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The final fate of spherical inhomogeneous dust collapse in higher dimensional spacetime

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Abstract : The work given in this paper searches for the occurrence of naked singularities in higher dimensional Tolman type spherically symmetric spacetime and, if they exist, to investigate whether the dimensionality of spacetime has any role in the nature of singularities. We show that dimensionality of spacetime does not essentially change the basic nature of singularity of an inhomogeneous dust collapse. We examine here the nature of the central singularity forming in the spherically symmetric collapse of dust cloud and it is shown that this is always a strong curvature singularity where gravitational tidal force diverge powerfully.

Keywords : Higher dimensional spacetime, naked singularity, cosmic censorship hypothesis, black hole, gravitational collapse

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1. Introduction

An outstanding problem in gravitation theory and relativistic astrophysics today is to understand the final outcome of an endless gravitational collapse. Such a continual collapse would take place when stars more massive than few times the mass of the sun collapse under their own gravity on exhausting their nuclear fuel. According to the general theory of relativity, this results either in a black hole or a naked singularity which can communicate with faraway observers in the universe. Various models of spherical collapse have been studied over the last few years, and these show that both black holes and naked singularities arise during gravitational collapse.

The models studied so far includes collapse of dust [1], radiation [2], perfect fluids [3] and imperfect fluids [4]. In each of these cases, the formation of covered as well as naked singularities has been observed.

Nowadays there has been a growing interest in studying gravitational collapse in higher dimensions [5-10]. Many works on higher dimensional solutions have appeared recently in literaure because of their implications in astrophysics, cosmology, string theory and particle physics [6].

The results of gravitational collapse in higher dimensions are of interest in the view of current possibilities being explored for higher dimensional gravity. An interesting problem that arises is the effect that higher dimensions can have on the formation of naked singularities. Sil and Chatterjee [8] studied dust collapse in five dimensional spacetime. By considering a self-similar Tolman type model in higher dimensional spacetime, they showed the occurrence of a naked shell focusing singularity which may develop into a strong curvature singularity.

In this paper, we consider nature and structure of singularities in both marginally as well as non-marginally bound dust collapse in 5-D. We show that the central singularity of collapse may indeed be a (strong or weak curvature) naked one depending on the conditions on initial density distribution.

The focus of our investigation will be the singularity that may possibly form during the collapse, at the origin $r = 0$ of the spherical coordinates. This is the so-called central shell-focussing singularity. Hence, we will specify initial conditions only in a small neighbourhood of the center and investigate the nature of collapse in that region, without considering the evolution in the other regions of the spherical object. We will also assume that the initial conditions are such that shell-crossing singularities do not form during evolution.

The organization of the paper is as below.

In Sction 2, the basic parameter of the Tolman-Bondi models in 5D describing the inhomogeneous dust collapse are specified. The existence and nature of the naked singularity is analyzed in Section 3. Strength of this singularity is examined in Section 4. We end the paper by giving concluding remarks in Section 5.

2. Tolman type model in higher dimensional spacetime

The metric of a spherically symmetric inhomogeneous dust cloud in five dimensional spacetime [8] is given by

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)} dr^2 + R^2\left(d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2\right), \qquad (1)$$

where $f(r)$ is an arbitrary function of comoving coordinate r, satisfying $f > -1$. $R(t,r)$ is the physical radius at a time t of the shell labeled by r, in the sense that $4\pi R^2(r,t)$ is the proper area of the shell at time t. A prime denotes the partial derivative with respect to r.

The energy momentum tensor is given by

$$T^{ij} = \varepsilon \delta_t^i \delta_t^j, \qquad (2)$$

where $\varepsilon(r,t)$ is the energy density of a cloud of radius r and is given by

$$\varepsilon(r,t) = \frac{3F'}{2R^3 R'}. \qquad (3)$$

The function $R(r,t)$ is the solution of

$$\dot{R}^2 = \frac{F(r)}{R^2} + f(r), \qquad (4)$$

where a dot denotes the partial derivative with respect to t. the function $f(r)$, $F(r)$ are arbitrary, and the results from integration of the field equations.

As we are only concerned with gravitational collapse of dust, we require

$$\dot{R}(t,r) < 0.$$

We have used units which fix the speed of light and the gravitational constant *via* $8\pi G = c^4 = 1$.

For physical reasons, one assumes that the energy density $\varepsilon(r,t)$ is everywhere non-negative.

Eq. (4) after integration yields

$$t - t_s(r) = \frac{-R^2}{\sqrt{F}} G\left(\frac{f R^2}{F}\right), \qquad (5)$$

where $G(y)$ is a strictly real positive function and is given by

$$G(y) = \sqrt{1+y} \qquad y \neq 0,$$

$$= 1/2, \qquad y = 0, \qquad (6)$$

and $t_s(r)$ is a constant of integration which can be fixed by the choice of scaling on the initial surface ($t = 0$). Using this scaling freedom, if we choose R so that $R(0, r) = r$ then

$$t_s(r) = \frac{r^2}{\sqrt{F}} G\left(\frac{r^2 f}{F}\right) \qquad (7)$$

$t_s(r)$ gives the time at which the physical radius (R) of the shell labeled by r becomes zero.

Thus, the range of coordinates is given by

$$0 \le r < r_c, \qquad -\infty < t < t_s(r), \qquad (8)$$

where $r = r_c$ denotes the boundary of the dust cloud.

It has been shown [11] that shell crossing singularities (characterized by $R' > 0$ and $R > 0$) are gravitationally weak and hence such singularities need not be considered seriously

We therefore consider only the shell focusing singularity We thus assume in the following discussion.

From eqs. (3) - (7), we can write

$$R' = r^{\alpha-1}\left\{\frac{1}{2}(\eta-\beta)X + \left[\Theta - \left(\frac{\eta}{2}-\beta\right)X^2 G(PX^2)\right]\left(P + \frac{1}{X^2}\right)^{1/2}\right\}$$

$$= r^{\alpha-1} H(X,r), \qquad (9)$$

where we have used the following notations

$$u = r^\alpha, \quad X = \frac{R}{u}, \quad \eta(r) = \frac{rF'}{F}, \quad \beta(r) = \frac{rf'}{f},$$

$$p(r) = \frac{r^2 f}{F}, \quad \Lambda = \frac{\sqrt{F}}{u}, \quad P = \frac{f u^2}{F},$$

$$\Theta = \frac{1}{r^{2(\alpha-1)}}\left[G(p)\left(\frac{\eta}{2}-\beta\right) + \frac{2+\beta-\eta}{2\sqrt{1+p}}\right], \qquad (11)$$

$$H(X,r) = \frac{1}{2}(\eta-\beta)X + \left[\Theta - \left(\frac{\eta}{2}-\beta\right)X^2 G(PX^2)\right]\left(P + \frac{1}{X^2}\right)^{1/2}. \qquad (12)$$

The function $\beta(r)$ is defined to be zero when $f(r)$ is zero (marginally bound case).

The parameter α (which satisfies $\alpha \geq 1$) is introduced here for examining the structure of the central singularity at $r = 0$.

One can write the energy density as

$$\varepsilon = \frac{3\eta\Lambda^2 r^\alpha}{2R^3 H(X,r)} . \tag{13}$$

Since $F' = \eta\Lambda^2 r^{2\alpha-1}$ it follows from above that everywhere $H(X,r) \geq 0$ and $\eta\Lambda^2 \geq 0$ as a consequence of the weak energy condition.

Kretschmann scalar ($K = R_{abcd} R^{abcd}$) for the metric (1) is given by

$$K = \frac{28F'^2}{R^6 R'^2} \quad \frac{144FF'}{R^7 R'} + \frac{288F^2}{R^8} \tag{14}$$

In Tolman-Bondi spacetimes, singularities occur, as one can see from eqs. (3) and (14), at points where R = 0, which are called shell focusing singularities.

3. Existence of naked singularity

In order to check whether the singularity is naked, we examine the null geodesic equations for the tangent vector $K^a = dx^a / dk$, where k is an affine parameter along the geodesics.

For radial null geodesics, these are

$$K^t = \frac{dt}{dk} = \frac{P}{R}, \tag{15}$$

$$K^r = \frac{dr}{dk} = \frac{K^t\sqrt{1+f}}{R'} = \frac{P\sqrt{1+f}}{RR'}, \tag{16}$$

where the fuction $P(t,r)$ satisfies the differential equation

$$\frac{dP}{dk} + P^2 \left(\frac{\dot{R}'}{R'R} \quad \frac{\dot{R}}{R^2} \right) \sqrt{1+f} = 0. \tag{17}$$

Let $u = r^\alpha (\alpha \geq 1)$, then

$$\frac{dR}{du} = \frac{1}{\alpha r^{\alpha-1}}\left[R' + \dot{R}\frac{dt}{dr}\right]. \tag{18}$$

From eq. (1) we see that for outgoing radial null geodesic,

$$\frac{dt}{dr} \quad \frac{R'}{\sqrt{1+f}} \tag{19}$$

Since we are considering collapsing case, we require

$$\dot{R} = -\sqrt{\frac{F}{R^2} + f} . \tag{20}$$

Using eqs. (19) and (20), eq. (18) becomes

$$\frac{dR}{du} = \frac{R'}{\alpha r^{\alpha-1}}\left[1 - \frac{\sqrt{\frac{F}{R^2} + f}}{\sqrt{1+f}}\right] \tag{21}$$

$$= \frac{H(X,u)}{\alpha}\left[1 - \frac{\sqrt{\frac{\Lambda^2}{X^2} + f}}{\sqrt{1+f}}\right] = U(X,u). \tag{22}$$

The function $H(X,r)$ in the above equation is strictly positive and nonzero for all $r > 0$. If the null geodesics terminate in the past at the singularity with a definite tangent, then at the singularity, the tangent to the geodesic dR/du is positive and must have a finite value. From eq. (21), we note that dR/du is positive if $R^2 > F$. Thus boundary of the trapped surface *i.e.* apparent horizon is given by

$$R = \sqrt{F} . \tag{23}$$

Using above equation, we find from eq. (5) that

$$t_{ah}(r) = t_s(r) - \sqrt{F}\, G(f), \tag{24}$$

where $t_{ah}(r)$ denotes the time at which apparent horizon forms.

It can be easily seen from the above equation that all the points on the singularity curve $t_s(r)$, other than the central point ($r = 0$) are covered by the apparent horizon. This is because, since both the functions $F(r)$ and $G(r)$ are strictly positive for $r > 0$, with $F(r) = 0$ at $r = 0$, therefore for all $r > 0$

$$t_s(r) > t_{ah}(r) \quad \text{and} \quad t_s(0) = t_{ah}(0) .$$

Thus only central singularity could be naked while non-central singularities are covered.

After simplifying differential eq. (22), we see that dR/du is of the form 0/0 in the limit approach to the singularity ($R = 0$, $u = 0$). The point $u = r^\alpha = 0$, $R = 0$ is thus a singularity of the differential equation (22).

Hence we study the detailed behaviour of the characteristic curves in the vicinity of the singularity. Defining quantity

$$X_0 = \lim_{\substack{R\to 0\\ r\to 0}} \frac{R}{r^\alpha} = \lim_{\substack{R\to 0\\ u\to 0}} \frac{R}{u} = \lim_{\substack{R\to 0\\ r\to 0}} \frac{dR}{du} \tag{25}$$

which gives the tangent to characteristics which terminate in the past at the singularity, it can be shown [12] that if the equation

$$V(x) = U(X,0) - X = \frac{H(X,0)}{\alpha}\left[1 - \frac{X^{\frac{0}{2}} + f_0}{\sqrt{1+f_0}}\right] - X = 0, \quad (26)$$

admits a real positive root, then the central singularity at $r = 0$, $R = 0$ is naked.

In other cases a black hole will be formed as the end product of collapse. We extend the earlier examples given in Ref. [12] to 5D case.

A. Marginally bound collapse :

Let us consider first the marginally bound spacetime ($f = 0$) characterized by the function $F(r)$ and $f(r)$ as

$$f(r) = 0, \; F(r) = F_0 r^n, \quad n \neq 4, \quad n \geq 2. \quad (27)$$

Here, F_0 is a constant.

Thus,

$$n_0 = n, \; \beta = 0, \; p = 0, \; P = 0, \; \Theta = \frac{1}{r^{2(\alpha-1)}}\left(1 - \frac{n}{4}\right). \quad (28)$$

For Θ to be finite we choose $\alpha = 1$, hence we get

$$\Theta = 1 - \frac{n}{4}.$$

Also

$$H(X, r) = \frac{nX}{4} + \frac{4-n}{4X}, \quad (29)$$

and

$$\Lambda(r) = \frac{\sqrt{F}}{u} = \sqrt{F_0}\, r^{\frac{n}{2}-1}.$$

Limiting value of $\Lambda(r)$ in the neighbourhood of $r = 0$ is given by

$$\Lambda(0) = \sqrt{F_0} \qquad \text{for } n = 2$$

$$= 0 \qquad \text{for } n > 2. \quad (30)$$

With the help of above functions, eq. (26) becomes

$$V(X) = \left[1 - \frac{\Lambda_0}{X}\right]\left[\frac{nX}{4} + \frac{4-n}{4X}\right] - X = 0$$

i.e.

$$\frac{nX}{4} + \frac{4-n}{4X} - \frac{n\Lambda_0}{4} - \frac{(4-n)\Lambda_0}{4X^2} - X = 0. \quad (31)$$

In the case $n > 2$ where $\Lambda_0 = 0$, the above equation becomes

$$(n-4)(X^2 - 1) = 0. \quad (32)$$

But since $n \neq 4$, we can write $X = \pm 1$.

Thus eq. (31) has only one positive root $X = 1$ which satisfies the equation $V(X) = 0$ for all $n > 2$, thus establishing the existence of naked singularity for all these spacetimes.

In the case $n = 2$, $\Lambda_0 = \sqrt{F_0}$, hence eq. (31) becomes

$$X^3 + \sqrt{F_0}\, X^2 - X + \sqrt{F_0} = 0. \quad (33)$$

Numerical calculations show that the above equation has real and positive root if

$$F_0 < -11 + 5\sqrt{5} \qquad i.e. \; F_0 < 0.09017. \quad (34)$$

For example, for $F_0 = 0.08$, there are two positive roots $X = 0.5984$ and 0.3759. Hence for all such values given by eq (34), the singularity is naked. On the other hand, if the inequality is reversed, *i.e.* $F_0 < \frac{-11+5\sqrt{5}}{2}$, no naked singularity occurs and gravitational collapse of dust results in a black hole. In the analogous 4-D case, one gets quartic equation and the shell focusing singularity is naked if and only if $F_0 < 0.1809$ [12].

B. Non-marginally bound collapse :

Next, consider non-marginally bound spacetimes ($f(r) \neq 0$) characterized by the functions $F(r)$ and $f(r)$ as

$$f(r) = f_0 r^2 (1 + f_1 r^2),$$

$$F(r) = F_0 r^4,$$

$$\frac{f_0}{F_0} = p_0 > -1. \quad (35)$$

where, f_0, F_0, and f_1 are constants.

Hence, we get

$$\beta_0 = 2, \quad \eta(r) = 4, \quad p(r) = p_0(1 + f_1 r^2), \quad \alpha = 2,$$

$$\Theta_0 = f_1\left[\frac{1}{\sqrt{1+p_0}} - 2G(p_0)\right. \qquad \Lambda(r) = \sqrt{F_0}$$

$$H(X,0) = X + \frac{\Theta_0}{X}. \tag{36}$$

Hence, eq. (26) becomes

$$V(X) = \frac{1}{2}\left[1 - \frac{\Lambda_0}{X}\right]\left[X + \frac{\Theta_0}{X}\right] - X = 0$$

$$X^3 + \sqrt{F_0}\,X^2 - \Theta_0 X + \Theta_0\sqrt{F_0} = 0. \tag{37}$$

Numerical calculations show that eq. (37) has positive real roots if

$$\frac{F_0}{\Theta_0} < \frac{-11 + 5\sqrt{5}}{2}. \tag{38}$$

4. Curvature strength

We now discuss the strength of the singularity by considering the curvature growth near it.

Following Clarke and Krolak [13], a sufficient condition for a singularity to be a strong in the sense of Tipler [14] is that at least along one null geodesic (with affine parameter k), we should have in the limit of approach to the singularity

$$\lim_{k \to 0} k^2 R_{ab} K^a K^b > 0, \tag{39}$$

where K^a is the tangent to null geodesics.

For the Tolman type model in five dimension, the condition (39) for radial null geodesics becomes

$$\lim_{k \to 0} k^2 R_{ab} K^a K^b = \lim_{k \to 0} k^2 \frac{3F'}{2R^3 R'}(K^t)^2$$

$$= \frac{3\eta_0 \Lambda_0^2}{2\alpha X_0^6} \lim\left(\frac{kP}{{}^{2\alpha}} \right. \tag{40}$$

where

$$K^t = \frac{P}{R}, \quad K^r = \frac{\sqrt{1+f}\;K^t}{R'} \tag{41}$$

and P satisfies the differential eq. (17).

For radial null geodesics, using L-Hospitals' rule and eqs. (4)-(9) and (15)-(17) and the fact that at the singularity $r \to 0$, $X \to X_0$ we find that

$$\lim_{k \to 0}\left(\frac{kP}{r^{2\alpha}}\right) = \frac{2\alpha X_0^3}{2\alpha(X_0 + 2\Lambda_0) - \Lambda_0 \eta_0}$$

$$\text{if } P_0 = \lim_{k \to 0} P = 0, \infty;$$

$$= \frac{X_0^2}{2}, \text{ elsewhere.} \tag{42}$$

Hence, we get

$$\lim_{k \to 0} k^2 R_{ab} K^a K^b = \frac{6\eta_0 \Lambda_0^2 \alpha}{\left[2\alpha X_0 + \Lambda_0(4\alpha - \eta_0)\right]^2},$$

$$\text{if } \lim_{k \to 0} P = P_0 = 0, \infty;$$

$$= \frac{3\eta_0 \Lambda_0^2}{8\alpha X_0^2}, \text{ elsewhere,} \tag{43}$$

indicating that the naked singularity is strong curvature one.

5. Conclusion

The Tolman-Bondi metric in the 4-D case has been extensively used to study the formation of naked singularities in spherical gravitational collapse. We have extended this study to higher dimensional Tolman-bondi metric and found that strong curvature naked singularities do arise in these spacetimes. In this work we have shown that the dimensionality of spacetime does not essentially change the basic nature of the singularity of an inhomogeneous dust collapse.

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