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Double-diffusive convection in compressible Walters' B′ elastico-viscous fluid in hydromagnetics

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Abstract A layer of compressible, electrically conducting Walters' B′ elastico-viscous fluid heated and soluted from below, in presence of magnetic field is considered. The presence of viscoelasticity, magnetic field and stable solute gradient introduce oscillatory modes in the system which were non-existent in their absence. The sufficient conditions for non-existence of overstability are obtained. For the case of stationary convection, the Walters' elastico-viscous fluid behaves like a Newtonian fluid and compressibility, magnetic field and stable solute gradient have stabilizing effects on the system

Keywords : Double-diffusive convection, Walters' B′ fluid, hydromagnetics, compressibility

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1. Introduction

Chandrasekhar [1] has given a detailed account of thermal convection in Newtonian fluid layer in the presence of magnetic field. Veronis [2] has investigated the problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient. The buoyancy forces can arise not only from density differences due to variations in temperature but also from those due to variations in solute concentration. Thermosolutal convection problems arise in oceanography, limnology and engineering. Examples of particular interest are provided by ponds built to trap solar heat [3] and some Antarctic lakes [4]. Bhatia and Steiner [5] have studied the problem of thermal instability of a Maxwellian viscoelastic fluid in the presence of rotation and have found that the rotation has a destabilizing effect in contrast to the stabilizing effect on an ordinary (Newtonian) viscous fluid. Bhatia and Steiner [6] have also studied the thermal instability of a Maxwellian viscoelastic fluid in presence of magnetic field while the thermal convection in Oldroydian viscoelastic fluid in hydromagnetics has been considered by Sharma [7].

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There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's constitutive relations. One such class of elastico-viscous fluids is Walters' B′ fluid (Walters [8]). Sharma and Kumar [9] have studied the stability of two superposed Walters' B′ viscoelastic fluids.

Sharma *et al* have [10] studied the double-diffusive convection in Walters' B′ visco-elastico fluid in porous medium in presence of uniform rotation.

Keeping in mind, the growing importance of non-Newtonian and compressible fluids in chemical technology, industry and geophysical fluid dynamics, the present paper attempts to study the double-diffusive convection in compressible Walters' viscoelastic fluid B′ in the presence of uniform magnetic field.

2. Perturbation equations and dispersion relation

Consider an infinite compressible layer of Walters' elastico-viscous, electrically conducting fluid B′ confined between the planes $z = 0$ and $z = d$, acted on by gravity force $\boldsymbol{g}$ (0, 0,$-g$) and a uniform vertical magnetic field $\boldsymbol{H}$ (0,0, H). This layer is heated and soluted from below such that a steady adverse temperature gradient $\beta(=|dT/dz|)$ and a solute concentration gradient $\beta'(=|dC/dz|)$ are maintained.

For double-diffusive convection problem, the Boussinesq approximation has been used, which is well justified in the case of incompressible fluids.

When the fluids are compressible, the equations governing the system become quite complicated. Spiegel and Veronis [11] have simplified the set of equations governing the flow of compressible fluids under the assumption that the depth of the fluid layer is much smaller than the scale height as defined by them, if only motions of infinitesimal amplitude are considered. Spiegel and Veronis [11] defined f as any one of the state variables (pressure (p), density(ρ) or temperature(T)) and expressed in the form

$$f(x, y, z, t) = f_m + f_0(z) + f'(x, y, z, t),$$

where f_m is the constant space average of f, f_0 is the variation in the absence of motion and f' is the fluctuation resulting from motion. The thermal instability in compressible fluids in presence of rotation and magnetic field has been studied by Sharma [12].

The linearized hydromagnetic perturbation equations for thermosolutal convection in Walters' elastico-viscous fluid B′ [1,2,8,12] are

$$\frac{\partial \boldsymbol{q}}{\partial t} = -\frac{1}{\rho_m}\nabla\delta p + \boldsymbol{g}\frac{\delta\rho}{\rho_m} + \frac{\mu_e}{4\pi\rho_m}(\nabla\times\boldsymbol{h})\times\boldsymbol{H} + \left(\upsilon - \upsilon'\frac{\partial}{\partial t}\right)\nabla^2\boldsymbol{q}, \tag{1}$$

$$\nabla\cdot\boldsymbol{q} = 0, \tag{2}$$

$$\nabla\cdot\boldsymbol{h} = 0, \tag{3}$$

$$\frac{\partial \boldsymbol{h}}{\partial t} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{q} + \eta\nabla^2\boldsymbol{h}, \tag{4}$$

$$\frac{\partial\theta}{\partial t} = \left(\beta - \frac{g}{c_p}\right)w + \kappa\nabla^2\theta, \tag{5}$$

$$\frac{\partial\gamma}{\partial t} = \beta' w + \kappa'\nabla^2\gamma. \tag{6}$$

Here $\boldsymbol{q}$ (u, v, w), $\boldsymbol{h}$ (h_x, h_y, h_z), $\delta p, \delta\rho, \theta$ and γ denote respectively the perturbations in velocity (0, 0 , 0), magnetic field $\boldsymbol{H}$ (0 , 0 , H), pressure p, density ρ, temperature T and solute concentration C. $\upsilon, \upsilon', \kappa, \kappa', \mu_e$ and η stand for kinematic viscosity, kinematic viscoelasticity, thermal diffusivity, solute diffusivity, magnetic permeability and electrical resistivity respectively. The equation of state is

$$\rho = \rho_m\left[1 - \alpha(T - T_0) + \alpha'(C - C_0)\right], \tag{7}$$

where the suffix zero refers to values at the reference level $z = 0$, α is the coefficient of thermal expansion and α' is the analogous solvent expansion. Therefore, the change in density $\delta\rho$ caused by the perturbations θ and γ in temperature and solute concentration is given by

$$\delta\rho = -\rho_m(\alpha\theta - \alpha'\gamma). \tag{8}$$

Eqs. (1)–(6) and (8) yield

$$\frac{\partial}{\partial t}\nabla^2 w - \left(\upsilon - \upsilon'\frac{\partial}{\partial t}\right)\nabla^4 w - \frac{\mu_e H}{4\pi\rho_0}\frac{\partial}{\partial z}\nabla^2 h_z - g\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\alpha\theta - \alpha'\gamma) = 0, \tag{9}$$

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\theta = \left(\beta - \frac{g}{c_p}\right)w, \tag{10}$$

$$\left(\frac{\partial}{\partial t} - \kappa'\nabla^2\right)\gamma = \beta' w, \tag{11}$$

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)h_z = H\frac{\partial w}{\partial z}, \tag{12}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We now analyze the disturbances into normal modes, assuming that the perturbation quantities have the space and time dependence of the form

$$[w, \theta, h_z, \gamma] = [W(z), \Theta(z), K(z), \Gamma(z)]\exp\left(ik_x x + ik_y y + nt\right), \tag{13}$$

where k_x, k_y are the wave numbers along the x- and y- directions respectively. $k\left(=\sqrt{k_x^2 + k_y^2}\right)$ is the resultant wave number and n is, in general, a complex constant.

The dimensionless forms of eqs. (9)–(12), using expression (13), are

$$\sigma\left(D^2 - a^2\right)W - (1 - F\sigma)\left(D^2 - a^2\right)^2 W - \frac{\mu_e H d}{4\pi\rho_m \nu}\left(D^2 - a^2\right)DK + \frac{g d^2 a^2}{\nu}(\alpha\Theta - \alpha'\Gamma) = 0, \tag{14}$$

$$\left(D^2 - a^2 - p_2\sigma\right)K = -\frac{na}{\eta}DW, \tag{15}$$

$$\left(D^2 - a^2 - p_1\sigma\right)\Theta = -\left(\frac{G-1}{G}\right)\frac{\beta d^2}{\kappa}W, \tag{16}$$

and
$$\left(D^2 - a^2 - q\sigma\right)\Gamma = -\frac{\beta' d^2}{\kappa'}W, \tag{17}$$

where $G = c_p\beta/g$, $a = kd$, $\sigma = nd^2/\nu$, $x^* = x/d$, $y^* = y/d$, $z^* = z/d$ and $D = d/dz^*$, $p_2 = \upsilon/\eta$ is the magnetic Prandtl number, $p_1 = \upsilon/\kappa$ is the Prandtl number, $q = \upsilon/\kappa'$ is the Schmidt number and $F = \upsilon'/d^2$ is the dimensionless kinematic viscoelasticity.

Eliminating Θ, Γ, K between eqs. (14 – 17), we obtain

$$(D^2 - a^2)(D^2 - a^2 - p_1\sigma)(D^2 - a^2 - q\sigma)$$

$$\left[\sigma(D^2 - a^2 - p_2\sigma) - (1 - F\sigma)(D^2 - a^2)(D^2 - a^2 - p_2\sigma)\right.$$

$$\left.+QD^2\right]W = (D^2 - a^2 - p_2\sigma)\left[\left(\frac{G-1}{G}\right)Ra^2(D^2 - a^2 - q\sigma)\right.$$

$$\left.-Sa^2(D^2 - a^2 - p_1\sigma)\right]W, \tag{18}$$

where

$R = \dfrac{g\alpha\beta d^4}{\upsilon\kappa}$ is the Rayleigh number,

$S = \dfrac{g\alpha'\beta' d^4}{\upsilon\kappa'}$ is analogous solute Rayleigh number,

and $Q = \dfrac{\mu_e H^2 d^2}{4\pi\rho_m \nu\eta}$ is the Chandrasekhar number.

Consider the case in which both the boundaries are free, the medium adjoining the fluid is perfectly conducting and temperatures, solute concentrations at the boundaries are kept fixed. The boundary conditions, appropriate to the problem, are

$$W = D^2W = 0, K = 0, \Theta = 0, \Gamma = 0 \text{ at } z = 0 \text{ and } 1. \tag{19}$$

The proper solution of the eq. (18) characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{20}$$

where W_0 is constant.

Substituting eq. (20) in (18), we get

$$R_1 = \left(\frac{G}{G-1}\right)\frac{(1+x)(1+x+ip_1\sigma_1)}{x}\left[(1+x)(1 - i\pi^2 F\sigma_1) + i\sigma_1\right]$$

$$\left(\frac{G}{G-1}\right)Q_1\left(\frac{1+x}{x}\right)\times\frac{(1+x+ip_1\sigma_1)}{(1+x+ip_2\sigma_1)}$$

$$\left(\frac{G}{G-1}\right)S_1\frac{(1+x+ip_1\sigma_1)}{(1+x+iq\sigma_1)}, \tag{21}$$

where $R_1 = R/\pi^4$, $S_1 = S/\pi^4$, $Q_1 = Q/\pi^2$, $x = a^2/\pi^2$ and $i\sigma_1 = \sigma/\pi^2$ (where σ can be complex).

3. The stationary convection

For stationary convection ($\sigma = 0$), eq. (21) reduces to

$$R_1 = \left(\frac{G}{G-1}\right)\frac{(1+x)}{x}\left[(1+x)^2 + Q_1\right] + S_1\left(\frac{G}{G-1}\right). \tag{22}$$

Eq. (22) implies that for stationary convection, compressible Walters' elastico-viscous fluid B' behaves like compressible Newtonian fluid.

$$\frac{dR_1}{dS_1} \quad G-1 \tag{23}$$

and

$$\frac{dR_1}{dQ_1} = \left(\frac{G}{G-1}\right)\left(\frac{1+x}{x}\right) \tag{24}$$

The stable solute gradient and magnetic field have stabilizing effects on the system.

For fixed Q_1 and S_1, let G (accounting for the compressibility effects) be also kept fixed in eq. (22). Then we find

$$\bar{R}_c = \frac{G}{G-1}\,R_c, \tag{25}$$

where $\bar{R}_c$ and R_c denote respectively the critical Rayleigh numbers in the presence and absence of compressibility. The effect of compressibility is thus to postpone the onset of double-diffusive convection. The compressibility, therefore, has a stabilizing effect on the thermosolutal convection.

4. Stability of the system and oscillatory modes

Multiplying eq. (14) by W^*, the complex conjugate of W, integrating over the range of z and making use of eqs. (15) -(17) together with the boundary conditions (19), we obtain

$$\sigma I_1 - \left(\frac{G}{G-1}\right)\left(\frac{ga^2\alpha\kappa}{\nu\beta}\right)(I_2 + p_1\sigma^* I_3) + \left(\frac{ga^2\alpha'\kappa'}{\nu\beta'}\right)$$

$$(I_4 + q\sigma^* I_5) + \frac{\mu_e\eta}{4\pi\rho_0\nu}(I_6 + p_2\sigma^* I_7) + (1 - F\sigma)I_8 = 0. \tag{26}$$

where σ^* is the complex conjugate of σ

and $I_1 = \int_0^1 \left(|DW|^2 + a^2|W|^2\right)dz,$

$$I_2 = \int_0^1 \left(|D\Theta|^2 + a^2|\Theta|^2\right)dz,$$

$$I_3 = \int_0^1 |\Theta|^2 dz,$$

$$I_4 = \int_0^1 \left(|D\Gamma|^2 + a^2|\Gamma|^2\right)dz, \tag{27}$$

$$I_5 = \int_0^1 |\Gamma|^2 dz,$$

$$I_6 = \int_0^1 \left(\left|D^2 K\right|^2 + 2a^2|DK|^2 + a^4|K|^2\right)dz,$$

$$I_7 = \int_0^1 \left(|DK|^2 + a^2|K|^2\right)dz,$$

$$I_8 = \int_0^1 \left(\left|D^2 W\right|^2 + 2a^2|DW|^2 + a^4|W|^2\right)dz.$$

The integrals $I_1 - I_8$ are all positive definite. Putting $\sigma = \sigma_r + i\sigma_i$ and then equating real and imaginary parts of eq. (26), we obtain.

$$\left[I_1 - \left(\frac{G}{G-1}\right)\frac{ga^2\alpha\kappa}{\nu\beta} p_1 I_3 + \frac{ga^2\alpha'\kappa'}{\nu\beta'} q I_5 + \frac{\mu_e\eta}{4\pi\rho_m\nu} p_2 I_7 - F I_8\right]\sigma_r$$

$$= -\left[I_2 - \left(\frac{G}{G-1}\right)\left(\frac{ga^2\alpha\kappa}{\nu\beta}\right) I_2 + \frac{ga^2\alpha'\kappa'}{\nu\beta'} I_4 + \left(\frac{\mu_e\eta}{4\pi\rho_m\nu}\right) p_2 I_6\right] \tag{28}$$

and

$$\left[I_1 - F I_8 + \left(\frac{G}{G-1}\right)\left(\frac{ga^2\alpha\kappa}{\nu\beta}\right) p_1 I_3 - \left(\frac{ga^2\alpha'\kappa'}{\nu\beta'}\right) q I_5 - \left(\frac{\mu_e\eta}{4\pi\rho_m\nu}\right) p_2 I_7\right]\sigma_i = 0. \tag{29}$$

It is clear from eq. (28) that σ_r is positive or negative which means that the system is stable or unstable. It is clear from (29) that σ_2 may be zero or nonzero, meaning that the modes may be non-oscillatory or oscillatory. The oscillatory modes are introduced due to the presence of viscoelasticity, stable solute gradient and magnetic field, which were non-existent in their absence.

5. The case of overstability

Here, we consider the possibility of whether instability may occur as an overstability. Since we wish to determine the critical Rayleigh number for overstability, it suffices to find conditions for which eq. (21) will admit of solution with σ_1 real.

Equating the real and imaginary parts of eq. (21) and eliminating R_1 between them, we obtain

$$Ac_1^2 + Bc_1 + C = 0, \tag{30}$$

where $A = p_2^2 q^2 \alpha\left(1 + p_1 - \pi^2 F\alpha\right)$,

$$B = \left(p_2^2 + q^2\right)\alpha^3\left(1 + p_1 - \pi^2 F\alpha\right) + Q_1 q^2 \alpha (p_1 - p_2) + S_1 p_2^2(\alpha - 1)(p_1 - q), \tag{31}$$

$$C = \alpha^5\left(1 + p_1 - \pi^2 F\alpha\right) + Q_1\alpha^3(p_1 - p_2) + S_1\alpha^2(\alpha - 1)(p_1 - q),$$

and $c_1 = \sigma_1^2, \alpha = (1 + x)$.

Since σ_1 is real for overstability, both the values of c_1 are positive. Eq. (30) is quadratic in c_1 and does not involve any of its roots to be positive if

$$p_1 > p_2,\ p_1 > q \text{ and } p_1 > \pi^2 F\alpha, \tag{32}$$

which implies that

$$\kappa < \eta,\ \kappa < \kappa' \text{ and } \kappa(\upsilon'/d^2)(\pi^2 + k^2 d^2) < \upsilon. \tag{33}$$

Thus, $\kappa < \eta, \kappa < \kappa'$ and $\kappa(\upsilon'/d^2)(\pi^2 + k^2 d^2) < \upsilon$ are the sufficient conditions for non-existence of overstability, the violation of which does not necessarily imply occurrence of overstability.

6. Conclusion

With the growing importance of non-Newtonian fluids in chemical engineering, modern technology and industry, the investigations on such fluids are desirable. The Walters elastico-viscous fluid (model B′) is one such fluid. Walters [14] reported that the mixture of polymethyl methacrylate and pyridine at 25°C containing 30.5 g of polymer per litre with density 0.98 g per litre behaves very nearly as the Walters (model B′) elastico-viscous fluid.

A detailed account of the thermal instability in Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [1]. The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been investigated by Veronis [2].

Elastico-viscous fluids may have different role as compared to Newtonian fluids, on the stability problems. For example, the effect of a uniform rotation on the thermal instability of Maxwellian viscoelastic fluid is destabilizing [5] whereas the uniform rotation has a stabilizing effect on the thermal instability of Newtonian fluid. Similarly, for the case of two superposed Walters B′ viscoelastic fluids in porous medium [13], the system is found to be stable or unstable if the kinematic viscoelasticity (assumed equal for both fluids) is less than or greater than the medium permeability divided by medium porosity, for the potentially stable arrangement. This is in contrast to the stability of two superposed Newtonian fluids in porous medium where the system is stable for the stable configuration.

A layer of compressible, electrically conducting Walters' B$'$ elastico-viscous fluid heated and soluted from below has been considered in the presence of a uniform vertical magnetic field. For stationary convection, the Walters' B$'$ elastico-viscous fluid behaves like a Newtonian fluid and compressibility, magnetic field and stable solute gradient have stabilizing effects on the system. The presence of viscoelasticity, magnetic field and stable solute gradient introduces oscillatory modes in the system which were non-existent in their absence. The sufficient conditions for non-existence of overstability are obtained, the violation of which does not necessarily imply occurrence of overstability.

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