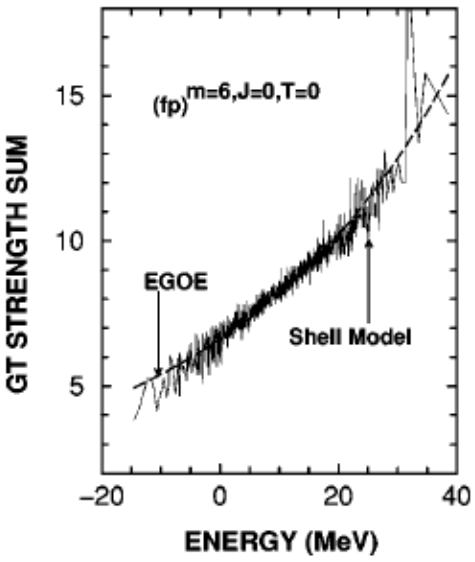

Figure 3.7: Gamow-Teller (GT) strength sum versus excitation en- ergy (E) for the 814 dimensional six particle (fp)-shell space with J = 0, T =0. The exact shell-model results for the realistic KB3 interaction are compared with the EGOE predictions given by Eq.(3.40).

Chapter 4

Fluctuation-free Nuclear Spectroscopy

4.1 Introduction

As far as a finite nucleus is concerned, it consists of a fixed number of nucleons each of which moves in the average one-body field generated by all other nucleons. In addition to this the nucleons also interact with each other through a residual two-body interaction. In such a many-body system, the wave-functions for the system of nucleons are usually constructed as linear combinations of the anti-symmetrised products of single-particle wave-functions. The Hilbert space for these many-particle states is in principle infinite: however for practical reasons, calculations are carried out in finite spaces defined by a set of single-particle states. Since for a nuclear system, it is usually the lower energy part of the spectrum which is of concern, so only a limited number of single-particle states near the Fermi energy are considerd to be active. There cannot be excitations in the states which are filled by particles, below the active ones as the excitations would require large amount of energy. Similarly, excitations of particles into the states above the active ones are forbidden. Except for their influence on the effective Hamiltonian in the active space, all the single-particle states other than the active ones can therefore essentially be ignored.

Within this finite many-particle space, calculations of a physical quantity are restricted to its contributions in the space used. For example, the density of states is generally an increasing function of energy simply from the fact that more single-particle states are accessible to the system at higher energies. On the other hand, a calculation of density of states using a finite space will produce a function that must eventually decrease with

increasing energy and go to zero asymptotically since the total number of states in the finite space is limited. This unrealistic feature of calculations using a finite space causes no problem when we compare the results with experiment if we assume that the space used is sufficiently large to encompass the region of interest. It is therefore understood that the results we calculate are always the partial results in a finite space and this part of the Hilbert space is usually referred as the active space or the space for short.

In statistical spectroscopy we deal with the generalised function, or the distribution, that describes the dependence of a physical quantity on energy and the other variables. This is different from the usual approach in which the calculated results are the expectation values of the correspondng operator over specific states or the transition strengths between particular pair of states obtained by solving an eigenvalue problem. The advantage of using the distributions is that, since the partial result of a physical quantity in a finite space is bounded, the energy and other dependencies can be expressed in terms of moments. If the expansions are restricted to lower order, the moments involved are then traces of simple products of operators and they are in general far easier to obtain than actually used to solve the eigen-value problem in a large space. The common aim of most of the studies in nuclear physics is to understand the nucleus starting from the fundamental nucleon-nucleon interaction. One of the major problems encountered in this regard is that very large space must be used before the results can be compared with the experiments. But on the other hand, most of the work involved may be superfluous since only a small part of the information generated by such calculations is actually used. For example, when the Hamiltonian matrix is diagonalised in a space of several thousand basis states, often only the lowest few eigenstates are of interest. Furthermore, the eigenvectors, each consisting of thousands of components, are used in general to obtain only a few expectation values and transition matrix elements. Instead of discarding the details at the end, which one cannot make use of, it would be more profitable, advisable and judicious not to calculate them and such procedure is essential from a practical point of view. It is well established experimentally observed fact that the need to increase the the number of single-particle states is felt because of the fact that the many-particle space grows exponentially with the addition of single-particle states to the active space. No improvement computational techniques can hope to cope with the problem of exponential growth in the dimension of many-particle space unless a new approach is taken and statistical methods represents one such attempt. In case of nucleus energy dependence of expectation values

or the excitation strengths of an operator can be separated into two parts; a secular part corresponding to the slow changes that are noticeable only over a distance of many states, and fluctuations corresponding to differences between neighbouring states. In statistical spectroscopy, the same separation can also be characterized in terms of the moments of the corresponding distributions, the low-order describing the secular variation and higher-order ones, the fluctuations. The expected economy comes from the belief that the slow variations of a distribution are the important features of the system and that, consequently, an expansion of the distribution can be limited to low-order moments. The justification for adopting such a scheme comes from the studies using ensembles of random matrices. It has been shown that the fluctuations in the distributions of a physical quantity are the properties common to many systems and therefore are not useful for understanding specific systems such as nuclei. As a result large part of the complexity in microscopic calculations in large spaces can be avoided without any loss of essential information. The reason for the success of statistical spectroscopy in nuclear systems is the presence of large number of degrees of freedom present in them. In such systems central limit theorem dominates and hence the distribution of the most observables are essentially Gaussian and are determined by few low order moments. The role played by the higher order moments, conveyed by details generated in large microscopic calculations, is reduced when the system is dominated by the statistical properties. Consequently, it is the low-order moments of the distribution that can tell us something about the nucleus. Certainly the statistical point of view cannot be taken to the extreme. For example, aspects of nucleus which involve the coherent motion of nucleons cannot be treated with advantage using statistical spectroscopy. Also, if the interest is in some particular state because of certain distinguishing features that distinguish this state from other states a statistical treatment cannot be used. To understand such features many models have been designed from time to time to understand such features successfully and statistical spectroscopy is incompetent to provide explanation to such features. The statistical approach to nuclear structute is based on the two premises. The first one is that a separation can be made of the roles played by the low-order and high-order moments, the low-order being responsible for the slow variations in the distribution as a function of energy and the higher order ones for the fluctuations. The second one is that the information of interest mainly lies in the smooth variations. Neither of the two premise can be established firmly, but random matrix studies provide strong support for their validity under a reasonable set of assumptions. In a given space, the properties of a system are governed by the effective Hamiltonian operating in the space. However, since the interaction between the nucleons inside the nucleus

is not completely known, one must refrain from drawing conclusions based on a particular Hamiltonian. Furthermore, since we are here interested in the general features of the nuclei, we are not concerned here with the special characteristics of a few nucleon properties resulting from the peculiarities of the effective Hamiltonian operating in the region. This calls for the introduction of ensembles in analogy with the ensembles used in statistical mechanics. If for example, Q is the physical quantity of interest, it is calculated with the eigenvectors obtained from solving the Schroedinger with all reasonable Hamiltonians. Reasonable Hamiltonian stands for the one which satisfies all the well known properties of a nuclear Hamiltonian, such as time-reversal invariance, rotational symmetry, and consisting of one-body and two-body interactions. This term includes also realistic Hamiltonians which usually means either Hamiltonians derived from nucleon-nucleon scattering data, or Hamiltonians whose defining matrix elements are obtained by fitting to experimental information of nuclei. Let $\hat{Q}$ represents the operator corresponding to the physical quantity Q of interest. In general, the values of Q obtained, say, in the form of expectation values of $\hat{Q}$, are different for different Hamiltonians. If the results calculated with the eigenvectors of all reasonable Hamiltonians are clustered in a narrow region, we can safely assume that the average over the collection, or ensemble, of results provides a good estimate of the value of Q calculated using eigenvectors obtained with the true Hamiltonian. This approach is, however, different from the conventional statistical mechanics which works with the time development of a system under the action of a fixed hamiltonian. Here in statistical nuclear spectroscopy the system is fixed but different Hamiltonians. Instead of assuming the system ergodic in time, i.e., given sufficient time the system will, with equal probability, be in all possible states each of which is represented by the member of the ensemble, it is assumed here in statistical spectroscopy that each reasonable Hamiltonian used to calculate the ensemble result of Q is equally representative of the true Hamiltonian. The proof for this type of ergodicity is not any easier than in statisticaol mechanics. On the other, hand, if the ensemble distribution is narrow, it is highly probable that the ensemble-averaged value is representative of what one would obtain using the true Hamiltonian, since all the reasonable Hamiltonians give the similar results in this case. If the ensemble distribution is flat, the ensemble-averge does not provide us with any clue concerning the possible outcome with the true Hamiltonian. There may be several reasons for this failure and one of them may well be that we had choosen the wrong ensemble. It is, therefore, extremely important to examine the following two points before drawing any conclusion on the basis of ensemble results. Firstly, we must ensure that the ensemble distribution is narrow so that it is unlikely to find values

far away from the average. This can be done by evaluating in addition to the mean, also the ensemble variance distribution, which is the ensemble average of the square of the quantity minus the square of the ensemble mean. A small variance indicates thet the ensemble distribution is narrow. However, as usual, higher moments are required to specify the shape of the distribution, but these are generally much harder to obtain in the case of ensemble distributions. As the nuclear Hamiltonian is not completely known it is difficult to ensure it to be the member of the assumed ensemble. As a result it may be required to enlarge the ensemble as much as possible for relaxing the conditions for a Hamiltonian to be a resonable one. Furthermore, the ensemble must be mathematically manageable. In fact the requirements of mathematical convenience and of reasonable Hamiltonians for the nuclear system do not necessarily coincide and the requirement here is again to enlarge the ensemble so as to accomodate both the requirements. But, at the same time if the ensemble is too large, the proportion of truly reasonable Hamiltonians may become so small that the ensemble averaged results will no longer be representative of the nuclear system. If the ensemble is dominated by unreasonable members, the average may not be physically meaningful even if the ensemble distribution is narrow.

The above lines can be made more clear if we consider the typical spectrum of a heavy nucleus as shown schematically in the fig.(4.1). This complex spectrum of heavy nucleus can be divided into four distinct spectral regions:

$\mathbf{Groundstate\quad Region(D_1)}$: This region which begins at the ground state and extends upto 2 MeV excitation energy. This region is extraordinarily rich in experimental data ahd this region has been studied in great detail through shell model, the extensions of shell model and other various microscopic models to determine the ground state energies, low lying spectra, transition strengths and goodness of symmetries among various quatities of interest. This conventional spectroscopy which has a very broad domain has been successful in the study of low-lying states of light and intermediate nuclei. However, this type of spectroscopy is constrained by the dimensionality of the spectroscopic space that can be considered here, so the range of applicability gets limited and large class of problems remain unaddressed in the domain of conventional spectroscopy. Also, it is generally considered because of the restricted dimensionality defined by single-particle orbits and applicable exact symmetries, it could only be valid at relatively low energies or close to the yrast line.

$\mathbf{D_2\quad Region}$: This region contains close-lying bound sates in the excitation energy

range of approximately (2-6)MeV. About this region very little is known and may contain an extremely large number of levels. The level spacing decreases with increasing excitation energy and it becomes more or less impossible to distinguish between the levels upto 6 MeV in a heavy nucleus.

$\mathbf{D_3}$ **Region**: This is the region of slow neutron resonances with excitation energy in the range 6-6.002 MeV. It is possible to identify the individual levels in this region by bombardment with slow neutrons at an excitation energy at which neutron decay becomes feasible and in a small region (few KeV) from the threshold of neutron decay. The resonances which are formed by the strong reflection of the slow neutrons at the surface by the deep and narrow nuclear potential are sharp because of the spreading of the target plus neutron giant-resonance state over a large number of eigenstates. In this region, neutron resonance spectroscopy has helped in the measurement of complete spectra containing hundred of individual resolved and measured neutron resonances. Further, advancement in charged particle spectroscopy have made the possible the measurement of similar spectra for proton resonances. The importance of $\mathbf{D_3}$ region has been emphasised by Bohr and Mottelson (1973) by saying that whole of the nuclear physics has been decisively influenced by the existence of small window, in the region of neutron binding energy, within which slow neutron reactions provide a probe of enormously great resolving power. The earlier experiments with slow neutrons revealed, very unexpectedly a dense spectrum of resonances and this discovery has led to the strong coupling between the motion of the incident neutron and many degrees of freedom the target. This coupling has given rise to the formation of a compound system with a lifetime very long as compared with the one-particle periods.

$\mathbf{D_4}$ **Region**: This is the region which lies in the energy window greater than 6.002 MeV and contains overlapping levels.

The characteristic features of different regions has led to study of the different quantities of interest. In the ground-state domain where Hamiltonian is known in detail, so we are not interest in calculating the averaged quantities such as level densities. In the neutron threshold region and at higher excitation energies, the quantities of interest are averages and fluctuations about them. Furthermore, the mathematical approaches and the underlying assumptions vary from region to region. For example, we go from a detailed Hamiltonian in $\mathbf{D_1}$, to random matrices in $\mathbf{D_3}$, to using just single-particle energies in $\mathbf{D_4}$.

The interesting point to be noted here is that there has been a little overlap between the assumptions and mathematical methods used in the various regions of the typical spectrum of a heavy nucleus. For example, the parameters for high excitation enegies have not been evaluated in terms of more fundamental quantities and conversely, the great body of level density and similar data has not been used to study or the test the effective interaction. In fact there has been least interest between those involved in the ground-sate region $\mathbf{D_1}$ and those interested in the higher energy regions $\mathbf{D_2}$, $\mathbf{D_3}$ and $\mathbf{D_4}$.

Statistical spectroscopy which is based on statistical laws and operating in model spaces unifies the different approaches used in the ground state and higher energy regions and makes clear the connection between the domains arising from the effective nuclear interaction. Although it works in the model spaces of conventional spectroscopy because it does not deal with the construction and diagonalisation of Hamiltonian matrices and hence is not constrained by the dimensionalities of these spaces. Furthermore, statistical methods are applicable in a wide range of circumstances. The objective of statistical spectroscopy is to deal with the general features of the complex nuclei keeping in mind statistical behaviour observed at high excitations extends right down to the ground state and hence it seems appropriate to use the same methods for all parts of the spectrum and in fact over the whole periodic table.

4.2 Laws Of Statistical Spectroscopy

Statistical spectroscopy deals with spectroscopy in terms of the complete set of correlation functions of various orders (k-point functions). The one-point function defines the state density and by parametric differentiation or otherwise we obtain occupancies, spin cut-off factors and expectation values more generally. There is a natural extension to the two-point correlation function which give a theory for transition strength and symmetry breaking (time reversal, parity, isospin, etc). Moreover, the $(k > 2)$-point functions carry essentially no information so that we have an economical structure. The five laws of statistical laws are [197]

: 1. There is in the model space, a microscopic simplicity, derived from the action of central limit theorems. The smoothed eigenvalue density is close to a characteristic form, usually Gaussian, describable therefore in terms of a small number of low-order Hamiltonian traces (moments).

2. There is a microscopic simplicity corresponding to a remarkable spectral rigidity which extends over the whole spectrum and ensures that the fluctuations are small and for the most carry little information.
3.There is indeed, as implied above, a sharp separation between the secular behaviour of the spectrum and its fluctuations, so that the two can be treated separately and by different methods. This separation also arises from the action of central limit theorem action.
4.There is a propagation of information (i.e., of traces) throughout the set of model spaces defined by N (the number of single particle states) and symmetry that label the spaces. This enables us to express either exactly or approximately, depending on the symmetries involve, the many -particle traces as linear combination of the few particle input traces.
5. The ensembles that one uses have a strong ergodic behaviour.
Similar laws apply to expectation value of operators and transition strength distributions, i.e., for all spectroscopic observables generating spectral distribution theory. Statistical nuclear spectroscopy was initiated by French during 1966-1967 with a series of five papers on trace propagation [198, 199, 200, 201, 202]. The early papers dealing with the trace propagation by [203, 204, 205] were in the context of atomic physics. A first detailed account of spectral distribution in nuclei was given by [206, 207, 208]. In addition there are articles by [209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220] published in various conference proceedings describing different aspects of statistical spectroscopy. There were also three review articles [197, 221, 222] and books [223, 224, 225, 226] on the subject.

4.3 Moments Of a Distribution

To approach the study of distributions an averages in a more quantitative manner, we shall first define the moments that characterise a distribution in general. For that consider various operators denoted by G, one of which is for example Hamiltonian defined in m-particle model space. It implies that, for $\psi_\alpha \in m$, G ψ_α is also a vector in m though this relation does not necessarily hold for the subspaces of m. The eigenvectors of H are denoted by $\psi_{r,i}$ where $i = 1, 2, ..., g_r$ distinguishes between degenerate states. The eigenvalues are E_r with $E_r < E_{r+1}$ and r= 1,2, ...,l, and the model space dimensionality is $d = \sum g_r$. The microscopic density I(x), its normalised counterpart $\rho(x)$ and the

distribution function F(x) are

$$\begin{aligned} I^H(x) \Leftrightarrow I(x) = d\rho(x) & \\ &= \sum_{r=1}^{l} g_r \delta(x - E_r) \\ &= \sum_{r,i} < ri|\delta(H-x)|ri > \\ &= << \delta(H-x) >>^m \end{aligned} \tag{4.1}$$

$$\int_{-\infty}^{+\infty} I_x dx = d; \int_{-\infty}^{+\infty} \rho(x) = 1, F(x) = \int_{-\infty}^{x} \rho(z)dz; F(-\infty) = 0, F(+\infty) = 1, \tag{4.2}$$

The above can be generalised for $d \longrightarrow \infty$. F(x) is a staircase function with jumps $d^{-1}g_r$ at eigenvalues E_r. Thus,

$$F(x) = F(E_r) = d^{-1} \sum_{k=1}^{r} g_k, E_r \leq x < E_{r+1} \tag{4.3}$$

Hereafter, it is assumed that if degeneracy $g_r = 1$. The moments M_p of ρ and the characteristic function ϕ are,

$$\begin{aligned} M_p &= \int \rho(x) x^p dx \\ &= d^{-1} \sum_r g_r (E_r)^p \\ &= < H^p >^m \end{aligned} \tag{4.4}$$

The characteristic function ϕ are

$$\begin{aligned} \phi(t) &= \int \rho(x) e^{itx} dx \\ &= \int \rho(x)[1 + itx + \frac{(itx)^2}{2!} + \frac{(itx)^3}{3!} + \ldots\ldots]dx \\ \phi(t) &= \int \rho(x)dx + it \int x\rho(x)dx \\ &\quad + \frac{(it)^2}{2!} \int x^2 \rho(x)dx + \frac{(it)^3}{3!} \int x^3 \rho(x)dx + \ldots\ldots \end{aligned}$$

$$\begin{aligned}
\phi(t) &= 1 + itM_1 + \frac{(it)^2}{2!}M_2 + \frac{(it)^3}{3!}M_3 \\
&\quad + \ldots\ldots \\
\phi(t) &= M_0 + itM_1 + \frac{(it)^2}{2!}M_2 + \frac{(it)^3}{3!}M_3 \ldots\ldots \\
&= \sum_{p=0}^{\infty} \frac{(it)^p}{p!}M_p \\
&= < exp(itH) >^m, \qquad (4.5)
\end{aligned}$$

$M_0 = 1, M_1 = \zeta, M_2 - (M_1)^2 = \sigma^2, \phi(0) = 1$
For a Gaussian density $\rho_G(x) = \frac{1}{\sqrt{2\pi}\sigma} exp - \frac{1}{2}\left(\frac{x-\zeta}{\sigma}\right)^2$ the characteristic function is given by

$$\begin{aligned}
\phi_G(t;\zeta,\sigma^2) &= \int e^{itx}\rho_G(x)dx \\
&= e^{itx}\frac{1}{\sqrt{2\pi}\sigma}e^{-}\frac{1}{2}(\frac{x-\zeta}{\sigma})^2 dx \\
&= \int \frac{1}{\sqrt{2\pi}} \int e^{itx}e^{-}\frac{1}{2}(\frac{x-\zeta}{\sigma})^2 dx \\
&= \frac{1}{\sqrt{2\pi}} \int e^{itx - \frac{1}{2}\left(\frac{x-\zeta}{\sigma}\right)^2} \qquad (4.6)
\end{aligned}$$

In order to simplify the above equation, let us put $\frac{x-\zeta}{\sigma} = z \Rightarrow dx = \sigma dz$

$$\begin{aligned}
\phi_G(t;\zeta,\sigma^2) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{it(\zeta+z\sigma)}e^{-\frac{z^2}{2}}\sigma dz \\
&= e^{it\zeta}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{itz-\frac{z^2}{2}} \\
&= e^{it\zeta}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(z-it\sigma)^2}{2}}e^{-\frac{t^2\sigma^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}}e^{it\zeta-\frac{t^2\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{(z-it\sigma)^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}}e^{it\zeta-\frac{t^2\sigma^2}{2}}\sqrt{2\pi} \\
&= e^{it\zeta-\frac{t^2\sigma^2}{2}} \qquad (4.7)
\end{aligned}$$

The characteristic function (which always exists) uniquely determines the distribution and in general the moments of the distribution does not determine the distribution and

may not even exist. In case of finite-dimensional spaces and in most of the cases which we encounter in statistical spectroscopy and for operators with bounded eigenvalues, the moments exists. The centroid ζ fixes the location of the distribution and width σ defines the scale. The translational invariant central moments (taken about the centroid as origin) are $\mathcal{M}$. The first few central moments are given by

$$\mathcal{M}_p = \int (x-\zeta)^p \rho(x) dx \tag{4.8}$$

$$\mathcal{M}_0 = \int \rho(x) dx = 1 \tag{4.9}$$

$$\begin{aligned} \mathcal{M}_1 &= \int (x-\zeta)\rho(x)dx \\ &= \int x\rho(x)dx - \zeta \int \rho(x)dx \\ &= M_1 - \zeta \\ &= 0 \end{aligned} \tag{4.10}$$

$$\begin{aligned} \mathcal{M}_2 &= \int (x-\zeta)^2 \rho(x)dx \\ &= \int (x^2+\zeta^2-2x\zeta)\rho(x)dx \\ &= \int x^2\rho(x)dx + \zeta^2 \int \rho(x)dx - 2\zeta \int x\rho(x)dx \\ &= M_2 + \zeta^2 - 2\zeta.M_1 \\ &= M_2 + \zeta^2 - 2\zeta\zeta \\ &= M_2 - \zeta^2 \\ &= M_2 - M_1^2 \\ &= \sigma^2 \end{aligned} \tag{4.11}$$

The set of $M_p(p \leq s)$fixes the$\mathcal{M}_p(p \leq s)$, and vice-versa, by the homogeneous expressions,

$$\mathcal{M}_p = \sum_r (-1)^r \binom{p}{r} M_{p-r}\zeta^r; M_p = \sum_r \binom{p}{r} \mathcal{M}_{p-r}\zeta^r \tag{4.12}$$

The reduced central moments $\mu_p = \frac{\mathcal{M}_p}{\sigma^p}$ are 1,0,1 for p=0,1,2. Since the energy E and centroid ζ can occur only as $(E-\zeta^2)$ in the density and since σ fixes the scale and the total integral of $\rho(E)$ is unity, then $\rho(x)$ must have the form $\rho(x) = \sigma^{-1}\eta^(\hat{x}); \rho(x)dx = \eta(\hat{x})d\hat{x}; \hat{x} = \frac{(x-\zeta)}{\sigma}$ where $\hat{x}$ is the standardised form of x, i.e., x measured with respect to ζ as the origin and re-normalised to unit variance. It follows then that

$$\frac{\partial\rho(x)}{\partial\zeta} = -\frac{\partial\rho(x)}{\partial x}; \frac{\partial\rho(x)}{\partial\sigma^2} = \frac{\partial}{\partial x}\Big[\frac{x-\zeta}{2\sigma^2}\rho(x)\Big] \tag{4.13}$$

Moreover, if $P_\nu^{(\rho)}(x)$ and$P_\nu^{(\eta)}(x)$, with $\nu = 0, 1, 2, ...$, are orthonormal polynomials defined respectively with $\rho(x)$ and $\eta(x)$ as weight functions we have $P_\nu^{(\rho)}(x) = P_\nu^{(\eta)}(\hat{x})$. Most of the distributions that we deal with are are continuous rather than discrete and we encounter several of them in the limit d $\to \infty$ or as a result of otherwise smoothing a discrete distribution. The polynomial excitation functions oscillate about zero and are not true probability densities. For $p \geq 3$, μ_p are translation and scale invariant and hence determine the shape of the distribution. In general terms, we can describe the homogeneous combinations of the $\mu_{p\geq 3}$ as shape parameters and write them as S_ν with $\nu \geq 3$ and it will be sometimes convenient to write $\zeta = S_1$, $\sigma^2 = S_2$. A particular set of distribution parameters, the reduced cumulants $k_p = K_p/\sigma^p$, which are non-trivial shape parameters for $p \geq 3$ are of considerable importance. Just as the moments enter into the Taylor expansion of $\phi(t)$, the cumulants K_p enter into the expansion of its logarithm,

$$log\phi(t) = \sum_{p=1}^{\infty}\frac{(it)^p}{p!}K_p; \phi(t) = exp\left(\frac{(it)^p}{p!}K_p\right) \tag{4.14}$$

$$\begin{aligned} log\phi(t) &= \frac{(it)}{1!}K_1 + \frac{(it)^2}{2!}K_2 + \frac{(it)^3}{3!}K_3 + ... \\ log\phi(t) &= itK_1 - \frac{t^2}{2!}K_2 - \frac{(it)^3}{3!}K_3 + ... \\ log\Big(\sum_{p=0}^{\infty}\frac{(it)^p}{p!}M_p\Big) &= itK_1 - \frac{t^2}{2!}K_2 - \frac{(it)^3}{3!}K_3 + ... \end{aligned} \tag{4.15}$$

Hence, expanding log on the left hand side, we can write the above equation

$$log\Big(M_0 + \frac{(it)}{1!}M_1 + \frac{(it)^2}{2!}M_2 + \frac{(it)^3}{3!}M_3 + ...\Big) = itK_1 - \frac{t^2}{2!}K_2 - \frac{(it)^3}{3!}K_3 + ...$$

Now using the expansion of $log(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

$$log\Big(1 + \frac{(it)}{1!}M_1 + \frac{(it)^2}{2!}M_2 + \frac{(it)^3}{3!}M_3 + \ldots\ldots\Big) = itK_1 - \frac{t^2}{2!}K_2 - \frac{(it)^3}{3!}K_3 + \ldots\ldots$$

$$\Big(itM_1 + \frac{(it)^2}{2!}M_2 + \ldots\Big) - \frac{1}{2}\Big(itM_1 + \frac{(it)^2}{2!}M_2 + \ldots\Big)^2 + \frac{1}{3}\Big(itM_1 + \frac{(it)^2}{2!}M_2 + \ldots\Big)^3$$
$$-\frac{1}{4}\Big(itM_1 + \frac{(it)^2}{2!}M_2 + \ldots\Big)^4 + \ldots\ldots - \ldots\ldots = itK_1 - \frac{t^2}{2!}K_2 - \frac{(it)^3}{3!}K_3 + \ldots\ldots$$

Comparing the coefficients of like powers of it or from eq.(4.5), it can be easily seen that $log\phi(t) = i\zeta t - \sigma^2 t^2/2 + ...$ which combined with eq.(4.14) $K_1 = M_1 = \zeta$, $K_2 = M_2 - M_1^2 = \sigma^2$ (as long as these moments). From eq.(4.7) it follows that the Gaussian cumulants K_p vanish for $p > 2$, which obviously is a defining property of the Gaussian distribution. In general, using the two expansion forms in eqs.(4.14) and (4.5), one finds that [227, 228] that, when $p \geq 2$, the reduced cumulants k_p are given in terms of the reduced central moments μ_p. The shape parameter$k_1 = \gamma_1$ is called skewness and $k_4 = \gamma_2$, the excess. Broadly speaking k_3 defines a distribution which extends more in the $(x > \zeta)$ domain than in the $(x < \zeta)$ and k_4 a distribution more sharply than the Gaussian. In general, the expression for the reduced cumulants in terms of the reduced central moments is,

$$k_p = P! \sum_{[P']} \left[\left\{ \prod_{i=1}^{l} \left(\frac{\mu_{p_i}}{P_i!} \right)^{S_i} \frac{1}{S_i!} \right\} (-1)^{S-1}(S-1)! \right] \tag{4.16}$$

and this gives

$$k_1 = \frac{K_1}{\sigma} = \frac{M_1}{\sigma} = \frac{\zeta}{\sigma}, k_2 = \frac{K_2}{\sigma^2} = \frac{M_2 - M_1^2}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1 \tag{4.17}$$

$$k_3 = \frac{K_3}{\sigma^3} = \frac{M_3 - 3M_1M_2 + 2M_1^3}{\sigma^3} = \mu_3(m) = \gamma_1 \tag{4.18}$$

where $\mu_p = \frac{\mathcal{M}_p}{\sigma_p}$ are the reduced central moments. In the eq. (4.18)k_3 can be shown be equal to μ_3 as follows.

$$\begin{aligned} \mu_3 &= \frac{\mathcal{M}_3}{\sigma_3} \\ &= \frac{1}{\sigma^3} \int (x-\zeta)^3 \rho(x) dx \\ &= \frac{1}{\sigma^3} \left[\int \left(x^3 - \zeta^3 - 3x\zeta(x-\zeta)\rho(x)dx \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sigma^3}\left[\int x^3\rho(x)dx - \zeta^3\int\rho(x)dx - 3\zeta^2\int x^2\rho(x)dx + 3\zeta^2\int x\rho(x)dx\right] \\
&= \frac{1}{\sigma^3}\left[M_3 - 3\zeta M_2 - \zeta^3 - 3\zeta^2 M_1\right] \\
&= \frac{1}{\sigma^3}\left[M_3 - M_1^3 - 3M_1M_2 + 3M_1^3\right] \\
&= \frac{M_3 - 3M_1M_2 + 2M_1^3}{\sigma^3}
\end{aligned}
$$

$$
k_4 = \frac{K_4}{\sigma^4} = \frac{M_4 - 3M_2^2 - 4M_1M_3 - 6M_1^4 + 12M_1^2M_2}{\sigma^4} \tag{4.19}
$$

Hence $\mu_4 = \frac{\mathcal{M}_4}{\sigma_4}$. Now

$$
\begin{aligned}
\mathcal{M}_4 &= \int (x-\zeta)^4\rho(x)dx \\
&= \int\left[(x^2+\zeta^2-2x\zeta)(x^2+\zeta^2-2x\zeta)\right]\rho(x)dx \\
&= \int\left(x^4 + x^2\zeta^2 - 2x^3\zeta + x^2\zeta^2 + \zeta^4 - 2x\zeta^3 - 2x^3\zeta - 2x\zeta^3 + 4x^2\zeta^2\right)\rho(x)dx
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_4 &= \int x^4\rho(x)dx + 6\zeta^2\int x^2\rho(x)dx + \zeta^4\int\rho(x)dx - 4\zeta^3\int x\rho(x)dx - 4\zeta\int x^3\rho(x)dx \\
&= M_4 + 6M_1^2M_2 + \zeta^4 - 4\zeta^4 - 4\zeta M_3 \\
&= M_4 + 6M_1^2M_2 + M_1^4 - 4M_1^4 - 4M_1M_3 \\
&= M_4 + 6M_1^2M_2 - 3M_1^4 - 4M_1M_3
\end{aligned} \tag{4.20}
$$

$$
\begin{aligned}
k_4 - \mu_4 &= \frac{K_4}{\sigma^4} - \frac{\mathcal{M}_4}{\sigma^4} \\
&= \frac{(M_4 - 3M_2^2 - 4M_1M_3 - 6M_1^4 + 12M_1^2M_2) - (M_4 + 6M_1^2M_2 - 3M_1^4 - 4M_1M_3)}{\sigma^4} \\
&= \frac{M_4 - 3M_2^2 - 4M_1M_3 - 6M_1^4 + 12M_1^2M_2 - M_4 - 6M_1^2M_2 + 3M_1^4 + 4M_1M_3}{\sigma^4} \\
&= \frac{-3M_2^2 - 3M_1^4 + 6M_1^2M_2}{\sigma^4} \\
&= \frac{-3M_2^2 - 3(M_2-\sigma^2)^2 + 6M_1^2M_2}{\sigma^4} \\
&= \frac{-3M_2^2 - 3M_2^2 - 3\sigma^4 + 6M_2\sigma^2 + 6(M_2-\sigma^2)M_2}{\sigma^4}
\end{aligned}
$$

$$
\begin{aligned}
k_4 - \mu_4 &= \frac{-6M_2^2 - 3\sigma^4 + 6M_2\sigma^2 + 6(M_2 - 6M_2^2\sigma^2}{\sigma^4} \\
k_4 - \mu_4 &= -\frac{3\sigma^4}{\sigma^4} = -3 \\
k_4 &= \mu_4(m) - 3 \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
k_5 - \mu_5 &= \frac{K_5}{\sigma^5} - \frac{\mathcal{M}_5}{\sigma^5} \\
&= \frac{M_5 - 5M_1M_4 - 10M_2M_3 + 20M_3M_1^2 + 30M_2^2M_1 - 60M_2M_1^3 + 24M_1^5 + x}{\sigma^5}
\end{aligned}
$$

where, $x = -M_5 - 10M_1^2M_3 + 5M_1M_4 + 10M_1^3M_2 - 4M_1^5$

$$
\begin{aligned}
k_5 - \mu_5 &= -10\frac{[M_2M_3 - M_3M_1^2 - 3M_2^2M_1 + 5M_2M_1^3 - 2M_1^5]}{\sigma^5} \\
&= -10\frac{[M_2M_3 - M_3(M_2 - \sigma^2) - 3M_2^2M_1 + 5M_1M_2(M_2 - \sigma^2) - 2M_1^3(M_2 - \sigma^2)]}{\sigma^5} \\
&= -10\frac{[M_2M_3 - M_3M_2 + M_3\sigma^2 - 3M_2^2M_1 + 5M_1M_2^2 - 5M_1M_2\sigma^2 - 2M_1^3M_2 - x']}{\sigma^5}
\end{aligned}
$$

where $x' = -2M_1^3\sigma^2$

$$
\begin{aligned}
k_5 - \mu_5 &= -10\frac{[M_3\sigma^2 + 2M_1M_2^2 - 5M_1M_2\sigma^2 - 2M_1^3M_2 + 2M_1^3\sigma^2]}{\sigma^5} \\
&= -10\frac{[M_3\sigma^2 + 2M_1^3\sigma^2 - 5M_1M_2(M_2 - M_1^2) + 2M_1M_2^2 - 2M_1^3M_2]}{\sigma^5} \\
&= -10\frac{[M_3\sigma^2 + 2M_1^3\sigma^2 - 5M_1^2M_2 + 5M_1^3M_2 + 2M_1M_2^2 - 2M_1^3M_2]}{\sigma^5} \\
&= -10\frac{[M_3\sigma^2 + 2M_1^3\sigma^2 - 3M_1M_2^2 + 3M_1^3M_2]}{\sigma^5} \\
&= -10\frac{[M_3\sigma^2 + 2M_1^3\sigma^2 + 3M_1M_2(M_1^2 - M_2)]}{\sigma^5} \\
&= -10\frac{[M_3\sigma^2 + 2M_1^3\sigma^2 - 3M_1M_2\sigma^2]}{\sigma^5} \\
&= -10\frac{[M_3 + 2M_1^3 - 3M_1M_2]}{\sigma^3} \\
&= -10\frac{K_3}{\sigma^3}
\end{aligned}
$$

$$
\begin{aligned}
&= -10\frac{\mathcal{M}_3}{\sigma^3} \\
&= -10k_3 \\
k_5 &= \mu_5(m) - 10k_3 \qquad (4.22)
\end{aligned}
$$

Similarly,

$$k_6 = \mu_6(m) - 15k_4(m) - 10\left[k_3(m)\right]^2 - 15 \qquad (4.23)$$

In eqn. (4.16) $[P']$ are all partitions of P such that$P_i \geq 2$ for allP_i.

The polynomial expansions for the densities is described in detail [229, 230, 227, 228]. There are two problems of immediate interest. The first is to find an adequate expansion for the density in terms of the asymptotic density and the polynomials defined by it. Given that density, the second problem is that of deriving the first order corrections to it when we add a small term to the Hamiltonian; with the solution to this (the problem of linear response), we shall be able to derive expectation values and sum rules. Only in the case of simplest textbook cases do we have any prospect of evaluating a complete set of moments; usually we must settle for a small number of lower order moments. These of course place constraints on the distribution [227] in accordance with the so called Principle of Moments, and they determine inequalities, as for example Chebyshev inequality [227, 228] on various quantities defined by the distribution [231]. These inequalities however are indequate for our purpose. But we are saved by the CLT generation of a close-to-Gaussian smoothed density. Then it will turn that, " to within fluctuations", calculable (low order) shape corrections will give adequate results for particle numbers and Hamiltonians of interest. As a consequence, we seek expansions of $\rho(x)$ around an asymptotic shape which we take to be a Gaussian (other forms will arise in special cases; the modifications needed or extensions will be straight forward). The general nature of this expansion will be in terms of a sequence of polynomial excitations of the asymptotic density (see ahead for GC, ED and CF expansions). Given a density ρ with central moments $\mathcal{M}_r$, it is possible to write orthogonal polynomials $P_\mu(x)$ as follows. The orthogonal polynomial $P_\mu(x)$ is

defined as

$$P_\mu(x) = [D_\mu D_{\mu-1}]^{-\frac{1}{2}} \begin{vmatrix} 1 & M_1 & - & - & - & M_\mu \\ M_1 & M_2 & - & - & - & M_{\mu+1} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ M_{\mu-1} & M_\mu & - & - & - & M_{2\mu-1} \\ 1 & x & x^2 & - & - & x^\mu \end{vmatrix}.$$

where

$$D_\mu = \begin{vmatrix} 1 & M_1 & - & - & - & M_\mu \\ M_1 & M_2 & - & - & - & M_{\mu+1} \\ - & - & - & - & - & M_{2\mu-1} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ M_\mu & M_{\mu+1} & - & - & - & M_{2\mu} \end{vmatrix}.$$

Hence $P_0(x) = 1$

$$\begin{aligned} P_1(x) &= [D_1 D_0]^{-\frac{1}{2}} \begin{vmatrix} 1 & M_1 \\ 1 & x \end{vmatrix} \\ &= \frac{(x - M_1)}{[D_1 D_0]^{\frac{1}{2}}} \\ &= \frac{(x - M_1)}{\left[\begin{vmatrix} 1 & M_1 \\ M_1 & M_2 \end{vmatrix} \times \begin{vmatrix} 1 & M_0 \\ M_0 & M_1 \end{vmatrix} \right]^{\frac{1}{2}}} \\ &= \frac{(x - M_1)}{[(M_2 - M_1^2)(M_1 - M_0^2)]^{\frac{1}{2}}} \end{aligned} \tag{4.24}$$

$$P_2(x) = \frac{\begin{vmatrix} 1 & M_1 & M_2 \\ M_1 & M_2 & M_3 \\ 1 & x & x^2 \end{vmatrix}}{\left[\begin{vmatrix} 1 & M_1 & M_2 \\ M_1 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{vmatrix} \right]^{\frac{1}{2}} \times \left[\begin{vmatrix} 1 & M_1 \\ M_1 & M_2 \end{vmatrix} \right]^{\frac{1}{2}}}$$

$$
\begin{aligned}
&= \frac{(M_2x^2 - M_3x) - M_1(M_1x^2 - M_3) + M_2(M_1x - M_2)}{[(M_2M_4 - M_3^2 - M_1(M_1M_4 - M_2M_3) + M_2(M_1M_3 - M_2^2)]^{\frac{1}{2}} \times (M_2 - M_1^2)^{\frac{1}{2}}} \\
&= \frac{M_2x^2 - M_3x - M_1^2x^2 + M_1M_3 + M_1M_2x - M_2^2}{[(M_2M_4 - M_3^2) - M_1^2M_4 + 2M_1M_2M_3 - M_2^3]^{\frac{1}{2}} \times \mathcal{M}_2^{\frac{1}{2}}} \qquad (4.25)
\end{aligned}
$$

Now let us try to first of all simplify the numerator of above eq.(4.25).

$$
\begin{aligned}
& M_2x^2 - M_3x - M_1^2x^2 + M_1M_3 \\
+ \;& M_1M_2x - M_2^2 \\
= \;& (M_2 - M_1^2)x^2 + (M_1M_2 - M_3)x + M_1M_3 - M_2^2 \\
= \;& \sigma^2x^2 + (M_1M_2 - M_3)x + M_1M_3 - M_2^2 \\
= \;& \sigma^2(\sigma\hat{x} + \zeta)^2 + (M_1M_2 - M_3)x + M_1M_3 - M_2^2 \\
= \;& \sigma^2(\sigma^2\hat{x}^2 + \zeta^2 + 2\sigma\hat{x}\zeta) + (M_1M_2 - M_3)x + M_1M_3 - M_2^2 \\
= \;& \sigma^4\hat{x}^2 + \sigma^2\zeta^2 + 2\sigma^3\hat{x}\zeta + (M_1M_2\sigma\hat{x} + (M_1M_2\zeta - M_3\sigma\hat{x} - M_3\zeta \\
+ \;& M_1M_3 - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 + \sigma^2\zeta^2 + 2\sigma^3\hat{x}\zeta + \zeta M_2\sigma\hat{x} + M_2\zeta^2 - M_3\sigma\hat{x} - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 + \sigma^2\zeta^2 + 2\sigma^3\hat{x}\zeta + \zeta M_2\sigma\hat{x} + M_2\zeta^2 \\
- \;& (\mathcal{M}_3 - 2\zeta^3 + 3M_2\zeta)\sigma\hat{x} - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 + \sigma^2\zeta^2 + 2\sigma^3\hat{x}\zeta + \zeta M_2\sigma\hat{x} + M_2\zeta^2 \\
- \;& \mathcal{M}_3\sigma\hat{x} + 2\zeta^3\sigma\hat{x} - 3M_2\zeta\sigma\hat{x} - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} + \sigma^2\zeta^2 + 2\sigma^3\hat{x}\zeta - 2\zeta M_2\sigma\hat{x} + M_2\zeta^2 \\
+ \;& 2\zeta^3\sigma\hat{x} - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} + \sigma^2\zeta^2 + M_2\zeta^2 + 2\sigma\hat{x}\zeta(\sigma^2 + \zeta^2) \\
- \;& 2\zeta M_2\sigma\hat{x} - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} + \sigma^2\zeta^2 + M_2\zeta^2 + 2M_2\sigma\hat{x}\zeta - 2M_2\sigma\hat{x}\zeta - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} + (M_2 + \sigma^2)\zeta^2 - M_2^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} + (M_2 + \sigma^2)\zeta^2 - (\sigma^2 + M_1^2) \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} + M_2\zeta^2 + \sigma^2\zeta^2 - \sigma^4 - M_1^4 - 2M_1^2\zeta^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} + M_2\zeta^2 + \sigma^2\zeta^2 - \mathcal{M}_2^2 - \zeta^4 - 2\sigma^2\zeta^2 \\
= \;& \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} - \mathcal{M}_2^2 + M_2\zeta^2 - \sigma^2\zeta^2 - \zeta^4
\end{aligned}
\qquad (4.26)
$$

$$\begin{aligned} &= \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} - \mathcal{M}_2^2 + \zeta^2(M_2 - \zeta^2) - \sigma^2\zeta^2 \\ &= \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} - \mathcal{M}_2^2 + \sigma^2\zeta^2 - \sigma^2\zeta^2 \\ &= \mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} - \mathcal{M}_2^2 \end{aligned} \tag{4.27}$$

Now, let us try to simplify the denominator of eq.(4.25).

$$\begin{aligned} & [(M_2M_4 - M_3^2) - M_1^2M_4 + 2M_1M_2M_3 - M_2^3]^{\frac{1}{2}} \\ \times\ & \mathcal{M}_2^{\frac{1}{2}} \\ = \ & \mathcal{M}_2^{\frac{1}{2}}[(M_2M_4 - M_3^2) - M_4\zeta^2 + 2\zeta M_2M_3 - M_2^3]^{\frac{1}{2}} \\ = \ & \mathcal{M}_2^{\frac{1}{2}}[(M_4(M_1^2 + \mathcal{M}_2) - M_3^2 - M_4\zeta^2 + 2\zeta M_2M_3 - M_2^3]^{\frac{1}{2}} \\ = \ & \mathcal{M}_2^{\frac{1}{2}}[M_4\zeta^2 + \mathcal{M}_2M_4 - M_3^2 - M_4 + 2\zeta M_2M_3 - (\zeta^2 + \mathcal{M}_2)^3]^{\frac{1}{2}} \\ = \ & \mathcal{M}_2^{\frac{1}{2}}[\mathcal{M}_2M_4 - M_3^2 + 2\zeta(\zeta^2 + \mathcal{M}_2)M_3 - \zeta^6 - \mathcal{M}_2^6 - 3\zeta^2\mathcal{M}_2^2 - 3\zeta^4\mathcal{M}_2]^{\frac{1}{2}} \\ = \ & \mathcal{M}_2^{\frac{1}{2}}[\mathcal{M}_2M_4 - M_3^2 + 2\zeta^3M_32\zeta\mathcal{M}_2M_3 - \zeta^6 - \mathcal{M}_2^6 - 3\zeta^2\mathcal{M}_2^2 - 3\zeta^4\mathcal{M}_2]^{\frac{1}{2}} \end{aligned} \tag{4.28}$$

Now,

$$\begin{aligned} \mathcal{M}_4 &= \int (x-\zeta)^4\rho(x)dx \\ &= \int \Big[(x^2 + \zeta^2 - 2x\zeta)(x^2 + \zeta^2 - 2x\zeta)\rho(x)dx\Big] \\ &= \int (x^4 + x^2\zeta^2 - 2x^3\zeta + x^2\zeta^2 + \zeta^4 - 2x\zeta^3 - 2x^3\zeta - 2x\zeta^3 + 4x^2\zeta^2)\rho(x)dx \\ &= \int x^4\rho(x)dx + 6\zeta^2\int x^2\rho(x)dx + \zeta^4\int \rho(x)dx - 4\zeta^3\int x\rho(x)dx - 4\zeta\int x^3\rho(x)dx \\ &= M_4 + 6\zeta^2M_2 + \zeta^4 - 4\zeta^3M_1 - 4\zeta M_3 \\ &= M_4 + 6\zeta^2M_2 + \zeta^4 - 4\zeta^4 - 4\zeta M_3 \\ &= M_4 + 6\zeta^2M_2 - 4\zeta M_3 - 3\zeta^4 \\ &= M_4 + 6\zeta^2(\zeta^2 + \mathcal{M}_2) - 4\zeta M_3 - 3\zeta^4 \\ &= M_4 + 6\zeta^4 + 6\zeta^2\mathcal{M}_2 - 4\zeta M_3 - 3\zeta^4 \\ &= M_4 + 6\zeta^2\mathcal{M}_2 - 4\zeta M_3 + 3\zeta^4 \end{aligned} \tag{4.29}$$

Also, the third central moment is given by

$$\begin{aligned}
\mathcal{M}_3 &= \int (x-\zeta)^3 \rho(x) dx \\
&= \int \Big(x^3 - \zeta^3 - 3x\zeta(x-\zeta) \Big) \rho(x) dx \\
&= \int x^3 \rho(x) dx - \zeta^3 \int \rho(x) dx - 3\zeta \int x^2 \rho(x) dx + 3\zeta^2 \int x \rho(x) dx \\
&= M_3 - \zeta^3 - 3\zeta M_2 + 3\zeta^3 \\
&= M_3 - 3\zeta M_2 + 2\zeta^3 \\
\implies M_3 &= \mathcal{M}_3 + 3\zeta M_2 - 2\zeta^3 \qquad (4.30)
\end{aligned}$$

From eq.(4.29), M_4 can be written as

$$\begin{aligned}
M_4 &= \mathcal{M}_4 - 6\zeta^2 \mathcal{M}_2 - 3\zeta^4 + 4\zeta M_3 \\
&= \mathcal{M}_4 - 3\zeta^4 - 6\zeta^2 \mathcal{M}_2 + 4\zeta(\mathcal{M}_3 + 3\zeta \mathcal{M}_2 + \zeta^3) \\
&= \mathcal{M}_4 - 3\zeta^4 - 6\zeta^2 \mathcal{M}_2 + 4\zeta \mathcal{M}_3 + 12\zeta^2 \mathcal{M}_2 + 4\zeta^4 \\
&= \mathcal{M}_4 + \zeta^4 + 4\zeta \mathcal{M}_3 - 6\zeta^2 \mathcal{M}_2 \qquad (4.31)
\end{aligned}$$

Hence, the equation becomes

$$\begin{aligned}
& [\mathcal{M}_2 M_4 - M_3^2 + 2\zeta^3 M_3 2\zeta \mathcal{M}_2 M_3 - \zeta^6 - \mathcal{M}_2^6 - 3\zeta^2 \mathcal{M}_2^2 - 3\zeta^4 \mathcal{M}_2]^{\frac{1}{2}} \\
\times\ & \mathcal{M}_2^{\frac{1}{2}} \\
=\ & [\mathcal{M}_2(\mathcal{M}_4 + \zeta^4 + 4\zeta \mathcal{M}_3 + 6\zeta^2 \mathcal{M}_2) - (\mathcal{M}_3 + 3\zeta M_2 + \zeta^3)^2 + 2\zeta^3(\mathcal{M}_3 + 3\zeta M_2 + \zeta^3) \\
+\ & 2\zeta \mathcal{M}_2 - \zeta^6 - \mathcal{M}_2^3 - 3\zeta^2 \mathcal{M}_2^2 - 3\zeta^4 \mathcal{M}_2]^{\frac{1}{2}} \mathcal{M}_2^{\frac{1}{2}} \\
=\ & [\mathcal{M}_2 \mathcal{M}_4 + \mathcal{M}_2 \zeta^4 + 6\zeta^2 \mathcal{M}_2^2 + 4\zeta \mathcal{M}_2 \mathcal{M}_3 - (\mathcal{M}_3^2 + 9\zeta^2 \mathcal{M}_2^2 + \zeta^6 + 6\zeta \mathcal{M}_2 \mathcal{M}_3 + 6\zeta^4 \mathcal{M}_2 \\
+\ & 2\zeta^3 \mathcal{M}_3) + 2\zeta^3 \mathcal{M}_3 + 6\zeta^4 \mathcal{M}_2 + 2\zeta^6 + 2\zeta \mathcal{M}_2 \mathcal{M}_3 + 6\zeta^2 \mathcal{M}_2^2 + 2\zeta^4 \mathcal{M}_2 - \zeta^6 - \mathcal{M}_2^3 \\
-\ & 3\zeta^2 \mathcal{M}_2^2 - 3\zeta^4 \mathcal{M}_2]^{\frac{1}{2}} \mathcal{M}_2^{\frac{1}{2}} \\
=\ & [\mathcal{M}_2 \mathcal{M}_4 + \mathcal{M}_2 \zeta^4 + 6\zeta^2 \mathcal{M}_2^2 + 4\zeta \mathcal{M}_2 \mathcal{M}_3 - \mathcal{M}_3^2 - 9\zeta^2 \mathcal{M}_2^2 - \zeta^6 - 6\zeta \mathcal{M}_2 \mathcal{M}_3 - 6\zeta^4 \mathcal{M}_2 \\
-\ & 2\zeta^3 \mathcal{M}_3) + 2\zeta^3 \mathcal{M}_3 + 6\zeta^4 \mathcal{M}_2 + 2\zeta^6 + 2\zeta \mathcal{M}_2 \mathcal{M}_3 + 6\zeta^2 \mathcal{M}_2^2 + 2\zeta^4 \mathcal{M}_2 - \zeta^6 - \mathcal{M}_2^3 - 3\zeta^2 \mathcal{M}_2^2 \\
-\ & 3\zeta^4 \mathcal{M}_2]^{\frac{1}{2}} \mathcal{M}_2^{\frac{1}{2}} \\
=\ & [\mathcal{M}_2 \mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3]^{\frac{1}{2}} \mathcal{M}_2^{\frac{1}{2}} \\
=\ & [\mathcal{M}_2(\mathcal{M}_2 \mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}} \qquad (4.32)
\end{aligned}$$

Therefore, $P_2(x)$ can be written as

$$P_2(x) = \frac{\mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} - \mathcal{M}_2^2}{[\mathcal{M}_2(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}}} \tag{4.33}$$

Alternatively, $P_2(x)$ can be written in the form as follows.

$$\begin{aligned}
P_2(x) &= \frac{\mathcal{M}_2^2\hat{x}^2 - \mathcal{M}_3\sigma\hat{x} - \mathcal{M}_2^2}{[\mathcal{M}_2(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}}} \\
&= \frac{\mathcal{M}_2^2\Big(\hat{x}^2 - \frac{\mathcal{M}_3}{\mathcal{M}_2^2}\sigma\hat{x} - 1\Big)}{[\mathcal{M}_2(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}}} \\
&= \frac{\mathcal{M}_2^2\Big(\hat{x}^2 - \frac{\mathcal{M}_3}{\sigma^4}\sigma\hat{x} - 1\Big)}{[\mathcal{M}_2(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}}} \\
&= \frac{\mathcal{M}_2^2\Big(\hat{x}^2 - \frac{\mathcal{M}_3}{\sigma^3}\hat{x} - 1\Big)}{[\mathcal{M}_2(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}}} \\
&= \frac{\mathcal{M}_2^2\Big(\hat{x}^2 - \mu_3\hat{x} - 1\Big)}{[\mathcal{M}_2(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}}} \\
&= \frac{\mathcal{M}_2^2\Big(\hat{x}^2 - \gamma_1\hat{x} - 1\Big)}{[\mathcal{M}_2(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)]^{\frac{1}{2}}} \\
&= \frac{\hat{x}^2 - \gamma_1\hat{x} - 1}{\Big[\frac{\mathcal{M}_2}{\mathcal{M}_2^4}(\mathcal{M}_2\mathcal{M}_4 - \mathcal{M}_3^2 - \mathcal{M}_2^3)\Big]^{\frac{1}{2}}} \\
&= \frac{\hat{x}^2 - \gamma_1\hat{x}}{\Big[\frac{\mathcal{M}_4}{\mathcal{M}_2^2} - \frac{\mathcal{M}_3^2}{\mathcal{M}_2^3} - 1\Big]^{\frac{1}{2}}} \\
&= \frac{\hat{x}^2 - \gamma_1\hat{x} - 1}{\Big[\frac{\mathcal{M}_4}{\sigma^4} - \frac{\mathcal{M}_3^2}{\mathcal{M}_2^3} - 1\Big]^{\frac{1}{2}}} \\
&= \frac{\hat{x}^2 - \gamma_1\hat{x} - 1}{\Big[\mu_4 - \frac{\mathcal{M}_3^2}{\sigma^6} - 1\Big]^{\frac{1}{2}}} \\
&= \frac{\hat{x}^2 - \gamma_1\hat{x} - 1}{\Big[\mu_4 - \mu_3^2 - 1\Big]^{\frac{1}{2}}}
\end{aligned} \tag{4.34}$$

$$
\begin{aligned}
&= \frac{\hat{x}^2 - \gamma_1 \hat{x} - 1}{\left[(\gamma_2 + 3) - \mu_3^2 - 1\right]^{\frac{1}{2}}} \\
&= \frac{\hat{x}^2 - \gamma_1 \hat{x} - 1}{\left[2 + \gamma_2 - \mu_3^2\right]^{\frac{1}{2}}} \\
&= \frac{\hat{x}^2 - \gamma_1 \hat{x} - 1}{\left[2 + \gamma_2 - \gamma_1^2\right]^{\frac{1}{2}}}
\end{aligned} \tag{4.35}
$$

Eq.(4.35) gives the general formula. In order to be useful, the first few terms of the polynomial expansion must give a satisfactory representation of the density. Compatible with this are the facts that our interest is in fluctuation free densities and there is a wide gap in the spectrum between the first few long wavelength excitations of present interest and the short wavelength ones which describes the fluctuations. It follows then that a finite expansion will often be useful even when the complete expansion is not formally point convergent. This is what we call " convergence to within fluctuations". In fact, statistical methods are valid only for strong interactions; when they are weak, perturbation theory is appropriate. The situation is somewhat more complex when the interactions are of intermediate strength.

Given the standardized variable $\hat{x}$ and the corresponding Gaussian density, it was argued by Edgeworth that $\eta(x)$

$$
\eta(\hat{x}) = exp. \left\{ \sum_{\nu \geq 3} (-1)^{\nu} \frac{k_v}{\nu!} \frac{\partial \nu}{\partial \hat{x}^{\nu}} \right\} \eta_G(\hat{x}); \eta_G(\hat{x}) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{\hat{x}^2}{2}\right) \tag{4.36}
$$

is a true and unique law that represents the frequency curve of a magnitude that depends on on a number of independent elements [Bowley (1972)]. If the numberγ of such events varies, then $k_\nu \propto \gamma^{-\frac{\nu}{2}}$, and that the appropriate method of approximation is by truncating the ν series. The k_ν are reduced cumulants and $k_\nu(m) = k_\nu(1)/\gamma^{\frac{\nu}{2}-1}$ if the independent variables are similarly distributed. Thus on the one hand Edgeworth argument is a statement of the CLT and, on the other, an argument that uniformity with regard to γ (particle number in our examples) is a good guide to a method of approximation. Expanding the exponential in eq.(4.36) and collecting all the terms that behave as $\gamma^{-\frac{P}{2}}$, $P = 1, 2, ...$, a

compact form for the Edgeworth(ED) expansion is

$$\eta_{ED}(\hat{x}) = \eta_G(\hat{x}) \left\{ 1 + \sum_{P=1}^{\infty} \sum_{[P]} \left[\prod_{i=1}^{l} \left(\frac{k_{P_i+2}}{(P_i+2)!} \right)^{S_i} \frac{1}{S_i!} \right] He_{P+2S}(\hat{x}) \right\} \tag{4.37}$$

Here $[P] = [P_1^{S_1}, P_2^{S_2}, ...P_l^{S_l}]$ is a partition of the integer P such that $P_1 \geq P_2 \geq .....P_l \geq 0$, $S_i > 0$, $S = \sum_{i=1}^{l}$ and $\sum_{i=1}^{l} = P$. Feller (1971) has shown that if the moments μ_3, $\mu_4, ...\mu_r$ exist and $|\phi(t)|^s$ is integrable for some s ≥ 1, then $\eta(\hat{x})$ exists for m $\geq$ s and the ED representation given by eqn.(4.37) is asymptotically convergent, i.e. as m $\to \infty$, the series converges to the first $r = P + 1$ terms. The ED expansion to order $P = 6$ is, with $k'_\nu = \frac{k_\nu}{\nu!}$.

$$\begin{aligned}
\eta_{ED}(\hat{x}) = \eta_G(\hat{x}) \Bigg\{ & 1 + [k'_3 He_3(\hat{x})] + \left[k'_4 He_4(\hat{x}) + \frac{(k'_3)^2}{2!} He_6(\hat{x}) \right] + [k'_5 He_5(\hat{x}) + k'_3 k'_4 He_7(\hat{x}) \\
& + \frac{(k'_3)^3}{3!} He_9(\hat{x}) \Bigg] + \left[k'_6 He_6(\hat{x}) + \left(\frac{(k'_4)^2}{2!} + k'_3 k'_5 \right) He_8(\hat{x}) + \frac{(k'_3)^2 k'_4}{2!} He_{10}(\hat{x}) \right. \\
& \left. + \frac{(k'_3)^4}{4!} He_{12}(\hat{x}) \right] + [k'_7 He_7(\hat{x}) + (k'_3 k'_6 + k'_4 k'_5)\, He_9(\hat{x}) \\
& + \left(\frac{(k'_3)^2 k'_5}{2!} + \frac{(k'_3 k'_4)^2}{2!} \right) He_{11}(\hat{x}) + \frac{(k'_3)^3 k'_4}{3!} He_{13}(\hat{x}) + \frac{(k'_3)^5}{5!} He_{15}(\hat{x})] + [k'_8 He_8(\hat{x}) \\
& + \left(\frac{(k'_5)^2}{2!} + k'_3 k'_7 + k'_4 k'_6 \right) He_{10}(\hat{x}) + \left(\frac{(k'_4)^3}{3!} + \frac{(k'_3)^2 k'_6}{2!} + k'_3 k'_4 k'_5 \right) He_{12}(\hat{x}) \\
& + \left(\frac{(k'_3)^2 (k'_4)^2}{2!2!} + \frac{(k'_3)^3 k'_5)}{3!} \right) He_{14}(\hat{x}) + \frac{(k'_3)^4 (k'_4)}{4!} He_{16}(\hat{x}) + \frac{(k'_3)^6}{6!} He_{18}(\hat{x})] \Bigg\}
\end{aligned} \tag{4.38}$$

The Hermite polynomials $He_r(\hat{x})$ satisfy the recursion relation $He_r(\hat{x}) = \hat{x} He_r(\hat{x}) - r He_{r-1}(\hat{x})$ and explicit expressions for the lowest six polynomials are,

$$\left.\begin{aligned}
He_0(\hat{x}) &= 1 \\
He_1(\hat{x}) &= \hat{x} \\
He_2(\hat{x}) &= \hat{x}^2 - 1 \\
He_3(\hat{x}) &= \hat{x}^3 - 3x \\
He_4(\hat{x}) &= \hat{x}^4 - 6\hat{x}^2 + 3 \\
He_5(\hat{x}) &= \hat{x}^5 - 10\hat{x}^3 + 15\hat{x} \\
He_5(\hat{x}) &= \hat{x}^6 - 15\hat{x}^4 + 45\hat{x}^2 - 15
\end{aligned}\right\}. \tag{4.39}$$

Here it should be noted that the centroid and width of $\rho(x)$ and $\rho_G(x)$ that correspond to $\eta_{ED}(\hat{x})$ are identical in the above ED expansion.

4.4 State density and Nuclear Partition function

The origins of statistical nuclear theory can be traced back to Bethe's derivation of level density. His calculation were based on statistical mechanics of essentially non-interacting particles in an unbound single-energy spectrum. This along with the Wigner's introduction of Hamiltonian random matrix ensembles are two landmarks in the in statistical spectroscopy. It is well known fact that the state density $\rho(A, E)$, the number of states per unit energy for a nucleus made of A nucleons, increases roughly with the squareroot of the excitation energy. Bethe(1937) derived the relation

$$\rho(A, E) = \frac{1}{12a^{\frac{1}{4}}E^{\frac{5}{4}}} exp 2\sqrt{aE} \tag{4.40}$$

using statistical arguments. This expression is often referred top as the Fermi gas model since the nucleons inside a nucleus are treated essentially as non-interacting Fermi particles. A brief review of the derivation of Bethe's level density for is useful, as a background for statistical approach to this problem. The Hamiltonian used in the derivation of Bethe level density formula is taken to be purely one-body and is given by a set of set of single-particle energies ϵ_i. This is one of the major assumptions made in the derivation of the formula. The ignored two-body part of the Hamiltonian is important since it depresses the ground-state energy from the excitation energy E in eq.(4.40) is measured.
For a one-body Hamiltonian, density of levels as a function of ϵ and particle number is given by

$$\rho(A, \epsilon) = \sum_{n,i} \delta(A - n)\delta(\mathcal{E} - \mathcal{E}_i(n)) \tag{4.41}$$

where $\epsilon_i(n)$ is the energy of the ith quantum state of the n-particle system. In the independent particle approximation, we can write

$$n = \sum_{\nu} n(\nu)_i; \epsilon_i = \sum_{\nu} (n(\nu))_i \epsilon(\nu) \tag{4.42}$$

Each single-particle orbit here consists of only one state so that $(n(\nu_i))_i$ is either 0, if the states is occupied, or 0 if it is occupied as per Pauli's exclusion principle. The eq.(4.41)

has singularities at each of the eigenvalues (4.42), but, the interest is in the average value of this function when integrated over an interval in A and ϵ. Because of the additive nature of the relations (4.42) which determine the eigenvalue of A and ϵ, it is convenient to work with the Laplace Transform

$$z(\alpha,\beta) = \int_0^{+\infty}\int_0^{+\infty} \rho(A,\epsilon)e^{(\alpha A-\beta\epsilon)}dAd\epsilon$$

The parameters α and β correspond to the chemical potential μ and temperature in statistical mechanics.

$$\begin{aligned} z(\alpha,\beta) &= \int_0^{+\infty}\int_0^{+\infty}\sum_{n,i}\delta(A-n)\delta(\mathcal{E}-\mathcal{E}_i(n))e^{(\alpha A-\beta\mathcal{E})}dAd\mathcal{E} \\ &= \sum_{n,i} e^{(\alpha n-\beta\mathcal{E}_i(n))} \\ &= \prod_{\nu}(1+e^{\alpha-\beta\epsilon(\nu)}) \end{aligned} \tag{4.43}$$

In the above equation the term 1 comes from $n(\nu)=0$ and the exponential term comes from $n(\nu)=1$. In order to evaluate the product in eq.(4.43) in terms of a sum over the one-particle states, we take logarithm on both sides of the equation

$$\begin{aligned} lnz(\alpha,\beta) &= ln\Big[\prod_{\nu}(1+e^{\alpha-\beta\epsilon(\nu)})\Big] \\ &= ln\Big[(1+e^{\alpha-\beta\epsilon(0)})(1+e^{\alpha-\beta\epsilon(1)})(1+e^{\alpha-\beta\epsilon(2)})+.....\Big] \\ &= ln(1+e^{\alpha-\beta\epsilon(0)})+ln(1+e^{\alpha-\beta\epsilon(1)})+ln(1+e^{\alpha-\beta\epsilon(2)})+..... \\ &= \sum_{\nu} ln(1+e^{\alpha-\beta\epsilon(\nu)}) \end{aligned} \tag{4.44}$$

The second assumption made to derive the Bethe's level density formula is that the single-particle spectrum,

$$g(\epsilon) = \sum_{\nu}\delta(\epsilon-\epsilon(\nu)), \tag{4.45}$$

can be approximated by a continuous distribution and we use an energy scale such that $\epsilon(\nu)\geq 0$. for all This is true if the single-particle states are closely spaced. In practice this assumption does not seem to affect the state density for large A. Using eq.(4.45),

eq.(4.44) can be written as

$$lnZ(\alpha,\beta) = \int_0^\infty g(\epsilon) ln(1+e^{\alpha-\beta\epsilon}) d\epsilon \tag{4.46}$$

This can be checked by substituting $g(\epsilon)$ in the above equation.

$$\begin{aligned} lnz(\alpha,\beta) &= \int_0^\infty \sum_\nu \delta(\epsilon-\epsilon(\nu)) ln(1+e^{\alpha-\beta\epsilon}) d\epsilon \\ &= \sum_\nu ln(1+e^{\alpha-\beta\epsilon(\nu)}) \int_0^\infty \delta(\epsilon-\epsilon(\nu)) d\epsilon \\ &= \sum_\nu ln(1+e^{\alpha-\beta\epsilon(\nu)}).1 \\ &= \sum_\nu ln(1+e^{\alpha-\beta\epsilon(\nu)}) \end{aligned} \tag{4.47}$$

In equation (4.26), the logarithmic factor approaches zero for $\epsilon > \frac{\alpha}{\beta}$, while for $\epsilon < \frac{\alpha}{\beta}$, it approaches the value $(\alpha - \beta(\epsilon))$ as can be easily seen below. If $\epsilon > \frac{\alpha}{\beta}$, then

$$\begin{aligned} ln(1+e^{\alpha-\beta\epsilon(\nu)}) &= ln(1+\frac{1}{e^{\beta(\epsilon(\nu)-\frac{\alpha}{\beta})}}) \\ &= ln1 \\ &= 0 \end{aligned}$$

If, however, $\epsilon < \frac{\alpha}{\beta}$,

$$ln(1+e^{\alpha-\beta\epsilon(\nu)}) = ln(1+e^{\beta(\frac{\alpha}{\beta}-\epsilon(\nu)}) = ln(e^{\alpha-\beta\epsilon(\nu)}) = \alpha-\beta\epsilon$$

Thus, we can write the integral (4.26) in the form

$$\begin{aligned} lnZ(\alpha,\beta) &= \int_0^{\frac{\alpha}{\beta}} g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)}) d\epsilon + \int_{\frac{\alpha}{\beta}}^\infty g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)}) \\ &= \int_0^{\frac{\alpha}{\beta}} g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)}) d\epsilon - \int_0^{\frac{\alpha}{\beta}} g(\epsilon)(\alpha-\beta(\epsilon)) d\epsilon \\ &\quad + \int_0^{\frac{\alpha}{\beta}} g(\epsilon)(\alpha-\beta(\epsilon)) d\epsilon + \int_{\frac{\alpha}{\beta}}^\infty g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)}) d\epsilon \end{aligned} \tag{4.48}$$

$$
\begin{aligned}
&= \int_0^{\frac{\alpha}{\beta}} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + \int_0^{\frac{\alpha}{\beta}} g(\epsilon)[ln(1+e^{\alpha-\beta\epsilon}) - (\alpha-\beta\epsilon)]d\epsilon \\
&\quad + \int_{\frac{\alpha}{\beta}}^{\infty} g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)})d\epsilon \\
&= \int_0^{\frac{\alpha}{\beta}} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon \\
&\quad + \int_0^{\frac{\alpha}{\beta}} g(\epsilon)[ln(1+e^{\alpha-\beta\epsilon}) - lne^{(\alpha-\beta\epsilon)}]d\epsilon \\
&\quad + \int_{\frac{\alpha}{\beta}}^{\infty} g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)})d\epsilon \\
&= \int_0^{\frac{\alpha}{\beta}} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon \\
&\quad + \int_0^{\frac{\alpha}{\beta}} g(\epsilon) ln[\frac{1+e^{\alpha-\beta\epsilon}}{e^{\alpha-\beta\epsilon}}]d\epsilon \\
&\quad + \int_{\frac{\alpha}{\beta}}^{\infty} g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)})d\epsilon \\
&= \int_0^{\frac{\alpha}{\beta}} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + \int_0^{\frac{\alpha}{\beta}} g(\epsilon) ln(1+e^{-\alpha+\beta\epsilon})d\epsilon \\
&\quad + \int_{\frac{\alpha}{\beta}}^{\infty} g(\epsilon) ln(1+e^{\alpha-\beta\epsilon(\nu)})d\epsilon
\end{aligned}
\tag{4.49}
$$

By a change of variable we can combine the last two integrals,

$$
\int_0^{\frac{\alpha}{\beta}} g(\epsilon) ln(1+e^{-\alpha+\beta\epsilon})d\epsilon + \int_{\frac{\alpha}{\beta}}^{\infty} g(\epsilon) ln(1+e^{\alpha-\beta\epsilon})d\epsilon = \int_0^{\infty} [g(\alpha/\beta+x)+g(\alpha/\beta-x)] ln(1+e^{-\beta x}) \tag{4.50}
$$

since $g(\epsilon) = 0$ for $\epsilon < 0$. The logarithm in this integral vanishes except in an interval of width $\sim \frac{1}{\beta}$ around x = 0. If this interval is wide compared with the spacing of the single-particle levels $\epsilon(\nu)$, we can treat the density function g in eq.(4.50) as smooth functions equal to the average of the expression (4.45). Hence the eq.(4.50) becomes

$$
lnz(\alpha,\beta) = \int_0^{\alpha/\beta} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + \int_0^{\infty} [g(\alpha/\beta+x)+g(\alpha/\beta-x)] ln(1+e^{-\beta x}) \tag{4.51}
$$

If, at the same time, the interval is small compared with the region over which g varies, we may expand the g functions in a power series in x and carry out the the integration,

term by term, to obtain

$$\begin{aligned} lnz(\alpha,\beta) &= \int_0^{\alpha/\beta} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + \int_0^{\infty}[g(\alpha/\beta)+xg'(\alpha/\beta)+\frac{x^2}{2!}g^{''}(\alpha/\beta).....\\ &\quad +g(\alpha/\beta)-xg'(\alpha/\beta)+\frac{x^2}{2!}g^{''}(\alpha/\beta)+.....]ln(1+e^{-\beta x})dx \\ &= \int_0^{\alpha/\beta} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + 2\int_0^{\infty} ln(1+e^{-\beta x})dx \\ &\quad +\int_0^{\infty} x^2 g^{,,}(\alpha/\beta)ln(1+e^{-\beta x})dx+..... \\ &= \int_0^{\alpha/\beta} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + 2g(\alpha/\beta)I_0 + g^{''}(\alpha/\beta)I_2 \end{aligned} \tag{4.52}$$

Where $g^{''}$is the second derivative of g. The integrals to be evaluated are of the form

$$I_n = \int_0^{\infty} x^n ln(1+e^{-\beta x})dx \tag{4.53}$$

with n an even intger. Further, using the expansion of $ln(1+x)$.

$$\begin{aligned} I_n &= \int_0^{\infty} x^n \left(\sum_{m=1}^{\infty}\frac{(-1)^{m-1}e^{-m\beta x}}{m}\right)dx \\ &= \frac{(-1)^{m-1}}{m}\int_0^{\infty} x^n e^{-m\beta x}dx \\ &= \frac{(-1)^{m-1}}{m}\frac{n!}{(m\beta)^{n+1}} \\ &= \frac{n!}{\beta^{n+1}}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^{n+2}} \\ &= n!\big(1-\frac{1}{2^{n+1}}\big)\zeta(n+2)\big) \end{aligned} \tag{4.54}$$

where $\zeta(x)$ is the Riemann zeta function. The Riemann-zeta function is defined as

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p} \tag{4.55}$$

hence,

$$\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\frac{1}{4}+\frac{1}{9}+..... = \frac{\pi^2}{6} \tag{4.56}$$

Also,

$$\zeta(4) = \sum_{n=1}^{\infty} n^{-4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots = \frac{\pi^4}{90} \tag{4.57}$$

the integrals I_0 and I_2 can be calculated from then,

$$I_n = n!(1 - \frac{1}{2^{n+1}})\zeta(n+2) \tag{4.58}$$

where $\zeta(x)$ is the Riemann Zeta function. For even integer n, this function can be expressed in nterms of the Bernoulli numbers as follows. We know that Taylor series expansion of some function f(x) about $x = 0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^n(0) \tag{4.59}$$

For example, the Taylor series expansion of $\frac{x}{e^x - 1}$ is given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) |_{x=0} \tag{4.60}$$

The Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{4.61}$$

Now, changing the x $\longrightarrow$ z, we have

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \tag{4.62}$$

So, the Taylor series expansion for f(z) is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0)$$

If the expansion is done around $z_0 = 0$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} f^{(n)}(0) \tag{4.63}$$

comparing eqns.(4.62) and (4.63), we get

$$\begin{aligned} B_n &= f^{(n)}(0) \\ &= \frac{n!}{2\pi i}\oint_c \frac{f(z)dz}{z^{n+1}} \\ &= \frac{n!}{2\pi i}\oint_c \frac{z/e^z-1}{z^{n+1}}dz \\ &= \frac{n!}{2\pi i}\oint_C \frac{z}{e^z-1}\frac{dz}{z^{n+1}} \end{aligned} \tag{4.64}$$

where the contour C_0 is around the origin counterclockwise with $|z| < 2\pi$ to avoid the poles at $2\pi in$. For n=0,

$$\begin{aligned} B_0 &= \frac{1}{2\pi i}\oint_{C_0} \frac{z}{e^z-1}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\oint_c \frac{dz}{e^z-1} \end{aligned} \tag{4.65}$$

has a pole of order at $z = 0$.

$$\begin{aligned} Res.(f(z)) &= \lim_{z\to 0} zf(z) \\ &= \lim_{z\to 0} z\frac{1}{e^z-1} \\ &= \lim_{z\to 0} \frac{z}{1+z+\frac{(z^2}{2!}+\ldots\ldots-1)} \\ &= \lim_{z\to 0} \frac{z}{z(1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots\ldots)} \\ &= 1 \end{aligned} \tag{4.66}$$

Hence from eqn.(4.65), we have B_0 given by

$$\begin{aligned} B_0 &= \frac{1}{2\pi i}\times 2\pi i(sumofresidues) \\ &= \frac{1}{2\pi i}\times 2\pi i\times 1 \\ &= 1 \end{aligned} \tag{4.67}$$

Similarly, for n=1, the singularity at z = 0 becomes a second-order pole. The residue can be shown to be - $\frac{1}{2}$ by series expansion, followed by the binomial expansion as follows.

$$\begin{aligned}
\frac{1}{z(e^z-1)} &= -\frac{1}{z}(1-e^z)^{-1} \\
&= -\frac{1}{z}\left[1-(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\ldots)\right]^{-1} \\
&= \frac{1}{z}\left[(z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\ldots)\right]^{-1} \\
&= \frac{1}{z^2}\left[1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots\ldots\right]^{-1} \\
&= \frac{1}{z^2}\left[1-(\frac{z}{2!}+\frac{z^2}{3!}+\ldots)+(\frac{z}{2!}+\frac{z^2}{3!}+\ldots)^2+\ldots\ldots\right] \\
&= \frac{1}{z^2}-\frac{1}{z}\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots
\end{aligned} \tag{4.68}$$

The coefficient of z^{-1}, which is the residue is $-\frac{1}{2!}$. Hence,

$$B_1 = \frac{1}{2\pi i}\times 2\pi i(-1/2) = -\frac{1}{2} \tag{4.69}$$

For $n \geq 2$, this procedure become rather tedious and one has to resort to different means for evaluating eqn.(4.64). The contour is deformed as shown in the fig.(4.2). The new contour 'C' still encircles the origin, as required but now it also encloses (in a negative direction) an infinite series of singular points along the imaginary axis at$z = \pm 2\pi ip; p = 1, 2, 3$..... The integration back and forth along the x-axis cancels out, and for R $\to \infty$, the integration over the infinite circle yields zero. It is to be noted here that $n \geq 2$. Therefore,

$$\oint_{C_0}\frac{z}{e^z-1}\frac{dz}{z^{n+1}} = -2\pi i\sum_{p=1}^{\infty} Residues(z=\pm 2\pi i) \tag{4.70}$$

$$\begin{aligned}
B_1 &= \frac{1}{2\pi i}\oint_C \frac{z}{(e^z-1)}\times\frac{dz}{z^2} \\
&= \frac{1}{2\pi i}\oint_C \frac{dz}{z(e^z-1)}
\end{aligned} \tag{4.71}$$

At $z = p2\pi i$, we have a simple pole with a residue $(p2\pi i)^{-n}$. When n is odd, the residue from $p = 2\pi ip$ exactly cancels that from $z = -p2\pi i$ and $B_n = 0$, n= 3, 5, 7, and so on.

For even n, the residues add, giving

$$\begin{aligned} B_n &= \frac{n!}{2\pi i}(-2\pi i) \times 2\sum_{p=1}^{\infty}\frac{1}{p^n(2\pi i)^n} \\ &= -\frac{(-1)^{n/2}2 \times n!}{(2\pi)^n}\sum_{p=1}^{\infty}P^{-n} \\ &= -\frac{(-1)^{n/2}2 \times n!}{(2\pi)^n}\zeta(n)(neven) \end{aligned} \tag{4.72}$$

where $\zeta(n)$ is the Riemann-zeta function. Hence eqn.(4.58) can be expressed in terms of Bernoulli numbers as

$$I_n = \frac{2^{n+1}-1}{(n+1)(n+2)}\pi^{n+2}|B_{n+2}| \tag{4.73}$$

Where $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, ... From the complex algebra, the residue at a pole of order m at $z = z_0$ for a function $f(z)$ is given by

$$Res.(f(z)) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]_{z=z_0} \tag{4.74}$$

$$I_0 = 0!(1-\frac{1}{2^1})\zeta(2) = \frac{1}{2}\zeta(2) = \frac{1}{2}\frac{\pi^2}{6} = \frac{\pi^2}{12} \tag{4.75}$$

$$I_2 = 2!(1-\frac{1}{2^3})\zeta(4) = \frac{7}{4}\zeta(4) = \frac{7}{4}\frac{\pi^4}{90} = \frac{7\pi^4}{360} \tag{4.76}$$

Hence, eqn.(4.52) can be written as

$$\begin{aligned} lnz(\alpha,\beta) &= \int_0^{\alpha/\beta} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + 2g(\alpha/\beta)\frac{\pi^2}{12\beta} + g^{''}(\alpha/\beta)\frac{7\pi^4}{360\beta^3} \\ &= \int_0^{\alpha/\beta} g(\epsilon)(\alpha-\beta(\epsilon))d\epsilon + \frac{\pi^2}{6\beta}g(\alpha/\beta) + \frac{7\pi^4}{360\beta^3}g^{''}(\alpha/\beta) + ... \end{aligned} \tag{4.77}$$

Having obtained an expression for Z, we now invert the Laplace transform in order to obtain the level density

$$\rho(A,\mathcal{E}) = (\frac{1}{2\pi i})^2\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}Z(\alpha,\beta)exp\{-\alpha A + \beta\mathcal{E}\}d\alpha d\beta \tag{4.78}$$

In evaluating the above expression, we shall employ the saddle point approximation exploiting the fact that the integrand is a rapidly varying function of α and β. Thus, the main contribution to the integral comes from a small region around the point $(\alpha_0\beta_0)$, where the

integrand is stationary. The conditions that determine this stationary point are

$$\frac{\partial lnZ}{\partial \alpha} - A = 0 \tag{4.79a}$$

$$\frac{\partial lnZ}{\partial \beta} + \mathcal{E} = 0 \tag{4.79b}$$

Expanding the exponent in the integrand to second order around the point determined by the conditions (4.79a) and (4.79b) we obtain a Gaussian integral which can be evaluated to yield

$$\rho(A, \mathcal{E}) = \frac{Z(\alpha_0, \beta_0 exp\{-\alpha_0 A + \beta_0 \mathcal{E}\})}{2\pi |D|^{1/2}} \tag{4.80}$$

where the determinant D is given by

$$D = \left| \begin{array}{cc} \frac{\partial^2 lnZ}{\partial \alpha^2} & \frac{\partial^2 lnZ}{\partial \alpha \partial \beta} \\ \frac{\partial^2 lnZ}{\partial \beta \partial \alpha} & \frac{\partial^2 lnZ}{\partial \beta^2} \end{array} \right|_{\alpha=\alpha_0, \beta=\beta_0} \tag{4.81}$$

In differentiating the function 4.77 to obtain the stationary point determined by be equations 4.79a and 4.79b, we shall cosistently neglect all the terms depending on the derivatives of g. Thus, we obtain

$$A = \int_0^{(\frac{\alpha}{\beta})_0} g(\epsilon) d\epsilon \tag{4.82a}$$

$$E = \int_0^{(\frac{\alpha}{\beta})_0} \epsilon g(\epsilon) d\epsilon + \frac{\pi^2}{6\beta_0^2} g(\alpha_0/\beta_0) \tag{4.82b}$$

The relations (4.42) imply that in the ground-state

$$\int_0^{\epsilon_F} g(\epsilon) d\epsilon = A \tag{4.83a}$$

$$\int_0^{\epsilon_F} \epsilon g(\epsilon) d\epsilon = E_0 \tag{4.83b}$$

where ϵ_F is the Fermi energy. Thus, the conditions can be written as

$$\alpha_0 = \beta_0 \epsilon_F \tag{4.84a}$$

$$E = \mathcal{E} - \mathcal{E}_0 = \frac{\pi^2}{6\beta_0^2} g(\epsilon_F) \tag{4.84b}$$

Introducing these relations into the expression 4.80, and carrying out the evaluation of the determinant 4.81, we finally obtain the level density as a function of A and the excitation energy E,

$$\rho(A,E) = \frac{1}{\sqrt{48}}\frac{1}{E}exp\Big\{2\left(\frac{\pi^2}{6}g(\epsilon_F)E\right)^{1/2}\Big\} \tag{4.85}$$

The derivation of the above result involves the following approximations.

1. The replacement of $g(\epsilon)$ by a smooth function in the evaluation of the integral 4.50. This approximation is valid, provided

$$\beta_0^{-1}g(\epsilon_F) >> 1 \tag{4.86}$$

which on, account of the relations 4.84a and 4.84b, is equivalent to

$$g(\epsilon_F) >> 1 \tag{4.87}$$

This condition simply reflects the fact that the average level density ρ is not defined until we come to excitation energies E large compared with the energy, g^{-1}, of the first excited state.

2. The neglect of terms depending on derivatives of g. The last term in eqn. 4.77 is typical of these contributions. From the relations 4.84a and 4.84b, we find that this term may be neglected, provided

$$\frac{\left(g^{''}(\epsilon_F)\right)^2 E^3}{\left(g(\epsilon_F)\right)^3} << 1 \tag{4.88}$$

For a Fermi gas, $g \sim A\epsilon^{1/2}\epsilon_F^{-3/2}$, and thus the condition 4.88 becomes

$$E << \epsilon_F A^{1/3} \tag{4.89}$$

The neglect of the higher-order terms in β^{-1} amounts to treating the Fermi gas as degenerate. Thus, one might have expected that much weaker condition $E << \epsilon_F A$ which, indeed, is sufficient to ensure that the exponent in the level density is accurate to within a factor 2. However, to obtain ρ itself to such an accuracy, we must estimate the exponent with an accuracy of one unit, and then the region of validity of the expression 4.85 is restricted by the more severe condition 4.89. For a system exhibiting the shell structure, the one-particle level density may vary much more rapidly and irregularly than for a Fermi gas, and it may be important to improve on the present approximation.

3. The use of the saddle-point approximation in evaluating the inverse Laplace transfom 4.78. The accuracy of this approximation may be estimated from the magnitude of the neglected terms in the expansion of the integrand. These terms are small provided the condition 4.86 is fulfilled.

4.5 Distribution of Eigenvalues

Let us start with the distribution of eigenvalues, also referred to as the density function or density of states. By the central limit theorem for a system of m particles in a space consisting of N single-particle states, in the limit $N >> m$, the eigenvalue distribution is Gaussian for a Hamiltonian with low $(<< m)$particle ranks. As such limiting conditions are not always satisfied in realistic situations, but we should be close to fulfill them, and the eigenvalue distributions are expected to be approximately Gaussian in general.

Given a set of moments defining a distribution that is nearly Gaussian, a question that arises is that how to find a way to realize the distribution itself, i.e., to reconstruct the distribution from the given moments. In otherwords, what we shall be seeking for is to find a distribution having the same moments as the given set. If only an incomplete set of moments is available, there is some ambiguity in reconstructing thr distribution and a model is required. For a nearly Gaussian distribution the most direct method is to use the Gram-Charlier series [229]. Let us derive the Gram-Charlier series from first principles. Any arbitrary density function $\eta(x)$ can be expanded in terms of polynomial excitations of a given density $\eta_0(x)$, i.e., in terms of the polynomials that are orthonormal $\eta_0(x)$ as the weight function as follows.

$$\eta(x) = \sum_{j=0}^{\infty} C_j He_j(x)\eta_0(x) \tag{4.90}$$

where $He_j(x)$ stands for Hermite Polynomials and $\eta_0(x)$ is the given weight function. Multiplying the above equation on both sides with $He_i(x)$ and integrating from $-\infty$ to $+\infty$, we get

$$\int_{-\infty}^{\infty} \eta(x)He_i(x)dx = \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} C_j He_i(x)He_j(x)\eta_0(x)dx \tag{4.91}$$

Now, from the orthogonal property of Hermite polynomials, we have

$$\begin{aligned}\int_{-\infty}^{\infty} He_i(x)He_j(x)\eta_0(x)dx &= 0; i \neq j \\ &= i!; i = j \end{aligned} \tag{4.92}$$

Using the eqn.(4.92) in eqn.(4.91), we obtain

$$i! \times C_i = \int_{-\infty}^{\infty} \eta(x)He_i(x)dx \tag{4.93}$$

Also, the explicit form of Hermite polynomials is given by

$$He_i(x) = x^i - \frac{i^{[2]}}{2.1!}x^{i-2} + \frac{i^{[4]}}{2^2.2!}x^{i-4} - \frac{i^{[6]}}{2^3.3!}x^{i-6} + \ldots\ldots - \ldots\ldots \tag{4.94}$$

Therefore, eqn.(4.93) becomes,

$$\begin{aligned} C_i &= \frac{1}{i!}\int_{-\infty}^{+\infty} \eta(x)\left[x^i - \frac{i^{[2]}}{2.1!}x^{i-2} + \frac{i^{[4]}}{2^2.2!}x^{i-4} - \frac{i^{[6]}}{2^3.3!}x^{i-6} + \ldots\ldots - \ldots\ldots\right]dx \\ &= \frac{1}{i!}\left[\mu_i' - \frac{i^{[2]}}{2.1!}\mu_{i-2}' + \frac{i^{[4]}}{2^2.2!}\mu_{i-4}' - \ldots + \ldots\right] \end{aligned} \tag{4.95}$$

For moments about the mean,

$$C_0 = 1 \tag{4.96}$$

and

$$C_1 = 0 \tag{4.97}$$

as can be shown below.

$$\begin{aligned} C_0 &= \frac{1}{0!}\int_{-\infty}^{+\infty}(x-\mu)^0\eta(x)dx = 1 \\ C_1 &= \frac{1}{1!}\int_{-\infty}^{+\infty}(x-\mu)^1\eta(x)dx \\ &= \frac{1}{1!}\int_{-\infty}^{+\infty}x\eta(x)dx - \mu\int_{-\infty}^{+\infty}\eta(x)dx \\ &= \mu - \mu \\ &= 0 \end{aligned}$$

If we talk in terms of central moments (moments about about the mean), the eqn.(4.95) becomes then

$$C_i = \frac{1}{i!}\left[\mu_i - \frac{i^{[2]}}{2.1!}\mu_{i-2} + \frac{i^{[4]}}{2^2.2!}\mu_{i-4} - \ldots + \ldots\right] \tag{4.98}$$

$$\begin{aligned} C_2 &= \frac{1}{2!}\left[\int_{-\infty}^{+\infty}(x-\mu)^2\eta(x)dx - \frac{2(2-1)}{2.1!}\int_{-\infty}^{+\infty}(x-\mu)^0\eta(x)dx\right] \\ &= \frac{1}{2!}\left[\int_{-\infty}^{+\infty}(x-\mu)^2\eta(x)dx - \int_{-\infty}^{+\infty}\eta(x)dx\right] \\ &= \frac{1}{2}\left[\int_{-\infty}^{+\infty}(x-\mu)^2\eta(x)dx - 1\right] \\ &= \frac{1}{2}(\mu_2 - 1) \end{aligned} \tag{4.99}$$

similarly,

$$C_3 = \frac{1}{3!}\left[(\mu_3 - \frac{3(3-1)}{2.1!}\mu_1\right] = \frac{1}{6}\mu_3 \tag{4.100}$$

$$\begin{aligned} C_4 &= \frac{1}{4!}\left[\mu_4 - \frac{4(4-1)}{2.1!}\mu_2 + \frac{4(4-1)(4-2)(4-3)}{2^2.2!}\mu_0\right] \\ &= \frac{1}{4!}\left[\mu_4 - 6\mu_2 + 3\right] \end{aligned} \tag{4.101}$$

, Similarly,

$$C_5 = \frac{1}{120}\left(\mu_5 - 10\mu_3\right) \tag{4.102}$$

,

$$C_6 = \frac{1}{720}\left(\mu_6 - 15\mu_4 + 45\mu_2 - 15\right) \tag{4.103}$$

,

$$C_7 = \frac{1}{5040}\left(\mu_7 - 21\mu_5 + 105\mu_3\right) \tag{4.104}$$

,

$$C_8 = \frac{1}{40320}\left(\mu_8 - 28\mu_6 + 210\mu_4 - 420\mu_2 + 105\right) \tag{4.105}$$

Substituting all the values of C's in eqn.(4.90), we get

$$\eta(x) = \eta_0(x)\left[1 + \frac{1}{2}(\mu_2 - 1)He_2x + \frac{1}{6}\mu_3 He_3x + \frac{1}{24}(\mu_4 - 6\mu_2 + 3)He_4x + \ldots\ldots\right] \tag{4.106}$$

If $\eta(x)$ is defined in standard measure, then [228, 229]

$$\left.\begin{array}{l} \eta_{GC}(\hat{x}) = \eta_G(\hat{x})\left[1+\frac{1}{6}\mu_3 He_3(\hat{x})+\frac{1}{24}(\mu_4+3)He_4(\hat{x})\right] \\ \eta_{GC}(\hat{x}) = \eta(\hat{x})\left[1+\sum_{\nu\geq 3}\frac{C_\nu}{\nu!}He_\nu\hat{x}\right] \\ C_\nu = \left\langle He_\nu(\hat{K})\right\rangle^m. \end{array}\right\}. \tag{4.107}$$

In the eqn.(4.107) the density is for the eigenvalues of an operator $\hat{K}$ and in most of the cases $\hat{K} = H$. Thus, the shape parameters here are the Hermite polynomials. The Gram-Charlier expansion truncated to include $\gamma_1 = k_3$ and $\gamma_2 = k_4$ corrections is

$$\begin{aligned} \eta_{GC}(\hat{x}) &= \eta_G(\hat{x})\left[1+\frac{C_3}{3!}He_3(\hat{x})+\frac{C_4}{4!}He_4(\hat{x})\right] \\ &= \eta_G(\hat{x})\left[1+\frac{\mu_3}{6}He_3(\hat{x})+\frac{(\mu_4-3)}{24}He_4(\hat{x})\right] \\ &= \eta_G(\hat{x})\left[1+\frac{k_3}{6}He_3(\hat{x})+\frac{k_4}{24}He_4(\hat{x})\right] \\ &= \eta_G(\hat{x})\left[1+\frac{\gamma_1}{6}He_3(\hat{x})+\frac{\gamma_2}{24}He_4(\hat{x})\right] \end{aligned} \tag{4.108}$$

Instead of expanding the density in terms of an asymptotic (often assumed to be Gaussian), it is sometimes useful to consider an expansion of the variable following the priciple used in the Edgeworth expansion. This gives the Cornish-Fischer (CF) expansion [227, 228] for the density. Including only γ_1 and γ_2 corrections, the CF expansion is [227, 237]

$$\begin{aligned} \eta_{CF}(\hat{x}) &= \frac{1}{\sqrt{2\pi}}\left[1-\frac{\gamma_1}{3}(\hat{x})-\frac{\gamma_2}{8}(x^2\hat{}-1)+\frac{\gamma_1^2}{36}\left(12\hat{x}^2-7\right)\right] \\ &\times \quad exp\left\{-\frac{1}{2}\left(\hat{x}-\frac{\gamma_1}{6}\left(\hat{x}^2-1\right)-\frac{\gamma_2}{24}\left(\hat{x}^3-3\hat{x}\right)+\frac{\gamma_1^2}{36}\left(4\hat{x}^3-7\hat{x}\right)\right)^2\right\} \end{aligned} \tag{4.109}$$

The truncation of ED, GC, or CF expansions to a finite number of terms of correction terms commonly that include γ_1 and γ_2 may give rise to the negative density distribution particularly in the distribution tail, which is very important for example for locating the locating the ground-state. This problem is avoided by partitioning of space which also generates new information about partitioning symmetry. Experience indicates thatthe partitioning is needed, if the ground state is 3.5 σ or more below the centroid which cor-

responds to a dimensionality of few thousand. The same problem, but less severe arises may arises with partitioning. It is to be mentioned here for the sake of completeness that the domain of validity i.e., giving positive densities for all the values of the variable.

4.6 Distribution Of Expectation values

Besides eigenvalue study, it is also of considerable interest to study the distribution of the expectation value $< E|\hat{O}|E >$ of an operator $\hat{O}$. In addition to the familiar electromagnetic moments of a nucleus, sum-rule quantities are also examples of expectation values. In otherwords, fluctuation free expectation values $K(E) = \ < K >^E$ of an operator K in the H eigenstates are encountered for example in calculating occupation probabilities, electromagnetic moments, in the study of symmetries (Where K might a function of Casimir operators), in calculating spin cut-off factors or J decomposition of state densities, in evaluating strength sums for excitations, for example Gamow Teller (GT) strength sums that are important in beta decay rates calculations and so on. The most commonly encountered situation is when it is required to calculate the expectation value of Hamiltonian in the eigenstates of another operator, and the case where neither of the two operators is the Hamiltonian.

The non-energy weighted sum rule quantity $G_O(E)$ is the sum of excitation strengths $R(E', E)$ defined by $R(E', E) = | < E'|\hat{O}|E > |^2 = \ < E|\hat{O}^\dagger|E' >< E'|\hat{O}|E >$ from a given state at energy at E to all final states E' and can be written in the form

$$G_O(E) = \sum_{E'} < E|\hat{O}^\dagger|E' >< E'|\hat{O}|E > \ = \ < E|\hat{O}^\dagger\hat{O}|E > \tag{4.110}$$

where in obtaining the final result closure relation is used. Since summation is over the final states, $G_O(E)$ depends only on the energy E of the initial state. In general, the energy weighted sum rule of order p is defined as

$$G_p(E) = \sum_{E'} E'^p R(E', E) = \sum_{E'} < E|\hat{O}^\dagger H^p|E' >< E'|\hat{O}|E > \ = \ < E|\hat{O}^\dagger H^p\hat{O}|E > \tag{4.111}$$

The linear and quadratic energy-weighted sum rules are the most common ones encountered in nuclear physics applications. For the sake of simplicity, we shall use the notation

$$K(E) = \quad < E|K|E > \tag{4.112}$$

for the expectation value of an operator $\hat{K}$ as a function of energy. For static moments, for example, $\hat{K}$ is the electromagnetic multipole operator while for the sum rule quantities, $\hat{K} = \hat{O}^{\dagger}\hat{O}$ for $G_0(E)$, and $\hat{K} = \hat{O}^{\dagger}H^p\hat{O}$ for $G_p(E)$. In order to take the advantage of the statistical spectroscopy approach, it is necessary to express eqns.(4.110) and (4.111) in terms of traces. For the sake of convenience, we shall make use of the average traces, traces divided by the number of states in the space. To distinguish between the two quantities, we shall use $<< \hat{O} >>$ for the trace of an operator $\hat{O}$, and $< \hat{O} > \quad = \frac{1}{d} << \hat{O} >>$ for the average trace. The trace of $\delta(H-E)$ is the number of states per unit energy interval at energy E, $<< \delta(H-E) >>= I(E)$ where$I(E) = d\rho(x)$ and is in general different from unity. The delta function can be expanded in terms of orthogonal polynomials $P_\mu(x)$ in the form

$$\delta(x-y) = \rho(x)\sum_{\mu=0}^{\infty} P_\mu(x)P_\nu(y) \tag{4.113}$$

Where the polynomials $P_\mu(x)$ satisfy the relation

$$\int_{-\infty}^{+\infty} P_\mu(x)P_\nu(y)\rho(x)dx = \delta_{\mu\nu} \tag{4.114}$$

Where the density function $\rho(x)$ is used as the weight function. When the density $\rho(x)$ is Gaussian, we have

$$P_\mu(x) \xrightarrow{\rho\to\rho_G} \frac{1}{\sqrt{\mu!}} He_\mu(x) \tag{4.115}$$

as can be seen by comparing eqns.(4.92) and (4.114).

A polynomial of order μ is a power series of the argument upto a maximum μ. If the moments $\mathcal{M}_\nu$ of $\rho(x)$ are known upto order 2μ, we can find all the polynomials $P_\nu(x)$ upto order μ. Let us first illustrate how the polynomials are obtained by working out the explicitly the lowest few orders. since the $P_\mu(x)$ are normalized according to eqn.(4.114).

$$P_0(x) = 1 \tag{4.116}$$

Next we can find $P_1(x)$ using the orthogonality condition (4.114)

$$\int_{-\infty}^{+\infty} P_1(x)P_0(y)\rho(x)dx = 0, \int_{-\infty}^{+\infty} P_1(x)P_1(x)\rho(x)dx = 1 \tag{4.117}$$

Since $\rho(x)$ is centered, $\int x\rho(x)$ = 0, and we obtain

$$P_1(x) = x \tag{4.118}$$

The second-order polynomial has the form

$$P_2(x) = a + bx + cx^2 \tag{4.119}$$

where a, b, and c are the coefficients to be determined by using eqn.(4.114). The orthogonality to $P_0(x)$ yields

$$\int_{-\infty}^{+\infty} P_2(x)P_0(x)\rho(x)dx = a + bM_1 + cM_2 = a + c = 0, \tag{4.120}$$

and with $P_1(x)$

$$\int_{-\infty}^{+\infty} P_2(x)P_1(x)\rho(x)dx = aM_1 + bM_2 + cM_3 = b + c\mathcal{M}_3 \tag{4.121}$$

the eqns.(4.120) and (4.121) provide two of the three equations required to determine the three unknown coefficients. The third equation comes from the normalization of $P_2(x)$,

$$\begin{aligned}\int_{-\infty}^{+\infty} (a + bx + cx^2)^2\rho(x)dx &= a^2 + 2abM_1 + (b^2 + 2ac)M_2 + 2bcM_3 + c^2M_4 \\ &= a^2 + (b^2 + 2ac) + 2bc\mathcal{M}_3 + c^2\mathcal{M}_4 = 1 \end{aligned} \tag{4.122}$$

from above eqn.(4.122), we see that the moments up to $\mathcal{M}_{2\mu}$ are needed to determine $P_\mu(x)$. In general, we can express a polynomial of arbitrary order in the form of a deter-

minant [229]

$$[D_\mu D_{\mu-1}]^{-\frac{1}{2}} P_\mu(x) = \begin{vmatrix} 1 & \mathcal{M}_1 & \mathcal{M}_2 & \dots & \mathcal{M}_\mu \\ \mathcal{M}_1 & \mathcal{M}_2 & \mathcal{M}_3 & \dots & \mathcal{M}_{\mu+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_{\mu-1} & \mathcal{M}_\mu & \mathcal{M}_{\mu+1} & \dots & \mathcal{M}_{2\mu-1} \\ 1 & x & x^2 & \dots & x^\mu \end{vmatrix}.$$

where D_μ is the same determinant as on the right side of the equation except that the last row is replaced by $(\mathcal{M}_\mu, \mathcal{M}_{\mu+1}, \dots, \mathcal{M}_{2\mu})$.
Using eqn.(4.113) we can now express $K(E)$ in terms of traces. Starting from eqn.(4.112), we have

$$K(E) = \frac{1}{I(E)} \sum_W < W|\hat{K}\delta(H-E)|W >= \frac{1}{I(E)} << \hat{K}\delta(H-E) >> \qquad (4.123)$$

With the help of eqn.(4.113), the delta function is replaced by an orthogonal expansion,

$$K(E) = \frac{1}{d} \sum_\mu << \hat{K}P_\mu(H) >> P_\mu(E) = \sum_\mu < \hat{K}P_\mu(H) > P_\mu(E) \qquad (4.124)$$

In the last step, we have absorbed the dimension d by replacing the trace of an average, and it is understood that both H and E are measured in the units of σ and with the origin at the distribution centroid. It perhaps becomes easier to see the implications of eqn.(4.123) by writing out the first few terms explicitly,

$$K(E) =< \hat{K} > + < \hat{K}H > E + < \hat{K}P_2(H) > P_2(E) + \dots \qquad (4.125)$$

The first term is the average of the operator over the entire space and it is the best possible estimate for the expectation value of $\hat{K}$ for an arbitrary energy E unless one has some further knowledge of the distribution. An improved value can be obtained by adding a linear energy independence if the correlation of $\hat{K}$ with H is known. If $\hat{K}$ is only weakly correlated with H, we do not expect $\hat{K}(E)$ to vary appreciably with E. On the other hand, if $< \hat{K}H >$ is negative, we expect an increase of $\hat{K}(E)$ at low energies (below the centroid) over and above $\hat{K}$ with a corresponding decrease at higher energy side. Conversely, a positive correlation between $\hat{K}$ and H moves the strength from low to high energy regions. The quadratic energy dependence is contained in the third term in the

form of second-order polynomial $P_2(E)$, and the more complicated energy dependences are provided by the higher-order correlations in the subsequent terms. The use of an orthogonal polynomial expansion normalized with the density distribution as the weight function ensures that the expansion is rapidly convergent.

4.7 Distribution Of Excitation Strengths

The excitation strength $R(E, E')$, is a function of both starting energy E and the final state energy E' and hence its distribution is a two-dimensional one in the variables E and E'. However, here our interest lies only in the smooth variation of the distribution with the state-to-state fluctuations removed, for example, by a running or local average.
In addition to the dependence of $R(E, E')$ itself on the averages, the number of states I(E) in the initial space, and $I'(E')$, also changes with energy because of variations in the state densities. Hence the strength function, the total strength measured between two given energy intervals,

$$S(E', E) = I(E)I'(E')R(E', E) \tag{4.126}$$

varies with the energies in a way that is in general different from$R(E', E)$.
Given the density distributions, the conversion between $R(E', E)$ and $S(E', E)$ is straightforward. However, in statistical spectroscopy $R(E', E)$ is the quantity that is calculated and $S(E', E)$ is obtained from it via eqn.(4.126). There is occasional confusion between the two quantities since $S(E', E)$ is the quantity usually measured in experiments.
Since it depends on both E and E', the distribution of $R(E', E)$ requires a double orthogonal polynomial expansion, one in E and one in E'. We can take the same approach as for the expectation value by using eqn.(4.113). However, before carrying out the expansion, we must first express the square of a matrix element as an expectation value, again by the use of a delta function,

$$\begin{aligned} R(E', E) &= < E|\hat{O}^\dagger|E' >< E'|\hat{O}|E > \\ &= \frac{1}{I'(E')} \sum_W < E|\hat{O}^\dagger \delta(H - E')|W >< W||\hat{O}|E > \\ &= \frac{1}{I'(E')} < E|\hat{O}^\dagger \delta(H - E')\hat{O}|E > \end{aligned} \tag{4.127}$$

The expectation value can be transformed, in turn, into a trace with the help of second delta function, and then into a polynomial series in the same way as in eqn.(4.123).

$$\begin{aligned} R(E',E) &= \frac{1}{I(E)I'(E')} << \hat{O}^{\dagger}\delta(H-E')\hat{O}\delta(H-E) >> \\ &= \frac{1}{dd'}\sum_{\mu\nu} << \hat{O}^{\dagger}P'_{\mu}(H)\hat{O}P_{\nu}(H) >> P'_{\mu}(E')P_{\nu}(E) \\ &= \frac{1}{d'}\sum_{\mu\nu} < \hat{O}^{\dagger}P'_{\mu}(H)\hat{O}P_{\nu}(H) > P'_{\mu}(E')P_{\nu}(E) \end{aligned} \tag{4.128}$$

Where $P_{\nu}(E)$ is the polynomial of order ν defined in the E space, and $P_{\nu}(E')$ is defined in the E' space.

The first term in eqn.(4.128) is $< \hat{O}^{\dagger}\hat{O} >$, the average strength in the space. The linear energy dependences of $\hat{O}$ and $\hat{O}^{\dagger}$ with the Hamiltonian, $< \hat{O}^{\dagger}H\hat{O} > E'$, $< \hat{O}^{\dagger}\hat{O}H > E$, and $< \hat{O}^{\dagger}H\hat{O}H > EE'$. Let us examine one of these coefficients in more detail, for example $< \hat{O}^{\dagger}H\hat{O}H >$. The average trace is taken over the product of four operators. On the extreme right, we have the Hamiltonian acting in the subspace containing the initial state. The intermediate states generated by the action of this Hamiltonian remain, in general, in the same space, the E space in this case. The effect of this H, therefore, provides the mutual interference between a pair of states in the starting space. The excitation operator $\hat{O}$ to its left takes the system into the final or E' space and the second H supplies the interaction between a pair of states in the final space before $\hat{O}^{\dagger}$ brings the system back to the starting space. More complicated interplays between the initial and final spaces are described by the high-order polynomial terms. Again we expect that the action of the first few terms in eqn.(4.128) contains enough mutual influences between the operators and spaces to give an adequate description of $R(E',E)$.

4.8 Conclusions

By examining the typical spectrum spectrum of a heavy nucleus, the fact that comes to the surface is that this complex spectrum may be broadly classified into four distinct regions: D_1 (ground-state domain) which begins at the groun-state domain and extends upto 2 MeV excitation energy; (D_2) close-lying bound states in the excitation energy range of approximately (2-6) MeV; slow neutron resonances in the energy window (6-6.002)MeV; (D_4) overlapping levels for excitation energies greater than 6.002 MeV. The character-

istic nature of the different regions has largely defined the quantities of interest. In the ground-state domain, there is a little interest in average properties such as level densities, because of the detailed knowledge of the Hamiltonian available. The study of average and fluctuations around them is of major interest at the neutron threshold energy energy and higher excitation energies. Further, the mathematical approaches and the underlying assumptions vary greatly from region to region. For example, we go from a detailed Hamiltonian in D_1, to random matrices in D_3, to using single-particle energies in D_4 and the interesting feature is that there has been a little overlap between the assumptions made and the methods used in the various regions.

Statistical spectroscopy, based on statistical laws [197] operating in model spaces, unifies the very different approaches used in the ground-state and higher energy region and makes clear the connection between the domains arising from the effective nuclear interaction. It works in the model spaces of conventional spectroscopy and is not constrained by the dimensionality of theses spaces because it does not deal with the construction and diagonalization of Hamiltonian matrices. Further, the statistical methods are applicable in wide range of circumstances and the main aim of statistical spectroscopy is to deal with the general features of complex nuclei keeping in mind that the statistical behaviour observed at high excitations extends right down to the ground-state domain. The origins of statistical nuclear theory can perhaps be traced back to the Bethe [232] derivation of level density and his calculations were based on statistical mechanics of essentially non-interacting particles (NIP) in an unbound single-particle spectrum.

The starting point in statistical spectroscopy is the density of states $\rho(E)$ arising from a Hamiltonian H acting in a spectroscopic space of m-particles (nucleons) distributed in N single-particle states. This density may be regarded as composed of two distinct parts: an average or smooth density and fluctuations around this average. The exact density may be written as

$$\rho(E) = \rho_s(E) + \rho_f(E) \tag{4.129}$$

where $\rho_s(E)$ and $\rho_f(E)$ refer to the smooth and fluctuation parts respectively. The basic questions that arise are: (i) Is there a clear separation between the average behaviour and fluctuation? this separation would allow treating the two phenomena by different methods, spectral distributions for the average behaviour and the random matrix ensembles for the fluctuations. (ii) What is the nature and magnitude of fluctuations? The magnitude

provides an estimate of the irreducible errors (limitations) implicit in spectral distributions and the nature of the fluctuations is linked to the amount of information carried by them. Other important questions that arise are: (i) how do the calculated measures used to define fluctuations agree with the experiment, (ii) are the fluctuations universal and if so what is the origin of the universality, (iii) what mechanisms affect fluctuations and what limits may be imposed by agreement with tha data?.

The separation of information into two distinct parts i.e., averages and fluctuations provides a physical basis for statistical spectroscopy of finite quantum systems with interactions such as nuclei, atoms and molecules. The decoupling arises due to the actions of central limit theorems. It has been well established by [108] that the eigenvalue density generated by a two-body Hamiltonian in a many-body space gets smoother as more particles are added to the system and converges to a Gaussian distribution. The density is then describable in terms of low-order moments of the Hamiltonian, defined by traces of powers of H, and all information contained in higher-order moments, being of little importance is washed away by the CLT. This CLT smoothing function does not affect the spectral fluctuations when renormalized according to the local density since as particle number increases, the local spacing must rapidly decrease.

If we represent the exact density in terms of excitations built upon a specific shape, a normal decomposition, a sharp separation implies that in the power spectrum, we have (i) a few excitations with wavelengths comparable to the spectrum span that care of the secular variation, (ii) no excitations of intermediate wavelengths because of CLT and (iii) short wavelength excitations of the order of the mean spacing responsible for fluctuations. This is indeed seen in seen shell model examples [217, 2, 233] and in random matrix ensembles, generated by random two-body interactions, for many particle (fermions or bosons) systems [234, 235, 233]. The first attempt demonstrating the average-fluctuation separation is due to [236].

As far as fluctuations are concerned, these deal with the deviations from local uniformity. The modelling of fluctuations by GOE of random matrices was introduced by Wigner [50, 1] has been very successful (along with GUE and GSE). The aim is not to calculate individual fluctuations but to understand the general nature of the fluctuation patterns and to calculate physically relevant statistics measures such as spacing distribu-

tion, number variances, spectral correlation functions etc.

The theoretical framework for studying fluctuations is provided by random matrix theory through the appropriate ensembles introduced by Wigner and others and the fluctuation measures are derived from the set of k-order correlation functions by averaging over the Hamiltonian ensemble [238, 239]. These ensembles have been shown to have proper ergodic behaviour [240] so that the results of the ensemble averaging apply to individual spectra. The two-point fluctuations [241] are dominated by short-range Von-Neumann Wigner level repulsion [242] and Dyson-Mehta long-range order [243]. Similarly, the strength fluctuations follow the Porter-Thomas distribution [244]. The best experimental evidences of GOE fluctuations has been provided by nuclear data based on slow neutron resonances in heavy nuclei and proton resonances in intermediate nuclei [245, 246, 247]. The essential requirement of the theory is to have a complete set of levels with the same quantum numbers (spin, parity etc.) thereby ensurung that there are no missing or spurious (those having different quantum numbers) levels. by combining all the available high quality into a nuclear data ensemble (NDE) and introducing new spectral measures, Haq, Pandey and Bohigas [248], Bohigas, Haq and Pandey [249, 250]; Lombardi, Bohigas and Seligman [251] have found remarkably close agreement between the predictions of random matrix theory and experiment confirming Wigner's suggestion that, " the Hamiltonian which governs the behaviour of a complicated system is a random symmetric matrix with no particular properties except for its symmetric nature.

The evidences of GOE fluctuations came also from a variety of quantum systems and beyond [252, 253] and same fluctuations are found in shell model spectra and also in many EGOE spectra. However, there are open questions about the fluctuation properties and ergodicity of EGOEs [254] and in fact a major gap is that the two-point function is not yet available even for EGOE(2) for spinless fermion systems [277, 279, 255, 256]. Historically during and after the Albany conference (1971) there was a confusion regarding the possibility of distinguishing between GOE and TBRE through the study of spacing distributions based on some preliminary work. However, very soon it was established [257, 274] that both GOE and TBRE essentially give the same spacing distributions and are not dependent on the rank of the interaction. Further, it was shown by Pandey [258] that how the perturbation of non-random matrix corresponding to a given Hamiltonian by a random GOE matrix leads quickly to the GOE fluctuations thereby establishing the fact

the fluctuations are unaffected by the specific features of the Hamiltonian. Also there are evidences that the two-body ensembles in general contain local GOE structure giving rise to GOE fluctuations [259, 260].

Fluctuations patterns are stable under a wide class of changes in the system's hamiltonian and only under the gradual breaking of a good symmetry they change from one pattern to another. One of the interesting problems is that has been addressed [261, 262, 263, 264] is the breaking of time-reversal invariance under which GOE changes to a more rigid GUE, the model for time-reversal non-invaraince (TRNI). The problem then is to determine from data a value or an upper bound to the TRNI nucleon-nucleon interaction. The first step of this problem involves the calculation of GOE to GUE transition curve and thereby determining the RMS value of the symmetry breaking element in the complex system and the final step involves the determining the magnitude V (TRNI) part of the nucleon-nucleon interaction. A similar study has been carried out for parity breaking by [265].

The fact that such a wide variety of systems as discussed in above paragraphs shows the same fluctuation patterns as those of a parameter free theory and points towards the existence of a universal law. The belief in the universality of GOE fluctuations has been strengthened by the connection between chaos in classical systems and fluctuations properties of their quantum analogues. This led Bohigas, Giannoni and Schmidt [4] to conjecture that, " the fluctuation properties of a generic quantum system with (without) time-reversal symmetry, which in the classical limit are fully chaotic, coincide with those of GOE(GUE). This link between the fluctuations and chaos has provided a much deeper understanding of the fluctuations especially for the nucleus [266, 267, 268] and has also lead to a study of of relationship between chaos and statistical spectroscopy [39, 269, 7]. Paraphrasing, Papenbrock and Weidenmuller [281] chaos in quantum systems implies if the statistical properties of the eigenvalue spectrum coincide with the predictions of random matrx theory and is a typical feature of atomic nuclei and other self-bound Fermi systems. Similarly Berry and Tabor [5] conjectured that, with certain exceptions completely integrable systems should lead to Poisson fluctuations which are much larger than GOE fluctuations. All this is well summarized by Altshhuler in tha abstract of colloquium he gave in the memory of J. B. French in Rochester 2004: Classical dynamical systems can be separated into two classes- integrable and chaotic. for quantum systems this distinction

manifests itself, e.g., in spectral statistics. Roughly speaking integrability leads to Poisson distribution for the energies while chaos implies Wigner-Dyson statistics of levels, which are the charateristic for the ensemble of random matrices.... the onset of chaotic behaviour for rather a broad class of systems can be understood as a delocalization in the space of quantum numbers that characterize the original integrable system.

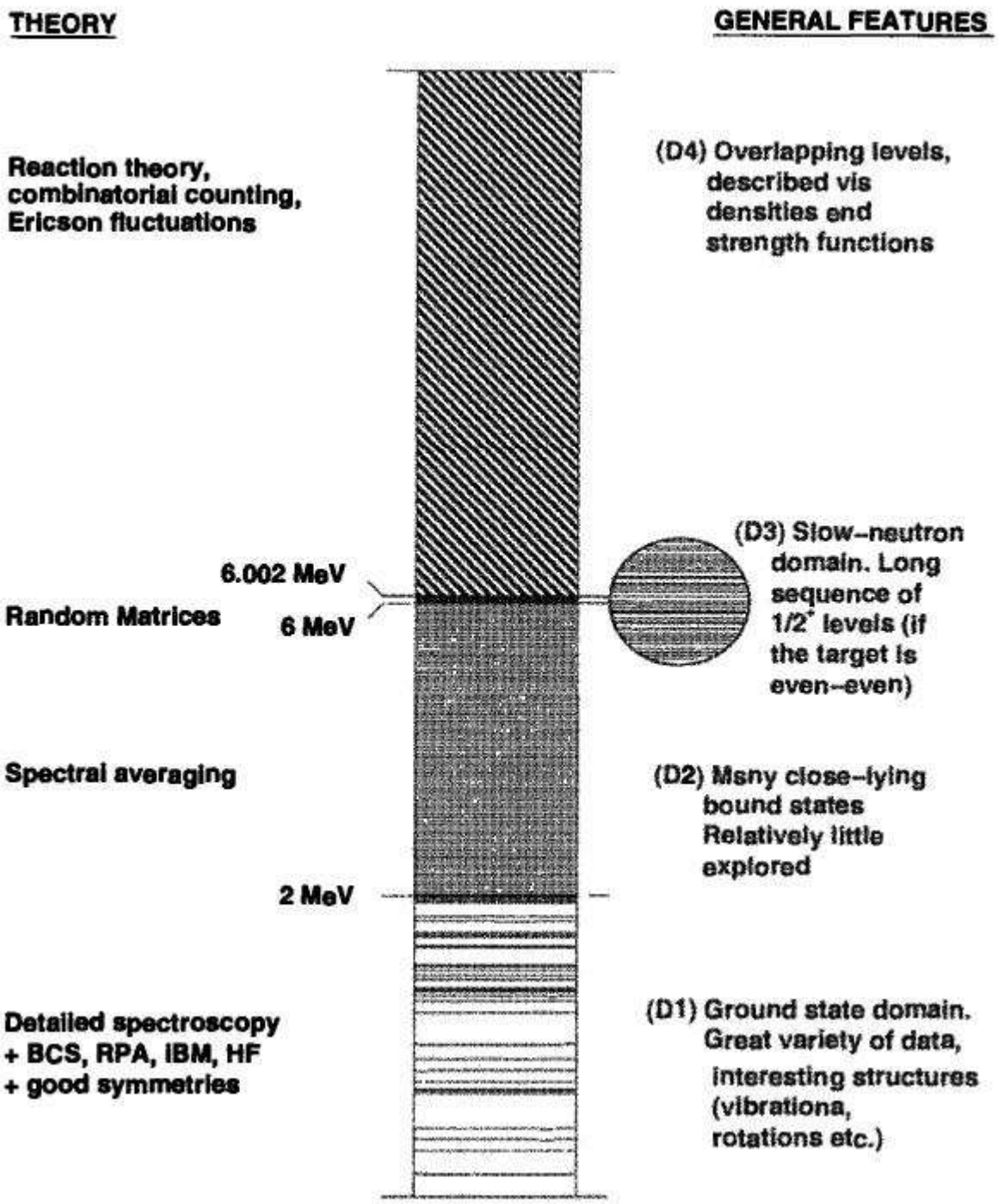

Figure 4.1: Typical spectrum of a heavy nucleus such as ^{169}Er. Figure adapted from (French and Kota (1989b)).

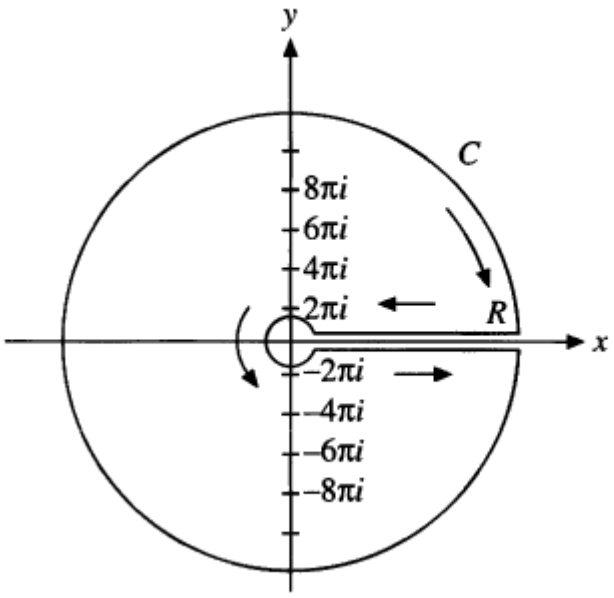

Figure 4.2: Contour of integration for Bernoulli numbers

Chapter 5

Summary

As far as quantum chaos is concerned, there is no general consensus on its definition. The question that arises then is how to identify signatures of quantum chaos. The various signatures of quantum chaos that had been identified are the spectral properties of the generating Hamiltonian [4], phase space scarring [18], hypersensitivity to perturbation [19] and fidelity decay [20] which indicate the chaos in corresponding classical systems. Recent studies have shown that entanglement in chaotic systems is a signature of quantum chaos by demonstrating that it can serve as a good indicator for the transition from regular to chaotic regimes [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. A different phenomenon characteristic of quantum chaos and without a classical counterpart is the dynamical localization of wavefunctions. In chaotic quantum systems, driven by an external dependent force, the wavepacket spreads diffusively in momentum space only upto a characteristic time, and then stops spreading. This behaviour is completely different in classical physics, since the occupation probability in phase space, characterizing the state of the classical analogue, spreads for ever diffusively [270]. Dynamical localizations also affects the statistical properties of energy levels. The RMT predictions depend only on the system symmetries and not on the specific nature of the system but dynamic localization leads to spectral statistics that depend on the degree of localization and not on the system symmetries [130]. Therefore, dynamical localization introduces non-universal features and a more complex scenario than predicted by RMT.

The atomic nucleus is a paradigmatic system to study many-body chaos and RMT has proven to be a very valuable tool in understanding the various aspects of nuclear physics.

In 1984 BGS proposed the conjecture that links quantum chaos to RMT spectral fluctuations and this discovery boosted a lot of experimental and theoretical research on the statistical properties of energy levels and wave functions in quantum systems, and in particular in complex many-body quantum systems. Random matrix theory has been applied to a huge number of fields with considerable success as described in chapter 1 by means of a figure. As far as physics is concerned, and in particular to nuclear physics, it has produced results and inferences that are quite consistent with the predictions from shell model calculations. In the first chapter of this dissertation the random matrix theory with its intimate relations with other branches of science and in particular, nuclear physics has been discussed at length. The main motivation behind the introduction of random matrix theory in nuclear physics by Wigner in 1955 was to get an understanding about level and strength fluctuations. Another apparent reason for the use of RMT in nuclear physics one can cite, is that at higher excitation energies the level density becomes very high as is clear from equation (1) so that by the time one reaches, for example, at neutron threshold, E $\sim$ 6 MeV the nuclear models fail to provide finer details about the individual states of a quantum many-body system like atomic nuclei. Paraphrasing Wigner, the assumption made while applying random matrix theory to nuclear physics is that Hamiltonians which govern the behaviour of a complicated system is a random syymetric matrix with no particular properties except for its symmetric nature. The tripartite classification of random matrix ensembles i.e., GOE, GUE and GSE given by Dyson have been discussed in the same unit along with their domain of applicability. Although the GOE which corresponds to an ensemble of asymptotically large real symmetric matrices apart from having rotational and time-reversal invarince with no other specific features, it describes simultaneous interactions between all particles because of the statistical independence of the matrix elements which is not physically significant as realistic hamiltonians are, in general, two-body in nature. In order to keep the generality of GOE and to conform to reasonable Hamiltonians, such as those used in shell model calculations, on the other, statistical extensions of the shell model has been proposed. This drawback of GOE provided a guiding clue and hence, necessitated the formulation of embedded ensembles. A matrix ensemble of random two-body Hamiltonians, with shell model angular momentum J (and isospin T) symmetry, called as two-body random ensemble was first introduced by French and Wong [271, 272] and Bohigas and Flores [273, 274]. These embedded ensembles are defined by representing the two-particle Hamiltonian by one of the three classical ensembles and then the many-particle Hamiltonian ($m > 2$) is generated by exploiting the direct product structure of the m-particle Hilbert space. As a random matrix

ensemble in the two-particle spaces is embedded in the many-particle Hamiltonian, that is why the name embedded ensembles. With GOE embedded in them, they are called as EGOEs. The distinguishing features of embedded ensembles compared to GOE are that it doesnot share any of the properties of GOE i.e., neither it is invariant under unitary transformations, nor ergodic. Till date, it is also not clear whether it is universal or not i.e., the TBRE yields results that does not depend on the assumed Guassian distribution of matrix elements v_α and how the non-Gaussian distribution of matrix elements is going to affect the spectral fluctuation properties of TBRE. The set of matrices $C_{\mu\nu}(J,\alpha)$ is fixed and under unitary transformation generate another representation rather than generating another memeber of the ensemble thereby making TBRE not invariant under unitary or orthogonal transformation, which further adds to the mathematical complexity of TBRE. In case of EGOE the correlations between between many particle 'H' matrix elements are responsible for generating results different from GOE. In case of EGOE the correlations between between many particle 'H' matrix elements are responsible for generating results different from GOE. The 'H' operators can have a wide variety of symmetries such as spin (s), spin-isospin SU(4), parity(π) etc for fermion systems or the fermions can be spinless. These give, EGOE(2), EGOE(2-**s**), EGOE(2)-SU(4), EGOE(2)-(π) and so on. Similarly, for boson systems it is possible that 'H' operator carry F-spin (as in proton-neutron IBM) or spin 1 (as spin T = 1 in IBM-3 Model) degree of freedom or the bosons can be spinless. Then, we have BEGOE(2), BEGOE(2)-F and BEGOE(2)-S1 ensembles where 'B' stands for boson. However, in reality in addition to two-body interactions, realistic systems also have a mean-field one-body part in the Hamiltonian, so that $H(1+2) = h(1) + \lambda V(2)$ where $h(1)$ is the one-body part, which is defined by single particle energies and V(2) is the two-body part. It is assumed that V(2) in particle spaces is represented by GOE (it is also possible to consider GUE representation and then we have EGUE and similarly EGSE) with matrix element variance unity, which is 2 for diagonal matrix elements. λ is the strength of interaction in terms of Δ and it is set equal to 1, without loss of generality. With H(1+2), we have EGOE(1+ 2), EGOE(1+ 2)-s etc, and these are one plus two-body embedded random matrix ensembles. By denoting the rank of interaction by k, in am m-particle space for spinless fermion systems, we have GOE for m = k and the embedded GOE, EGOE(k) for $m >> k$ [275]. For k = 2, $m >> 2$ and H preserving the shell model J symmetry we have TBRE or EGOE(2)-J and the ensemble averaged level density for GOE is semi-circular while for TBRE or EGOE(2) it is close to Gaussian. The reasons which have to do with the rank of the Hamiltonian (for m particles GOE implies that the rank of H is m) have been established in [276, 277, 278, 279]. The transition from

semi-circle to Gaussian has been studied in great detail by Mon and french [278] and by Benet, Rupp and Weidenmuller [279, 280]. The Mon and French [279] method is based on evaluating the ensemble averaged moments of the Hamiltonian in the two limits m = k and $m >> k$. By choosing matrix elements from a zero-centered distribution, all odd moments vanish. For even-order moments, the binary (pair-wise) correlations dominate, higher order correlations being smaller by a factor $\frac{1}{N}$, where N is the number of single-particle states, and go to zero in the limit $N \rightarrow \infty$. The binary correlations are of two kinds, linked and unlinked. For $m >> k$ all terms contribute while for m = k, the linked correlations goes to zero. this then gives either moments of a Gaussian or a semicircle. Going beyond this , Benet, Rupp and Weidenmuller [280] proved that the semi-circle to Gaussian transition point is m = 2k.

EGOEs with group symmetries provide a complete statistical description, including both spectral averages and fluctuations, of interacting finite many particle systems such as nuclei. However, GOE is sufficient if the point of focus is local fluctuations in a given spectrum. The EGOEs generate forms for spectral distributions for various observables such as density of states, occupancies, transition strengths, strength sums and so on, although we need to apply corrections to the EGOE forms. More significantly, EGOEs generate correlations between many particle states with different quantum numbers including particle number (spectra of different nuclei and or with different J or JT values) and these cross-correlations will be zero in a GOE description [281, 282, 283, 284, 285]. Experimental tests of this feature are yet not available and a detailed account of EGOEs with group symmetries is given [2, 7, 286, 282, 287].

Study of chaos measures like number of principal components and localization length in wave-functions and transition strengths has lead to a burst of spectroscopic activity in nuclear statistical spectroscopy. They measure the fragmentation of transition strengths and are reliable measures of chaos and complexity in the system. An attempt is made in chapter 3 to rederive the formulae for NPC and information entropy in wavefunctions and transition strengths. The re-derivation of these measures involves the following steps: (i) The EGOE exhibits average-fluctuation separation (with little communication between the two) is used. (ii) the second step involves locally renormalized amplitudes are Gaussian distributed with zero center and unit variance i.e. , local strength fluctuations follow Porter-Thomas distribution. Studying these chaos measures in transition strengths are of

more importance as compared in wave-functions because transition strengths are observables while wave-functions are not. For example, the predictions of EGOE for the chaos and complexity measures, number of principal components and localization length in transition strengths originating from an eigenstate with energy E have been tested sucessfully for E_2 and M_1 transition strengths and shell model results for the 2p1f-shell nucleus ^{46}V are used as the example [43]. The shell model results for electric quadrupole (E_2), magnetic dipole (M_1), Gamow-Teller strength sums and occupation numbers calculated using different valence spaces, when compared to predictions from EGOE have established the fact that transition strength sums can serve as a new statistic able to distinguish between regular and chaotic motion. These studies have further confirmed the fact that EGOE provides the good description of shell-model strength sums in chaotic domain and for obtaining these results the study of behaviour of strength sums from order to chaos transitions generated by means of a family of Hamiltonians $H(\lambda) = h(1) + \lambda V(2)$, built from realistic one- and two-body interactions [42]. The transition strength as a new statistic to measure chaos have been established also from the fact that for EGOE of random matrices, the strength sums generated by a transition operator acting on an eigenstate vary with the excitation energy as the ratio of two Gaussians and this general result when compared to exact shell model calculations of Gamow-Teller strength sums in nuclei and good agreement is obtained in the chaotic domain of the spectrum, and strong deviations are observed as nuclear motion approaches a regular regime [41].

There are several open questions and directions for future research as far as quantum chaos in atomic nuclei is concerned: (i) A deep analytical understanding of TBRE is lacking. The analytical approach should be based on properties of the matrices $C_{\mu\nu}(J, \alpha)$ and theoretical description for shells with several subshells is difficult and focussing on single j shell might simplify the problem. The TBRE predicts correlations between spectra with different quantum numbers (e.g., different masses, spins, or isospins) for nuclei within a major shell. Experimental verification is difficult due to limitations in length and completeness of observed nuclear spectra, but other Fermi systems might be more acessible. (iii) The correlations between spectra with different quantum numbers might also affect the scattering matrix, more precisely, such correlations might induce among S-matrix elements carrying different total spin quantum numbers.
The major objective to be achieved in future is to investigate quantum chaos using recently introduced embedded random matrix theory measures for transition densities and this will

be studied using the simplified particle-rotor model picture and the realistic approach of projected shell model. Further, the connection between rotational damping and statistical distributions in high-spin phenomena employing projected shell model shall also be explored.

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