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Planetary orbits around a spinning gravitating star

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Abstract It is an empirical fact based on solar physics of sunspots, that our sun and generally stars are spinning gravitational sources for bodies in orbit in any such solar system. Thus, a planetary theory for the gravitational field of a star or a satellite dynamics around a (spinning) planet – including the problem of artificial satellites – should in principle, take into account the possible effect of axial symmetry. But to the best of our knowledge, existing formalisms of gravitation physics do not address the problem of spin referred to above This paper proposes the following strategy to handle this kind of computational physics. Our calculations suggest that this kind of spin should give rise to a slight residual perturbation on constant areal velocity, computed by the standard model of orbital theory. In particular, the second law of planetary motion might require revision. Also, it turns out that the classical result of Kepler is recoverable from our result as a special case

Keywords : Spin of a star, planetary orbits, Kepler's laws, areal velocity

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1. Introduction

Spin of physical systems, even in the classical sense, continues to offer newer surprises to workers in the field of physics. One of these phenomena is related to solar physics. After having been practically ignored for more than two centuries, a long series of observations of the apparent motion of sunspots, beginning somewhere around 1850s, has established the problem of solar spin as an independent active area of study in its own right. It is by now confirmed that the outer visible envelope of the sun does not rotate like a solid body. Astrophysical observations show that a typical sunspot takes about fourteen days to cross the solar disc and that this time is the same whether the spot passes through the centre of the solar disc or along a shorter path at some distance from the centre. Of course, the rate of motion of a particular spot is by no means uniform, it always appearing much slower when near the solar limb than when near the centre. As such, different sunspots give different periods of spin and the period of solar spin varies as a function of heliocentric latitude φ, given by $\xi \approx (14.37 - \sin^{7/5} \varphi)$ degree / day for the diurnal angle ξ of solar spin, the spin period being minimum at the equator and increasing gradually towards the poles. After correction for the annual motion of the earth around the sun in the ecliptic, a mean period of 24.96 days at the solar equator has been derived.

The existence of solar spots [1, 2] convincingly establishes the phenomenon of axial spin of our sun. This solar physics of spin gives birth to an important problem in celestial mechanics which lies heretofore unattended. Thus, a planetary theory for the gravitational field of a star or a satellite dynamics around a (spinning) planet – including the problem of artificial satellites, should in principle, take into account the possible effect of axial symmetry. Such a relativistic correction of standard theories of classical mechanics can no more be ignored [3]. This is amply demonstrated the series of 1997 resolutions and decisions of the Commission 7 of International Astronomical Union (IAU) on Celestial Mechanics and Dynamical Astronomy [4].

But to the best of our knowledge, existing formalisms of gravitation physics do not address the problem of spin referred to above. The work reported here proposes the following strategy to handle this kind of computational physics. As is well known, from the viewpoint of symmetry structure of space-

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times, effects of gravitational field of any spinning source on its attendant satellites in their respective orbits may be computed and modelled using a class of axisymmetric manifolds possessing stationarity.

The plan of the paper is as follows. In Section 2, considering a specific class of axisymmetric stationary solutions of Einstein field equations, we work out orbit equations in the context of the space-time manifolds mentioned. Next, in Section 3, we derive an interesting application of the modified Keplerian second law (MKSL) in a central force field. Finally, Section 4 comprises a summary and further research directions.

2. Perturbations on orbital motion due to the spin of the central gravitating source

An incorporation of the spin correction in orbital theory needs to be carried out in the framework of modern gravitation physics [5, 6]. This means what would solve the problem is the choice of the class of all axisymmetric spacetimes. As a test case and for the sake of mathematical tractability, in an earlier paper we considered the needed incorporation of spin correction [7] in the class of axisymmetric space-time manifolds possessing staticity, first studied in [8]. To be able to interpret things physically, we carried out the analysis of orbital evolution equations by invoking the concept of relativistic multipole moments of the gravitating source as a perturbation on static axisymmetric space-times.

However, what is nearer the ground reality is the class of axisymmetric stationary fields studied in relativistic gravitodynamics (RGD) [9]. Now an attempt to incorporate the effects of the spin of the central body of any solar system into orbit calculations of a satellite or planet immediately confronts us with the problem of seeking an exact solution of Einstein field equations of gravitation for a reasonable stress-energy tensor $T_{\mu\nu}\left(x^{\alpha}\right)$ figuring in these field equations. In RGD, a determination of such an exact solution, either the interior or the exterior gravitational field representing even a uniformly rotating homogeneous inviscid fluid mass, presents formidable problems. Even today, one is far from finding such an exact solution, if one exists in the case in question. In general, finding exact solutions of Einstein field equations for well-defined physical situations is extremely difficult and few such solutions are known. In fact, gravitational field of a uniformly rotating bounded source must depend on at least two variables. But finding any solutions of Einstein field equations depending on two or more variables is quite difficult, let alone a physically interesting one.

An important aspect of this difficulty is this. An essential property of a physically significant solution representing the exterior field of a bounded spinning source is that it should be asymptotically flat. It is because the gravitational field (due to a source with nonzero mass) tends to zero mass as we move further and further away from the source. However, no interior solution has yet been found which matches, on the boundary of the gravitating source, smoothly onto the exterior solution possessing the property of asymptotic flatness. Although several stationary axially symmetric exact solutions of Einstein equations are known, very few of these are asymptotically flat. Thus their physical interpretation is uncertain.

Another desirable property of a physically relevant asymptotically flat solution of field equations of gravitation is that when the angular momentum of the source producing the field tends to zero, the solution tends to Schwarzschild solution, representing the exterior field of a spherically symmetric source. This behaviour is what we would expect for relativistic stars, because for the latter the departure from spherical symmetry is usually caused by spin and if the spin vanishes we would get a spherically symmetric star, whose exterior field is a Schwarzschild solution. What we, therefore, need is a vacuum asymptotically flat rotating solution of Einstein field equations possessing the property that it reduces to a class of spherically symmetric metrics in the limit of zero angular momentum.

The stationary spinning vacuum space-time in Boyer and Lindquist coordinates [10] is

$$ds \otimes ds \equiv \left\{1 - 2Mx^1 \Lambda^{-1}\right\} dx^0 \otimes dx^0 - 4aMx^1 \sin^2 x^3 \Lambda^{-1} dx^0 \otimes dx$$

$$- \left\{2a^2 Mx^1 \sin^2 x^3 \Lambda^{-1} + \left(x^1\right)^2 + a^2\right\} \sin^2 x^3 dx^2 \otimes dx^2$$

$$- \Lambda(\Delta^{-1} dx^1 \otimes dx^1 + dx^3 \otimes dx^3),$$

where

$$\Lambda \equiv (x^1)^2 + a^2 \cos^2 x^3, \quad \Delta \equiv (x^1)^2 - 2Mx^1 + a^2, \tag{2}$$

and the coordinate identification goes as follows :

$$\equiv t, \quad x^1 \equiv r, \quad x^2 \equiv \varphi, \quad x^3 \equiv \theta. \tag{3}$$

In eq (1), we interpret M and Ma as mass and angular momentum. The above class of space-times was first obtained in Kerr [11] by an approach involving a formal classification of symmetries [12-15]; a simpler approach to this kind of classification is provided in [16, 17].

Let us now consider the motion of a test object, with a rest-mass m, along a trajectory in the external gravitational field of a spinning gravitating source. In general, time-like geodesics, even in the space-time slice t = [constant] are fairly complicated because the field has no spherical symmetry. Evolution equations for test objects are Hamilton-Jacobi equations,

$$-\frac{\partial S}{\partial \tau} = H\left(x^{\alpha}, \frac{\partial S}{\partial x^{\beta}}\right) = \frac{1}{2} g^{\alpha\beta} \frac{\partial S}{\partial x^{\alpha}} \frac{\partial S}{\partial x^{\beta}}, \tag{4}$$

H being relativistic superhamiltonian, $S(x^{\alpha})$ relativistic Hamilton-Jacobi function, and the rest of the nomenclature having the conventional connotation. Plugging metric components from eq. (1) in the above equation yields

$$\frac{\partial S}{\partial \tau} = \frac{1}{2}\frac{1}{\Delta \Lambda^2}\left\{\left(x^1\right)^2+a^2\right\}\frac{\partial S}{\partial x^1}+a\frac{\partial S}{\partial x^2}$$

$$\frac{1}{2}\frac{1}{\Lambda^2 \sin^2 x^2}\frac{\partial S}{\partial x^2}+a\sin^2 x^3\frac{\partial S}{\partial x^0}$$

$$+\frac{1}{2}\frac{\Delta}{\Lambda^2}\left(\frac{\partial S}{\partial x^2}\right)^2+\frac{1}{2}\frac{1}{\Lambda^2}\left(\frac{\partial S}{\partial x^3}\right)^2 \quad (5)$$

Now, this equation does not have any explicit dependence on τ, x^0 or x^2, and so the solution $S(x^\alpha)$ of the equation must assume the following form :

$$S\left(x^1, x^3\right)=\frac{1}{2}m^2\tau - Ex^0 + L_z x^2 + S_1(x^1)+S_3(x^3), \quad (6)$$

where the integral of motion E is energy of the orbiting celestial body relative to an observer at large distance from the gravitating source being considered and L_z is the integral of motion representing the projection of angular momentum of the celestial object onto the spin axis of our gravitating source. Plugging the S just written into the preceding equation and invoking the familiar principle of separation of variables leads to the following two separated equations :

$$S_1(x^1)=\int \Delta^{-1}\left(\left[E\left\{\left(x^1\right)^2+a^2\right\}-L_z a\right]^2\right.$$

$$\left.-\Delta\left\{m^2\left(x^1\right)^2+\left(L_z-aE\right)^2+C\right\}\right)^{1/2}dx^1, \quad (7)$$

$$S_3(x^3)=\int\left[C-\cos^2 x^3\left\{a^2\left(m-E^2\right)+\frac{L_z^2}{\sin^2\theta}\right\}\right]^{1/2}dx^3, \quad (8)$$

where

$$C=p^2+\cos^2 x^3\left[a^2(m^2-E^2)+\sin^{-2}x^3 L_z^2\right] \quad (9)$$

is separation-of-variables constant, p being the x^3-component of momentum of the heavenly body in question. Thus, feeding these values of S in eq. (6), equations describing the orbit of the celestial-object or satellite under investigation are [18]

$$\frac{\partial S}{\partial\left[C+\left(L_z-aE\right)^2\right]}=0,\ \frac{\partial S}{\partial m^2}=0,\ \frac{\partial S}{\partial E}=0,\ \frac{\partial S}{\partial L_z}=0. \quad (10)$$

Performing the needed differentiation and judiciously combining things, yields the following four evolution equations for motion in space-time regions possessing axisymmetry and stationarity :

$$\Lambda^2\frac{dx^3}{d\tau}=\left[C-\cos^2 x^3\left\{a^2(m-E^2)+\frac{L_z}{\sin^2\theta}\right\}\right]^{1/2}, \quad (11a)$$

$$\Lambda^2\frac{dx^1}{d\tau}=\left(\left[E\left\{\left(x^1\right)^2+a^2\right\}-L_z a\right]^2\right.$$

$$\left.-\Delta\left\{m^2\left(x^1\right)^2+\left(L_z-aE\right)^2+C\right\}\right)^{1/2}, \quad (11b)$$

$$\frac{dx^2}{d\tau}=aE-\frac{L_z}{\sin x^3}\ \frac{\left(x^1\right)^2+a}{\Delta}\left[E\left\{\left(x^1\right)^2+a^2\right\}L_z a\right], \quad (11c)$$

$$\Lambda^2\frac{dx^0}{d\tau}=-a(aE\sin^2 x^3-L_z)+\frac{a}{\Delta}\left[E\left\{\left(x^1\right)^2+a^2\right\}L_z a\right]. \quad (11d)$$

Taking into account the space-time slicing (*i.e.* setting t = [constant]), let us now consider the motion of the satellite object with respect to a rigid lattice of the chronometric reference frame described by the space-time chart (3). We can assume that motion takes place in the equatorial plane ($x^3 \equiv \theta = \pi/2$). In this case, eq. (11a) does not give us any worthwhile information except saying that the constant C gets a specific value. On the other hand, eqs. (11d, c) can be employed to look into the structure of orbits of objects executing motion in the gravitational field described by eq. (1). Work on this aspect of the problem is in progress at present. As regards eq. (11b), as we shall see in Section 3, it sheds new light regarding conclusions of Galilei-Newtonian (GN) orbital theory.

3. Modification of Kepler's second law and applications

Let us look at possible consequences of the modification of classical orbital theory obtained in Section 2. Notice that in the equatorial plane, eq. (11b) assumes the following form :

$$\left(x^1\right)^2\frac{dx^0}{d\tau}=x^1\frac{\left\{x^1-2M\right\}L_z+2aME}{\Delta} \quad (12)$$

This shows that aerial velocity is not, in general, constant due to spinning gravitating sources as described by vacuum gravitational field solutions given by eq (1), again in contrast to the case of Schwarzschild class of space-times currently used for orbit calculations, for instance those required in the construction of ephemerides and nautical almanacs. If we set a = 0 in the preceding equation, the result is

$$\left(x^1\right)^2\frac{dx^2}{d\tau}=L_z. \quad (13)$$

In other words, areal velocity becomes constant on setting solar angular momentum equal to zero, a situation not corroborated by astrophysical observations and experimentation. The calculations therefore, show that if we incorporate the contribution of spin of the central gravitating body in orbital calculations, a residual slight perturbation on the standard constant areal velocity should exist. In particular, the second law of planetary motion requires a revision. However, it turns out that the classical result of Kepler is recoverable from our result as a special case.

To be able to appreciate the need for the revision suggested by the new perturbation considered here, we need to look into the genesis of orbital theory. As we know, orbital mechanics of solar systems, stellar systems (like our own Milky-Way Galaxy), galaxy clusters, *etc* is essentially based on celestial mechanics. But this whole edifice presently rests on the conventional (empirical) Keplerian laws, based on Brahe's planetary observations that describe motion in unperturbed planetary orbits [19]. Moreover, when we refer to Keplerian orbits, we implicitly assume that masses of planets are truly negligible and that Kepler's so-called laws are exact. In fact, however, with the exception of two-body motion (an n-body problem for the specific subcase of $n = 2$), astrodynamical problems are, generally, incapable of exact analytical solutions [20].

Due to the difficulty of absence of an exact solution to 3-body and, generally, n-body problem, one often tries to exploit the method of two-body problem ; this is particularly true for applications of standard GN-theory. The same difficulty is perhaps also responsible for the popular misconception that planets of our solar system have constant areal velocity. In fact, however, staying right in the framework of GN-theory, if we switch from the two-body problem to even the restricted three-body problem, areal velocity turns out to be nonconstant in general. This situation nicely compares with our finding of the general nonconstancy of the aerial velocity (MKSL) in relativistic astrodynamics.

Confining, for the moment, again to the specific context of solar system, note that eq (12) can be written as

$$\frac{dx^2}{d\tau} = \frac{\left\{x^1 - 2M\right\}L_z + 2aME}{x^1\Delta} \tag{14}$$

Again, as we know, the radial and transverse components of the central force are as follows :

$$F_R = m\left[\frac{d^2x^1}{d\tau^2} - x^1\left(\frac{dx^2}{d\tau}\right)^2\right] \tag{15}$$

$$F_T = \frac{m}{x^1}\frac{d}{d\tau}\left(\left(x^1\right)^2\frac{dx^2}{d\tau}\right). \tag{16}$$

Sandwiching eqs. (14) and (16) together then yields

$$F_T \neq 0. \tag{17}$$

This shows that in the new development presented in Section 2, the transverse component of the force field is nonzero, in contrast to the GN-physics wherein such a component vanishes. In particular, the transverse component of the central force field does vanish if we neglect the spin of the gravitating source. It suggests that the inverse square law, as generally employed in GN-physics, needs a modification (even in non-quantum settings and situations where in Yukawa potentials may not apparently have applications).

To sum up, the preceding calculations and inferences flowing from them suggest the need for a new experimental test of the GN-orbital physics to check the contribution of axisymmetric nature of the gravitational field to the overall gravity sensed by an orbiting object. Recent advances in technology promise to provide high enough precision of the order of 10^{-8} [21], to design such experiments.

4. Summary and future directions

As shown in Section 1, the issue of enhancing our ability to carry operations into the interplanetary environment for jobs like utilising the material and energy resources of space and improving our ecosphere and biosphere, signals a paradigm shift in the current space science programmes. Clearly, this rests among other factors on the fundamental framework for launching artificial satellites and space probes in various kinds of orbits as envisaged by the current and future space science missions of various countries. As shown in Section 2, although Keplerian laws of GN-physics have had fundamental place so far, they are not altogether immutable and require a revision especially in the light of modern gravitation physics and in particular, the new perturbation, the spin of the central gravitating body, controlling the orbital motion of the attendant objects.

Section 3 provides an immediate application of the modification (MKSL) obtained in Section 2. Yet another facet of MKSL emerges from the fact that the existence of non-zero transverse components of the force is mathematically equivalent to the existence of a third body in the physical system being considered (a problem which is examined in another paper by us). The work presented here could also be of relevance to things like Global Positioning System (GPS) research and to the methodology of preparing more reliable nautical almanacs and ephemerides for geophysical surveys, geodesy work, terrestrial and space navigation, which are currently prepared at world observatories without taking the mentioned perturbation into account. More fundamentally, cumulative effects due to spin may make motion of an orbiting object chaotic, stochastic, cascade, dump, or oscillatory in nature, which needs to be investigated. Spin may influence the velocity, distance from the central source, and perhpas length of the year too for the test object. Even phenomena like precession and mutation of celestial objects and prediction of eclipses might need a fresh check.

Thus, apart from various lines of investigation already indicated in Section 3, there are a host of further open problems to look into for a sensible rehashing of existing basis of space science investigations under newly emerging economic secnarios around to globe.

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