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Average number of photons present in a single-mode squeezed vacuum state as function of the mean free path

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Abstract We devote this research paper to construct a relationship between the average of the total number of quanta in a single-mode squeezed vacuum state and the mean free path of atoms of a material through which the natural light passes This relationship provides a method to measure experimentally the mean number of photons present in this mode of the quantum radiation field.

Keywords : Squeezed vacuum state, near free path, average number of photons

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1. Introduction

The first scientific light scattering experiment was performed by Tyndall [1], who observed the natural light scattered by particles whose size was small compared with the incident wavelength and noticed the appearance of a bluish blue hue in the scattered radiation. Rayleigh [2] gave the theoretical explanation of this fact, showing that the intensity of light scattered by such particles, considered non-interacting, is inversely proportional to the forth power of the wavelength. This effect accounts in particular, for the blue color of sky, once the scattering centers are assumed to be the vapour molecules constituting the atmosphere, possessing a mean free path larger than the optical wavelength.

In 1998, the author established an interesting relationship, which relates the mean number of photons (symbolized by ε) occupied by a mode of the natural light to the mean free path of the atoms of the material through which the light passes [3]. Owning to this relationship, one can measure experimentally the mean occupation number of photons present in a chaotic field mode by measuring the mean free path (symbolized by T).

Recently, we have defined by applying the Boltzmann integrodifferential equation, a fully quantum-mechanical P function, [4]. By means of this function, we introduce here a mixed-state density operator for a single-mode squeezed vacuum state. This operator enabled us to define a relationship between the total mean number of quanta present in a single-mode squeezed vacuum state of the mode and the parameter ε that depends on T.

In fact, we have evaluated, for two cases, the mean value of the number operator $a^+ a$ (a^+ and a are respectively the creation and annihilation operators of a photon) [5]. It is found that for a single-mode pure squeezed vacuum state, this mean value is equal to $\sinh^2 r$, while it is equal to the statistical average of this function ($\sinh^2 r$) for the mixed squeezed vacuum state, (r is called the squeeze factor). This leads us to construct an interesting relationship that relates the average number of photons present in a single-mode squeezed vaccum state and the parameter ε which depends on the mean free path T.

In fact, if one could measure experimentally the mean free path T, then we can formulate the number ε, which is the main factor in the theory. Once this number is fixed, we can determine owing to the constructed relationship, the mean occupation number of the thermal and squeezed states of the mode we are studying.

2. Definitions and notations

(a) Single-mode squeezed vacuum state :

The single-mode squeeze operator is defined by

$$S(r,\varphi)=\exp\left[\frac{1}{2}z^{*}a^{2}-\frac{1}{2}z(a^{+})^{2}\right], \quad (1)$$

where $z=r\exp(2i\varphi)$ is called the complex factor of the squeezed state. The real numbers r and φ are known as squeeze factor and squeeze angle respectively. These numbers are defined in the range $0\le r<\infty$, $-\pi/2<\varphi\le\pi/2$. The single-mode squeezed state $|r,\varphi\rangle$, results by acting operator (1) on the ground state $|0\rangle$ of the mode [6-9],

$$|r,\varphi\rangle=S(r,\varphi)|0\rangle. \quad (2)$$

The normal-ordered form of $S(r,\varphi)$ can be calculated. The most straightforward method is to apply McCoy's theorem. This allows an exponential of a polynomial in a^{+} and a up to powers of two to be written in the normal-ordered form. The result is [10],

$$S(r,\varphi)=\{\cosh r\}^{-1/2}\exp\left[-\frac{1}{2}e^{2i\varphi}\{\tanh r\}\left(a^{+}\right)^{2}\right]$$

$$\times\exp\left[-\ln\{\cosh r\}a^{+}a\right]\exp\left[\frac{1}{2}e^{-2i\varphi}\{\tanh r\}\left(a^{+}\right)^{2}\right]. \quad (3)$$

If we insert operator (3) into (2) and use the identity $a|0\rangle=0$, then we obtain

$$|r,\varphi\rangle=\{\cosh r\}^{-1/2}\exp\left[-\frac{1}{2}e^{2i\varphi}\{\tanh r\}\left(a^{+}\right)^{2}\right]|0\rangle. \quad (4)$$

Now, knowing that the energy eigenstates $|n\rangle$ ($n=0,1,2,3,\ldots\ldots$) of a harmonic oscillator are constructed from its vacuum state $|0\rangle$ by repeated application of creation operator as

$$|n\rangle=\frac{(a^{+})^{n}}{(n!)^{1/2}}|0\rangle, \quad (5)$$

we can write (4) in the form

$$|r,\varphi\rangle=\{\cosh r\}^{-1/2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}e^{2i\varphi}\tanh r\right)^{n}}{n!}\sqrt{(2n)!}|2n\rangle;$$

$$0\le r<\infty,\ -\pi/2<\varphi\le\pi/2. \quad (6)$$

Since $|r,\phi\rangle$ is an eigenstate of the unitary operator (1), it is normalized to unity, such as

$$\langle r,\varphi|r,\varphi\rangle=1. \quad (7)$$

(b) Mean number of photons present in a thermal mode :

Let $\boldsymbol{\Omega}$ be a unit vector coinciding with the direction of the velocity of a particle when its energy is u. We introduce the function $g(\mu_0,u-u')$ to represent the relative probability of a particle being left with parameters $(\boldsymbol{\Omega},u)$ as a result of a collision before which it was characterized by the pair $(\boldsymbol{\Omega}',u')$ ($\mu_0=\boldsymbol{\Omega}.\boldsymbol{\Omega}'$ is the cosine of the angle though which the particle is scattered). We assume that $g(\mu_0,u-u')$ may be expressed in terms of Legendre polynomials $P_j(\mu_0)$ such as

$$g(\mu_0,u-u')=(1/4\pi)\sum_{n=0}^{\infty}(2n+1)\,g_n(u-u')\,P_n(\mu_0) \quad (8)$$

with

$$g_n(u-u')=2\pi\int_{-1}^{+1}d\mu_0 g(\mu_0,u-u')\,P_n(\mu_0), \quad (9)$$

where we have used the normalization condition of Legendre polynomials. Now, we introduce the symbols α and β to denote the integrals

$$\alpha=2\pi\int_{u-\gamma}^{u}du'\,(u-u')g_0(u-u') \quad (10)$$

and

$$\beta=2\pi\int_{u-\gamma}^{u}du'\,g_1(u-u') \quad (11)$$

with $g_0(u-u')$ and $g_1(u-u')$ are defined by eq. (9) for $n=0$ and 1 respectively, while γ represents the maximum energy loss (the maximum energy loss occurs when the particle is scattered through an angle of 180°). Accordingly, the average number of photons present in a thermal field mode as function of the mean free path $T(u)$, is given by [3]

$$\varepsilon=\frac{}{3\alpha(1-\beta)}\int^{u}du'\,T^{2}(u'). \quad (12)$$

3. P-representation for a single-mode squeezed vacuum state

For a system in the pure state (6) corresponding to no statistical indetermination, the density operator $\hat{\rho}$ is defined by

$$\rho=|r,\varphi\rangle\langle\varphi,r \quad (13)$$

which is a hermitian operator. The quantum statistical expectation value of the number operator $a^{+}a$ is given by [5]

$$<a^{+}a>=\mathrm{Tr}\,(\hat{\rho}\,a^{+}\,a). \quad (14)$$

This determines the average number of photons present in state (6). If we now substitute (6) into (13), we get

$$\hat{\rho} = \frac{1}{\cosh r}\sum_{m,n=0}^{\infty}(-1)^{m+n}e^{2i(m-n)\varphi}\left(\frac{1}{2}\tanh^2 r\right)^{(m+n)/2}$$

$$\times\left[\frac{(2m-1)!!(2n-1)!!}{m!n!}\right]^{1/2}|2m\rangle\langle 2n|. \quad (15)$$

If we insert (15) into (14) and use the eigenvalue equation $a|k\rangle=\sqrt{k}\,|k-1\rangle$ as well as the completeness property of the number states $|k\rangle$ $(k = 0, 1, 2)$, then we obtain

$$<a^{+}a>=\frac{2}{\cosh r}\sum_{n=0}n\frac{(2n-1)!!}{n!}\left(\frac{1}{2}\tanh^2 r\right. \quad (16)$$

The sum over n can now be carried out. By differentiating both sides of the following relation

$$\sum_{n=0}^{\infty}n\frac{(2n-1)!!}{n!}\left(\frac{1}{2}\tanh^2 r\right)^n=\left(1-\tanh^2 r\right)^{-1/2} \quad (17)$$

with respect to $\tanh^2 r$, we get

$$\sum^{\infty}n\frac{(2n-1)!!}{n!}\left(\frac{1}{2}\tanh^2 r\right)^n=\frac{1}{2}\tanh^2 r\left(1-\tanh^2 r\right)^{-3/2}. \quad (18)$$

Now, a substitution from (18) into (16) leads immediately to

$$<a^{+}a>=\sinh^2 r. \quad (19)$$

We consider a linear superposition of operator (13) in the form

$$\rho=\int_0^{\infty}\int_{-\pi/2}^{\pi/2}dr\,d\varphi\,rQ(r,\varphi;\varepsilon)\,|r,\varphi\rangle\langle\varphi,r|, \quad (20)$$

where $r\,dr\,d\varphi$ is the element of area in the plane, while the weighting factor $Q(r,\varphi;\varepsilon)$ assumes the form

$$Q(r,\varphi;\varepsilon)=-(\pi\varepsilon)^{-1/2}\cos\varphi\frac{d}{dr}\left[\exp\left(-r^2/\varepsilon\right)\right] \quad (21)$$

with ε defined by (12). Expansion (20) is known as the P-representation for the density operator in terms of r and φ. This representation was introduced by Glauber [11] and Sudarshan [12], independently, in 1963.

In order to preserve the Hermitian and unitary characters of ρ, the P-representation (20) must be real and satisfy the normalization condition

$$\int_0^{\infty}\int_{-\pi/2}^{\pi/2}dr\,d\varphi\,r\,Q(r,\varphi;\varepsilon)=1. \quad (22)$$

In fact, owing to successful mathematical manipulations in addition to some physical restrictive conditions, the steady-state Boltzmann integrodifferential equation [13], has been reduced to a suitable form of a familiar second-order partial differential equation in the two-dimensional space. The solution of this equation is taken as a combination of two terms. Accordingly, it is found that the first term has the same physical and mathematical characters of the form of function (21). Details are given in an earlier publication [4].

Indeed, it is easy to show that for all value of r, the real weight function (21) takes positive definite values in the interval $-\pi/2<\varphi\le\pi/2$, while it takes negative values otherwise in the plane. This function distributes in phase through its dependence on function $\cos\varphi$. Despite of $Q(r,\varphi;\varepsilon)$ being a non-singular function, it may be considered more singular than δ - *function*, because $\cos\varphi$ projects all the vector states in the plane into the direction of the beam represented by the weight function $Q(r,\varphi;\varepsilon)$.

According to (6) and (21), operator (20) can be expressed in terms of the number states, as follows

$$\rho=\frac{2}{(\pi\varepsilon^3)^{1/2}}\sum_{m,n=0}^{\infty}\int_0^{\infty}\int_{-\pi/2}^{\pi/2}\cos\varphi\exp\left[2i(m-n)\varphi\right]$$

$$\times\frac{\left(-\frac{1}{2}\tanh r\right)^{m+n}}{m!n!}(\cosh r)^{-1}re^{-r^2/\varepsilon}\sqrt{(2m)!(2n)!}$$

$$\times|2m\rangle\langle 2n|\,r\,dr\,d\varphi. \quad (23)$$

Actually, the integral with respect to φ can be easily carried out by applying the rule of integration by parts, the result is

$$\int_{-\pi/2}^{\pi/2}d\varphi\cos\varphi\exp\left[2i(m-n)\varphi\right]=\frac{2(-1)^n}{1-4(m-n)^2} \quad (24)$$

If we insert the value of integral (24) into (23), then we get after a little manipulation, the following simple form

$$\rho=4(\pi\varepsilon^3)^{-1/2}\sum_{m,n=0}^{\infty}\left[\frac{(2m-1)!!(2n-1)!!}{m!n!}\right]^{1/2}$$

$$\frac{I_{mn}(\varepsilon)}{1-4(m-n)}|2m\rangle\langle 2n|. \quad (25)$$

where

$$I_{mn}(\varepsilon)=\int_0^{\infty}dr\,r^2(\cosh r)^{-1}\exp(-r^2/\varepsilon)\left(\frac{1}{2}\tanh^2 r\right)^{(m+n)/2} \quad (26)$$

The properties of ρ will be discussed in the next section. Indeed, we shall see that this operator verifies all the properties of the density operators.

4. Some properties of ρ

The first simple property of ρ is that its trace is unity *i.e.* $\mathrm{Tr}(\rho) = 1$. According to (25), we have

$$\mathrm{Tr}(\rho) = 4(\pi\varepsilon^3)^{-1/2} \sum_{m,n=0}^{\infty} \frac{(2m-1)!!(2n-1)!!}{m!n!}^{1/2} \frac{I_{mn}(\varepsilon)}{1-4(m-n)^2} \langle 2n|2m\rangle.$$

Since the number states $|k\rangle$ $(k = 0, 1, 2, \ldots)$ form a complete orthogonal set of vectors, the above relation reduces to

$$\mathrm{Tr}(\rho) = 4(\pi\varepsilon^3)^{-1/2} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!} I_{nn}(\varepsilon), \tag{27}$$

where

$$I_{nn}(\varepsilon) = \int_0^{\infty} dr\, r^2 (\cosh r)^{-1} \exp(-r^2/\varepsilon) \left(\frac{1}{2}\tanh^2 r\right)^n \tag{28}$$

which follows from (26) for $m = n$. Now, we insert (28) into the right side of (27) and then we use expansion (17), to obtain

$$\mathrm{Tr}(\rho) = 4(\pi\varepsilon^3)^{-1/2} \int_0^{\infty} dr\, r^2 \exp(-r^2/\varepsilon). \tag{29}$$

Using integration by parts, the right side of (29) becomes

$$\int_0^{\infty} dr\, r^2 \exp(-r^2/\varepsilon) = (\pi\varepsilon^3)^{1/2} \tag{30}$$

This allows us to write (29) as

$$\mathrm{Tr}(\rho) = 1. \tag{31}$$

Thus, the sum of the eigenvalues of ρ (which represents probabilities) must be equal to 1. This result emphasizes the important role of the weight function (21) in the theory of quantum radiation field.

The matrix elements of operator (25) are given by

$$\langle 2l|\rho|2k\rangle = 4(\pi\varepsilon^3)^{-1/2} \sum_{m,n=0}^{\infty} \left[\frac{(2m-1)!!(2n-1)!!}{m!n!}\right]^{1/2} \times \frac{I_{mn}(\varepsilon)}{1-4(m-n)^2} \delta_{lm}\,\delta_{nk}. \tag{32}$$

If we use the properties of δ *– function*, (32) reduces to the form

$$\langle 2l|\rho|2k\rangle = 4(\pi\varepsilon^3)^{-1/2} \left[\frac{(2l-1)!!(2k-1)!!}{l!k!}\right]^{1/2} \frac{I_{lk}(\varepsilon)}{1-4(l-k)^2}. \tag{33}$$

Since each of l and k takes only integer values or zero, the quantity $1-4(l-k)^2$ is always negative for $l \neq k$. This indicates that the off diagonal elements of the matrix $\langle 2l|\rho|2k\rangle$ are all negative and the absolute values of these elements decreases by increasing the value of the difference between l and k. On the other hand, since $1 - 4(l-k)^2$ reduces to 1 for $l = k$, the diagonal elements of matrix (33) are given by

$$\langle 2l|\rho|2k\rangle = 4(\pi\varepsilon^3)^{-1/2} \frac{(2k-1)!!}{k!} I_{kk}(\varepsilon). \tag{34}$$

Expression (34) shows that the diagonal elements of matrix (33) are all positive definite numbers as they must be.

Since the weighting factor $Q(r, \varphi; \varepsilon)$ is subject to the normalization condition (22), operator (20) is hermitian. In addition, the trace of this operator is equal to 1 and its diagonal elements are all positive definite. Thus, operator (20) possesses the necessary characters of the density operators [14, 15].

5. The mean of photons present in a quantum state

As we know, the operator that represents the number of photons present in a quantum state is a^+a [5]. Therefore, the mean number of photons present in the mixed-state (25) denoted by $\langle N\rangle$, is given by

$$\langle N \rangle = \mathrm{Tr}(\rho\, a^+ a). \tag{35}$$

If we substitute (25) into (35), then we get

$$\langle N \rangle = 4(\pi\varepsilon^3)^{-1/2} \sum_{m,n=0}^{\infty} \frac{(2m-1)!!(2n-1)!!}{m!n!}^{1/2} \times \frac{I_{mn}(\varepsilon)}{1-4(m-n)^2} (2m)\delta_{mn}.$$

where the eigenvalue equation $a|k\rangle = \sqrt{k}|k\rangle$ has been used. Also, if we use the properties of δ_{mn}, the mean number $\langle N\rangle$ reduces to

$$\langle N \rangle = 8(\pi\varepsilon^3)^{-1/2} \sum_{n=0} n \frac{(2n-1)!!}{n!} I_{nn}(\varepsilon). \tag{36}$$

Inserting the value of $I_{nn}(\varepsilon)$ given by (28), into the right hand side of (36), we obtain the expression

$$\langle N \rangle = 8(\pi\varepsilon^3)^{-1/2} \int_0^{\infty} dr \frac{r^2}{\cosh r} \exp(-r^2/\varepsilon) \sum_{n=0}^{\infty} n \frac{(2n-1)!!}{n!} \left(\frac{1}{2}\tanh^2 r\right)^n. \tag{37}$$

The sum over n in the right hand side of (37) may be carried out by using (18), which allows us to have

$$\langle N \rangle = 4(\pi\varepsilon^3)^{-1/2} \int_0^{\infty} dr\, r^2 \exp(-r^2/\varepsilon) \sinh^2 r. \tag{38}$$

Now, integral (30) assures the normalization condition $4(\pi\varepsilon^3)^{-1/2}\int_0^\infty dr\, r^2 \exp(-r^2/\varepsilon) = 1$, which means that the function $4(\pi\varepsilon^3)^{-1/2} r^2 \exp(-r^2/\varepsilon)$ may be considered as a normalized density. Thus, expectation value (38) may be written as [16],

$$< N > = \left\langle \sinh^2 r \right\rangle, \tag{39}$$

where $\left\langle \sinh^2 r \right\rangle$ denotes the mean value of $\sinh^2 r$.

Relation (19) shows that $\sinh^2 r$, which still depends on r (that varies from 0 to ∞), defines the mean number of photons present in pure state (13). Whereas, according to (39), the mean number of photons present in the mixed state (20) is the expectation value of $\sinh^2 r$, which is simply a number as it must be. Thus, while the mixed states supply us some interesting information about the structure of the field of photons, the pure state fails to do so.

Indeed, since the function $\sinh^2 r$ is independent of the phase angle φ, relation (38) can be written as

$$< N > = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^\infty dr\, r\, Q(r,\varphi;\varepsilon) \sinh^2 r. \tag{40}$$

The weighting factor $Q(r,\varphi;\varepsilon)$ is given by (21) and satisfies the normalization condition (22). In fact, relations (38) and (40) are identical. To see this, it is sufficient to insert the value of $Q(r,\varphi;\varepsilon)$ given by (21), into the right hand side of (40) and then integrate with respect to φ to obtain back the relation (38)

6. Mean number of photons as function of the mean free path

In order to find the relationship between the average of the total quanta $<N>$ present in the mixed-state (25) and the mean free path T, it is sufficient to carry out integral (38). This integral may be written again in the equivalent form

$$< N > = 2(\pi\varepsilon^3)^{-1/2} \int_0^\infty dr\, r^2 \exp(-r^2/\varepsilon)(\cosh 2r - 1). \tag{41}$$

The integral in the right hand side of (41) can be easily carried out, to give

$$< N > = \left(\varepsilon + \frac{1}{2}\right)\exp(\varepsilon) - \frac{1}{2}, \tag{42}$$

where ε is defined by (12) *i.e.*

$$\varepsilon = \frac{4}{3\alpha(1-\beta)}\int_0^u du'\, T(u'); \quad 0 \le \beta \prec 1.$$

Thus, (42) with ε given by the above integral is the desired relationship that relates the mean number of quanta present in a single-mode squeezed vacuum state and the mean free path $T(u)$

In fact, the objective of this job is to construct relationship (42), which allows fixing the mean number of photons in a single-mode squeezed vacuum state as function of ε that depends on the mean free path T, which can be measured experimentally.

7. Concluding remarks

From the results obtained above, we can conclude some interesting remarks as follows : (i) A relation between the weighting factor (21) and Glauber-Sudarshan's P function [15-17], can be easily found. For this purpose, let us write here the P function as

$$f(r) = \frac{1}{\pi\varepsilon}\exp(-r^2/\varepsilon); \quad 0 \le r \prec \infty, \tag{43}$$

where ε denotes the mean number of photons occupying this mode [3]. Accordingly, we can write function (21) in the form

$$Q(r,\varphi;\varepsilon) = -(\pi\varepsilon)^{-1/2}\cos\varphi\, f'(r). \tag{44}$$

Function $f'(r)$ denotes the derivative of $f(r)$, defined by (43), with respect to r. Relation (44) relates $Q(r,\varphi;\varepsilon)$ with the derivative of Glauber-Sudarshan's P-function.

(ii) According to (42), the mean number of photons present in a single-mode squeezed vacuum state is $(\varepsilon + 1/2)\exp(\varepsilon) - 1/2$, where ε represents the mean number of photons present in the chaotic or thermal state of the mode. This shows clearly, that a squeezed field is very dense comparing with a chaotic field. Thus, squeezing is an effect, which may occur in fields with high intensity. In this sense one may also say that the squeezing effect is a macroscopic quantum-mechanical effect. This fact has been shown by Walls [6] and recently by Daoud [18].

(iii) It is clear that when T vanishes, the parameter

$$\varepsilon = \frac{1}{3\alpha(1-\beta)}\int_0^u du'\, T(u')$$

tends to zero. In this case, the mean occupation numbers of both chaotic and squeezed vacuum states of the mode vanish.

(iv) According to (42), the expectation value of energy contained in the squeezed vacuum state is equal $\hbar\omega(\varepsilon + 1/2)\exp(\varepsilon)$ (ω denotes the angular frequency of the field mode). This energy tends to $\hbar\omega/2$ as ε tends to zero. This limit is the energy contained in the vacuum state of a quantum field mode. This assures the fact that the squeezing is a pure quantum effect.

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