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Comparative study of mixed product and quaternion product

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Abstract Mixed number is the sum of a scalar and a vector The quaternion can also be written as the sum of a scalar and a vector but the product of mixed numbers and the product of quaternions are different. Here, we studied the mixed product which is derived from the product of mixed numbers and the quaternion product which is derived from the product of quaternions It was observed that mixed product is more consistent with Physics than that of quaternion product.

Keywords . Mixed number, mixed product, quaternion product

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1. Introduction

1.1. Mixed product :

Mixed number [1] α is the sum of a scalar x and a vector $\boldsymbol{A}$:

i.e. $\alpha = x + \boldsymbol{A}$.

The product of two mixed numbers is defined as [1]

$$\alpha\beta = (x + \boldsymbol{A})(y + \boldsymbol{B}) = xy + \boldsymbol{A}.\boldsymbol{B} + x\boldsymbol{B} + y\boldsymbol{A} + i\boldsymbol{A} \times \boldsymbol{B}. \quad (1)$$

Taking $x = y = 0$, we get from eq. (1)

$$\boldsymbol{A} \otimes \boldsymbol{B} = \boldsymbol{A}.\boldsymbol{B} + i\boldsymbol{A} \times \boldsymbol{B}. \quad (2)$$

This product is called mixed product [2] and the symbol $\otimes$ is chosen for it.

1.2 Quaternion product :

The quaternion can also be written as the sum of scalar and a vector [3]

i.e. $\underline{A} = a + \boldsymbol{A}$.

The multiplication of any two quaternions $\underline{A} = a + \boldsymbol{A}$ and $\underline{B} = b + \boldsymbol{B}$ is given by [3-5]

$$\underline{\boldsymbol{AB}} = (a + \boldsymbol{A})(b + \boldsymbol{B}) = ab - \boldsymbol{A}.\boldsymbol{B} + a\boldsymbol{B} + b\boldsymbol{A} + \boldsymbol{A} \times \boldsymbol{B}. \quad (3)$$

Taking $a = b = 0$, we get from eq. (3)

$$\underline{\boldsymbol{AB}} = -\boldsymbol{A}.\boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{B}. \quad (4)$$

This product we call quaternion product.

2. Consistency of mixed product and quaternion product with physics

2.1 Consistency with Pauli matrix algebra :

It can be shown that [6]

$$(\sigma.\boldsymbol{A})(\sigma.\boldsymbol{B}) = \boldsymbol{A}.\boldsymbol{B} + i\sigma.(\boldsymbol{A} \times \boldsymbol{B}), \quad (5)$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are two vectors and σ is the Pauli matrix.

From eqs. (2) and (5), we can say that the mixed product is directly consistent with Pauli matrix algebra. From eqs. (4) and (5), we can also say that the quaternion product is not directly consistent with Pauli matrix algebra.

2.2 Consistency with Dirac equation :

Dirac equation $(E - \alpha.\boldsymbol{P} - \beta m)\psi = 0$ can be operated by the Dirac operator $(t - \alpha.\boldsymbol{V} - \beta n)$, then we get

$$(t - \alpha.\boldsymbol{V} - \beta n)\{(E - \alpha.\boldsymbol{P} - \beta m)\psi\} = 0. \quad (6)$$

Corresponding Author

For mass-less particles *i.e.* for $m = n = 0$, we get [7]

$$(t-\alpha.V)(E-\alpha.P)\psi = \{tE + V.P + i\sigma.(V\times P)\}\psi_1$$

$$+\{-(t\sigma.P + E\sigma.V)\}\psi_2 = 0, \quad (7)$$

where ψ is the wave function and ψ_1 and ψ_2 are the components of ψ.

Putting $t = 0$ and $E = 0$ in eq. (7), we get

$$(\alpha.V)(\alpha.P)\psi = \{V.P + i\sigma.(V\times P)\}\psi_1 = 0. \quad (8)$$

From eqs. (2) and (8), it is clear that Mixed product is consistent with Dirac equation.

From eqs. (4) and (8), it is also clear that quaternion product is not consistent with Dirac equation.

3. Applications of these products in dealing with differential operators

In region of space where there is no charge or current, Maxwell's equation can be written as

(i) $\nabla.\boldsymbol{E} = 0$, (ii) $\nabla\times\boldsymbol{E} = -(\partial\boldsymbol{B})/(\partial t)$,

(iii) $\nabla.\boldsymbol{B} = 0$, (iv) $\nabla\times\boldsymbol{B} = \mu_0\varepsilon_0(\partial\boldsymbol{E})/(\partial t)$. (9)

From these equations, it can be written as [8]

$$\nabla^2\boldsymbol{E} = \mu_0\varepsilon_0(\partial^2\boldsymbol{E})/(\partial t^2),$$

$$\nabla^2\boldsymbol{B} = \mu_0\varepsilon_0(\partial^2\boldsymbol{B})/(\partial t^2). \quad (10)$$

Using eqs. (2) and (9), we can write

$$\nabla\otimes\boldsymbol{E} = \nabla.\boldsymbol{E} + i\nabla\times\boldsymbol{E}$$

$$= 0 + -i(\partial B)/(\partial t),$$

or $$\nabla\otimes\boldsymbol{E} = -i(\partial\boldsymbol{B})/(\partial t), \quad (11)$$

or $$\nabla\otimes(\nabla\otimes\boldsymbol{E}) = \nabla\otimes\{-i(\partial\boldsymbol{B})/(\partial t)\}$$

$$= -i(\partial/\partial t)\{\nabla\otimes\boldsymbol{B}\}$$

$$= -i(\partial/\partial t)\{\nabla.\boldsymbol{B} + i\nabla\times\boldsymbol{B}\}$$

$$= -i(\partial/\partial t)\{0 + i\mu_0\varepsilon_0(\partial\boldsymbol{E})/(\partial t)\},$$

or $$\nabla\otimes(\nabla\otimes\boldsymbol{E}) = \mu_0\varepsilon_0(\partial^2\boldsymbol{E})/(\partial t^2). \quad (12)$$

It can be shown that $\nabla\otimes(\nabla\otimes\boldsymbol{E}) = \nabla^2\boldsymbol{E}$. (13)

From eqs. (12) and (13), we can write

$$\nabla^2\boldsymbol{E} = \mu_0\varepsilon_0(\partial^2\boldsymbol{E})/(\partial t^2)$$

which is exactly same as shown in eq. (10).

Similarly using mixed product, it can also be shown that

$$\nabla^2\boldsymbol{B} = \mu_0\varepsilon_0(\partial^2\boldsymbol{B})/(\partial t^2).$$

Therefore, mixed product can be used successfully in dealing with differential operators. Using the definition of quaternion product (eq. 4), it can be shown that quaternion product can not be used in dealing with differential operators.

4. Elementary properties of these products

4.1 Elementary properties of mixed product :

(i) Mixed product of two perpendicular vectors is equal to the imaginary of the vector product of the vectors

(ii) Mixed product of two parallel vectors is simply the scalar product of the vectors.

(iii) It satisfies the distribution law of multiplication.

(iv) It is associative.

4.2 Elementary properties of quaternion product :

(i) Quaternion product of two perpendicular vectors is simply the vector product of the vectors.

(ii) Quaternion product of two parallel vectors is negative of the scalar product of the vectors.

(iii) It satisfies the distribution law of multiplication.

(iv) It is non-associative.

5. Conclusion

A comparison of Quaternion and mixed products is given in Table 1. From the table we may conclude that mixed product is more consistent with different laws of Physics than that Quaternion product. From the table we may conclude that mixed product is more consistent with different laws of Physics than the Quaternion product.

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Table 1. Comparison of quaternion product and mixed product

	Quaternion product	Mixed product
1. Mathematical expression	$\underline{AB} = A.B - A \times B$	$A \otimes B = A.B + iA \times B$
2. Consistency with Pauli matrix algebra	It is not directly consistent with Pauli matrix algebra	It is directly consistent with Pauli matrix algebra
3. Consistency with Dirac equation	It is not consistent with Dirac equation	It is consistent with Dirac equation
4. In dealing with differential operators	It can not be used in dealing with differential operators	It can be used successfully in dealing with differential operators
5. Elementary properties	(i) Quaternion product of two perpendicular vectors is simply the vector product of the vectors.	(i) Mixed product of two perpendicular vectors is equal to the imaginary of the vector product of the vectors
	(ii) Quaternion product of two parallel vectors is negative of the scalar product of the vectors.	(ii) Mixed product of two parallel vectors is simply the scalar product of the vectors.
	(iii) It satisfies the distribution law of multiplication.	(iii) It satisfies the distribution law of multiplication.
	(iv) It is non-associative.	(iv) It is associative

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