BOUNDS FOR THE MODULI OF THE ZEROS OF A POLYNOMIAL

(A STUDY)

M.Phil. Dissertation

Submitted by

Faroz Ahmad Bhat

Under the supervision of

Prof. Nisar Ahmad Rather

Post-Graduate Department Of Mathematics,

Faculity of Physical and Material Sciences

University Of Kashmir, Hazratbal

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DEDICATED

TO MY

BELOVED PARENTS

CERTIFICATE

This is to certify that the scholar Mr. Faroz Ahmad Bhat has completed this dissertation entitled, "BOUNDS FOR THE MODULI OF THE ZEROS OF A POLYNOMIAL" under my supervision. The dissertation has not so far been submitted for the award of M.Phil or any other degree or diploma.

(Supervisor)

Prof. Nisar Ahmad Rather

Department of Mathematics

University of Kashmir.

This dissertation which is of an expository type is required in partial fulfillment of the award of M.Phil degree of the University of Kashmir.

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Faroz Ahmad Bhat

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INTRODUCTION

Polynomials pervade mathematics and much that is beautiful in mathematics is related to polynomials, virtually every branch of mathematics, from Algebraic number theory and Algebraic Geometry to Applied Analysis, Fourier analysis and Computer sciences, has its corpus of theory arising from the study of polynomials. Historically, give rise to some important problems of the day. The subject is now much too large to attempt an encyclopaedic coverage.

The most complicated problems of trade and industry called for the solutions of equations and the introduction of literal symbols thus arose algebra, which at the time amounted to a science of equations. Even in antiquity, solutions had been for equations of first order and for quadratic equations, those stumbling blocks of today school children.

We recall here that an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 ,$$

where $a_n, a_{n-1}, \cdots, a_0$ are real or complex numbers with $a_n \neq 0$, is called a polynomial of degree n. If there is a value of $z = \alpha$ say, such that $P(\alpha) = 0$, then α is called the zero of polynomial $P(z)$. Enormous efforts were put into solving polynomial equations of degree higher than the second and only in sixteenth century were such solutions forthcoming for equations of the third and fourth degrees. Another three centauries were spent in vain efforts to get the solutions of polynomial equations of degree higher than the fourth. It required the geneous of Abel and Galois to resolve this problem in it entirely. At the beginning of the nineteenth century, a young Norweigian mathematician, Neil Henrik Abel mediated long and Painstakingly on the problem and finally came to the conviction that equations of degree higher than fourth cannot, generally speaking, be solved by radicals. At about this time, another young mathematician Evarista Galois of France took a new approach and proved a similar result.

The problems of obtaining exact new bounds, the improvements and generalisations of some older results for the location of the zeros of a polynomial are still of considerable interest. In view of this fact and that of many as yet unsettled questions this subject continues be an active field of research.

The aim of this dissertation is to present a survey of certain results concerning the bounds for the moduli of the zeros of a polynomial. We have divided this material into four chapters.

We start chapter 1 by presenting two independent proofs of a classical result due to Cauchy concerning on upper bounds for the moduli of the zeros of a polynomial $P(z)$. Next we shall study an improvement of Cauchyφs theorem due to A. Joyal, G. Labelle and Q. I. Rahman and Cauchyφs theorem for class of lacunary type polynomials due to A. Aziz and B. A. Zargar. In the same chapter, besides studying other related results, we shall state a result of Datt and Govil and present its generalisation due to A. Aziz and B.A. Zargar. We shall also state a result of J.L. Diaz-Barrero and present its generalisation due to A. Aziz and Aliya Qayoom which includes a variety of interesting results as a special case. Finally in this chapter we shall present an interesting refinement of Cauahyφs theorem due to A. Aziz and Aliya Qayoom.

In chapter 2 we shall discuss a well-known theorem of Eneström and Kakeya and present some of its extensions due to A. Joyal, G. Labelle and Q.I. Rahman, N.K. Govil and Prof. Rahman, A. Aziz and Mohammad and A. Aziz and B.A. Zargar. We shall also present a recent generalisation of Eneström-Kakeya theorem due to A. Aziz and B.A. Zargar and finally conclude this chapter by presenting a more recent generalisation of Eneström-Kakeya theorem by the same authors.

We shall start chapter 3 by proving a celebrated result invoked as Guass-Lucas theorem concerning the zeros of the derivative of a polynomial. Next we shall present a simple and purely analytical proof of Laguerrerφs theorem due to A. Aziz. We shall study Graces theorem and present its generalisation due to A. Aziz and its application to Walshφs coincidence theorem. We shall also present some results due to A. Aziz which are the generalisation of a result due to Szegö and DeBruijn. Finally we close this chapter by presenting a compact generalisation of Walshφs two-circle theorem for the polynomials and rational functions A. Aziz and N.A. Rather.

Finally we shall start chapter 4 by stating a conjecture of B. L. Sendove concerning the zeros of the derivative of a polynomial and present its proof for the boundary case and for the polynomials of degree 3 or 4 due to Z. Rubinstein. We shall also state a conjecture due to Goodman, Roth, Rahman and Schmeisser and present a result due to A. Meir and A.Sharma which is a generalisation of a result due to Goodman, Rahman and Ratti. Finally we close this chapter by presenting a result which is also due to A. Meir and A. Sharma concerning the zeros of $(n-2)th$ order derivative of a polynomial.

Before proceeding to the study of such specific results, we shall find it useful to mention certain general theorems to which we shall make references whenever required.

1. Fundamental Theorem Of Algebra [44, p.118]

Every polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 ,$$

where $a_n, a_{n-1}, \cdots, a_0$ are complex numbers with $a_n \neq 0$, of degree $n \geq 1$ has n zeros.

2. Rouches Theorem [37, p.2]

If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) \pm g(z)$ has the same number of zeros inside C.

3. Gauss-Lucas Theorem [37, p.22]

Any circle C which encloses all the zeros of a polynomial $P(z)$ also encloses all the zeros of its derivative $P'(z)$.

4. Maximum Modulus Principle [44, p.165]

If $f(z)$ is analytic and non-constant in region D, then its absolute value $|f(z)|$ is maximum on D but not inside D.

5. Schwarz Lemma [44, p.168]

If $f(z)$ is analytic function, regular for $|z| \leq R$ and $|f(z)| \leq M$ for $|z| = R$ and $f(0) = 0$, then

$$|f(re^{i\theta})| \leq \frac{Mr}{R}, \qquad 0 \leq r \leq R \text{ and } 0 \leq \theta \leq 2\pi.$$

Chapter - 1

Estimates For The Moduli Of The Zeros Of A Polynomial

We start this chapter by presenting the following classical result due to Cauchy $[37,\ p.123]$ concerning the bounds for the moduli of the zeros of a polynomial.

In order to emphasize the methods and techniques used for studying the location of the zeros of a polynomial, we shall give two independent proofs of this result.

THEOREM 1.1. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n and

$$M = \max\left\{\frac{|a_{n-1}|}{|a_n|}, \frac{|a_{n-2}|}{|a_n|}, \ldots, \frac{|a_1|}{|a_n|}, \frac{|a_0|}{|a_n|}\right\},$$

then all the zeros of $P(z)$ lie in the circle

$$|z| < 1 + M.$$

FIRST PROOF OF THEOREM 1.1. We have

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

so that

$$|P(z)| = |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$$

$$(1.1) \qquad \geq |a_n||z|^n\left\{1-\left(\frac{|a_{n-1}|}{|a_n|}\frac{1}{|z|}+\cdots\frac{|a_1|}{|a_n|}\frac{1}{|z|^{n-1}}+\frac{|a_0|}{|a_n|}\frac{1}{|z|^n}\right)\right\}.$$

Since

$$M = Max\left\{\frac{|a_{n-1}|}{|a_n|},\frac{|a_{n-2}|}{|a_n|},\cdots,\frac{|a_1|}{|a_n|},\frac{|a_0|}{|a_n|}\right\},$$

therefore,

$$\frac{|a_j|}{|a_n|}\leq M, \qquad \text{for all } j=0,1,\cdots,n-1.$$

Using it in (1.1), we get

$$(1.2) \qquad |P(z)| \geq |a_n||z|^n\left[1-M\left(\frac{1}{|z|}+\frac{1}{|z|^2}+\cdots+\frac{1}{|z|^n}\right)\right].$$

Now if $|z|>1$, then $\frac{1}{|z|}<1$ and we have

$$\frac{1}{|z|}+\frac{1}{|z|^2}+\cdots+\frac{1}{|z|^n}<\frac{1}{|z|}+\frac{1}{|z|^2}+\cdots+\frac{1}{|z|^n}+\cdots$$

$$=\frac{1}{|z|-1}.$$

Hence from (1.2), we get

$$|P(z)| > |a_n||z|^n\left[1-\frac{M}{|z|-1}\right]$$

and therefore,

$$|P(z)|>0. \quad \text{if} \quad |z|-1>M$$

that is, if

$$|z|\geq 1+M.$$

Thus $P(z)$ does not vanish for

$$|z|\geq 1+M.$$

Therefore, those zeros of $P(z)$ whose modulus is greater than 1 lie in $|z| < 1 + M$. But those zeros of $P(z)$ whose modulus is less than or equal to 1 already satisfy the inequality

$$|z| < 1 + M.$$

Hence all the zeros of $P(z)$ lie in

$$|z| < 1 + M.$$

This completes the first proof of Theorem 1.1.

Before presenting the second proof of this classical result, we shall first define the companion matrix of a polynomial.

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, be a polynomial of degree n, then the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{a_1}{a_n} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{a_2}{a_n} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix},$$

is called the companion matrix of a polynomial $P(z)$. The characteristic polynomial of C is the polynomial $P(z)$.

For the second proof of Theorem 1.1, we need the following results [37, p. 140].

LEMMA 1.1 (HARDMARDS INEQUALITY). If

$$A = [a_{ij}]$$

is an $n \times n$ matrix, then

$$|A| \neq 0,$$

if

$$|a_{ii}| > p_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1,2,\cdots,n.$$

LEMMA 1.2 (GRESHGORIAN DISK THEOREM). Let

$$A = [a_{ij}],$$

be $n \times n$ complex matrix. Then the characteristics roots of A lie in the union of disks

$$|z - a_{ii}| \leq p_i, \qquad i = 1,2,\cdots,n$$

where

$$p_i = \sum_{\substack{j=1 \\ J \neq i}}^{n} |a_{ij}| \qquad i = 1,2,\cdots,n,$$

that is, p_i is the sum of the moduli of the off diagonal elements in the ith row of A.

PROOF OF LEMMA 1.2. We have

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}.$$

The characteristic equation of A is given by

$$|A - zI| - 0,$$

that is,

$$\begin{vmatrix} a_{11}-z & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22}-z & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ii}-z & \cdots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn}-z \end{vmatrix} = 0.$$

By Lemma 1.1,

$$|A - zI| \neq 0,$$

if $$|z - a_{ii}| > |a_{i1}| + |a_{i2}| + \cdots + |a_{ii-1}| + |a_{ii-2}| + \cdots + |a_{in}|, \text{ for } i = 1,2,\cdots,n.$$

Therefore, it follows that

$$|A - zI| = 0,$$

if $$|z - a_{ii}| \leq p_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \qquad i = 1,2,\dots,n.$$

Hence we conclude that the characteristics roots of A lie in the union of the disks

$$|z - a_{ii}| \leq p_i, \qquad i = 1,2,\cdots,n$$

where $$p_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \qquad i = 1,2,\cdots,n.$$

SECOND PROOF OF THEOREM 1.1. The companion matrix of the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

of degree n is

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{a_1}{a_n} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{a_2}{a_n} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix}.$$

Applying Lemma 1.2, it follows that all the characteristics roots of C lie in the union of circles

$$|z| \leq \left|\frac{a_0}{a_n}\right|, \quad |z| \leq 1 + \left|\frac{a_j}{a_n}\right|, \quad j = 1,2,\cdots,n-2$$

and

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq 1.$$

This gives that all the characteristics roots of C lie in the union of circles

$$|z| \leq 1 + \left|\frac{a_j}{a_n}\right|, \quad j = 1,2,\cdots,n-1.$$

But all these circles are contained in the circle

$$|z| \leq 1 + \mathrm{M},$$

where

$$\mathrm{M} = \max_{0 \leq j \leq n-1} \left|\frac{a_j}{a_n}\right|,$$

it follows that all the characteristics roots of C lie in the circle

$$|z| \leq 1 + \mathrm{M}.$$

But the characteristic roots of C are the zeros of the polynomial $P(z)$. We therefore conclude that all zeros of $P(z)$ lie in the circle

$$|z| \leq 1 + \mathrm{M},$$

where

$$M = \max_{0 \le j \le n-1} \left| \frac{a_j}{a_n} \right| \cdot$$

This completes the second proof of Theorem 1.1.

Quite a few results giving bounds for the zeros depending on the moduli of the coefficients of a polynomial $P(z)$ and the improvements of Theorem 1.1 may be found in [37] and [46]. Here we shall present the following improvement of Cauchyǒs theorem which is due A. joyal, G. Labelle and Q. I. Rahman [31].

THEOREM 1.2. If

$$(1.3) \qquad B = \max_{0 \le j \le n-2} |a_j|,$$

then all the zeros of the polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

of degree n are contained in the circle

$$(1.4) \qquad |z| \le \frac{1}{2}\left\{ 1 + |a_{n-1}| + \sqrt{(1 - |a_{n-1}|)^2 + 4B} \right\} \cdot$$

PROOF OF THEOREM 1.2. Since

$$\sqrt{(1 - |a_{n-1}|)^2 + 4B} \ge 1 - |a_{n-1}|,$$

we have

$$1 + |a_{n-1}| + \sqrt{(1 - |a_{n-1}|)^2 + 4B} \ge 2.$$

Now if

$$|z| > \frac{1}{2}\left\{ 1 + |a_{n-1}| + \sqrt{(1 - |a_{n-1}|)^2 + 4B} \right\},$$

then $|z| > 1$, and

$$|z|^2 - (1 + |a_{n-1}|)|z| + |a_{n-1}| - B > 0.$$

This gives

$$(|z| - 1)(|z| - |a_{n-1}|) - B > 0,$$

or

$$|z|^{n-1}(|z|-1)(|z|-|a_{n-1}|)-B|z|^{n-1}>0.$$

Since $|z|>1$, therefore, on dividing both sides of this inequality by $|z|-1$, we get

$$|z|^n-|a_{n-1}||z|^{n-1}>\frac{B|z|^{n-1}}{|z|-1}\cdot$$

But

$$\frac{B|z|^{n-1}}{|z|-1}>\frac{B(|z|^{n-1}\quad 1)}{|z|-1}=B(1+|z|+|z|^2+\cdots+|z|^{n-2}),$$

and

$$|a_{n-2}z^{n-2}+\cdots+a_1z+a_0|\leq|a_{n-2}||z^{n-2}|+\cdots+|a_1||z|+|a_0|$$

$$\leq B(|z|^{n-2}+\cdots+|z|+1)\quad \text{by } (1.3),$$

therefore, it follows that

$$(1.5)\qquad |z|^n-|a_{n-1}||z|^{n-1}>|a_{n-2}z^{n-2}+\cdots+a_1z+a_0|.$$

Hence if

$$|z|>\frac{1}{2}\{1+|a_{n-1}|+\sqrt{(1-|a_{n-1}|)^2+4B}\},$$

then from (1.5), we get

$$|P(z)|=|z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0|$$

$$\geq(|z|^n-|a_{n-1}||z|^{n-1})-|a_{n-2}z^{n-2}+\cdots+a_1z+a_0|$$

$$>0\cdot$$

This shows that all the zeros of $P(z)$ lie in the circle defined by (1.4) and hence completes the proof of Theorem 1.2.

Recently A. Aziz and B. A. Zargar [11] have extended Cauchy theorem for a class of lacunary type polynomials and have proved the following generalization of Theorem 1.1.

THEOREM 1.3. If

$$P(z)=a_nz^n+a_pz^p+\cdots+a_1z+a_0\,,\qquad 0\leq p\leq n-1$$

is a polynomial of degree n and

$$M = \max_{0 \le j \le p} \left| \frac{c_j}{a_n} \right| ,$$

then all the zeros of $P(z)$ lie in $|z| < k,$ where k is the unique positive root of the trinomial equation

$$x^{n-p} - x^{n-p-1} - M = 0.$$

PROOF OF THEOREM 1.3. We have

$$P(z) = a_n z^n + c_p z^p + \cdots + a_1 z + a_0, \quad 0 \le p \le n-1$$

so that, for $|z| > 1$

$$|P(z)| \ge |a_n||z|^n \left\{ 1 - \left(\frac{|a_p|}{|a_n|} \frac{1}{|z|^{n-p}} + \cdots + \frac{|a_1|}{|a_n|} \frac{1}{|z|^{n-1}} + \frac{|a_0|}{|a_n|} \frac{1}{|z|} \right) \right\} .$$

Since $\frac{|a_j|}{|a_n|} \le M,$ for all $j = 1,2,\cdots n,$ it follows that

$$|P(z)| \ge |a_n||z|^n \left\{ 1 - \frac{M}{|z|^{n-p}} \left(1 + \frac{1}{|z|} + \frac{1}{|z|^2} + \cdots + \frac{1}{|z|^p} \right) \right\}$$

$$> |a_n||z|^n \left\{ 1 - \frac{M}{|z|^{n-p}} \left(1 + \frac{1}{|z|} + \frac{1}{|z|^2} + \cdots \right) \right\}$$

$$= |a_n||z|^n \left\{ 1 - \frac{M}{|z|^{n-p-1}} \frac{1}{(|z|-1)} \right\}$$

$$\ge 0,$$

if

$$|z|^{n-p} - |z|^{n-p-1} - M \ge 0.$$

This implies

$$|P(z)| > 0, \qquad \text{for} \quad |z| \ge k$$

where k is the (unique) positive root of the trinomial equation defined by

$$x^{n-p} - x^{n-p-1} - M = 0$$

in the interval $(1,\infty)$. Hence all the zeros of $P(z)$ whose modulus is greater than 1 lie in $|z| < k$. Since all those zeros of $P(z)$ whose modulus is less then or equal to 1 already lie in $|z| < k$. Hence it follows that all the zeros of $P(z)$ lie in $|z| < k$. This completes the proof of Theorem1.3.

REMARK 1.1. If we take $p = n - 1$ in Theorem 1.3, we get Theorem 1.1.

The following corollary is obtained by taking $p = n \quad 2$ in Theorem 1.3.

COROLLARY 1.1. If

$$P(z) = a_n z^n + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n and

$$M = \max_{0 \le j \le n-2} \left|\frac{a_j}{a_n}\right|,$$

then all zeros of $P(z)$ lie in the circle

$$|z| < \frac{1 + \sqrt{1 + 4M}}{2}.$$

From Corollary 1.1, we can easily deduce

COROLLARY 1.2. If

$$P(z) = a_n z^n + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n, such that

$$|a_j| \le |a_n|, \qquad j = 0,1,\cdots,n-2,$$

then all zeros of $P(z)$ lie in

$$|z| < \frac{1 + \sqrt{5}}{2}.$$

In the Literature there exists several improvements of the generalization of Cauchyês theorem. In this direction Datt and Govil [22] have obtained the following improvement of Theorem 1.1.

THEOREM 1.4. If

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 ,$$

is a polynomial of degree n and

$$A = \max_{0 \le j \le n-1} |a_j|,$$

then all the zeros of $P(z)$ lie in a ring shaped region

$$(1.6) \qquad \frac{|a_0|}{2(1+A)^{n-1}(1+An)} \le |z| \le 1 + \lambda_0 A,$$

where λ_0 is the unique root of the equation

$$x = 1 - \frac{1}{(1+Ax)^n}$$

in the interval $(0,1)$. The upper bound in (1.6) is best possible and is attained for the polynomial

$$P(z) = z^n - A(z^{n-1} + z^{n-2} + \cdots + z + 1).$$

Here we present the following generalization of Theorem 1.4 to lacunary type polynomials recently proved by A. Aziz and B. A. Zargar [11].

THEOREM 1.5. If

$$P(z) = z^n + a_p z^p + \cdots + a_1 z + a_0, \qquad 0 \le p \le n-1$$

is a polynomial of degree n and $A = \max|a_j|$, $j = 0,1,\cdots,p$, then $P(z)$ has all its zeros in the ring shaped region

$$\frac{|a_0|}{2(1+A)^{n-1}\{1+(p+1)A\}} \le |z| \le 1 + \alpha_0 A,$$

where α_0 is the unique root of the equation

$$x = 1 - \frac{1}{(1+Ax)^{p+1}} \qquad \text{in the interval } (0,1).$$

The upper bound $1 + \alpha_0 A$ in (1.7) is best possible and is attained for the polynomial

$$P(z) = z^n - A(z^p + z^{p-1} + \cdots + z + 1).$$

For the proof of Theorem 1.5, we need the following lemma.

LEMMA 1.3. Let

$$f(x) = x - \left[\frac{1}{(1+Ax)^{n-p-1}} - \frac{1}{(1+Ax)^n}\right],$$

where n is a positive integer and $A > 0$. If $(p+1)A > 1$, then $f(x)$ has a unique root in the interval $(0,1)$.

PROOF OF LEMMA 3. Consider

$$(1+Ax)^n f(x) = (1+Ax)^n x - (1+Ax)^{p+1} + 1, \qquad \text{where} \quad p \le n-1$$

$$= \left\{\binom{n}{0} + \binom{n}{1}Ax + \binom{n}{2}A^2x^2 + \cdots + \binom{n}{n}A^nx^n\right\}x$$

$$-\left\{\binom{p+1}{0} + \binom{p+1}{1}Ax + \binom{p+1}{2}A^2x^2 + \cdots + \binom{p+1}{p}A^px^p + \binom{p+1}{p+1}A^{p+1}x^{p+1}\right\} + 1$$

$$= (1-(p+1)A)x$$

$$+ \sum_{k=2}^{p+1} \frac{(p+1)!\,A^{k-1}x^k}{k!\,(p-k+2)!}\left\{k\left(\frac{n(n-1)\cdots(n-k+2)}{(p+1)p\cdots(p-k+3)} + A\right) - (p+2)A\right\}$$

$$+ \sum_{k=p+2}^{n+1} \binom{n}{k-1} A^{k-1}x^k$$

(1.8) $$= (1-(p+1)A)x + \sum_{k=2}^{p+1} \frac{(p+1)!\,A^{k-1}x^k}{k!\,(p-k+2)!}\{k(A+I_k) - (p+2)A\}$$

$$+ \sum_{k=p+2}^{n+1} \binom{n}{k-1} A^{k-1}x^k$$

where $I_k = \dfrac{n(n-1)\cdots(n-k+2)}{(p+1)p\cdots(p-k+3)} \ge 1$ for all $k = 2,3,\cdots,p+1$, as $p \le n-1$.

Since $1-(p+1)A<0$, the coefficients of $x^{p+2},\cdots,x^{n+1}$ are positive and $k(A+I_k)-(p+2)A$ are monotonically increasing for $k=2,3,\cdots,p+1$, it follows from Descartes rule of signs that $(1+Ax)^n f(x)=0$ has exactly one positive root. Since

$$f(x)=x-\left[\frac{1}{(1+Ax)^{n-p-1}}-\frac{1}{(1+Ax)^n}\right],$$

then

$$f'(x)=1-\left[\frac{(1+p-n)A}{(1+Ax)^{n-p}}+\frac{nA}{(1+Ax)^{n+1}}\right]. \tag{1.9}$$

If $(p+1)A>1$, then it is clear from (1.9) that $f'(0)<0$. Thus there exists a $\delta>0$ such that $f'(x)<0$ in $(0,\delta)$. Also $f(1)>0$, hence $f(x)=0$ has one and only one positive root in $(0,1)$ and the lemma follows.

PROOF OF THEOREM 1.5. We shall first prove that $P(z)$ has all its zeros in $|z|\le 1+\alpha_0 A$, and for this it is sufficient to consider the case when $(p+1)A>1$ (for if $(p+1)A\le 1$, then on $|z|=R>1$, $|P(z)|\ge R^n-(P+1)A|z|^p\ge R^n-R^p>0$).

If

$$A=\max_{0\le j\le p}|a_j|,$$

and

$$P(z)=z^n+a_pz^p+a_{p-1}z^{p-1}+\cdots+a_0\,,\qquad 0\le p\le n-1,$$

then,

$$|P(z)|\ge |z|^n\left\{1-\left(\frac{|a_p|}{|z|^{n-p}}+\frac{|a_{p-1}|}{|z|^{n-p+1}}+\cdots+\frac{|a_0|}{|z|^n}\right)\right\}$$

$$\ge |z|^n\left\{1-\frac{A}{|z|^{n-p}}\left(\frac{|z|^{p+1}-1}{(|z|-1)|z|^p}\right)\right\}$$

$$\ge |z|^n-A\left\{\frac{|z|^{p+1}-1}{|z|-1}\right\}.$$

Hence for every $\alpha > 0$, we have on $|z| = 1 + A\alpha$,

$$|P(z)| \geq (1 + A\alpha)^n - \frac{(1 + A\alpha)^{p+1} - 1}{\alpha} > 0,$$

if

$$\alpha(1 + A\alpha)^n > (1 + A\alpha)^{p+1} - 1,$$

which implies

$$\alpha > \frac{(1 + A\alpha)^{p+1}}{(1 + A\alpha)^n} - \frac{1}{(1 + A\alpha)^n}$$

$$(1.10) \qquad = \frac{1}{(1 + A\alpha)^{n-p-1}} - \frac{1}{(1 + A\alpha)^n}.$$

Thus by above lemma it follows that, if α_0 is the unique root of the equation

$$x = \frac{1}{(1 + Ax)^{n-p-1}} - \frac{1}{(1 + Ax)^n}$$

in $(0,1)$, then every $\alpha > \alpha_0$ satisfies (1.10) and hence $|P(z)| > 0$, on $|z| = 1 + A\alpha$ which implies that $P(z)$ has all its zeros in

$$(1.11) \qquad |z| \leq 1 + A\alpha_0.$$

Next we prove that $P(z)$ has no zero in

$$|z| < \frac{|a_0|}{2(1 + A)^{n-1}\{1 + (p + 1)A\}}.$$

Let us denote the polynomial $(1 - z)P(z)$ by $g(z)$, then

$$g(z) = a_0 + \sum_{j=1}^{p}(a_j - a_{j-1})\, z^j + z^n - a_p z^{p+1} - z^{n+1}$$

$$= a_0 + h(z), \qquad \text{(say)}.$$

Now if

$$R = 1 + A,$$

then

$$\max_{|z|=R}|h(z)| \leq R^{n+1} + R^n + |a_p|R^{p+1} + \sum_{j=1}^{p}|a_j - a_{j-1}|R^j$$

$$\leq R^n[R + 1 + A + 2AP]$$

$$= 2R^n[R + AP]$$

$$(1.12) \qquad = 2(1 + A)^n[1 + (p + 1)A].$$

Hence by Schwarz lemma we have on $|z| \leq R$,

$$|g(z)| = |a_0 + h(z)| \geq |a_0| - |h(z)|$$

$$\geq |a_0| \quad \frac{|z|}{(1+A)} \max_{|z|=R>1}|h(z)|$$

$$\geq |a_0| - \frac{|z|}{(1+A)} 2(1+A)^n\{1 + (p+1)A\} \quad \text{by} \quad (1.12)$$

$$> 0,$$

if

$$|z| < \frac{|a_0|}{2(1+A)^{n-1}[1 + (p+1)A]}.$$

Hence all the zeros of $P(z)$ lie in

$$(1.13) \qquad |z| \geq \frac{|a_0|}{2(1+A)^{n-1}[1 + (p+1)A]}.$$

Combining (1.11) and (1.13), we get all the zeros of $P(z)$ to be in the ring shaped region

$$\frac{|a_0|}{2(1+A)^{n-1}[1 + (p+1)A]} \leq |z| \leq 1 + A\alpha_0.$$

This completes the proof of Theorem 1.5.

Recently J.L. Diaz ï Barreror [24] has obtain an annulus containing all the zeros of a polynomial involving binomial coefficients and Fibonacciös numbers

$$F_k = F_{k-1} + F_{k-2} \quad \text{for} \quad k \geq 2,$$

where $F_0 = 0$ *and* $F_1 = 1$. In fact he has proved the following result.

THEOREM1.6. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a non ï constant polynomial of degree n. Then all the zeros of $P(z)$ lie in the annulus $r_1 \le |z| \le r_2$, where

$$r_1 = \frac{3}{2} \min_{1 \le k \le n} \left\{ \frac{2^n F_k \; C(n,k)}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \quad \text{and}$$

$$r_2 = \frac{2}{3} \max_{1 \le k \le n} \left\{ \frac{F_{4n}}{2^n F_k \; C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

More recently A. Aziz and Aliya Qayoom [4] have proved the following more general result which includes not only Theorem 1.6 as a special case but also a variety of other interesting results can be established from Theorem 1.7 by a fairly uniform proceedure.

THEOREM 1.7. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a non- constant complex polynomial of degree n. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ is any set of n real or complex numbers such that

$$\sum_{K=1}^{n} |\lambda_K| \le 1,$$

then all the zeros of $P(z)$ lie in the annuals $R = \{z \in C : r_1 \le \; |z| \le r_2\}$, where

$$r_1 = \min_{1 \le k \le n} \left| \lambda_k \frac{a_0}{a_k} \right|^{\frac{1}{k}} \quad \text{and}$$

$$r_2 = \max_{1 \le k \le n} \left| \frac{1}{\lambda_k} \frac{a_{n-k}}{a_n} \right|^{\frac{1}{k}}.$$

PROOF OF THEOREM 1.7. We first show that all the zeros of $P(z)$ lie in

$$(1.14) \qquad |z| \le r_2 = \max_{1\le k\le n} \left|\frac{1}{\lambda_k}\frac{a_{n-k}}{a_n}\right|^{\frac{1}{k}}.$$

From (1.14) it follows that

$$\left|\frac{a_{n-k}}{a_n}\right| \le |\lambda_k| r_2^k, \qquad k = 1,2,\cdots,n$$

and hence

$$(1.15) \qquad \sum_{k=1}^{n}\left|\frac{a_{n-k}}{a_n}\right|\frac{1}{r_2^k} \le \sum_{k=1}^{n}|\lambda_k| \ .$$

Now for $|z| > r_2$, we have

$$|P(z)| = |a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|$$

$$\ge |a_n||z|^n\left\{1 - \left(\left|\frac{a_{n-1}}{a_n}\right|\frac{1}{|z|} + \left|\frac{a_{n-2}}{a_n}\right|\frac{1}{|z|^2} + \cdots + \left|\frac{a_1}{a_n}\right|\frac{1}{|z|^{n-1}} + \left|\frac{a_0}{a_1}\right|\frac{1}{|z|}\right)\right\}$$

$$> |a_n||z|^n\left\{1 - \sum_{k=1}^{n}\left|\frac{a_{n-k}}{a_n}\right|\frac{1}{{r_2}^k}\right\}.$$

Using (1.15) and noting that by hypothesis

$$\sum_{k=1}^{n}|\lambda_k| \le 1,$$

we obtain for $|z| > r_2$,

$$|P(z)| > |a_n||z|^n\left\{1 - \sum_{k=1}^{n}|\lambda_k|\right\} \ge 0\cdot$$

Thus $|P(z)| > 0$ for $|z| > r_2$, consequently all the zeros of $P(z)$ lie in $|z| \le r_2$ and this proves the second part of Theorem 1.7.

To prove the first part of this theorem, we shall use the second part. If $a_0 = 0$, then clearly

$$r_1 = \max_{1\le k\le n}\left|\lambda_k\frac{a_0}{a_k}\right| = 0$$

and there is nothing to prove. So we assume that $a_0 \ne 0$. Consider the polynomial

$$Q(z) = z^n P\left(\frac{1}{z}\right) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n .$$

By the second part of the theorem, all the zeros of the polynomial $Q(z)$ lie in

$$|z| \le \max_{1\le k\le n} \left|\frac{1}{\lambda_k}\frac{a_k}{a_0}\right|^{\frac{1}{k}}$$

$$= \max_{1\le k\le n} \left|\frac{1}{\lambda_k}\frac{1}{\frac{a_0}{a_k}}\right|^{\frac{1}{k}}$$

$$= \max_{1\le k\le n} \left\{\frac{1}{\left|\lambda_k \frac{a_0}{a_k}\right|^{\frac{1}{k}}}\right\}$$

$$= \frac{1}{\min_{1\le k\le n} \left|\lambda_k \frac{a_0}{a_k}\right|^{\frac{1}{k}}} .$$

Replacing z by $\frac{1}{z}$ and observing that

$$P(z) = z^n\, Q\left(\frac{1}{z}\right),$$

we conclude that all the zeros of $P(z)$ lie in

$$(1.15) \qquad |z| \ge r_1 = \min_{1\le k\le n} \left|\lambda_k \frac{a_0}{a_k}\right|^{\frac{1}{k}} .$$

The desired result follows by combining (1.14) and (1.15) .

We conclude Chapter 1 by presenting the following interesting and significant refinement of Theorem 1.1 due to A. Aziz and Aliya Qayoom [4].

THEOREM 1.8. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} \ldots + a_1 z + a_0,$$

be a non constant polynomial of degree n, then all the zeros of $P(z)$ lie in the disk

$$|z| \le \{(1+M)^n - 1\}^{\frac{1}{n}},$$

where

$$M = \max_{1 \le k \le n} \left| \frac{a_{n-k}}{a_n} \right| .$$

PROOF OF THEOREM1.8. Since

$$M = \max_{1 \le k \le n} \left| \frac{a_{n-k}}{a_n} \right| .$$

We have

$$(1.16) \qquad \left| \frac{a_{n-k}}{a_n} \right| \le M, \qquad k = 1,2,\cdots,n$$

we take

$$(1.17) \qquad \lambda_k = \left\{ \frac{(1+M)^n}{(1+M)^n - 1} \right\} \left\{ \frac{a_{n-k}}{a_n (1+M)^k} \right\}, \quad k = 1,2,\cdots,n$$

Then with the help of (1.16), we get

$$\sum_{k=1}^{n} |\lambda_k| = \frac{(1+M)^n}{(1+M)^n - 1} \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{(1+M)^k}$$

$$(1.18) \qquad \le \left\{ \frac{(1+M)^n}{(1+M)^n - 1} \right\} \sum_{k=1}^{n} \frac{M}{(1+M)^k} .$$

Now

$$\sum_{k=1}^{n} \frac{M}{(1+M)^k} = \frac{M}{1+M} \left\{ 1 + \frac{1}{(1+M)} + \cdots + \frac{1}{(1+M)^{n-1}} \right\}$$

$$= \frac{M}{(1+M)} \left\{ \frac{1 - \frac{1}{(1+M)^n}}{1 - \frac{1}{(1+M)}} \right\}$$

$$= \frac{(1+M)^n - 1}{(1+M)^n} .$$

Using this in (1.18), we see that

$$\sum_{k=1}^{n} |\lambda_k| \le 1.$$

Applying Theorem 1.7 with λ_k defined by (1.17), it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leq r_2 = \max_{1<k<n} \left|\frac{a_{n-k}}{\lambda_k a_n}\right|^{\frac{1}{k}}$$

$$= \max_{1\leq k\leq n} \left\{\frac{(1+M)^n - 1}{(1+M)^n}\right\}^{\frac{1}{k}} (1+M)$$

$$= (1+M) \max_{1\leq k\leq n} \left\{1 - \frac{1}{(1+M)^n}\right\}^{\frac{1}{k}}$$

$$= (1+M) \left\{1 - \frac{1}{(1+M)^n}\right\}^{\frac{1}{n}}$$

$$= \{(1+M)^n - 1\}^{\frac{1}{n}}.$$

This completes the proof of Theorem 1.8.

Chapter - 2

Eneström – Kakeya Theorem And Some Of Its Generalizations

We start this chapter with the following result known as Eneström- Kakeya theorem [37, p.136] on the distribution of zeros of polynomials with real coefficients.

THEOREM 2.1. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n, such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 > 0\,, \tag{2.1}$$

then all the zeros of $P(z)$ lie in $|z| \leq 1$.

PROOF OF THEOREM 2.1. Consider the polynomial

$$F(z) = (1 - z)P(z)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0\,,$$

then

$$|F(z)| \geq |a_n||z|^{n+1} - \{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_1 - a_0||z| + |a_0|\}$$

(2.2)
$$= |z|^{n+1}\left\{|a_n| - |a_n - a_{n-1}|\frac{1}{|z|} - |a_{n-1} - a_{n-2}|\frac{1}{|z|^2} - \cdots - |a_1 - a_0|\frac{1}{|z|^n} - \frac{|a_0|}{|z|^{n+1}}\right\}.$$

Now let $|z| > 1$, then $\frac{1}{|z|^j} < 1$ $\quad j = 1,2,\cdots,n+1,$

which gives

$$-\frac{1}{|z|^j} > -1, \qquad j = 1,2,\cdots,n+1.$$

Therefore,

$$|F(z)| > |z|^{n+1}\{|a_n| - |a_n - a_{n-1}| - |a_{n-1} - a_{n-2}| - \cdots - |a_1 - a_0| - |a_0|\}$$

$$= |z|^{n+1}\{a_n \quad (a_n \quad a_{n-1}) \quad (a_{n-1} \quad a_{n-2}) \quad \cdots \quad (a_1 \quad a_0) \quad a_0\}$$

$$= |z|^{n+1}\{a_n - a_n + a_{n-1} - a_{n-1} + a_{n-2} - \cdots - a_1 + a_0 - a_0\}$$

$$= 0.$$

Thus for $|z| > 1$, $|F(z)| > 0$, that is, $F(z)$ does not vanish for $|z| > 1$. This implies that all the zeros of $F(z)$ lie in $|z| \leq 1$. Since all the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that $P(z)$ has all its zeros in $|z| \leq 1$. This completes the proof of Theorem 2.1.

If

$$t = max\left\{\frac{a_0}{a_1}, \frac{a_1}{a_2}, \cdots, \frac{a_{n-1}}{a_n}\right\},$$

then

$$a_1 t \geq a_0, \quad a_2 t \geq a_1, \quad \cdots \quad, a_n t \geq a_{n-1},$$

and hence

$$a_n t^n \geq a_{n-1} t^{n-1} \geq \cdots \geq a_2 t^2 \geq a_1 t \geq a_0 \geq 0 \cdot$$

This shows that the polynomial

$$F(z) = P(tz)$$

$$= a_n t^n z^n + a_{n-1} t^{n-1} z^{n-1} + \cdots + a_1 t z + a_0,$$

satisfies the conditions of Theorem 2.1 and therefore, it follows that all the zeros of $F(z)$ lie in $|z| \leq 1$.

Replacing z *by* z/t and noting that $F(z/t) = P(z)$, we get the following more general result

COROLLORY 2.1. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n with real and positive coefficients, then all the zeros of $P(z)$ lie in

$$|z| \leq t = max\left\{\frac{a_0}{a_1}, \frac{a_1}{a_2}, \cdots, \frac{a_{n-1}}{a_n}\right\}.$$

If in the Eneström-Kakeya theorem (Theorem 2.1) we do not assume coefficients to be non-negative, then the conclusion does not hold, for example we may consider the second degree polynomial

$$P(z) = z^2 + z - 1.$$

Here

$$a_2 = 1, \quad a_1 = 1, \qquad a_0 = -1$$

and so

$$a_2 \geq a_1 \geq a_0 \,.$$

But the zeros of $P(z)$ are

$$z_1 = \frac{-1-\sqrt{5}}{2}, \quad z_2 = \frac{-1+\sqrt{5}}{2}$$

and we have

$$|z_1| = \frac{1+\sqrt{5}}{2} > 1 .$$

However in this case A. Joyal, G. Labelle and Q. I. Rahman [31] have obtained the following extension of Eneström ï Kakeya theorem.

THEOREM 2.2. If

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0 ,$$

then the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

of degree n has all its zeros in the circle

$$(2.3) \qquad |z| \leq \frac{1}{|a_0|}\{a_n - a_0 + |a_0|\} \cdot$$

If $a_0 > 0$, this result reduces to Eneström ï Kakeya theorem.

PROOF OF THEOREM 2.2. Consider the polynomial

$$F(z) = (1-z)P(z)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \cdots + (a_1 - a_0) z + a_0$$

$$= -a_n z^{n+1} + G(z) \qquad \text{(say)},$$

where

$$G(z) = (a_n - a_{n-1}) z^n + \cdots + (a_1 - a_0) z + a_0 .$$

If $|z| = 1,$ then

$$|G(z)| \leq |a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots$$

$$+ |a_1 - a_0||z| + |a_0|$$

$$= |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_1 - a_0| + |a_0|$$

$$= a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_1 - a_0 + |a_0|$$

$$= a_n - a_0 + |a_0|.$$

Since for $|z| = 1$,

$$\left|z^n\overline{G(1/\bar{z})}\right| = |G(z)|,$$

therefore, we have

$$\left|z^n\overline{G(1/\bar{z})}\right| < a_n - a_0 + |a_0| \qquad \text{for} \quad |z| = 1. \tag{2.4}$$

But the polynomial $z^n\overline{G(1/\bar{z})}$ is analytic in $|z| < 1$. Therefore, it follows by maximum modulus principle that the inequality (2.4) holds inside the unit circle also, that is

$$\left|z^n\overline{G(1/\bar{z})}\right| \le a_n - a_0 + |a_0| \qquad \text{for} \quad |z| \le 1 .$$

Replacing z by $1/z$, we get

$$|G(z)| \le (a_n - a_0 + |a_0|)|z|^n \quad \text{for} \quad |z| \ge 1. \tag{2.5}$$

Since by hypothesis

$$a_n - a_0 = |a_n - a_0| \ge |a_n| - |a_0| ,$$

therefore,

$$a_n - a_0 + |a_0| \ge |a_n|.$$

Now if

$$|z| > \frac{a_n - a_0 + |a_0|}{|a_n|},$$

then $|z| > 1$, and we have

$$|F(z)| = |-z^{n+1} + G(z)|$$

$$\ge |a_n||z|^{n+1} - |G(z)|$$

$$\ge |z|^n\{|a_n||z| - (a_n - a_0 + |a_0|)\} \qquad (\text{by } (2.5))$$

$$> 0.$$

This shows that all the zeros of $F(z)$ lie in the circle defined by (2.3), Since all the zeros $P(z)$ are also the zeros of $F(z)$, we conclude that all the zeros of $P(z)$ lie in the circle defined by (2.3). This completes the proof of Theorem 2.2.

In the literature for references see [5, 19, 31, 34, 39], there exists some extensions of the Eneström-Kakeya theorem. As regards to this theorem, it was asked by N. K. Govil and Prof. Rahman [28], what can be said if we drop the restriction that coefficients of the polynomial $P(z)$ are all positive and instead assume monotoncity to hold for the modulli of the coefficients of a polynomial $P(z)$? As an answer to this question they have established the following result.

THEOREM 2.3. Let

$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$,

be a polynomial of degree n such that for some $t > 0$

(2.6) $$|a_n| \geq t|a_{n-1}| \geq t^2|a_{n-2}| \geq \cdots \geq t^n|a_0|,$$

then $P(z)$ has all its zeros in

$$|z| \leq \left(\frac{1}{t}\right) k_1$$

where k_1 is the greatest positive root of the trinomial equation

$$k^{n+1} - 2k^n + 1 = 0.$$

It was shown by A. Aziz and Q. G. Mohammad [5] that the assertion of Theorem 2.3 holds under much weaker assumptions. In this connection, they have established the following result.

THEOREM 2.4. Let

$$P(z) = a_n z^n + a_p z^p + \cdots + a_1 z + a_0, \quad a_p \neq 0, \quad 0 \leq p \leq n-1$$

be a polynomial of degree n with complex coefficients such that for some $t > 0$

$$|a_n| \geq t^{n-j}|a_j|, \qquad j = 1, 2, \cdots, p .$$

Then $P(z)$ has all its zeros in

$$|z| \leq \left(\frac{1}{t}\right) k_1$$

where k_1 is the greatest positive root of the equation

$$(2.7) \qquad k^{n+1} - k^n - k^{p+1} + 1 = 0.$$

The polynomial

$$P(tz) = (tz)^n - (tz)^p - \cdots - (tz) - 1,$$

shows that the result is best possible.

For the proof of Theorem 2.4, we shall use the following result [5, Lemma 1].

LEMMA 2.1. Let

$$P(z) = a_n z^n + a_p z^p + \cdots + a_1 z + a_0 , \quad 0 \leq p \leq n-1$$

be a polynomial of degree n with complex coefficients. Then for every positive real number r, all the zeros of $P(z)$ lie in the circle

$$|z| \leq max\left[r, \sum_{j=0}^{p} \left|\frac{a_j}{a_n}\right| \frac{1}{r^{n-j-1}}\right].$$

PROOF OF LEMMA 2.1. The companion matrix of the polynomial $P(z)$ is

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{a_1}{a_n} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{a_2}{a_n} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & -\frac{a_p}{a_n} \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

We take a matrix

$$P = \mathrm{diag}(r^{n-1}, r^{n-2}, \cdots, r, 1),$$

where r is a positive real number and form the matrix

$$P^{-1}CP = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{a_0}{a_n r^{n-1}} \\ r & 0 & 0 & \cdots & 0 & -\frac{a_1}{a_n r^{n-2}} \\ 0 & r & 0 & \cdots & 0 & -\frac{a_2}{a_n r^{n-3}} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & r & -\frac{a_p}{a_n r^{n-p+1}} \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Applying Greshgorian theorem (Lemma 1.2) to the columns of the matrix $P^{-1}CP$, it follows that all the eigen values of $P^{-1}CP$ lie in the circle

$$|z| \leq max\left[r, \sum_{j=0}^{p}\left|\frac{a_j}{a_n}\right|\frac{1}{r^{n-j-1}}\right]. \tag{2.8}$$

Since the matrix $P^{-1}CP$ is similar to the matrix C and the eigen values of C are the zeros of $P(z)$, it follows that all the zeros of $P(z)$ lie in the circle defined by (2.8). This completes the proof of the Lemma 2.1.

PROOF OF THEOREM 2.4. Since by hypothesis

$$\left|\frac{a_j}{a_n}\right| \le \frac{1}{t^{n-j}}, \qquad j = 0,1,\cdots,p$$

it follows by Lemma 2.1 that for every positive real number r, all the zeros of $P(z)$ lie in the circle

$$(2.9) \qquad |z| \le max\left[r, r\sum_{j=0}^{p}\frac{1}{(rt)^{n-j}}\right].$$

We choose r such that

$$\sum_{j=0}^{p}\frac{1}{(rt)^{n-j}} = 1,$$

which gives,

$$(rt)^p + (rt)^{p-1} + \cdots + (rt) + 1 = (rt)^n,$$

so that

$$(rt-1)[(rt)^p + (rt)^{p-1} + \cdots + (rt) + 1] = (rt-1)(rt)^n$$

equivalently,

$$(rt)^{n+1} - (rt)^n - (rt)^{p+1} + 1 = 0 \cdot$$

Replacing (rt) by k, it follows from (2.8) that all the zeros of $P(z)$ lie in

$$|z| \le \left(\frac{1}{t}\right)k_1,$$

where k_1 is the greatest positive root of the equation defined by (2.7). This completes the proof of the Theorem 2.4.

Taking $p = n-1$ in Theorem 2.4, we get the following result.

COROLLORY 2.2. Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n with complex coefficients such that for some $t > 0$

$$(2.10) \qquad |a_n| \ge t^{n-j}|a_j|, \qquad j = 0,1,\cdots,n.$$

Then all the zeros of $P(z)$ lie in

$$|z| \leq \left(\frac{1}{t}\right) k_1,$$

where k_1 is the greatest positive root of the equation

$$k^{n+1} - 2k^n + 1 = 0.$$

REMARK 2.1. If a polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

satisfies the condition (2.6) of Theorem 2.3, then it also satisfies the conditions (2.10) of Corollary 2.2. This shows that Theorem 2.3 holds under much weaker conditions than the Theorem 2.4 for $p = n - 1$.

Next we present the following generalization of Eneström-Kakeya theorem proved by A. Aziz and Q. G. Mohammad [6] by using Schwartz lemma.

THEOREM 2.5. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n with real and positive coefficients. If $t_1 > t_2 \geq 0$ can be found such that

$$(2.11) \qquad a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \qquad r = 0,1,\cdots,n+1.$$

$$(a_{-2} = a_{-1} = a_{n+1} = 0)$$

Then all the zeros of $P(z)$ lie in

$$|z| \leq t_1.$$

For $t_1 = 1$ and $t_2 = 0$, this result reduces to Eneström - Kakeya theorem (Theorem 2.1).

PROOF OF THEOREM 2.5. Consider the polynomial

$$F(z) = (t_1 - z)(t_2 + z)P(z)$$

$$= (t_1 t_2 + (t_1 - t_2)z - z^2)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)$$

$$= -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\} z^{n+1} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\} z^n + \cdots + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\} z + a_0 t_1 t_2 .$$

Let

$$G(z) = z^{n+2} F\left(\frac{1}{z}\right)$$

$$= z^{n+2}\Big[-a_n \frac{1}{z^{n+2}} + \{a_n(t_1 - t_2) - a_{n-1}\}\frac{1}{z^{n+1}} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}\frac{1}{z^n} + \cdots + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\}\frac{1}{z} + a_0 t_1 t_2\Big]$$

$$= [-a_n + \{a_n(t_1 - t_2) - a_{n-1}\} z + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\} z^2 + \cdots + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\} z^{n+1} + a_0 t_1 t_2 z^{n+2}]$$

$$= -a_n + D(z) \qquad \text{(say)},$$

where

$$D(z) = \{a_n(t_1 - t_2) - a_{n-1}\} z + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\} z^2 + \cdots + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\} z^{n+1} + a_0 t_1 t_2 z^{n+2},$$

then

$$D(0) = 0,$$

and

$$D\left(\frac{1}{t_1}\right) = \Big[\{a_n(t_1 - t_2) - a_{n-1}\}\frac{1}{t_1} + \{a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2}\}\frac{1}{t_1^2} + \cdots + \{a_1 t_1 t_2 + a_0(t_1 - t_2)\}\frac{1}{t_1^{n+1}} + a_0 t_1 t_2 \frac{1}{t_1^{n+2}}\Big]$$

$$= a_n - a_n\frac{t_2}{t_1} - \frac{a_{n-1}}{t_1} + \frac{a_n t_2}{t_1} + \frac{a_{n-1}}{t_1} - \frac{a_{n-1} t_2}{t_1^2} - \frac{a_{n-2}}{t_1^2} + \cdots + \frac{a_1 t_2}{t_1^n} + \frac{a_0}{t_1^n} - \frac{a_0 t_2}{t_1^{n+1}} + \frac{a_0 t_2}{t_1^{n+1}}$$

$$= a_n. \tag{2.12}$$

Now by (2.11) all the coefficients of $D(z)$ are positive, therefore

$$\max_{|z|=\frac{1}{t_1}} |D(z)| = D\left(\frac{1}{t}\right) = a_n . \qquad \text{(by 2.12)}$$

Thus $P(z)$ satisfies all the conditions of Schwartz lemma and therefore,

$$|D(z)| \leq a_n |z| t_1 \quad \text{for} \quad |z| \leq \frac{1}{t_1}.$$

Hence for $|z| \leq \frac{1}{t_1}$,

$$|G(z)| = |-a_n + D(z)|$$

$$\geq |a_n| - |D(z)|$$

$$\geq a_n (1 - |z| t_1)$$

$$> 0,$$

if

$$1 - |z| t_1 > 0,$$

that is, if

$$|z| < \frac{1}{t_1}.$$

Thus in $|z| \leq \frac{1}{t_1}$, $|G(z)| > 0$ if $|z| < \frac{1}{t_1}$. Consequently, all the zeros of $G(z)$ lie in $|z| \geq \frac{1}{t_1}$. As

$$F(z) = z^{n+2} G\left(\frac{1}{z}\right),$$

we conclude that all the zeros $F(z)$ and hence all the zeros of $P(z)$ lie in $|z| \leq t_1$. This completes the proof of Theorem 2.5.

A. Aziz and Q. G. Mohammad have obtained several interesting generalizations of Eneström-Kakeya theorem. Here we shall mention the following result due to A. Aziz and Q. G. Mohammad [5].

THEOREM 2.6. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 ,$$

be a polynomial of degree n with complex coefficients such that

$$(2.13) \qquad |arg\, a_j - \beta| \leq \alpha \leq \frac{\pi}{2} , \qquad J = 0,1,\cdots,n \text{ for some real } \beta.$$

If for some $t > 0$ and $k = 0,1,2,\ldots,n$,

$$(2.14) \qquad t^n|a_n| \leq t^{n-1}|a_{n-1}| \leq \cdots \leq t^k|a_k|,$$

$$t^k|a_k| \geq t^{k-1}|a_{k-1}| \geq \cdots \geq t|a_1| \geq |a_0| ,$$

then $P(z)$ has all its zeros in the circle

$$(2.15) \qquad |z| \leq t\left[\left(\frac{2t^k|a_k|}{t^n|a_n|} - 1\right)cos\alpha + sin\alpha\right] + sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{t^{n-j-1}|a_n|} .$$

Recently A. Aziz and B. A Zargar have proved several Eneström- Kakeya type theorems which generalize some known results by putting less restrictive conditions on the coefficients of the polynomial. Here we present a few of them.

THEOREM 2.7. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n such that for some $K \geq 1$,

$$K a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then $P(z)$ has all its zeros in

$$|z + K - 1| \leq K.$$

We may apply Theorem 2.7 to the polynomial $P(tz)$ to obtain the following result.

COROLLORY 2.3. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 ,$$

is a polynomial of degree n such that for some $K \geq 1$,

$$K t^n a_n \geq t^{n-1} a_{n-1} \geq \cdots \geq t a_1 \geq a_0 > 0 ,$$

then $P(z)$ has all its zeros in

$$|z + Kt - t| \leq Kt.$$

The next corollary is obtained by taking $K = \frac{a_{n-1}}{a_n} \geq 1$ in Theorem 2.7.

COROLLORY 2.4. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial of degree n such that

$$a_n \leq a_{n-1} \geq a_{n-2} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in the circle

$$\left|z + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{a_{n-1}}{a_n}.$$

Instead of proving Theorem 2.7, we present the following more general result.

THEOREM 2.8. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 ,$$

is a polynomial of degree n such that for some $K \geq 1$

$$K a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 ,$$

then all the zeros of $P(z)$ lie in

$$(2.16) \qquad |z + (K-1)| \leq \frac{K a_n - a_0 + |a_0|}{|a_n|}.$$

PROOF OF THEOREM 2.8. Consider

$$F(z) = (1 - z)P(z)$$

$$= (1 - z)(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \cdots + (a_1 - a_0) z + a_0 ,$$

then for $|z| > 1$, we have

$$|F(z)| = |-a_n z^{n+1} + (a_n - a_{n-1}) z^n + \cdots + (a_1 - a_0) z + a_0|$$

$$= |-a_n z^{n+1} + a_n z^n - Ka_n z^n + (Ka_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0|$$

$$\geq |a_n||z|^n|z + K - 1| - |z|^n \Big|(Ka_n - a_{n-1})\frac{1}{z} + \cdots$$

$$+(a_1 - a_0)\frac{1}{z^{n-1}} + a_0\frac{1}{z^n}\Big|$$

$$\geq |z|^n \Big[\, |a_n||z + K - 1| - \Big\{|Ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}|\frac{1}{|z|} + \cdots +$$
$$|a_1 - a_0|\frac{1}{|z^{n-1}|} + |a_0|\frac{1}{|z|^n}\Big\}\Big]$$

$$> |z|^n[\, |a_n||z + K - 1| - \{Ka_n - a_0 + |a_0|\}\,]$$
$$> 0,$$

if

$$|z + K - 1| > \frac{Ka_n - a_0 + |a_0|}{|c_n|} .$$

Hence all the zeros of $F(z)$ whose modulus is greater than 1 lie in the circle

$$(2.17) \qquad |z + K - 1| \leq \frac{Ka_n - a_0 + |a_0|}{|a_n|} .$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfies the inequality (2.17). Hence all the zeros of $F(z)$ lie in the circle defined by (2.16).

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, therefore it follows that all the zeros of $P(z)$ lie in the circle defined by (2.8).

REMARK 2.2. For $a_0 \geq 0$, Theorem 2.8 reduces to the Theorem 2.7.

In [12] the authorŝs have relaxed the hypothesis of Theorem 2.1 by assuming alternate coefficients of polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 ,$$

satisfies condition (2.1), in fact they proved the following result.

THEOREM 2.9. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial of degree n such that either

$$a_n \geq a_{n-2} \geq \cdots \geq a_3 \geq a_1 > 0 \quad \text{if} \quad n \quad \text{is odd}$$

or $$a_{n-1} \geq a_{n-3} \geq \cdots \geq a_2 \geq a_0 > 0 \quad \text{if} \quad n \quad \text{is even} ,$$

then all the zeros of $P(z)$ lie in the circle

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq 1 + \frac{a_{n-1}}{a_n}.$$

PROOF OF THE THEOREM 2.9. Consider

$$F(z) = (1 - z^2)P(z)$$

$$= -a_n z^{n+2} - a_{n-1} z^{n+1} + (a_n - a_{n-1}) z^n + \cdots + (a_2 - a_0) z^2$$

$$+ a_1 z + a_0.$$

Then for $|z| > 1$, we have

$$|F(z)| = \; |-(a_n z + a_{n-1}) z^{n+1} + (a_n - a_{n-2}) z^n +$$

$$(a_{n-1} - a_{n-3}) z^{n-1} + \cdots + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z + a_0|$$

$$\geq |z|^n \left[|z||a_n z + a_{n-1}| - \left\{ |a_n - a_{n-2}| + |a_{n-1} - a_{n-3}| \frac{1}{|z|} + \cdots \right.\right.$$

$$\left.\left. + |a_2 - a_0| \frac{1}{|z|^{n-2}} + a_1 \frac{1}{|z|^{n-1}} + a_0 \frac{1}{|z|^n} \right\} \right]$$

$>$

$|a_n z + a_{n-1}| - |a_n - a_{n-2}| - |a_{n-1} - a_{n-3}| - \cdots - |a_3 - a_1| -$
$|a_2 \quad a_0| \quad a_1 \quad a_0$

$$= |a_n z + a_{n-1}| - (a_n - a_{n-2}) - (a_{n-1} - a_{n-3}) - \cdots - (a_3 - a_1)$$

$$-(a_2 - a_0) - a_1 - a_0$$

$$= \; |a_n z + a_{n-1}| - (a_n + a_{n-1})$$

$$> 0,$$

if

$$|a_n z + a_{n-1}| > a_n + a_{n-1}\,.$$

Hence $F(z)$ does not vanish for

$$\left|z + \frac{a_{n-1}}{a_n}\right| > 1 + \frac{a_{n-1}}{a_n}.$$

Therefore those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\text{(2.18)} \qquad \left|z + \frac{a_{n-1}}{a_n}\right| \leq 1 + \frac{a_{n-1}}{a_n}.$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the inequality (2.18). Hence we conclude that all the zeros of $F(z)$ and hence those of $P(z)$ lie in the circle

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq 1 + \frac{a_{n-1}}{a_n}.$$

This completes the proof of Theorem 2.9.

If we apply Theorem 2.9 to the polynomial $P(tz)$, we get the following result.

CORROLORY 2.5. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\,,$$

is a polynomial of degree n such that for some $t > 0$, either

$$a_n t^n \geq a_{n-2} t^{n-2} \geq \cdots \geq a_3 t^3 \geq a_1 t > 0,$$

and

$$a_{n-1} t^{n-1} \geq a_{n-3} t^{n-3} \geq \cdots \geq a_2 t^2 \geq a_0 > 0 \qquad \text{if } n \text{ is odd}$$

or

$$a_n t^n \geq a_{n-2} t^{n-2} \geq \cdots \geq a_3 t^3 \geq a_1 t > 0,$$

and

$$a_{n\ 1} t^{n-1} > a_{n\ 3} t^{n-3} > \cdots > a_1 t > 0 \qquad \text{if } n \text{ is even}\,,$$

then all the zeros of $P(z)$ lie in the circle

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le t+\frac{a_{n-1}}{a_n}.$$

We next present the following generalization of Enestrom-Kakeya theorem (Theorem 2.1) due to Govil and Rahman [28].

THEOREM 2.10. Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 ,$$

be a polynomial of degree n. If

(2.19) $\quad Re(a_j) = \alpha_j,\ \ Im(a_j) = \beta_j,\ \ j = 0,1,\cdots,n,$ and for some $t > 0$

$$0 < t^n\alpha_n \le \cdots \le t^k\alpha_k \ge t^{k-1}\alpha_{k-1} \ge \cdots \ge t\alpha_1 \ge \alpha_0 \ge 0,$$

where $0 \le k \le n$, then all the zeros of $P(z)$ lie in the circle

$$|z| \le t\left(\frac{2t^k\alpha_k}{t^n\alpha_n} - 1\right) + \frac{2}{\alpha_n}\sum_{j=0}^{n}\frac{|\beta_j|}{t^{n-j-1}}. \tag{2.20}$$

PROOF OF THEOREM 2.10. Consider the polynomial

$$F(z) = (t-z)P(z)$$

$$= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + (ta_{n-1} - a_{n-2})z^{n-1} + \cdots + (ta_1 - a_0)z + ta_0.$$

Applying Lemma 2.1 to the polynomial $F(z)$ which is of degree $n+1$, with $p = n$ and $r = t$, it follows that all the zeros of $F(z)$ lie in the circle

$$|z| \le Max\left[t, \sum_{j=0}^{n}\frac{|ta_j - a_{j-1}|}{t^{n-j}|a_n|}\right]. \qquad (a_{-1} = 0) \tag{2.21}$$

Since

$$\left|\sum_{j=0}^{n}\frac{(ta_j-a_{j-1})}{t^{n-j}|a_n|}\right| = \left|\frac{ta_0}{t^n a_n}+\frac{ta_1-a_0}{t^{n-1}a_n}+\frac{ta_2-a_1}{t^{n-2}a_n}+\cdots+\frac{ta_n-a_{n-1}}{a_n}\right|$$

$$=\frac{1}{t^n|a_n|}|ta_0+t(ta_1-a_0)+t^2(ta_2-a_1)+\cdots+t^n(ta_n-a_{n-1})|$$

$$=\frac{1}{t^n|a_n|}t^{n+1}|a_n|$$

$$=t,$$

therefore,

$$t=\left|\sum_{j=0}^{n}\frac{(ta_j-a_{j-1})}{t^{n-j}|a_n|}\right|$$

$$\leq\sum_{j=0}^{n}\frac{|ta_j-a_{j-1}|}{|t^{n-j}|a_n||}.$$

Using it in (2.21), we get

$$|z|\leq\sum_{j=0}^{n}\frac{|ta_j-a_{j-1}|}{t^{n-j}|a_n|}$$

$$=\sum_{j=0}^{n}\frac{|t(\alpha_j+i\beta_j)-(\alpha_{j-1}+i\beta_{j-1})|}{t^{n-j}|a_n|}$$

$$=\sum_{j=0}^{n}\frac{|(t\alpha_j-\alpha_{j-1})+i(t\beta_j-\beta_{j-1})|}{t^{n-j}|a_n|}$$

$$\leq\sum_{j=0}^{n}\frac{|t\alpha_j-\alpha_{j-1}|}{t^{n-j}\alpha_n}+\sum_{j=0}^{n}\frac{|t\beta_j-\beta_{j-1}|}{t^{n-j}\alpha_n}$$

$$=\sum_{j=0}^{k}\frac{(t\alpha_j-\alpha_{j-1})}{t^{n-j}\alpha_n}+\sum_{j=k+1}^{n}\frac{(t\alpha_j-\alpha_{j-1})}{t^{n-j}\alpha_n}+\sum_{j=0}^{n}\frac{|t\beta_j-\beta_{j-1}|}{t^{n-j}\alpha_n}$$

$$\leq\sum_{j=0}^{k}\frac{(t\alpha_j-\alpha_{j-1})}{t^{n-j}\alpha_n}+\sum_{j=k+1}^{n}\frac{(t\alpha_j-\alpha_{j-1})}{t^{n-j}\alpha_n}+\sum_{j=0}^{n}\frac{|t\beta_j|}{t^{n-j}\alpha_n}+\sum_{j=0}^{n}\frac{|\beta_{j-1}|}{t^{n-j}\alpha_n}$$

$$-\left[\frac{2t\alpha_k}{t^{n-k}\alpha_n}-\iota\right]+\sum_{j=0}^{n}\frac{|t\beta_j|}{t^{n-j}\alpha_n}+\sum_{j=0}^{n}\frac{|\beta_{j-1}|}{t^{n-j}\alpha_n}$$

$$\leq t\left(\frac{2t^k\alpha_k}{t^n\alpha_n}-1\right)+\sum_{j=0}^{n}\frac{2|\beta_j|}{t^{n-j-1}\alpha_n}.$$

Thus all the zeros of $F(z)$ lie in the circle

$$|z|\leq t\left(\frac{2t^k\alpha_k}{t^n\alpha_n}-1\right)+\sum_{j=0}^{n}\frac{2|\beta_j|}{t^{n-j-1}\alpha_n}.$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $P(z)$ lie in the circle defined by (2.20).

REMARK 2.3. If we take $t=1,\ k=n$ and $\beta_j=0,\ j=0,1,2,\dots,n,$ in Theorem 2.10, then we get the Eneström- Kakeya theorem (Theorem 2.1).

We shall conclude this chapter by presenting the following two results more recently proved by A. Aziz and B. A Zargar [13] which among other things yield some interesting refinements of Theorem 2.6 for $0\leq k\leq n-2.$

THEOREM 2.11. Let

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0,$$

be a polynomial of degree $n\geq 2$ with complex coefficients. If for some real β

$$|arg\,a_j-\beta|\leq\alpha\leq\frac{\pi}{2}\ \text{ for }\ 0\leq j\leq n\ \text{ and for some } t>0$$

$$0<|a_0|\leq t|a_1|\leq\cdots\leq t^{k-1}|a_{k-1}|\leq t^k|a_k|$$

$$t^k|a_k|\geq t^{k+1}|a_{k+1}|\geq\cdots\geq t^{n-1}|a_{n-1}|\geq t^n|a_n|>0,$$

where $0\leq k\leq n-2,$ then all the zeros of $P(z)$ lie in the circle

$$\left|z+\frac{a_{n-1}}{a_n}-t\right|\leq\frac{t}{|a_n|}\left\{\frac{2|a_k|}{t^{n-k}}\cos\alpha-\frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)\right\}\tag{2.22}$$

$$+2\sin\alpha\sum_{j=0}^{n}\frac{|a_j|}{|a_n|t^{n-j-1}}.$$

For the proof of Theorem 2.11, we need the following lemma.

LEMMA 2.2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n. If for some real β ,

$$|arg\, a_j - \beta| \le \alpha \le \frac{\pi}{2}\ , a_j \ne 0\ , j = 0,1,\cdots,n\,.$$

Then for each $t > 0$,

$$|ta_j - a_{j-1}| \le \big|t|a_j| - |a_{j-1}|\big| cos\alpha + (t|a_j| + |a_{j-1}|) sin\alpha.$$

PROOF OF THEOREM 2.11. Consider the polynomial

$$F(z) = (t - z)P(z)$$

$$= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + (ta_{n-1} - a_{n-2})z^{n-1} + \cdots$$

$$+ (ta_1 - a_0)z + ta_0$$

$$= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + \sum_{j=0}^{n-1} (ta_j - a_{j-1})z^j\ . \quad (a_{-1} = 0)$$

Let $|z| > t$, then

$$|F(z)| = \left| -a_n z^n \left\{ \left(z + \frac{a_{n-1}}{a_n} - t \right) - \frac{1}{a_n} \sum_{j=0}^{n-1} (ta_j - a_{j-1}) \frac{1}{z^{n-j}} \right\} \right|$$

$$\ge |a_n||z|^n \left\{ \left| z + \frac{a_{n-1}}{a_n} - t \right| - \frac{1}{|a_n|} \sum_{j=0}^{n-1} |ta_j - a_{j-1}| \frac{1}{|z|^{n-j}} \right\}$$

$$> |a_n||z|^n \left\{ \left| z + \frac{a_{n-1}}{a_n} - t \right| - \frac{1}{|a_n|} \sum_{j=0}^{n-1} |ta_j - a_{j-1}| \frac{1}{t^{n-j}} \right\}$$

(2.23)
$$= |a_n||z|^n \left\{ \left| z + \frac{a_{n-1}}{a_n} - t \right| - \frac{1}{|a_n| t^n} \sum_{j=0}^{n-1} (ta_j - a_{j-1}) t^j \right\}.$$

Now by Lemma 2.2

$$|ta_j - a_{j-1}| \le \left| t|a_j| - |a_{j-1}| \right| cos\alpha + (t|a_j| + |a_{j-1}|)sin\alpha$$

for all $j = 0,1,\cdots,n$. This gives

$$\sum_{j=0}^{n-1}|ta_j - a_{j-1}|t^j \le \sum_{j=0}^{n-1}\left|t|a_j| - |a_{j-1}|\right| t^j cos\alpha + \sum_{j=0}^{n-1}(t|a_j| + |a_{j-1}|)t^j sin\alpha$$

$$\le \sum_{j=0}^{k}(t|a_j| - |a_{j-1}|)t^j cos\alpha + \sum_{j=k+1}^{n-1}(|a_{j-1}| - t|a_j|)\, t^j cos\alpha$$

$$+\sum_{j=0}^{n-1}(t|a_j| + |a_{j-1}|)t^j sin\alpha, \qquad (0 \le k \le n-1)$$

$$\le \sum_{j=0}^{k}(|a_j|t^{j+1} - |a_{j-1}|t^j)cos\alpha + \sum_{j=k+1}^{n-1}(|a_{j-1}|t^{j+1} - |a_j|t^{j+1})\, cos\alpha$$

$$+\sum_{j=0}^{n-1}(|a_j|t^{j+1} + |a_{j-1}|t^j)sin\alpha$$

$$= 2|a_k|t^{k+1}cos\alpha - t^n|a_{n-1}|cos\alpha - t^n|a_{n-1}|sin\alpha + 2tsin\alpha\sum_{j=0}^{n-1}|a_j|t^j$$

$$= 2|a_k|t^{k+1}cos\alpha - t^n|a_{n-1}|(cos\alpha + sin\alpha) + 2tsin\alpha\sum_{j=0}^{n-1}|a_j|t^j.$$

Hence

$$\frac{1}{|a_n|t^n}\sum_{j=0}^{n-1}|ta_j - a_{j-1}|t^j$$
$$\le \frac{t}{|a_n|}\left\{\frac{2|a_k|cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(cos\alpha + sin\alpha) + 2sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right\}.$$

Using this in (2.23), we obtain for $|z| > t$,

$$|F(z)| > |a_n||z|^n\left[\left|z + \frac{a_{n-1}}{a_n} - t\right| - \frac{t}{|a_n|}\left\{\frac{2|a_k|cos\alpha}{t^{n-k}} - \frac{|a_{n-1}|}{t}(cos\alpha + sin\alpha)\right.\right.$$

$$-2sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\Bigg\}\Bigg] > 0,$$

whenever

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| > \frac{t}{|a_n|}\left\{\frac{2|a_k|cos\alpha}{t^{n-k}}-\frac{|a_{n-1}|}{t}(cos\alpha+sin\alpha)-2sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right\}.$$

Hence all those zeros of $F(z)$ whose modulus is greater than t lie in the circle

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \leq \frac{t}{|a_n|}\left\{\frac{2|a_k|cos\alpha}{t^{n-k}}-\frac{|a_{n-1}|}{t}(cos\alpha+sin\alpha)\ +2sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right\}.$$

We now show that all those zeros of $F(z)$ whose modulus is less than or equal to t also satisfies (2.22) for $0\leq k\leq n-1$. Let $|z|\leq t$, then by using Lemma 2.2 and hypothesis, we get

$$\left|z+\frac{a_{n-1}}{a_n}-t\right| \leq |z|+\left|\frac{a_{n-1}}{a_n}-t\right|$$

$$\leq t+\frac{|ta_n-a_{n-1}|}{|a_n|}$$

$$\leq t+\frac{\big|t|a_n|-|a_{n-1}|\big|cos\alpha+(t|a_n|+|a_{n-1}|)sin\alpha}{|a_n|}$$

$$= t+\left(\left|\frac{a_{n-1}}{a_n}\right|-t\right)cos\alpha+\left(t+\left|\frac{a_{n-1}}{a_n}\right|\right)sin\alpha$$

$$\leq \frac{t}{|a_n|}\left\{\frac{2|a_k|cos\alpha}{t^{n-k}}-\frac{|a_{n-1}|}{t}(cos\alpha+sin\alpha)\ +2sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right\},$$

whenever

$$(2.24)\qquad t(1-cos\alpha)+tsin\alpha+2\left|\frac{a_{n-1}}{a_n}\right|cos\alpha$$

$$\leq 2\left|\frac{a_k}{a_n}\right|\frac{1}{t^{n-k-1}}cos\alpha+2sin\alpha\sum_{j=0}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n}.$$

Since by hypothesis

$$t^n|a_{n-1}| \leq |a_k|t^{k+1},$$

it follows that (2.24) is true, if

$$(2.25) \qquad t(1-\cos\alpha)+t\sin\alpha \leq 2\sin\alpha\sum_{j=0}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n}\cdot$$

Again by hypothesis and noting that $0 \leq k \leq n-2$, we get

$$\sum_{j=0}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n} = \sum_{j=0}^{k-1}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n} + \sum_{j=k}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n}$$

$$\geq \sum_{j=k}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^{j+1}}{t^n}$$

$$= t\sum_{j=k}^{n-2}\left|\frac{a_j}{a_n}\right|\frac{t^j}{t^n}$$

$$\geq t\sum_{j=k}^{n-2}1 = t(n-k-1)$$

$$\geq t.$$

Hence (2.25) holds, whenever

$$t(1-\cos\alpha)+t\sin\alpha < 2t\sin\alpha, \qquad 0 < \alpha < \frac{\pi}{2}$$

or

$$(2.26) \qquad \cos\alpha+\sin\alpha \geq 1 \qquad \text{whenever } 0 \leq \alpha \leq \tfrac{\pi}{2}\cdot$$

But we note that when $0 \leq \alpha \leq \frac{\pi}{2}$

$$\cos\alpha+\sin\alpha = \sqrt{2}\sin\left(\alpha+\frac{\pi}{4}\right) \geq \sqrt{2}\frac{1}{\sqrt{2}} = 1,$$

so (2.26) holds. Thus we have shown that if $|z| \leq t$, then for $0 \leq k \leq n-2$,

$$\left|z+\frac{a_{n-1}}{a_n}-t\right|\leq\frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}}-\frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right\}.$$

Hence all the zeros of $F(z)$ lie in the circle defined by (2.22). But all the zeros of $P(z)$ are also the zeros of $F(z)$, we conclude that all the zeros of $P(z)$ lie in the circle defined by (2.22). This completes the proof of Theorem 2.11.

REMARK 2.4. To see that Theorem 2.11 is an improvement of Theorem 2.6 for $0\leq k\leq n-2$. We show that the circle defined by (2.22) is contained in the circle defined by (2.15). For this let $z=w$ be any point belonging to the circle defined by (2.22), then

$$\left|w+\frac{a_{n-1}}{a_n}-t\right|\leq\frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}}-\frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right\}.$$

This implies

$$|w|=\left|w+\frac{a_{n-1}}{a_n}-t+t-\frac{a_{n-1}}{a_n}\right|$$

$$\leq\left|w+\frac{a_{n-1}}{a_n}-t\right|+\left|t-\frac{a_{n-1}}{a_n}\right|$$

$$\leq\frac{|a_nt-a_{n-1}|}{|a_n|} \tag{2.26}$$

$$+\frac{t}{|a_n|}\left\{\frac{2|a_k|\cos\alpha}{t^{n-k}}-\frac{|a_{n-1}|}{t}(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}\right\}.$$

Since $|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2}$, $j=0,1,\cdots,n,$ it follows that (by Lemma 2.2)

$$|ta_n-a_{n-1}|\leq\big|t|a_n|-|a_{n-1}|\big|\cos\alpha+(t|a_n|+|a_{n-1}|)\sin\alpha \quad \text{for all } j=0,1,2,\cdots,n.$$

$$=(|a_{n-1}|-t|a_n|)\cos\alpha+(t|a_n|+|a_{n-1}|)\sin\alpha \quad \text{(by hypothesis)}$$

Using this in (2.26), we get

$$|w|\leq\frac{(|a_{n-1}|-t|a_n|)}{|a_n|}\cos\alpha+\frac{(t|a_n|+|a_{n-1}|)}{|a_n|}\sin\alpha+t\left[\frac{2t^k|a_k|\cos\alpha}{t^n|a_n|}\right]$$

$$-\frac{|a_{n-1}|}{|a_n|}(\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}$$

$$=t\left\{\left[\frac{2t^k|a_k|}{t^n|a_n|}-1\right]\cos\alpha+\sin\alpha\right\}+2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|t^{n-j-1}}.$$

This shows that the point $z = w$ belongs to the circle defined by (2.15). Hence the circle defined by (2.22) is contained in the circle defined by (2.15).

REMARK 2.5. If we take $k = n,\ \alpha = \beta = 0$ and $t = 1$ in Theorem 2.11, then we get the Eneström- Kakeya theorem (Theorem 2.1).

Finally we present the following result which considerably improves up on the Theorem 2.10 for $0 \le k \le n-1$.

THEOREM 2.12. Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n with complex coefficients. If

$$Re(a_j) = \alpha_j, Im(a_j) = \beta_j,\quad J = 0,1,\cdots,n \quad \text{and for some } t > 0$$

(2.28) $$0 < t^n\alpha_n \le t^{n-1}\alpha_{n-1} \le \cdots \le t^k\alpha_k,$$

$$t^k\alpha_k \ge t^{k-1}\alpha_{k-1} \ge \cdots \ge t\alpha_1 \ge \alpha_0 > 0,$$

where $0 \le k \le n-1$, then all the zeros of $P(z)$ lie in the circle

(2.29) $$\left|z+\frac{\alpha_{n-1}-t\alpha_n}{\alpha_n}\right| \le \frac{1}{|a_n|}\left\{\left(\frac{2\alpha_k t^{k+1}}{t^n}-\alpha_{n-1}\right)+t|\beta_n|+\frac{2}{t^{n-j}}\sum_{j=0}^{n-1}|\beta_j|t^j\right\}.$$

PROOF OF THEOREM 2.12. Consider the polynomial

$$F(z) = (t-z)P(z)$$

$$= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + (ta_{n-1} - a_{n-2})z^{n-1} + \cdots$$

$$+(ta_{k+1} - a_k)z^{k+1} + (ta_k - a_{k-1})z^k + \cdots + (ta_2 - a_1)z^2$$

$$+(ta_1 - a_0)z + ta_0.$$

$$= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + \sum_{j=0}^{n-1}(ta_j - a_{j-1})z^j, \qquad (a_{-1} = 0)$$

Let $|z| > t,$ then we have

$$|F(z)| > |z|^n \left\{ |a_n z + a_{n-1} - ta_n| - \sum_{j=0}^{n-1} |ta_j - a_{j-1}| \frac{1}{|z|^{n-j}} \right\}$$

$$= |z|^n \left\{ |\alpha_n z + \alpha_{n-1} - t\alpha_n + i(\beta_{n-1} - t\beta_n)| - \sum_{j=0}^{n-1} |ta_j - a_{j-1}| \frac{1}{|z|^{n-j}} \right\}$$

(2.30) $$\geq |z|^n \left\{ |a_n z + \alpha_{n-1} - t\alpha_n| - |\beta_{n-1}| - t|\beta_n| - \sum_{j=0}^{n-1} |ta_j - a_{j-1}| \frac{1}{|z|^{n-j}} \right\}.$$

Now by hypothesis

$$\sum_{j=0}^{n-1} |ta_j - a_{j-1}| t^j \leq \sum_{j=0}^{n-1} |t\alpha_j - \alpha_{j-1}| t^j + \sum_{j=0}^{n-1} (|\beta_{j-1}| + t|\beta_j|) t^j$$

$$= \sum_{j=0}^{k} |t\alpha_j - \alpha_{j-1}| t^j + \sum_{j=k+1}^{n-1} |t\alpha_j - \alpha_{j-1}| t^j + \sum_{j=0}^{n-1} (|\beta_{j-1}| + t|\beta_j|) t^j$$

$$= \sum_{j=0}^{k} (t\alpha_j - \alpha_{j-1}) t^j + \sum_{j=k+1}^{n-1} (\alpha_{j-1} - t\alpha_j) t^j + \sum_{j=0}^{n-1} (|\beta_{j-1}| + t|\beta_j|) t^j$$

$$= \sum_{j=0}^{k} (t^{j+1}\alpha_j - t^j \alpha_{j-1}) + \sum_{j=k+1}^{n-1} (t^j \alpha_{j-1} - t^{j+1}\alpha_j)$$

$$+ \sum_{j=0}^{n-1} (t^j |\beta_{j-1}| + t^{j+1}|\beta_j|)$$

$$= 2t^{k+1}\alpha_k - t^n \alpha_{n-1} + 2t \sum_{j=0}^{n-1} t^j |\beta_j| + t^n |\beta_{n-1}|$$

Using this in (2.30), we get for $|z| > t,$

$$|F(z)| \geq |z|^n\{|a_n z + \alpha_{n-1} - t\alpha_n| - |\beta_{n-1}| - t|\beta_n| - \frac{1}{t^n}[2t^{k+1}\alpha_k - t^n\alpha_{n-1}$$

$$+ 2t\sum_{j=0}^{n-1} t^j|\beta_j| + t^n|\beta_{n-1}|]\}$$

$$= |z|^n\{|a_n z + \alpha_{n-1} - t\alpha_n| - |\beta_{n-1}| - t|\beta_n| - \frac{2t^{k+1}\alpha_k}{t^n} + \alpha_{n-1}$$

$$-\frac{2t}{t^n}\sum_{j=0}^{n-1} t^j|\beta_j| - |\beta_{n-1}|\}$$

$$> 0,$$

whenever

$$|a_n z + \alpha_{n-1} - t\alpha_n| > \frac{2t^{k+1}\alpha_k}{t^n} + (t|\beta_n| - \alpha_{n-1}) + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|.$$

Hence all the zeros of $F(z)$ whose modulus is greater than t lie in the circle

$$\left|z + \frac{\alpha_{n-1} - t\alpha_n}{a_n}\right| \leq \frac{1}{|a_n|}\left\{\left(\frac{2\alpha_k t^k}{t^n} - \alpha_n\right) + t|\beta_n| + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|\right\}. \tag{2.31}$$

Now, if $|z| \leq t$, then we have

$$|a_n z + \alpha_{n-1} - t\alpha_n| \leq |a_n|t + |\alpha_{n-1} - t\alpha_n|$$

$$\leq t\alpha_n + t|\beta_n| + \alpha_{n-1} - t\alpha_n \qquad \text{(by 2.28)}$$

$$= t|\beta_n| + \alpha_{n-1}$$

$$= t|\beta_n| + 2\alpha_{n-1} - \alpha_{n-1}$$

$$\leq t|\beta_n| + \frac{2t^{k+1}\alpha_k}{t^n} - \alpha_{n-1} \qquad \text{(by 2.28)}$$

$$\leq t|\beta_n| + \frac{2t^{k+1}\alpha_k}{t^n} - \alpha_{n-1} + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|$$

$$= \left(\frac{2t^{k+1}\alpha_k}{t^n} - \alpha_{n-1}\right) + t|\beta_n| + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|.$$

This implies,

$$\left|z + \frac{\alpha_{n-1} - t\alpha_n}{a_n}\right| \le \frac{1}{|a_n|}\left\{\left(\frac{2t^{k+1}\alpha_k}{t^n} - \alpha_{n-1}\right) + t|\beta_n| + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|\right\}.$$

This shows that all the zeros of $F(z)$ whose modules is less than or equal to t also satisfy the inequality (2.29). Thus we conclude that all the zeros of $F(z)$ lie in the circle defined by (2.29). Since all the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $P(z)$ lie in the circle defined by (2.29). This completes the proof of Theorem 2.18.

REMARK 2.6. To verify that Theorem 2.12 is an improvement of Theorem 2.10 for $0 \le k \le n \quad 1$. we have to show the circle defined by (2.29) is contained in the circle defined by (2.20). For this let $z = w$ be any point belonging to the circle defined by (2.29), then we have

$$\left|w + \frac{\alpha_{n-1} - t\alpha_n}{a_n}\right| \le \frac{1}{|a_n|}\left\{\left(\frac{2t^{k+1}\alpha_k}{t^n} - \alpha_{n-1}\right) + t|\beta_n| + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|\right\}.$$

This implies

$$|w| = \left|w + \frac{\alpha_{n-1} - t\alpha_n}{a_n} - \frac{\alpha_{n-1} - t\alpha_n}{a_n}\right|$$

$$\le \frac{|\alpha_{n-1} \quad t\alpha_n|}{|a_n|} + \left|w + \frac{\alpha_{n-1} \quad t\alpha_n}{a_n}\right|$$

$$\le \frac{1}{|a_n|}\left\{|\alpha_{n-1} - t\alpha_n| + \frac{2\alpha_k}{t^{n-k-1}} - \alpha_{n-1} + t|\beta_n| + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|\right\}$$

$$= \frac{1}{|\alpha_n|}\left\{\alpha_{n-1} - t\alpha_n + \frac{2\alpha_k}{t^{n-k-1}} - \alpha_{n-1} + t|\beta_n| + \frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|\right\}$$

$$\leq t\left(\frac{2\alpha_k}{\alpha_n t^{n-k}}-1\right)+\frac{1}{\alpha_n}\left[t|\beta_n|+\frac{2}{t^{n-1}}\sum_{j=0}^{n-1}|\beta_j|\right]$$

$$= t\left(\frac{2t^k\alpha_k}{\alpha_n t^n}-1\right)+\frac{2}{\alpha_n}\sum_{j=0}^{n}\frac{|\beta_j|}{t^{n-j-1}}.$$

Hence the point $z = w$ belongs to the circle defined by (2.20) and therefore, the circle defined by (2.29) is contained in the circle defined by (2.20).

Chapter - 3

Composite Polynomials and Some Generalizations of Graces Theorem

The relative position of the real zeros and the zeros of the derivative of a real differentiable function is described in the well known Rolleӧs theorem which states that, ñBetween any two zeros of a real differentiable function $f(x)$ lies at least one zero of its derivativeò. It is a theorem which one meets in any introductory course of calculus. Yet its extension to the complex plane is by no means trivial. In fact Rolle ӧs theorem is not generally true for arbitrary analytic function of a complex variable. For example, consider the function

$$f(z) = e^{2\pi iz} - 1.$$

We have

$$f(1) = e^{2\pi i} - 1 = 0, \qquad \text{and} \quad f(0) = 1 - 1 = 0,$$

that is, $f(z)$ vanishes for $z = 0$ and $z = 1$, but its derivative

$$f'(z) = 2\pi i e^{2\pi iz}$$

never vanishes. This leads to the question as to what generalizations or analogues of Rolleӧs theorem are valid for at least a suitably restricted class of analytic functions such as polynomials in a complex variable. This question is answered not with respect to Rolleӧs theorem, but rather with respect to a particular corollary of Rolleӧs theorem. It says that, any interval of the real axis which contains all the zeros of a polynomial $P(z)$, also contains all the zeros of its derivative $P'(z)$. But this result is only a special case of the following result which is known as Gauss-Lucas theorem [37, p.22].

THEOREM 3.1. If all the zeros of the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

of degree n, lie in the circle $|z| \le R$, then all the zeros of its derivative $P'(z)$ also lie in

$$|z| \le R.$$

PROOF OF THEOREM 3.1. Let $z_1, z_2, \cdots, z_n$ be the zeros of $P(z)$, then

$$|z_j| \le R, \quad \text{for all } j = 0,1,\cdots,n$$

and

$$P(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n)$$

$$= a_n \prod_{j=1}^{n} (z - z_j).$$

This gives,

$$\mathrm{Log} P(z) = \mathrm{Log} a_n + \sum_{j=1}^{n} \mathrm{Log}(z - z_j).$$

Differentiating both sides with respect to z, we get

$$\text{(3.1)} \qquad \frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{(z - z_j)}.$$

Now let w be a zero of $P'(z)$. If w is also a zero of $P(z)$, then $|w| \le R$ and therefore the result follows. So we suppose that w is not a zero of $P(z)$, then $P(w) \ne 0$, so that $w \ne z_j$, for any $j = 1,2,\cdots,n$, and $P'(w) = 0$. Now from (3.1) we have

$$\sum_{j=1}^{n} \frac{1}{(w - z_j)} = \frac{P'(w)}{P(w)} = 0.$$

This gives

$$\sum_{j=1}^{n} \frac{1}{\overline{w} - \overline{z}_j} = 0,$$

which implies

$$\sum_{j=1}^{n} \frac{(w - z_j)}{(\overline{w} - \overline{z}_j)(w - z_j)} = 0,$$

or equivalently

$$\sum_{j=1}^{n} \frac{(w - z_j)}{|w - z_j|^2} = 0,$$

so that

$$\sum_{j=1}^{n} \frac{w}{|w - z_j|^2} = \sum_{j=1}^{n} \frac{z_j}{|w - z_j|^2},$$

which gives

$$\left| w \sum_{j=1}^{n} \frac{1}{|w - z_j|^2} \right| = \left| \sum_{j=1}^{n} \frac{z_j}{|w - z_j|^2} \right|$$

and hence

$$|w| \sum_{j=1}^{n} \frac{1}{|w - z_j|^2} \leq \sum_{j=1}^{n} \frac{|z_j|}{|w - z_j|^2}$$

$$\leq R \sum_{j=1}^{n} \frac{1}{|w - z_j|^2},$$

from which we conclude that $|w| \leq R$. Since w is an arbitrary zero of $P'(z)$, it follows that all the zeros of $P'(z)$ lie in the circle $|w| \leq R$ and this completes the proof of Theorem 3.1.

COROLLORY 3.1. If all the zeros of the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

of degree n lie in the circle $|z-c| \leq R$, then all the zeros of $P'(z)$ also lie in the circle

$$|z-c| \leq R.$$

PROOF OF COROLLORY 3.1. Since the polynomial $P(z)$ has all its zeros in $|z-c| \leq R$, it follows that all the zeros of the polynomial $P(z+c)$ lie in $|z| \leq R$. Hence by Theorem 3.1, all the zeros of $P'(z+c)$ also lie in $|z| \leq R$. Replacing z by $z-c$, we conclude that all the zeros $P'(z)$ lie in $|z-c| \leq R$, which is the desired result.

POLAR DERIVATIVE OF A POLYNOMAIL.

Let $P(z)$ be a polynomial of degree n and α be real or complex number, then the polar derivative $D_\alpha P(z)$ of $P(z)$ with respect to α is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Clearly the polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha - z} = P'(z).$$

Now an nth degree polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

may be regarded as a rational function which has an nth order pole at infinity. A natural generalization of Gauss-Lucas theorem is therefore, the following, given by Laguerre in 1880 which concerns with the zeros of the derivative of the function

$$F(z) - \frac{P(z)}{(z-\alpha)^n}.$$

THEOREM 3.2. If all the zeros $z_1, z_2, \cdots, z_n$, of a polynomial $P(z)$, of degree n lie in a circular region C and if w is any zero of $D_\alpha P(z)$, the polar derivative of

$P(z)$, then not both points w and α may lie outside C. Further, if $P(w) \neq 0$, any circle K through w and α either pass through all these zeros or separates these zeros.

Here by a circular region we mean the closure of not merely the interior of a circle but also the exterior of a circle or a half plane.

Several proofs of Laguerreĝs theorem can be found in [36], [37] and [41]. But these are based mainly on considerations from mechanics (spherical and plane fields of forces, points of equilibrium, centre of mass etc). Here we shall present a new simple and purely analytic proof of this theorem, due to A. Aziz [1] which in essence involves no considerations from mechanics. This proof is based on the following lemma which is also of independent interest.

LEMMA 3.1. If $z_1, z_2, \cdots, z_n$ are the zeros of a polynomial of degree n and if w is any zeros of $D_\alpha P(z)$ such that $P(w) \neq 0$, then for every complex number c

$$(3.2) \qquad (w-c)\left[\sum_{j=1}^{n} \frac{1}{|w-z_j|^2} - \frac{n}{|w-\alpha|^2}\right] = \left[\sum_{j=1}^{n} \frac{z_j - c}{|w-z_j|^2} - \frac{n(\alpha - c)}{|w-\alpha|^2}\right]$$

and

$$(3.3) \qquad |w-c|^2\left[\sum_{j=1}^{n} \frac{1}{|w-z_j|^2} - \frac{n}{|w-\alpha|^2}\right] = \left[\sum_{j=1}^{n} \frac{|z_j - c|^2}{|w-z_j|^2} - \frac{n|\alpha - c|^2}{|w-\alpha|^2}\right].$$

PROOF OF LEMMA 3.1. Let w be a zero of $D_\alpha P(z)$, then we have

$$(3.4) \qquad [D_\alpha P(z)]_w = nP(w) + (\alpha - w)P'(w) = 0.$$

Since $P(w) \neq 0$, therefore $w \neq \alpha$ and from (3.4), we get

$$(3.5) \qquad \frac{P'(w)}{P(W)} = \frac{n}{w-\alpha}.$$

If $z_1, z_2, \cdots, z_n$ are the zeros of $P(z)$, then $w \neq z_j$, $j = 1, 2, \cdots, n$ and (3.5) implies

$$(3.6) \qquad \sum_{j=1}^{n} \frac{1}{(w-z_j)} = \frac{n}{w-\alpha},$$

which can be written as

$$\sum_{j=1}^{n}\frac{1}{(w-c)-(z_j-c)}=\frac{n}{(w\quad c)\quad(\alpha\quad c)}$$

where c is given real or complex number. That is,

$$\text{(3.7)}\qquad \sum_{j=1}^{n}\frac{1}{(W-Z_j)}=\frac{n}{W-\beta}$$

where $\quad W=w-c,\quad Z_j=z_j-c,\ j=1,2,\cdots,n,\ \beta=\alpha-c\ \ and\ \ W\neq\beta.$

This gives

$$\sum_{j=1}^{n}\frac{1}{\overline{W-Z_j}}=\frac{n}{\overline{W-\beta}}$$

so that

$$\sum_{j=1}^{n}\frac{W-Z_j}{\overline{(W-Z_j)}(W-Z_j)}=\frac{n(W-\beta)}{\overline{(W-\beta)}(W-\beta)},$$

or equivalently

$$\text{(3.8)}\qquad W\left[\sum_{j=1}^{n}\frac{1}{|W-Z_j|^2}-\frac{n}{|W-\beta|^2}\right]=\left[\sum_{j=1}^{n}\frac{Z_j}{|W-Z_j|^2}-\frac{n\beta}{|W-\beta|^2}\right].$$

Now replacing W by $w-c$, Z_j by z_j-c and β by $\alpha-c$ in (3.8), we obtain (3.2) and this proves the first part of the Lemma 3.1.

To prove the second part of the lemma, we write (3.7) in the form

$$\sum_{j=1}^{n}\left(\frac{1}{W-Z_j}-\frac{1}{W-\beta}\right)=0.$$

This gives

$$\sum_{j=1}^{n}\frac{Z_j-\beta}{W-Z_j}=0,\qquad\qquad (\text{Since } W \neq\beta)$$

which implies by (3.7)

$$\text{(3.9)} \qquad \sum_{j=1}^{n} \frac{Z_j(\bar{W}-\bar{Z}_j)}{|W-Z_j|^2} = \sum_{j=1}^{n} \frac{\beta}{W \quad Z_j} = \frac{n\beta}{W \quad \beta}.$$

Now (3.9) can be written as

$$\sum_{j=1}^{n} \frac{\bar{W}Z_j}{|W-Z_j|^2} - \frac{n\beta(\bar{W}-\bar{\beta})}{|W-\beta|^2} = \sum_{j=1}^{n} \frac{|Z_j|^2}{|W-Z_j|^2}.$$

This gives

$$\text{(3.10)} \qquad \bar{W}\left[\sum_{j=1}^{n} \frac{Z_j}{|W-Z_j|^2} - \frac{n\beta}{|W-\beta|^2}\right] = \left[\sum_{j=1}^{n} \frac{|Z_j|^2}{|W-Z_j|^2} - \frac{n|\beta|^2}{|W-\beta|^2}\right].$$

Now multiplying the two sides of (3.8) by $\bar{W}$, we get

$$|W|^2\left[\sum_{j=1}^{n} \frac{1}{|W-Z_j|^2} - \frac{n}{|W-\beta|^2}\right] = \bar{W}\left[\sum_{j=1}^{n} \frac{Z_j}{|W-Z_j|^2} - \frac{n\beta}{|W-\beta|^2}\right].$$

using this in (3.10), we obtain

$$\text{(3.11)} \qquad |W|^2\left[\sum_{j=1}^{n} \frac{1}{|W-Z_j|^2} - \frac{n}{|W-\beta|^2}\right] = \left[\sum_{j=1}^{n} \frac{|Z_j|^2}{|W-Z_j|^2} - \frac{n|\beta|^2}{|W-\beta|^2}\right].$$

Replacing W by $w-c$, Z_j by $z_j - c$ and β by $\alpha - c$ in (3.11), we obtain (3.3) and the lemma is proved completely.

PROOF OF THEOREM 3.2. If

$$P(z) = (z-\alpha)^n,$$

then

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)P'(z) \\ &= nP(z) + (\alpha - z)n(z-\alpha)^{n-1} \\ &= n(z-\alpha)^n - n(z-\alpha)^n \\ &= 0. \end{aligned}$$

Therefore, we suppose that $P(z) \neq (z-\alpha)^n$, so that $D_\alpha P(z) \neq 0$. Assume that all the zeros $z_1, z_2, \cdots, z_n$ of $P(z)$ lie in a circular region C and let w be a zero of $D_\alpha P(z)$. If w is also zero of $P(z)$, then w lies in C and the result of first part of the theorem follows. Henceforth we assume that $P(w) \neq 0$. Now from (3.6) we have

$$\left|\frac{n}{w-\alpha}\right| = \left|\sum_{j=1}^{n} \frac{1}{W-Z_j}\right|$$

$$\leq \sum_{j=1}^{n} \frac{1}{|W-Z_j|},$$

with equality sign holding only if

$$\frac{1}{W-Z_j} = \left|\frac{1}{W-Z_j}\right| e^{i\theta} = \frac{1}{|W-Z_j|} e^{i\theta} = r_j e^{i\theta}, \qquad \text{(say)}$$

for $j = 1,2,\cdots,n$, θ real. Using Cauchy-Schwartz inequality, it follows that

$$\frac{n^2}{|w-\alpha|^2} \leq \left[\sum_{j=1}^{n} 1.\frac{1}{|W-Z_j|}\right]^2$$

$$\leq \sum_{j=1}^{n} 1^2 . \sum_{j=1}^{n} \frac{1}{|W-Z_j|^2}$$

$$= n \sum_{j=1}^{n} \frac{1}{|W-Z_j|^2},$$

where now equality holds on the right hand side of inequality only if

$$r_1 = r_2 = \cdots = r_n = r \qquad \text{(say)}.$$

Thus equality sign in both the inequalities holds only if

$$\frac{1}{W-Z_1} = \frac{1}{W-Z_2} = \cdots = \frac{1}{W-Z_n} = re^{i\theta}.$$

This gives with the help of (3.6) that

$$Z_1 = Z_2 = \cdots = Z_n = \alpha,$$

So that $P(z) = (z-\alpha)^n$, which is not the case. Hence in fact, we have

$$(3.12) \qquad B = \sum_{j=1}^{n} \frac{1}{|W - Z_j|^2} \quad \frac{n}{|W-\alpha|^2} > 0.$$

To prove the result, we shall consider the three cases of C separately.

Case 1. Let $C\colon |z-c| \le R$, $R > 0$.

By second part of Lemma 3.1, we have

$$(3.13) \qquad B|w-c|^2 = \sum_{j=1}^{n} \frac{|z_j - c|^2}{|W - Z_j|^2} - \frac{n|\alpha - c|^2}{|W-\alpha|^2}.$$

Now let us assume that w lies exterior to $C\colon |z-c| \le R$, then $|w-c| > R$. Since $|z_j - c| \le R,\ j = 1,2,\cdots,n,$ it follows from (3.13) that

$$BR^2 < B|w-c|^2$$

$$= \sum_{j=1}^{n} \frac{|z_j - c|^2}{|W - Z_j|^2} - \frac{n|\alpha - c|^2}{|W-\alpha|^2}$$

or

$$\frac{n|\alpha - c|^2}{|w-\alpha|^2} < \sum_{j=1}^{n} \frac{|z_j - c|^2}{|W - Z_j|^2} - BR^2,$$

which gives with the help of (3.12) that

$$\frac{n|\alpha - c|^2}{|w-\alpha|^2} < \sum_{j-1}^{n} \frac{R^2}{|W - Z_j|^2} - \sum_{j-1}^{n} \frac{R^2}{|W - Z_j|^2} + \frac{nR^2}{|W-\alpha|^2}$$

and this implies

$$|\alpha - c| < R.$$

Similarly if $|\alpha - c| > R$, that is, if α lies exterior to $C\colon |z-c| \le R$, then from (3.13), we get

$$B|w-c|^2 < \sum_{j=1}^{n} \frac{R^2}{|W-Z_j|^2} - \frac{nR^2}{|W-\alpha|^2},$$

which gives with the help of (3.12) that

$$B|w-c|^2 < R^2 \left[\sum_{j=1}^{n} \frac{1}{|W-Z_j|^2} - \frac{n}{|W-\alpha|^2} \right]$$

$$= R^2 B,$$

and hence

$$|w-c| < R.$$

Thus in this case, not both points α and w may lie outside of $C: |z-c| \le R$.

Case 2. Let now $C: |z-c| \ge r, \quad r > 0.$

Since all the zeros of $P(z)$ lie in C, therefore,

$$|z_j - c| \ge r, \qquad j = 1,2, \cdots, n.$$

If w lies exterior to $C: |z-c| \ge r$, then $|w-c| < r$ and from (3.13), we get with the help of (3.12) that

$$Br^2 > B|w-c|^2$$

$$\ge \sum_{j=1}^{n} \frac{r^2}{|W-Z_j|^2} - \frac{n|\alpha-c|^2}{|W-\alpha|^2}$$

$$= Br^2 + \frac{nr^2}{|W-\alpha|^2} - \frac{n|\alpha-c|^2}{|W-\alpha|^2}.$$

This gives

$$|\alpha - c| > r.$$

If now α lies exterior to $C: |z-c| \ge r$, that is, if $|\alpha - c| < r$, then from (3.13) and with the help of (3.12) we get

$$B|w-c|^2 \geq \sum_{j=1}^{n} \frac{r^2}{|W-Z_j|^2} - \frac{\text{n}|\alpha-c|^2}{|W-\alpha|^2}$$

$$> \sum_{j=1}^{n} \frac{r^2}{|W-Z_j|^2} - \frac{\text{n}r^2}{|W-\alpha|^2}$$

$$= r^2 \left[\sum_{j=1}^{n} \frac{1}{|W-Z_j|^2} - \frac{\text{n}}{|W-\alpha|^2} \right]$$

$$= r^2 B,$$

which implies that $|w-c| > r$. Thus in this case also not both the points α and w may lie outside $|z-c| \geq r$.

Case 3. Finally let C be a half plane, that is, let $Re(z) \leq a$, or $C: Re(z) \geq b$, or $C: Im(z) \leq a'$, or $C: Im(z) \geq b'$, where a, b, a', b' are real numbers . We will prove the result for one of these four cases, say $C: \text{Re}\,(z) \leq a$. The remaining three cases will follow in a similar way. Now we have $C: Re(z) \leq a,$ Therefore, $Re(z_j) \leq a$, $j = 1,2,\cdots,n.$ Since all the zeros of $P(z)$ lie in C. Now by the first part of Lemma 3.1 (with $c = 0$), we have

$$(3.14) \qquad wB = \sum_{j=1}^{n} \frac{Z_j}{|W-Z_j|^2} - \frac{n}{|w-\alpha|^2},$$

where $B > 0$ is defined by (3.13).

We now assume that $\text{Re}\,(w) > a$, that is, w lies exterior to $C: Re(z) \leq a$, then from (3.14) we obtain

$$aB \leq \text{Re}\,(w)B$$

$$\leq \sum_{j=1}^{n} \frac{Re(Z_j)}{|W-Z_j|^2} - \frac{nRe(\alpha)}{|w-\alpha|^2}$$

$$\leq a\left(\sum_{j=1}^{n} \frac{1}{|W-Z_j|^2} \right) - \frac{nRe(\alpha)}{|w-\alpha|^2}, \quad (\text{ since } Rez_j \leq a\,)$$

$$\frac{nRe(\alpha)}{|w-\alpha|^2} < a\sum_{j=1}^{n}\frac{1}{|W-Z_j|^2} - aB$$

$$= a\left[\sum_{j=1}^{n}\frac{1}{|W-Z_j|^2} - B\right].$$

This gives with the help of (3.12) that

$$\frac{nRe(\alpha)}{|w-\alpha|^2} < \frac{na}{|w-\alpha|^2},$$

so that $Re(z) \le a$, that is α lies exterior to C. Now if we assume that $Re(\alpha) > a$, that is α lies exterior to C: $Re(z) \le a$, then again from (3.14) we have

$$\text{Re}(w)B = \sum_{j=1}^{n}\frac{Re(Z_j)}{|W-Z_j|^2} - \frac{nRe(\alpha)}{|w \quad \alpha|^2}$$

$$< a\sum_{j=1}^{n}\frac{1}{|W-Z_j|^2} - \frac{na}{|w-\alpha|^2}$$

$$= a\left[\sum_{j=1}^{n}\frac{1}{|W-Z_j|^2} - \frac{n}{|w-\alpha|^2}\right]$$

$$= aB. \qquad\qquad \text{by} \quad (3.13)$$

This gives

$$Re(\alpha) < a.$$

This shows not both points w and α may lie outside of C: $Re(z) \le a$. Hence the first part of the theorem is completely proved.

To prove the second part of the theorem, let us suppose first that a circle K: $|z-c| \le r$, through w and α has atleast one z_j in its interior, no z_j in its exterior and the remaining z_j's on its circumference.Then we have

$$|w-c| = r, \qquad |\alpha - c| = r, \qquad |z_j - c| < r,$$

for at least one j and $|z_j - c| = r$ for the remaining z_j's. This gives

$$\sum_{j=1}^{n} \frac{|z_j - c|^2}{|W - Z_j|^2} < \sum_{j=1}^{n} \frac{r^2}{|W - Z_j|^2},$$

Using this in (3.13), we obtain

$$Br^2 < \sum_{j=1}^{n} \frac{r^2}{|W - Z_j|^2} - \frac{nr^2}{|W - \alpha|^2}$$

$$= r^2 \left[\sum_{j=1}^{n} \frac{1}{|W - Z_j|^2} - \frac{n}{|w - \alpha|^2} \right]$$

$$= r^2 B,$$

which is obviously a contradiction. Since we also get a contradiction if we assume that a circle K through w and α has atleast one z_j in its exterior, no z_j in its interior and the remaining z_j's on its circumference, we conclude that any circle through α and w either passes through all the zeros of $P(z)$ or separates these zeros. This completes the proof of the theorem in full.

As an application of Laguerrerês theorem we shall next present a result which is known as Graces theorem and which concerns with the relative location of the zeros of two apolar polynomials $P(z)$ and $Q(z)$. But before we state this result, we shall define first apolar polynomials.

DEFINITION. Two polynomials

$$P(z) = \sum_{j=0}^{n} C(n,j) A_j z^j$$

$$= C(n,0)A_0 + C(n,1)A_1 z^1 + \cdots + C(n,n-1)A_{n-1} z^{n-1} + C(n,n)A_n z^n$$

and

$$Q(z) = \sum_{j=0}^{n} C(n,j) B_j z^j$$

$$- C(n,0)B_0 + C(n,1)B_1 z^1 + \cdots + C(n,n-1)B_{n-1} z^{n-1} + C(n,n)B_n z^n$$

$A_nB_n \neq 0$ where $C(n,j)$ denote the binomial coefficient $\frac{n!}{(n-j)!j!}$, of the same degree n are said to be apolar if their coefficients satisfies the relation

$$(3.15) \qquad C(n,0)A_oB_n - C(n,1)A_1B_{n-1}z^1 + C(n,2)A_2B_{n-2}z^2 + \cdots + (-1)^{n-1}C(n,n-1)A_{n-1}B_1z^{n-1} + (-1)^nC(n,n)A_nB_0z^n = 0.$$

Clearly, there are infinite number of polynomials which are apolar to a given polynomial, for example, the polynomial z^3+1 is apolar to the polynomial $z^3+3\alpha z^2+3\beta z+1$ for any choice of constants α and β. Since the condition of apolarity namely

$$C(3,0)1.1 - C(3,1).0.\beta.z^1 + C(3,2).0.\alpha.z^2 + C(3,3)1.1$$

$$= 1 - 0 + 0 - 1 = 0$$

is satisfied for any choice of the constants α and β.

As to the relative location of the zeros of the polynomials $P(z)$ and $Q(z)$ we have the following fundamental result due to Grace $[37,\ p.81]$ and $[41,\ p.57]$.

THEOREM 3.3 (GRACES THEORM). If

$$P(z) = \sum_{j=0}^{n} C(n,j)A_jz^j$$

and

$$Q(z) = \sum_{j=0}^{n} C(n,j)B_jz^j \quad , \qquad A_nB_n \neq 0$$

are apolar polynomials and if one of them has all its zeros in a circular region C, then the other will also have atleast one zero in C.

For the proof of Theorem 3.3 we need the following lemma.

LEMMA 3.2. If

$$P(z) = \sum_{j=0}^{n} C(n,j)A_jz^j,$$

is a polynomial of degree n and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are n arbitrary real or complex numbers, then the kth polar derivative

$$P_k(z) = (n-k+1)P_{k-1}(z) + (\lambda_k - z)P'_{k-1}(z),$$

of $P(z)$ with $P_0(z) = P(z)$, can be written in the form

$$P_k(z) = \sum_{j=0}^{n-k} C(n-k, j)A_j^{(k)} z^j,$$

where

$$A_j^{(k)} = n(n-1)(n-2)\cdots(n-k+1)\sum_{i=0}^{k} S(k,i)A_{i+j}$$

and $S(k,i)$ being the symmetric function consisting of the sum of all products of $\lambda_1, \lambda_2, \cdots, \lambda_n$ taken i at a time.

PROOF OF LEMMA 3.2. We have

$$P(z) = C(n,0)A_0 + C(n,1)A_1 z^1 + \cdots + C(n,j)A_j z^j + \cdots + C(n,n)A_n z^n$$

$$(3.16) \qquad P_k(z) = C(n-k,0)A_0^{(k)} + C(n-k,1)A_1^{(k)} z + \cdots + C(n-k, n-k)A_{n-k}^{(k)} z^{n-k}.$$

Also

$$(3.17) \qquad P_k(z) = (n-k+1)P_{k-1}(z) + (\lambda_k - z)P'_{k-1}(z).$$

Substituting the value of $P_{k-1}(z)$ and $P_k(z)$ from (3.16) in (3.17), we get

$$\sum_{j=0}^{n-k} C(n-k,j)A_j^{(k)} z^j = (n-k+1)\sum_{j=0}^{n-k+1} C(n-k+1,j)A_j^{(k-1)} z^j + (\lambda_k - z)\sum_{j=0}^{n-k+1} jC(n-k+1,j)A_j^{(k-1)} z^{j-1}.$$

Equating the coefficients of z^j, we obtain

$$C(n-k,j)A_j^{(k)} = (n-k+1)C(n-k+1,j)A_j^{(k-1)} - jC(n-k+1,j)A_j^{(k-1)}$$

$$+\lambda_k(j+1)C(n-k+1,j+1)A_{j+1}^{(k-1)}.$$

or $$C(n-k,j)A_j^{(k)} = A_j^{(k-1)}(n-k+1-j)\frac{(n-k+1)!}{j!\,(n-k+1-j)!}$$

$$+\lambda_k(j+1)A_{j+1}^{(k-1)}\frac{(n-k+1)!}{(j+1)!\,(n-k-j)!}$$

$$= (n-k+1)C(n-k,j)A_j^{(k-1)} + \lambda_k(n-k+1)A_{j+1}^{(k-1)}C(n-k,j),$$

which yields ,

(3.18) $$A_j^{(k)} = (n-k+1)A_j^{(k-1)} + \lambda_k(n-k+1)A_{j+1}^{(k-1)}.$$

Let us now show by repeated applications of (3.18), we may derive the formula

(3.19) $$A_j^{(k)} = n(n-1)(n-2)\cdots(n-k+1)\sum_{i=1}^{k} S(k,i)A_{i+j}\,,$$

where $S(k,i)$ is the symmetric function consisting of the sum of all possible products $\lambda_1,\lambda_2,\cdots,\lambda_n$ taken i at a time. First we note that for $k=1$, we have

$$A_j^{(1)} = n\sum_{i=0}^{1} s(1,i)A_{i+j}$$

$$= n\left[s(1,0)A_j + s(1,1)A_{1+j}\right]$$

$$= n\left[A_j + \lambda_1 A_{1+j}\right],$$

that is

$$A_j^{(1)} = nA_j + n\lambda_1 A_{1+j}.$$

Also from (3.18), we have

$$A_j^{(1)} = n\left[A_j^{(0)} + \lambda_1 A_{j+1}^{(0)}\right]$$

$$= nA_j + n\lambda_1 A_{j+1},$$

which shows that the result is true for $k = 1$. Supposing that the result to be true for k, we prove that the result holds for $k+1$ also. From (3.18), with k replaced by $k+1$, we get

$$A_j^{(k+1)} = (n-k)\left[A_j^{(k)} + \lambda_{k+1} A_{j+1}^{(k)}\right]$$

$$- (n-k)A_j^{(k)} + (n-k)\lambda_{k+1} A_{j+1}^{(k)}.$$

Using induction hypothesis, we get

$$A_j^{(k+1)} - (n-k)\left[n(n-1)\cdots(n-k+1)\sum_{i=0}^{k} s(k,i)A_{i+j}\right]$$

$$+\lambda_{k+1}(n-k)\left[n(n-1)\cdots(n-k+1)\sum_{i=0}^{k} s(k,i)A_{i+j+1}\right]$$

$$= n(n-1)\cdots(n-k+1)(n-k)\left[\sum_{i=0}^{k} s(k,i)A_{i+j} + \lambda_{k+1}\sum_{i=0}^{k} s(k,i)A_{i+j+1}\right]$$

$$= n(n-1)\cdots(n-k+1)(n-k)\left[\sum_{i=0}^{k+1} s(k,i)A_{i+j} + \lambda_{k+1}\sum_{i=1}^{k+1} s(k,i-1)A_{i+j}\right],$$

(since in the first sum $s(k,k+1) = 0$

and in the second sum replace i by $i-1$)

or

$$A_j^{(k+1)} = n(n-1)\cdots(n-k+1)(n-k)$$

$$\left[\sum_{i=0}^{k+1} s(k,i)A_{i+j} + \lambda_{k+1}\sum_{i=0}^{k+1} s(k,i-1)A_{i+j}\right]$$

(Since in the second sum $s(k,-1) = 0$)

$$= n(n-1)\cdots(n-k)\left[\sum_{i=0}^{k+1}(s(k,i) + \lambda_{k+1}s(k,i-1))A_{i+j}\right]$$

$$= n(n-1)\cdots(n-k)\sum_{i=0}^{k+1} s(k+1,i)A_{i+j}\ ,$$

(Since $s(k,i) + \lambda_{k+1} s(k, i-1) = s(k+1, i)$)

which shows that the result is true for $k+1$ as well. Since the result is already true for $k=1$, it follows by mathematical induction that the result is true for all integers $k \geq 1$. This completes the proof of Lemma 3.2.

PROOF OF THEOREM 3.3. Since

$$P(z) = \sum_{j=0}^{n} C(n,j) A_j z^j$$

$$= C(n,0)A_0 + C(n,1)A_1 z^1 + \cdots + C(n,n-1)A_{n-1}z^{n-1} + C(n,n)A_n z^n.$$

And

$$Q(z) = \sum_{j=0}^{n} C(n,j) B_j z^j$$

$$= C(n,0)B_0 + C(n,1)B_1 z^1 + \cdots + C(n,n-1)B_{n-1}z^{n-1} + C(n,n)B_n z^n$$

$A_n B_n \neq 0$, therefore, if $w_1, w_2, \cdots, w_n$ are the zeros of $Q(z)$, then

$$Q(z) = B_n(z-w_1)(z-w_2)\cdots(z-w_n),$$

so that

$$B_n(z-w_1)(z-w_2)\cdots(z-w_n) = \sum_{j=0}^{n} C(n,j) B_j z^j.$$

Equating the coefficients of like powers of z on the two sides, we get

$$(3.20) \quad \left[\begin{array}{l} S(n,1) = \sum w_1 = -C(n,n-1)\dfrac{B_{n-1}}{B_n} \\ S(n,2 = \sum w_1 w_2 = C(n,n-2)\dfrac{B_{n-2}}{B_n} \\ \quad\vdots \\ S(n,j) = \sum w_1 w_2 \cdots w_j = (-1)^j C(n,j)\dfrac{B_{n-j}}{B_n} \end{array} \right. ,$$

where $j = 1,2,\cdots,n$ and $S(n,j)$ is the symmetric function consisting of the sum of all possible products of $w_1, w_2, \cdots, w_n$ taken j at a time. Since $P(z)$ and $Q(z)$ are apolar, we have

$$(3.21) \qquad C(n,0)A_o B_n - C(n,1)A_1 B_{n-1} z^1 + C(n,2)A_2 B_{n-2} z^2 + \cdots +$$

$$(-1)^{n-1} C(n,n-1) A_{n-1} B_1 z^{n-1} + (-1)^n C(n,n) A_n B_0 z^n = 0$$

To prove the theorem, we suppose that all the zeros of $P(z)$ lie in a circular region C. We have to show atleast one zero of $Q(z)$ lie in C. If possible suppose that the zeros $w_1, w_2, \cdots, w_n$ lie exterior to C. Then all the zeros of the first polar derivative of $P(z)$ with respect to w_1 lies in C, that is, all the zeros of

$$P_1(z) = nP(z) + (w_1 - z)P'(z),$$

lie in C. Since w_2 lies exterior to C, it follows that all the zeros of

$$P_2(z) = (n-1)P_1(z) + (w_2 - z)P_1'(z),$$

lie in C. Proceeding like this it follows by the repeated application of Lagurrerὂs theorem that all the zeros of

$$P_k(z) = (n-k+1)P_{k-1}(z) + (w_k - z)P_k'(z), \qquad k = 1,2,\cdots,n-1,n$$

lie in C. In particular all the zeros of

$$P_{n-1}(z) = 2P_{n-2}(z) + (w_{n-1} - z)P_{n-2}'(z)$$

lie in C. But by Lemma 3.2, $P_{n-1}(z)$ can be written in the form

$$P_{n-1}(z) = \sum_{j=0}^{1} C(1,j) A_j^{(n-1)} z^j$$

$$= A_0^{(n-1)} + A_1^{(n-1)} z,$$

where

$$A_0^{(n-1)} = n! \sum_{i=0}^{n-1} s(n-1,i) A_i$$

(3.22) $$= n!\,[s(n-1,0)A_0 + s(n-1,1)A_1 + \cdots + s(n-1,n-2)A_{n-2}$$

$$+ s(n-1,n-1)A_{n-1}]$$

and

(3.23) $$A_1^{(n-1)} = n! \sum_{i=0}^{n-1} s(n-1,i)A_{i+1}$$

$$= n!\,[s(n-1,0)A_1 + s(n-1,1)A_2 + \cdots + s(n-1,n-2)A_{n-1}$$

$$+ s(n-1,n)A_n.]$$

By (3.22) and (3.23), we have

$$P_{n-1}(w_n) = A_0^{(n-1)} + A_1^{(n-1)} w_n$$

$$= n!\,[\, s(n-1,0)A_0 + s(n-1,1)A_1 + \cdots + s(n-1,n-2)A_{n-2}$$

$$+ s(n-1,n)A_{n-1} + w_n\{s(n-1,0)A_1 + s(n-1,1)A_2 + \cdots$$

$$+ s(n-1,n-2)A_{n-1} + s(n-1,n)A_n\}\,]$$

$$= n!\,[\, s(n-1,0)A_0 + \{s(n-1,1) + w_n s(n-1,0)\}A_1$$

$$+ \{s(n-1,2) + w_n s(n-1,1)\}A_2 + \cdots + \{s(n-1,n-1)$$

$$+ w_n s(n-1,n-2)\}A_{n-1} + w_n s(n-1,n-1)A_n\,].$$

Now we use the fact that

$$s(n-1,k) + w_n s(n-1,k-1) = s(n,k),\ k = 1,2,\ldots,n-1.$$

$$s(n-1,0) = s(n,0)$$

and $$w_n s(n-1,n-1) = s(n,n),$$

so that we get from above

$$P_{n-1}(w_n) = n!\,[s(n,0)A_0 + s(n,1)A_1 + \cdots +$$

$$s(n,n-1)A_{n-1} + s(n,n)A_n]$$

By using (3.20), we get

$$P_{n-1}(w_n) = n!\left[A_0 - c(n,1)\frac{B_{n-1}}{B_n}A_1 + \cdots + (-1)^{n-1}c(n,n-1)\frac{B_1}{B_n}A_{n-1}\right.$$

$$\left.+(-1)^n c(n,n)\frac{B_0}{B_n}A_n\right]$$

$$= \frac{n!}{B_n}[A_o B_n - C(n,1)A_1 B_{n-1} + \cdots + (-1)^{n-1}C(n,n-1)A_{n-1}B_1$$

$$+(-1)^n C(n,n)A_n B_0]$$

$$= 0.$$

The point w_n is therefore the zero of $P_{n-1}(z)$ and must lie in C, which is a contradiction. Hence our supposition that $Q(z)$ has all its zeros exterior to C is wrong.Therefore, it follows that $Q(z)$ must have atleast one zero in C. This completes the proof of Theorem 3.3.

Recently in a series of papers A. Aziz has studied the relative location of the zeros of polynomials

$$P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j, \qquad A_0 A_n \neq 0$$

and

$$Q(z) = \sum_{j=0}^{m} C(n,j)B_j z^j, \qquad B_0 B_m \neq 0$$

of arbitrary degree m and n respectively, $m \leq n$, when their coefficients satisfy an apolar type relation. Here we first present the following result due to A. Aziz which is a generalization of his earlier result [3].

THEOREM 3.4. If

$$P(z) = \sum_{j=0}^{n} C(n,j)A_j z^j, \qquad A_0 A_n \neq 0$$

and

$$Q(z) = \sum_{j=0}^{m} C(m,j)B_j z^j, \qquad B_0 B_m \neq 0$$

are two polynomials degree n and m respectively $m \leq n$ such that

$$c(m,0)B_0A_n - c(m,1)B_1A_{n-1} + \cdots + (-1)^m c(m,m)B_mA_{n-m} = 0,$$

then the following holds.

(i) $Q(z)$ has all its zeros in $|z-c| \geq r$, then $P(z)$ has at least one zero in $|z-c| \geq r$.

(ii) $P(z)$ has all its zeros in $|z-c| \leq r$, then $Q(z)$ has at least one zero in $|z-c| \geq r$.

For the proof of the Theorem 3.4, we need the following lemma, which is a generalization of a result due to Markovitch [37, p. 64].

LEMMA 3.3. Let

$$P(z) = \sum_{j=0}^{n} C(n,j)A_jz^j$$

and

$$Q(z) = \sum_{j=0}^{m} C(n,j)B_jz^j$$

be two polynomials of degree n and m, respectively $n \leq m$. If we form

$$U(z) = \sum_{j=0}^{n} (-1)^j P^{(n-j)}(z)Q^{(j)}(z),$$

then

$$U(z) = n! \sum_{j=0}^{n} (-1)^j c(m,j)\, A_{n-j}B_j\,.$$

PROOF OF LEMMA 3.3. Since $P(z)$ and $Q(z)$ are two polynomials of degree n and m respectively, we have

$$P^{(n+1)}(z) = 0 \qquad \text{and} \qquad Q^{(m+1)} = 0.$$

Now we can write

$$P(z) = \sum_{j=0}^{n} C(n,j)A_jz^j = \sum_{j=0}^{n} \frac{P^{(j)}(0)}{j!} z^j$$

and $$Q(z) = \sum_{j=0}^{n} C(m,j) B_j z^j = \sum_{j=0}^{m} \frac{Q^{(j)}(0)}{j!} z^j,$$

from which it follows that,

$$(3.25) \qquad P^{(j)}(0) = C(n,j) j!\, A_j, \qquad j = 0,1,\cdots,n.$$

and

$$(3.26) \qquad Q^{(j)}(0) = C(m,j) j!\, B_j, \qquad j = 0,1,\cdots,m.$$

Now

$$U'(z) = \sum_{j=0}^{n} (-1)^j P^{(n-j+1)}(z) Q^{(j)}(z)$$

$$+ \sum_{j=0}^{n} (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z),$$

so that by (3.24), we have

$$U'(z) - \sum_{j=1}^{m} (-1)^j P^{(n-j+1)}(z) Q^{(j)}(z) + \sum_{j=0}^{m-1} (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z),$$

$$= \sum_{j=0}^{m-1} (-1)^{j+1} P^{(n-j)}(z) Q^{(j+1)}(z) + \sum_{j=0}^{m-1} (-1)^j P^{(n-j)}(z) Q^{(j+1)}(z),$$

$$= 0,$$

therefore $U(z)$ is constant and thus

$$U(z) = U(0) = \sum_{j=0}^{m} (-1)^j P^{(n-j)}(0) Q^{(j)}(0).$$

Using (3.25) and (3.26), we obtain

$$U(z) = n! \sum_{j=0}^{n} (-1)^j c(m,j)\, A_{n-j} B_j.$$

This proves the lemma.

PROOF OF THEOREM 3.4. Consider

$$H(z) = P(z+c) = \sum_{j=0}^{n} \frac{P^{(j)}(c)}{j!} z^j = \sum_{j=0}^{n} c(n,j) A_j^* z^j \qquad \text{(say)}$$

and

$$G(z) = Q(z+c) = \sum_{j=0}^{m} \frac{Q^{(j)}(c)}{j!} z^j = \sum_{j=0}^{m} c(m,j) B_j^* z^j \qquad \text{(say)}.$$

$H(z)$ and $G(z)$ are two apolar polynomials of degree n and m, respectively, $m \leq n$. Now we have

$$\sum_{j=0}^{m} (-1)^j c(m,j) A_{n-j}^* B_j^*$$

$$= \sum_{j=0}^{m} (-1)^j c(m,j) \frac{P^{n-j}(c)}{c(n,n-j)(n-j)!} \cdot \frac{Q^j(c)}{c(m,j) j!}$$

$$= \frac{1}{n!} \sum_{j=0}^{m} (-1)^j P^{n-j}(c) Q^j(c).$$

Using lemma 3.3, we obtain

$$\sum_{j=0}^{m} (-1)^j c(m,j) A_{n-j}^* B_j^* = \sum_{j=0}^{m} (-1)^j c(m,j) A_{n-j} B_j$$

$$= 0, \qquad \text{(by hypothesis)}$$

this shows that the coefficients of the polynomials $H(z)$ and $G(z)$ satisfy the apolarity condition. Hence it follows from (i) and (ii) that if all the zeros of $G(z)$ lie in $|z| \geq r$, then $H(z)$ has at least one zero in $|z| \geq r$ and if all the zeros of $H(z)$ lie in $|z| \leq r$, then $G(z)$ has atleast one zero in $|z| \leq r$. Replacing z by $z - c$ and noting that $P(z) = H(z-c)$ and $Q(z) = G(z-c)$ the conclusion of Theorem 3.4 follows immediately.

As an application of Theorem 3.4, we shall present the following result, which is a generalization of Walshôs coincidence theorem [45] for the case when the circular region C is a circle $|z-c| = r$.

THEOREM 3.5. Let $G(z_1, z_2, \cdots, z_n)$ be a symmetric n −linear form of total degree m, $m \le n$, in $z_1, z_2, \cdots, z_n$ and let $C: |z-c| \le r$ be a circle containing the n points $w_1, w_2, \cdots, w_n$. Then in C, there exists at least one point w such that

$$G(w, w, \ldots, w) = G(w_1, w_2, \ldots, w_n).$$

PROOF OF THEOREM 3.5. We write

$$P(z) = \prod_{j=1}^{n} (z - z_j) = \sum_{j=0}^{n} C(n,j) A_j z^j, \qquad \text{(say)}$$

so that

$$C(n,j) = (-1)^j S(n,j) A_j z^j \qquad (3.27)$$

where $S(n,j)$ are the symmetric functions consisting of the sum of all possible products of $z_1, z_2, \cdots, z_n$, taken j at a time. Now if $G(w_1, w_2, \cdots, w_n) = G^*$. Then the difference $G(z_1, z_2, \cdots, z_n) - G^*$ is linear, symmetric and of total degree $m \le n$, in the variables $z_1, z_2, \cdots, z_n$, so that by well known theorem of algebra, $G(z_1, z_2, \cdots, z_n) - G^*$ can be expressed as a linear combination of the elementary symmetric functions $S(n,j)$, $j = 0, 1, \cdots, m$ that is, there exists B_j such that

$$G(z_1, z_2, \ldots, z_n) - G^* = B_0 + S(n,1)B_1 + +S(n,2)B_2 + \cdots + S(n,m)B_m$$

$$= \frac{1}{A_n}\{ B_0 A_n - c(n,1) B_1 A_{n-1} + \cdots + (-1)^n c(n,m) B_m A_{n-m} \}.$$

If we define the polynomial $Q(z)$ by

$$Q(z) - \sum_{j=0}^{m} c(m,j) \frac{c(n,j)}{c(m,j)} B_j z^j$$

$$= G(z_1, z_2, \ldots, z_n) - G^*.$$

Then the relation

$$G(w_1, w_2, \dots, w_n) - G^* = 0$$

shows that the polynomial $P(z)$ and $Q(z)$ satisfy the condition of Theorem 3.4. Since the zeros of $P(z)$ lie in $|z - c| \leq r$, we conclude from second part of Theorem 3.4 that $Q(z) = G(z, z, \cdots, z) - G^*$ has at least one zero in the circle $|z - c| \leq r$, that is, there exists at least one complex number w in C such that

$$Q(w) = 0,$$

that is

$$G(w, w, \cdots, w) = G(w_1, w_2, \cdots, w_n).$$

This completes the proof of theorem 3.5.

Next we present the following theorem due to A. Aziz [2] which is a partial generalization of a result due to Szegö [37, p. 65]

THEOREM 3.6. If all the zeros of a polynomial

$$Q(z) - \sum_{j=0}^{m} C(m, j) B_j z^j$$

of degree m lie in $|z \quad c| \geq r$ and if α is a zero of the polynomial

$$P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j, \qquad A_0 A_n \neq 0$$

of degree n, $m \leq n$, then every zero w of the polynomial

$$R(z) = \sum_{j=0}^{m} C(m, j) A_j B_j z^j$$

of degree m, has the form $w = -\alpha\beta$, where β is a suitably chosen point in $|z - c| > r$.

PROOF OF THEOREM 3.6. If w is a zero of

$$R(z) = \sum_{j=0}^{m} C(m, j) A_j B_j z^j,$$

then

$$(3.28) \qquad R(w) = \sum_{j=0}^{m} C(m,j)A_jB_jw^j = 0.$$

Equation (3.28) shows that the polynomials

$$z^nP\left(-\frac{w}{z}\right) = c(n,0)(-1)^nA_nw^n + \cdots + c(n,m)(-1)^mA_mw^mz^{n-m}$$
$$+ \cdots + c(n,n)A_0z^n,$$

and

$$Q(z) = C(m,0)B_0 + C(m,1)B_1z + \cdots + C(m,m)B_nz^n$$

satisfy the condition of Theorem 3.4, since all the zeros of $Q(z)$ lie in $|z-c| \geq r$, it follows from first part of Theorem 3.4 that $z^nP\left(-\frac{w}{z}\right)$ has at least one zero in $|z-c| \geq r$. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the zeros of $P(z)$, then the zeros of $z^nP\left(-\frac{w}{z}\right)$ are $-\frac{w}{\alpha_1}, -\frac{w}{\alpha_2}, \cdots, -\frac{w}{\alpha_n}$. Therefore one of these zeros must be some β satisfying $|\beta - c| \geq r$. Hence we must have $w = -\alpha_j\beta$ for some $j = 1,2,\cdots,n$. This complete the proof of Theorem 3.6.

The next theorem which is also due to A. Aziz [2] is a generalization of a result due to DeBruijn [20].

THEOREM 3.7. From the two given polynomials

$$P(z) = \sum_{j=0}^{n} C(n,j)A_jz^j$$

and

$$Q(z) = \sum_{j=0}^{m} C(m,j)B_jz^j,$$

of degree n and m, respectively $m \leq n$. Let us form the third polynomial

$$R(z) = \sum_{j=0}^{m} C(m,j)A_jB_jz^j,$$

of degree m. Given a subset S of the w plane, let $P(z) \in S$ for $|z| \leq r$ and $Q(z) \neq 0$ for $|z| < 1$. Then $R(z) \in B_0S$ for $|z| \leq r$ where

$$B_0S = \{B_0s : s \in S\}.$$

PROOF OF THEOREM 3.7. Let s be a real or complex number, we replace the polynomial $P(z)$ by the polynomial $F(z) = P(z) - \delta$ and hence $R(z)$ by $H(z) = R(z) - \delta B_0$. If δ does not belong to S, then $F(z)$ does not vanish in $|z| \leq r$. So all the zeros of $F(z)$ lie in $|z| > r$. By hypothesis, the zeros of $Q(z)$ lie $|z| \geq 1$. Now if w is a zero of $H(z)$, then by Theorem 2 of [3], w has the form $w = -\alpha\beta$ where α is suitably chosen point in $|z| > r$ and β is a zero of $Q(z)$. Hence $|w| = |\alpha||\beta| \geq |\alpha| > r$. This shows $H(z)$ does not vanish in $|z| \leq r$. If therefore, δB_0 is a value assumed by $R(z)$ in $|z| \leq r$, then δ is a value assumed by $P(z)$ in $|z| \leq r$. Since $P(z) \in S$ for $|z| \leq r$, it follows that $R(z) \in B_0S$ for $|z| \leq r$ and this completes the proof.

If $P(z)$ is a polynomial of degree n having all its zeros in the circle $C_1: |z - a| \leq r_1$ and $Q(z)$ is a polynomial of degree m having all its zeros in the circle $C_2: |z - a| \leq r_2$, then according to Walshôs two circle theorem ([37, p. 89], [41, p. 57]) all the zeros of the polynomial

$$F(z) = P'(z)Q(z) + P(z)Q'(z),$$

lie in C_1, C_2 and a third circle

$$C: \left|z - \frac{nc_2 + mc_1}{n+m}\right| \leq \frac{nr_2 + mr_1}{n+m}.$$

In the literature for example [37], [41], there exists some implicit extensions of Walshôs two circle theorem. Here in this chapter we finally present the following compact generalization of Walshôs two-circle theorem for the polynomials and rational function due to A. Aziz and N. A. Rather [7].

THEOREM 3.8. If the locus of the zeros of a polynomial $P(z)$ of degree n is the closed interior of the circle C_1 with centre c_1 and radius r_1 and the locus of the zeros of a polynomial $Q(z)$ of degree m is the closed interior of a circle C_2 with centre c_2 and radius r_2. Then for every non-zero complex number $\beta \neq -\frac{n}{m}$, the locus of the zeros of the polynomial

$$F(z) = P'(z)Q(z) + \beta P(z)Q'(z)$$

consists of the closed interior of C_1 if $n > 1$, of C_2 if $m > 1$ and a third circle C with centre c and radius r where

$$c = \frac{nc_2 + \beta mc_1}{n + m} \text{ and } r = \frac{nr_2 + |\beta| mr_1}{|n + \beta m|}.$$

Under the hypothesis of Theorem 3.8 if $\beta \neq -\frac{n}{m}$ and if the closed interior of C_1 and C_2 have no point in common, then these two circles contain all the zeros of $F(z)$.

For the proof of Theorem 3.8, we need the following result known as Walshös coincidence Theorem [45].

LEMMA 3.4. Let $G(z_1, z_2, \cdots, z_n)$ be a symmetric n-linear form of totel degree n in $z_1, z_2, \cdots, z_n$ and let K be a circle containing the n points $w_1, w_2, \cdots, w_n$. Then there exists at least one point α belonging to K such that

$$G(\alpha, \alpha, \cdots, \alpha) = G(w_1, w_2, \cdots, w_n).$$

PROOF OF THEOREM 3.8. If w is any zero of $F(z)$, then

$$F(w) = P'(w)Q(w) + \beta P(w)Q'(w) = 0. \tag{3.29}$$

This is an equation which is linear and symmetric in the zeros of $P(z)$ and in the zeros of $Q(z)$. By above lemma, w will also satisfy the equation obtained by substituting into equation (3.89)

$$P(z) = (z - \alpha_1)^n \text{ and } Q(z) = (z - \alpha_2)^m,$$

where α_1 is suitably chosen point in C_1 and where α_2 is suitably chosen point in C_2. That is w satisfies the equation

$$n(w-\alpha_1)^{n-1}(w-\alpha_2)^m+\beta\, m(w-\alpha_1)^n(w-\alpha_2)^{m-1}=0,$$

equivalently

$$(3.30)\qquad (w-\alpha_1)^{n-1}(w-\alpha_2)^{m-1}\{(n+\beta m)w-(n\alpha_2+\beta m\alpha_1)\}=0.$$

First suppose that $\beta\neq-\frac{n}{m}$ and that the circles C_1 and C_2 have no point in common, then clearly $\alpha_1\neq\alpha_2$ and so that $n\alpha_2+\beta m\alpha_1\neq 0$. Hence from (3.30) we get

$$w=\alpha_1,\ \text{if } n>1,\quad or\quad w=\alpha_2,\ \text{if } m>1.$$

Since α_1 is a point in C_1, α_2 is a point in C_2 and w is an arbitrary zero of $F(z)$, it follows that, in this case the two circles C_1 and C_2 contain all the zeros of $F(z)$. Henceforth we assume $\beta\neq-\frac{n}{m}$, so that $F(z)$ is a polynomial of degree $n+m-1$. Now from (3.30), it follows that w has the values

$$w=\alpha_1\ \text{ if } n>1,\qquad w=\alpha_2\ \text{ if } m>1,\quad w=\frac{n\alpha_2+\beta m\alpha_1}{n+\beta m}.$$

Clearly the first w is a point in C_1, the second w is a point in C_2 and that the third w is a point in the circle

$$C:\left|z-\frac{nc_2+\beta mc_1}{n+\beta m}\right|\leq\frac{nr_2+|\beta|mr_1}{|n+\beta m|}$$

follows from the fact that

$$\left|\frac{nc_2+\beta mc_1}{n+\beta m}-\frac{nc_2+\beta mc_1}{n+\beta m}\right|=\left|\frac{n(\alpha_2-c_2)+\beta m(\alpha_1-c_1)}{n+\beta m}\right|$$

$$\leq\frac{n|\alpha_2-c_2|+|\beta|m|\alpha_1-c_1|}{|n+\beta m|}$$

$$\leq\frac{nr_2+|\beta|mr_1}{|n+\beta m|}.$$

Since w is arbitrary zero of $F(z)$, it follows that every zero $F(z)$ lies in at least one of the circles C_1, C_2 and C.

Conversely we now show that any point w in or on the circles C_1, C_2 or C is a possible zero of $F(z)$. For, we may take $P(z) = (z - \alpha_1)^m$ and choose α_1 and α_2 as follows. If $n > 1$ and if w lies in C_1, we choose $\alpha_1 = w$ and α_2 as arbitrary point in C_2. Similarly, If $m > 1$ and w lies in C_2, we choose $\alpha_2 = w$ and α_1 as an arbitrary point in C_1. If however, w is any point in or on C, then we may write

$$w = c + \alpha r e^{i\theta}$$

$$= \frac{n}{n+\beta m} c_2 + \frac{\beta m}{n+\beta m} c_1 + \alpha e^{i\theta} \left\{ \frac{n r_2}{|n+\beta m|} + \frac{|\beta| m r_1}{|n+\beta m|} \right\} \text{ for } 0 \le \alpha \le 1$$

and associate with this w,

$$\alpha_1 = c_1 + \alpha \left| \frac{\beta}{n+\beta m} \right| \frac{(n+\beta m)}{\beta} r_1 \text{ and } \alpha_2 = c_2 + \alpha \frac{(n+\beta m)}{|n+\beta m|} r_2 e^{i\theta},$$

then $|\alpha_1 - c_1| \le r_1$ and $|\alpha_2 - c_2| \le r_2$, so that α_1 is a point in C_1, α_2 is a point in C_2 and

$$w = c + \alpha r e^{i\theta} = \frac{n \alpha_2}{n+\beta m} + \frac{\beta m \alpha_1}{n+\beta m}.$$

This completes the proof.

The Sendove Ilieff's Conjecture Concerning The Critical Points Of A Polynomial

A very familiar theorem of real analysis which is often introduced in our first calculus courses known as Rolles theorem states that ñBetween any two real zeros of a differentiable real function f lies atleast one critical point of f ò.

As discussed by Morris Marden [38] this theorem first appeared in a book published in 1691 by the French mathematician Michel Rolle. Its publication had predated the adoption of the geometric representation of complex numbers by about 140 years. For, though this representation was devised by the Norwegian Cartographer Casper Wessel in 1797 and again by Swiss mathematician Jean Argand in 1806. Its universal acceptance had to await its invention in 1831 by none other than the great Karl Friedrich Gauss.

With this representation came the concepts of a complex variable $z = x + iy$ and a function of a complex variable. It is not surprising that some early attention was directed towards constructing in the complex domain counterparts to well known theorems of real analysis such as Rolleôs theorem. However, the generalization of Rolleôs theorem to the complex plane is not obvious or trivial, as the following two example show.

First, take the function $f(z) = e^{iz} - 1$ which has zero at $z = 0$ and at $z = 2\pi$. If the Rolleôs theorem were valid, at least one critical point would be situated on the interval $0 < x < 2\pi$. But $f^{'}(z) = ie^{iz}$, so that $f^{'}$ has no zero whatsoever.

Secondly, take the polynomial $f(z) = (z^2 - 1)(z - i\sqrt{3})$ which has zero at the vertices $z = \pm 1$, $z = i\sqrt{3}$ of an isosceles triangle. If Rolleôs theorem were valid, f would have a critical point on each side of this triangle. But $f^{'}(z) = 3\left(z - \frac{i}{\sqrt{3}}\right)^2$ so that $f^{'}$ has a single zero at $z = \frac{i}{\sqrt{3}}$, a point interior to the triangle.

As second example shows, the concept of a critical point lying between two real zeros (i.e. on the line segment joining the two zeros) generally is replaced in the complex plane by the concept of a critical point situated in some region containing the zeros of the given function. In the second example that region is a closed triangle, but in the later examples a polygon or a circular disk may be the most convenient choice.

As to the importance of locating the critical points of a given function, it is to be recalled that, whereas for a function of a real variable the determination of the critical points helps in locating the maxima or minima, for a function of a complex variable f analytic in a region T, finding the critical points aids in determining where the map of T by $w = f(z)$ fails to be conformal.

During the past dozen years considerable interest has been aroused regarding the location of the critical points of any polynomial f all of whose zeros lie in the unit disk $|z| \le 1$. By Gauss-Lucas theorem we know that all the critical points of f also lie on the unit disk. The question recently raised is how close to each zero do the critical points lie [see 40].

The following conjecture was made by the Bulgarian mathematician B. L. Sendov in 1962 but became later known as the ñIlieff Conjectureò.

CONJECTURE 1. If all the zeros of an nth degree polynomial $P(z)$ lie in the disk $|z| \le 1$ and z_0 is any one of the zeros, then atleast one critical point of $P(z)$ lies within the unit distance from z_0.

The constant 1 in the above conjecture is best possible upon considering the polynomial $P(z) = z^n - 1$. This conjecture has been open since appearing in Haymanôs Research Problem in Function Theory [30]. It has been verified for $n = 3,4$ see [16, 42], for $n = 5$ see [35] and for $n = 6$ see [14, 33] and for $n = 7$ see [15, 17]. It has also been verified for some special classes of polynomials [see Schmeisser 43]. The proof for 6 and 7 degree polynomials where obtained through slightly different estimates with some involved computations. Johnny E. Brown and Guangping Xiang have presented a unified method for investigating the Sendov conjecture. As an application, they have proved the conjecture 1 for polynomials of degree at most 8, for references see [18].

Since in 1962, when conjecture 1 first became known, it has been the subject to more than 70 articles. However, it was fully verified for the polynomials of degree at most 8. A variety of special cases have been dealt with over the years (see [42, 43] for references). Among these results we mention Millerôs qualitative result [27] according to which those roots of $P(z)$ lying sufficiently close to the unit circle satisfy an even stronger condition than the one stated in conjecture 1.

Here we shall present the following result due to Zalman Rubinstein which concerns the boundary case, that is when $|a| = 1$, of conjecture 1.

THEOREM 4.1. If $P(z)$ is a polynomial of degree n which has all its zeros in $|z| \leq 1$ and $P(1) = 0$, then $P'(z)$ has atleast one zero in the circle

$$(4.1) \qquad |z - 1| \leq 1.$$

The result is best possible as shown by the example $P(z) = z^n - 1$.

PROOF OF THEOREM 4.1 Since

$$P(1) = 0,$$

we write

$$(4.2) \qquad P(z) = (z - 1)Q(z),$$

where $Q(z)$ is a polynomial of degree $n - 1$, whose all zeros lie in $|z| \leq 1$. Then from (4.2) we get

$$(4.3) \qquad P'(z) = (z - 1)Q'(z) + Q(z)$$

and

$$(4.4) \qquad P''(z) = (z - 1)Q''(z) + 2Q'(z).$$

If $z - 1$ is a multiple zero of $P(z)$, then $z = 1$ is also a zero of $P'(z)$. Since this zero lie in (4.1), the result follows. Hence we suppose that $z = 1$ is a simple zero of $P(z)$. From (4.3) and (4.4), we have

$$(4.5) \qquad \frac{P''(1)}{P'(1)} = \frac{2Q'(1)}{Q(1)},$$

If $z_1, z_2, \cdots, z_{n-1}$ are the zeros of $Q(z)$, then

$$|z_j| \leq 1, \quad j = 1,2, \cdots, n - 1,$$

and from (4.5), we get

$$(4.6) \qquad \frac{P''(1)}{P'(1)} = 2\sum_{j=1}^{n-1} \frac{1}{1-z_j}.$$

Let $w_1, w_2, \cdots, w_{n-1}$ be the zeros of $P'(z)$, then from (4.6), we get

$$\sum_{j=1}^{n-1} \frac{1}{1-w_j} = 2\sum_{j=1}^{n-1} \frac{1}{1-z_j},$$

so that

$$(4.7) \qquad \sum_{j=1}^{n-1} Re\left(\frac{1}{1-w_j}\right) = 2\sum_{j=1}^{n-1} Re\left(\frac{1}{1-z_j}\right)$$

Since, $|z_j| \le 1, \quad J = 1,2,\cdots,n-1$. It follows that

$$Re\frac{1}{1-z_j} \ge \frac{1}{2} \quad \text{for all} \quad j = 1,2,\ldots,n-1,$$

and therefore from (4.7), we have

$$(4.8) \qquad \sum_{j=1}^{n-1} Re\left(\frac{1}{1-w_j}\right) \ge 2\sum_{j=1}^{n-1} \frac{1}{2}$$

$$= n-1.$$

If

$$Re\left(\frac{1}{1-w}\right) = \max_{1\le j\le n-1} Re\left(\frac{1}{1-w_j}\right),$$

then from (4.8), we see that

$$Re\left(\frac{1}{1-w}\right) \ge \frac{1}{n-1}\sum_{j=1}^{n-1} \frac{1}{1-w_j}$$

$$\ge 1.$$

This implies

$$\left|\frac{1}{1-w}\right| \geq Re\,\frac{1}{1-w} \geq 1,$$

equivalently

$$|w-1| \leq 1,$$

which is equivalent to (4.1) and the required result follows.

Next we shall present a proof of Sendov-Ilieff conjecture for polynomials of degree 3 and 4 which is also due to Zalman Rubenstein.

THEOREM 4.2. Let $P(z)$ be a polynomial of degree 3 or 4 whose zeros lie in the closed unit disk, $|z| \leq 1$. If $P(a) = 0$, then $P^{'}(z)$ always has a zero in the circle $|z-a| \leq 1$.

For the proof of this theorem we need the following result which is known as Szegöôs composition theorem.

LEMMA 4.1. From the given polynomials

$$f(z) = \sum_{j=0}^{n} C(n,j)A_j z^j$$

and

$$g(z) = \sum_{j=0}^{n} C(n,j)B_j z^j,$$

let us form a third polynomial

$$h(z) = \sum_{k=0}^{n} C(n,k)\, A_k B_k z^k.$$

If all the zeros of $f(z)$ lie in a circular region A, then every zero γ of $h(z)$ has the form

$$\gamma = -\alpha\beta,$$

where α is suitably chosen point in A and β is a zero of $g(z)$.

PROOF OF LEMMA 4.1. We have

$$f(z) = \binom{n}{0}A_0 + \binom{n}{1}A_1 z + \binom{n}{2}A_2 z^2 + \cdots + \binom{n}{n}A_n z^n,$$

$$g(z) = \binom{n}{0}B_0 + \binom{n}{1}B_1 z + \binom{n}{2}B_2 z^2 + \cdots + \binom{n}{n}B_n z^n$$

and $$h(z) = \binom{n}{0}A_0 B_0 + \binom{n}{1}A_1 B_1 z + \binom{n}{2}A_2 B_2 z^2 + \cdots + \binom{n}{n}A_n B_n z^n,$$

Let γ be any zero of $h(z)$, then

$$h(\gamma) = 0,$$

this gives

$$\binom{n}{0}A_0 B_0 + \binom{n}{1}A_1 B_1 \gamma + \binom{n}{2}A_2 B_2 \gamma^2 + \cdots + \binom{n}{n}A_n B_n \gamma^n - 0. \tag{4.9}$$

Consider the polynomial

$$z^n g\left(-\frac{\gamma}{z}\right) = z^n\left\{\binom{n}{0}B_0 - \binom{n}{1}B_1\left(-\frac{\gamma}{z}\right) + \binom{n}{2}B_2\left(-\frac{\gamma}{z}\right)^2 - \cdots\right.$$

$$\left. +(-1)^n\binom{n}{n}B_n\left(-\frac{\gamma}{z}\right)^n\right\}$$

$$= \binom{n}{0}B_0 z^n - \binom{n}{1}B_1 \gamma z^{n-1} + \cdots + (-1)^n\binom{n}{n}B_n \gamma^n.$$

Now the polynomial

$$f(z) = \binom{n}{0}A_0 + \binom{n}{1}A_1 z + \binom{n}{2}A_2 z^2 + \cdots + \binom{n}{n}A_n z^n,$$

will be apolar to the polynomial $z^n g\left(-\frac{\gamma}{z}\right)$ if

$$\binom{n}{0}A_0 B_0 + \binom{n}{1}A_1 B_1 \gamma + \binom{n}{2}A_2 B_2 \gamma^2 + \cdots + \binom{n}{n}A_n B_n \gamma^n = 0$$

which is true by (4.9). Hence $f(z)$ and $z^n g\left(-\frac{\gamma}{z}\right)$ are apolar polynomials. Since all the zeros of $f(z)$ lie in the circular region A, therefore, it follows by Graces theorem that $z^n g\left(-\frac{\gamma}{z}\right)$ has at least one zero in A. Now if $\beta_1, \beta_2, \cdots, \beta_n$ are the zeros of $g(z)$, then

$$g(z) = \beta_n(z \quad \beta_1)(z \quad \beta_2)\cdots(z \quad \beta_n)\,.$$

This implies

$$z^n g\left(-\frac{\gamma}{z}\right) = z^n \beta_n \left(-\frac{\gamma}{z} - \beta_1\right)\left(-\frac{\gamma}{z} - \beta_2\right)\cdots\left(-\frac{\gamma}{z} - \beta_n\right)$$

or

$$z^n g\left(-\frac{\gamma}{z}\right) = (-1)^n \beta_n (\gamma + z\beta_1)(\gamma + z\beta_2)\dots(\gamma + z\beta_n)$$

This shows that the zeros of $z^n g\left(-\frac{\gamma}{z}\right)$ are $-\frac{\gamma}{\beta_1}, -\frac{\gamma}{\beta_2}, -\frac{\gamma}{\beta_3}, \cdots, -\frac{\gamma}{\beta_n}$.

Since $z^n g\left(-\frac{\gamma}{z}\right)$ has atleast one zero in A, therefore $-\frac{\gamma}{\beta_i} = \alpha$, for atleast one $i = 1,2,\cdots,n$ where $\alpha \epsilon A$. This gives $\gamma = -\alpha\beta_i$, for at least one $i = 1,2,\cdots,n$, which shows $\gamma = -\alpha\beta$, when $\alpha \epsilon A$, and β is a zero of $g(z)$. This completes the proof of Lemma 4.1.

PROOF OF THEOREM 4.2. We may assume that

$$P(z) = (z - x)Q(z), \qquad \text{where} \qquad 0 < x < 1$$

and the zeros z_k , $k = 1,2,\cdots,n$ of $Q(z)$ lie in $|z| < 1$. We shall prove the polynomial $f(z) - P'(z + x)$ has a zero in $|z| < 1$.

Consider the polynomials

$$f(z) = \sum_{k=0}^{n} (k+1) \frac{Q^k(x)}{k!} z^k$$

$$g(z) = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} z^k$$

$$\text{and} \qquad h(z) = \sum_{k=0}^{n} \frac{Q^k(x)}{k!} z^k,$$

by Lemma 4.1 every zero γ of $h(z)$ has the form

$$\gamma = -\alpha\beta,$$

where β is a zero of $g(z)$and α is a point belonging to a circular region containing all the zeros of $f(z)$. The zeros of $g(z)$ have the form

$$\beta = -1 + \sqrt[n+1]{1}$$

such that $\beta \neq 0$. For $n = 2,3$, $|\beta| \geq \sqrt{2}$. If $f(z) \neq 0$ in $|z| < 1$. We may choose α such that $|\alpha| \geq 1$. Thus $|\gamma| \geq \sqrt{2}$.

Since

$$h(z) = Q(z + x)$$

$$\text{and} \qquad f(z) = P'(z + x),$$

it follows that all the zeros of $Q(z)$ satisfy $|z| \leq 1$and $|z - x| \geq \sqrt{2}$ and no zero of $P'(z)$ lies in

$$|z - x| < 1.$$

Consider the polynomial

$$R(z) = P(z - 1 + x) = (z - 1)Q_1(z),$$

where,

$$Q_1(z) = Q(z - 1 + x).$$

No zero of $R'(z)$ lies in $|z - x| < 1$, by Theorem 4.1, we shall obtain a contradiction if we show that all the zeros of $Q(z)$ lie in $|z| < 1$. Indeed the zeros of $Q_1(z)$ satisfies the inequality

$$|z - 1 + x| \leq 1 \qquad \text{and} \qquad |z - 1| \geq \sqrt{2},$$

where

$$z = u + iv.$$

Now

$$|u - 1 + x + iv| \leq 1$$

which gives

$$\sqrt{(u - 1 + x)^2 + v^2} \leq 1.$$

Squaring both sides, we get

$$(u-1+x)^2+v^2 \leq 1,$$

it follows that

$$\text{or} \qquad u^2+x^2+2u(x-1)-2x+v^2 \leq 1$$

$$(4.10) \qquad u^2+x^2-2u(1-x)+(x-1) \leq 3x-1-x^2 \quad \text{for} \quad 0<x<1$$

and

$$|z-1| \geq \sqrt{2}$$

which gives

$$(u-1)^2+v^2 \geq 2$$

or

$$u^2+v^2-2u \geq 1.$$

Since $0<x<1$, therefore $(1-x)>0$, so that $x-1<0$. Multiplying both sides of this inequality by $(x-1)$, we have

$$(4.11) \qquad (x-1)(u^2+v^2)-2u(x-1) \leq x-1.$$

Combining (4.10) and (4.11), we get

$$x(u^2+v^2)+x-1 \leq x-1+3x-1-x^2$$

or

$$x(u^2+v^2) \leq 3x-(1+x^2)$$

which implies

$$u^2+v^2 \leq 3-\left(x+\frac{1}{x}\right)$$

$$<1, \qquad \text{for} \quad 0<x<1.$$

This completes the proof of the Theorem 4.2.

In connection with conjecture 1, the following conjecture was made in 1969 by the American mathematicians Goodman and Roth, the Canadian-Indian mathematician Rahman and the German mathematician Schmeisser.

CONJECTURE 2. Under the same hypothesis as for conjecture 1, at least one critical point of f lies on the disk

$$\left|z-\frac{z_0}{2}\right| \le 1-\frac{|z_0|}{2}.$$

The boundary case, that is when $|z_0| = 1$, of this conjecture was proved by A. W. Goodman, Q. I. Rahman and J. S. Rath [38].

We present the following result due to A. Meir and A. Sharma [35] which is an extension of a result due to Goodman, Rahman and Ratti [27] for the boundary case of conjecture 2 for the zeros of higher derivatives of polynomials having multiple zeros.

THEOREM 4.3. Let

$$P(z) = (z-z_0)^k Q(z),$$

where

$$Q(z) = \prod_{j-1}^{n-k}(z-z_j)$$

with $|z_0| = 1$, and $|z_j| \le 1,\ \ z_j \ne z_0$ for $J = 1,2,\cdots,n-k$.

Then at least one zero of $P^r(z)$ $(1 \le r \le n-1)$ lies in the disk

$$\left|z-\frac{kz_0}{r+1}\right| \le 1-\frac{k}{r+1}. \tag{4.12}$$

For $r > k$, strictly inequality will hold in (4.12) except when $r = n-1$ and $P(z) = (z-z_0)^k(z-z_1)^{n-k}$ with $|z_0| = |z_1| = 1$.

PROOF OF THEOREM 4.3. We have

$$P(z) = (z-z_0)^k Q(z),$$

where

$$Q(z) = \prod_{j=1}^{n-k} (z - z_j).$$

Without loss of generality, we may suppose that $z_0 = 1$ and $r > k$. Now

(4.13)
$$P^r(z) = \left\{\frac{d^r}{dz^r}(z-1)^k\right\} Q(z) + \binom{r}{1}\left\{\frac{d^{r-1}}{dz^{r-1}}(z-1)^k\right\} Q'(z) + \cdots + \binom{r}{j}\left\{\frac{d^{r-j}}{dz^{r-j}}(z-1)^k\right\} Q^j(z) + \cdots + \binom{r}{r}(z-1)^k Q^r(z).$$

Now

$$\frac{d^k}{dz^k}(z-1)^k = k(k-1)(k-2)\cdots 2\cdot 1 = k!\,.$$

Therefore

$$\frac{d^{r-j}}{dz^{r-j}}(z-1)^k = 0 \qquad \text{for} \quad r-j > k.$$

Thus from (4.3), we have

$$\frac{P^{r+1}(1)}{P^r(1)} = \frac{\binom{r+1}{k} k!\, Q^{r-k+1}(1)}{\binom{r}{k} k!\, Q^{r-k}(1)}$$

$$= \frac{r+1}{r-k+1} \frac{Q^{r-k+1}(1)}{Q^{r-k}(1)}.$$

Since $P(z)$ is a polynomial of degree n, it follows that $P^r(z)$ is a polynomial of degree $n-r$. Also $Q(z)$ is a polynomial of degree $n-k$, therefore $Q^{(r-k)}(z)$ is a polynomial of degree $n-r$.

If $z_1, z_2, \cdots, z_{n-r}$ are the zeros of $P^r(z)$ and $w_1, w_2, \cdots, w_{n-r}$ are the zeros of $Q^{(r-k)}(z)$. Then from (4.13), we have

$$\sum_{j=1}^{n-r} \frac{1}{1-z_j} = \frac{r+1}{r-k+1} \sum_{j=1}^{n-r} \frac{1}{1-w_j}$$

So that

$$(4.14)\qquad Re\sum_{j=1}^{n-r}\frac{1}{1-z_j}=\frac{r+1}{r-k+1}Re\sum_{j=1}^{n-r}\frac{1}{1-w_j}$$

Since the zeros of $Q(z)$ lie in $|z|\le 1$, it follows by Gauss-Lucas theorem that the zeros of $Q^{r-k}(z)$ also lie in $|z|<1$ and therefore $|w_j|\le 1$ for all $j=1,2,\cdots,n-r$, this implies

$$Re\frac{1}{1-w_j}\ge\frac{1}{2},\qquad \text{for}\quad j=1,2,\cdots,n-r.$$

Hence from (4.14), we get

$$\sum_{j=1}^{n-r}Re\frac{1}{1-z_j}\ge\frac{r+1}{r-k+1}\sum_{j=1}^{n-r}\frac{1}{2}$$

$$(4.15)\qquad =\frac{(r+1)(n-r)}{2(r-k+1)}.$$

Now if

$$\max_{1\le j\le n-1}Re\frac{1}{1-z_j}=Re\frac{1}{1-\alpha}\cdot$$

Then from (4.15), we have

$$Re\frac{1}{1-\alpha}\ge\frac{1}{n-r}\sum_{j=1}^{n-r}Re\frac{1}{1-z_j}$$

$$>\frac{(r+1)}{2(r-k+1)}.$$

equivalently

$$Re\frac{(r-k+1)}{(r+1)(1-\alpha)}\ge\frac{1}{2}\cdot$$

This gives

$$\left|1-\frac{(r-k+1)}{(r+1)(1-\alpha)}\right|\le\frac{r-k+1}{r+1}\left|\frac{1}{1-\alpha}\right|.$$

that is

$$\left|1-\alpha-\frac{r-k+1}{r+1}\right| \le \frac{r-k+1}{r+1}$$

or

$$\left|\alpha-\frac{k}{r+1}\right| \le 1-\frac{k}{r+1},$$

which is equivalent to (4.12) and this proves the result.

REMAK 4.1. The conjectured result of Goodman, Rahman and Ratti [27] for zeros on the boundary is included in Theorem 4.3 as a special case when $k=1, r=1$.

Finally we conclude this chapter by present the following result which is also due to A. Meir and A. Sharma [35] concerning the zeros of $(n-2)th$ order derivative of a polynomial.

THEOREM 4.4. If

$$P(z)=(z-z_0)^k Q(z), \qquad (k \ge 1, \;\; n \ge 2+k)$$

where

$$|z_0| \le 1 \quad \text{and} \quad Q(z)=\prod_{j=1}^{n-k}(z-z_j)$$

$z \ne z_0$, $|z_j| \le 1$ for all $j=1,2,\cdots,n-k$, then at least one zero of $P^{n-2}(z)$ lies in the closed disk

$$(4.16) \qquad |z-z_0| \le \frac{2(n-k-1)}{n-1}\sqrt{\frac{n-1+|z_0|}{n}}.$$

For the proof of this result we need the following lemmas.

LEMMA 4.2. Let

$$f(z)-\sum_{j=0}^{n}\binom{n}{j}a_j z^j,$$

$$g(z) = \sum_{j=0}^{n} \binom{n}{j} b_j z^j,$$

$$h(z) = \sum_{j=0}^{n} \binom{n}{j} a_j b_j z^j,$$

and suppose that the zeros of $f(z)$ lie in the annulus $p \le |z| \le q$, and those zeros of $g(z)$ lie in $r \le |z| \le s$, then the zeros of $h(z)$ lie in $pr \le |z| \le qs$.

If $R(t)$ is a polynomial of degree $n-k$, and $f(t) = \frac{d^r}{dt^r}\{t^k R(t)\}$ and $h(t) = R(t)^{(r-k)}$ $(r \ge k)$, then the polynomial $g(t)$ of the above lemma may be chosen, except for a constant factor, as follows,

$$g(t) = \sum_{j=0}^{n-r} \frac{\binom{n-r}{j}\binom{n}{k}}{\binom{r+j}{k}} t^j, \tag{4.17}$$

LEMMA 4.3. Let $0 < \alpha \le 1$ and suppose w is a point in the closed unit disk. Then

$$Re\frac{1}{\alpha - w} \ge \frac{1}{2\alpha} - \frac{1-\alpha^2}{2\alpha} \cdot \frac{1}{r^2} \qquad r = |\alpha - w|$$

PROOF OF THEOREM 4.4. Without loss of generality, we may take $z_0 = \alpha$, $0 \le \alpha \le 1$. Setting in Lemma 4.2,

$$f(t) = P^{(n-2)}(\alpha + t) = \frac{d^{n-2}}{dt^{n-2}}\{t^k Q(\alpha + t)\}$$

and

$$h(t) = Q^{(n-2-k)}(\alpha + t),$$

we have by (4.17)

$$g(t) = t^2 + \frac{2n}{n-k} t + \frac{n(n-1)}{(n-k-1)(n-k)}.$$

The zeros of $g(t)$ are given by

$$\beta = \frac{-\frac{2n}{n-k} \pm \sqrt{\frac{4n^2}{(n-k)^2} - \frac{4n(n-1)}{(n-k)(n-k-1)}}}{2}$$

$$-\frac{n}{n-k} \pm \sqrt{\frac{n^2(n-k-1)-(n^2-n)(n-k)}{(n-k)^2(n-k-1)}}$$

$$= \frac{1}{n-k}\left(-n \pm i\sqrt{\frac{nk}{n-k-1}}\right).$$

Therefore if β_1 and β_2 are the zeros of $g(t)$, then

$$|\beta_1| = |\beta_2| = \frac{n(n-1)}{(n-k)(n-k-1)}.$$

Assuming that $\rho_1 \leq \rho_2$ and $r_1 \leq r_2$, where

$$\rho_1 = |z_0 - \beta_1|, \qquad \rho_2 = |z_0 - \beta_2|, \qquad r_1 = |z_0 - w_1| \text{ and}$$

$$r_2 = |z_0 - w_2|.$$

we have by lemma 4.2

$$(4.18) \qquad \rho_1\sqrt{\frac{n(n-1)}{(n-k)(n-k-1)}} \leq r_1 \leq r_2,$$

$$\text{whence} \qquad \frac{1}{r_1^2} + \frac{1}{r_2^2} \leq \frac{(n-K)(n-k-1)}{n(n-1)} \cdot \frac{2}{\rho_1^2}.$$

Suppose now the theorem is false. Then

$$(4.19) \qquad \rho_2 \geq \rho_1 > 2\frac{(n-k-1)}{n-1}\sqrt{\frac{n-1+\alpha}{n}},$$

and thus

$$(4.20) \qquad \frac{1}{r_1^2} + \frac{1}{r_2^2} < \frac{1}{2}\frac{(n-1)(n-k)}{(n-k-1)(n-1+\alpha)}.$$

Also from (4.18) and (4.19) and from $n-k \geq 2$, we have $r_1 > 1$, which for $\alpha = 0$ yields the desired contradiction. If $\alpha \neq 0$, then since

$$\sum_{j=1}^{2}\frac{1}{\alpha-\xi_j}=\frac{n-1}{n-k-1}\sum_{j=1}^{2}\frac{1}{\alpha-w_j},$$

we get from (4.19) and Lemma 4.3,

$$\frac{1}{2}Re\sum_{j=1}^{2}\frac{1}{\alpha-\xi_j}>\frac{n-1}{2(n-k-1)}\left\{\frac{1}{\alpha}-\frac{1-\alpha^2}{4\alpha}\cdot\frac{(n-1)(n-k)}{(n-k-1)(n-1+\alpha)}\right\}$$

$$(4.21)\qquad \geq\frac{n-1}{2(n-k-1)}\cdot\frac{1}{2\alpha}\left\{2-\frac{(1-\alpha^2)(n-1)}{n-1+\alpha}\right\}$$

$$>\frac{n\quad 1}{4\alpha(n-k-1)}\left\{1+\alpha^2\frac{n}{n-1+\alpha}\right\}$$

$$\geq\frac{n-1}{2(n-k-1)}\sqrt{\frac{n}{n-1+\alpha}}.$$

Observing that $n-k\leq 2(n-k-1)$. Since $\frac{1}{\rho_j}\geq Re\frac{1}{\alpha-\xi_j}$ $j=1,2,$ (4.21) yields a contradiction to (4.19) which completes the proof of the theorem.

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