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An optimal angle of launching a point mass in a medium with quadratic drag force

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Abstract : The problem of finding an optimal angle of launching a point mass in a medium with quadratic drag force is considered. An equation for determining an approximate value of this angle is obtained. After finding the optimal angle of launching, eight main parameters of the particle motion are analytically determined. These parameters are used to analytically construct six main functional relationships of the problem. An example is given.

Keywords : Analytical formulae, functional relationships, kinematical parameters

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The problem of the motion of a point mass under a drag force is considered in innumerable papers, beginning with Euler's [1-10]. Most of them solve the problem numerically. Analytical approaches to the solution of the problem are developed insufficiently. Meanwhile, analytical solutions are compact and convenient for a direct use in applied problems and for a qualitative analysis. Comparatively simple approximate analytic formulae to study point mass motion with quadratic drag force were obtained within the framework of this approach in [5], [6]. One of the most important aspects of the problem is the determination of the optimal angle of throwing which provides the maximum range of throw [7], [8]. This paper shows, how the formulae [5, 6] are used to find the optimal angle of throwing and other kinematical parameters of the motion. The given example makes a comparison between analytically obtained characteristics of motion and those calculated numerically. Numeric values of parameters are obtained by integration of point mass motion equations.

The problem of the motion of a point mass under a drag force with a number of conventional assumptions, in the case of the drag force proportional to the square of the velocity, $R = mgkV^2$, boils down to the solution of the differential system [9] :

$$\dot{V} = -g\sin\theta - gkV^2, \quad \dot{\theta} = -g\cos\theta / V,$$
$$\dot{x} = V\cos\theta, \quad \dot{y} = V\sin\theta. \qquad (1)$$

Here V is the speed of the point mass, m is the mass, θ is the trajectory slope angle to the horizontal, g is the acceleration due to gravity, x, y are the Cartesian coordinates of the point, and k is a proportionality factor (Figure 1).

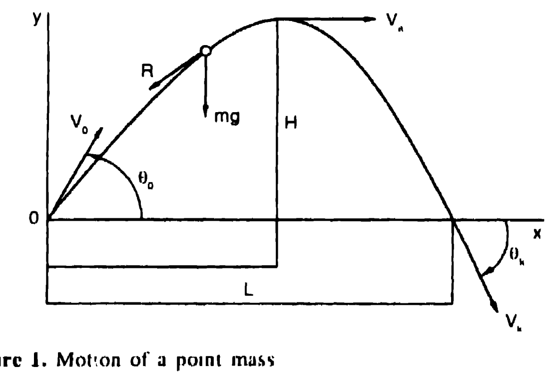

Figure 1. Motion of a point mass

The well-known solution of eqs. (1) consists of an explicit analytical dependence of the velocity on the slope angle of the trajectory and three quadratures [9]

$$V(\theta) = \frac{V_0 \cos\theta_0}{\cos\theta\sqrt{1 + kV_0^2 \cos^2\theta_0 (f(\theta_0) - f(\theta))}},$$

$$f(\theta) = \frac{\sin\theta}{\cos^2\theta} + \ln tg\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \tag{2}$$

$$t = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta} \frac{V}{\cos\theta}d\theta, \quad x = x_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta,$$

$$y = y_0 - \int_{\theta_0}^{\theta} V^2 tg\theta\, d\theta. \tag{3}$$

Here, V_0 and θ_0 are the initial values of the velocity and the slope of the trajectory respectively, t_0 is the initial value of the time, x_0 and y_0 are the initial values of the coordinates of the point mass.

The integrals on the right-hand sides of (3) are not taken in finite form. Hence, to determine the variables t, x and y we must either integrate (1) numerically or evaluate the definite integrals (3).

Using the integration of quadratures (3) by parts for enough small interval $[\theta_0, \theta]$, the variables t, x and y can be written in the form [5]

$$t(\theta) = t_0 + \frac{2(V_0 \sin\theta_0 - V\sin\theta)}{g(2+\varepsilon)},$$

$$x(\theta) = x_0 + \frac{V_0^2 \sin 2\theta_0 - V^2 \sin 2\theta}{2g(1+\varepsilon)}$$

$$y(\theta) = y_0 + \frac{V_0^2 \sin^2\theta_0 - V^2 \sin^2\theta}{g(2+\varepsilon)},$$

$$= k\left(V_0^2 \sin\theta_0 + V^2 \sin^2\theta\right). \tag{4}$$

Hence, for a small interval $[\theta_0, \theta]$, the trajectory of the point mass can be approximated by eq. (4). These formulae have a local form. We can construct the whole trajectory very accurately in steps. We can calculate $V(\theta)$, $t(\theta)$, $x(\theta)$, $y(\theta)$ using eqs (2) and (4) at the right-hand end of the interval $[\theta_0, \theta]$ and take them as the initial values for the following step

$$V_0 = V(\theta),\ t_0 = t(\theta),\ x_0 = x(\theta),\ y_0 = y(\theta).$$

This cyclical procedure replaces both numerical integration of system (1) and the evaluation of the integrals (3). The decrease of k leads to the increased range $[\theta_0, \theta]$ of applicability of the formulae obtained. When $k = 0$, *i.e.* when there is no drag force, eqs. (4) reduce to the well-known accurate formulae of the theory of the parabolic motion of a point mass and became valid for any θ_0 and θ values. As calculations show [5], the trajectory obtained by integrating system of eq. (1) and the trajectory constructed using formulae (2) and (4), are identical. Here to construct the trajectory it is sufficient to use a step $\Delta\theta = \theta - \theta_0$ of the order of $0.1°$.

Eqs. (4) enable us to obtain comparatively simple approximate analytical formulae for the main parameters of motion of the point mass [6]. We will give a complete summary of the formulae for the maximum ascent height H, time of flight T, particle speed V_a at apex, flight distance L, time of ascent t_a, trajectory horizontal coordinate at apex x_a, the angle of incidence θ_k and the final velocity V_k (see Figure 1) :

$$H = \frac{V_0^2 \sin^2\theta_0}{g\left(2 + kV_0^2 \sin\theta_0\right)}, \quad T = 2$$

$$V_a = \frac{V_0 \cos\theta_0}{\sqrt{1 + kV_0^2 \cos^2\theta_0 f(\theta_0)}}, \quad L = V_a T,$$

$$t_a = \frac{T - kHV_a}{2}, \quad x_a = \sqrt{LH.ctg\theta_0},$$

$$\theta_k = -arctg\frac{LH}{(L - x_a)^2} \quad V_k = V(\theta_k) \tag{5}$$

The function $V(\theta)$ in (5) is defined by relation (2).

In the case of no drag ($k = 0$), eqs. (5) are reduced to the respective relationships of the point mass parabolic motion theory. All parameters defined by formulae (5) are functions of the initial conditions of throwing V_0, θ_0.

In turn, eqs (5) make it possible to obtain simple analytic formulae for basic functional relationships of the problem $y(x)$, $y(t)$, $x(t)$, $x(\theta)$, $y(\theta)$, $t(\theta)$ [6] :

$$y(x) = \frac{Hx(L-x)}{x_a^2 + (L - 2x_a)x},$$

$$y(t) = \frac{Ht(T-t)}{t_a^2 + (T - 2t_a)\cdot t},$$

$$x(t) = \frac{L\left(w_1^2 + w_2 + w_1\sqrt{w_1^2 + \eta w_2}\right)}{2w_1^2 + nw_2},$$

$$x(\theta) = a\left(1 + \frac{1-n}{\sqrt{1 + b.tg\theta}}\right),$$

$$y(\theta) = c\left(d - \frac{2 + b.tg\theta}{\sqrt{1 + b\cdot tg\theta}}\right),$$

$$t(\theta) = a_1\left(1 + \frac{1-\mu}{\sqrt{1 + b_1 V\sin\theta}}\right). \tag{6}$$

Here, $n = L / x_a$, $a = x_a / (2 - n)$,

$b = (L - 2x_a) / H$, $c = H(n-1)/(2-n)^2$,

$d = 2 + H / c$, $w_1(t) = t - t_a$,

$w_2(t) = 2t(T - t) / n$, $\eta = 2(n-1)/n$,

$\mu = T / t_a$, $a_1 = t_a / (2 - \mu)$,

$b_1 = (T - 2t_a) / H$.

The function $V(\theta)$ in (6) is defined by relation (2). The functions $x(\theta)$, $y(\theta)$, $t(\theta)$ are defined on the interval $\theta_k < \theta \leq \theta_0$. Thus, with the known motion parameters H, L, T, x_a, t_a, formulae (6) make it possible to construct functions $y(x)$, $y(t)$, $x(t)$, $x(\theta)$, $y(\theta)$, $t(\theta)$.

The formula for the range of throw is written as

$$L(\theta_0) = V_a(\theta_0) \cdot T(\theta_0).$$

The optimal angle of throwing α, which provides the maximum distance of flight, is a root of equation

$$\frac{dL(\theta_0)}{d\theta_0} = 0.$$

Differentiating the $L(\theta_0)$ function with respect to θ_0 and after certain transformations, we obtain the equation for finding the angle α

$$tg^2\alpha + \frac{p\sin\alpha}{4 + 2p\sin\alpha} = \frac{1 + p\lambda}{1 + p\left(\sin\alpha + \lambda\cos^2\alpha\right)}. \qquad (7)$$

Here $p = kV_0^2$, $\lambda(\alpha) = \ln tg\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

With $k = 0$, the eq. (7) gives the known solution $\alpha = 45^\circ$. With $k \neq 0$, the eq. (7) is easy to solve graphically or numerically.

With a condition

$$p = kV_0^2 = \text{constant},$$

it is possible to change values k and V_0 simultaneously. The optimal angle of throwing α will be alike. But main parameters of motion H, L, T will change as it follows from the formulae (5). Let the values of motion parameters H_0, L_0, T_0 correspond to drag coefficient k_0, and values H_1, L_1, T_1 to drag coefficient $k_1 = qk_0$. Then with the condition

$$p = k_0V_0^2 = k_1V_1^2 = \text{constant},$$

we get the correlations

$$H_1 = \frac{H_0}{q}, \quad L_1 = \frac{L_0}{q}, \quad T_1 = \frac{T_0}{\sqrt{q}}.$$

Assume that with the given values of k and V_0, the angle α is found. Substituting it in formulae (5), we determine eight main parameters of the optimal trajectory. Using the parameters H, L, T, x_a, t_a, we construct with the help of eq. (6), six main functional relationships of the problem.

Let us find the optimal angle of launching and other characteristics of motion for the following parameters of the baseball [10]:

$$k = 0.000548\ s^2/m^2,\ V_0 = 40\ m/s,\ g = 9.81\ m/s^2.$$

The results of calculations are shown in Table 1 and in Figures 2-7. The exact values of the motion parameters are obtained by the numeric integration of eqs. (1). The relationships $y(x)$, $y(t)$, $x(t)$, $x(\theta)$, $y(\theta)$, $t(\theta)$ are presented in Figure 2 through Figure 7, respectively.

Table 1. Optimal trajectory parameters

parameter	exact value	calculated value by means (2 5)	error (%)
α (deg)	41.2°	40.8°	-1.0
L (m)	101.4	101.5	0.1
H (m)	27.0	27.1	0.4
T (s)	4.68	4.70	0.4
V_a (m/s)	21.4	21.6	0.9
t_a (s)	2.22	2.19	-1.3
x_a (m)	55.5	56.4	1.6
θ_k (deg)	-52.5°	-53.5°	1.9
V_k (m/s)	26.2	26.6	1.5

Each of the Figures displays two functions. Thin solid lines show functions obtained by numeric integration of eqs. (1). Broken curves show the same functions constructed from formulae (5) and (6).

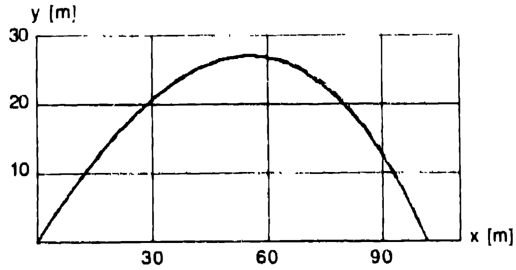

Figure 2. Graph of the dependence $y(x)$

The tabulated data show that analytically calculated parameters of the optimal trajectory are determined with errors of 1 – 2 %.

Analysis of the curves in Figures 2 through 7 shows that numerically obtained functions $y(x)$, $y(t)$, $x(t)$, $x(\theta)$, $y(\theta)$, $t(\theta)$ practically coincide with the same functions constructed from formulae (5) and (6).

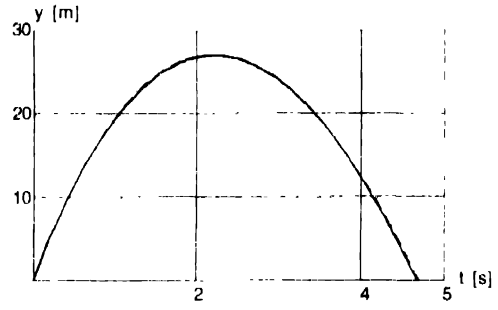

Figure 3. Graph of the dependence $y(t)$

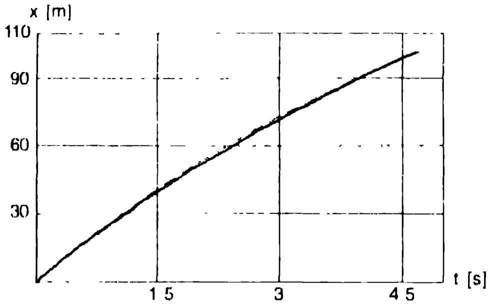

Figure 4. Graph of the dependence $x(t)$

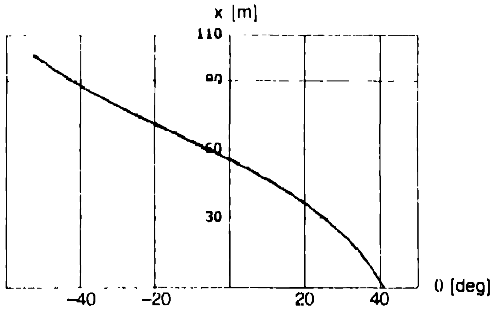

Figure 5. Graph of the dependence $x(\theta)$

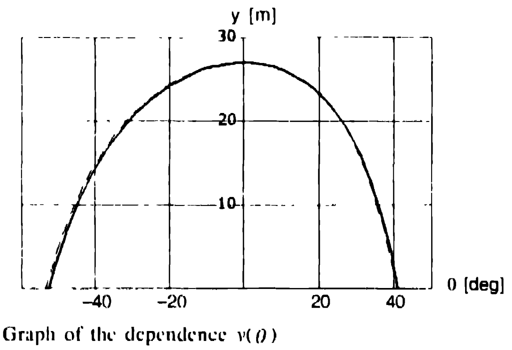

Figure 6. Graph of the dependence $y(\theta)$

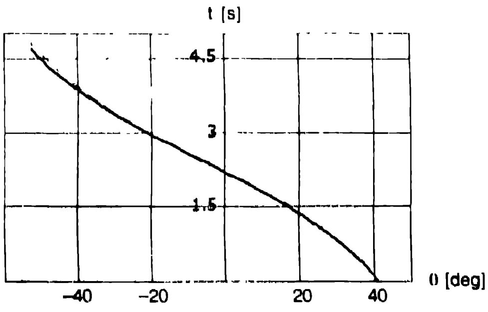

Figure 7. Graph of the dependence $t(\theta)$

Figure 8 is an interesting geometric picture. If we use motion parameters L, H, x_a to construct the ABC triangle with the height $BB_1 = LH$, $AB_1 = x_a^2$, $CB_1 = (L - x_a)^2$, then in this triangle $\angle A \cong \theta_0$, $\angle C \cong \theta_k$.

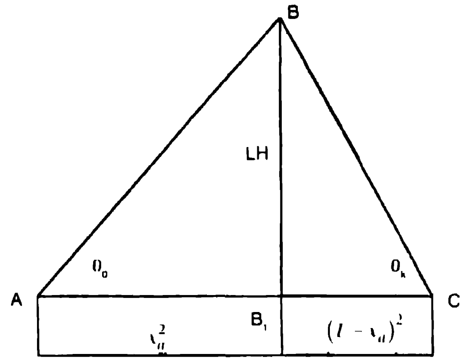

Figure 8. Motion parameters

Thus, formulae (5) and (6) make it possible to calculate main kinematical parameters of motion and analytically construct the most significant functional relationships of the problem directly from the known initial conditions of throwing V_0, θ_0 and factor k with an acceptable accuracy.

As a whole, the set of formulae (4–7) considerably widens the possibilities of studying the problem of point mass motion with quadratic drag force and supplement numeric methods with the analytic ones.

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