Department of Data Analysis and Mathematical Modelling

A Kaleidoscopic View of Multivariate Copulas and Quasi-copulas

José De Jesús Arias García

Thesis submitted in fulfillment of the requirements for the degree of Doctor (Ph.D.) of Science: Mathematics

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English version

For everyone else who does not speak the other languages, sorry four is enough.

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José De Jesús Arias García
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Nederlandstalige versie

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Version en français

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José De Jesús Arias García
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Contents

General Introduction

Background

Copulas have become very popular over the last years since they facilitate the study of the relationship between a given n-distribution function and its one-dimensional marginals. The word “copula” was not used until it was introduced by A. Sklar [189] in 1959, although the ‘idea’ of a copula already appeared in previous works such as [80]. In [189], the famous Sklar theorem was formulated, which allows one to build an n-dimensional distribution function with given marginals.

Unfortunately, at that time the statistics community paid little attention to the concept of an n-copula. In fact, during a long period of time, Chapter 6 of the book of Schweizer and Sklar [185] was the only available reference for results about copulas, even though Schweizer and Wolff had already written some articles [186, 187] that were relevant to the statistics community.

As a consequence of the lack of interest in n-copulas from the statistics community, most of the early results about n-copulas were obtained in the framework of probabilistic metric spaces and distribution functions with given marginals.

The poor diffusion of the concept of an n-copula gradually changed in the nineties. The first edition of the book of Nelsen [152] and the book of Joe [102] helped to diffuse the topic of copulas to a wider audience. However, the major reason for this increase of interest is due to the successful application of copulas in several fields as stated in the words of Nelsen [62] when he was asked the question “When did you realize that copulas have become so popular?”: “ About the time when I began to see papers with applications in finance, actuarial science, hydrology, etc.”. Some examples of fields where n-copulas have been applied are finance [86, 139], actuarial science [82], hydrology [178, 179], biostatistics [24, 96], machine learning [70] and imprecise probability theory [149].

As a consequence, several researchers started to study n-copulas more extensively and, as is usual in mathematics, found that there are several previously studied topics that could be analysed from a copula-perspective, mainly because of the flexibility that n-copulas provide to model dependence concepts, as Sklar’s theorem allows one to separate the effect of the marginal distributions from the multivariate dependence structure. The use of n-copulas helped to bring new insights to the concept of dependence of random variables, which unlike the concept of independence had not been properly studied before, as remarked by Nelsen [62]: “For me it’s like the *negative* definition of an irrational number, it’s a real number that is not rational. Similarly dependence has a *negative* definition, it is any relation between random variables other than independence". Hereunder are some examples of dependence concepts that can be studied using copulas.

(i) Exchangeable random variables and other types of symmetry, such as radial symmetry, can be characterized in terms of n-copulas [152].

(ii) Orthant dependence (known as quadrant dependence in the bivariate case), sometimes called concordance order [102], can be described using n-copulas equipped with the pointwise ordering of functions.

(iii) Several concordance dependence measures have been studied using n-copulas. In 1981 Schweizer et al. [186], have already obtained analytical expressions for the well-known Spearman rho and Kendall tau that only depend on the associated 2-copula of the random variables. Some examples of further studies of measures of concordance in the framework of n-copulas can be found in [47, 83, 182, 190, 191].

(iv) Tail dependence between two random variables was studied in the framework of 2-copulas by Joe [103] by using the tail dependence coefficients. These coefficients only depend on the associated 2-copula of the random variables and can be computed in terms of what it is called the diagonal section of a 2-copula [152]. More recently, several authors have studied possible generalizations to higher dimensions, see for example [104, 105, 127].

(v) Darsow et. al [27] proved a characterization of Markov process in terms of 2-copulas. This characterization has been generalized to higher order Markov processes [98] and multi-dimensional Markov processes [129].

Due to the relevance of n-copulas in the framework of dependence modelling, it is deemed important to have a wide range of families of n-copulas available, in order to have more tools in practice. There are several methods to obtain families of n-copulas. One such approach to construct families of n-copulas is to use Sklar's theorem in order to identify the copula of a well-known probability distribution. Elliptical n-copulas [69] and Marshal-Olkin n-copulas [134] are obtained by using this approach.

Another approach, which is also inspired by Sklar's theorem, is to apply some construction methods of probability distributions to n-copulas. Extreme-value n-copulas [69] and vine copulas [1, 14] can be obtained by using this approach.

Other families of n-copulas were introduced in the framework of probabilistic metric spaces and originally did not have a probabilistic interpretation. The most famous example is the family of Archimedean n-copulas, that latter became quite popular [69] due to their simple form, nice properties and the wide range of dependence structures that can be modelled with them.

Other construction methods of n-copulas focus on obtaining copulas with given analytical properties, for example, the construction of n-copulas with a given set of values [117] or n-copulas that are obtained through linearly interpolating between a given set of values [56].

Finally, other methods consist in applying a transformation to one or more n-copulas in order to obtain a new one. Some examples of such transformations are the ordinal sums of n-copulas [143], shuffles of n-copulas [66], patchworks of copulas [51, 65, 90], orthogonal grid constructions [30], among others.

As a mathematical object, n-copulas have been studied from different angles. For example, Olsen et. al [163] proved that Markov operators and 2-copulas are isomorphic. Studying the set of n-copulas with different metrics has also been a topic of interest due to the relevance of the use of approximations of n-copulas in applications [126, 147, 148]. Other authors have studied an algebraic group of transformations of n-copulas that are induced by certain types of transformations on random variables [84, 85]. The lattice structure of the set of n-copulas has been studied in [72, 161]. Finally, several authors have focused on the statistical properties of n-copulas, such as methods to simulate them [134] and parametric and non-parametric inference methods [101].

Closely related to the concept of an n-copula is that of an n-quasi-copula. Before the nineties, the characterization of a certain class of (bivariate) operations on distribution functions was deemed to be of great interest (see, for example, [5, 184, 185]). The concept of a 2-quasi-copula was introduced in [4] in order to characterize such operations. In 1996, Nelsen et al. [160] generalized the concept of a 2-quasi-copula to the higher-dimensional case. However, little attention was paid to n-quasi-copulas, just as in the case of n-copulas. Additionally, the original definition of an n-quasi-copula was too impractical to use, making it hard to study their properties.

However, in 1999 Genest et al. [88] proved a purely algebraic characterization of 2-quasi-copulas that has become the *de facto* definition of a 2-quasi-copula and the most natural way of studying them. Two years later, this result was generalized to the higher-dimensional case by Cuculescu et al. [26].

With the help of the previous characterization, some authors began studying properties of n-quasi-copulas. For example, Nelsen et al. [159] studied additional algebraic properties of n-quasi-copulas (see also [172]). Later, Nelsen et al. [161] and Fernández-Sánchez et al. [72] were interested in studying the lattice structure of the set of n-quasi-copulas. Fernández-Sánchez et al. [73, 75], Nelsen et al. [158] and Durante et al. [53] studied whether or not n-quasi-copulas induce signed stochastic measures, while Nelsen et al. [159] and De Baets et al. [36] studied how negative the volume of an n-box induced by an n-quasi-copula can be.

The main application of n-quasi-copulas in the field of n-copulas has been to derive bounds on sets of n-copulas. In 2004, Nelsen et al. [155] proved that the pointwise supremum and pointwise infimum of any set of 2(-quasi)-copulas are 2-quasi-copulas. One year later, this result was generalized to the higher-dimensional case by Rodríguez-Lallena et al. [170]. Inspired by the previous two articles, several other authors began to study bounds on sets of n-copulas with a given set of values with the help of n-quasi-copulas (see, for example, [36, 37, 117, 136, 168, 177]). Other authors studied similar bounds while paying more attention to financial applications [16, 17, 130, 131, 167, 192]. However, there are several other applications of n-quasi-copulas outside the framework of n-copulas, for example, they have become increasingly popular in fuzzy set theory and aggregation function theory due to their 1-Lipschitz continuity property (see, for example, [31, 39, 78, 99, 122, 124]).

As can be seen, n-copulas and n-quasi-copulas provide a new way to study several probabilistic and statistic concepts from different perspectives. Conversely, it is also possible that one can also look at n-copulas and n-quasi-copulas from different angles. It is the main purpose of this dissertation to present several new and relevant results in the theory of n-copulas and n-quasi-copulas that were obtained while observing n-copulas and n-quasi-copulas from different points of view, i.e., as if one would be looking at them through a kaleidoscope.

Structure of the dissertation

Hereunder we give a more detailed description of the structure of this dissertation.

This dissertation consists of two parts. The first part of this dissertation consists of Chapters 1, 2 and 3. In these chapters we focus our attention on the study of **n-copulas**.

More specifically, in Chapter 1 we recall some **concepts** and review several **results** that are necessary for the rest of this dissertation. We start by recalling the concepts of an **n-copula**, survival copula and diagonal sections of an n-copula. Additionally, we review important properties and results about n-copulas that are relevant to the development of this dissertation, such as some basic dependence concepts and several families of n-copulas.

In the following two chapters, instead of studying n-copulas in a general framework, we study two construction methods: one of them is a generalization to the multivariate case of an algebraic construction method of 2-copulas, while the other has the objective of constructing n-copulas that have a special type of symmetry, more specifically, radial symmetry.

In Chapter 2, we focus our attention on the construction of an n-copula given its **diagonal section**, more specifically we propose a generalization of the well-known class of **upper semilinear** 2-copulas [60] to higher dimensions. We start by recalling the definition of upper and lower semilinear 2-copulas and we propose a generalization of this construction method to the multivariate case. Next, we provide necessary and sufficient conditions on these diagonal sections that guarantee that the upper semilinear construction method yields an n-copula. Additionally, we provide some examples of the upper semilinear construction method.

Next, we change our attention to the property of **radial symmetry** in Chapter 3 and propose a construction method of radially symmetric n-copulas. To this end, we first prove a representation theorem for symmetric and radially symmetric n-copulas. Thereafter, with the help of the representation theorem, we propose a construction method for higher-dimensional radially symmetric copulas using an auxiliary function. Afterwards, we provide several examples of our construction method in the trivariate case when the auxiliary function is obtained using the nesting of 2-copulas, when the auxiliary function is constructed using a generalization of the $*$-product of copulas or when the auxiliary function is a product-type extension of a 2-copula.

In the second part of this dissertation we focus our attention to the concept of an **n-quasi-copula**.

Going into detail, the purpose of Chapter 4 is to recapitulate the various results that have been proven in the literature about **n-quasi-copulas**. First, we present the concept of an n-quasi-copula as it was originally introduced. Subsequently we review all the **characterizations** of n-quasi-copulas that have been proved in the literature, while stressing the differences that occur between the case $n = 2$ and $n \geqslant 3$. Next, we discuss the lattice structure of the set of n-quasi-copulas and its relationship to n-copulas and note that there are several results that cannot be extended to higher dimensions $(n \geqslant 3)$. Finally, we recall results in the literature related to the mass distribution of n-quasi-copulas and stochastic signed measures.

In Chapter 5 we look through the kaleidoscope of n-quasi-copulas and 'observe an image that has never been considered before', more specifically, we study **supermodular n-quasi-copulas** and we propose a generalization of supermodularity for quasi-copulas in higher dimensions. In the bivariate case, it is known that 2-increasingness is equivalent to supermodularity. However, we show that this is no longer true for $n \geqslant 3$ and, as a consequence, the class of supermodular n-copulas is different from n-copulas. We study the properties of the newly introduced class of supermodular n-quasi-copulas. Next, we recall a characterization of 2-quasi-copulas and an open problem on a certain class of n-quasi-copulas that inspired us to introduce other new classes of n-quasi-copulas. We characterize the new classes of n-quasi-copulas and use them to solve the previously mentioned open problem.

In the following chapters, we continue to show that n-quasi-copulas are more closely related to supermodular functions than to n-copulas. In particular, we focus our attention to the smallest n-quasi-copula with a given diagonal section, to the set of supermodular n-quasi-copulas when endowed with the uniform metric and to the lattice structure of the set of supermodular n-quasi-copulas.

Chapter 6 consists of two main parts. First, we work in the more general framework of **aggregation functions** to study the analytical expressions to compute the smallest and the greatest M-Lipschitz continuous n-ary aggregation functions with a given diagonal section and provide some examples. Thereafter, we identify necessary and sufficient conditions to guarantee that the smallest and the greatest M-Lipschitz continuous n-ary aggregation functions have a neutral element and an absorbing element. We then study the supermodularity and submodularity properties of the smallest and the greatest M-Lipschitz continuous n-ary aggregation functions and as a byproduct we prove that the **smallest** n-quasi-copula with a **given diagonal section**, called the **Bertino n-quasi-copula**, is supermodular for any $n \geqslant 2$.

In the second part of Chapter 6 we work again in the framework of n-quasi-copulas. We present some complementary results on the **marginal k-quasi-copulas** of a Bertino n-quasi-copula. Next, we introduce the concept of a regular n-diagonal function. The rest of Chapter 6 is concerned with the **characterisation** of the sets of regular n-diagonal functions for which there exists an n-dimensional Bertino copula whose diagonal section coincides with the given n-diagonal function.

Chapter 7 also consists of two parts. First, we study the set of supermodular n-quasi-copulas when endowed with the **uniform metric** and we show that it has similar properties as the metric space of n-copulas endowed with the uniform metric. The second part of Chapter 7 studies the **lattice structure** of the set of supermodular n-quasi-copulas. This part contains the most relevant results of this chapter, namely that the set of supermodular n-quasi-copulas is **join-dense** in the set of n-quasi-copulas, even though the set of n-quasi-copulas is not isomorphic to the **Dedekind-MacNeille completion** of the poset of supermodular n-quasi-copulas. To this end, we first develop some additional results that are needed to prove the main theorem of this chapter, before studying the lattice structure of the poset of supermodular n-quasi-copulas.

Finally, in Chapter 8 we present the general conclusions of this dissertation, discuss some open questions and identify future research lines.

It is important to remark that it is not necessary to read this dissertation sequentially, as shown in Figure I.1. Chapter 1 is necessary for those readers who are not familiar with the concept of an n-copula, as the rest of the chapters of this dissertation require a good understanding of the concept of an n-copula. The results of Chapters 2 and 3 are independent from the following chapters of this dissertation and, consequently, Chapters 2 and 3 can be read separately from the following chapters of this dissertation. Chapter 4 is required for those readers who are not familiar with the concept of an n-quasi-copula, as the second part of the dissertations focuses heavily on this concept. Chapter 5 is required to read the following two chapters of this dissertation, since Chapter 5 introduces a new class of n-quasi-copulas that is further studied in Chapters 6 and 7. Finally, the results of Chapters 6 and 7 are independent from each other.

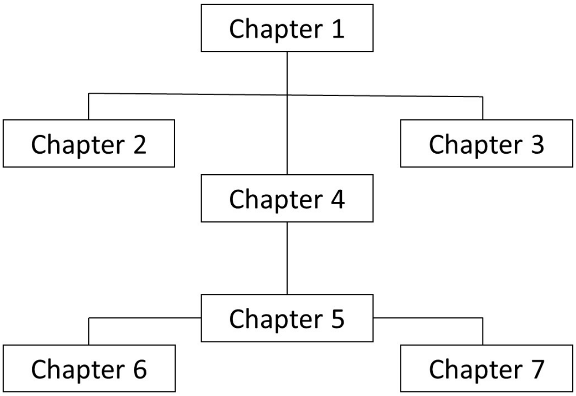


Figure I.1: Structure of this dissertation

PART I

COPULAS

1 Copulas: Basic definitions and properties

In this chapter we provide all tools from copula theory that will be necessary for this dissertation. First, we give the definition of an n-copula, as well as some of its basic properties that are relevant for this dissertation. Next, we give some examples of well-known copula families. Finally, we provide some examples of dependence concepts that can be studied using of copulas.

1.1. First definitions

Definition 1.1. [152] A function $F : D \subset [0,1]^n \to [0,1]$ is called n-increasing if for any n-box $\mathbf{P} = \times_{j=1}^n [x_j, y_j] \subseteq [0,1]^n$ such that $[x_i, y_j] \in D$ for any $i, j \in \{1, 2, \ldots, n\}$, it holds that

$$V_F(\mathbf{P}) = \sum_{\mathbf{z} \in \text{vertices}(\mathbf{P})} (-1)^{S(\mathbf{z})} F(\mathbf{z}) \geqslant 0\,, \tag{1.1}$$

where $S(\mathbf{z}) = \#\{j \in \{1, 2, ..., n\} \mid z_j = x_j\}$. $V_F(\mathbf{P})$ is called the F-volume of $\mathbf{P}$. Here $\#A$ denotes the cardinality of the set A.

Note that an n-increasing function with $n = 1$ is simply an increasing function.

We now recall the formal definition of a distribution function and a survival function.

Definition 1.2. A function $F : [-\infty, \infty]^n \to [0,1]$ (resp. $\bar{F} : [-\infty, \infty]^n \to [0,1]$) that satisfies

(1) $F(\infty, \ldots, \infty) = 1$ (resp. $\bar{F}(-\infty, \ldots, -\infty) = 1$).

(2) $F(\mathbf{x}) = 0$ (resp. $\bar{F}(\mathbf{x}) = 0$) if $\mathbf{x}$ is such that $x_j = -\infty$ (resp. $x_j = \infty$) for some $j \in \{1, 2, ..., n\}$.

(3) F (resp. $\bar{F}$) is right continuous in each argument.

(4) F (resp. $(-1)^n \bar{F}$) is n-increasing.

is called a joint distribution function (resp. joint survival function).

Now we recall the definition of an n-copula.

Definition 1.3. [152] An n-copula C_n is a $[0,1]^n \to [0,1]$ function that satisfies (c1) $C_n(\mathbf{x}) = 0$ if $\mathbf{x}$ is such that $x_j = 0$ for some $j \in \{1, 2, ..., n\}$.

(c2) $C_n(\mathbf{x}) = x_j$ if $\mathbf{x}$ is such that $x_i = 1$ for all $i \neq j$.

(c3) C_n is n-increasing.

We will denote by $\mathcal{C}_n$ the set of all n-copulas.

Note that by properly extending an n-copula to a function on $\mathbb{R}^n$, we obtain an n-dimensional distribution function. If $X_1, \ldots, X_n$ are random variables that have C_n as their joint distribution function, then X_j has a uniform distribution on $[0, 1]$ for any $j \in \{1, \ldots, n\}$.

When $n - k$ of the arguments of an n-copula are set equal to 1, we obtain a k-dimensional marginal of the n-copula, which itself is a k-copula. More generally, we have the following definition.

Definition 1.4. For any $\mathbf{a} \in [0, 1]^n$ and any set of indices $A \subset \{1, 2, \ldots, n\}$ with $0 < \#A = k < n$, the k-dimensional section of an n-copula C_n with fixed values given by $\mathbf{a}$ in the positions determined by A is the function $C_{\mathbf{a},A} : [0, 1]^k \to [0, 1]$ given by $C_{\mathbf{a},A}(\mathbf{x}) = C_n(\mathbf{y})$, where $y_j = x_j$ if $j \in A$ and $y_j = a_j$ if $j \notin A$.

Note that only the coordinates of $\mathbf{a}$ whose indices are not in A are relevant to the definition of a section.

Remark 1.1. If C is an n-copula, then it is easy to see that for any $\mathbf{a} \in [0, 1]^n$ and any set of indices $A \subset \{1, 2, \ldots, n\}$ with $0 < \#A = k < n$, the k-dimensional section $C_{\mathbf{a},A}$ is k-increasing. For any $\mathbf{x} \in [0, 1]^r$, $C_{\mathbf{a},A}$ represents both the $C_{\mathbf{a},A}$-volume of $\times_{j=1}^{r}[0, x_j]$ and the C-volume of the n-box $\mathbf{P'} = \times_{j=1}^{n}[0, x'_j]$ where $[0, x'_j] = [0, x_j]$ if $j \in A$ and $[0, x_j] = [0, a_j]$ if $j \notin A$.

For any permutation σ on the set $\{1, \ldots, n\}$ and for any copula C_n, we will denote by C_n^σ the n-copula given by

$$C_n^\sigma(\mathbf{x}) = C_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) .$$

The probabilistic interpretation of C_n^σ is simple: if the copula of the random vector $(X_1, X_2, \ldots, X_n)$ is C_n, then C_n^σ is the copula of the random vector $(X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, \ldots, X_{\sigma^{-1}(n)})$. The following definition concerns n-copulas that are invariant under permutations.

Definition 1.5. An n-copula C_n is called symmetric if for any permutation σ of $\{1, 2, \ldots, n\}$ and for any $\mathbf{x} \in [0, 1]^n$ it holds that

$$C_n(\mathbf{x}) = C_n^\sigma(\mathbf{x}) .$$

Symmetric copulas are the copulas associated to exchangeable random variables. Note that if an n-copula is symmetric, then all of its k-dimensional marginals coincide for any $k \in \{1, 2, \ldots n - 1\}$.

For any copula C_n and for any $i \in \{1, \ldots, n\}$, the copula C_n^i given by

$$C_n^i(\mathbf{x}) = C_n(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - C_n(x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n)$$

is called the reflection of C_n in the i-th argument. The reflection of n-copulas has a probabilistic interpretation: if $(X_1, X_2, \ldots, X_n)$ has the n-copula C_n as its joint distribution function, then the joint distribution function of the random vector $(X_1, X_2, \ldots X_{i-1}, 1 - X_i, X_{i+1}, \ldots, X_n)$ is the reflection of C_n in the i-th argument. The probabilistic interpretation of reflections and permutations of n-copulas has led to several studies of the transformations of copulas that are induced by certain types of transformations on random variables [84, 85, 120].

For a given n-copula C_n, $\bar{C}_n$ denotes the associated survival n-copula, which itself is an n-copula, and is given by

$$\begin{aligned}\bar{C}_n(x_1, \ldots, x_n) = & \sum_{i=1}^{n} x_i - (n-1) + \sum_{i<j}^{n} C_n(1, \ldots, 1 - x_i, \ldots, 1 - x_j, \ldots, 1) \\ & - \sum_{i<j<k}^{n} C_n(1, \ldots, 1 - x_i, \ldots, 1 - x_j, \ldots 1 - x_k, \ldots, 1) + \ldots \\ & + (-1)^n C_n(1 - x_1, 1 - x_2, \ldots, 1 - x_n) \,. \end{aligned} \tag{1.2}$$

Survival copulas have a clear probabilistic interpretation: if the joint distribution function of the random vector $(X_1, \ldots X_n)$ is the copula C_n , then $\bar{C}_n$ is the joint distribution function of the random vector $(1 - X_1, \ldots, 1 - X_n)$, i.e., the result of applying the reflection transformation to all of its arguments.

The following theorem explains the importance of copulas in the framework of dependence modelling [189].

Theorem 1.1. *Let G_n be an n-dimensional joint distribution function with margins $F_{1,1}, \ldots, F_{1,n}$. Then there exists an n-copula C_n such that for all $\mathbf{x} \in [-\infty, \infty]$ it holds that*

$$G_n(\mathbf{x}) = C_n(F_{1,1}(x_1), \ldots, F_{1,n}(x_n)) \,. \tag{1.3}$$

If $F_{1,j}$ is continuous for all $j \in \{1, \ldots, n\}$, then C_n is unique; otherwise, it is unique on $\times_{j=1}^{n} \operatorname{Ran} F_{1,j}$, where Ran denotes the range.
Conversely, if C_n is an n-copula and $(F_{1,j})_{j=1}^n$ are univariate distribution functions, then G_n defined as

$$G_n(\mathbf{x}) = C_n(F_{1,1}(x_1), \ldots, F_{1,n}(x_n))$$

is an n-dimensional joint distribution function.

This theorem is known as Sklar's theorem. Note that Sklar's theorem states that a continuous multivariate distribution function can be expressed in terms of its n univariate marginals by means of a unique n-copula. Sklar's theorem can also be reformulated in terms of survival functions.

Theorem 1.2. *Let $\bar{G}_n$ be an n-dimensional joint survival function with margins $\bar{F}_{1,1}, \ldots, \bar{F}_{1,n}$. Then there exists an n-copula C_n such that for all $\mathbf{x} \in [-\infty, \infty]$ it holds that*

$$\bar{G}_n(\mathbf{x}) = C_n(\bar{F}_{1,1}(x_1), \ldots, \bar{F}_{1,n}(x_n)) . \tag{1.4}$$

If $\bar{F}_{1,j}$ is continuous for all $j \in \{1, \ldots, n\}$, then C_n is unique; otherwise, it is unique on $\times_{j=1}^{n}$ Ran $\bar{F}_{1,j}$.
Conversely, if C_n is an n-copula and $(\bar{F}_{1,j})_{j=1}^{n}$ are univariate survival functions, then $\bar{G}_n$ defined as

$$\bar{G}_n(\mathbf{x}) = \bar{C}_n(F_{1,1}(x_1), \ldots, \bar{F}_{1,n}(x_n))$$

is an n-dimensional joint survival function.

Some well-known examples of n-copulas are:

(i) The product copula $\Pi_n(\mathbf{x}) = \prod_{i=1}^{n} x_i$. Given a random vector $(X_1, \ldots, X_n)$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the copula of the random variables $X_1, \ldots, X_n$ is Π_n if and only if the random variables $X_1, \ldots, X_n$ are independent.

(ii) The comonotonic copula $M_n(\mathbf{x}) = \min(x_1, \ldots, x_n)$. Given a random vector $(X_1, \ldots, X_n)$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the copula of the random variables $X_1, \ldots, X_n$ is M_n if and only if there exist a random variable X defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and n strictly increasing functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ such that the equality $f_1(X) = f_2(X) = \cdots = f_n(X)$ holds almost surely.

(iii) The countermonotonic copula for $n = 2$, $W_2(x_1, x_2) = (x_1 + x_2 - 1)^+$, where $u^+ = \max(u, 0)$. Given a random vector (X_1, X_2) defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the copula of the random variables X_1, X_2 is W_2 if and only if there exists a strictly decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that the equality $X_1 = f(X_2)$ holds almost surely.

As we will see in Section 1.4 and in Chapter 4, the comonotonic copula represents the strongest notion of positive dependence, while for $n = 2$ the countermonotonoic copula represents the strongest notion of negative dependence.

1.2. The diagonal section of an n-copula

The concept of the diagonal section of an n-copula will be a topic of interest for this dissertation. For any n-copula C_n, its diagonal section is the function

$d : [0,1] \to [0,1]$ given by $d(x) = C_n(x, x, \ldots, x)$. The diagonal section of an n-copula satisfies some nice properties, as is shown in the proposition below.

Proposition 1.1. *Let C_n be an n-copula and d its diagonal section. Then*

(d1) d is increasing.

(d2) $(nx - (n-1))^+ \leqslant d(x) \leqslant x$.

(d3) d is n-Lipschitz continuous, i.e.,

$$|d(x) - d(y)| \leqslant n|x - y| \,.$$

Note that condition (d2) implies that $d(0) = 0$ and $d(1) = 1$.

The following definition is based on the properties that a diagonal section satisfies.

Definition 1.6. [69] A function $d : [0,1] \to [0,1]$ that satisfies (d1), (d2) and (d3), for some positive integer n, is called an n-diagonal function.

It can be shown that for any n-diagonal function d, there exists an n-copula that has d as diagonal section [100].

The diagonal section has an interesting probabilistic interpretation: it is the distribution function of the maximum of the uniform random variables on $[0,1]$ that have the given n-copula as joint distribution function. Also the probabilistic concept of tail dependence between random variables can be related to the diagonal section of their joint distribution (see [152]).

Therefore, many studies have been devoted to the construction of n-copulas with given diagonal section. These studies have mainly focused on the 2-dimensional case and have led to a rich variety of construction techniques (see e.g. [61, 81, 154, 157]). Unfortunately, for $n \geqslant 3$, the situation is more complicated, as it is no longer easy to propose construction techniques that work for any diagonal section(see [71]). Indeed, to the authors' knowledge, only in [100] a universal construction method (given a diagonal section) with an explicit form of an n-copula has been given. In Chapters 2 and 6 we will study a generalization of two methods for constructing an n-copula given its diagonal section.

1.3. Some families of copulas

A family of n-copulas is usually described as a mapping from a set Θ to the set of n-copulas $\mathcal{C}_n$ [69]. The elements of Θ are called parameters. According to Joe [102], it is desirable that a family of n-copulas has the following properties that we now describe in order to be used in statistical applications. First, the family of n-copulas should have a probabilistic interpretation. Second, the family of copulas should be able to describe a wide range of dependence structures. Finally, it should be 'manageable', in the sense that it ought to have a closed analytical form, or at least numerically tractable. Hereunder, we recall some families of n-copulas.

Archimedean copulas

The class of Archimedean n-copulas is a well-known class of copulas. Their popularity is due to their simple form and nice properties. We recall some definitions that are needed in order to give the definition of an Archimedean n-copula.

Definition 1.7. Let $\varphi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function such that $\varphi(1) = 0$. The pseudo-inverse of φ is the function $\varphi^{[-1]} : [0,\infty] \to [0,1]$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & , \text{if } 0 \leqslant t < \varphi(0)\,, \\ 0 & , \text{if } \varphi(0) \leqslant t \leqslant \infty\,, \end{cases} \tag{1.5}$$

where φ^{-1} is the usual inverse function.

The function $\varphi^{[-1]}$ has some interesting properties, for example, $\varphi^{[-1]}$ is a continuous decreasing function and for any $t \in [0,1]$ it holds that $\varphi^{[-1]}(\varphi(t)) = t$. Furthermore, it holds that $\varphi(\varphi^{[-1]}(t)) = \min(t, \varphi(0))$, and, if $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{-1}$. The following result can be found in [156] (see also [152]).

Lemma 1.1. *Let $\varphi, \varphi^{[-1]}$ be defined as in Definition 1.7. Define the function $C_{n,\varphi} : [0,1]^n \to [0,1]$ as:*

$$C_{n,\varphi}(\mathbf{x}) = \varphi^{[-1]}\left(\sum_{j=1}^{n} \varphi(x_j)\right).$$

Then $C_{n,\varphi}$ satisfies conditions (c1) and (c2).

$C_{n,\varphi}$ is the n-ary form of an Archimedean continuous t-norm (see [121] for more details). If $C_{n,\varphi}$ is an n-copula, then we say that $C_{n,\varphi}$ is an Archimedean n-copula. In such case, φ is called an additive generator of $C_{n,\varphi}$ (which is unique up to a strictly positive multiplicative constant). Note that Archimedean n-copulas are symmetric.

Archimedean n-copulas are also associative. Hence, for any $n \geqslant 2$ and $\mathbf{x} \in [0,1]^{n+1}$, it holds that

$$\begin{aligned} C_{n+1,\varphi}(\mathbf{x}) &= C_{2,\varphi}(x_1, C_{n,\varphi}(x_2, \ldots, x_{n+1})) \\ &= C_{2,\varphi}(C_{n,\varphi}(x_1, \ldots, x_n), x_{n+1})\,. \end{aligned}$$

Before introducing the characterization of the additive generators of Archimedean n-copulas, let us recall the concept of an n-monotone function [140].

Definition 1.8. A function $f : [a, b] \to \mathbb{R}$ is called n-monotone if it is differentiable up to the order $n-2$ and its derivatives satisfy $(-1)^k f^{(k)}(t) \geqslant 0$ for any $t \in]a, b[$ and $k \in \{0, 1, \dots, n-2\}$ and $(-1)^{n-2} f^{(n-2)}$ is decreasing and convex on $[0, 1]$.

A function is $f : [a, b] \to \mathbb{R}$ is called completely monotone if f is n-monotone for any $n \geqslant 2$.

Remark 1.2. Note that the concepts of n-monotonicity and n-increasingness should not be confused, since n-increasingness is a property of an n-dimensional function, while n-monotonicity is a property of a univariate function.

The generators of Archimedean n-copulas were characterized in [140].

Theorem 1.3. *Let $\varphi, \varphi^{[-1]}$ be defined as in Definition 1.7. Then, $C_{n,\varphi}$ is an n-copula if and only if $\varphi^{[-1]}$ is n-monotone on $[0, 1]$.*

It is well known that an Archimedean 2-copula is also an Archimedean continuous t-norm, and the 1-Lipschitz continuity is equivalent to the convexity of the generator; for further details we refer to [6, 144].

Hereunder we give some examples of Archimedean n-copulas [69, 152].

(i) The Frank family of n-copulas,

$$C_n(\mathbf{x}) = -\frac{1}{\alpha} \ln \left(1 + \frac{\prod_{i=1}^n (e^{-\alpha x_i} - 1)}{(e^{-\alpha} - 1)}^{n-1} \right) .$$

The additive generator of the Frank n-copula is given by

$$\varphi(t) = \frac{1}{\alpha} \ln \left(1 - (1 - e^{-\alpha}) e^{-t}\right) .$$

Its additive generator is completely monotone if $\alpha \geqslant 0$.

(ii) The Gumbel family of n-copulas, given by

$$C_n(\mathbf{x}) = \exp \left(- \left(\sum_{i=1}^n (-\ln(x_i))^{\alpha} \right) \right) .$$

The additive generator of the Gumbel family of n-copulas is given by

$$\varphi(t) = \exp \left(\exp(-t)^{1/\alpha} \right) .$$

Its additive generator is completely monotone if $\alpha \geqslant 1$.

(iii) The Clayton family of n-copulas, given by

$$C_n(\mathbf{x}) = \left(\sum_{i=1}^n x_i^{-\alpha} - (n-1) \right)^{+} .$$

The additive generator of the Clayton family of n-copulas is given by

$$\varphi(t) = \Big((1 + \alpha t)^+\Big)^{-1/\alpha} .$$

Its additive generator is completely monotone if $\alpha \geqslant 0$.

Elliptical copulas

Elliptical n-copulas are constructed from a direct application of Sklar's theorem, in the sense that one starts with a continuous random vector with a given joint distribution function, and then one identifies the copula associated to the joint distribution function of the random vector.

We first recall that a random vector $(X_1, \ldots, X_n)$ is said to have an elliptical distribution if $(X_1, \ldots, X_n)$ has the same distribution as $\mu + R\mathbf{A}\mathbf{U}$, with

(i) $\mu \in \mathbb{R}^n$.

(ii) $\mathbf{A}$ an $n \times k$ matrix, such that $\operatorname{rank}(\Sigma) = k \leqslant n$ with $\Sigma = \mathbf{A}\mathbf{A}^T$;

(iii) $\mathbf{U} = (U_1, \ldots, U_n)$ a random vector such that it has uniform distribution on the sphere $\{\mathbf{x} \in \mathbb{R}^n \mid \sum_{j=1}^n u_j^2 = 1\}$.

(iv) R a positive random variable that is independent of $(U_1, \ldots, U_n)$.

We can now recall the definition of an elliptical n-copula.

Definition 1.9. An n-copula C_n is said to be elliptical, if there exists a random vector $(X_1, \ldots, X_n)$ which has an elliptical distribution and is such that C_n is the n-copula associated to the joint distribution function of the random vector $(X_1, \ldots, X_n)$.

Two of the most well-known elliptical copulas are the Gaussian copula and the t-copula. The Gaussian copula is the n-copula associated to a random vector $(X_1, \ldots, X_n)$ which follows a Gaussian distribution, while the t-copula is the n-copula associated to a random vector $(X_1, \ldots, X_n)$ which follows a multivariate Student t-distribution. For both the Gaussian copula and the t-copula it is not possible to write their expression in closed form.

Extreme-value copulas

Extreme-value (EV) n-copulas have a probabilistic interpretation in terms of the componentwise maximum of a stationary stochastic process [69]. They can be easily characterised in analytical terms as follows.

Definition 1.10. An n-copula C_n is called an extreme-value copula if there exists an n-copula D_n such that for any $\mathbf{x} \in [0,1]^n$ it holds that

$$C_n(\mathbf{x}) = \lim_{m\to\infty} \left(D_n(x_1^{1/m}, \ldots, x_n^{1/m}) \right)^m .$$

We have the following alternative characterisation.

Proposition 1.2. *An n-copula C_n is an EV n-copula if and only if for any $\mathbf{x} \in [0,1]^n$ and $m \geqslant 1$ it holds that*

$$C_n(\mathbf{x}) = \left(C_n(x_1^{1/m}, \ldots, x_n^{1/m}) \right)^m .$$

The above proposition describes the concept of a max-stable n-copula. EV n-copulas and max-stable n-copulas were introduced as different families of n-copulas, but it has recently been proved that they are the same family [93].

The most well-known EV n-copula is the Galambos copula, that can be computed as

$$C_n(\mathbf{x}) = \exp\left(\ell_\alpha(-\ln(x_1), \ldots, -\ln(x_n)\right) ,$$

with

$$\ell_\alpha = \sum_{\varnothing \neq A \subseteq \{1,\ldots,n\}} (-1)^{\#A+1} \left(\sum_{i\in A} x_i^{-\alpha} \right) .$$

Marshall-Olkin copulas

Marshall-Olkin n-copulas arise naturally in the framework of exchangeable exogenous shock models, i.e., models that consider the arrival times of shocks that can affect one or more components of a given system [134]. More specifically, consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and for every non-empty subset $A \subseteq \{1, \ldots, n\}$ let Z_A be an exponentially distributed random variable with parameter $\lambda_A > 0$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Additionally, suppose that the $2^n - 1$ random variables are independent. For any $i \in \{1, \ldots, n\}$ define X_i as

$$X_i = \min\left(Z_A \mid i \in A\right) .$$

Then the survival n-copula associated to the joint survival function of the random vector $(X_1, \ldots, X_n)$ is given by [134]

$$\bar{C}_n(\mathbf{x}) = \prod_{\varnothing \neq A \subseteq \{1,\ldots,n\}} \min\left(x_i^{\frac{\lambda_A}{\sum_{B:i\in B} \lambda_B}} \mid i \in A \right) . \tag{1.6}$$

Definition 1.11. *An n-copula C_n belongs to the Marshal-Olkin family if C_n has Eq. (1.6) as its analytical expression.*

1.4. Some basic dependence concepts

In this section, we give some examples of how n-copulas can be used to study multivariate dependence structures. Since independence is a concept that has been extensively studied in Probability Theory, it is not a surprise that the product copula is often used as a reference point for various notions of dependence. The following concepts of dependence compare pointwisely a given n-copula with the product copula.

Definition 1.12. An n-copula C_n is said to be

(1) Positively lower orthant dependent, PLOD (resp. negatively lower orthant dependent, NLOD) if for any $\mathbf{x} \in [0,1]^n$ it holds that $C_n(\mathbf{x}) \geqslant \Pi_n((x))$ (resp. $C_n(\mathbf{x}) \leqslant \Pi_n((x)))$.

(2) Positively upper orthant dependent, PUOD (resp. negatively upper orthant dependent, NUOD) if for any $\mathbf{x} \in [0,1]^n$ it holds that $\bar{C}_n(\mathbf{x}) \geqslant \Pi_n((x))$ (resp. $\bar{C}_n(\mathbf{x}) \leqslant \Pi_n((x)))$.

(3) Positively orthant dependent, POD (resp. negatively orthant dependent, NOD) if it is both PLOD and PUOD (resp. NLOD and NUOD).

In the bivariate case, it is common to find the previous concepts using the term 'quadrant' instead of 'orthant'. It is also important to remark that in the bivariate case positively lower quadrant dependent and positively upper quadrant dependent are equivalent (resp. negatively lower quadrant dependent and negatively upper quadrant dependent).

Sometimes it is also useful to compare two n-copulas, instead of only comparing one n-copula with the product copula.

Definition 1.13. Let $C_{n,1}, C_{n,2}$ be two n-copulas.

(1) $C_{n,1}$ is more PLOD than $C_{n,2}$ if for any $\mathbf{x} \in [0,1]^n$ it holds that $C_{n,1}(\mathbf{x}) \geqslant C_{n,2}(\mathbf{x})$.

(2) $C_{n,1}$ is more PUOD than $C_{n,2}$ if for any $\mathbf{x} \in [0,1]^n$ it holds that $\bar{C}_{n,1}(\mathbf{x}) \geqslant \bar{C}_{n,2}(\mathbf{x})$.

(3) $C_{n,1}$ is more POD than $C_{n,2}$ if $C_{n,1}$ is more PLOD than $C_{n,2}$ and $C_{n,1}$ is more PUOD than $C_{n,2}$.

Since M_n is associated to the strongest notion of positive dependence, it is not difficult to see that M_n is more POD than C_n, for any n-copula C_n. Analogously, for the bivariate case, C_2 is more POD than W_2 for any 2-copula C_2.

Another way to measure dependence is through the use of measures of concordance. In the bivariate case, the measures of concordance provide an idea of how 'large' (resp. 'small') values of one variable are associated with the 'large' (resp. 'small') values of the other variable. We now give the definition of a bivariate measure of concordance.

Definition 1.14. A measure of concordance is a mapping $\mu : \mathcal{C}_2 \to [-1, 1]$ that satisfies the following conditions:

(1) $\mu(M_2) = 1$ and $\mu(\Pi_2) = 0$.

(2) $\mu(C_{2,1}) \geqslant \mu(C_{2,2})$ if $C_{2,1}$ is more PLOD than $C_{2,2}$.

(3) $\mu(C_2^1) = \mu(C_2^2) = -\mu(C_2)$.

(4) $\mu(C_2) = \mu(\bar{C}_2)$.

(5) $\mu(C_2) = \mu(C_2^\sigma)$ for the only permutation σ on $\{1, 2\}$ different from the identity.

(6) For any sequence of copulas $(C_{2,i})_{i=1}^\infty$ that converges uniformly to a copula $C_{2,L}$, the following equality holds: $\lim_{i\to\infty} \mu(C_{2,i}) = \mu(C_{2,L})$.

Hereunder we present a list of the most well-known measures of concordance in the bivariate case [152].

(1) Spearman's rho:

$$\rho(C_2) = 12 \int_{[0,1]^2} (C(x, y))\, \mathrm{d}x\mathrm{d}y - 3\,.$$

(2) Kendall's tau:

$$\tau(C_2) = 4 \int_{[0,1]^2} C(x, y)\mathrm{d}C(x, y) - 1\,.$$

(3) Gini's gamma:

$$\gamma(C_2) = 4 \left(\int_{[0,1]} C(x, 1 - x)\mathrm{d}x - \int_{[0,1]} (x - C(x, x))\, \mathrm{d}x \right)\,.$$

(4) Blomqvist's beta:

$$\beta(C_2) = 4C(\frac{1}{2}, \frac{1}{2}) - 1\,.$$

In the multivariate case the situation is more complicated. Several studies have been made in order to better understand multivariate measures of concordance. For example, in [47, 190] a copula-based definition of measure of concordance was introduced, while a construction method of multivariate measures of condordance was proposed in [83].

Additionally, some multivariate generalizations of Spearman's rho, Kendall's tau, Gini's gamma and Blomqvist's beta were studied in [182] and in the references therein. However, there are still several relevant questions that have to be answered on this topic, as remarked in [191].

Other measures of dependence that are commonly used are the tail dependence coefficients. For convenience, we will give first the definition in the bivariate case [102, 103].

Definition 1.15. Let (X, Y) be two random variables with joint distribution function given by the 2-copula C_2.

(1) The lower tail dependence coefficient λ_L of (X, Y) is computed as

$$\lambda_L = \lim_{t \to 0^+} \mathbb{P}(X \leqslant t \mid Y \leqslant t) = \lim_{t \to 0^+} \frac{C_2(t,t)}{t},$$

provided that the limit exists.

(2) The upper tail dependence coefficient λ_U of (X, Y) is computed as

$$\lambda_U = \lim_{t \to 1^-} \mathbb{P}(X > t \mid Y > t) = \lim_{t \to 0^+} \frac{\bar{C}_2(t,t)}{t},$$

provided that the limit exists.

The coefficient λ_L (resp. λ_U) represents the probability that one variable is smaller (resp. larger) than a given small (resp. large) value, given that the other variable is already smaller (resp. larger) than the given value. Hence, these coefficients have been used to study extreme events [183].

In higher dimensions, the situation is more complicated since more variables are involved. A common approach is to use the lower and upper tail dependences functions [104, 105, 127] that we define hereunder.

Definition 1.16. Let C_n be an n-copula.

(1) The lower tail dependence function $b(\cdot, C_n)$ is given by

$$b(\mathbf{w}, C_n) = \lim_{t \to 0^+} \frac{C_n(w_1 t, w_2 t, \ldots, w_n t)}{t},$$

for the values of $\mathbf{w} \in \mathbb{R}^n$ such that the limit exists.

(2) The upper tail dependence function $b^*(\cdot, C_n)$ is given by

$$b^*(\mathbf{w}, C_n) = \lim_{t \to 0^+} \frac{\bar{C}_n(w_1 t, w_2 t, \ldots, w_n t)}{t},$$

for the values of $\mathbf{w} \in \mathbb{R}^n$ such that the limit exists.

When we evaluate the lower tail dependence function (resp. upper tail dependence function) at the point $\mathbf{w} = \mathbf{1}$ we obtain the multivariate generalization of the lower tail dependence coefficient λ_L (resp. upper tail dependence coefficient λ_U).

2 A multivariate generalization of upper semilinear copulas

2.1. Introduction

Recall that for any n-diagonal function d, there exists an n-copula that has d as diagonal section. In the bivariate case, several methods are available to construct copulas with a given diagonal section. Some examples are:

(i) The diagonal copula [154], given by:

$$D_{d_2}(x_1, x_2) = \min\left(x_1, x_2, \frac{d_2(x) + d_2(y)}{2}\right),$$

is the greatest (pointwisely) symmetric 2-copula with given diagonal section d_2.

(ii) The Bertino copula [81], given by:

$$B_{d_2}(x_1, x_2) = \min(x_1, x_2) + \sup\{d_2(t) - t \mid t \in [\min(x_1, x_2), \max(x_1, x_2)]\},$$

is the smallest 2-copula with given diagonal section d_2.

As mentioned in Chapter 1, there exist several other ways of constructing bivariate copulas with a given diagonal section, such as the upper semilinear construction method, which is the main source of inspiration for the results in the present chapter. Semilinear copulas were first introduced by Durante et al. in [56]. These copulas are constructed by linearly interpolating between the values at the lower boundaries (condition (c1)) or upper boundaries (condition (c2)) of the unit square and the values at the diagonal given by the 2-diagonal function d_2. Several generalizations of this approach have been proposed, for example, construction methods that linearly interpolate on other segments of the unit square ([34, 41, 106, 107, 108, 110, 112]) or construction methods that use a polynomial interpolation ([77, 109, 111]); see also [76] for other generalizations.

The aim of this chapter is to study the possible generalization of the upper semilinear 2-copulas to the n-dimensional case. First, we recall the definition of upper and lower semilinear 2-copulas and later, we present a construction method for the multivariate case. Several of the following results can also be found in [9].

2.2. The construction method

Semilinear copulas were first introduced by Durante et al. in [60]. A 2-copula C is called upper semilinear if the mappings

$$h_1 : [t_0, 1] \to [0, 1], \quad h_1(x) = C(x, t_0)$$

$$v_1 : [t_0, 1] \to [0, 1], \quad v_1(x) = C(t_0, x)$$

are linear for all $t_0 \in [0, 1]$. It has been proven that the bivariate function U_{d_2}, defined by

$$U_{d_2}(x_1, x_2) = \frac{(x_{(2)} - x_{(1)})x_{(1)} + (1 - x_{(2)})d_2(x_{(1)})}{1 - x_{(1)}}\,, \tag{2.1}$$

where $x_{(1)} = \min(x_1, x_2)$ and $x_{(2)} = \max(x_1, x_2)$ and the convention $0/0 = 1$ is adopted, is the upper semilinear 2-copula with diagonal section d_2, provided that the following conditions hold:

(i) The function $\nu_{d_2} : [0, 1[\,, \to [0, \infty[$, defined by $\nu_{d_2}(x) = (x - d_2(x))/(1 - x)$, is increasing.

(ii) The function $\phi_{d_2} : [0, 1[\,, \to [0, \infty[$, defined by $\phi_{d_2}(x) = (1 - 2x + d_2(x))/(1 - x)^2$, is increasing.

Analogously, a 2-copula C is called lower semilinear if the mappings

$$h_2 : [0, t_0] \to [0, 1], \quad h_2(x) = C(x, t_0)$$

$$v_2 : [0, t_0] \to [0, 1], \quad v_2(x) = C(t_0, x)$$

are linear for all $t_0 \in [0, 1]$. The bivariate function L_{d_2}, defined by

$$L_{d_2}(x_1, x_2) = \frac{x_{(1)}}{x_{(2)}} d_2(x_{(2)})\,,$$

where the convention $0/0 = 0$ is adopted, is the lower semilinear 2-copula with diagonal section d_2, provided that the following conditions hold:

(i) The function $\nu^*_{d_2} :]0, 1] \to [0, \infty[$, defined by $\nu^*_{d_2}(x) = d_2(x)/x$, is increasing.

(ii) The function $\phi^*_{d_2} :]0, 1] \to [0, \infty[$, defined by $\phi^*_{d_2}(x) = d_2(x)/x^2$, is decreasing.

Following the idea underlying upper semilinear 2-copulas, we propose a similar construction method for higher dimensions. Given a vector $\mathbf{x}$, we denote by $x_{(j)}$ its j-th ordered component, i.e.,

$$\min(x_1, x_2, ..., x_n) = x_{(1)} \leqslant x_{(2)} \leqslant ... \leqslant x_{(n)} = \max(x_1, x_2, ..., x_n)\,.$$

We illustrate the construction method in the 3-dimensional case. Suppose we are given a 3-diagonal function d_3 and the upper semilinear 2-copula U_{d_2} with diagonal section d_2. Let $\mathbf{a} = (a_1, a_2, a_3)$ be a point in $[0,1]^3$ such that $a_1 \leqslant a_2 \leqslant a_3$. The parametric equation of the line that passes through the point $\mathbf{a}$ and the point $\mathbf{b} = (a_1, a_1, a_1)$ is given by:

$$(x_1, x_2, x_3) = (a_1, a_1, a_1) + \lambda \left(0, \frac{(1-a_1)(a_2-a_1)}{a_3-a_1}, (1-a_1) \right) .$$

Note that the given point $\mathbf{a}$ is obtained by choosing $\lambda = (a_3 - a_1)/(1-a_1)$. Setting $\lambda = 1$ we obtain the point $\mathbf{q} = (a_1, a_1 + (1-a_1)(a_2-a_1)/(a_3-a_1), 1)$, i.e., the intersection point of the line with a face of the cube. To the point $\mathbf{a}$, we attribute the value that is obtained by linearly interpolating between the values in the points $\mathbf{b}$ and $\mathbf{q}$, respectively situated on the main diagonal and on an upper face of the cube. In this way, by symmetrization, we build the so-called upper semilinear function given by

$$U_{\mathcal{D}_3}(\mathbf{x}) = \frac{1-x_{(3)}}{1-x_{(1)}} d_3(x_{(1)}) + \frac{x_{(3)}-x_{(1)}}{1-x_{(1)}} U_{d_2}(\mathbf{x}^*) ,$$

where $\mathcal{D}_3$ is shorthand for (d_2, d_3) and $\mathbf{x}^*$ is a 2-dimensional vector with j-th coordinate given by

$$x_j^* = x_{(1)} + \frac{(x_{(j)} - x_{(1)})(1-x_{(1)})}{x_{(3)} - x_{(1)}} .$$

Substituting U_{d_2} by its analytical expression (2.1), we can rewrite $U_{\mathcal{D}_3}$ as:

$$U_{\mathcal{D}_3}(\mathbf{x}) = \frac{(1-x_{(3)})d_3(x_{(1)}) + (x_{(3)}-x_{(2)})d_2(x_{(1)}) + (x_{(2)}-x_{(1)})x_{(1)}}{1-x_{(1)}} .$$

We call $U_{\mathcal{D}_3}$ the upper semilinear 3-variate function with given diagonal sections d_2 and d_3. By repeating this procedure recursively in higher dimensions, we get the following expression for the upper semilinear function $U_{\mathcal{D}_n}$ in n dimensions:

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{1-x_{(n)}}{1-x_{(1)}} d_n(x_{(1)}) + \frac{x_{(n)}-x_{(1)}}{1-x_{(n)}} U_{\mathcal{D}_{n-1}}(\mathbf{x}^*) ,$$

where $\mathcal{D}_n$ is shorthand for $(d_2, d_3, \ldots, d_n)$ and $\mathbf{x}^*$ is an $(n-1)$-dimensional vector with j-th coordinate given by:

$$x_j^* = x_{(1)} + \frac{(x_{(j+1)} - x_{(1)})(1-x_{(1)})}{x_{(n)} - x_{(1)}} .$$

Solving the recurrence equation gives us the following explicit expression for $U_{\mathcal{D}_n}$:

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{1}{1-x_{(1)}}\left[(1-x_{(n)})\,d_n(x_{(1)}) + \sum_{j=1}^{n-1}\left(x_{(j+1)}-x_{(j)}\right)d_j(x_{(1)})\right], \qquad (2.2)$$

where the conventions $d_1(x) = x$ and $0/0 = 1$ are adopted. Note that $U_{\mathcal{D}_n}$ is a symmetric function by construction. In the following section, we identify necessary and sufficient conditions that guarantee this function to be an n-copula, and as a byproduct, guarantee the compatibility of the given diagonal functions, i.e., when there exists an n-copula such that the diagonal sections of its lower-dimensional marginals are the given diagonal functions.

Remark 2.1. Note that the expression in Eq. (2.2) is a weighted average. For a given point $\mathbf{x} \in [0,1]^n$, the values $x_{(1)}, d_2(x_{(1)}), \ldots, d_{n-1}(x_{(1)}), d_n(x_{(1)})$ have corresponding weights $\frac{x_{(2)}-x_{(1)}}{1-x_{(1)}}, \frac{x_{(3)}-x_{(2)}}{1-x_{(1)}}, \ldots, \frac{x_{(n)}-x_{(n-1)}}{1-x_{(1)}}, \frac{1-x_{(n)}}{1-x_{(1)}}$, which are easily verified to add up to one. Hence, when $U_{\mathcal{D}_n}$ is an n-copula, we can interprete Eq. (2.2) as a mixture of distributions. Indeed, if $X_1, ..., X_n$ are random variables with joint cumulative distribution function given by Eq. (2.2), then the variables are exchangeable, due the symmetry of $U_{\mathcal{D}_n}$. Hence, for any $\mathbf{x} \in [0,1]^n$, it holds

$$\begin{aligned}\mathbb{P}(X_1 \leqslant x_1, \ldots, X_n \leqslant x_n) = {} & \frac{1-x_{(n)}}{1-x_{(1)}}\mathbb{P}(\max(X_1,\ldots,X_n) \leqslant x_{(1)}) + \\ & \sum_{j=1}^{n-1}\frac{x_{(j+1)}-x_{(j)}}{1-x_{(1)}}\mathbb{P}(\max(X_1,\ldots,X_j) \leqslant x_{(1)})\,.\end{aligned}$$

Note that this is not a mixture of distributions of the order statistics associated to $X_1, ..., X_n$.

Remark 2.2. The above construction yields a generalization of upper semilinear 2-copulas. Following the same reasoning for the lower semilinear case, it turns out that the lower semilinear function $L_{\mathcal{D}_n}$ in n dimensions is given by:

$$L_{\mathcal{D}_n}(\mathbf{x}) = \frac{x_{(1)}d_n(x_{(n)})}{x_{(n)}}\,.$$

Note that this expression only depends on d_n. There is a simple probabilistic argument that proves that this lower semilinear function is an n-copula if and only if $d_n(x) = x$, whence $L_{\mathcal{D}_n} = M_n$. Indeed, from the expression of $L_{\mathcal{D}_n}$ it follows that the $(n-1)$-marginal $L_{\mathcal{D}_{n-1}}$ (putting $x_{(n)} = 1$) is M_{n-1}. This implies that if $X_1, ..., X_n$ are random variables with joint distribution $L_{\mathcal{D}_n}$, then the equalities $X_1 = X_2 = \cdots = X_{n-1}$ hold almost surely (a.s.) and since $L_{\mathcal{D}_n}$ is a symmetric function, it follows that the equalities $X_1 = X_2 = \cdots = X_n$ hold a.s., i.e., the random variables are perfectly positive dependent and, hence, the n-copula of the random vector $(X_1, X_2, ..., X_n)$ must be M_n. It thus follows that the only lower semilinear n-copula is M_n if $n \geqslant 3$.

The above observation is not in conflict with the result obtained in [54], where Durante et al. also constructed an n-dimensional copula that reduces to the bivariate lower semilinear copula when setting $n = 2$. However, their construction is not based on linear interpolation between the diagonal and the lower faces.

Remark 2.3. Another generalization of lower bivariate semilinear copulas was proposed in [136]. It is not based on linear interpolation between the diagonal and the lower faces, but rather on the study of exogenous shock models. More precisely, the survival copula corresponding to an exchangeable exogenous shock model is characterized in terms of functions that can be regarded as quotients of diagonal functions. As a consequence, the construction method in [133] is one of the first attempts to build an n-copula given both the diagonal section of the n-copula and the diagonal sections of all of its marginals.

2.3. Characterization

For the characterization of upper semilinear n-copulas, we need some combinatorial identities. First, recall that for $n, k \in \mathbb{Z}$, the binomial coefficient is defined as:

$$\binom{n}{k} = \begin{cases} \dfrac{1}{k!}\displaystyle\prod_{j=0}^{k-1}(n-j) & \text{, if } k > 0, \\ 1 & \text{, if } k = 0, \\ 0 & \text{, if } k < 0. \end{cases}$$

Note that if n is a positive integer and $k > n$, it follows from this definition that $\binom{n}{k} = 0$. The following identities, which we will use in the proof of the next theorem, were taken from [91].

Lemma 2.1. *For any positive integers n and k, it holds that:*

$$\sum_{j=0}^{k}(-1)^j\binom{n}{j} = (-1)^k\binom{n-1}{k}.$$

Note that for $k = n$, we retrieve the well-known identity $\sum\limits_{j=0}^{n}(-1)^j\binom{n}{j} = 0$.

Remark 2.4. Note that any n-box can be decomposed in smaller boxes of the form:

$$\bigtimes_{j=1}^{n}[a_j, b_j],$$

where for all $i \neq j$ either $a_j \geqslant b_i$, $a_i \geqslant b_j$ or $[a_i, b_i] = [a_j, b_j]$ holds. Due to the symmetry of $U_{\mathcal{D}_n}$, the $U_{\mathcal{D}_n}$-volume of any such box is equal to the $U_{\mathcal{D}_n}$-volume of a box of the type:

$$\bigtimes_{j=1}^{r} [a_j, b_j]^{m_j} \quad \text{with} \quad \sum_{j=1}^{r} m_j = n\,, \tag{2.3}$$

where for all $i < j$, the inequality $a_j \geqslant b_i$ holds.

Theorem 2.1. *The function $U_{\mathcal{D}_n}$ defined in Eq.* (2.2) *is an n-copula if and only if:*

(i) For any $m \in \{1, 2, ..., n-1\}$, the function $\nu_{\mathcal{D}_n}^{(m)} : [0, 1[\to [0, \infty[$, defined by:

$$\nu_{\mathcal{D}_n}^{(m)}(x) = \frac{1}{1-x} \sum_{j=0}^{m} (-1)^j \binom{m}{j} d_{n-m+j}(x)\,,$$

is increasing. Note that $(1-x)\nu_{\mathcal{D}_n}^{(m)}(x)$ represents the $U_{\mathcal{D}_n}$-volume of the n-box $[0, x]^{n-m} \times [x, 1]^m$.

(ii) The function $\zeta_{\mathcal{D}_n} : [0, 1] \to [0, 1]$, defined by

$$\zeta_{\mathcal{D}_n}(x) = 1 + \sum_{j=1}^{n} (-1)^j \binom{n}{j} d_j(x)$$

is decreasing. Note that $\zeta_{\mathcal{D}_n}(x)$ represents the $U_{\mathcal{D}_n}$-volume of the n-box $[x, 1]^n$.

(iii) The inequality

$$\left(\frac{d_n(x)}{1-x}\right)' \geqslant \frac{1 - \zeta_{\mathcal{D}_n}(x)}{(1-x)^2} \tag{2.4}$$

holds almost everywhere with respect to the Lebesgue measure on $]0, 1[$.

Proof. It suffices to compute the volume of n-boxes of the form described in Eq. (2.3). We split the proof in three parts. In the first part, we will compute the volume of n-boxes of the form $\mathbf{P} = [a_1, b_1]^{n-m} \times [a_2, b_2]^m$ with $a_2 \geqslant b_1$, and we will prove that the positivity of the volume of this type of n-boxes is equivalent to the monotonicity of $\nu_{\mathcal{D}_n}^{(m)}$. In the second part, we will prove that the other types of n-boxes that are asymmetric, and are not of the form described in the first part, have $U_{\mathcal{D}_n}$-volume zero. Finally, in the third part, we will compute the volume of n-boxes that are centered around the main diagonal, where we will deduce the necessity and sufficiency of conditions (ii) and (iii).

Part 1

In this case, we will compute the $U_{\mathcal{D}_n}$-volume of asymmetric n-boxes of the type $\mathbf{P} = [a_1, b_1]^{n-m} \times [a_2, b_2]^m$ with $a_2 \geqslant b_1$. Note that for the vertex $\mathbf{x}$ such that

$S(\mathbf{x}) = 0$ (all the coordinates are of the type b_j), it holds that

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{(1-b_2)d_n(b_1) + (b_2-b_1)d_{n-m}(b_1)}{1-b_1}\,.$$

Among the vertices $\mathbf{x}$ such that $S(\mathbf{x}) = 1$ (just one coordinate is of the type a_j), there are $\binom{n-m}{0}\binom{m}{1}$ vertices that have one coordinate equal to a_2; for these vertices, it holds that

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{(1-b_2)d_n(b_1) + (b_2-a_2)d_{n-m+1}(b_1) + (a_2-b_1)d_{n-m}(b_1)}{1-b_1}\,.$$

The remaining $\binom{n-m}{1}\binom{m}{0}$ vertices such that $S(\mathbf{x}) = 1$ have one coordinate equal to a_1, and hence, for those vertices it holds that

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{(1-b_2)d_n(a_1) + (b_2-b_1)d_{n-m}(a_1) + (b_1-a_1)a_1}{1-a_1}\,.$$

For the vertices such that $S(\mathbf{x}) = 2$ (exactly two coordinates are of the type a_j), the value

$$\frac{(1-b_2)d_n(b_1) + (b_2-a_2)d_{n-m+2}(b_1) + (a_2-b_1)d_{n-m}(b_1)}{1-b_1}$$

is assigned to $\binom{n-m}{0}\binom{m}{2}$ vertices, the value

$$\frac{(1-b_2)d_n(a_1) + (b_2-a_2)d_{n-m+1}(b_1) + (a_2-b_1)d_{n-m}(a_1) + (b_1-a_1)a_1}{1-a_1}$$

to $\binom{n-m}{1}\binom{m}{1}$ vertices, whereas in the remaining $\binom{n-m}{2}\binom{m}{0}$ vertices the value is

$$\frac{(1-b_2)d_n(a_1) + (b_2-b_1)d_{n-m}(a_1) + (b_1-a_1)d_2(a_1)}{1-a_1}\,.$$

Continuing this procedure, we obtain that for vertices $\mathbf{x}$ such that $S(\mathbf{x}) = s$ for any $0 \leqslant s \leqslant n$, there are $\binom{n-m}{0}\binom{m}{s}$ vertices where $U_{\mathcal{D}_n}$ takes the form

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{(1-b_2)d_n(b_1) + (b_2-a_2)d_{n-m+s}(b_1) + (a_2-b_1)d_{n-m}(b_1)}{1-b_1}\,,$$

whereas in the remaining $\binom{n-m}{k}\binom{m}{s-k}$ vertices ($k \in \{1, 2, \ldots, m\}$), $U_{\mathcal{D}_n}$ is given by

$$\begin{aligned} U_{\mathcal{D}_n}(\mathbf{x}) \;=\; & \frac{(1-b_2)d_n(a_1) + (b_2-a_2)d_{n-m+s-k}(a_1)}{1-a_1} \\ & + \frac{(a_2-b_1)d_{n-m}(a_1) + (b_1-a_1)d_k(a_1)}{1-a_1}\,. \end{aligned}$$

Hence, the $U_{\mathcal{D}_n}$-volume of the n-box $\mathbf{P}$ is given by

$$
\begin{aligned}
V_{U_{\mathcal{D}_n}}(\mathbf{P}) &= \sum_{\mathbf{x}\in \text{vertices}(\mathbf{P})} (-1)^{S(\mathbf{x})} U_{\mathcal{D}_n}(\mathbf{x}) \\
&= \sum_{s=0}^{n} (-1)^s \binom{m}{s} \frac{(1-b_2)d_n(b_1) + (b_2-a_2)d_{n-m+s}(b_1) + (a_2-b_1)d_{n-m}(b_1)}{1-b_1} \\
&\quad + \sum_{k=1}^{n}\sum_{s=k}^{n} (-1)^s \binom{n-m}{k}\binom{m}{s-k} \left[\frac{(1-b_2)d_n(a_1) + (b_2-a_2)d_{n-m+s-k}(a_1)}{1-a_1}\right] \\
&\quad + \sum_{k=1}^{n}\sum_{s=k}^{n} (-1)^s \binom{n-m}{k}\binom{m}{s-k} \left[\frac{(a_2-b_1)d_{n-m}(a_1) + (b_1-a_1)d_k(a_1)}{1-a_1}\right].
\end{aligned}
$$

Observe that in the first summation, the terms $(1-b_2)d_n(b_1)/(1-b_1)$ and $(a_2 - b_1)d_{n-m}(b_1)/1-b_1$ are constant with respect to the summation index and, hence, have a coefficient equal to zero due to Lemma 2.1. Next, we note that in the second and third summations, if the coefficients k and s are such that at least one of the inequalities $k > n-m$ or $s > m+k$ holds, then the respective combinatorial coefficient is zero, and hence:

$$
\begin{aligned}
V_{U_{\mathcal{D}_n}}(\mathbf{P}) &= \sum_{\mathbf{x}\in \text{vertices}(\mathbf{P})} (-1)^{S(\mathbf{x})} U_{\mathcal{D}_n}(\mathbf{x}) \\
&= \sum_{s=0}^{n} (-1)^s \binom{m}{s} \frac{(b_2-a_2)d_{n-m+s}(b_1)}{1-b_1} \\
&\quad + \sum_{k=1}^{n-m}\sum_{s=k}^{m+k} (-1)^s \binom{n-m}{k}\binom{m}{s-k} \left[\frac{(1-b_2)d_n(a_1) + (b_2-a_2)d_{n-m+s-k}(a_1)}{1-a_1}\right] \\
&\quad + \sum_{k=1}^{n-m}\sum_{s=k}^{m+k} (-1)^s \binom{n-m}{k}\binom{m}{s-k} \left[\frac{(a_2-b_1)d_{n-m}(a_1) + (b_1-a_1)d_k(a_1)}{1-a_1}\right].
\end{aligned}
$$

By applying the change of variable $t = s-k$ in the second and third summations, we obtain

$$
\begin{aligned}
V_{U_{\mathcal{D}_n}}(\mathbf{P}) &= \sum_{s=0}^{m} (-1)^s \binom{m}{s} \frac{(b_2-a_2)d_{n-m+s}(b_1)}{1-b_1} \\
&\quad + \sum_{k=1}^{n-m}\sum_{t=0}^{m} (-1)^{s+t} \binom{n-m}{k}\binom{m}{t} \frac{(b_2-a_2)d_{n-m+t}(a_1)}{1-a_1} \\
&= (b_2-a_2)\left[\nu_{\mathcal{D}_n}^{(m)}(b_1) + \sum_{k=1}^{n-m} (-1)^k \binom{n-m}{k} \nu_{\mathcal{D}_n}^{(m)}(a_1)\right].
\end{aligned}
$$

Since due to Lemma 2.1 it holds that

$$
0 = \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} = 1 + \sum_{k=1}^{n-m} (-1)^k \binom{n-m}{k},
$$

we get

$$V_{U_{\mathcal{D}_n}}(\mathbf{P}) = (b_2 - a_2)\left[\nu^{(m)}_{\mathcal{D}_n}(b_1) - \nu^{(m)}_{\mathcal{D}_n}(a_1)\right].$$

From this equality, it follows that $V_{U_{\mathcal{D}_n}}(\mathbf{P})$ is positive if and only if $\nu^{(m)}_{\mathcal{D}_n}$ is increasing for all $m \in \{1, 2, ..., n-1\}$.

Part 2

Next, we prove that the other types of asymmetric n-boxes, which are not of the form described in the first part, have $U_{\mathcal{D}_n}$-volume zero. Let $\mathbf{P}$ be an n-box of the form described in Eq. (2.3), with $r \geqslant 3$ (the case $r = 2$ is treated in Part 1). Using a similar reasoning as in the previous part, we can show that for the $\binom{n}{s}$ vertices such that $S(\mathbf{x}) = s$, the value

$$\frac{(1-b_r)d_n(a_1) + \sum_{j=1}^{r}(b_j - a_j)d_{\sum_{k=1}^{j-1} m_k + r_j}(a_1) + \sum_{j=1}^{r}(a_{j+1} - b_j)d_{\sum_{k=1}^{j} m_k}(a_1)}{1 - a_1}$$

is assigned to

$$\prod_{j=1}^{r}\binom{m_j}{i_j} \quad \text{with} \quad \sum_{j=1}^{r} i_j = s \quad \text{and} \quad i_1 > 0\,,$$

vertices. Similarly, the value

$$\frac{(1-b_r)d_n(b_1) + \sum_{j=1}^{r}(b_j - a_j)d_{\sum_{k=1}^{j-1} m_k + r_j}(b_1) + \sum_{j=1}^{r}(a_{j+1} - b_j)d_{\sum_{k=1}^{j} m_k}(b_1)}{1 - b_1}$$

is assigned to

$$\prod_{j=2}^{r}\binom{m_j}{i_j} \quad \text{with} \quad \sum_{j=2}^{r} i_j = s\,,$$

vertices. Hence, the $U_{\mathcal{D}_n}$-volume of the n-box $\mathbf{P}$ is given by:

$$V_{U_{\mathcal{D}_n}}(\mathbf{P}) = V^{(1)}_{U_{\mathcal{D}_n}}(\mathbf{P}) + V^{(2)}_{U_{\mathcal{D}_n}}(\mathbf{P})\,,$$

where

$$V^{(1)}_{U_{\mathcal{D}_n}}(\mathbf{P}) = \sum_{s=0}^{n}(-1)^s \sum_{\mathbf{i} \in A_s}\left[\prod_{k=1}^{r}\binom{m_k}{i_k}\right] U_{\mathcal{D}_n}(\mathbf{x_i})\,,$$

with

$$A_s = \{(i_1, i_2, \ldots, i_r) \in \{0, 1, 2, \ldots, s\}^r \mid i_1 > 0 \text{ and } \sum_{j=1}^{r} i_j = s\}\,,$$

and

$$V^{(2)}_{U_{\mathcal{D}_n}}(\mathbf{P}) = \sum_{s=0}^{n}(-1)^s \sum_{\mathbf{i} \in B_s}\left[\prod_{k=1}^{r}\binom{m_k}{i_k}\right] U_{\mathcal{D}_n}(\mathbf{x_i})\,,$$

with

$$B_s = \{(i_1, i_2, \ldots, i_r) \in \{0, 1, 2, \ldots, s\}^r \mid i_1 = 0 \text{ and } \sum_{j=1}^{r} i_j = s\}\,.$$

For the first term, $V^{(1)}_{U_{\mathcal{D}_n}}(\mathbf{P})$, note that there are r summations involved, since $\mathbf{i}$ is of dimension r, and so, by rearranging the terms in a similar way as in the first part, we get:

$$V^{(1)}_{U_{\mathcal{D}_n}}(\mathbf{P}) = \sum_{i_1=1}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{s=m_r-\sum_{j=1}^{r-1} i_j}^{n} (-1)^s \left[\prod_{k=1}^{r-1} \binom{m_k}{i_k}\right] \binom{m_r}{s-\sum_{j=1}^{r-1} i_j} U_{\mathcal{D}_n}(\mathbf{x}_i)\,.$$

By applying the change of variable $i_r = s - \sum_{j=1}^{r-1} i_j$, we can rewrite $V^{(1)}_{U_{\mathcal{D}_n}}(\mathbf{P})$ as

$$\begin{aligned}
V^{(1)}_{U_{\mathcal{D}_n}}(\mathbf{P}) &= \sum_{i_1=1}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1+i_2+\ldots+i_r} \left[\prod_{k=1}^{r} \binom{m_k}{i_k}\right] \frac{(1-b_r)d_n(a_1)}{1-a_1} \\
&+ \sum_{i_1=1}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1+i_2+\ldots+i_r} \left[\prod_{k=1}^{r} \binom{m_k}{i_k}\right] \sum_{j=1}^{r} \frac{(b_j-a_j)d_{\sum_{k=1}^{j-1} m_k + r_j}(a_1)}{1-a_1} \\
&+ \sum_{i_1=1}^{m_1} \sum_{i_2=0}^{m_2} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1+i_2+\ldots+i_r} \left[\prod_{k=1}^{r} \binom{m_k}{i_k}\right] \sum_{j=1}^{r} \frac{(a_{j+1}-b_j)d_{\sum_{k=1}^{j} m_k}(a_1)}{1-a_1}\,.
\end{aligned}$$

From this expression, it obviously follows with Lemma 2.1 that $V^{(1)}_{U_{\mathcal{D}_n}}(\mathbf{P})$ equals zero. Using a similar reasoning, it can be proven that also $V^{(2)}_{U_{\mathcal{D}_n}}(\mathbf{P})$ equals to zero.

Part 3

Finally, we will compute the volume of an n-box $\mathbf{P} = [a, b]^n$ centered around the main diagonal. It is easy to see that in this case, if a vertex is such that $S(\mathbf{x}) = s$, with $s \in \{1, 2, ..., n-1\}$, then

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{(1-b)d_n(a) + (b-a)d_s(a)}{1-a}\,.$$

It then follows immediately that

$$V_{U_{\mathcal{D}_n}}(\mathbf{P}) = d_n(b) + (-1)^n d_n(a) + \sum_{j=1}^{n-1} (-1)^j \binom{n}{j} \frac{(1-b)d_n(a) + (b-a)d_j(a)}{1-a}\,,$$

which can be rewritten as

$$V_{U_{\mathcal{D}_n}}(\mathbf{P}) = d_n(b) + (-1)^n d_n(a) - (1+(-1)^n)\frac{(1-b)d_n(a)}{1-a} + \sum_{j=1}^{n-1} (-1)^j \binom{n}{j} \frac{(b-a)d_j(a)}{1-a}\,,$$

which reduces to

$$d_n(b) - \frac{(1-b)d_n(a)}{1-a} + \sum_{j=1}^{n}(-1)^j \binom{n}{j} \frac{(b-a)d_j(a)}{1-a} \,.$$

Hence, the positivity of the $U_{\mathcal{D}_n}$-volume of the n-box $\mathbf{P}$ is equivalent to

$$\frac{d_n(b)}{1-b} - \frac{d_n(a)}{1-a} \geqslant -\sum_{j=1}^{n}(-1)^j \binom{n}{j} \frac{(b-a)d_j(a)}{(1-a)(1-b)} \,,$$

which can be rewritten as

$$\frac{d_n(b)}{1-b} - \frac{d_n(a)}{1-a} \geqslant \frac{(b-a)(1-\zeta_{\mathcal{D}_n}(a))}{(1-a)(1-b)} \,. \tag{2.5}$$

Now, if $U_{\mathcal{D}_n}$ is a copula, then it is clear that (ii) holds, and by dividing both sides of Eq. (2.5) by $b-a$ and taking the limit as $b \to a$, we get that condition (iii) holds.

For the sufficiency, suppose that conditions (i), (ii) and (iii) hold. Clearly, the function $\zeta_{\mathcal{D}_n}$ is absolutely continuous, since it is a linear combination of diagonal functions, which are absolutely continuous. Hence, by integrating from a to b in both sides of Eq. (2.4), we get

$$\frac{d_n(b)}{1-b} - \frac{d_n(a)}{1-a} \geqslant \int_a^b \frac{1-\zeta_{\mathcal{D}_n}(t)}{(1-t)^2} dt \geqslant \int_a^b \frac{1-\zeta_{\mathcal{D}_n}(a)}{(1-t)^2} dt = \frac{(b-a)(1-\zeta_{\mathcal{D}_n}(a))}{(1-a)(1-b)} \,.$$

The second inequality is justified by condition (ii). Note that this last expression implies that Eq. (2.5) holds, which is in turn equivalent to $V_{U_{\mathcal{D}_n}}(\mathbf{P}) \geqslant 0$. □

Remark 2.5. Recall that the function $\zeta_{\mathcal{D}_n}(x)$ represents the $U_{\mathcal{D}_n}$-volume of the n-box $[x,1]^n$. This function can also be used to compute one of the tail dependence coefficients. Some elementary computations show that $\lambda_L = d_n'(0^+)$ and $\lambda_U = \zeta_{\mathcal{D}_n}'(1^-)$, provided that the limits exist.

We now study the particular case when the n-copula $U_{\mathcal{D}_n}$ and all of its lower dimensional marginals have the same diagonal section, i.e., when $d_2 = d_3 = ... = d_n = d$. This case also appeared in [8], where it was proven that for a specific type of diagonal functions, the associated n-dimensional Bertino copula is such that all of its lower dimensional marginals have the same diagonal section as the Bertino n-copula itself.

Note that if $d_2 = d_3 = ... = d_n = d$, then the expression in Eq. (2.2) reduces to:

$$U_{\mathcal{D}_n}(\mathbf{x}) = \frac{\left(1-x_{(2)}\right) d\left(x_{(1)}\right) + \left(x_{(2)} - x_{(1)}\right) x_{(1)}}{1-x_{(1)}} \,. \tag{2.6}$$

For the expression given in Eq. (2.6), the conditions of Theorem 2.1 can be reformulated as follows.

Corollary 2.1. *Let $U_{\mathcal{D}_n}$ be defined as in Eq.* (2.6)*. Then $U_{\mathcal{D}_n}$ is an n-copula if and only if the function $\zeta_{n,d} : [0,1] \to [0,1]$ is decreasing and the functions $\nu_{n,d}$, $\phi_{n,d} : [0,1[\to \mathbb{R}$ are increasing, where*

$$\begin{aligned} \zeta_{n,d}(x) &= 1 - nx + (n-1)d(x) \\ \phi_{n,d}(x) &= \frac{1 - nx + (n-1)d(x)}{(1-x)^n} \\ \nu_{n,d}(x) &= \frac{x - d(x)}{1-x} \,. \end{aligned}$$

Proof. Some elementary calculations show that $\phi_{n,d}$ is increasing if and only if the inequality

$$\frac{n-1}{(1-x)^{n-1}} \left(\frac{d'(x)}{1-x} + n \frac{d(x)-x}{(1-x)^2} \right) \geqslant 0 \tag{2.7}$$

holds almost everywhere with respect the Lebesgue measure in $]0,1[$. The latter inequality is also equivalent to

$$\frac{d'(x)}{1-x} + \frac{d(x)}{(1-x)^2} - \frac{1-\zeta_{n,d}(x)}{(1-x)^2} \geqslant 0 \,,$$

which is precisely condition (iii) of Theorem 2.1. The other conditions follow immediately from Theorem 2.1. □

2.4. Examples

Example 2.1. Consider the function $d(a,\lambda;x)$ given by

$$d(a,\lambda;x) = \begin{cases} \lambda x & , \text{if } 0 \leqslant x \leqslant a \,, \\ \dfrac{(1-\lambda a)x - a(1-\lambda)}{1-a} & , \text{if } a < x \leqslant 1 \,. \end{cases}$$

This function is an n-diagonal function if and only if $\lambda \in [0,1]$ and $a \in [0,(n-1)/(n-\lambda)]$.

Let $\lambda_1 = 1$. It is clear that, independently of the value of a, $d(a,\lambda_1;x) = x$. Now choose $\lambda_2, \lambda_3, \ldots, \lambda_n$ such that the following inequalities are satisfied:

(a) For any $m \in \{1,2,\ldots,m-1\}$, it holds that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \lambda_{n-m+j} \geqslant 0 \,, \tag{2.8}$$

(b) The parameter a is upper bounded as follows

$$a \leqslant \frac{\lambda_n}{\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\lambda_j}\,, \tag{2.9}$$

Then $\mathcal{D}_n = (d(a,\lambda_2;\cdot), d(a,\lambda_3;\cdot), ..., d(a,\lambda_n;\cdot))$ satisfies the conditions of Theorem 2.1. Indeed, note that the function $\nu_{\mathcal{D}_n}^{(m)}$ is given by

$$\nu_{\mathcal{D}_n}^{(m)}(x) = \min\left(\frac{x}{1-x}, \frac{a}{1-a}\right)\sum_{j=0}^{m}(-1)^j\binom{m}{j}\,,$$

which is clearly increasing if inequality (2.8) holds. Now, note that

$$\zeta_{\mathcal{D}_n}(x) = \begin{cases} 1 + x\sum_{j=1}^{n}(-1)^j\binom{n}{j}\lambda_j & \text{, if } 0 \leqslant x \leqslant a\,,\\ 1 - nx + \sum_{j=2}^{n}(-1)^j\binom{n}{j}\left(\frac{(1-\lambda_j a)x - a(1-\lambda_j)}{1-a}\right) & \text{, if } a < x \leqslant 1\,. \end{cases}$$

We will show that this function is decreasing. Clearly

$$\zeta'_{\mathcal{D}_n}(x) = \sum_{j=1}^{n}(-1)^j\binom{n}{j}\lambda_j\,, \tag{2.10}$$

on the interval $]0,a[$. We will show by induction that this expression is negative. For $n=2$, it is clear that Eq. (2.10) reduces to $-2+\lambda_2$, which is always negative. For $n=3$, we have

$$\zeta'_{\mathcal{D}_3}(x) = -3 + 3\lambda_2 - \lambda_3 = -(1 - 2\lambda_2 + \lambda_3) + (\lambda_2 - 2)\,.$$

The first term is negative due to Eq. (2.8), while the second term is negative due to the previous step. In general, by using the identity $\binom{n-1}{j-1} + \binom{n-1}{j} = \binom{n}{j}$, we can rewrite Eq. (2.10) as

$$\zeta'_{\mathcal{D}_n}(x) = \sum_{j=1}^{n}(-1)^j\binom{n}{j}\lambda_j = \sum_{j=0}^{n-1}(-1)^{j+1}\binom{n-1}{j}\lambda_j + \sum_{j=1}^{n-1}(-1)^j\binom{n-1}{j}\lambda_j\,.$$

Once again, the first summation is negative due to Eq. (2.8) and the second summation is negative due to the induction hypothesis. Next, for $x \in]a,1[$, we have

$$\zeta'_{\mathcal{D}_n}(x) = -n + \sum_{j=2}^{n}(-1)^j\binom{n}{j}\frac{(1-\lambda_j a)}{1-a} = -n + \frac{n - 1 + a\sum_{j=2}^{n}(-1)^{j+1}\binom{n}{j}\lambda_j}{1-a}\,.$$

Using inequality (2.9), we get

$$\zeta'_{\mathcal{D}_n}(x) \leqslant -n + \frac{n-1+\lambda_n - na}{1-a} = \frac{\lambda_n - 1}{1-a} \leqslant 0\,.$$

Hence, it follows that $\zeta'_{\mathcal{D}_n} \leqslant 0$ almost everywhere in $]0,1[$ and since $\zeta_{\mathcal{D}_n}$ is continuous, we can conclude that $\zeta_{\mathcal{D}_n}$ is decreasing. We now verify condition (iii) of Theorem 2.1. This condition states that for $x \in]0,a]$, it should hold that

$$\frac{\lambda_n}{1-x} + \frac{\lambda_n x}{(1-x)^2} \geqslant \frac{\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\lambda_j x}{(1-x)^2}\,.$$

Some simple computations show that this is equivalent to inequality (2.9). Furthermore, for $x \in]a,1]$, it should hold that

$$\begin{aligned}\frac{1-\lambda_n a}{1-x} + \frac{(1-\lambda_n a)x - a(1-\lambda_n)}{(1-x)^2} &\geqslant \frac{\sum_{j=1}^{n}(1-\lambda_j a)x - a(1-\lambda_j)}{(1-x)^2}\\ &= \frac{x-a+a(1-x)\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\lambda_j}{(1-x)^2}\,.\end{aligned}$$

After some elementary computations, this last inequality is seen to be equivalent to

$$a \leqslant \frac{1}{\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\lambda_j}\,,$$

which is satisfied since inequality (2.9) holds.

Example 2.2. Let $d_n(x) = x^\alpha$. Some elementary calculations show that the conditions of Corollary 2.1 are fulfilled if and only if $\alpha \in [1, n/(n-1)]$.

Example 2.3. Consider the n-diagonal function defined by $d_n(x) = \frac{\lambda x}{1-(1-\lambda)x}$, with $\lambda \in [1/2, 1]$. The conditions of Corollary 2.1 are satisfied if and only if $\lambda \in [(n-1)/n, 1]$. Indeed, some simple computations show that the sign of the derivative of ϕ_{n,d_n} only depends on

$$\lambda - nx(1-\lambda)(1-(1-\lambda)x)\,.$$

The latter expression is always decreasing if $\lambda > 1/2$. Hence, the minimum is attained at $x = 1$, from which we get the condition that ϕ_{n,d_n} is increasing if and only if $\lambda \geqslant (n-1)/n$. If $\lambda \leqslant 1/2$, then the function has a minimum at $1/[2(1-\lambda)]$, where it is easy to see that the derivative is negative in an open interval centered around this value. The other conditions are trivially satisfied.

Example 2.4. In this example, we find the smallest d_n such that the function ϕ_{n_d} from Corollary 2.1 is increasing. From Eq. (2.7) it follows that $\phi_{n,d}$ is increasing if and only if

$$\frac{d_n'(t)}{(1-t)^n} + \frac{d_n(t)}{(1-t)^{n+1}} - \frac{nt}{(1-t)^{n+1}} \geqslant 0$$

holds almost everywhere. Integrating the left and right side of this last inequality from 0 to x, we get that the following inequality holds almost everywhere in $]0,1[$:

$$d_n(x) \geqslant \frac{(1-x)^n + nx - 1}{n-1} = d_{n,0}(x)\,.$$

Since both $d_{n,0}$ and d_n are continuous, the inequality holds for all $x \in [0,1]$. Some easy computations show that $d_{n,0}$ is an n-diagonal function, $\zeta_{n,d_{n,0}}$ is a decreasing function and the functions $\phi_{n,d_{n,0}}$ and $\nu_{n,d_{n,0}}$ are increasing. Hence, by construction, $d_{n,0}$ is the smallest n-diagonal function such that $U_{\mathcal{D}_n}$ defined as in Eq. (2.6) is an n-copula.

Remark 2.6. Note that $d_{n,0}(x) \geqslant x^n$. To see this, note that if $n \geqslant 2$ is an integer, then for any $x \in [0,1]$ it holds that

$$(1-x)^{n-1}\left[(n-2)x+1\right] \leqslant (1-x)^{n-2}\left[(n-3)x+1\right]\,.$$

This inequality is equivalent to:

$$d_{n-1,0}(x) = \frac{(1-x)^{n-1} + (n-1)x - 1}{n-2} \leqslant \frac{(1-x)^n + nx - 1}{n-1} = d_{n,0}(x)\,.$$

Since $d_{2,0}(x) = x^2$, it follows from the last expression that $d_{n,0}(x) \geqslant x^n$. As a consequence, for any d_n that satisfies the conditions of Corollary 2.1, the inequality

$$U_{\mathcal{D}_n}(\mathbf{x}) \geqslant \Pi_n(\mathbf{x})$$

holds for any $\mathbf{x} \in [0,1]^n$, where $U_{\mathcal{D}_n}$ is defined as in Eq. (2.6). This generalizes the well-known result that for $n = 2$, the upper semilinear 2-copulas are positively quadrant dependent.

3 A construction method for radially symmetric copulas in higher dimensions

3.1. Introduction

For this chapter, we rotate the kaleidoscope of copulas and switch our attention from the diagonal section of an n-copula to the concept of radial symmetry.

The concept of symmetry of a random variable is uniquely defined. A random variable X is said to be symmetric about a if $X - a$ has the same distribution as $a - X$. In the multivariate case, the situation is more complicated, as there are several ways to generalize the notion of univariate symmetry (see, for example, [188]). One such possible generalization is the concept of radial symmetry. An n-dimensional random vector $(X_1, \ldots, X_n)$ is said to be radially symmetric about $(a_1, \ldots, a_n)$ if the random vector $(X_1 - a_1, \ldots, X_n - a_n)$ has the same distribution as the random vector $(a_1 - X_1, \ldots, a_n - X_n)$.

One of the advantages of radial symmetry is that it is a rank invariant property [153] and, as a consequence, it can be studied on the basis of the associated n-copula of the random vector. It can be easily shown that a continuous random vector $(X_1, \ldots, X_n)$ is radially symmetric about $(a_1, \ldots, a_n)$ if and only if, for any $j \in \{1, \ldots, n\}$, $X_j - a_j$ has the same distribution as $a_j - X_j$, and $C_n = \bar{C}_n$, where C_n is the copula associated to the random vector $(X_1, \ldots, X_n)$ and $\bar{C}_n$ its survival copula as defined in Chapter 1. Due to this characterization, we say that an n-copula C_n is radially symmetric if it satisfies the identity $C_n = \bar{C}_n$. Clearly, radial symmetry of an n-copula implies the radial symmetry of its lower dimensional marginals, however, the converse statement is not true. For example, the Frank 3-copula is not radially symmetric, even though all its 2-dimensional marginals are radially symmetric.

Radially symmetric copulas have a particular importance in stochastic simulation and statistics, as they can be used, in certain situations, in the multivariate version of the antithetic variates method, which is a variance reduction technique used in Monte Carlo methods [134]. Additionally, there has also been a growing interest in developing statistical tests for testing the presence of radial symmetry [2, 19, 42, 87, 95, 162, 173].

In the bivariate case, well-known examples of families of copulas that are radially symmetric are the Frank family and the Farlie-Gumbel-Morgenstern (FGM) family [152]. However, there are only a few families of n-copulas that are radially symmetric for $n \geqslant 3$, elliptical copulas being the best known [134].

Even well-known methods to construct n-copulas in higher dimensions are not useful to construct radially symmetric n-copulas, such as, for example, associative extensions of 2-copulas. Recall that a 2-copula is associative if for any $x, y, z \in [0,1]$ the equality $C_2(x, C_2(y,z)) = C_2(C_2(x,y),z)$ holds, thus allowing to extend it recursively to higher dimensions by defining for any $n \geqslant 2$ and $\mathbf{x} \in [0,1]^{n+1}$,

$$C_{n+1}(\mathbf{x}) = C_2(x_1, C_n(x_2, \ldots, x_{n+1})) = C_2(C_n(x_1, \ldots, x_n), x_{n+1}) .$$

However, we now show that radial symmetry can be rather restrictive if we focus only on associative copulas. To see this, note that if a 2-copula is radially symmetric, then for any $x, y \in [0,1]$, it holds that

$$C_2(x,y) + (1 - C_2(1-x, 1-y)) = x + y . \tag{3.1}$$

If C_2 is an associative copula, the latter equation is a particular case of a functional equation studied by Frank in [79], where it is proven that the only functions $F : [0,1]^2 \to [0,1]$ that satisfy conditions (c1) and (c2) of a 2-copula and are such that both $F(x,y)$ and $G(x,y) = x + y - F(x,y)$ are associative, are the members of the Frank family of 2-copulas or ordinal sums of members of this family. We first recall the concept of an ordinal sum. Let $(C_{n,j})_{j \in J}$ be a family of n–copulas. For any $\mathbf{x}$, denote by $\mathbf{x}^j$ the point in $[0,1]^n$ given by

$$\mathbf{x}^j = \left(\frac{((x_1 \wedge b_j) - a_j)^+}{b_j - a_j}, \ldots, \frac{((x_n \wedge b_j) - a_j)^+}{b_j - a_j} \right) .$$

The ordinal sum C_n of $(C_{n,j})_{j \in J}$ with respect to the family of intervals $(]a_j, b_j[)_{j \in J}$ is defined for all $\mathbf{x} \in [0,1]^n$ as

$$C_n(\mathbf{x}) = \begin{cases} a_j + (b_j - a_j) C_{n,j}(\mathbf{x}^j) & \text{, if } M_n(\mathbf{x}) \in]a_j, b_j[\text{ for some } j \in J \,, \\ M_n(\mathbf{x}) & \text{, otherwise} \,. \end{cases}$$

Now, let us recall that the bivariate Frank family is given by:

$$F^{(\alpha)}(x,y) = -\frac{1}{\alpha} \ln \left(1 + \frac{(e^{-\alpha x} - 1)(e^{-\alpha y} - 1)}{e^{-\alpha} - 1} \right) ,$$

where $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$.

The latter result was complemented in [120] by showing that the only 2-copulas that are both associative and radially symmetric are the members of the Frank family of 2-copulas or ordinal sums of the form $C_2 = (\langle a_j, b_j, F_j^{(\alpha_j)} \rangle)_{j \in J}$, such that for any j, there exists i_j with the property that $\alpha_j = \alpha_{i_j}$, $a_j = 1 - b_{i_j}$ and $b_j = 1 - a_{i_j}$.

Note that if a radially symmetric 3-copula is an associative extension of a 2-copula, then its 2-dimensional marginals must also be radially symmetric. Unfortunately, as shown in [125], for $n \geqslant 3$ the only copulas that are obtained as an associative extension of a 2-copula and that are radially symmetric, are the product copula Π_n and the upper Fréchet-Hoeffding bound M_n or ordinal sums thereof. As a consequence, we must drop the associativity property, as simultaneously requiring associativity and radial symmetry is too restrictive. As a consequence, the well-known Archimedean n-copulas cannot be radially symmetric for $n \geqslant 3$, with the exception of the product copula.

The purpose of this chapter is to provide a representation theorem for symmetric and radially symmetric copulas. Then, we use this theorem to construct such copulas in higher dimensions, more specifically, given a symmetric and radially symmetric $(n-1)$-copula C_{n-1}, we construct a symmetric and radially symmetric n-copula C_n, with the property that its $(n-1)$-dimensional marginal coincides with C_{n-1}. Most of the results of this chapter can be found in [10].

3.2. The representation theorem and resulting construction method

The study of transformations of copulas has been a topic of great interest [84, 85, 121]. The following characterization of radially symmetric 2-copulas, which can be easily generalized to higher dimensions, was proven by Klement et al. in [120].

Theorem 3.1. *An n-copula C_n is radially symmetric if and only if there exists an n-copula D_n such that for any $\mathbf{x} \in [0,1]^n$ it holds that*

$$C_n(\mathbf{x}) = \frac{D_n(\mathbf{x}) + \bar{D}_n(\mathbf{x})}{2}\,. \tag{3.2}$$

The latter characterization has been used in financial applications [97, 165] in order to construct 2-copulas that are radially symmetric starting from well-known families of 2-copulas. We now give an alternative representation of symmetric and radially symmetric copulas in terms of a function that does not necessarily have to be a copula. While we will formulate the following characterization for symmetric copulas only, the results in the rest of this section can be extended to non-symmetric copulas easily, albeit by introducing more tedious notations and performing more extensive computations. In the following, $\mathbf{x}_A$ in $\mathbb{R}^{n-\#A}$ denotes the vector whose components take the values of the elements $x_1, \ldots, x_n$, except for those elements x_j for which j belongs to the index set A. We also remark that the notations C_2, C_3, $\ldots$, C_{n-1} used further on make sense, since we will construct symmetric copulas and, hence, all the k-dimensional marginals are equal for any $k \in \{2,3,\ldots,n-1\}$.

Theorem 3.2. Let C_n be a symmetric n-copula. Then C_n is radially symmetric if and only if there exists a symmetric function $H : [0,1]^n \to \mathbb{R}$ that satisfies the following four properties

(1) For any $\mathbf{x}$, the following equality holds

$$\begin{aligned} C_n(\mathbf{x}) \;=\; & \frac{1}{2}\Big[\sum_{i=1}^{n} x_i - n + 1 + \sum_{i<j}^{n} C_2(1-x_i, 1-x_j) \\ & - \sum_{i<j<k}^{n} C_3(1-x_i, 1-x_j, 1-x_k) + \ldots \\ & + (-1)^{n-1} \sum_{i=1}^{n} C_{n-1}(1-x_1, \ldots, 1-x_{i-1}, 1-x_{i+1}, \ldots, 1-x_n)\Big] \\ & + \frac{1}{2}\left[H(\mathbf{x}) + (-1)^n H(\mathbf{1}-\mathbf{x})\right] , \end{aligned} \tag{3.3}$$

where C_k denotes the k-dimensional marginal of C_n.

(2) If $\mathbf{x}$ is such that $x_j = 0$ for some $j \in \{1,2,...,n\}$, then $H(\mathbf{x}) = 0$.

(3) If $\mathbf{x}$ is such that $x_j = 1$ for some $j \in \{1,2,...,n\}$, then $H(\mathbf{x}) = C_{n-1}(\mathbf{x}_{\{j\}})$.

(4) For any n-box $\mathbf{P} = \times_{i=1}^{n}[a_i, b_i] \subseteq [0,1]^n$, it holds that $V_H(\mathbf{P}) + V_H(\mathbf{1}-\mathbf{P}) \geqslant 0$, where $\mathbf{1}-\mathbf{P} = \times_{i=1}^{n}[1-b_i, 1-a_i]$.

Remark 3.1. Note that the right-hand side of Eq. (3.3) can be written as

$$\frac{1}{2}\left(\bar{C}_n(\mathbf{x}) + (-1)^{n+1} C_n(\mathbf{1}-\mathbf{x}) + H(\mathbf{x}) + (-1)^n H(\mathbf{1}-\mathbf{x})\right) .$$

We will split the proof of Theorem 3.2 in several parts. First, we identify sufficient conditions that guarantee that the boundary conditions of an n-copula are satisfied.

Proposition 3.1. *Let C_{n-1} be a symmetric and radially symmetric $(n-1)$-copula and denote by C_k the k-dimensional marginal of C_{n-1} and let $H : [0,1]^n \to \mathbb{R}$ be a symmetric function. Define the function $S_{C_{n-1},H} : [0,1]^n \to \mathbb{R}$ as*

$$\begin{aligned} S_{C_{n-1},H}(\mathbf{x}) \;=\; & \frac{1}{2}\Big[\sum_{i=1}^{n} x_i - n + 1 + \sum_{i<j}^{n} C_2(1-x_i, 1-x_j) \\ & - \sum_{i<j<k}^{n} C_3(1-x_i, 1-x_j, 1-x_k) + \ldots \end{aligned}$$

$$+(-1)^{n-1}\sum_{i=1}^{n} C_{n-1}(1-x_1,\ldots,1-x_{i-1},1-x_{i+1},\ldots,1-x_n)\big]$$
$$+\frac{1}{2}\left[H(\mathbf{x})+(-1)^n H(\mathbf{1}-\mathbf{x})\right], \tag{3.4}$$

Then, $S_{C_{n-1},H}$ satisfies the boundary conditions of an n-copula and has C_{n-1} as its $(n-1)$-dimensional marginal if the following conditions hold:

(i) If $\mathbf{x}$ is such that $x_j = 0$ for some $j \in \{1,2,...,n\}$, then $H(\mathbf{x}) = 0$.

(ii) If $\mathbf{x}$ is such that $x_j = 1$ for some $j \in \{1,2,...,n\}$, then $H(\mathbf{x}) = C_{n-1}(\mathbf{x}_{\{j\}})$.

Proof. First, we prove condition (c1) of an n-copula. Without loss of generality, suppose that $\mathbf{x} \in [0,1]^n$ is such that $x_n = 0$. Note that

$$\begin{aligned}\sum_{i<j}^{n} C_2(1-x_i,1-x_j) &= \sum_{i<j}^{n-1} C_2(1-x_i,1-x_j)+\sum_{i=1}^{n-1} C_2(1-x_i,1-x_n)\\ &= \sum_{i<j}^{n-1} C_2(1-x_i,1-x_j)+\sum_{i=1}^{n-1}(1-x_i)\,.\end{aligned}$$

Analogously, we have

$$\begin{aligned}\sum_{i<j<k}^{n} C_3(1-x_i,1-x_j,1-x_k) &= \sum_{i<j<k}^{n-1} C_3(1-x_i,1-x_j,1-x_k)\\ &\quad+\sum_{i<j}^{n-1} C_3(1-x_i,1-x_j,1-x_n)\\ &= \sum_{i<j<k}^{n-1} C_3(1-x_i,1-x_j,1-x_k)\\ &\quad+\sum_{i<j}^{n-1} C_2(1-x_i,1-x_j)\,.\end{aligned}$$

Continuing this procedure, we can write $S_{C_{n-1},H}(\mathbf{x})$ as

$$\begin{aligned}S_{C_{n-1},H}(\mathbf{x}) &= \frac{1}{2}\Big[\sum_{i=1}^{n-1} x_i - n + 1 + \sum_{i=1}^{n-1}(1-x_i)+\sum_{i<j}^{n-1} C_2(1-x_i,1-x_j)\\ &\quad-\sum_{i<j<k}^{n-1} C_3(1-x_i,1-x_j,1-x_k)-\sum_{i<j}^{n-1} C_2(1-x_i,1-x_j)+\ldots\\ &\quad+(-1)^{n-1}\sum_{i=1}^{n-1} C_{n-2}(1-x_1,\ldots,1-x_{i-1},1-x_{i+1},\ldots,1-x_{n-1})\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{n-1}C_{n-1}(1-x_1,\dots,1-x_{n-1})\Big] \\
&+\frac{1}{2}\left[H(x_1,\dots,x_{n-1},0)+(-1)^n H(1-x_1,\dots,1-x_{n-1},1))\right] \\
= {}& (-1)^{n-1}\frac{1}{2}C_{n-1}(1-x_1,\dots,1-x_{n-1}) \\
&+(-1)^n\frac{1}{2}C_{n-1}(1-x_1,\dots,1-x_{n-1}) \\
= {}& 0\,,
\end{aligned}
$$

where the second equality follows from conditions (i) and (ii).

To prove that the $(n-1)$-dimensional marginal of $S_{C_{n-1},H}$ is C_{n-1}, suppose w.o.l.g. that $\mathbf{x}\in[0,1]^n$ is such that $x_n=1$. First note that

$$
\sum_{i<j}^{n} C_2(1-x_i,1-x_j)=\sum_{i<j}^{n-1} C_2(1-x_i,1-x_j)\,.
$$

$$
\sum_{i<j<k}^{n} C_3(1-x_i,1-x_j,1-x_k)=\sum_{i<j<k}^{n-1} C_3(1-x_i,1-x_j,1-x_k)\,.
$$

Continuing this procedure, we can write $S_{C_{n-1},H}(\mathbf{x})$ as

$$
\begin{aligned}
S_{C_{n-1},H}(\mathbf{x}) = {}& \frac{1}{2}\Big[\sum_{i=1}^{n-1} x_i-(n-2)+\sum_{i<j}^{n-1} C_2(1-x_i,1-x_j) \\
&-\sum_{i<j<k}^{n-1} C_3(1-x_i,1-x_j,1-x_k)+\dots \\
&+(-1)^{n-1}C_{n-1}(1-x_1,\dots,1-x_{n-1})\Big] \\
&+\frac{1}{2}\left[H(x_1,\dots,x_{n-1},1)+(-1)^n H(1-x_1,\dots,1-x_{n-1},0)\right] \\
= {}& \frac{1}{2}C_{n-1}(x_1,\dots,x_{n-1})+\frac{1}{2}C_{n-1}(x_1,\dots,x_{n-1}) \\
= {}& C_{n-1}(x_1,\dots,x_{n-1})\,,
\end{aligned}
$$

where the second equality follows from conditions (i) and (ii) and from the radial symmetry of C_{n-1}. □

Note that the second condition in Proposition 3.1 is more restrictive than the boundary condition (c2) of an n-copula. If H is a symmetric n-copula that has C_{n-1} as its $(n-1)$-dimensional marginal, then this condition is automatically satisfied.

Next, we identify necessary and sufficient conditions that guarantee that $S_{C_{n-1},H}$ as defined in Proposition 3.1 is an n-copula. For this purpose, we need to introduce the notion of vacuity of a function. While we introduce this notion here for functions with the unit hypercube $[0,1]^n$ as domain, it can be easily generalized to functions with $\mathbb{R}^n$ as domain.

Definition 3.1. A function $F : [0,1]^n \to [0,1]$ has the vacuity property if for any n-box $\mathbf{P} = \times_{j=1}^n [a_j, b_j] \subseteq [0,1]^n$ it holds that $V_F(\mathbf{P}) = 0$.

The following characterization of functions that have the vacuity property can be found in [32, 90].

Lemma 3.1. *A function $G : [0,1]^n \to [0,1]$ has the vacuity property if and only if there exist n functions $G_1, \ldots G_n : [0,1]^{n-1} \to [0,1]$ such that*

$$G(\mathbf{x}) = \sum_{i=1}^{n} G_i(\mathbf{x}_{\{i\}}) \,,$$

i.e., G can be decomposed as a sum of functions that each depend on all but one variable.

Remark 3.2. In [90] the term 'modular' is used for functions f that have the property that any n-box has f-volume equal to 0. However, we argue that the term 'modular' is not adequate to describe this property. In Chapter 5 we will see that for $n = 2$ the properties of supermodularity and 2-increasingness are equivalent, and also modularity and the vacuity property are equivalent. However, for $n \geqslant 3$, supermodularity and n-increasingness are no longer equivalent, nor modularity and the vacuity property. Since the concept of modularity is related to the concept of supermodularity, the term 'modular' is not appropriate to describe functions f that have the property that any n-box has f-volume equal to 0.

With the help of Lemma 3.1, we can provide a characterization of the functions H such that the function $S_{C_{n-1},H}$ defined in Eq. (3.4) yields an n-copula.

Proposition 3.2. *Let C_{n-1} be a symmetric and radially symmetric $(n-1)$-copula, $H : [0,1]^n \to \mathbb{R}$ be a symmetric function and $S_{C_{n-1},H}$ be defined as in Eq. (3.4). Suppose that H satisfies conditions (i) and (ii) of Proposition 3.1. Then $S_{C_{n-1},H}$ is an n-copula if and only if for any n-box $\mathbf{P} = \times_{i=1}^n [a_i, b_i] \subseteq [0,1]^n$ it holds that $V_H(\mathbf{P}) + V_H(\mathbf{1} - \mathbf{P}) \geqslant 0$.*

Proof. Consider the n-box $\mathbf{P} = \times_{j=1}^n [a_i, b_i] \subseteq [0,1]^n$. First, we will show that the $S_{C_{n-1},H}$-volume of $\mathbf{P}$ only depends on the terms containing the function H, as all other terms are a function of a proper subset of all n arguments and, as a consequence, they have the vacuity property. To formally prove the latter statement, note that for any $i \in \{1, 2, \ldots, n\}$ the projection $f(\mathbf{x}) = x_i$ has the vacuity property since it has the representation given by Lemma 3.1, by taking for a fixed $i \neq j$, $G_i(\mathbf{x}) = x_j$ and $G_k(\mathbf{x}) = 0$ for $k \neq i$. Analogously, for any $i, j \in \{1, 2, \ldots, n\}$ with $i < j$ the function $C_2(1 - x_i, 1 - x_j)$ has the vacuity property by taking for a fixed $k \notin \{i, j\}$, $G_k(\mathbf{x}) = C_2(1 - x_i, 1 - x_j)$ and $G_l(\mathbf{x}) = 0$ for $l \neq k$. Continuing this procedure, it follows that the $S_{C_{n-1},H}$-volume of $\mathbf{P}$ only depends on H. Hence, from the expression of $S_{C_{n-1},H}$, the only terms that do not have the vacuity property are $H(\mathbf{x})$ and $(-1)^n H(\mathbf{1}-\mathbf{x})$. Note that the $\tilde{H}$-volume of $\mathbf{P}$, with $\tilde{H}$ defined by $\tilde{H}(\mathbf{x}) = (-1)^n H(\mathbf{1}-\mathbf{x})$, is equal to $V_H(\mathbf{1}-\mathbf{P})$ since the function $x \to 1-x$ maps the interval $[a,b]$ into the interval $[1-b, 1-a]$, while the term $(-1)^n$ is a correction factor for odd n. As a consequence, the $S_{C_{n-1},H}$-volume of $\mathbf{P}$ is given by

$$V_{S_{C_{n-1},H}}(\mathbf{P}) = \frac{1}{2}(V_H(\mathbf{P}) + V_H(\mathbf{1}-\mathbf{P}))\,.$$

From the last equality, it follows that $V_{S_{C_{n-1},H}}(\mathbf{P}) \geqslant 0$ if and only if $V_H(\mathbf{P}) + V_H(\mathbf{1}-\mathbf{P}) \geqslant 0$. □

Finally, we will prove that our construction method yields a radially symmetric n-copula when $S_{C_{n-1},H}$ is an n-copula.

Proposition 3.3. *Let C_{n-1} be a symmetric and radially symmetric $(n-1)$-copula, $H : [0,1]^n \to \mathbb{R}$ be a symmetric function that satisfies conditions (i) and (ii) of Proposition 3.1 and and $S_{C_{n-1},H}$ be defined as in Eq. (3.4). If $S_{C_{n-1},H}$ is an n-copula, then it is radially symmetric.*

Proof. In this proof, we will write $\bar{S}_{C_n,H}$ as shorthand for $\overline{S_{C_n,H}}$. We start by computing $\bar{S}_{C_n,H}$:

$$\begin{aligned}\bar{S}_{C_{n-1},H}(\mathbf{x}) \;=\; & \sum_{i=1}^{n} x_i - n + 1 + \sum_{i<j}^{n} S_{C_{n-1},H}(1,\ldots,1-x_i,\ldots,1-x_j,\ldots,1)\\ & - \sum_{i<j<k}^{n} S_{C_{n-1},H}(1,\ldots,1-x_i,\ldots,1-x_j,\ldots 1-x_k,\ldots,1) + \ldots\\ & +(-1)^n S_{C_{n-1},H}(1-x_1, 1-x_2,\ldots,1-x_n)\,.\end{aligned}$$

Since H satisfies conditions (i) and (ii) of Proposition 3.1, it follows that it is possible to write all the lower-dimensional marginals of $S_{C_{n-1},H}$ in terms of the $(n-1)$-copula C_{n-1}. Denote by C_k the k-dimensional marginal of C_{n-1}. Note that

$$S_{C_{n-1},H}(1,\ldots,1-x_i,\ldots,1-x_j,\ldots,1) = C_2(1-x_i, 1-x_j)\,,$$

$$S_{C_{n-1},H}(1,\ldots,1-x_i,\ldots,1-x_j,\ldots,1-x_k,\ldots,1) = C_3(1-x_i,1-x_j,1-x_k)\,,$$

and so on. Hence,

$$\begin{aligned}\bar{S}_{C_{n-1},H}(\mathbf{x}) \;=\; & \sum_{i=1}^{n} x_i - n + 1 + \sum_{i<j}^{n} C_2(1-x_i, 1-x_j)\\ & - \sum_{i<j<k}^{n} C_3(1-x_i,1-x_j,1-x_k) + \ldots\end{aligned}$$

$$+(-1)^n S_{C_{n-1},H}(1-x_1,1-x_2,\ldots,1-x_n)\,. \tag{3.5}$$

Now, $S_{C_{n-1},H}(\mathbf{1}-\mathbf{x})$ can be written as

$$\begin{aligned} S_{C_{n-1},H}(\mathbf{1}-\mathbf{x}) &= \frac{1}{2}\big[1-\sum_{i=1}^{n} x_i + \sum_{i<j}^{n} C_2(x_i,x_j) - \sum_{i<j<k}^{n} C_3(x_i,x_j,x_k) + \ldots \\ &\quad +(-1)^{n-1}\sum_{i=1}^{n} C_{n-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\big] \\ &\quad +\frac{1}{2}\left[H(\mathbf{1}-\mathbf{x})+(-1)^n H(\mathbf{x})\right]. \end{aligned}$$

Since C_2 is radially symmetric, it holds that

$$\begin{aligned} \sum_{i<j}^{n} C_2(x_i,x_j) &= \sum_{i<j}^{n}(x_i+x_j-1+C_2(1-x_i,1-x_j)) \qquad (3.6)\\ &= \binom{n-1}{1}\sum_{i=1}^{n} x_i - \binom{n}{2} + \binom{n-2}{0}\sum_{i<j}^{n} C_2(1-x_i,1-x_j)\,, \end{aligned}$$

taking into account that there are $\binom{n}{2}$ terms in the sum in the right-hand side of Eq. (3.6) and for a fixed i, the term x_i appears in $n-1$ terms of the sum. Next,

$$\begin{aligned} \sum_{i<j<k}^{n} C_3(x_i,x_j,x_k) &= \sum_{i<j<k}^{n} (x_i+x_j+x_k-2+C_2(1-x_i,1-x_j) \\ &\quad +C_2(1-x_i,1-x_k)+C_2(1-x_j,1-x_k) \\ &\quad -C_3(1-x_i,1-x_j,1-x_k)) \qquad (3.7)\\ &= \binom{n-1}{2}\sum_{i=1}^{n} x_i - 2\binom{n}{3} \\ &\quad +\binom{n-2}{1}\sum_{i<j}^{n} C_2(1-x_i,1-x_j) \\ &\quad -\binom{n-3}{0}\sum_{i<j<k}^{n} C_3(1-x_i,1-x_j,1-x_k)\,, \end{aligned}$$

taking into account that there are $\binom{n}{3}$ terms in the sum in the right-hand side of Eq. (3.7) and for a fixed i, the term x_i appears in $\binom{n-1}{2}$ of the terms of the sum and the term $C_2(1-x_i,1-x_j)$ appears in $\binom{n-2}{1}$ terms of the sum. In general, the sum of all the k-dimensional marginals is equal to

$$
\begin{aligned}
\sum_{\substack{A\subseteq\{1,\ldots,n-1\}\\ \#A=n-k}} C_k(\mathbf{x}_A) \;=\; & \binom{n-1}{k-1}\sum_{i=1}^{n} x_i-(k-1)\binom{n}{k}\\
& +\binom{n-2}{k-2}\sum_{i<j}^{n} C_2(1-x_i,1-x_j)\\
& -\binom{n-3}{k-3}\sum_{i<j<k}^{n} C_3(1-x_i,1-x_j,1-x_k)+\ldots\\
& (-1)^k\binom{n-k}{0}\sum_{\substack{A\subseteq\{1,\ldots,n\}\\ \#A=n-k}} C_k(\mathbf{1}-\mathbf{x}_A)\,.
\end{aligned}
$$

Hence, we can rewrite $S_{C_{n-1},H}(\mathbf{1}-\mathbf{x})$ as

$$
\begin{aligned}
S_{C_{n-1},H}(\mathbf{1}-\mathbf{x}) \;=\; & \frac{1}{2}\Big[1-\sum_{i=1}^{n} x_i+\binom{n-1}{1}\sum_{i=1}^{n} x_i-\binom{n}{2}\\
& +\binom{n-2}{0}\sum_{i<j}^{n} C_2(1-x_i,1-x_j)\\
& -\binom{n-1}{2}\sum_{i=1}^{n} x_i+2\binom{n}{3}\\
& -\binom{n-2}{1}\sum_{i<j}^{n} C_2(1-x_i,1-x_j)\\
& +\binom{n-3}{0}\sum_{i<j<k}^{n} C_3(1-x_i,1-x_j,1-x_k)+\ldots\Big]\\
& +\frac{1}{2}\left[H(\mathbf{1}-\mathbf{x})+(-1)^n H(\mathbf{x})\right]. \qquad (3.8)
\end{aligned}
$$

With the help of Lemma 2.1, we note that the coefficient of the sum of the 2-dimensional marginals in Eq. (3.8) is given by

$$
\sum_{i=0}^{n-3}(-1)^i\binom{n-2}{i} = -(-1)^{n-2} = (-1)^{n-1}\,,
$$

while the coefficient of the sum of the 3-marginals in Eq. (3.8) is given by

$$
\sum_{i=0}^{n-4}(-1)^i\binom{n-3}{i} = (-1)^{n-2}\,.
$$

In general, for any $k \in \{2, 3, \ldots, n\}$, the coefficient of the sum of the k-dimensional marginals in Eq. (3.8) is given by

$$\sum_{i=0}^{n-k-1} (-1)^i \binom{n-k}{i} = (-1)^{n-k+1} .$$

Next, the coefficient of $\sum_{i=1}^{n} x_i$ in Eq. (3.8) is given by

$$-1 + \sum_{i=1}^{n-2} (-1)^i \binom{n-1}{i} = -1 + (1 + (-1)^{n-1}) = (-1)^{n-1} .$$

Finally, the sum of the independent terms is given by

$$\begin{aligned} 1 - \sum_{i=2}^{n-1} (-1)^i (i-1) \binom{n}{i} &= 1 + \sum_{i=2}^{n-1} (-1)^i \binom{n}{i} - \sum_{i=2}^{n-1} (-1)^i i \binom{n}{i} \\ &= 1 - (1 - n + (-1)^n) \\ &\quad -n \sum_{i=2}^{n-1} (-1)^i \binom{n-1}{i-1} \\ &= 1 + n - 1 + (-1)^{n-1} + n \sum_{i=1}^{n-2} (-1)^i \binom{n-1}{i} \\ &= n + (-1)^n - n(1 + (-1)^n) \\ &= (n-1)(-1)^n . \end{aligned}$$

Hence, substituting the last expressions in Eq. (3.8), we get

$$\begin{aligned} S_{C_{n-1},H}(\mathbf{1} - \mathbf{x}) &= \frac{1}{2}\Big[(n-1)(-1)^n + (-1)^{n-1} \sum_{i=1}^{n} x_i \\ &\quad + (-1)^n \sum_{i<j}^{n} C_2(1 - x_i, 1 - x_j) \\ &\quad + (-1)^{n-2} \sum_{i<j<k}^{n} C_3(1 - x_i, 1 - x_j, 1 - x_k) + \ldots \Big] \\ &\quad + \frac{1}{2} \left[H(\mathbf{1} - \mathbf{x}) + (-1)^n H(\mathbf{x})\right] . \end{aligned}$$

Multiplying both sides of the last equality by $(-1)^n$, we obtain

$$(-1)^n S_{C_{n-1},H}(\mathbf{1} - \mathbf{x}) = \frac{1}{2}\Big[n - 1 - \sum_{i=1}^{n} x_i$$

$$
\begin{aligned}
& - \sum_{i<j}^{n} C_2(1-x_i, 1-x_j) \\
& + \sum_{i<j<k}^{n} C_3(1-x_i, 1-x_j, 1-x_k) + \dots] \\
& + \frac{1}{2} \left[H(\mathbf{x}) + (-1)^n H(\mathbf{1}-\mathbf{x}) \right] . \qquad (3.9)
\end{aligned}
$$

Substituting Eq. (3.9) into Eq. (3.5), we get

$$
\begin{aligned}
\bar{S}_{C_{n-1},H}(\mathbf{x}) &= \frac{1}{2} [\sum_{i=1}^{n} x_i - n + 1 + \sum_{i<j}^{n} C_2(1-x_i, 1-x_j) \\
& - \sum_{i<j<k}^{n} C_3(1-x_i, 1-x_j, 1-x_k) + \dots \\
& + (-1)^{n-1} \sum_{j=1}^{n} C_{n-1}(1-x_1, \dots, 1-x_{j-1}, 1-x_{j+1}, \dots, 1-x_n)] \\
& + \frac{1}{2} \left[H(\mathbf{x}) + (-1)^n H(\mathbf{1}-\mathbf{x}) \right] \\
&= S_{C_{n-1},H}(\mathbf{x}) .
\end{aligned}
$$

□

Combining all the preceding results, we can now prove Theorem 3.2.

Proof of Theorem 3.2. First suppose that C_n is radially symmetric. Let $H = C_n$. Clearly H satisfies conditions (2)–(4), while condition (1) follows by realizing that $S_{C_{n-1},H}$ is simply the average of C_n and its survival copula, which is equal to C_n.

Conversely, suppose that there exists a symmetric function H that satisfies conditions (1)–(4). Then, by using Propositions 3.1, 3.2 and 3.3, it follows that C_n is radially symmetric. □

Remark 3.3. We note that if H satisfies conditions (i) and (ii) of Proposition 3.1 and H is n-increasing, i.e., H is an n-copula, then $S_{C_{n-1},H}$ has the form of radially symmetric copulas given in Theorem 3.1, since the right-hand side of Eq. (3.4) represents the average of an n-copula and its survival copula. However, Theorem 3.2 is more general than Theorem 3.1 because H does not necessarily need to be an n-copula, since for a given n-box $\mathbf{P}$, $V_{S_{C_{n-1},H}}(\mathbf{P}) \geqslant 0$ if and only if one of the following three conditions holds:

(i) $V_H(\mathbf{P}) \geqslant 0$ and $V_H(\mathbf{1}-\mathbf{P}) \geqslant 0$;

(ii) $V_H(\mathbf{P}) < 0$ and $V_H(\mathbf{1}-\mathbf{P}) \geqslant |V_H(\mathbf{P})|$;

(iii) $V_H(\mathbf{1}-\mathbf{P}) < 0$ and $V_H(\mathbf{P}) \geqslant |V_H(\mathbf{1}-\mathbf{P})|$.

Conditions (ii) and (iii) cannot be fulfilled if H is an n-copula. Further on, we will give an example of a function H that is not an n-copula, but satisfies the conditions of Proposition 3.2.

From Theorem 3.2 we have the following corollary, the proof of which is obvious.

Corollary 3.1. *Let C_n be a symmetric and radially symmetric n-copula. If there exists a symmetric function $H : [0,1]^n \to \mathbb{R}$ such that C_n can be written in terms of H as in Eq. (3.3), then for any n-box $\mathbf{P}$ it holds that $V_H(\mathbf{P}) + V_H(\mathbf{1}-\mathbf{P}) \geqslant 0$.*

Next, we give an example of the representation of radially symmetric copulas in terms of a function H as described in Theorem 3.2 and as a byproduct, we show that the function H is not unique.

Example 3.1. Consider the member of the Dirichlet family of 3-copulas [132, 181] given by

$$C_3(x,y,z) = \frac{x_{(1)}(x_{(2)}+1)(x_{(3)}+2)}{6},$$

where $x_{(1)} := \min(x,y,z)$, $x_{(2)} := \operatorname{med}(x,y,z)$ and $x_{(3)} := \max(x,y,z)$. It can be proven that this copula is radially symmetric [181]. Some elementary computations show that for any $c \in \mathbb{R}$, the symmetric function $H_c : [0,1]^3 \to \mathbb{R}$ defined by

$$H_c(x,y,z) = \frac{x_{(1)}x_{(2)}x_{(3)} + 2x_{(1)}x_{(2)} + 3x_{(1)} - cx_{(1)}(1-x_{(3)})}{6}$$

satisfies conditions (1), (2) and (3) of Theorem 3.2. Using Corollary 3.1, it follows that for any 3-box $\mathbf{P}$ and $c \in \mathbb{R}$ it holds that $V_{H_c}(\mathbf{P}) + V_{H_c}(\mathbf{1}-\mathbf{P}) \geqslant 0$. Some tedious computations show that H_c is a 3-copula if and only if $c \in [0,2]$.

Remark 3.4. From Example 3.1, one can see that the function H neither needs to be increasing nor 1-Lipschitz continuous by taking $-c$ large enough. Additionally, if we consider a 3-box $\mathbf{P}_0 = [x_1,x_2]^2 \times [y_1,y_2]$ such that $x_1 < x_2 < y_1 < y_2$, then for the function H_c of Example 3.1 it holds that

$$V_{H_c}(\mathbf{P}_0) = \frac{1}{6}\left((x_2-x_1)^2(y_2-y_1) + c(x_2-x_1)(y_2-y_1)\right).$$

Clearly, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(c) = V_{H_c}(\mathbf{P}_0)$ is neither bounded from above nor from below.

We now turn Theorem 3.2 into a construction method. Our objective is, given a symmetric and radially symmetric $(n-1)$-copula C_{n-1}, to construct a symmetric and radially symmetric n-copula, such that all of its $(n-1)$-dimensional marginals coincide with C_{n-1}. In view of Theorem 3.2, we will give some examples of functions H that satisfy the conditions of Proposition 3.2, so that the function $S_{C_{n-1},H}$ defined in Eq. (3.4)is a symmetric and radially symmetric n-copula.

3.3. Possible options for the auxiliary function

In this section, we will propose several ways of constructing the auxiliary function for H. The first one will be based on the nesting of 2-copulas, the second option will be based on the $\star_{\mathbf{D}}$ product of copulas [58, 175], while the last one will be based on the product of copulas.

3.3.1. An option based on the nesting of copulas

As mentioned before, requiring associativity (in particular Archimedeanity) and radial symmetry to construct copulas is too restrictive in higher dimensions. However, some further generalizations of Archimedean n-copulas have been proposed; for example in [138], nested Archimedean n-copulas (also called hierarchical Archimedean n-copulas) are studied, where several Archimedean 2-copulas are iterated to construct an n-copula. For example, in the trivariate case, $D_2(x, C_2(y, z))$ is an example of such a construction, where C_2 and D_2 are Archimedean 2-copulas.

Nested Archimedean copulas suggest a way to build the auxiliary function H in our construction method, in the sense that we can construct a trivariate function from two bivariate functions. If C_2 is a symmetric and radially symmetric 2-copula, and D_2 is a given symmetric 2-copula, we define H_{C_2,D_2} as

$$\begin{aligned} H_{C_2,D_2}(x,y,z) \;=\; & D_2(x, C_2(y,z)) + D_2(y, C_2(x,z)) + D_2(z, C_2(x,y)) \\ & -\frac{2}{3}\left[D_2(x, D_2(y,z)) + D_2(y, D_2(x,z)) + D_2(z, D_2(x,y))\right]. \end{aligned} \tag{3.10}$$

Since $D_2(x, C_2(y, z))$ may not be symmetric, in order to preserve the symmetry, we have considered the sum $D_2(x, C_2(y, z)) + D_2(y, C_2(x, z)) + D_2(z, C_2(x, y))$, while the correction term $-\frac{2}{3}[D_2(x, D_2(y, z)) + D_2(y, D_2(x, z)) + D_2(z, D_2(x, y))]$ guarantees that the function H_{C_2,D_2} has C_2 as its 2-dimensional marginal and satisfies the boundary conditions of Proposition 3.1. However, $S_{C_2,H_{C_2,D_2}}$ may even not be an increasing function. For example, if $C_2 = D_2 = F^{(-2)}$, i.e., the Frank 2-copula with parameter $\alpha = -2$, then

$$S_{F^{(-2)},H_{F^{(-2)},F^{(-2)}}}\left(\frac{1}{2}, \frac{1}{10}, \frac{1}{10}\right) < 0 = S_{F^{(-2)},H_{F^{(-2)},F^{(-2)}}}\left(0, \frac{1}{10}, \frac{1}{10}\right).$$

Note that if $C_2 = D_2$, the expression of H_{C_2,D_2} reduces to

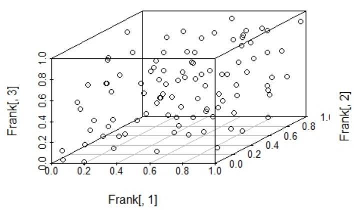
$$H_{C_2,C_2}(x,y,z) = \frac{1}{3}\left[C_2(x, C_2(y,z)) + C_2(y, C_2(x,z)) + C_2(z, C_2(x,y))\right].$$

We now provide some examples of copulas C_2 and D_2 such that the function H_{C_2,D_2}

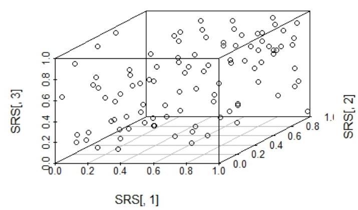
satisfies condition (4) of Theorem 3.2.

Example 3.2. Consider the Frank family of 2-copulas. From [140], we know that the 3-dimensional associative extension of the Frank 2-copula, given by $F_3^{(\alpha)}(x, y, z) = F^{(\alpha)}(x, F^{(\alpha)}(y, z))$, is a 3-copula if and only if $\alpha \geqslant -\ln(2)$. With the help of Mathematica, it can be shown that $S_{F^{(\alpha)}, H_{F^{(\alpha)}, F^{(\alpha)}}}$ is also a 3-copula for $\alpha \geqslant -\ln(3)$, showing that H indeed does not need to be a copula in our construction.

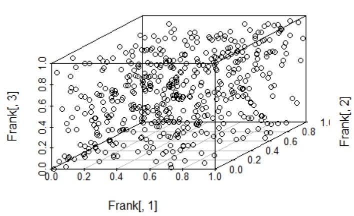
We now simulate some samples of the 3-copula given in Example 3.2, using the statistical software R, the R package "copula" and the functions *rCopula* and *frankCopula* for positive values of the parameter, since for the latter case we can simulate the resulting copula as a mixture of a Frank 3-copula and its survival 3-copula. In Figure 3.1 we show samples of size $m = 100$ and $m = 500$ for different parameters, with the objective of comparing the copula of Example 3.2 and the 3-dimensional associative extension of the Frank 2-copula.



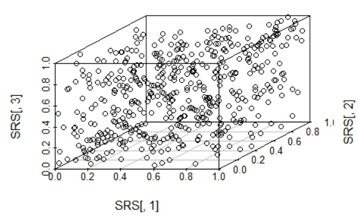
Frank 3-copula, $\alpha = 1$, $m = 100$



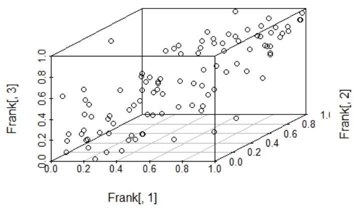
3-copula of Example 3.2, $\alpha = 1$, $m = 100$



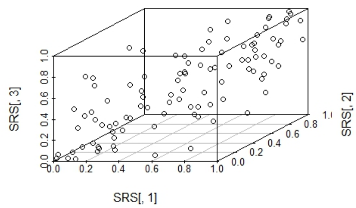
Frank 3-copula, $\alpha = 1$, $m = 500$



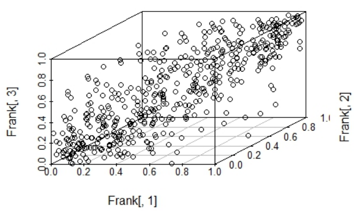
3-copula of Example 3.2, $\alpha = 1$, $m = 500$



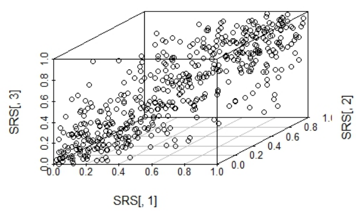
Frank 3-copula, $\alpha = 5$, $m = 100$



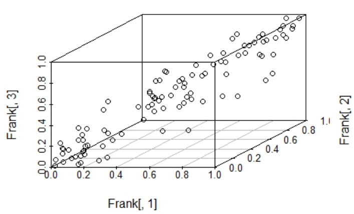
3-copula of Example 3.2, $\alpha = 5$, $m = 100$



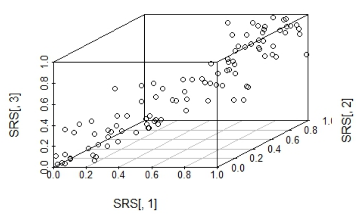
Frank 3-copula, $\alpha = 5$, $m = 500$



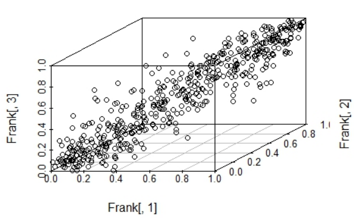
3-copula of Example 3.2 $\alpha = 5$, $m = 500$



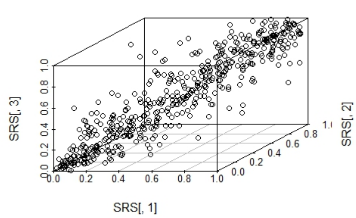
Frank 3-copula, $\alpha = 10$, $m = 100$



3-copula of Example 3.2 $\alpha = 10$, $m = 100$



Frank 3-copula, $\alpha = 10$, $m = 500$



3-copula of Example 3.2, $\alpha = 10$, $m = 500$

Figure 3.1: Simulations of the Frank 3-copula and the 3-copula of Example 3.2

As can be noted, the difference between the copula of Example 3.2 and the

3-dimensional associative extension of the Frank 2-copula cannot be easily distinguished in Figure 3.1. However, with the help of the R function *gofCopula*, we can indeed see that the bivariate marginals of the copula of Example 3.2 are Frank copulas, and while the goodness of fit test may not detect that the copula of Example 3.2 is statistically different from a Frank 3-copula, the expected number of rejections confirms that it does not correspond to the expected number of rejections under a 3-dimensional Frank distribution. Table 3.1 shows the results for 1000 simulations of samples of size $m = 100$. It is important to remark that the results reported for the bivariate marginals contain the information of the three bivariate marginals of the 3-copula.

Description	Bivariate marginals	3-copula
# times P-value $\leqslant .1$	311	477
# times P-value $\leqslant .05$	145	315
# times P-value $\leqslant .01$	30	79

Table 3.1: Results for $\alpha = 5$.

Example 3.3. Recall that the FGM family of 2-copulas is given by

$$F^{(\theta)}(x, y) = xy + \theta xy(1 - x)(1 - y)\,, \quad \theta \in [-1, 1]\,.$$

In this case, some tedious computations show that $S_{F^{(\theta)},H_{F^{(\theta)},F^{(\theta)}}}$ is a 3-copula if and only if $\theta \in [-1/2(3 - \sqrt{5}), 1/2(\sqrt{21} - 3)]$.

Example 3.4. In this example, we consider the case $D_2 = \Pi_2$. For any symmetric and radially symmetric 2-copula C_2, by using the vacuity property of the terms in $S_{C_2,H_{C_2,\Pi_2}}$ as in the proof of Proposition 3.2, some direct computations show that $S_{C_2,H_{C_2,\Pi_2}}$ is a 3-copula if and only if for any $x_1, x_2, y_1, y_2, z_1, z_2 \in [0, 1]$ such that $x_1 \leqslant x_2$, $y_1 \leqslant y_2$ and $z_1 \leqslant z_2$, it holds that

$$\begin{aligned}
&(x_2 - x_1)V_{C_2}([y_1, y_2] \times [z_1, z_2]) + (y_2 - y_1)V_{C_2}([x_1, x_2] \times [z_1, z_2]) \\
&+(z_2 - z_1)V_{C_2}([x_1, x_2] \times [y_1, y_2]) \\
&\geqslant 2(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)\,.
\end{aligned}$$

If C_2 is absolutely continuous, then the latter condition is equivalent to:

$$\frac{\partial^2 C_2}{\partial x \partial y}(x, y) + \frac{\partial^2 C_2}{\partial x \partial z}(x, z) + \frac{\partial^2 C_2}{\partial y \partial z}(y, z) \geqslant 2\,, \tag{3.11}$$

at the points (x, y, z) where the mixed partial derivative exists. An example of a family of 2-copulas that satisfy Eq. (3.11) are the FGM copulas $F^{(\theta)}$ for $\theta \in [-1/3, 1/3]$.

We note that Example 3.4 can be generalized to higher dimensions. If we define the n-ary function $H_{C_{n-1},\Pi}$ as

$$
\begin{aligned}
H_{C_{n-1},\Pi}(\mathbf{x}) = & \sum_{i=1}^{n} x_i C_{n-1}(\mathbf{x}_{\{j\}}) \\
& - \sum_{i<j}^{n} x_i x_j C_{n-2}(\mathbf{x}_{\{i,j\}}) \\
& \dots \\
& +(-1)^{n-1} \sum_{i<j}^{n} \left(\prod_{k \neq i,j} x_k \right) C_2(x_i, x_j) \\
& +(n-1)(-1)^n x_1 x_2 \dots x_n \,,
\end{aligned}
$$

then the characterization in the absolutely continuous case is also simple. Indeed, after doing some simple combinatorial analysis and assuming that C_{n-1} is absolutely continuous, it follows that $S_{C_{n-1},H_{C_{n-1},\Pi}}$ is an n-copula if and only if it holds that

$$
\begin{aligned}
& \sum_{i=1}^{n} \frac{\partial^{n-1} C_{n-1}}{\partial x_1 \dots \partial x_{i-1} \partial x_{i+1} \dots \partial x_n}(\mathbf{x}_{\{j\}}) \\
& - \sum_{i<j}^{n} \frac{\partial^{n-2} C_{n-2}}{\partial x_1 \dots \partial x_{i-1} \partial x_{i+1}, \dots, \partial x_{j-1} \partial x_{j+1}, \dots \partial x_n}(\mathbf{x}_{\{i,j\}}) \\
& \dots \\
& +(-1)^{n-1} \sum_{i<j}^{n} \frac{\partial^2 C_2}{\partial x_i \partial x_j}(x_i, x_j) + (n-1)(-1)^n \geqslant 0 \,.
\end{aligned}
$$

for any $\mathbf{x} \in [0,1]^n$ at which the mixed partial derivatives exist.

3.3.2. An option based on the $\star_{\mathbf{D}}$-product of copulas

In this subsection, we introduce a way of constructing the auxiliary function H using the $\star_{\mathbf{D}}$-product of copulas. First, we recall the $\star_{\mathbf{D}}$-product of copulas [57].

Theorem 3.3. *Let $C_{2,1}$ and $C_{2,2}$ be two 2-copulas and $\mathbf{D} = (D_{2,t})_{t \in [0,1]}$ be a family of 2-copulas. If $(D_{2,t})_{t \in [0,1]}$ is such that for any $x, y \in [0,1]$ the function $G_{x,y}(t) : [0,1] \to [0,1]$ given by*

$$
G_{x,y}(t) = D_{2,t}\left(\frac{\partial C_{2,1}}{\partial t}(x,t), \frac{\partial C_{2,2}}{\partial t}(t,y) \right)
$$

is Lebesgue measurable, then the function $C_{2,1} *_{\mathbf{D}} C_{2,2} : [0,1]^3 \to [0,1]$ *given by*

$$(C_{2,1} \star_{\mathbf{D}} C_{2,2})(x,y,z) = \int_0^z D_{2,t}\left(\frac{\partial C_{2,1}}{\partial t}(x,t), \frac{\partial C_{2,2}}{\partial t}(t,y)\right) \mathrm{d}t$$

is a 3*-copula.*

Note that $(C_{2,1} \star_{\mathbf{D}} C_{2,2})$ has a nice probabilistic interpretation in terms of conditional distributions: if the joint distribution function of the random vector (X,Y,Z) is given by $(C_{2,1} \star_{\mathbf{D}} C_{2,2})$, then the copula of X and Y given $Z = z$ is $D_{2,z}$. In fact, any 3-copula that has $C_{2,1}$ and $C_{2,2}$ as two of its bivariate marginals can be decomposed in the form given by Theorem 3.3 by means of suitable family of copulas $\mathbf{D}$. Theorem 3.3 also gives a generalization of the well-known $*$-product of 2-copulas, which is also called the Darsow-Nguyen-Olsen product. We refer to [27, 28, 98, 147, 148] and the references therein for more details on the $*$-product.

We now give some examples that can be found in [58].

(i) $(C_2 \star_{\mathbf{D}} M_2)(x,y,z) = C_2(x, \min(y,z))$;

(ii) $(M_2 \star_{\mathbf{D}} C_2)(x,y,z) = C_2(\min(x,z), y)$;

(iii) $(C_2 \star_{\mathbf{D}} W_2)(x,y,z) = \max(C_2(x,z) - C_2(x, 1-y), 0)$;

(iv) $(W_2 \star_{\mathbf{D}} C_2)(x,y,z) = \max(C_2(z,y) - C_2(1-x, y), 0)$;

(v) $(\Pi_2 \star_{\Pi_2} C_2)(x,y,z) = xC_2(z,y)$;

(vi) $(C_2 \star_{\Pi_2} \Pi_2)(x,y,z) = yC_2(x,z)$.

The $\star_{\mathbf{D}}$ operation provides a way of constructing 3-copulas such that the 2-dimensional marginals are easy to compute and as a consequence provides another possibility to construct the function H in order to construct radially symmetric 3-copulas. Hence, using a similar explanation as the one given for the development of the expression in Eq. (3.10), if C_2 is a symmetric and radially symmetric 2-copula, then for a given family of symmetric 2-copulas $(D_{2,t})_{t\in[0,1]}$, we define $H_{\mathbf{D}}$ as

$$\begin{aligned} H_{\mathbf{D}}(x,y,z) = \; & \frac{1}{2}\big[(C_2 \star_{\mathbf{D}} C_2)(x,y,z) + (C_2 \star_{\mathbf{D}} C_2)(x,z,y) + (C_2 \star_{\mathbf{D}} C_2)(y,x,z) \\ & -y(C_2 \star_{\mathbf{D}} C_2)(x,1,z) - z(C_2 \star_{\mathbf{D}} C_2)(x,1,y) \\ & -x(C_2 \star_{\mathbf{D}} C_2)(y,1,z) + 2xyz\big]. \end{aligned} \tag{3.12}$$

In the last expression we are assuming that all the integrals are well defined, i.e., that the conditions of measurability of Theorem 3.3 are satisfied. We give an example of this construction method.

Example 3.5. Let C_2 be a member of the FGM family of 2-copulas. Suppose that for any $t \in [0,1]$, $D_{2,t} = \Pi_2$. Some simple computations show that

$$\partial_2 F^{(\theta)}(x,t) = x + \theta x(1-x)(1-2t)$$

and

$$\begin{aligned}(F^{(\theta)} \star_{\mathbf{D}} F^{(\theta)})(x,y,z) &= xyz + \theta xyz(1-x)(1-y) + \theta xyz(1-y)(1-z) \\ &\quad + \theta^2 xyz(1-x)(1-z)(1-2y+\frac{4}{3}y^2)\,.\end{aligned}$$

Note that $(F^{(\theta)} \star_{\mathbf{D}} F^{(\theta)})(x,1,z)$ is a FGM copula with parameter $\theta^2/3$. Then, substituting $(F^{(\theta)} \star_{\mathbf{D}} F^{(\theta)})$ in Eq. (3.12), it follows that $H_{\mathbf{D}}(x,y,z)$ is given by

$$\begin{aligned}H_{\mathbf{D}}(x,y,z) &= xyz + \theta x(y-y^2)(z-z^2) + \theta y(x-x^2)(z-z^2) \\ &\quad + \theta z(x-x^2)(y-y^2) + \frac{\theta^2}{2}(x-x^2)(y-y^2)(z-2z^2+\frac{4}{3}z^3) \\ &\quad + \frac{\theta^2}{2}(x-x^2)(z-z^2)(y-2y^2+\frac{4}{3}y^3) \\ &\quad + \frac{\theta^2}{2}(y-y^2)(z-z^2)(x-2x^2+\frac{4}{3}x^3) - \frac{\theta^2}{6}x(y-y^2)(z-z^2) \\ &\quad - \frac{\theta^2}{6}y(x-x^2)(z-z^2) - \frac{\theta^2}{6}z(x-x^2)(y-y^2)\,.\end{aligned}$$

Note that the volume condition $V_H(\mathbf{P}) + V_H(\mathbf{1}-\mathbf{P}) \geqslant 0$ is equivalent to the 3-increasingness of the function $H_{\mathbf{D}}(x,y,z) - H_{\mathbf{D}}(1-x,1-y,1-z)$. Since $H_{\mathbf{D}}(x,y,z)$ is absolutely continuous with respect to the Lebesgue measure on $[0,1]^3$, the latter holds if and only if the third mixed partial of $H_{\mathbf{D}}(x,y,z) - H_{\mathbf{D}}(1-x,1-y,1-z)$, which is given by

$$\begin{aligned}&2 + (2\theta - \frac{\theta^2}{3})((1-2x)(1-2y) + (1-2x)(1-2z) + (1-2y)(1-2z)) \\ &+ \theta^2(1-2x)(1-2y)(1-2z)(3-2x-2y-2z)\,,\end{aligned}$$

is positive for any $x,y,z \in [0,1]$. Some simple computations show that the latter expression is positive if and only if $\theta \in [3-\sqrt{15}, (-3+\sqrt{21})/2]$.

3.3.3. An option based on the product of copulas

Another possible way of constructing the auxiliary function H is a product-type construction, inspired by the results in [137], in the following way

$$H(x,y,z) = \begin{cases} \frac{C_2(x,y)C_2(x,z)C_2(y,z)}{xyz} & , \text{if } \min(x,y,z) > 0\,, \\ 0 & , \text{otherwise}\,. \end{cases} \tag{3.13}$$

We note that this choice of H requires some additional conditions, such as the fact that for any $y, z \in [0,1]$ it must hold that

$$\lim_{x \to 0} \frac{C_2(x,y)C_2(x,z)}{x} = 0\,. \tag{3.14}$$

The latter condition may not hold, for example, if the lower tail dependence coefficient of C_2 does not exist. A family of 2-copulas that satisfy Eq. (3.14) is the FGM family of 2-copulas.

Example 3.6. Let C_2 be a member of the FGM family of copulas, then H as defined in Eq. (3.13) is given by

$$H(x,y,z) = (x + \theta x(1-x)(1-y))\,(y + \theta y(1-y)(1-z))\,(z + \theta z(1-x)(1-z))\;.$$

It is easy to see that the function $H(x,y,z)$ is absolutely continuous with respect to the Lebesgue measure on $[0,1]^3$, and as a consequence $H(x,y,z) - H(1-x, 1-y, 1-z)$ is 3-increasing if and only if if and only if the third mixed partial of $H(x,y,z) - H(1-x,1-y,1-z)$, which is given by,

$$\begin{aligned}
&2 + 2\theta((1-2x)(1-2y) + (1-2x)(1-2z) + (1-2y)(1-2z)) \\
&+\theta^2(1-2x)(1-2y)(1-6z+6z^2) + \theta^2(1-2x)(1-2z)(1-6y+6y^2) \\
&+\theta^2(1-2y)(1-2z)(1-6x+6x^2) - \theta^3(2x-3x^2)(2y-3y^2)(2z-3z^2) \\
&+\theta^3(1-4x+3x^2)(1-4y+3y^2)(1-4z+3z^2)\,,
\end{aligned}$$

is positive for any $x, y, z \in [0,1]$. The last condition is satisfied if and only if $\theta \in [r, \sqrt{3} - 1/2]$, where r is the only real root of the polynomial $t^3 + 3t^2 + 6t + 2$.

3.3.4. Extensions to higher dimensions

Next, we discuss one of the difficulties that arises when trying to generalize the above three possible options for H to higher dimensions. The main difficulty is that there may not be a unique way to choose H. In order to illustrate the latter problem in the case $n = 4$, define the functions G_1, G_2, G_3 and G_4 as

$$\begin{aligned} G_1(x,y,z,w) &= C_2(x,C_3(y,z,w)) + C_2(y,C_3(x,z,w)) \\ &\quad + C_2(z,C_3(x,y,w)) + C_2(w,C_3(x,y,z)), \end{aligned}$$

$$\begin{aligned} G_2(x,y,z,w) &= C_3(x,y,C_2(z,w)) + C_3(x,z,C_2(y,w)) + C_3(x,w,C_2(y,z)) \\ &\quad + C_3(y,z,C_2(x,w)) + C_3(y,w,C_2(x,z)) + C_3(z,w,C_2(x,y)), \end{aligned}$$

$$\begin{aligned} G_3(x,y,z,w) &= C_2(C_2(x,y),C_2(z,w)) + C_2(C_2(x,z),C_2(y,w)) \\ &\quad + C_2(C_2(x,w),C_2(y,z)), \end{aligned}$$

$$\begin{aligned} G_4(x,y,z,w) &= C_2(x,C_2(y,C_2(z,w))) + C_2(x,C_2(z,C_2(y,w))) + C_2(x,C_2(w,C_2(y,z))) \\ &\quad + C_2(y,C_2(x,C_2(z,w))) + C_2(y,C_2(z,C_2(x,w))) + C_2(y,C_2(w,C_2(x,z))) \\ &\quad + C_2(z,C_2(x,C_2(y,z))) + C_2(z,C_2(y,C_2(x,w))) + C_2(z,C_2(w,C_2(x,y))) \\ &\quad + C_2(w,C_2(x,C_2(y,z))) + C_2(w,C_2(y,C_2(x,z))) + C_2(w,C_2(z,C_2(x,y))). \end{aligned}$$

Then the functions H_1, H_2, H_3, H_4 defined by

$$\begin{aligned} H_1(x,y,z,w) &= \frac{1}{2}(G_2(x,y,z,w) - G_1(x,y,z,w)), \\ H_2(x,y,z,w) &= \frac{1}{3}(G_2(x,y,z,w) - G_3(x,y,z,w)), \\ H_3(x,y,z,w) &= \frac{1}{4}(4G_1(x,y,z,w) - G_4(x,y,z,w)), \\ H_4(x,y,z,w) &= \frac{1}{12}(4G_2(x,y,z,w) - G_4(x,y,z,w)) \end{aligned}$$

satisfy the conditions of Proposition 3.1. Each of these functions, as well as convex linear combinations thereof, can be regarded as a 4-dimensional generalization of Eq. (3.10), in the sense that each of these functions is obtained by nesting the copula C_3 or its 2-dimensional marginals and by subtracting a correction term in order to guarantee that the 3-dimensional marginal of H_j is C_3.

Obviously, as the dimensionality increases, the number of possibilities also increases. A similar problem occurs with Eq. (3.12), even in the three-dimensional case, as there are different ways to choose the family of 2-copulas $\mathbf{D}$. In the case of the multidimensional generalization of Eq. (3.13), we can define H as

$$H(\mathbf{x}) = \frac{\prod_{i=1}^{n} C_{n-1}(\mathbf{x}_{\{i\}})}{\prod_{i<j}^{n} C_{n-2}(\mathbf{x}_{\{i,j\}})} \cdots \left(\frac{\prod_{i<j}^{n} C_2(x_i, x_j)}{\prod_{i=1}^{n} x_i} \right)^{(-1)^{n-1}}$$

for the points $\mathbf{x} \in [0,1]^n$ such that $\prod_{i<j}^{n} C_{n-2}(\mathbf{x}_{\{i,j\}}) \neq 0$, otherwise we define it as zero.

The latter generalizations satisfy the boundary conditions described in Proposition 3.1. However, it is not a simple task to show that the latter generalizations satisfy the volume condition of Proposition 3.2. A first step in this direction would be to characterize all the n-ary functions that satisfy the volume property of Proposition 3.2.

PART II

QUASI-COPULAS

4 A multi-faced view of quasi-copulas

4.1. Introduction

In this chapter we rotate the kaleidoscope in order to study the concept of an n-quasi-copula, a concept that is closely related to the concept of an n-copula. The concept of quasi-copula was introduced in 1993 by Alsina et al. [4] in order to characterize a certain class of operations on univariate distribution functions that can be derived from corresponding operations on random variables. The original definition of an n-quasi-copula was given in terms of tracks and n-copulas, which was too impractical to use. It was in [88] and in [160], for the bivariate and multivariate case respectively, where an alternative characterization of n-quasi-copulas in terms of their analytical properties was proven. This characterization has become the *de facto* definition of n-quasi-copula.

Since then there has been a growing interest in the concept of an n-quasi-copula, for example, several other characterizations have been proven in [159, 161, 172], as well as several interesting properties have been studied [36, 73, 157].

While n-quasi-copulas have been used extensively in the field of n-copulas, mainly to derive bounds on sets of n-copulas (see, for example, [4, 57, 72, 161, 170, 193]), they have also become increasingly popular in fuzzy set theory and aggregation theory due to their 1-Lipschitz continuity property. For example, 2-quasi-copulas were studied in [99] as a particular case of conjuntors that satisfy the Bell inequalities, in [56] the residual implicators of several classes of conjunctors, including 2-quasi-copulas, were studied. Bivariate quasi-copulas have also been used in the framework of fuzzy preference modelling and to extend a fuzzy measure on $N = \{1, 2, \ldots n\}$ to an n-ary aggregation function [124]. For other applications of n-quasi-copulas, we refer to [31, 39, 40, 43, 44, 45, 78, 94, 122].

The aim of this chapter is to give an overview to the various results that have been proven in the literature on the topic of n-quasi-copulas: from the different characterizations, highlighting which ones cannot be extended to higher dimensions $n \geqslant 3$, to the different properties that have been studied, such as the lattice structure of the set of n-quasi-copulas.

First, we recall the concept of an n-quasi-copula as it was originally introduced. Next, we review several characterizations of n-quasi-copulas, while stressing the differences that occur between the case $n = 2$ and $n \geqslant 3$. Thereafter, we discuss the lattice structure of the set of n-quasi-copulas and its relationship to n-copulas, while once again highlighting the difference between the cases $n = 2$ and $n \geqslant 3$. Finally, we recall results in the literature related to the mass distribution of n-quasi-copulas and stochastic signed measures.

4.2. The concept of a quasi-copula as it was originally introduced

We start by recalling some definitions that were used to introduce the concept of an n-quasi-copula. In the following, let $\mathscr{D}$ denote the space of univariate distribution functions.

Definition 4.1. An n-ary operation Φ on $\mathscr{D}$ is a function $\Phi : \mathscr{D}^n \to \mathscr{D}$.

We start with a definition of a specific type of n-ary operation on $\mathscr{D}$ [160].

Definition 4.2. An n-ary operation Φ on $\mathscr{D}$ is said to be derivable from a function of random variables if there exists a Borel measurable function $H : [-\infty, \infty]^n \to [-\infty, \infty]$ such that for any family of n univariate distribution functions $(F_{1,j})_{j=1}^n$, there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an n-dimensional random vector $(X_1, \ldots, X_n)$ such that for any $j \in \{1, \ldots, n\}$ the distribution function of X_j is $F_{1,j}$ and such that the distribution function of the random variable $H(X_1, \ldots, X_n)$ is $\Phi(F_{1,1}, \ldots, F_{1,n})$.

A well-known example of an n-ary operation that it is derivable from a function of random variables is the convolution operation, since for any family of n univariate distribution functions $(F_{1,j})_{j=1}^n$, we can construct a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and random variables $X_1, \ldots, X_n$ such that for any $j \in \{1, \ldots, n\}$ the distribution function of X_j is $F_{1,j}$ and such that $X_1, \ldots, X_n$ are independent random variables. For such a space, the convolution of $(F_{1,j})_{j=1}^n$, i.e., $F_{1,1} * F_{1,2} * \cdots * F_{1,n}$, is the distribution function of the random variable $X_1 + X_2 + \cdots + X_n$.

We can give another example of an n-ary operation that it is derivable from a function of random variables by using Sklar's theorem, more specifically, any n-copula C_n is derivable from a function H on random variables defined on a common probability space. Indeed, clearly $H(\mathbf{x}) = \max(x_1, \ldots, x_n)$ is a Borel measurable function, and for any family of univariate distribution functions $(F_{1,j})_{j=1}^n$, by using Sklar's theorem, we can construct a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and random variables $X_1, \ldots, X_n$ such that for any $j \in \{1, \ldots, n\}$ $F_{1,j}$ is the distribution function of X_j and the joint distribution function of $(X_1, \ldots, X_n)$ is given by $C_n(F_{1,1}, \ldots, F_{1,n})$. Then it follows that

$$\begin{aligned}
\mathbb{P}\left(\max(X_1, \ldots X_n) \leqslant t\right) &= \mathbb{P}\left(X_1 \leqslant t, \ldots, X_n \leqslant t\right) \\
&= C_n(F_{1,1}(t), \ldots, F_{1,n}(t)) \\
&= C_n(F_{1,1}, \ldots, F_{1,n})(t)\,.
\end{aligned}$$

We now give the definition of a second type of n-ary operation on $\mathscr{D}$ [160].

Definition 4.3. An n-ary operation Φ on $\mathscr{D}$ is said to be induced pointwisely by an n-dimensional function $\zeta : [0,1]^n \to [0,1]$ if for any $t \in [-\infty, \infty]$ and any $(F_{1,j})_{j=1}^n$ in $\mathscr{D}$ it holds that

$$\Phi(F_{1,1}, \dots, F_{1,n})(t) = \zeta(F_{1,1}(t), \dots, F_{1,n}(t)) \,.$$

An example of an n-ary operation Φ on $\mathscr{D}$ that is induced pointwisely by a function is the mixture of distributions, since it is clearly induced pointwisely by the function $\zeta_{\mathbf{a}}(\mathbf{x}) = \sum_{j=1}^n a_j x_j$, where $\mathbf{a} \in [0,1]^n$ and $\sum_{j=1}^n a_j = 1$.

The properties of an n-ary operation 'being derivable from a function of random variables' and 'being induced pointwisely by an n-dimensional function' are independent in the sense that neither of them implies the other one. For example, it can be easily seen that the convolution of functions is not induced pointwisely by any function, while the mixture of distributions is not derivable (see [4, 5, 184]). It was deemed to be of great interest to characterize n-ary operations that are both derivable and induced pointwisely by a function (see [5, 184, 185] and the references therein). In order to answer this question, the concept of n-quasi-copula was introduced.

Another important concept that is required for the original definition of an n-quasi-copula is that of a track. It was introduced in [4] for the bivariate case and in [160] for the higher-dimensional case.

Definition 4.4. A subset B of $[0,1]^n$ is called a track on $[0,1]^n$ if it can be written as

$$B = \{(F_{1,1}(t), F_{1,2}(t), \dots F_{1,n}(t)) \mid t \in [0,1]\}$$

where $(F_{1,j})_{j=1}^n$ is a family of univariate distribution functions that satisfy $F_{1,j}(0) = 0$ and $F_{1,j}(1) = 1$ for any $j \in \{1, 2, \dots, n\}$.

We are now ready to give the original definition of an n-quasi-copula, as it was introduced in [4, 160].

Definition 4.5. An n-quasi-copula is a function $Q_n : [0,1]^n \to [0,1]$ such that for every track B on $[0,1]^n$ there exists an n-copula $C_{n,B}$ that coincides with Q_n on B, i.e., for any $\mathbf{x} \in B$ it holds that $Q_n(\mathbf{x}) = C_{n,B}(\mathbf{x})$.

Alsina et al. [4] have proven the following characterization for $n = 2$.

Theorem 4.1. *Let Φ be a bivariate operation on $\mathscr{D}$ that it is induced pointwisely by a 2-dimensional function $\zeta : [0,1]^2 \to [0,1]$ and that is derivable from a function H on random variables defined on a common probability space. Then precisely one of the following holds:*

(i) $H(x,y) = \max(x,y)$ and $\zeta(x,y) = Q_2(x,y)$ for all $x, y \in [0,1]$, where Q_2 is

a 2-quasi-copula.

(ii) $H(x,y) = \min(x,y)$ *and* $\zeta(x,y) = x + y - Q_2(x,y)$ *for all* $x, y \in [0,1]$*, where* Q_2 *is a 2-quasi-copula.*

(iii) $H(x,y) = x$ *and* $\zeta(x,y) = x$ *for all* $x, y \in [0,1]$.

(iv) $H(x,y) = y$ *and* $\zeta(x,y) = y$ *for all* $x, y \in [0,1]$.

For the higher-dimensional case, we refer to [160], since the characterization requires to introduce more concepts that are out of the scope of this dissertation.

4.3. Some characterizations of quasi-copulas

It is clear that the original definition of an n-quasi-copula is too impractical to use, making it hard to study properties of n-quasi-copulas or to prove that a given function is an n-quasi-copula. Fortunately, the concepts of n-quasi-copula and n-copula have drawn a lot of attention of researchers, and as a consequence, several characterizations have been provided.

The first characterization and the relationship between quasi-copulas and copulas

The first characterization of n-quasi-copulas was proven by Genest et al. [88] for $n = 2$, and it has become the most natural way of proving that a given function is a 2-quasi-copula.

Theorem 4.2. *A 2-quasi-copula* Q_2 *is a* $[0,1]^2 \to [0,1]$ *function that satisfies the following conditions:*

(i) For any $x \in [0,1]$ *it holds that* $Q_2(x,0) = Q_2(0,x) = 0$.

(ii) For any $x \in [0,1]$ *it holds that* $Q_2(x,1) = Q_2(1,x) = x$.

(iii) Q_2 *is increasing, i.e., for any* $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ *such that* $x_1 \leqslant x_2$ *and* $y_1 \leqslant y_2$ *it holds that*

$$Q_2(x_1,y_1) \leqslant Q_2(x_2,y_2)\,.$$

(4) Q_2 *is 1-Lipschitz continuous with respect to the* L^1 *norm on* $[0,1]^2$*, i.e., for any* $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ *it holds that*

$$|Q_2(x_2,y_2) - Q_2(x_1,y_1)| \leqslant |x_2 - x_1| + |y_2 - y_1|\,.$$

At the time of its publication in 1999, a generalization of Theorem 4.2 and Theorem 4.6, which is formulated later on, for $n \geqslant 3$ was not evident. As stated by

Genest et al. [88] “Many of the arguments presented herein extend almost immediately to the multivariate case; Proposition 1 provides an example. At the time of publication, however, it was not clear to the authors how the proof given in the appendix could be generalized to characterize quasi-copulas in higher dimensions. This will be the object of future research”. It was two years later, in 2001, when a generalization of Theorem 4.2 to higher dimensions was proven in [26].

Theorem 4.3. *An n-quasi-copula Q_n is a $[0,1]^n \to [0,1]$ function that satisfies the following conditions:*

(q1) $Q_n(\mathbf{x}) = 0$ *if* $\mathbf{x}$ *is such that* $x_j = 0$ *for some* $j \in \{1, 2, ..., n\}$.

(q2) $Q_n(\mathbf{x}) = x_j$ *if* $\mathbf{x}$ *is such that* $x_i = 1$ *for all* $i \neq j$ *and some* $j \in \{1, 2, ..., n\}$.

(q3) Q_n *is increasing.*

(q4) Q_n *is 1-Lipschitz continuous with respect to the* L^1 *norm on* $[0,1]^n$*, i.e., for any* $\mathbf{x}, \mathbf{y} \in [0,1]^n$ *it holds that*

$$|Q_n(\mathbf{x}) - Q_n(\mathbf{y})| \leqslant \sum_{j=1}^{n} |x_j - y_j| \,.$$

As in the case $n = 2$, the characterization given by Theorem 4.3 has become the usual procedure to prove that a given function is an n-quasi-copula and it is even sometimes given as the definition of an n-quasi-copula. We can see from the original definition of an n-quasi-copula that any n-copula is an n-quasi-copula. However, Theorem 4.3 shows that n-quasi-copulas and n-copulas satisfy ‘similar’ conditions further justifying the word ‘quasi’. Indeed, it can be shown easily that every n-copula is an n-quasi-copula, since condition (c3) of Definition 1.3 implies conditions (q3) and (q4) of Theorem 4.3 when supposing the validity of conditions (q1) and (q2). However, the converse is not true, i.e., there exist n-quasi-copulas that are not n-copulas; such n-quasi-copulas are called *proper n-quasi-copulas.* For example, consider the function $Q_{2,\mathrm{pr}} : [0,1]^2 \to [0,1]$ given by

$$Q_{2,\mathrm{pr}}(x,y) = \begin{cases} \min(x, y, \frac{1}{3}, x + y - \frac{2}{3}) & , \text{if } \frac{2}{3} \leqslant x + y \leqslant \frac{4}{3}\,, \\ \max(x + y - 1, 0) & , \text{otherwise}\,. \end{cases} \tag{4.1}$$

It can be shown (see [152], Exercise 2.11) that $Q_{2,\mathrm{pr}}$ is a 2-quasi-copula, but not a 2-copula, since $V_{Q_{2,\mathrm{pr}}}([1/3, 2/3]^2) = -1/3$. For $n \geqslant 3$, the function $Q_{n,\mathrm{pr}} : [0,1]^n \to [0,1]$ given by:

$$Q_{n,\mathrm{pr}}(x_1, x_2, \ldots, x_n) = Q_{2,\mathrm{pr}}(x_1, x_2) \prod_{k=3}^{n} x_k \tag{4.2}$$

is a proper n-quasi-copula.

Quasi-copulas as aggregation functions

We first give some definitions related to n-ary aggregation function on $[0,1]^n$ that will also be useful in Chapter 6.

Definition 4.6. A function $A : [0,1]^n \to [0,1]$ is called an n-ary aggregation function if it satisfies:

(a1) $A(\mathbf{0}) = 0$.

(a2) $A(\mathbf{1}) = 1$.

(a3) A is increasing in each argument.

Well-known examples of aggregation functions are the arithmetic mean, the minimum and the maximum operations. We refer to [15, 22, 92] for more details on aggregation functions.

We recall the definition of the dual of an n-ary aggregation function.

Definition 4.7. Let A be an n-ary aggregation function. The dual of A is the n-ary aggregation function A^* defined by

$$A^*(\mathbf{x}) = 1 - A(\mathbf{1} - \mathbf{x})\,.$$

Remark 4.1. In the literature on aggregation functions, it is common to denote the dual of an aggregation funcion A as A^d. However, for the purpose of this paper we will not use such notation to avoid confusion with the notation of the diagonal function.

We now recall the concepts of neutral element and absorbing element.

Definition 4.8. An element $e \in [0,1]$ is a neutral element of an n-ary aggregation function A if $A(\mathbf{x}) = x_j$ if $\mathbf{x}$ is such that $x_i = e$ for all $i \neq j$ and some $j \in \{1, 2, ..., n\}$.

Definition 4.9. An element $a \in [0,1]$ is an absorbing element of an n-ary aggregation function A if $A(\mathbf{x}) = a$ if $\mathbf{x}$ is such that $x_j = a$ for some $j \in \{1, 2, ..., n\}$.

It can be easily shown that if an n-ary aggregation function has a neutral element (resp. absorbing element), then this element is unique.

It follows from Theorem 4.3 that any n-quasi-copula is an n-ary aggregation function with 0 as its absorbing element and 1 as its neutral element. We now present other characterizations that relate both concepts. Alsina has shown the following characterization of 2-quasi-copulas [3].

Theorem 4.4. *A 2-quasi-copula Q_2 is a $[0,1]^2 \to [0,1]$ function that satisfies the following conditions:*

(i) $Q_2(1,1) = 1$.

(ii) For any $x, y \in [0,1]$, *it holds that* $Q_2(x,y) \leqslant \min(x,y)$.

(iii) For any $x, y_1, y_2 \in [0,1]$, *it holds that* $Q_2(x,y_2) - Q_2(x,y_1) \leqslant \max(0, y_2 - y_1)$.

(iv) For any $y, x_1, x_2 \in [0,1]$, *it holds that* $Q_2(x_2,y) - Q_2(x_1,y) \leqslant \max(0, x_2 - x_1)$.

A generalization of the above theorem was given in [172].

Theorem 4.5. *An* n*-quasi-copula* Q_n *is a* $[0,1]^n \to [0,1]$ *function that satisfies the following conditions:*

(i) $Q_n(1,1,\ldots,1) = 1$.

(ii) For any $x_1, \ldots, x_n \in [0,1]$, *it holds that* $Q_n(x_1, x_2, \ldots, x_n) \leqslant \min(x_1, x_2, \ldots, x_n)$.

(iii) For any $x_1, \ldots, x_n, y_k \in [0,1]$ *and* $k \in \{1, \ldots, n\}$, *it holds that* $Q_n(x_1, \ldots, x_k, \ldots, x_n) - Q_n(x_1, \ldots, y_k, \ldots, x_n) \leqslant \max(0, x_k - y_k)$.

Aggregation functions that satisfy condition (ii) of Theorem 4.5 are called conjunctive [15]. Note that 1-Lipschitz continuity and increasingness follow from condition (iii) of Theorem 4.5. As a consequence, Theorems 4.4 and 4.5 state that the set of n-ary aggregation functions that are conjunctive and 1-Lipschitz continuous coincides with the set of n-quasi-copulas. For other studies of quasi-copulas as 1-Lipschitz aggregation functions, see [12, 113, 115, 116, 117, 122, 123].

Characterization using volumes

Even though an n-quasi-copula may not be n-increasing, there exists some specific type of n-box that always has a positive Q_n-volume. The following characterization of bivariate quasi-copulas states that the definition of a 2-quasi-copula is equivalent to the positivity of the volume of a certain type of 2-box [88].

Theorem 4.6. *A function* $Q_2 : [0,1]^2 \to [0,1]$ *is a* 2*-quasi-copula if and only if it satisfies conditions (i) and (ii) of Theorem 4.2 and for any* 2*-box* $\mathbf{P} = [x_1, x_2] \times [y_1, y_2] \subseteq [0,1]^2$ *such that* $\{x_1, x_2, y_1, y_2\} \cap \{0,1\} \neq \varnothing$ *it holds that* $V_{Q_2}(\mathbf{P}) \geqslant 0$.

For applications of this characterization, we refer to [46, 124]. We note that, unlike the preceding two characterizations, Theorem 4.6 cannot be extended to higher dimensions, as we will see in the next chapter.

Even though it is not possible to generalize Theorem 4.6 straightforwardly, it is possible to relate the concept of an n-quasi-copula to the positivity of the volume of an even more restrictive type of n-box, as was shown in [172].

Theorem 4.7. *A function* $Q_n : [0,1]^n \to [0,1]$ *is an* n*-quasi-copula if and only if it satisfies conditions (q1) and (q2) of Theorem 4.3 and for any* n*-box* $\mathbf{P} = \times_{j=1}^{n} [x_j, y_j]$ *with the property that there exists* $k \in \{1, 2, \ldots, n\}$ *such that for* *any $j \neq k$ it holds that $x_j = 0$, the Q_n-volume of* $\mathbf{P}$ *is non-negative and bounded by $y_k - x_k$, i.e.,*

$$0 \leqslant V_{Q_n}(\mathbf{P}) \leqslant y_k - x_k \,.$$

Characterization using partial derivatives

Since an n-quasi-copula is an increasing and 1-Lipschitz continuous function, its partial derivatives must be well behaved. This was proven in [159] (see also [169]), where bivariate copulas are characterized in terms of their partial derivatives.

Theorem 4.8. *Let $Q_2 : [0,1]^2 \to [0,1]$ be a function satisfying the boundary conditions (i) and (ii) of Theorem 4.2. Then Q_2 is a 2-quasi-copula if and only if Q is absolutely continuous in each argument and:*

(i) For any $y \in [0,1]$, the partial derivative $\frac{\partial Q_2}{\partial x}(x,y)$ exists for almost all x, and for such x and y it holds that $0 \leqslant \frac{\partial Q_2}{\partial x}(x,y) \leqslant 1$.

(ii) For any $x \in [0,1]$, the partial derivative $\frac{\partial Q_2}{\partial y}(x,y)$ exists for almost all y, and for such x and y it holds that $0 \leqslant \frac{\partial Q_2}{\partial y}(x,y) \leqslant 1$.

The generalization of Theorem 4.8 to higher dimensions is straightforward and was proven in [172]. To that end, we need to introduce a more compact notation. For a given function $f : \mathbb{R}^n \to \mathbb{R}$, we write $\partial_j f(x_1, \ldots, x_j, \ldots, x_n)$ as a short-hand notation for $\frac{\partial f}{\partial x_j}(x_1, \ldots, x_j, \ldots, x_n)$.

Theorem 4.9. *Let $Q_n : [0,1]^n \to [0,1]$ be a function satisfying the boundary conditions (q1) and (q2) of Theorem 4.3. Then Q_n is an n-quasi-copula if and only if Q_n is absolutely continuous in each argument and for any $j \in \{1,2,\ldots,n\}$ and $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in [0,1]^{n-1}$, the partial derivative*

$$\partial_j Q_n(x_1, \ldots, x_j, \ldots, x_n)$$

exists for almost all $x_j \in [0,1]$ and belongs to the interval $[0,1]$.

Characterization using non-increasing tracks

Based on the concept of a decreasing set, in [172] the concept of a non-increasing track was introduced for $n = 2$.

Definition 4.10. A subset B of $[0,1]^2$ is called a non-increasing track on $[0,1]^2$ if it can be written as

$$B = \{(F_{1,1}(t), 1 - F_{1,2}(t)) \mid t \in [0,1]\}\,,$$

where $F_{1,1}, F_{1,2} : [0,1] \to [0,1]$ are continuous distribution functions that satisfy $F_{1,j}(0) = 0$ and $F_{1,j}(1) = 1$. The function $G_{1,j}(t) = 1 - F_{1,j}(t)$ is the continuous survival function associated with $F_{1,j}$, and satisfies $G_{1,j}(0) = 1$ and $G_{1,j}(1) = 0$.

Note that the main difference between this definition and the original definition of a track is that, for one of the arguments, we are considering the survival function instead of the distribution function. This definition was inspired by the fact that if C_2 is a 2-(quasi)-copula, then $x - C_2(x, 1-y)$, $y - C_2(1-x, y)$ and $x + y - 1 + C_2(1-x, 1-y)$ are also 2-(quasi)-copulas. The transformation $x+y-1+C_2(1-x, 1-y)$ is called the survival transformation of a 2(-quasi)-copula and the result of the latter transformation is called the survival 2(-quasi)-copula associated with C_2. Based on the latter idea, in [172] the following characterization was proved.

Theorem 4.10. *A function $Q_2 : [0,1]^2 \to [0,1]$ is a 2-quasi-copula if and only if for every non-increasing track B on $[0,1]^2$ there exists a copula $C_{2,B}$ such that for any $(x,y) \in B$ it holds that $Q_2(x,y) = C_{2,B}(x,y)$.*

Note that it is not clear how to extend Definition 4.10 to higher dimensions, since the definition of a non-increasing set is not obvious for $n \geqslant 3$. Additionally, in higher dimensions, the result of applying the survival transformation to an n-quasi-copula is not necessarily an n-quasi-copula. As a consequence, it is not trivial to characterize higher-dimensional quasi-copulas in a way similar to the one given by Theorem 4.10.

4.4. Quasi-copulas, bounds and lattice theory

In this section we recall some results related to the lattice structure on the sets of n-copulas and n-quasi-copulas. The properties studied here show the relevance of n-quasi-copulas in the study of bounds on sets of n-copulas. We start by recalling some notions from lattice theory that are needed for the rest of the section.

Basic definitions from lattice theory

For a given poset $(\Omega, \preccurlyeq)$ and a subset $A \subseteq \Omega$, we will denote by $\bigvee A$ the supremum of A (if it exists) and by $\bigwedge A$ the infimum of A (if it exists). Sometimes we will write $\bigvee_\Omega A$ or $\bigwedge_\Omega A$ to make explicit in which set the computations take place.

Definition 4.11. (i) A poset $(\Omega, \preccurlyeq)$ is called a lattice if for any x, y in Ω it holds that both $x \vee y := \bigvee\{x,y\}$ and $x \wedge y := \bigwedge\{x,y\}$ exist.

(ii) A poset $(\Omega, \preccurlyeq)$ is called a complete lattice if for any $A \subseteq \Omega$ it holds that both $\bigvee A$ and $\bigwedge A$ exist. In particular, it then follows that Ω has a greatest element and a smallest element.

Definition 4.12. A subset A of a poset B is said to be join-dense (resp. meet-dense) in B if for any d in B there exists a set $S_d \subseteq A$ such that $d = \bigvee_B S_d$ (resp. $d = \bigwedge_B S_d$).

If $(P, \leqslant)$ is a poset and $\phi : P \to \Omega$ is an order-preserving injection, where $(\Omega, \leqslant)$ is a complete lattice, then $(\Omega, \leqslant)$ is called a completion of $(P, \leqslant)$. Furthermore, if ϕ maps Ω onto L, then ϕ is called an order-isomorphism. A well-known procedure leading to a minimal completion of a poset is due to Dedekind and MacNeille. Any complete lattice $(\Omega, \leqslant)$ in which $(P, \leqslant)$ is both join-dense and meet-dense is order-isomorphic to the Dedekind-MacNeille completion of $(P, \leqslant)$. For more details on lattice theory, we refer to [29].

4.4.1. The lattice structure of the set of quasi-copulas

We now proceed to the study the lattice structure of the set of n-quasi-copulas equipped with the pointwise order, i.e., for two n-quasi-copulas $Q_{n,1}, Q_{n,2}$ we say that $Q_{n,1} \leqslant Q_{n,2}$ if for any $\mathbf{x} \in [0,1]^n$ it holds that $Q_{n,1}(\mathbf{x}) \leqslant Q_{n,2}(\mathbf{x})$. In the following, $\mathcal{C}_n$ denotes the set of all n-copulas, while $\mathcal{Q}_n$ denotes the set of all n-quasi-copulas. Obviously, $\mathcal{Q}_n \backslash \mathcal{C}_n$ denotes the set of proper n-quasi-copulas.

We start by studying the bounds. The following result can be found in [155] for the bivariate case and in [170] for the higher-dimensional case.

Theorem 4.11. *Let $\mathscr{Q} \subseteq \mathcal{Q}_n$ be a set of n-quasi-copulas. For any $\mathbf{x} \in [0,1]^n$, we define $Q_{n,u}(\mathbf{x})$ and $Q_{n,l}(\mathbf{x})$ as*

$$Q_{n,u}(\mathbf{x}) = \sup\{Q_n(\mathbf{x}) \mid Q_n \in \mathscr{Q}\}$$

and

$$Q_{n,l}(\mathbf{x}) = \inf\{Q_n(\mathbf{x}) \mid Q_n \in \mathscr{Q}\}\,.$$

Then $Q_{n,u}$ and $Q_{n,l}$ are n-quasi-copulas.

Theorem 4.12. *$\mathcal{Q}_n$ is a complete lattice. However, neither $\mathcal{C}_n$ nor $\mathcal{Q}_n \backslash \mathcal{C}_n$ is a complete lattice.*

From the above theorem, it follows that both the pointwise supremum and the pointwiseinfimum of a set of n-quasi-copulas are n-quasi-copulas. However, the pointwise supremum and pointwise infimum of a set of n-copulas may not be n-copulas. Similarly, the pointwise supremum and pointwise infimum of a set of proper n-quasi-copulas may not be proper n-quasi-copulas. As a particular case, it holds that the pointwise supremum and pointwise infimum of the set of all n-quasi-copulas are n-quasi-copulas. In fact, it is possible to explicitly compute this supremum and infimum, resulting in the expressions

$$M_n(\mathbf{x}) = \sup\{Q_n(\mathbf{x}) \mid Q_n \in \mathcal{Q}_n\} = \min(x_1, x_2, \ldots, x_n)\,,$$

$$W_n(\mathbf{x}) = \inf\{Q_n(\mathbf{x}) \mid Q_n \in \mathcal{Q}_n\} = \left(\sum_{j=1}^{n} x_j - (n-1)\right)^+ ,$$

where $u^+ = \max(u, 0)$. Recall from Chapter 1 that M_n is called the upper Fréchet-Hoeffding bound and it is always an n-copula. The n-quasi-copula W_n is called the lower Fréchet-Hoeffding bound, and it is an n-copula only for $n = 2$.

Due to Theorem 4.11, n-quasi-copulas naturally appear when studying bounds on sets of n-copulas. One such example is the study of bounds on sets of n-copulas with a given set of values. The first result was obtained by Nelsen [152], proving the existence of a 2-copula with a given value at a single point and deriving some bounds on the set of 2-copulas with a given value at a single point. This result was generalized for n-quasi-copulas by Rodríguez Lallena and Úbeda Flores in [170].

Theorem 4.13. *Let $\mathbf{z} \in [0,1]^n$ and $a \in [W_n(\mathbf{z}), M_n(\mathbf{z})]$. Then for any n-quasi-copula Q_n such that $Q_n(\mathbf{z}) = a$, it holds that:*

$$Q_{n,l,\mathbf{z},a}(\mathbf{x}) \leqslant Q_n(\mathbf{x}) \leqslant Q_{n,u,\mathbf{z},a}(\mathbf{x}) ,$$

for any $\mathbf{x} \in [0,1]^n$, where

$$Q_{n,l,\mathbf{z},a}(\mathbf{x}) = \max\left(W_n(\mathbf{x}), a - \sum_{j=1}^{n} (z_j - x_j)^+\right)$$

and

$$Q_{n,u,\mathbf{z},a}(\mathbf{x}) = \min\left(M_n(\mathbf{x}), a + \sum_{j=1}^{n} (x_j - z_j)^+\right) .$$

It is important to remark that $Q_{n,l,\mathbf{z},a}$ and $Q_{n,u,\mathbf{z},a}$ are n-quasi-copulas such that $Q_{n,u,\mathbf{z},a}(\mathbf{z}) = Q_{n,l,\mathbf{z},a}(\mathbf{z}) = a$. If $n = 2$, then both $Q_{2,l,\mathbf{z},a}$ and $Q_{2,u,\mathbf{z},a}$ are 2-copulas, making the bounds the best possible. However, if $n \geqslant 3$, then $Q_{n,u,\mathbf{z},a}$ and $Q_{n,u,\mathbf{z},a}$ are proper n-quasi-copulas and it is still an open problem whether they are the best bounds possible on the set of n-copulas; at present, it is known that they are the best possible on the region $\times_{j=1}^{n}[0, a_j] \bigcup \times_{j=1}^{n}[a_j, 1]$ (see [170]).

The above result was generalized by Mardani-Fard et al. [136] who proved that there exists a 2-copula with given values of a 2-quasi-copula at two or three arbitrary points, while showing that this is no longer true for four or more points. Additionally, they derived bounds similar to those of Theorem 4.13, and these bounds are 2-quasi-copulas but not necessarily 2-copulas (the same authors further studied this topic for some specific configurations of points in [177]).

Another generalization of these results was obtained by De Baets et al. [36] who proved the existence of a 3-copula with given values of a 3-quasi-copula at two arbitrary points in the unit cube, while showing that this is no longer true for three or more points.

The previous results were further generalized by Tankov in [192] for the bivariate case as expressed in the following theorem.

Theorem 4.14. *Let S be a compact subset of $[0,1]^2$ and let Q^* be a 2-quasi-copula. Let the set $\mathcal{C}_{Q*,S}$ be defined as*

$$\mathcal{C}_{Q*,S} = \{C \in \mathcal{C}_2 \mid C(x,y) = Q^*(x,y) \text{ for any } (x,y) \in S\}\,.$$

Suppose that $\mathcal{C}_{Q*,S}$ is not empty. Then for any $C \in \mathcal{C}_{Q*,S}$ and any $(x,y) \in [0,1]^2$, it holds that:

$$L_{Q*,S}(x,y) \leqslant C(x,y) \leqslant U_{Q*,S}(x,y)\,,$$

where

$$L_{Q*,S}(x,y) = \max\left(0, x+y-1, \max_{(u,v)\in S}(Q^*(u,v) - (u-x)^+ - (v-y)^+)\right)$$

and

$$U_{Q*,S}(x,y) = \min\left(x, y, \min_{(u,v)\in S}(Q^*(u,v) + (x-u)^+ - (y-v)^+)\right)\,.$$

Moreover, $L_{Q*,S}$ and $U_{Q*,S}$ are 2-quasi-copulas.

It is important to remark that $L_{Q*,S}$ and $U_{Q*,S}$ may not be 2-copulas, hence the bounds may not be the best possible. However, if we consider the set $\mathcal{Q}_{Q*,S} = \{Q \in \mathcal{Q}_2 \mid Q(x,y) = Q^*(x,y) \text{ for any } (x,y) \in S\}$, then for any $Q \in \mathcal{C}_{Q*,S}$ and any $(x,y) \in [0,1]^2$, it holds that $L_{Q*,S}(x,y) \leqslant Q(x,y) \leqslant U_{Q*,S}(x,y)$, i.e., the same bounds hold and if we consider the bounds on the set of 2-quasi-copulas, then the bounds are sharp. In [192], the following conditions were stated in order to guarantee that $L_{Q*,S}$ and $U_{Q*,S}$ are 2-copulas.

Theorem 4.15. *Let $S, Q^*, \mathcal{C}_{Q*,S}, L_{Q*,S}, U_{Q*,S}$ be defined as in Theorem 4.14.*

(i) If the set S is increasing, i.e., for any $(u_1,v_1),(u_2,v_2) \in S$ either $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$ or $u_1 \geqslant u_2$ and $v_1 \geqslant v_2$ (i.e., the pairs (u_1,v_1) and (u_2,v_2) are comonotone), then $L_{Q*,S}$ is a 2-copula.

(ii) If the set S is decreasing, i.e., for any $(u_1,v_1),(u_2,v_2) \in S$ either $u_1 \leqslant u_2$ and $v_1 \geqslant v_2$ or $u_1 \geqslant u_2$ and $v_1 \leqslant v_2$ (i.e., the pairs (u_1,v_1) and (u_2,v_2) are countercomonotone), then $U_{Q*,S}$ is a 2-copula.

Note that the conditions of Theorem 4.15 are quite restrictive. Bernard et al. [16] found less restrictive conditions, which we recall below. To that end, given a compact set $S \in [0,1]^2$, we define the following functions

(i) $\gamma_1 : \mathrm{proj}_1(S) \to \mathrm{proj}_2(S)$ given by $\gamma_1(x) = \min\{y \mid (x,y) \in S\}$.

(ii) $\gamma_2 : \text{proj}_1(S) \to \text{proj}_2(S)$ given by $\gamma_2(x) = \max\{y \mid (x, y) \in S\}$.

(iii) $\gamma_3 : \text{proj}_2(S) \to \text{proj}_1(S)$ given by $\gamma_3(y) = \min\{x \mid (x, y) \in S\}$.

(iv) $\gamma_4 : \text{proj}_2(S) \to \text{proj}_1(S)$ given by $\gamma_4(y) = \max\{x \mid (x, y) \in S\}$.

We recall the following result from [16].

Theorem 4.16. *Let $S, Q^*, \mathcal{C}_{Q*,S}, L_{Q*,S}, U_{Q*,S}$ be defined as in Theorem 4.14. Suppose that Q^* is also a 2-copula and let $\gamma_1, \gamma_2, \gamma_3$ and γ_4 be as defined above. Then*

(i) If γ_1 and γ_2 are increasing and for any $(x, y_1), (x, y_2) \in S$ it holds that $(x, \frac{y_1+y_2}{2}) \in S$, then $L_{Q*,S}$ is a 2-copula.

(ii) If γ_3 and γ_4 are increasing and for any $(x_1, y), (x_2, y) \in S$ it holds that $(\frac{x_1+x_2}{2}, y) \in S$, then $L_{Q*,S}$ is a 2-copula.

(iii) If γ_1 and γ_2 are decreasing and for any $(x, y_1), (x, y_2) \in S$ it holds that $(x, \frac{y_1+y_2}{2}) \in S$, then $U_{Q*,S}$ is a 2-copula.

(iv) If γ_3 and γ_4 are decreasing and for any $(x_1, y), (x_2, y) \in S$ it holds that $(\frac{x_1+x_2}{2}, y) \in S$, then $U_{Q*,S}$ is a 2-copula.

A different condition was given later in [17], which is stated below

Theorem 4.17. *Let $S, Q^*, \mathcal{C}_{Q*,S}, L_{Q*,S}, U_{Q*,S}$ be defined as in Theorem 4.14. Suppose that for any $(x_1, y_1), (x_2, y_2) \in S$ it holds that $(x_1, y_2), (x_2, y_1) \in S$ and Q^* is such that for any $x_1, x_2, y_1, y_2 \in S$ with $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ it holds that*

$$Q^*(x_2, y_2) - Q^*(x_2, y_1) - Q^*(x_1, y_2) + Q^*(x_1, y_1) \geqslant 0\,,$$

then $L_{Q*,S}$ and $U_{Q*,S}$ are 2-copulas.

The following generalization of Theorem 4.14 to the higher-dimensional case can be found in [130] (see also [135, 167]).

Theorem 4.18. *Let S be a compact subset of $[0, 1]^n$ and let Q^* be an n-quasi-copula. Let the set $\mathcal{Q}_{Q*,S}$ be defined as*

$$\mathcal{Q}_{Q*,S} = \{Q \in \mathcal{Q}_n \mid Q(\mathbf{x}) = Q^*(\mathbf{x}) \textit{ for any } \mathbf{x} \in S\}\,.$$

Then for any $Q \in \mathcal{Q}_{Q*,S}$ and any $\mathbf{x} \in [0, 1]^n$ it holds that:

$$L_{Q*,S}(\mathbf{x}) \leqslant Q(\mathbf{x}) \leqslant U_{Q*,S}(\mathbf{x})\,,$$

where

$$L_{Q*,S}(\mathbf{x}) = \max\left(W_n(\mathbf{x}), \max_{\mathbf{u}\in S}\,(Q^*(\mathbf{u}) - \sum_{i=1}^{n}(u_i - x_i)^+)\right)$$

and

$$U_{Q*,S}(\mathbf{x}) = \min\left(M_n(\mathbf{x}), \min_{\mathbf{u}\in S}\left(Q^*(\mathbf{u}) + \sum_{i=1}^{n}(x_i - u_i)^+\right)\right).$$

Moreover, $L_{Q*,S}$ and $U_{Q*,S}$ are n-quasi-copulas.

However, unlike in the bivariate case, $L_{Q*,S}$ and $U_{Q*,S}$ are proper n-quasi-copulas except for trivial cases, as shown in Theorem 4.2 of [130].

Theorems 4.14 and 4.18 have been used in order the compute the bounds of functionals $\rho : \mathcal{Q}_n \to \mathbb{R}$ that are increasing w.r.t. the pointwise order of n-quasi-copulas, i.e., if $Q_1 \leqslant Q_2$, then $\rho(Q_1) \leqslant \rho(Q_2)$. The latter result is useful to compute bounds on the set of n-copulas with a specified value of Spearman's rho or Kendall's tau [130, 192]. To see how these bounds have been used in financial applications, we refer to [16, 17, 130, 131, 167, 192].

There are several other applications in which n-quasi-copulas have been useful to study bounds on a set of n-copulas. For example, in the bivariate case there are studies of 2-quasi-copulas with a given opposite diagonal section [37], a given subdiagonal section [168] and a given affine section [117]. Additionally, 2-quasi-copulas were used in the context of imprecise probabilities, where one specifies a (coherent) probability interval (i.e., bounds) instead of a single value. In the multivariate case, the study of bounds on sets of n-copulas with a given set of marginals was studied in [130].

4.4.2. A lattice-theorical characterization of quasi-copulas

As mentioned before, n-quasi-copulas are well behaved when considering the pointwise supremum or the pointwise infimum of a given set of n(-quasi)-copulas, since such supremum and infimum are n-quasi-copulas. It is not a surprise that another characterization of bivariate quasi-copulas can be given in terms of the lattice structure of the set of 2-copulas. To be more precise, we recall that any 2-quasi-copula can be regarded as the pointwise supremum (or pointwise infimum) of a set of 2-copulas.

We start with the following results, which are the main contribution of [161].

Theorem 4.19. *$\mathcal{Q}_2$ is order-isomorphic to the Dedekind-MacNeille completion of $\mathcal{C}_2$.*

This theorem has the following corollary, which in turn yields another characterization of bivariate quasi-copulas.

Corollary 4.1. *A function $Q_2 : [0,1]^2 \to [0,1]$ is a 2-quasi-copula if and only if there exist $A_{Q_2}, B_{Q_2} \subseteq \mathcal{C}_2$ such that $Q_2 = \bigvee_{\mathcal{Q}_2} A_{Q_2}$ and $Q_2 = \bigwedge_{\mathcal{Q}_2} B_{Q_2}$.*

Unfortunately, the latter results do not hold for $n \geqslant 3$, as was shown in [72].

Theorem 4.20. *For $n \geqslant 3$, $\mathcal{Q}_n$ is not order-isomorphic to the Dedekind-MacNeille completion of $\mathcal{C}_n$.*

Corollary 4.2. *For $n \geqslant 3$, there exist an n-quasi-copula $Q_{n,L}$ such that for any $A \subseteq \mathcal{C}_n$ it holds that $Q_{n,L} \neq \bigvee_{\mathcal{Q}_n} A$ and an n-quasi-copula $Q_{n,U}$ such that for any $A \subseteq \mathcal{C}_n$ it holds that $Q_{n,U} \neq \bigwedge_{\mathcal{Q}_n} A$.*

While it is obvious that for any $A \subseteq \mathcal{C}_n$, it holds that $W_n \neq \bigvee_{\mathcal{Q}_n} A$, the construction of an n-quasi-copula $Q_{n,L}$ such that for any $A \subseteq \mathcal{C}_n$, it holds that $Q_{n,L} \neq \bigvee_{\mathcal{Q}_n} A$ is not trivial. We shall only provide a sketch of how such an n-quasi-copula can be constructed (for further details, we refer to [23, 72]).

For $n = 3$, let $C_{3,1}$ be the 3-copula whose mass is distributed uniformly along the main diagonals of the 3-boxes $[0, 1/4]^3$, $[1/4, 1/2] \times [1/2, 3/4]^2$, $[1/2, 3/4] \times [1/4, 1/2]^2$ and $[3/4, 1]^3$; and let $C_{3,2}$ be the 3-copula whose mass is distributed uniformly along the main diagonals of the 3-boxes $[0, 1/4]^3$, $[1/4, 1/2] \times [1/2, 3/4] \times [1/4, 1/2]$, $[1/2, 3/4] \times [1/4, 1/2] \times [1/4, 1/2]$ and $[3/4, 1]^3$. Define $Q_{3,L}$ as $Q_{3,L} = C_{3,1} \vee C_{3,2}$. Then $Q_{3,L}$ is a proper 3-quasi-copula such that for any $A \subseteq \mathcal{C}_3$ it holds that $Q_{3,L} \neq \bigvee_{\mathcal{Q}_3} A$. For $n \geqslant 4$, the proper n-quasi-copula $Q_{n,L}$ given by

$$Q_{n,L}(x_1, x_2, \ldots, x_n) = Q_{3,L}(x_1, x_2, x_3) \prod_{k=4}^{n} x_k$$

is such that for any $A \subseteq \mathcal{C}_n$ it holds that $Q_{n,L} \neq \bigvee_{\mathcal{Q}_n} A$.

4.5. Quasi-copulas and measures

4.5.1. Measure theory

In this subsection, we recall several measure-theoretical notions that are used to investigate whether n-quasi-copulas induce signed measures. Recall that a measurable space consists of a non-empty set Ω and a sigma-algebra $\mathscr{F}$ of subsets of Ω.

Definition 4.13. [174] Let $(\Omega, \mathscr{F})$ be a measurable space. A signed measure is a function $\nu : \mathscr{F} \to [-\infty, \infty]$ that satisfies the following conditions

(i) ν takes at most one of the values $-\infty, \infty$.

(ii) $\nu(\varnothing) = 0$.

(iii) ν is σ-additive, i.e., for any family $(A_j)_{j=1}^{\infty}$ in $\mathscr{F}$ that satisfies the condition $A_i \cap A_j = \varnothing$ for any $i \neq j$, it holds that $\nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \nu(A_j)$.

When the co-domain of ν is $[0, \infty]$, ν is called a measure instead of signed measure.

A signed measure is sometimes called a charge [13]. Two measures μ_1 and μ_2 defined on the same measurable space $(\Omega, \mathscr{F})$ are called singular if there exists a set $A \in \mathscr{F}$ such that, for any set $D \in \mathscr{F}$, it holds that $\mu_1(D \cap A) = \mu_2(D \cap (\Omega \backslash A)) = 0$ and in such case we write $\mu_1 \perp \mu_2$. The following theorem, which is known as the Jordan decomposition, states that any signed measure is the difference of two measures [174].

Theorem 4.21. *Let ν be a signed measure on a measurable space $(\Omega, \mathscr{F})$. Then there exist two measures ν^+ and ν^- such that the following conditions hold:*

(i) $\nu^+ \perp \nu^-$.

(ii) For any $A \in \mathscr{F}$, it holds that $\nu(A) = \nu^+(A) - \nu^-(A)$.

(iii) At least one of the conditions $\nu^+(\Omega) < \infty$ or $\nu^-(\Omega) < \infty$ holds.

For more details on these concepts, we refer to [13, 174].

4.5.2. Quasi-copulas and signed measures

n this subsection, we recall that not every n-quasi-copula Q_n induces a signed measure on $\mathscr{B}([0,1]^n)$, the set of Borel sets of $[0,1]^n$, i.e., for a given n-quasi-copula Q_n, there may not exist a signed measure on $([0,1]^n, \mathscr{B}([0,1]^n))$ such that for any $\mathbf{x} \in [0,1]^n$ it holds that $\nu([0,\mathbf{x}]) = Q_n(\mathbf{x})$. A signed measure ν defined on the measurable space $([0,1]^n, \mathscr{B}([0,1]^n))$ is called stochastic if for any $k \in \{0, 1, \ldots, n-1\}$ and any $A \in \mathscr{B}([0,1])$ it holds that

$$\nu\left([0,1]^k \times A \times [0,1]^{n-k-1}\right) = \lambda_1(A)\,,$$

where λ_1 denotes the Lebesgue measure on $\mathbb{R}$.

It is well known that any n-copula induces a stochastic measure on $\mathscr{B}([0,1]^n)$ that can be extended to $\mathbb{R}^n$. A natural question is whether a similar result holds for n-quasi-copulas, i.e., whether any n-quasi-copula induces a signed stochastic measure on $\mathscr{B}([0,1]^n)$. The answer is negative and it was proven in [73] for the bivariate case and in [157] for $n \geqslant 3$ (another proof was presented in [75]). Interestingly, this result was first proven in the n-dimensional case ($n \geqslant 3$), and only later in the bivariate case.

Proposition 4.1. *There exists a proper n-quasi-copula Q_n that does not induce a stochastic signed measure on $([0,1]^n, \mathscr{B}([0,1]^n))$.*

4.5.3. Baire category results

Although the n-quasi-copulas that induce a stochastic measure on $\mathscr{B}([0,1]^n)$ have not been characterized yet, it was shown in [53] that they are 'small' from the Baire category point of view. Baire categories were first used in the framework of n-copulas and n-quasi-copulas in [52], in order to characterize how 'large' the set of exchangable copulas is. First, we need to recall some notions.

Definition 4.14. Let (Ω, d) be a metric space.

(i) A subset B of (Ω, d) is called nowhere dense if it is not dense in any non-degenerate open ball $B(\mathbf{x}, r)$ of radius $r > 0$.

(ii) A subset B is called of first category in (Ω, d) (also called meager) if there exists a countable family $(U_i)_{i=1}^{\infty}$ of nowhere dense sets such that $B \subseteq \bigcup_{i=1}^{\infty} U_i$.

(iii) A set is called of second category in (Ω, d) if it is not of first category.

(iv) A set B is called a residual set (or co-meager) if B^c is of first category.

Informally speaking, given a complete metric space, the sets of first category are the 'small ones', sometimes called 'atypical' from a topological point of view. We recall the main results in [53], which were only proven in the 2-dimensional case, but they can be extended easily to higher dimensions. The first result shows that the set of n-copulas is 'small' with respect to the set of n-quasi-copulas when considering the supremum distance d_∞ [53].

Theorem 4.22. *The set of n-copulas $\mathcal{C}_n$ is nowhere dense in $(\mathcal{Q}_n, d_\infty)$.*

In the following, $\mathcal{Q}_{n,M}$ denotes the set of n-quasi-copulas that induce a signed measure on $\mathscr{B}([0,1]^n)$. The following result shows that an n-quasi-copula that induces a signed measure on $\mathscr{B}([0,1]^n)$ is atypical from a topological point of view [53].

Theorem 4.23. *The set $\mathcal{Q}_{n,M}$ is of first category in $(\mathcal{Q}_n, d_\infty)$.*

Even though the set $\mathcal{Q}_{n,M}$ is 'small' in the set of n-quasi-copulas, it is dense as the following proposition shows [53].

Proposition 4.2. *The set $\mathcal{Q}_{n,M}$ is dense in $(\mathcal{Q}_n, d_\infty)$.*

The following definition was also introduced in [53].

Definition 4.15. An n-quasi-copula Q is called locally extendable if there exist $\mathbf{x} \in [0,1]^n$ and a positive constant $r > 0$ such that the volume induced by Q can be extended to a signed measure on $\mathscr{B}([0,1]^n) \cap B(\mathbf{x}, r)$.

In the following $\mathcal{Q}_{n,\mathrm{Loc}}$ denotes the class of all locally extendable n-quasi-copulas. We have the following negative result [53].

Theorem 4.24. *The set $\mathcal{Q}_{n,\mathrm{Loc}}$ is of first category in $(\mathcal{Q}_n, d_\infty)$.*

4.5.4. The mass distribution associated with an n-quasi-copula

Even though an n-quasi-copula may not induce a stochastic signed measure, a natural question is: 'how negative can the Q_n-volume of an n-box be?'. This question has been studied in the bivariate and trivariate case. We start by recalling the main results in the bivariate case, which can be found in [159].

Proposition 4.3. *For any bivariate quasi-copula Q_2 and any 2-box $\mathbf{P} = [x_1, x_2] \times [y_1, y_2]$, it holds that*

$$-1/3 \leqslant V_{Q_2}(\mathbf{P}) \leqslant 1\,.$$

Moreover, if $V_{Q_2}(\mathbf{P}) = -1/3$, then $\mathbf{P} = [1/3, 2/3]^2$, $Q_2(1/3, 1/3) = 0$ and $Q_2(1/3, 2/3) = Q_2(2/3, 1/3) = Q_2(2/3, 2/3) = 1/3$ and if $V_{Q_2}(\mathbf{P}) = 1$, then $\mathbf{P} = [0, 1]^2$.

It is easily verified that the bivariate quasi-copula $Q_{2,pr}$ defined in Eq. (4.1) attains the minimal volume at the 2-box $[1/3, 2/3]^2$. The generalization for $n = 3$ was studied in [36], where De Baets et al. proved the following result.

Proposition 4.4. *For any 3-quasi-copula Q_3 and any 3-box $\mathbf{P} = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$, it holds that*

$$-4/5 \leqslant V_{Q_3}(\mathbf{P}) \leqslant 1$$

Moreover, if $V_{Q_3}(\mathbf{P}) = -4/5$, then $\mathbf{P} = [2/5, 4/5]^3$ and if $V_{Q_3}(\mathbf{P}) = 1$, then there exists $a \in [0, 1/2]$ such that $\mathbf{P} = [a, 1]^3$.

De Baets et al. [36] gave the following example of a proper 3-quasi-copula such that the minimal volume $-4/5$ is attained: distribute uniformly a positive mass of 2/5 on each of the 3-boxes $[2/5, 4/5]^2 \times [0, 2/5]$, $[0, 2/5] \times [2/5, 4/5]^2$ and $[2/5, 4/5] \times [0, 2/5] \times [2/5, 4/5]$; distribute uniformly a positive mass of 1/5 on each of the 3-boxes $[2/5, 4/5] \times [4/5, 1] \times [2/5, 4/5]$,$[4/5, 1] \times [2/5, 4/5]^2$ and $[2/5, 4/5]^2 \times [4/5, 1]$; distribute uniformly a negative mass of 4/5 on the 3-box $[2/5, 4/5]^3$; and 0 mass on the remaining 3-boxes.

Note that unlike in the bivariate case, in the trivariate case there exists a 3-quasi-copula Q_3 such that there exists a 3-box $\mathbf{P} \neq [0, 1]^3$ $V_{Q_3}(\mathbf{P}) = 1$. One example of such a 3-quasi-copula where the maximal mass is attained is the following [36]: distribute uniformly a positive mass of 1 on the 3-box $[1/2, 1]^3$; distribute uniformly a positive mass of 1/2 on each of the 3-boxes $[0, 1/2]^2 \times [1/2, 1]$,$[0, 1/2] \times [1/2, 1] \times [0, 1/2]$ and $[1/2, 1] \times [0, 1/2]^2$; 0 on the 3-box $[0, 1/2]^3$; and a negative mass of 1/2 on each of the remaining 3-boxes. Clearly, for the this example the value of a from Proposition 4.4 equals 1/2.

At the time when the negative mass distribution of n-quasi-copulas was studied, it was not known that n-quasi-copulas do not induce signed measures in general. While further studies for higher dimensions still need to be done, such as the computation of the value of the minimum mass for any dimension, as well as the form of the n-boxes where the maximum and the minimum mass are attained, we believe that the solution of these problems is now rather a non-trivial mathematical exercise and would not bring new insight into the properties of n-quasi-copulas.

5 Intermediate classes between quasi-copulas and copulas

5.1. Introduction

In this chapter, we continue to observe n-quasi-copulas through the kaleidoscope. More specifically, we study supermodular n-quasi-copulas and we propose a generalization of supermodularity for quasi-copulas in higher dimensions. As a byproduct, we also solve an open problem posed in [143], which asks for a characterization of the subclass of n-quasi-copulas for which particular n-boxes have a positive volume. These n-quasi-copulas are crucial in a generalization of the Lovász extension [128] and of the Owen extension [164] of monotone games (see [124] for more details on these generalizations). Most of the results of this chapter can be found in [11].

It is important to remark that this is not the first time that supermodular functions appear in the framework of dependence modelling. There have been several studies devoted to the study of the supermodular order of random vectors (see [151] and the references therein). We recall that a random vector $\mathbf{X}$ is smaller than or equal to a random vector $\mathbf{Y}$ in the supermodular order if $\mathbb{E}(f(\mathbf{X})) \leqslant \mathbb{E}(f(\mathbf{Y}))$ holds for any supermodular function f such that the expectations exist. Rather than focusing on expectations of random vectors, in this chapter we focus our attention on the properties of supermodular n-quasi-copulas. We show that some properties of 2-copulas that cannot be generalized to higher-dimensional copulas, hold true for supermodular n-quasi-copulas.

5.2. Supermodular quasi-copulas

We start by recalling the definitions of supermodular, submodular and modular functions.

Definition 5.1. (i) A function $f : [0,1]^n \to [0,1]$ is called supermodular if for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$ it holds that

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geqslant f(\mathbf{x}) + f(\mathbf{y}).$$

(ii) A function $f : [0,1]^n \to [0,1]$ is called submodular if for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$ it holds that

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \leqslant f(\mathbf{x}) + f(\mathbf{y}).$$

(iii) A function $f : [0,1]^n \to [0,1]$ is called modular if for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$ it holds that
$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}).$$

As stated in [11], 2-increasingness is equivalent to supermodularity for bivariate functions that satisfy (q1) and (q2). However, this is no longer true for $n \geqslant 3$.

We now recall a characterization of supermodular n-ary functions that will be often used to prove the results further on this dissertation. The proof of the following characterization can be consulted in [18, 114]. Obviously, a similar characterization can be given for submodular functions.

Proposition 5.1. *A function $f : [0,1]^n \to [0,1]$ is supermodular if and only if all of its two-dimensional sections are supermodular.*

We now proceed to study the properties of supermodular n-quasi-copulas. We will first show that, as in the case of 2-copulas, supermodularity together with the boundary conditions of an n-copula implies increasingness and 1-Lipschitz continuity.

Proposition 5.2. *Let $Q : [0,1]^n \to [0,1]$ be a supermodular function satisfying conditions (c1) and (c2) of an n-copula. Then Q is an n-quasi-copula.*

Proof. We start by proving that Q is increasing in each argument. Without loss of generality, we will show that Q is increasing in the first argument. Let $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} = (x_1, z_2, \ldots, z_n)$ and $\mathbf{y} = (y_1, z_2, \ldots, z_n)$ with $x_1 \leqslant y_1$. Define $\mathbf{a} \in [0,1]^n$ as $\mathbf{a} = (y_1, 0, \ldots 0)$. Then, due to the supermodularity of Q, it holds that
$$Q(\mathbf{x} \vee \mathbf{a}) + Q(\mathbf{x} \wedge \mathbf{a}) \geqslant Q(\mathbf{x}) + Q(\mathbf{a}).$$
Note that $Q(\mathbf{a}) = 0$ and $Q(\mathbf{x} \wedge \mathbf{a}) = 0$. Since $\mathbf{x} \vee \mathbf{a} = \mathbf{y}$ it follows that $Q(\mathbf{x}) \leqslant Q(\mathbf{y})$.

Next, we will prove that Q is 1-Lipschitz continuous with respect to the L^1-norm on $[0,1]^n$. It suffices to prove that Q is 1-Lipschitz in each argument. Without loss of generality, let $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} = (x_1, z_2, \ldots, z_n)$ and $\mathbf{y} = (y_1, z_2, \ldots, z_n)$ with $x_1 \leqslant y_1$. Define $\mathbf{a} \in [0,1]^n$ as $\mathbf{a} = (x_1, 1, \ldots 1)$. Then, due to the supermodularity of Q, it holds that
$$Q(\mathbf{y} \vee \mathbf{a}) + Q(\mathbf{y} \wedge \mathbf{a}) \geqslant Q(\mathbf{y}) + Q(\mathbf{a}).$$
Note that $Q(\mathbf{y} \vee \mathbf{a}) = y_1$ and $Q(\mathbf{a}) = x_1$. Since $\mathbf{y} \wedge \mathbf{a} = \mathbf{x}$, it follows that $Q(\mathbf{y}) - Q(\mathbf{x}) \leqslant y_1 - x_1$. □

First, we turn our attention to the class of Archimedean n-quasi-copulas, and prove that they are also supermodular. Just as in the case of Archiemedean n-copulas, if $C_{n,\varphi}$ given in Lemma (1.1) is an n-quasi-copula, we say that $C_{n,\varphi}$ is an Archimedean

n-quasi-copula, and in such case, we also say that φ is a generator of the quasi-copula $C_{n,\varphi}$. Similarly as in the case of Archimedean n-copulas, the generator is unique up to a strictly positive multiplicative constant and all Archimedean n-quasi-copulas are symmetric and associative.

The generators of Archimedean n-quasi-copulas were characterized in [156].

Theorem 5.1. *Let $\varphi, \varphi^{[-1]}$ be defined as in Definition 1.7. Then $Q_{n,\varphi}$ is an n-quasi-copula if and only if $\varphi^{[-1]}$ is convex.*

Note that all Archimedean 2-quasi-copulas are 2-copulas, i.e., they are supermodular. We now show that this is also true in higher dimensions.

Theorem 5.2. *Let $Q_{n,\varphi}$ be an Archimedean n-quasi-copula. Then $Q_{n,\varphi}$ is a supermodular function.*

Proof. We will restrict our attention to the case $n \geqslant 3$, since the result for $n = 2$ is given by Theorem 5.1. Due to Proposition 5.1, we only need to prove that the two-dimensional sections are supermodular. Because of the symmetry property of $Q_{n,\varphi}$, it suffices, without loss of generality, to prove that for any $\mathbf{z} \in [0,1]^n$ the function $Q_{n,\varphi,\mathbf{z}} : [0,1]^2 \to [0,1]$ given by

$$Q_{n,\varphi,\mathbf{z},\{1,2\}}(x,y) = Q_{n,\varphi}(x,y,z_3,z_4,\dots,z_n)$$

is 2-increasing. Let $x_1, x_2, y_1, y_2 \in [0,1]$ such that $x_1 \leqslant x_2$, $y_1 \leqslant y_2$. Note that $Q_{n,\varphi,\mathbf{z}\{1,2\}}(0,y_2) = 0 \leqslant Q_{n,\varphi,\mathbf{z},\{1,2\}}(1,y_1) \leqslant Q_{n,\varphi,\mathbf{z},\{1,2\}}(1,y_2)$, since $Q_{n,\varphi}$ is an n-quasi-copula. Hence, since for any $y \in [0,1]$ the function $f(x) = Q_{n,\varphi,\mathbf{z},\{1,2\}}(x,y)$ is a continuous function, there exists $t \in [0,1]$ such that

$$Q_{n,\varphi,\mathbf{z},\{1,2\}}(t,y_2) = Q_{n,\varphi,\mathbf{z},\{1,2\}}(1,y_1)\,,$$

which in turn is equivalent to

$$\varphi^{[-1]}(\varphi(t) + \varphi(y_2) + \varphi(q_{\mathbf{z}})) = \varphi^{[-1]}(\varphi(y_1) + \varphi(q_{\mathbf{z}}))\,, \tag{5.1}$$

where $q_{\mathbf{z}}$ is a shorthand notation for $Q_{n,\varphi}(1,1,z_3,z_4,\dots,z_n)$. Note that the associativity of Archimedean n-quasi-copulas is used in Eq. (5.1). From the properties of $\varphi^{[-1]}$, the inequality $\varphi(t)+\varphi(y_2)+\varphi(q_{\mathbf{z}}) \geqslant \varphi(0)$ holds if and only if $\varphi(y_1)+\varphi(q_{\mathbf{z}}) \geqslant 0$ holds. We need to distinguish two cases now.

Case 1: If $\varphi(y_1) + \varphi(q_{\mathbf{z}}) < \varphi(0)$, then by applying φ to both sides of Eq. (5.1), we deduce that the equality $\varphi(t) + \varphi(y_2) = \varphi(y_1)$ holds. Hence,

$$\begin{aligned}
&Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_2,y_1) - Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_1,y_1)\\
&= \varphi^{[-1]}(\varphi(x_2) + \varphi(y_1) + \varphi(q_{\mathbf{z}})) - \varphi^{[-1]}(\varphi(x_1) + \varphi(y_1) + \varphi(q_{\mathbf{z}}))\\
&= \varphi^{[-1]}(\varphi(x_2) + \varphi(y_2) + \varphi(q_{\mathbf{z}}) + \varphi(t)) - \varphi^{[-1]}(\varphi(x_1) + \varphi(y_2) + \varphi(q_{\mathbf{z}}) + \varphi(t))
\end{aligned}$$

$$
\begin{aligned}
&= Q_{2,\varphi}(Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_2,y_2),t) - Q_{2,\varphi}(Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_1,y_2),t) \\
&\leqslant Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_2,y_2) - Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_1,y_2)\,.
\end{aligned}
$$

The last equality follows from the associativity of Archimedean n-quasi-copulas, while the last inequality holds due to the 1-Lipschitz continuity of quasi-copulas.

Case 2: If $\varphi(y_1) + \varphi(q_{\mathbf{z}}) \geqslant \varphi(0)$, then clearly the inequalities $\varphi(x_2) + \varphi(y_1) + \varphi(q_{\mathbf{z}}) \geqslant \varphi(0)$ and $\varphi(x_1) + \varphi(y_1) + \varphi(q_{\mathbf{z}}) \geqslant \varphi(0)$ hold, from which it follows that $Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_2,y_1) = Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_1,y_1) = 0$. Hence,

$$
\begin{aligned}
Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_2,y_1) - Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_1,y_1) &= 0 \\
&\leqslant Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_2,y_2) - Q_{n,\varphi,\mathbf{z},\{1,2\}}(x_1,y_2)\,,
\end{aligned}
$$

since $Q_{n,\varphi,\mathbf{z},\{1,2\}}$ is increasing. □

As mentioned before, the lower Fréchet-Hoeffding lower bound W_n is not an n-copula for $n \geqslant 3$, but it turns out to be a supermodular n-quasi-copula as the following corollary shows.

Corollary 5.1. *For any $n \geqslant 2$, the lower Fréchet-Hoeffding lower bound W_n is supermodular.*

Proof. It suffices to realize that W_n is an Archimedean n-quasi-copula by considering the generator φ given by $\varphi(x) = 1 - x$. □

In the following proposition we show that there exist supermodular n-quasi-copulas that are not n-copulas, and that there exist n-quasi-copulas that are not supermodular. In the following, $\mathcal{SQ}_n$ denotes the set of all supermodular n-quasi-copulas.

Proposition 5.3. *If $n \geqslant 3$, then $\mathcal{C}_n \subset \mathcal{SQ}_n$.*

Proof. From Proposition 5.1 and Remark 1.1 it follows that all the bivariate sections of an n-copula are supermodular. Hence, $\mathcal{C}_n \subseteq \mathcal{SQ}_n$. W_n is not an 3-copula for $n \geqslant 3$ but by Corollary 5.1 it is a supermodular n-quasi-copula. Hence, the set inclusion is strict.

The fact that the set inclusion is strict for $n \geqslant 3$ follows from Lemma 5.1, since W_n is not an n-copula for $n \geqslant 3$. □

5.3. A problem on the characterization of a certain class of quasi-copulas revisited

Recall from Chapter 4 that the characterization of 2-quasi-copulas in terms of the positivity of the volume of a certain type of 2-box given in Theorem 4.6 cannot be extended to higher dimensions. For example, consider $n = 3$ and suppose that Q_3 is a function that satisfies conditions (c1) and (c2) of an n-copula and such that Eq. (1.1) holds for any $x_1 \leqslant x_2$, $y_1 \leqslant y_2$, $z_1 \leqslant z_2$ such that $\{x_1, x_2, y_1, y_2, z_1, z_2\} \cap \{0, 1\} \neq \varnothing$. We now prove that Q_3 is a 3-quasi-copula.

To see that Q_3 is increasing it suffices to prove that Q_3 is increasing in each argument. Without loss of generality, we will prove that Q_3 is increasing in the first argument. Consider the 3-box given by $[x_1, x_2] \times [0, y] \times [0, z]$ with $x_1 \leqslant x_2$. Using Eq. (1.1) and condition (c1), it follows that $Q_3(x_2, y, z) - Q_3(x_1, y, z) \geqslant 0$.

To prove that Q_3 is 1-Lipschitz continuous with respect to the L^1-norm on $[0, 1]^3$, it suffices to prove that Q_3 is 1-Lipschitz in each argument. Without loss of generality, we will show that Q_3 is 1-Lipschitz continuous in the first argument. Consider $x_1, x_2, y, z \in [0, 1]$ with $x_1 \leqslant x_2$. Note that if Eq. (1.1) holds, then the 3-boxes $[x_1, x_2] \times [y, 1] \times [z, 1]$, $[x_1, x_2] \times [y, 1] \times [0, z]$ and $[x_1, x_2] \times [0, y] \times [z, 1]$ have a positive volume. Hence, the following inequalities hold:

$$\begin{aligned}&x_2 - x_1 - Q_3(x_2, y, 1) - Q_3(x_2, 1, z) + Q_3(x_1, y, 1) + Q_3(x_1, 1, z) \geqslant \\ &Q_3(x_1, y, z) - Q_3(x_2, y, z)\,,\end{aligned}$$

$$Q_3(x_2, y, 1) - Q_3(x_1, y, 1) \geqslant Q_3(x_2, y, z) - Q_3(x_1, y, z)\,,$$
$$Q_3(x_2, 1, z) - Q_3(x_1, 1, z) \geqslant Q_3(x_2, y, z) - Q_3(x_1, y, z)\,.$$

Adding up all the above inequalities side by side, we obtain

$$x_2 - x_1 \geqslant Q_3(x_2, y, z) - Q_3(x_1, y, z)\,.$$

Hence, Q_3 is 1-Lipschitz continuous with respect to the L^1-norm on $[0, 1]^3$.

However, note that if $x_1 = 0$, then it holds that

$$Q_3(x_2, y_2, z_2) - Q_3(x_2, y_1, z_2) - Q_3(x_2, y_2, z_1) + Q_3(x_2, y_1, z_1) \geqslant 0\,. \tag{5.2}$$

This last condition is not satisfied by all 3-quasi-copulas. For example, for the proper 3-quasi-copula $Q_{3,\mathrm{pr}}$ given in Eq. (4.1), it is clear that Eq. (5.2) does not hold, and, as a consequence, it does not satisfy Eq. (1.1). While the previous analysis was done for $n = 3$, the results can be easily extended to higher dimensions.

As mentioned in [143], the following constraint on n-quasi-copulas plays an important role in the extension of a fuzzy measure to an aggregation function [124]: the

n-quasi-copula Q has to satisfy $V_Q(\mathbf{P}) \geqslant 0$ for any n-box $\mathbf{P}$ such that at least one of its vertices is contained in the boundary $[0,1]^n \setminus]0,1[^n$ of the unit hypercube $[0,1]^n$. Clearly, the characterization of those n-quasi-copulas is related to the problem that Theorem 4.6 cannot be directly generalized to higher dimensions.

As a first observation in order to find a solution to the previous problem, we note that the left-hand side of Eq. (5.2) represents the $Q_{3,\mathbf{a},A}$-volume of the 2-box $[y_1, y_2] \times [z_1, z_2]$ with $\mathbf{a} = (x_2, 1, 1)$ and $A = \{2, 3\}$. This suggests that a 3-quasi-copula that satisfies Eq. (5.2) has to be supermodular. As we will show in the next section, for $n = 3$, supermodularity will indeed be a necessary condition for a trivariate quasi-copula to satisfy the technical conditions needed in [124], but it will not be a sufficient condition, since, for example, W_3 is supermodular and $V_{W_3}([1/2, 1]^3) = -1/2$.

5.4. Other classes that lie in between copulas and quasi-copulas

Inspired by the characterization of supermodularity given in Proposition 5.1, we propose the following definition that it is motivated by the characterization of supermodular functions in higher dimensions.

Definition 5.2. A function $F : [0,1]^n \to [0,1]$ is called k-dimensionally-increasing (k-dim-increasing, for short), with $k \in \{1, \ldots, n\}$, if any of its k-dimensional sections is k-increasing.

In the framework of dependence modelling, k-dim-increasing functions have been previously studied in [176], where they were called $\triangle$-antitonic functions. In particular, Rüschendorf found bounds for $E(f(\mathbf{X})))$ where $\mathbf{X}$ is a random vector and f a k-dim-increasing function.

Remark 5.1. For any n-quasi-copula Q_n that is 1-dim-increasing, Q_n is simply an n-quasi-copula. Any n-quasi-copula that is 2-dim-increasing is a supermodular n-quasi-copula, and obviously any n-dimensionally n-increasing n-quasi-copula is an n-copula. Additionally, we note that if an n-quasi-copula is k-dim-increasing, then all of its k-dimensional marginals are k-copulas.

We will now show that k-dim-increasing n-quasi-copulas have interesting properties.

Lemma 5.1. *Let Q_n be a k-dim-increasing n-quasi-copula with $k \in \{2, \ldots, n\}$. Then Q_n is r-dim-increasing for $r \in \{1, \ldots k-1\}$.*

Proof. The proof is immediate by realizing that for any $k \in \{2, \ldots, n\}$ and $r \in \{1, \ldots k-1\}$, any r-box can be also regarded as a k-box, as explained in Remark 1.1. $\square$

As a consequence of Lemma 5.1, any k-dim-increasing n-quasi-copula with $k \geqslant 2$ is also a supermodular n-quasi-copula. Hence, using Proposition 5.2 we get the following result.

Corollary 5.2. *Let $Q_n : [0,1]^n \to [0,1]$ be a function that satisfies conditions (c1) and (c2). If Q_n is k-dim-increasing with $k \in \{2,\ldots,n\}$, then Q_n is an n-quasi-copula.*

We now develop results similar to those given in Proposition 5.3 and Theorem 5.2. To do this we need the following proposition, which can be found in [140] as Proposition 2.2.

Proposition 5.4. *Let g be a real function on $[0,\infty[$ and $p \in [0,1]$. Define $\bar{G} : \mathbb{R}^n \to \mathbb{R}$ as*

$$\bar{G}(\mathbf{x}) = g(\|\mathbf{x} \vee \mathbf{0}\|_1) + (1-p)\mathbb{1}\{\mathbf{x} < \mathbf{0}\}\,.$$

Then $\bar{G}$ is an n-dimensional survival function on $\mathbb{R}^n$ if and only if g is n-monotone on $[0,\infty[$ and satisfies $g(0) = p$ and $\lim_{x\to\infty} g(x) = 0$.

With this proposition, we can obtain a generalization of Theorem 5.2 for k-dimensional k-increasing n-quasi-copulas. The proof of the following result is heavily inspired on the proof of the main result in [140].

Theorem 5.3. *Let $Q_{n,\varphi}$ be an Archimedean n-quasi-copula and $k \in \{2,\ldots,n-1\}$. Then $Q_{n,\varphi}$ is a k-dim-increasing n-quasi-copula if and only if $\varphi^{[-1]}$ is a k-monotone function.*

Proof. First suppose that $\varphi^{[-1]}$ is a k-monotone function. Due to the symmetry of $Q_{n,\varphi}$, it suffices, without loss of generality, to prove that for any $\mathbf{z} \in [0,1]^n$ the function $Q_{n,\varphi,\mathbf{z}} : [0,1]^k \to [0,1]$ given by

$$Q_{n,\varphi,\mathbf{z},A}(\mathbf{x}) = Q_{n,\varphi}(x_1, x_2, \ldots, x_k, z_{n-k+1}, z_{n-k+2}, \ldots, z_n)$$

is k-increasing, where $A = \{1,\ldots,k\}$. In the following, we will use $q_{\mathbf{z}}$ as a shorthand notation for $Q_{n,\varphi}(1,1,\ldots,1,z_{n-k+1},z_{n-k+2},\ldots,z_n)$.

It is clear that $Q_{n,\varphi,\mathbf{z},A}(\mathbf{1}) = q_{\mathbf{z}} \in [0,1]$. Also note that

$$\lim_{x\to\infty} \varphi^{[-1]}(x + q_{\mathbf{z}}) = 0\,,$$

due to the definition of $\varphi^{[-1]}$. Define $\bar{G} : [0,\infty[^{\,k} \to [0,1]$ as

$$\bar{G}(\mathbf{x}) = \varphi^{[-1]}(\|\mathbf{x} \vee \mathbf{0}\|_1 + q_{\mathbf{z}}) + (1-q_{\mathbf{z}})\mathbb{1}\{\mathbf{x} < \mathbf{0}\}\,.$$

Then, using Proposition 5.4, we deduce that $\bar{G}$ is a k-dimensional survival function. Using the classical arguments of probability theory, we can construct a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and k random variables $X_1, X_2, \ldots, X_k$ defined on this space such that their joint survival function is given by $\bar{G}$. Note that for any $\mathbf{x} \in [0,1]^k$ the equality

$$\varphi^{[-1]}\left(\sum_{j=1}^{k}\varphi(x_j)+q_{\mathbf{z}}\right)=\mathbb{P}\left(\bigcap_{j=1}^{k}\{X_j>\varphi(x_j)\}\right)$$

holds. Then, for any k-box $\mathbf{P}=\times_{j=1}^{k}[x_j,y_j]\subseteq[0,1]^k$, it holds that $\varphi(x_j)\geqslant\varphi(y_j)\geqslant 0$, and thus

$$\begin{aligned}V_{Q_{n,\varphi,\mathbf{z}}}(\mathbf{P}) &= V_{\bar{G}}\left(\bigtimes_{j=1}^{k}[\varphi(y_j),\varphi(x_j)]\right)\\ &= \mathbb{P}\left(\bigcap_{j=1}^{k}\{\varphi(y_j)<X_j\leqslant\varphi(x_j)\}\right)\\ &\geqslant 0\,.\end{aligned}$$

Hence, $Q_{n,\varphi}$ is k-dim-increasing.

Now suppose that $Q_{n,\varphi}$ is k-dim-increasing. Then its k-variate marginals are Archimedean k-copulas generated by φ and from Theorem 5.1 it follows that $\varphi^{[-1]}$ is k-monotone. □

The following result, which is a generalization of Proposition 5.3, is now immediate.

Corollary 5.3. *Let $\mathcal{DQ}_{n,k}$ denote the class of all k-dim-increasing n-quasi-copulas. Then it holds that*

$$\mathcal{C}_n\subset\mathcal{DQ}_{n,n-1}\subset\mathcal{DQ}_{n,n-2}\subset\cdots\subset\mathcal{DQ}_{n,3}\subset\mathcal{SQ}_n\subset\mathcal{Q}_n\,.$$

Proof. It follows immediatly from the fact that there exist generators that are k-monotone, but not $(k+1)$-monotone, as shown in [140]. □

We now proceed to develop the main result of this section. To that end, we need to recall some results regarding certain transformations of n-quasi-copulas. The following definition, theorem and lemma can be deduced from Definition 3.1, Theorem 3.1, Lemma 3.1 and Theorem 3.2 in [49] (see also [32, 35] for the case $n=2$).

Definition 5.3. Let Q_n be an n-quasi-copula and $i\in\{1,\dots,n\}$. The function $Q_n^i:[0,1]^n\to[0,1]$ given by

$$Q_n^i(\mathbf{x})=Q_n(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_n)-Q_n(x_1,\dots,x_{i-1},1-x_i,x_{i+1},\dots,x_n)$$

is called the flipping of Q_n in the i-th argument.

Remark 5.2. It is important to note that Q_n^i may not be increasing. However, if Q_n^i is an increasing function, then Q_n^i is an n-quasi-copula, as stated in Theorem 3.2

in [49]. In such case, it also holds that $(Q_n^i)^i = Q_n$.

The following theorem gives sufficient and necessary conditions for Q^i to be an n-quasi-copula.

Theorem 5.4. *Let Q_n be an n-quasi-copula and $i \in \{1, \ldots, n\}$. Define $\mathbf{a} \in [0,1]^n$ as the point such that $a_i = 1$ and $a_j = 0$ for any $j \neq i$. Then Q_n^i is an n-quasi-copula if and only if for any n-box $\mathbf{P} = \times_{j=1}^n [x_j, y_j]$ with the property that $\#\{j \in \{1, \ldots, n\} \mid x_j \neq a_j \text{ and } y_j \neq a_j\} \leqslant 1$ it holds that $V_{Q_n}(\mathbf{P}) \geqslant 0$.*

We obtain the following corollary from Theorem 5.4.

Corollary 5.4. *If Q_n is a supermodular n-quasi-copula, then for any $i \in \{1, \ldots, n\}$ Q_n^i is an n-quasi-copula.*

Proof. For any $i \in \{1, \ldots, n\}$, let $\mathbf{P} = \times_{j=1}^n [x_j, y_j]$ be an n-box such that $\#\{j \in \{1, \ldots, n\} \mid x_j \neq a_j \text{ and } y_j \neq a_j\} \leqslant 1$, where $\mathbf{a} \in [0,1]^n$ is the point such that $a_i = 1$ and $a_j = 0$ for any $j \neq i$. Note that the last condition implies that $\#\{j \in \{1, \ldots, n\} \mid x_j \neq 0\} \leqslant 2$, i.e., at least $n-2$ of the values $x_1, \ldots, x_n$ must be zero. Hence, using a similar argument as in the proof of Lemma 5.1, any supermodular n-quasi-copula Q_n satisfies the condition required in Theorem 5.4, from which the result follows. $\square$

The following lemma [49] will be useful for the main result of this section.

Lemma 5.2. *For any $i \in \{1, \ldots, n\}$ define $f_i : [0,1]^n \to [0,1]^n$ as*

$$f_i(\mathbf{x}) = (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n) .$$

Then for any n-quasi-copula Q_n and n-box $\mathbf{P} = \times_{j=1}^n [x_j, y_j] \subseteq [0,1]^n$ it holds that

$$V_{Q_n^i}(\mathbf{P}) = V_{Q_n}(f_i(\mathbf{P})) ,$$

where $f_i(\mathbf{P})$ is the n-box that is obtained by applying the function f_i to each of the vertices of $\mathbf{P}$, i.e., $f_i(\mathbf{P}) = \times_{j=1}^n [x'_j, y'_j]$ where $[x'_i, y'_i] = [1 - y_i, 1 - x_i]$ and $[x'_j, y'_j] = [x_j, y_j]$ for any $j \neq i$.

We now have all the tools that are necessary to solve Problem 2.8 in [143].

Theorem 5.5. *Let Q_n be a $[0,1]^n \to [0,1]$ function that satisfies conditions (c1) and (c2) of an n-copula. Then the following statements are equivalent:*

(i) Q_n is an $(n-1)$-dim-increasing n-quasi-copula such that Q_n^i is $(n-1)$-dim-increasing n-quasi-copula for any $i \in \{1, \ldots n\}$.

(ii) For any n-box $\mathbf{P}$ such that at least one of its vertices is contained in the boundary $[0,1]^n \backslash]0,1[^n$ of the unit hypercube $[0,1]^n$ it holds that $V_{Q_n}(\mathbf{P}) \geqslant 0$.

Proof. First suppose that Q_n is an $(n-1)$-dim-increasing n-quasi-copula such that Q_n^i is an $(n-1)$-dim-increasing n-quasi-copula for any $i \in \{1,\dots\}$. Let $\mathbf{P} = \times_{j=1}^{n}[x_j, y_j] \subseteq [0,1]^n$ be an n-box such that at least one of its vertices is contained in the boundary $[0,1]^n \setminus]0,1[^n$ of the unit hypercube $[0,1]^n$. Without loss of generality, we will suppose that $\{x_1, y_1\} \cap \{0,1\} \neq \varnothing$. We have to distinguish two cases.

Case 1: If $x_1 = 0$, then using the $(n-1)$-dim-increasingness of Q_n it follows that

$$V_{Q_n}(\mathbf{P}) = V_{Q_{\mathbf{a},\{1\}}}(\mathbf{P}') \geqslant 0\,,$$

where $\mathbf{P}' = \times_{j=2}^{n}[x_j, y_j]$ and $\mathbf{a} = (y_1, 1, \dots, 1)$.

Case 2: If $y_1 = 1$, then note that if $\mathbf{P}' = [0, 1-y_1] \times_{j=2}^{n}[x_j, y_j]$, then $f_1(\mathbf{P}') = \mathbf{P}$ where f_1 is given as in Lemma 5.2. Then, using Lemma 5.2, it follows that

$$V_{Q_n}(\mathbf{P}) = V_{Q_n^1}(f_1(\mathbf{P}')) \geqslant 0\,.$$

The last inequality holds due to the $(n-1)$-dim-increasingness of Q_n^1, using a similar argument as the one in the first case.

Now suppose that Q_n is such that for any n-box $\mathbf{P}$ with the property that if at least one of its vertices is contained in the boundary $[0,1]^n \setminus]0,1[^n$ of the unit hypercube $[0,1]^n$ it holds that $V_{Q_n}(\mathbf{P}) \geqslant 0$. Then, for any $y_1 \in [0,1]^n$ and $(n-1)$-box $\mathbf{P}' = \times_{j=2}^{n}[x_j, y_j]$, it holds that

$$V_{Q_{\mathbf{a},\{1\}}}(\mathbf{P}') = V_{Q_n}(\mathbf{P}) \geqslant 0\,,$$

where $\mathbf{a} = (y_1, 1, \dots, 1)$ and $\mathbf{P} = [0, y_1] \times_{j=2}^{n}[x_j, y_j]$. In a similar manner, we can prove that all the other $(n-1)$-sections are $(n-1)$-increasing, and by Corollary 5.2, Q_n is an $(n-1)$-dim-increasing n-quasi-copula.

Now, note that for any $i \in \{1, \dots, n\}$, Q_n^i is an n-quasi-copula as proven in Corollary 5.4. Without loss of generality, we will show that Q_n^1 is $(n-1)$-dim-increasing. We have to distinguish two cases.

Case 1: We will prove that the $(n-1)$-dimensional sections of Q_n^1, where the first coordinate is fixed, are $(n-1)$-increasing. For this, consider $\mathbf{P} = [0, y] \times_{j=2}^{n}[x_j, y_j]$, then $f_1(\mathbf{P}) = [y_1, 1] \times_{j=2}^{n}[x_j, y_j]$, where f_1 is given as in Lemma 5.2. Let $\mathbf{P}' = \times_{j=2}^{n}[x_j, y_j]$, then

$$V_{Q^1_{\mathbf{a},\{1\}}}(\mathbf{P}') = V_{Q_n^1}(\mathbf{P}) = V_{Q_n}(f_i(\mathbf{P})) \geqslant 0\,,$$

where $\mathbf{a} = (y_1, 1, \dots, 1)$. Hence, $Q^1_{\mathbf{a},\{1\}}$ is $(n-1)$-increasing.

Case 2: Now we will show that $(n-1)$-dimensional sections of Q_n^1, where another argument different from the first argument is fixed, are $(n-1)$-increasing. Without loss of generality we will fix the second argument. Consider $\mathbf{P} = [x_1, y_1] \times [0, y_2] \times_{j=3}^{n} [x_j, y_j]$, then $f_1(\mathbf{P}) = [1 - y_1, 1 - x_1] \times [0, y_2] \times_{j=3}^{n} [x_j, y_j]$. Let $\mathbf{P}' = [1 - y_1, 1 - x_1] \times_{j=2}^{n} [x_j, y_j]$, then

$$V_{Q^1_{\mathbf{a},\{2\}}}(\mathbf{P}') = V_{Q^1_n}(\mathbf{P}) = V_{Q_n}(f_i(\mathbf{P})) \geqslant 0\,,$$

where $\mathbf{a} = (1, y_2, 1, \ldots, 1)$. Hence, $Q^1_{\mathbf{a},\{2\}}$ is $(n-1)$-increasing, concluding the proof. $\square$

Recall that the trivariate quasi-copula W_3 is supermodular, and as a consequence of Corollary 5.4, it holds that W_3^1 is a 3-quasi-copula. Since W_3 does not satisfy condition (ii) of Theorem 5.5, it follows that W_3^1 cannot be supermodular.

Indeed, consider $\mathbf{x} = (0, 2, 0.7, 0.5)$ and $\mathbf{y} = (0.2, 0.5, 0.7)$. Then $\mathbf{x} \vee \mathbf{y} = (0.2, 0.7, 0.7)$ and $\mathbf{x} \wedge \mathbf{y} = (0.2, 0.5, 0.5)$. So $W_3^1(\mathbf{x} \vee \mathbf{y}) + W_3^1(\mathbf{x} \wedge \mathbf{y}) = 0.2 + 0 = 0.2$, while $W_3^1(\mathbf{x}) + W_3^1(\mathbf{y}) = 0.2 + 0.2 = 0.4$, from which it follows that W_3^1 is not supermodular.

6 The multivariate Bertino quasi-copula

6.1. Introduction

In this chapter, we study for a given Lipschitz-continuous diagonal function d, the pointwise infimum of all n-copulas having d as diagonal section. To this end, we first work in the framework of aggregation functions. Just as in the case of n-quasi-copulas and n-copulas, the diagonal section of an n-ary aggregation function A is the increasing function $d : [0,1] \to [0,1]$ defined by $d(x) = A(x, x, \ldots, x)$ and satisfies the boundary conditions $d(0) = 0$ and $d(1) = 1$. Conversely, for any increasing function $d : [0,1] \to [0,1]$ with $d(0) = 0$ and $d(1) = 1$, there exists at least one n-ary aggregation function that has d as diagonal section.

As previously mentioned, in the bivariate case, the pointwise infimum of all 2-copulas having d as diagonal section is a 2-copula, called the Bertino 2-copula and denoted by $B_{d,2}$; it also has d as diagonal section. For the pointwise supremum, the situation is more complicated, since it is always a 2-quasi-copula, yet only a 2-copula under very restrictive conditions [74, 157, 193]. In the more general n-ary case, there are only a few studies focusing on the diagonal section of n-copulas [8, 9, 21, 100]. However, there has recently been a growing interest to see whether the above-mentioned construction methods that were developed for the bivariate case can be generalized to the n-dimensional case. For instance, the concept of diagonal copula in n dimensions has been studied in [100], whereas we have investigated the generalization of upper semilinear copulas in Chapter 2.

We first study for a given Lipschitz-continuous diagonal function d, the smallest and greatest n-ary Lipschitz-continuous aggregation functions that have d as diagonal section. We show that some results extend naturally from the two-dimensional case to the n-dimensional case. As a byproduct, we show that the Bertino n-quasi-copula is a supermodular function, further strengthening the results obtained in Chapter 5.

In the second part, we partially solve the open problem recently posed by R. Mesiar and J. Kalická in [141], which can be rephrased as follows: find a characterization of the set of increasing n-Lipschitz-continuous functions d for which a Bertino n-copula with diagonal section d exists, i.e., for which the associated Bertino n-quasi-copula is an n-copula. The main result is that this characterization can be obtained by imposing a single condition on the Lipschitz constant of the n-diagonal functions d, along with some regularity conditions. The result that we obtain is very restrictive, in the sense that as the dimension increases, the set of n-diagonal functions for which there exists an n-dimensional Bertino copula, gets smaller.

The results of this chapter can also be found in [8, 12].

6.2. Computation of the extremal aggregation functions

We start by studying how to compute the smallest and the greatest M-Lipschitz n-ary aggregation function with a given diagonal section. In the following, we will denote by $\mathbf{T}$ the triplet (n, M, d), where n is the dimension, M a constant greater than or equal to $1/n$ and d a diagonal function that is nM-Lipschitz continuous.

In [15] it is stated that for $M \geqslant 1/n$, the greatest M-Lipschitz n-ary aggregation function with given diagonal section d can be computed as

$$U_{\mathbf{T}}(\mathbf{x}) = \inf\{d(t) + M\sum_{i=1}^{n}(x_i - t)^+ \mid t \in [0,1]\}\,, \tag{6.1}$$

while the smallest M-Lipschitz n-ary aggregation function with given diagonal section d can be computed as

$$L_{\mathbf{T}}(\mathbf{x}) = \sup\{d(t) - M\sum_{i=1}^{n}(t - x_i)^+ \mid t \in [0,1]\}\,. \tag{6.2}$$

We have the following proposition the proof of which is obvious.

Proposition 6.1. *For any* $\mathbf{T} = (n, M, d)$, *it holds that*

$$U_{\mathbf{T}} = (L_{\mathbf{S}})^*$$

and

$$L_{\mathbf{T}} = (U_{\mathbf{S}})^*,$$

where $\mathbf{S} = (n, M, d^*)$.

In [116] it was shown that for $n = 2$ and $M = 1$, it suffices to consider t belonging to the interval $[x_{(1)}, x_{(n)}]$ in Eqs. (6.1) and (6.2), where $x_{(j)}$ is the j-th ordered component of $\mathbf{x}$. We now show that this also holds true for any $n \geqslant 2$ and $M \geqslant 1/n$. The proof follows the same lines as the proof of Theorem 3.1 in [116], but we present it here for sake of completeness.

Theorem 6.1. *For any* $\mathbf{T} = (n, M, d)$, $U_{\mathbf{T}}$ *can be computed as*

$$U_{\mathbf{T}}(\mathbf{x}) = \inf\{d(t) + M\sum_{i=1}^{n}(x_i - t)^+ \mid t \in [x_{(1)}, x_{(n)}]\}\,, \tag{6.3}$$

and $L_{\mathbf{T}}$ can be computed as

$$L_{\mathbf{T}}(\mathbf{x}) = \sup\{d(t) - M\sum_{i=1}^{n}(t - x_i)^+ \mid t \in [x_{(1)}, x_{(n)}]\}\,. \tag{6.4}$$

Proof. We will prove the expression for $L_{\mathbf{T}}$, since the proof for $U_{\mathbf{T}}$ follows from Proposition 6.1. We first show that the right-hand side of Eq. (6.4), i.e.,

$$S_{\mathbf{T}}(\mathbf{x}) = \sup\{d(t) - M\sum_{i=1}^{n}(t - x_i)^+ \mid t \in [x_{(1)}, x_{(n)}]\}\,.$$

is an M-Lipschitz aggregation function with diagonal section d. Clearly, for any $x \in [0,1]$ it holds that $S_{\mathbf{T}}(x, x, \ldots, x) = d(x)$ and as a byproduct, $S_{\mathbf{T}}$ satisfies conditions (a1) and (a2). We now show that $S_{\mathbf{T}}$ is increasing in each argument. Without loss of generality, we will prove that $S_{\mathbf{T}}$ is increasing in the first argument. Consider the vectors $\mathbf{x} = (x, z_2, z_3, \ldots, z_n)$, $\mathbf{y} = (y, z_2, z_3, \ldots, z_n) \in [0,1]^n$ with $x \leqslant y$. Denote by $a = \min(z_2, \ldots, z_n)$ and by $b = \max(z_2, \ldots, z_n)$. We have to consider several cases.

Case 1: Suppose that $x \leqslant y \leqslant a \leqslant b$. Using the continuity of d, we have

$$\begin{aligned} S_{\mathbf{T}}(\mathbf{x}) &= \sup\{d(t) - M(t-x) - M\sum_{i=2}^{n}(t - z_i)^+ \mid t \in [x, b]\} \\ &= \sup\{d(t) - M(t-x) - M\sum_{i=2}^{n}(t - z_i)^+ \mid t \in [y, b]\} \\ &\quad \vee \sup\{d(t) - M(t-x) - M\sum_{i=2}^{n}(t - z_i)^+ \mid t \in [x, y]\}\,. \end{aligned}$$

Using elementary properties of the supremum, we obtain

$$\begin{aligned} &\sup\{d(t) - M(t-x) - M\sum_{i=2}^{n}(t - z_i)^+ \mid t \in [y, b]\} \\ &= M(x-y) + \sup\{d(t) - M(t-y) - M\sum_{i=2}^{n}(t - z_i)^+ \mid t \in [y, b]\} \\ &= M(x-y) + S_{\mathbf{T}}(\mathbf{y}) \end{aligned}$$

and

$$\sup\{d(t) - M(t-x) - M\sum_{i=2}^{n}(t - z_i)^+ \mid t \in [x, y]\} = \sup\{d(t) - M(t-x) \mid t \in [x, y]\}\,,$$

since $y < a = \min(z_2, \ldots z_n)$. Hence,

$$\begin{aligned}
S_{\mathbf{T}}(\mathbf{x}) &= \sup\{d(t) - M(t-x) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [y,b]\} \\
&\quad \vee \sup\{d(t) - M(t-x) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [x,y]\} \\
&= (M(x-y) + S_{\mathbf{T}}(\mathbf{y})) \vee \sup\{d(t) - M(t-x) \mid t \in [x,y]\} \\
&\leqslant S_{\mathbf{T}}(\mathbf{y}) \vee \sup\{d(t) \mid t \in [x,y]\} \\
&= S_{\mathbf{T}}(\mathbf{y}) \vee d(y) \\
&= S_{\mathbf{T}}(\mathbf{y})\,.
\end{aligned}$$

Case 2: Suppose that $a \leqslant x \leqslant y \leqslant b$. Since $x \leqslant y$, it holds that $(t-x)^+ \geqslant (t-y)^+$. Hence,

$$\begin{aligned}
S_{\mathbf{T}}(\mathbf{x}) &= \sup\{d(t) - M(t-x)^+ - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [a,b]\} \\
&\leqslant \sup\{d(t) - M(t-y)^+ - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [a,b]\} \\
&= S_{\mathbf{T}}(\mathbf{y})\,.
\end{aligned}$$

Case 3: Suppose that $a \leqslant b \leqslant x \leqslant y$. Clearly, $[a,x] \subseteq [a,y]$ and it follows that

$$\begin{aligned}
S_{\mathbf{T}}(\mathbf{x}) &= \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [a,x]\} \\
&\leqslant \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [a,y]\} \\
&= S_{\mathbf{T}}(\mathbf{y})\,.
\end{aligned}$$

Case 4: If $x < a \leqslant y \leqslant b$, the result follows from cases 1 and 2.

Case 5: If $x < a \leqslant b < y$, the result follows from cases 1, 2 and 3.

Case 6: If $a < x < b < y$, the result follows from cases 2 and 3.

Now we show that $S_{\mathbf{T}}$ is M-Lipschitz continuous. Again, without loss of generality, we will prove that $S_{\mathbf{T}}$ is M-Lipschitz continuous in the first argument. Consider again the vectors $\mathbf{x} = (x, z_2, z_3, \ldots, z_n)$, $\mathbf{y} = (y, z_2, z_3, \ldots, z_n) \in [0,1]^n$ with $x \leqslant y$. Denote by $a = \min(z_2, \ldots, z_n)$ and by $b = \max(z_2, \ldots, z_n)$. Once again, we have to consider several cases.

Case 1: Suppose that $x \leqslant y < a \leqslant b$. Using the same arguments as in Case 1 of the proof of the increasingness of $S_{\mathbf{T}}$, we have

$$\begin{aligned} S_{\mathbf{T}}(\mathbf{x}) &= (M(x-y) + S_{\mathbf{T}}(\mathbf{y})) \vee \sup\{d(t) - M(t-x) \mid t \in [x,y]\} \\ &\geqslant M(x-y) + S_{\mathbf{T}}(\mathbf{y})\,. \end{aligned}$$

Hence $S_{\mathbf{T}}(\mathbf{y}) - S_{\mathbf{T}}(\mathbf{x}) \leqslant M(y-x)$.

Case 2: Suppose that $a < x \leqslant y \leqslant b$. For any $t \in [a,b]$, it holds that

$$d(t) - M(t-y)^+ - M\sum_{i=2}^{n}(t-z_i)^+ = d(t) - M(t-x)^+ - M\sum_{i=2}^{n}(t-z_i)^+ + M(t-x)^+ - M(t-y)^+\,.$$

Note that $M(t-x)^+ - M(t-y)^+ \leqslant M(y-x)$ for any t since

$$(t-x)^+ - (t-y)^+ = \begin{cases} 0 & \text{, if } t < x, \\ t-x & \text{, if } x \leqslant t < y, \\ y-x & \text{, if } t \geqslant y. \end{cases}$$

Hence

$$\begin{aligned} d(t) & -M(t-y)^+ - M\sum_{i=2}^{n}(t-z_i)^+ \\ &= d(t) - M(t-x)^+ - M\sum_{i=2}^{n}(t-z_i)^+ + M(t-x)^+ - M(t-y)^+ \\ &\leqslant d(t) - M(t-x)^+ - M\sum_{i=2}^{n}(t-z_i)^+ + M(y-x) \\ &\leqslant \sup\{d(t) - M(t-x)^+ - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [a,b]\} + M(y-x) \\ &\leqslant S_{\mathbf{T}}(\mathbf{x}) + M(y-x)\,, \end{aligned}$$

from which it follows that $S_{\mathbf{T}}(\mathbf{y}) \leqslant S_{\mathbf{T}}(\mathbf{x}) + M(y-x)$.

Case 3: Suppose that $a \leqslant b \leqslant x \leqslant y$. Note that

$$S_{\mathbf{T}}(\mathbf{y}) = \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [a,y]\}$$

$$
\begin{aligned}
&= \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [a,x]\} \\
&\quad \vee \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [x,y]\} \\
&= S_{\mathbf{T}}(\mathbf{x}) \vee \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [x,y]\}\,.
\end{aligned}
$$

Now, observe that

$$
\sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [x,y]\} = \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i) \mid t \in [x,y]\}
$$

Hence, for any $t \in [x,y]$, it holds that

$$
d(t) - M\sum_{i=2}^{n}(t-z_i) \leqslant d(x) - M\sum_{i=2}^{n}(x-z_i) + M(y-x)\,, \tag{6.5}
$$

since the latter inequality is equivalent to

$$
d(t) - d(x) \leqslant (n-1)M(t-x) + (y-x)\,,
$$

which always holds true due to the nM-Lipschitz-continuity property of d. Since Eq. (6.5) holds true for any $t \in [x,y]$, it follows that

$$
\begin{aligned}
\sup\{d(t) - M\sum_{i=2}^{n}(t-z_i) \mid t \in [x,y]\} &\leqslant d(x) - M\sum_{i=2}^{n}(x-z_i) + M(y-x) \\
&\leqslant S_{\mathbf{T}}(\mathbf{y}) + M(y-x)\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
S_{\mathbf{T}}(\mathbf{y}) &= S_{\mathbf{T}}(\mathbf{x}) \vee \sup\{d(t) - M\sum_{i=2}^{n}(t-z_i)^+ \mid t \in [x,y]\} \\
&\leqslant S_{\mathbf{T}}(\mathbf{x}) \vee (S_{\mathbf{T}}(\mathbf{x}) + M(y-x)) \\
&= S_{\mathbf{T}}(\mathbf{x}) + M(y-x)\,,
\end{aligned}
$$

which concludes this case.

Case 4: If $x < a \leqslant y \leqslant b$, the result follows from cases 1 and 2.

Case 5: If $x < a \leqslant b < y$, the result follows from cases 1, 2 and 3.

Case 6: If $a < x < b < y$, the result follows from cases 2 and 3.

Finally, we prove that for any M-Lipschitz n-ary aggregation A with diagonal section d, it holds that $A \geqslant S_{\mathbf{T}}$. Let $\mathbf{x} \in [0,1]^n$ and without loss of generality, suppose that $x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_n$. For any $t \in [x_1, x_2]$, using the fact that A is increasing and M-Lipschitz continuous, it holds that

$$\begin{aligned} A(\mathbf{x}) - d(t) &\geqslant A(x_1, t, t, \ldots, t) - d(t) \\ &\geqslant M(x_1 - t) \\ &= -M(t - x_1)^+ \\ &= -\sum_{i=1}^{n} M(t - x_i)^+ \,. \end{aligned}$$

For any $t \in]x_2, x_3]$, it holds that

$$\begin{aligned} A(\mathbf{x}) - d(t) &\geqslant A(x_1, x_2, t, \ldots, t) - d(t) \\ &\geqslant M(x_1 - t) + M(x_2 - t) \\ &= -M(t - x_1)^+ - M(t - x_2)^+ \\ &= -\sum_{i=1}^{n} M(t - x_i)^+ \,. \end{aligned}$$

By repeating this procedure, we have that for any $t \in [x_1, x_n]$, it holds that

$$A(\mathbf{x}) - d(t) \geqslant -M \sum_{i=1}^{n} (t - x_i)^+ \,,$$

which implies $A(\mathbf{x}) \geqslant S_{\mathbf{T}}(\mathbf{x})$. Hence, $S_{\mathbf{T}} = L_{\mathbf{T}}$. □

Example 6.1. In this example, we use Theorem 6.1 to compute the extremal M-Lipschitz n-ary aggregation functions when considering the identity function as diagonal section. Let $\mathbf{T} = (n, M, d)$ where $d(x) = x$ and $M < 1$. Let $k \in \{2, ..., n\}$ be the integer such that $(k-1)M < 1$ and $kM \geqslant 1$. After some elementary computations, it can be shown that

$$L_{\mathbf{T}}(\mathbf{x}) = x_{(k)}(1 - (k-1)M) + M \sum_{i=1}^{k-1} x_{(i)}$$

and

$$U_{\mathbf{T}}(\mathbf{x}) = x_{(n-k+1)}(1 - (k-1)M) + M \sum_{i=2}^{k} x_{(n-k+i)} \,.$$

Example 6.2. In this example, we show how to use Theorem 6.1 in order to compute the extremal M-Lipschitz n-ary aggregation functions for an ordinal sum, when the number of intervals is finite. Let d be an n-Lipschitz continuous diagonal function such that there exists $t_0 \in]0,1[$ such that $d(t_0) = t_0$ and $d(t) \leqslant t$ for any $t \in [0,1]$. Denote by A the restriction of $L_{\mathbf{T}}$ to $[0, t_0[^n$, and denote by B the restriction of $L_{\mathbf{T}}$ to $[t_0, 1]^n$. Then

$$L_{\mathbf{T}}(\mathbf{x}) = \begin{cases} B(\mathbf{x}) & \text{, if } \min(x_1, x_2, \ldots, x_n) \geqslant t_0 \,, \\ A(\mathbf{x} \wedge (t_0, t_0, \ldots, t_0)) & \text{, otherwise} \,. \end{cases}$$

Note that the equality is obvious if $\mathbf{x} \in [0, t_0]^n$ or $\mathbf{x} \in [t_0, 1]^n$. For the other cases, note that if $\mathbf{x} \in [0,1]^n$ is such that there exists $k \in \{1, 2, \ldots, n-1\}$ with the property that $x_{(k)} < t_0$ but $x_{(k+1)} \geqslant t_0$, then for any $t \in [t_0, x_{(n)}]$ the following double inequality holds

$$d(t) \leqslant t \leqslant t_0 + k(t - t_0) + \sum_{i=k+1}^{n} (t - x_i)^+ \,.$$

The latter inequality implies

$$d(t) - \sum_{i=1}^{n} (t - x_i)^+ \leqslant d(t_0) - \sum_{i=1}^{n} (t - x_i)^+ \leqslant L_{\mathbf{T}}(x_{(1)}, x_{(2)}, \ldots, x_{(k)}, t_0, t_0, \ldots, t_0) \,.$$

Hence,

$$\sup\{d(t) - \sum_{i=1}^{n} (t - x_i)^+ \mid t \in [t_0, x_{(n)}]\} \leqslant L_{\mathbf{T}}(x_{(1)}, x_{(2)}, \ldots, x_{(k)}, t_0, t_0, \ldots, t_0) \,.$$

Since we can compute $L_{\mathbf{T}}(\mathbf{x})$ as

$$L_{\mathbf{T}}(\mathbf{x}) = \sup\{d(t) - \sum_{i=1}^{n} (t - x_i)^+ \mid t \in [x_{(1)}, t_0]\} \vee \sup\{d(t) - \sum_{i=1}^{n} (t - x_i)^+ \mid t \in [t_0, x_{(n)}]\}$$

and

$$\begin{aligned} \sup\{d(t) - \sum_{i=1}^{n} (t - x_i)^+ \mid t \in [x_{(1)}, t_0] &= \sup\{d(t) - \sum_{i=1}^{k} (t - x_i)^+ \mid t \in [x_{(1)}, t_0] \\ &= L_{\mathbf{T}}(x_{(1)}, x_{(2)}, \ldots, x_{(k)}, t_0, t_0, \ldots, t_0) \,, \end{aligned}$$

the result follows from the symmetry of $L_{\mathbf{T}}$. Obviously this result can be generalized to the case where there exists a finite number of points $t_0, t_1, \ldots, t_m$ such that $d(t_i) = t_i$ for any $i \in \{0, 1, 2, \ldots, m\}$.

Example 6.3. In this example, we use Theorem 6.1 to compute the extremal M-Lipschitz n-ary aggregation, when considering their diagonal section as a specific power of the identity. Let $\mathbf{T} = (n, M, d)$ where $d(x) = x^\alpha$ with $\alpha \leqslant M$. Let $\mathbf{x} \in [0,1]^n$ and note that for any $j \in \{1, 2, \ldots, n-1\}$ any $t \in [x_{(j)}, x_{(j+1)}]$, it holds that the function $d(t) - M\sum_{i=1}^n (t - x_i)^+$ is decreasing since its derivative on the interval $]x_{(j)}, x_{(j+1)}[$ is equal to $\alpha t^{\alpha-1} - jM$, which is negative. Hence, the minimum of the function $d(t) - M\sum_{i=1}^n (t - x_i)^+$ on the interval $t \in [x_{(1)}, x_{(n)}]$ is attained at $t = x_{(1)}$. From the latter it follows that $L_\mathbf{T}(\mathbf{x}) = x_{(1)}^\alpha$ and by doing a similar analysis, it can be shown that $U_\mathbf{T}(\mathbf{x}) = x_{(n)}^\alpha$. As a particular case, note that if $d(x) = x$, then for any $M \geqslant 1$ it holds that $L_\mathbf{T}(\mathbf{x}) = x_{(1)} = \min(x_1, \ldots, x_n)$.

The previous example can be further generalized as the following proposition shows.

Proposition 6.2. *Let $\mathbf{T} = (n, M, d)$ where d is a K-Lipschitz continuous diagonal function and K is such that $K \leqslant jM$ with $j \in \{1, 2, \ldots, n-1\}$. Let $\mathbf{S}_1 = (j, M, d)$ and $\mathbf{S}_2 = (n-j, M, d)$. Then for any $\mathbf{x}$, it holds that*

$$L_\mathbf{T}(\mathbf{x}) = L_{\mathbf{S}_1}(x_{(1)}, \ldots, x_{(j)})$$

and

$$U_\mathbf{T}(\mathbf{x}) = U_{\mathbf{S}_2}(x_{(n-j+1)}, \ldots, x_{(n)})$$

Proof. We will give the proof for $L_\mathbf{T}$. The result for $U_\mathbf{T}$ then follows from Proposition 6.1. Note that for any $\mathbf{x} \in [0,1]^n$ the function $q : [x_{(1)}, x_{(n)}] \to \mathbb{R}$ given by

$$q(t) = d(t) - M\sum_{j=1}^n (t - x_j)^+$$

is absolutely continuous and has a derivative of the form $d'(t) - Mr$ almost everywhere, with r a positive integer depending on the value of t. Since d is a K-Lipschitz continuous diagonal section with $K \leqslant jM$ for some $j \in \{1, 2, \ldots, n-1\}$, it follows that for almost every $t \in]x_j, x_{(n)}[$ the derivative is negative almost surely. Hence the maximum must be attained in the interval $t \in [x_1, x_{(j)}]$, from which the result follows. $\square$

Example 6.4. We give an easy example of how Proposition 6.2 simplifies the computations when the value of M is sufficiently large. Consider $\mathbf{T}_M = (2, M, d)$ where $d(x) = x^2$. Then, if $M \in [1, 2[$, it holds that

$$L_\mathbf{T}(x,y) = \begin{cases} x^2 & \text{, if } x \leqslant y \text{ and } x + y < M\,, \\ y^2 & \text{, if } x > y \text{ and } x + y < M\,, \\ y^2 - M(y - x) & \text{, if } x \leqslant y \text{ and } x + y \geqslant M\,, \\ x^2 - M(x - y) & \text{, if } x > y \text{ and } x + y \geqslant M\,, \end{cases}$$

whereas if $M \geqslant 2$, then
$$L_{\mathbf{T}}(x,y) = (\min(x,y))^2 \,.$$

6.3. Absorbing and neutral elements of the extremal functions

We now proceed to investigate whether the smallest Lipschitz-continuous aggregation function with a given diagonal section d has similar properties as the ones of quasi-copulas, more specifically, the existence of a neutral element and an absorbing element.

We start by studying the existence of an absorbing element.

Proposition 6.3. *Let $\mathbf{T} = (n, M, d)$. Then $L_{\mathbf{T}}$ has 0 as an absorbing element if and only if the following conditions hold*

(1) $M \geqslant 1$,

(2) for any $t \in [0,1]$ it holds that $d(t) \leqslant Mt$.

Proof. First, suppose that $L_{\mathbf{T}}$ has 0 as an absorbing element. Since L is M-Lipschitz continuous, it holds that
$$1 - 0 = L_{\mathbf{T}}(\mathbf{1}) - L_{\mathbf{T}}(0,1,1,\ldots,1) \leqslant M(1-0) = M \,.$$

Hence, $M \geqslant 1$. Now consider the point $(0,1,1,\ldots 1)$, then
$$\begin{aligned} 0 &= L_{\mathbf{T}}(0,1,1,\ldots,1) \\ &= \sup\{d(t) - M\sum_{i=1}^{n}(t-x_i)^+ \mid t \in [0,1]\} \\ &= \sup\{d(t) - Mt \mid t \in [0,1]\} \,. \end{aligned}$$

Consequently, for any $t \in [0,1]$, it holds that $d(t) - Mt \leqslant 0$, i.e., $d(t) \leqslant Mt$. Conversely, suppose that $M \geqslant 1$ and for any $t \in [0,1]$ it holds that $d(t) \leqslant Mt$. Let $\mathbf{x} \in [0,1]^n$ be such that $x_{(1)} = 0$. Then, for any $t \in [0, x_{(2)}]$, it holds that
$$d(t) - M\sum_{i=1}^{n}(t-x_i)^+ = d(t) - M(t-0) = d(t) - Mt \leqslant 0 \,.$$

Now, for any $t \in [x_{(2)}, x_{(3)}]$, it holds that
$$d(t) - M\sum_{i=1}^{n}(t-x_i)^+ = d(t) - M(t-0) - M(t - x_{(2)}) \leqslant 0 \,.$$

By repeating this procedure, we easily deduce that for any $t \in [x_{(1)}, x_{(n)}]$, it holds that

$$d(t) - \sum_{i=2}^{n} M(t - x_i)^+ \leqslant 0\,,$$

Since $d(0) - \sum_{i=2}^{n} M(0 - x_i)^+ = 0$, we conclude that $L_{\mathbf{T}}(\mathbf{x}) = 0$. □

Example 6.5. In this example we show that the condition $d(t) \leqslant Mt$ cannot be weakened. Let $\mathbf{T} = (2, 2, d)$, where $d(t) = \min(4t, 1)$. Then, for any $y \in [0, 1]$ a simple computation shows that $L_{\mathbf{T}}(0, y) = \min(2y, 1)$. Hence, 0 is not an absorbing element of $L_{\mathbf{T}}$.

Now, we turn our attention to the existence of a neutral element.

Proposition 6.4. *Let $\mathbf{T} = (n, M, d)$. Then $L_{\mathbf{T}}$ has 1 as a neutral element if and only if one of the two following conditions holds:*

(1) $M = 1$ and $d(t) \leqslant t$ for any $t \in [0, 1]$;

(2) $M > 1$ and $d(t) = t$ for any $t \in [0, 1]$.

Proof. Using the same argument as in the case of Proposition 6.3, we can show that if $M < 1$, then 1 cannot be a neutral element of $L_{\mathbf{T}}$. First suppose that $M = 1$ and 1 is a neutral element of $L_{\mathbf{T}}$. Let $\mathbf{x}$ be such that $x_1 = 0$ and $x_i = 1$ for all $i \neq 1$, then

$$0 = \sup\{d(t) - \sum_{i=1}^{n} (t - x_i)^+ \mid t \in [0, 1]\} = \sup\{d(t) - t \mid t \in [0, 1]\}\,.$$

Hence, $d(t) \leqslant t$. For the converse, suppose that $d(t) \leqslant t$ for any $t \in [0, 1]$ and without loss of generality, consider the point $\mathbf{x} = (x, 1, 1, \ldots, 1)$, then

$$L_{\mathbf{T}}(\mathbf{x}) = \sup\{d(t) - \sum_{i=1}^{n} (t - x_i)^+ \mid t \in [x, 1]\} = \sup\{d(t) - t + x \mid t \in [x, 1]\} = x\,,$$

since $d(t) - t \leqslant 0$ and $d(1) - 1 = 0$.

Now, we study the case $M > 1$. First suppose that 1 is a neutral element of $L_{\mathbf{T}}$. Then for any $x \in [0, 1]$, it holds that

$$x = L_{\mathbf{T}}(x, 1, 1, \ldots, 1) = \sup\{d(t) - M(t - x) \mid t \in [x, 1]\} \geqslant d(x)\,.$$

Hence, $d(x) \leqslant x$. Now, for a fixed x, the function $d(t) - M(t - x)$ is continuous on the interval $[x, 1]$. Since this interval is compact, there exists $r_x \in [x, 1]$ such that

$$d(r_x) - M(r_x - x) = \sup\{d(t) - M(t - x) \mid t \in [x, 1]\}\,.$$

Therefore,

$$\begin{aligned} x &= L_{\mathbf{T}}(x,1,1,\ldots,1) \\ &= \sup\{d(t) - M(t-x) \mid t \in [x,1]\} \\ &= d(r_x) - M(r_x - x) \\ &\leqslant r_x - M(r_x - x)\,. \end{aligned}$$

Hence, $x \leqslant r_x - M(r_x - x)$, which is equivalent to $(M-1)x \geqslant (M-1)r_x$, i.e., $x \geqslant r_x$. This implies that $r_x = x$ and

$$\begin{aligned} x &= L_{\mathbf{T}}(x,1,1,\ldots,1) \\ &= d(r_x) - M(r_x - x) \\ &= d(x)\,. \end{aligned}$$

Consequently, $d(x) = x$ for any $x \in [0,1]$.

For the converse, suppose that $M > 1$ and $d(x) = x$. Without loss of generality, consider the point $\mathbf{x} = (x,1,1,\ldots,1)$, then

$$\begin{aligned} L_{\mathbf{T}}(x,1,1,\ldots,1) &= \sup\{t - M(t-x) \mid t \in [x,1]\} \\ &= \sup\{(1-M)t + Mx \mid t \in [x,1]\} \\ &= x\,, \end{aligned}$$

since $(1-M)t + Mx$ is a decreasing function of t and as consequence the minimum is reached at $t = x$. Hence, 1 is a neutral element of $L_{\mathbf{T}}$. □

Remark 6.1. As a consequence of Example 6.3, the only setting where it is possible to have 1 as a neutral element of $L_{\mathbf{T}}$ is when working with Lipschitz constant $M = 1$ and with diagonal sections of quasi-copulas, as condition (2) of Proposition 6.4 reduces to the minimum operator independently of the choice of M.

Example 6.6. In this example we show that the condition $d(t) \leqslant t$ cannot be weakened in the case $M = 1$. Let $\mathbf{T} = (n,1,d)$, where $d(t) = \min(2t,1)$. Then, for any $x \in [0,1]$, a simple computation shows that $L_{\mathbf{T}}(x,1,1,\ldots,1) = \min(x + \frac{1}{2}, 1)$. Hence, 1 is not a neutral element of $L_{\mathbf{T}}$.

The following corollary follows immediately from Propositions 6.3 and 6.4, and Remark 6.1.

Corollary 6.1. *The function $L_{\mathbf{T}}$ has 0 as an absorbing element and 1 as a neutral element if and only if it is a quasi-copula.*

6.4. Supermodularity and submodularity of the extremal functions

It is well known that for a diagonal function d satisfying $d(t) \leqslant t$, the smallest 1-Lipschitz continuous aggregation function with given diagonal section d is the Bertino n-quasi-copula $B_{d,n}$. In the bivariate case, the Bertino 2-quasi-copula is always a 2-copula. However, this result is not universal, in the sense that for $n \geqslant 3$, the Bertino n-quasi-copula $B_{d,n}$ is not always an n-copula.

We now show that that the smallest (resp. greatest) M-Lipschitz continuous n-ary aggregation function with a given diagonal section is supermodular (resp. submodular). As a consequence, we will show that the Bertino n-quasi-copula is supermodular for any $n \geqslant 2$, giving another example of a property of 2-copulas that cannot be generalized to higher-dimensional copulas, holds true for supermodular n-quasi-copulas.

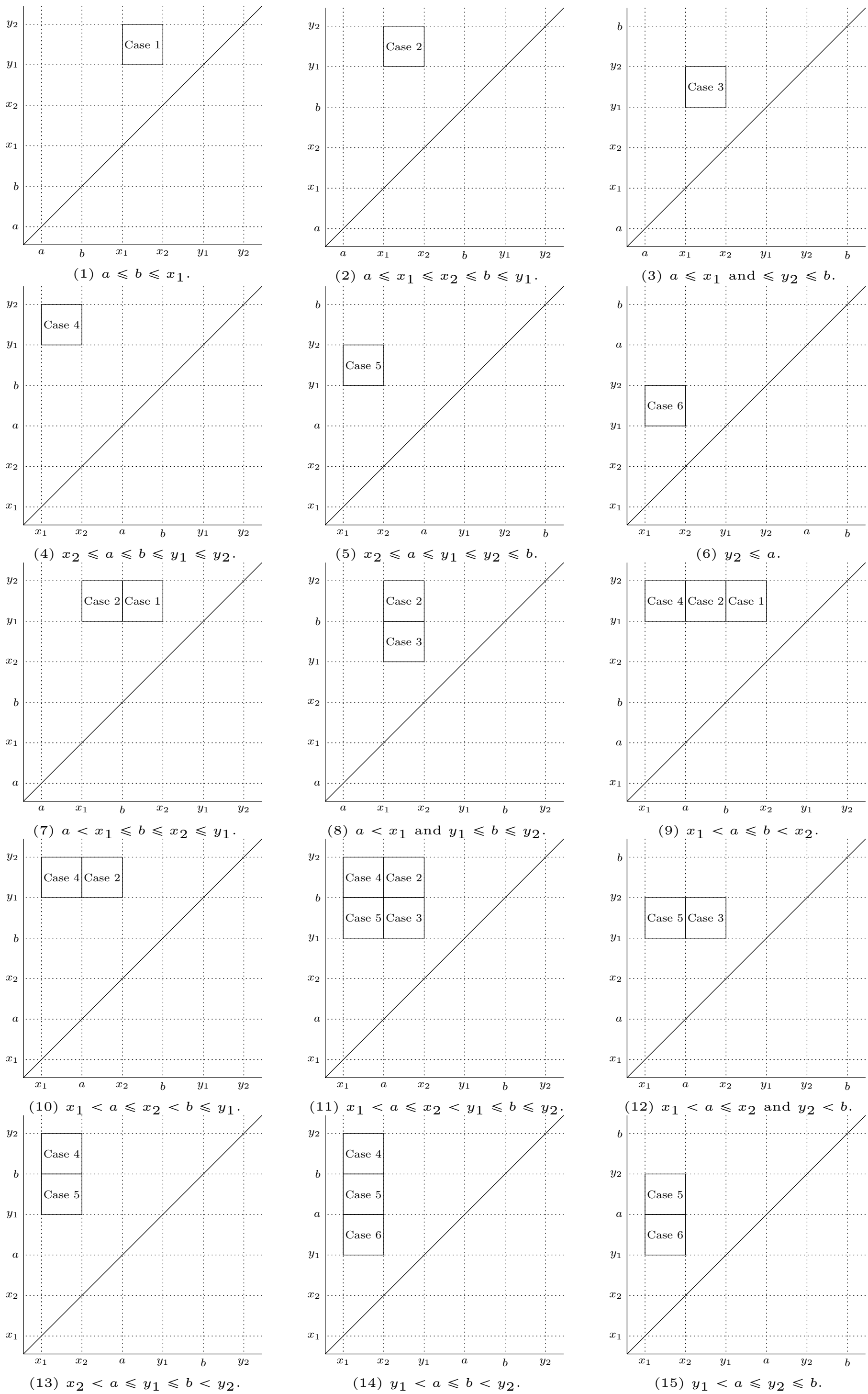
Theorem 6.2. *For any* $\mathbf{T} = (n, M, d)$*, the function* $U_{\mathbf{T}}$ *is submodular and the function* $L_{\mathbf{T}}$ *is supermodular.*

Proof. Due to Proposition 6.1, it suffices to show that $L_{\mathbf{T}}$ is supermodular. In view of Proposition 5.1 it suffices to show that the two-dimensional sections are supermodular. Without loss of generality, we only need to prove that for any $\mathbf{z} \in [0,1]^n$, the function $H : [0,1]^2 \to [0,1]$ defined by

$$H(x, y) = L_{\mathbf{T}}(x, y, z_3, z_4, \ldots, z_n)$$

is 2-increasing. Note that any 2-box contained in $[0,1]^2$ can be decomposed in boxes centred around the main diagonal (i.e., of the type $[x, y]^2$ with $x \leqslant y$), boxes above the main diagonal (i.e., of the type $[x_1, x_2] \times [y_1, y_2]$ with $x_1 \leqslant x_2 \leqslant y_1 \leqslant y_2$) or below the main diagonal (i.e., of the type $[x_1, x_2] \times [y_1, y_2]$ with $y_1 \leqslant y_2 \leqslant x_1 \leqslant x_2$). Let $a = \min(z_3, \ldots, z_n)$ and $b = \max(z_3, \ldots, z_n)$. Due to the symmetry of $L_{\mathbf{T}}$, we can suppose, without loss of generality, that $a = z_3$ and $b = z_4$.

We start by considering a box of the type $[x_1, x_2] \times [y_1, y_2]$ with $x_1 \leqslant x_2 \leqslant y_1 \leqslant y_2$. There are 15 subcases to consider, but most of them can be regarded as a consequence of the six main cases by using the additivity of the volume, as shown in Figure 6.1 (1)–(15).



(1) $a \leqslant b \leqslant x_1$. (2) $a \leqslant x_1 \leqslant x_2 \leqslant b \leqslant y_1$. (3) $a \leqslant x_1$ and $\leqslant y_2 \leqslant b$.

(4) $x_2 \leqslant a \leqslant b \leqslant y_1 \leqslant y_2$. (5) $x_2 \leqslant a \leqslant y_1 \leqslant y_2 \leqslant b$. (6) $y_2 \leqslant a$.

(7) $a < x_1 \leqslant b \leqslant x_2 \leqslant y_1$. (8) $a < x_1$ and $y_1 \leqslant b \leqslant y_2$. (9) $x_1 < a \leqslant b < x_2$.

(10) $x_1 < a \leqslant x_2 < b \leqslant y_1$. (11) $x_1 < a \leqslant x_2 < y_1 \leqslant b \leqslant y_2$. (12) $x_1 < a \leqslant x_2$ and $y_2 < b$.

(13) $x_2 < a \leqslant y_1 \leqslant b < y_2$. (14) $y_1 < a \leqslant b < y_2$. (15) $y_1 < a \leqslant y_2 \leqslant b$.

Figure 6.1: 2-boxes above the main diagonal

Case 1: Suppose that $a \leqslant b \leqslant x_1$. Since the functions

$$d(t) - M(t-a) - M(t-b)^+ - M(t-x_2)^+ - M\sum_{i=5}^{n}(t-z_j)^+$$

and

$$d(t) - M(t-a) - M(t-b)^+ - M(t-x_1)^+ - M\sum_{i=5}^{n}(t-z_j)^+$$

are continuous on the intervals $[a, y_1]$ and $[a, y_2]$, respectively, there exist two points $r \in [a, y_1]$ and $s \in [a, y_2]$ such that

$$\begin{aligned} H(x_2, y_1) &= \sup\{d(t) - M(t-a) - M(t-b)^+ - M(t-x_2)^+ \\ &\quad -M\sum_{i=5}^{n}(t-z_j)^+ \mid t \in [a, y_1]\} \\ &= d(r) - M(r-a) - M(r-b)^+ - M(r-x_2)^+ - M\sum_{i=5}^{n}(r-z_j)^+ , \end{aligned}$$

and

$$\begin{aligned} H(x_1, y_2) &= \sup\{d(t) - M(t-a) - M(t-b)^+ - M(t-x_1)^+ \\ &\quad -M\sum_{i=5}^{n}(t-z_j)^+ \mid t \in [a, y_2]\} \\ &= d(s) - M(s-a) - M(s-b)^+ - M(s-x_1)^+ - M\sum_{i=5}^{n}(s-z_j)^+ . \end{aligned}$$

Define the auxiliary function $F_{r,s} : [a, y_1] \times [a, y_2] \to \mathbb{R}$ as

$$\begin{aligned} F_{r,s}(t_1, t_2) &= d(t_2) - M(t_2-a)^+ - M(t_2-b)^+ - M(t_2-x_2)^+ - M(t_2-y_2)^+ \\ &\quad -M\sum_{i=5}^{n}(t_2-z_j)^+ - H(x_2, y_1) - H(x_1, y_2) + d(t_1) - M(t_1-a) \\ &\quad -M(t_1-b)^+ - M(t_1-x_1)^+ - M(t_1-y_1)^+ \\ &\quad -M\sum_{i=5}^{n}(t_1-z_j)^+ . \end{aligned} \tag{6.6}$$

Note that for any $(t_1, t_2) \in [a, y_1] \times [a, y_2]$, it holds that

$$F_{r,s}(t_1, t_2) \leqslant H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1) . \tag{6.7}$$

The previous inequality is justified by taking the supremum over all possible values of t_1 and t_2. Hence, it suffices to show that there exist t_1, t_2 such that $F_{r,s}(t_1, t_2) \geqslant 0$. Since $a \leqslant b \leqslant x_1$, $F_{r,s}(t_1, t_2)$ can be rewritten as

$$\begin{aligned} F_{r,s}(t_1,t_2) \;=\; & d(t_2) - Mt_2 + d(t_1) - Mt_1 - M(t_2-b)^+ - M(t_2-x_2)^+ \\ & -M\sum_{i=5}^{n}(t_2-z_j)^+ - M(t_1-b)^+ - M(t_1-x_1)^+ - M\sum_{i=5}^{n}(t_1-z_j)^+ \\ & +Mr - d(r) + Ms - d(s) + M(r-b)^+ + M(r-x_2)^+ + M\sum_{i=5}^{n}(r-z_j)^+ \\ & +M(s-b)^+ + M(s-x_1)^+ + M\sum_{i=5}^{n}(s-z_j)^+ \,. \end{aligned}$$

We now analyse two possible subcases. First, if $s \in [a, y_1]$, then $(s, r) \in [a, y_1] \times [a, y_2]$. Hence, $F_{r,s}(s, r)$ is well defined and after a simple computation, we get $F_{r,s}(r, s) = 0$ and from Eq. (6.7), it follows that

$$H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0\,.$$

On the other hand, if $s \in]y_1, y_2]$, then $(r, s) \in [a, y_1] \times [a, y_2]$ and after a simple computation, we get

$$F_{r,s}(r,s) = Mx_2 - Mx_1 + M(r-x_2)^+ - M(r-x_1)^+ = \begin{cases} M(x_2-x_1) & \text{, if } r < x_1, \\ M(r-x_1) & \text{, if } x_1 \leqslant r < x_2, \\ 0 & \text{, if } r \geqslant x_2. \end{cases}$$

Hence, $F_{r,s}(r, s) \geqslant 0$ and from Eq. (6.7), we obtain

$$H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0\,.$$

Case 2: The case $a \leqslant x_1 \leqslant x_2 \leqslant b \leqslant y_1$ can be proven using similar arguments as the ones given in Case 1.

Case 3: Suppose that $a \leqslant x_1$ and $y_2 \leqslant b$. Similarly as in Case 1, define the points $r, s \in [a, b]$ as the points such that the following equalities hold

$$H(x_2,y_1) = d(r) - M(r-a) - M(r-x_2)^+ - M(r-y_1)^+ - M\sum_{i=5}^{n}(r-z_j)^+\,,$$

$$H(x_1,y_2) = d(s) - M(s-a) - M(s-x_1)^+ - M(s-y_2)^+ - M\sum_{i=5}^{n}(s-z_j)^+\,.$$

Define the function $F_{r,s}$ as in Eq. (6.6), but now by considering the 2-box $[a, b]^2$ as its domain. Then

$$\begin{aligned}F_{r,s}(t_1,t_2) &= d(t_2) - Mt_2 + d(t_1) - Mt_1 - M(t_2 - x_2)^+ - M(t_2 - y_2)^+ \\ &\quad -M\sum_{i=5}^{n}(t_2 - z_j)^+ - M(t_1 - x_1)^+ - M(t_1 - y_1)^+ - M\sum_{i=5}^{n}(t_1 - z_j)^+ \\ &\quad +Mr - d(r) + Ms - d(s) + M(r - x_2)^+ + M(r - y_1)^+ + M\sum_{i=5}^{n}(r - z_j)^+ \\ &\quad +M(s - x_1)^+ + M(s - y_2)^+ + M\sum_{i=5}^{n}(s - z_j)^+ .\end{aligned}$$

We have to consider two subcases. First, if $r \geqslant s$, then consider the point $(s, r) \in [a, b]^2$. After a simple computation, we obtain the following simplified expression for $F_{r,s}(r, s)$

$$F_{r,s}(r,s) = M(r - y_1)^+ - M(r - y_2)^+ + M(s - y_2)^+ - M(s - y_1)^+ .$$

The latter expression can be rewritten as

$$F_{r,s}(r,s) = \begin{cases} 0 & \text{, if } r < y_1, \\ M(r - y_1) & \text{, if } s < y_1 \leqslant r < y_2, \\ M(r - s) & \text{, if } y_1 \leqslant s \leqslant r < y_2 < x_2, \\ M(y_2 - y_1) & \text{, if } s < y_1 \leqslant y_2, \leqslant r \\ M(y_2 - s) & \text{, if } y_1 \leqslant s < y_2 < r, \\ 0 & \text{, if } y_2 \leqslant s \leqslant r. \end{cases}$$

Hence, $H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1) \geqslant F_{r,s}(r, s) \geqslant 0$. The subcase when $s \geqslant r$ can be analogously proven by considering the point (r, s).

Case 4: Suppose that $x_2 \leqslant a$ and $b \leqslant y_1$. Similarly as in the previous cases, define the points $r \in [x_2, y_1]$ and $s \in [x_1, y_2]$ such that the following equalities hold

$$H(x_2, y_1) = d(r) - M(r - x_2) - M(r - a)^+ - M(r - b)^+ - M\sum_{i=5}^{n}(r - z_j)^+ ,$$

$$H(x_1, y_2) = d(s) - M(s - x_1) - M(s - a)^+ - M(s - b)^+ - M\sum_{i=5}^{n}(s - z_j)^+ .$$

Define the function $F_{r,s}$ as in Eq. (6.6), but now with the 2-box $[x_1, y_1] \times [x_2, y_2]$

as its domain. Then

$$\begin{aligned}F_{r,s}(t_1,t_2) \;=\; & d(t_2) - Mt_2 + d(t_1) - Mt_1 - M(t_2-a)^+ - M(t_2-b)^+ \\ & -M\sum_{i=5}^{n}(t_2-z_j)^+ - M(t_1-a)^+ - M(t_1-b)^+ - M\sum_{i=5}^{n}(t_1-z_j)^+ \\ & +Mr - d(r) + Ms - d(s) + M(r-a)^+ + M(r-b)^+ + M\sum_{i=5}^{n}(r-z_j)^+ \\ & +M(s-a)^+ + M(s-b)^+ + M\sum_{i=5}^{n}(s-z_j)^+ .\end{aligned}$$

We now analyse two possible subcases. First, if $s \in]y_1, y_2]$, then $(r,s) \in [x_1, y_1] \times [x_2, y_2]$ and $F_{r,s}(r,s) = 0$. Hence, $H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0$.

Second if $s \in [x_1, y_1]$, then $(s,r) \in [x_1,y_1] \times [x_2,y_2]$ and $F_{r,s}(r,s) = 0$. Hence, $H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0$.

Case 5: Suppose that $x_2 \leqslant a$ and $y_2 \leqslant b$. Similarly as in the previous cases, define the points $r \in [x_2, b]$ and $s \in [x_1, b]$ as the points such that the following equalities hold

$$H(x_2,y_1) = d(r) - M(r-x_2) - M(r-a)^+ - M(r-y_1)^+ - M\sum_{i=5}^{n}(r-z_j)^+ ,$$

$$H(x_1,y_2) = d(s) - M(s-x_1) - M(s-a)^+ - M(s-y_2)^+ - M\sum_{i=5}^{n}(s-z_j)^+ .$$

Define the function $F_{r,s}$ as in Eq. (6.6), but now with the 2-box $[x_1,b] \times [x_2,b]$ as its domain. Then

$$\begin{aligned}F_{r,s}(t_1,t_2) \;=\; & d(t_2) - Mt_2 + d(t_1) - Mt_1 - M(t_2-B)^+ - M(t_2-y_2)^+ \\ & -M\sum_{i=5}^{n}(t_2-z_j)^+ - M(t_1-b)^+ - M(t_1-y_1)^+ - M\sum_{i=5}^{n}(t_1-z_j)^+ \\ & +Mr - d(r) + Ms - d(s) + M(r-b)^+ + M(r-y_1)^+ + M\sum_{i=5}^{n}(r-z_j)^+ \\ & +M(s-b)^+ + M(s-y_1)^+ + M\sum_{i=5}^{n}(s-z_j)^+ .\end{aligned}$$

Once again, we have to analyse two subcases. First, if $s \in [x_2, b]$, then $(r,s) \in [x_1,b] \times [x_2,b]$ and $F_{r,s}(r,s) = 0$. Hence, $H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0$.

Second, if $s \in [x_1, x_2[$, then $(s, r) \in [x_1, b] \times [x_2, b]$, and

$$F_{r,s}(s,r) = M(r-y_1)^+ - M(r-y_2)^+ = \begin{cases} 0 & \text{, if } r < y_1, \\ M(r-y_1) & \text{, if } y_1 \leqslant r < y_2, \\ M(y_2-y_1) & \text{, if } r \geqslant y_2. \end{cases}$$

Hence, $H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1) \geqslant 0$.

Case 6: The case $y_2 \leqslant a$ can be proven using similar arguments as the ones given in Case 5.

We now prove that a box of the type $[x, y]^2$ with $x \leqslant y$ has a positive H-volume. There are 9 further subcases to consider, however, as in the case of asymmetric boxes, most of them can be regarded as a consequence of the three main cases, as shown in Figure 6.2 (16)–(21).

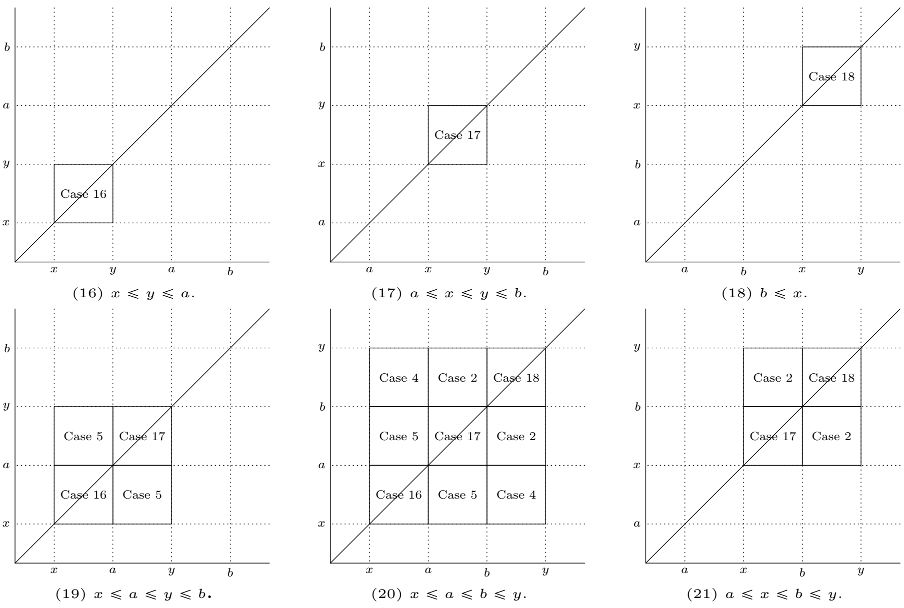


Figure 6.2: 2-boxes centred around the main diagonal

Case 16: Suppose that $x \leqslant y \leqslant a$. Let $r \in [x, b]$ be the point such that the following equality holds

$$H(x,y) = H(y,x) = d(r) - M(r-x) - M(r-a)^+ - M(r-y)^+ - M\sum_{i=5}^{n}(r-z_j)^+ .$$

Define the function $F_r : [x,b] \times [y,b] \to \mathbb{R}$ as

$$\begin{aligned} F_r(t_1,t_2) &= d(t_2) - 2M(t_2-a)^+ - 2M(t_2-b)^+ - 2M(t_2-y)^+ \\ &\quad -2M\sum_{i=5}^{n}(t_2-z_j)^+ - 2H(x,y) + d(t_1) - 2M(t_1-a)^+ \\ &\quad -2M(t_1-b)^+ - 2M(t_1-x)^+ - 2M\sum_{i=5}^{n}(t_1-z_j)^+ . \end{aligned}$$

Clearly, for any $(t_1,t_2) \in [x,b]\times[y,b]$, it holds that $F_r(t_1,t_2) \leqslant H(y,y) - H(x,y) - H(y,x) + H(x,x)$. Since $x \leqslant y \leqslant a$, $F_r(t_1,t_2)$ can be rewritten as

$$\begin{aligned} F_r(t_1,t_2) &= 2My + d(t_2) - 2Mt_2 + d(t_1) - 2Mt_1 - M(t_2-a)^+ \\ &\quad -M\sum_{i=5}^{n}(t_2-z_j)^+ - M(t_1-a)^+ - M\sum_{i=5}^{n}(t_1-z_j)^+ \\ &\quad +2Mr - 2d(r) + 2M(r-y)^+ + 2M(r-a)^+ + 2M\sum_{i=5}^{n}(r-z_j)^+ . \end{aligned}$$

Once again, we have to analyse two subcases. First, if $r \in [y,b]$, then $(r,r) \in [x,b]\times[y,b]$ and $F_r(r,r) = 0$. Hence, $H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0$.

Second, if $r \in [x,y[$, then $(r,y) \in [x,b]\times[y,b]$ and $F_r(r,y) = d(y) - d(r) \geqslant 0$, since d is increasing. Hence, $H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0$.

Case 17: Suppose that $a \leqslant x \leqslant y \leqslant b$. Let $r \in [a,b]$ be the point such that the following equality holds

$$H(x,y) = H(y,x) = d(r) - M(r-a) - M(r-x)^+ - M(r-y)^+ - M\sum_{i=5}^{n}(r-z_j)^+ .$$

Define the function $F_{r,s}$ as in Eq. (6.8), but now considering the 2-box $[a,b]^2$ as its domain. Then

$$\begin{aligned} F_r(t_1,t_2) &= d(t_2) - Mt_2 + d(t_1) - Mt_1 - 2M(t_2-y)^+ \\ &\quad -M\sum_{i=5}^{n}(t_2-z_j)^+ - 2M(t_1-x)^+ - M\sum_{i=5}^{n}(t_1-z_j)^+ \\ &\quad +2Mr - 2d(r) + 2M(r-x)^+ + 2M(r-y)^+ + 2M\sum_{i=5}^{n}(r-z_j)^+ . \end{aligned}$$

Note that $(r,r) \in [a,b]^2$ and that $F_r(r,r) = 0$. Hence, $H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0$.

Case 18: Suppose that $b \leqslant x$. Let $r \in [a, y]$ be the point such that the following equality holds

$$H(x,y) = H(y,x) = d(r) - M(r-a) - M(r-x)^+ - M(r-y)^+ - M\sum_{i=5}^{n}(r-z_j)^+ .$$

Define the function $F_{r,s}$ as in Eq. (6.8), but now considering the 2-box $[a,b]^2$ as its domain. Then

$$\begin{aligned}
F_r(t_1,t_2) &= d(t_2) - Mt_2 + d(t_1) - Mt_1 - M(t_2-b)^+ \\
&\quad -M\sum_{i=5}^{n}(t_2-z_j)^+ - M(t_1-b)^+ - M\sum_{i=5}^{n}(t_1-z_j)^+ \\
&\quad +2Mr - 2d(r) + 2M(r-b)^+ + 2M(r-x)^+ + 2M\sum_{i=5}^{n}(r-z_j)^+ .
\end{aligned}$$

Once again, we have to analyse two subcases. First, if $r \in [a, x[$, then $(r,r) \in [a,x] \times [a,y]$ and $F_r(r,r) = 0$. Hence, $H(x_2,y_2) - H(x_2,y_1) - H(x_1,y_2) + H(x_1,y_1) \geqslant 0$.

Second, if $r \in [x,y]$, then $(x,r) \in [a,x] \times [a,y]$ and

$$\begin{aligned}
F_r(x,r) &= d(x) - d(r) + 4M(r-x) + M\sum_{i=5}^{n}(r-z_j)^+ - M\sum_{i=5}^{n}(x-z_j)^+ \\
&= d(x) - d(r) + 4M(r-x) + M\sum_{i=5}^{n}(r-z_j) - M\sum_{i=5}^{n}(x-z_j) \\
&= d(x) - d(r) + 4M(r-x) + M\sum_{i=5}^{n}(r-z_j) - M\sum_{i=5}^{n}(r-x) \\
&= d(x) - d(r) + nM(r-x) + M\sum_{i=5}^{n}(r-z_j) \\
&\geqslant 0 ,
\end{aligned}$$

where the first inequality follows from the inequalities $r \geqslant x \geqslant b \geqslant z_j$ and the last inequality is justified by the nM-Lipschitz continuity of d. $\square$

Corollary 6.2. *For any $n \geqslant 2$, the Bertino n-quasi-copula with diagonal section d, given by*

$$B_{d,n}(\mathbf{x}) = \sup\{d(t) - \sum_{i=1}^{n}(t-x_i)^+ \mid t \in [0,1]\} ,$$

is supermodular.

Remark 6.2. In this remark we show that the "natural" n-dimensional extension of the diagonal copula [154] given by

$$D_n(\mathbf{x}) = \min\left(M_n(\mathbf{x}), \frac{1}{n}\sum_{i=1}^{n} d(x_i)\right)$$

is not supermodular, in general. To see this, consider the case $n = 3$ and $d(t) = t^3$. Clearly, $d(t)$ is the diagonal section of a 3-copula. Consider the points $\mathbf{x} = \left((\frac{5}{8})^{1/3}, (\frac{3}{4})^{1/3}, \frac{1}{24}\right)$ and $\mathbf{y} = \left((\frac{3}{4})^{1/3}, (\frac{5}{8})^{1/3}, \frac{1}{24}\right)$, then

$$D_n(\mathbf{x} \vee \mathbf{y}) - D_n(\mathbf{x}) - D_n(\mathbf{y}) + D_n(\mathbf{x} \wedge \mathbf{y}) = \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{11}{24} = -\frac{1}{24} < 0\,.$$

Note also that as a consequence, one of the results of [32], in which it is shown that a 2-copula can be constructed by truncating an appropriate modular function on the unit square with the upper Fréchet-Hoeffding bound, does not extend to higher dimensions, since the function $\frac{1}{n}\sum_{i=1}^{n} d(x_i)$ is modular, but by truncating it from above with the upper Fréchet-Hoeffding bound, the result is D_n, which is not a supermodular function. In Chapter 7 we will show that the other result of [32], in which it is shown that a 2-copula can be constructed by truncating an appropriate modular function on the unit square with the lower Fréchet-Hoeffding bound, extends to higher dimensions for supermodular functions.

6.5. Some analytical properties of the multivariate Bertino quasi-copula

We now change our attention to the Bertino n-quasi-copula. First, we present some properties of the Bertino n-quasi-copula, with special emphasis on the behaviour of its k-marginals, and on its diagonal section.

First, since the Bertino n-quasi-copula is symmetric, it follows that its k-marginals $H_k : [0,1]^k \to [0,1]$, $k \in \{2, 3, ..., n-1\}$, coincide and are given by

$$H_k(x_1, x_2, ..., x_k) = \sup\left\{ d(t) - \sum_{j=1}^{k} (t - x_j)^+ \mid t \in \left[x_{(1)}, 1\right] \right\}\,.$$

The diagonal section h_k of the k-marginal H_k of $B_{d,n}$ is given by

$$h_k(x) = \sup\left\{ d(t) - k(t-x) \mid t \in [x, 1] \right\}\,.$$

We now analyse the behaviour of h_k when d is k-Lipschitz continuous.

Proposition 6.5. *If d is k-Lipschitz continuous, then $h_k = d$.*

Proof. For any $t \in [x, 1]$, it holds that $d(t) - k(t-x) \leqslant d(x)$ due to the k-Lipschitz continuity. Evaluating $d(t) - k(t-x)$ at $t = x$, we get $d(x)$. From this, it follows that $\sup\{d(t) - k(t-x) \mid t \in [x,1]\} = d(x)$. $\square$

Example 6.7. Consider the n-diagonal function $d(x) = x^2$, then the associated Bertino n-quasi-copula is for any $n \geqslant 2$ given by

$$B_{d,n}(\mathbf{x}) = \max(x_{(1)}^2, x_{(2)}^2 - x_{(2)} + x_{(1)})\,.$$

By setting $x = x_{(1)} = x_{(2)}$ and $x_{(j)} = 1$ for $j \geqslant 3$, it is clear that $h_2(x) = x^2$.

Proposition 6.6. *If the diagonal section d of a Bertino n-quasi-copula $B_{d,n}$ is such that $d(x) \leqslant (kx-(k-1))^+$, then the diagonal section h_k of its k-marginals is given by $h_k(x) = (kx-(k-1))^+$.*

Proof. Let $t \in [x,1]$. If $t \leqslant (k-1)/k$, then

$$d(t) - k(t-x) = -k(t-x) \leqslant 0\,.$$

If t is such that $t > (k-1)/k$, then

$$d(t) - k(t-x) \leqslant kt - (k-1) - k(t-x) = kx - (k-1)\,.$$

Hence, for all $t \in [x,1]$, it holds that $d(t) - k(t-x) \leqslant (kx-(k-1))^+$. From this and property (d2), it follows that $h_k(x) = (kx-(k-1))^+$. $\square$

Example 6.8. Consider the n-diagonal function $d(x) = (\alpha x - \alpha + 1)^+$, with $\alpha \in]n-1, n]$. Clearly, $d(x) \leqslant (kx-(k-1))^+$ for any integer $k < n$. After some simple computations, we obtain the following expression for the associated Bertino n-quasi-copula

$$B_{d,n}(\mathbf{x}) = \left((\alpha - n + 1)x_{(n)} + \left(\sum_{j=1}^{n-1} x_{(j)}\right) - \alpha + 1\right)^+\,.$$

Setting $n-k$ arguments equal to 1, it follows that

$$H_k(\mathbf{x}) = \left(\sum_{j=1}^{k} x_{(j)} - k + 1\right)^+\,,$$

with diagonal section $h_k(x) = (kx-(k-1))^+$.

Examples 6.7 and 6.8 suggest that the k-marginal of a Bertino n-quasi-copula might be the Bertino k-quasi-copula associated with the k-diagonal function h_k. This is indeed true, as the following theorem shows.

Theorem 6.3. *The k-marginal of a Bertino n-quasi-copula $B_{d,n}$ is the Bertino k-quasi-copula $B_{h_k,k}$, with h_k the diagonal section of the k-marginal H_k of $B_{d,n}$.*

Proof. We need to show that for any $\mathbf{x} \in [0,1]^k$ it holds that $B_{h_k,k}(\mathbf{x}) = H_k(\mathbf{x})$, i.e.,

$$\sup\{h_k(t) - \sum_{j=1}^{k}(t - x_{(j)})^+ \mid t \in [x_{(1)}, x_{(k)}]\} = \sup\{d(s) - \sum_{j=1}^{k}(s - x_{(j)})^+ \mid s \in [x_{(1)}, 1]\}\,.$$

First, we prove that $B_{h_k,k}(\mathbf{x}) \leqslant H_k(\mathbf{x})$. Let $t \in [x_{(1)}, x_{(k)}]$ and let $m_0 = \min\{m \in \{1, 2..., k\} \mid x_{(m)} \geqslant t\}$, then

$$\begin{aligned} h_k(t) - \sum_{j=1}^{k}(t - x_{(j)})^+ &= \sup\{d(r) - k(r-t) \mid r \in [t,1]\} - \sum_{j=1}^{m_0-1}(t - x_{(j)}) \\ &= \sup\{d(r) - kr \mid r \in [t,1]\} + (k - m_0 + 1)t + \sum_{j=1}^{m_0-1} x_{(j)}\,. \end{aligned}$$

Since $[t,1]$ is compact, the continuous function $d(r) - kr$ attains a maximum in $[t,1]$. Assume that this maximum is attained at r^*, then we have

$$d(r^*) - kr^* + (k - m_0 + 1)t + \sum_{j=1}^{m_0-1} x_{(j)} = d(r^*) - \sum_{j=m_0}^{k}(r^* - t) - \sum_{j=1}^{m_0-1}(r^* - x_{(j)})\,.$$

Now observe that for any $m \in \{m_0+1, m_0+2, ..., k\}$, if $r^* \leqslant x_{(m)}$, then $-(r^* - t) \leqslant 0 = -(r^* - x_{(m)})^+$, and, if $r^* > x_{(m)}$, then $-(r^* - t) \leqslant -(r^* - x_{(m)}) = -(r^* - x_{(m)})^+$. Hence,

$$d(r^*) - \sum_{j=m_0}^{k}(r^* - t) - \sum_{j=1}^{m_0-1}(r^* - x_{(j)}) \leqslant d(r^*) - \sum_{j=1}^{k}(r^* - x_{(j)})^+\,,$$

from which it follows that

$$\begin{aligned} h_k(t) - \sum_{j=1}^{k}(t - x_{(j)})^+ &\leqslant d(r^*) - \sum_{j=1}^{k}(r^* - x_{(j)})^+ \\ &\leqslant \sup\{d(s) - \sum_{j=1}^{k}(s - x_{(j)})^+ \mid s \in [t,1]\} \\ &\leqslant \sup\{d(s) - \sum_{j=1}^{k}(s - x_{(j)})^+ \mid s \in [x_{(1)},1]\}\,. \end{aligned}$$

Hence,

$$\sup\{h_k(t)-\sum_{j=1}^{k}(t-x_{(j)})^+ \mid t\in[x_{(1)},x_{(k)}]\} \leqslant \sup\{d(s)-\sum_{j=1}^{k}(s-x_{(j)})^+ \mid s\in[x,1]\}\,.$$

Second, we prove that $B_{h_k,k}(\mathbf{x}) \geqslant H_k(\mathbf{x})$. Let $s\in[x_{(1)},1]$. We have to analyze two cases. If $s\in[x_{(1)},x_{(k)})$, then

$$\begin{aligned}
d(s)-\sum_{j=1}^{k}(s-x_{(j)})^+ &= d(s)-k(s-s)-\sum_{j=1}^{k}(s-x_{(j)})^+ \\
&\leqslant \sup\{d(r)-k(r-s) \mid r\in[s,1]\}-\sum_{j=1}^{k}(s-x_{(j)})^+ \\
&\leqslant h_k(s)-\sum_{j=1}^{k}(s-x_{(j)})^+ \\
&\leqslant \sup\{h_k(t)-\sum_{j=1}^{k}(t-x_{(j)})^+ \mid t\in[x_{(1)},x_{(n)}]\}\,.
\end{aligned}$$

Next, if $s\in[x_{(k)},1]$, then

$$\begin{aligned}
d(s)-\sum_{j=1}^{k}(s-x_{(j)})^+ &= d(s)-k(s-x_{(k)})-\sum_{j=1}^{k}(x_{(k)}-x_{(j)})^+ \\
&\leqslant h_k(x_{(k)})-\sum_{j=1}^{k}(x_{(k)}-x_{(j)})^+ \\
&\leqslant \sup\{h_k(t)-\sum_{j=1}^{k}(t-x_{(j)})^+ \mid t\in[x_{(1)},x_{(n)}]\}\,.
\end{aligned}$$

Hence, for any $s\geqslant x_{(1)}$, it holds that

$$d(s)-\sum_{j=1}^{k}(s-x_{(j)})^+ \leqslant \sup\{h_k(t)-\sum_{j=1}^{k}(t-x_{(j)})^+ \mid t\in[x_{(1)},x_{(n)}]\}\,,$$

from which it follows that

$$\sup\{d(s)-\sum_{j=1}^{k}(s-x_{(j)})^+ \mid s\in[x_{(1)},1]\} \leqslant \sup\{h_k(t)-\sum_{j=1}^{k}(t-x_{(j)})^+ \mid t\in[x_{(1)},x_{(n)}]\}\,.$$

$\square$

Example 6.9. Consider the n-diagonal function of Example 6.8. Clearly, the k-marginals H_k coincide with the lower Fréchet bound W_k.

Example 6.10. Consider the 2-diagonal function given by $h_2(x) = \max(x^3, 2x-1)$. The associated Bertino 2-copula is given by

$$B_{h_2,2}(x_1, x_2) = \max(x_{(1)}^3, x_{(2)}^3 - x_{(2)} + x_{(1)}, x_{(1)} + x_{(2)} - 1) .$$

Now consider the 3-diagonal function $d(x) = x^3$, whose associated Bertino 3-quasi-copula is given by

$$B_{d,3}(x_1, x_2, x_3) = \max(x_{(1)}^3, x_{(2)}^3 - x_{(2)} + x_{(1)}, x_{(3)}^3 - 2x_{(3)} + x_{(2)} + x_{(1)}) .$$

One can easily verify that its bivariate marginal is the Bertino 2-copula associated with h_2.

For the next results, we need to recall the concept and some properties of Dini derivatives (for other properties of Dini derivatives, the reader is referred to [89]). For any real-valued function defined on an open interval $]x, y[$, the upper right and lower right Dini derivatives at $t \in]x, y[$ are defined as

$$D^+f(t) = \limsup_{h\to 0^+} \frac{f(t+h) - f(t)}{h} \qquad D_+f(t) = \liminf_{h\to 0^+} \frac{f(t+h) - f(t)}{h} ,$$

while the upper left and lower left Dini derivatives are defined as

$$D^-f(t) = \limsup_{h\to 0^+} \frac{f(t) - f(t-h)}{h} \qquad D_-f(t) = \liminf_{h\to 0^+} \frac{f(t) - f(t-h)}{h} .$$

Clearly $D_+f(t) \leqslant D^+f(t)$ and $D_-f(t) \leqslant D^-f(t)$. A function is differentiable at t if and only if the four Dini derivatives are finite and equal, with common value the 'classical' derivative.

If a function is Lipschitz continuous with Lipschitz constant M, then the four Dini derivatives are finite and bounded by M. Conversely, if one of the Dini derivatives of a continuous function is bounded in an open interval $]a, b[$, then the function is Lipschitz continuous on $]a, b[$.

There also exist some results that characterize the monotonicity of a function in terms of its Dini derivatives. Indeed, if a function is continuous on an open interval $]a, b[$, then if for all $t \in]a, b[$ at least one $D^+f(t), D_+f(t), D^-f(t)$ or $D_-f(t)$ is (strictly) positive, then the function is (strictly) increasing on $]a, b[$.

Remark 6.3. Theorem 6.3 states that if the diagonal section d of a Bertino n-quasi-copula is 2-Lipschitz continuous, then all the k-marginals, with $k \in \{2, 3, ..., n-1\}$, have the same diagonal section d. In that case it is relatively simple to compute the Bertino n-quasi-copula by using Proposition 6.2. However, we now obtain the same result by using Dini derivatives. Note that the function

$$q(t) = d(t) - \sum_{j=1}^{n}(t - x_j)^+$$

is continuous and has upper right Dini derivative of the form $D^+d(t) - m$, with m a positive integer depending on the position of t. From this, it follows that $q(t)$ is strictly decreasing on $[x_{(3)}, x_{(n)}]$, since for those values of t we have $m \geqslant 3$, while for $t \in]x_{(2)}, x_{(3)}]$, we have $m = 2$ and the function is either decreasing or constant there. Hence the maximum of $q(t)$ is attained in $[x_{(1)}, x_{(2)}]$. This proves that $B_{d,n}(\mathbf{x})$ coincides with the value of the 2-dimensional Bertino copula (with diagonal section d) in the point $(x_{(1)}, x_{(2)})$, i.e., if d is 2-Lipschitz continuous, then $B_{d,n}(\mathbf{x}) = B_{d,2}(x_{(1)}, x_{(2)})$.

We conclude this section with a nice property expressing the stability of the Bertino n-quasi-copulas when considering limits of diagonal sections. This property will be useful further on when we will approximate an n-diagonal function by a piecewise linear function.

Proposition 6.7. *Consider a sequence $(d_m)_{m=1}^{\infty}$ of n-diagonal functions converging pointwisely to a function d. Then d is an n-diagonal function and the sequence $(B_{d_m,n})_{m=1}^{\infty}$ converges pointwisely to $B_{d,n}$.*

Proof. Since the n-diagonal functions d_m, $m = 1, 2, ...$, are all n-Lipschitz continuous, $(d_m)_{m=1}^{\infty}$ is a family of equicontinuous functions. Hence, due to the Arzela-Ascoli theorem (see [174]), the convergence must be uniform. Therefore, the limit function d is n-Lipschitz continuous and therefore also an n-diagonal function. Consider $\mathbf{x} \in [0,1]^n$ and $t \in [x_{(1)}, x_{(n)}]$. By definition, it holds that

$$d_m(t) - \sum_{j=1}^{n}(t - x_j)^+ \leqslant B_{d_m,n}(\mathbf{x})\,.$$

Hence,

$$d(t) - \sum_{j=1}^{n}(t - x_j)^+ \leqslant \liminf_{m\to\infty} B_{d_m,n}(\mathbf{x})\,.$$

Since this inequality holds for all $t \in [x_{(1)}, x_{(n)}]$, we have

$$B_{d,n}(\mathbf{x}) \leqslant \liminf_{m\to\infty} B_{d_m,n}(\mathbf{x})\,.$$

Now, for any $\epsilon > 0$, we know that there exists an integer N such that $d_m(t) < d(t)+\epsilon$ for all $m > N$ and all $t \in [0,1]$. Thus for any $t \in [x_{(1)}, x_{(n)}]$, the following double inequality holds

$$d_m(t) - \sum_{j=1}^{n}(t-x_j)^+ < d(t) - \sum_{j=1}^{n}(t-x_j)^+ + \epsilon \leqslant B_{d,n}(\mathbf{x}) + \epsilon\,.$$

Hence,

$$B_{d_m,n}(\mathbf{x}) \leqslant B_{d,n}(\mathbf{x}) + \epsilon\,.$$

This last inequality implies that

$$\limsup_{m\to\infty} B_{d_m,n}(\mathbf{x}) \leqslant B_{d,n}(\mathbf{x}) + \epsilon\,,$$

for any $\epsilon > 0$. Hence, by taking the limit $\epsilon \to 0$ we obtain the desired result

$$\limsup_{m\to\infty} B_{d_m,n}(\mathbf{x}) \leqslant B_{d,n}(\mathbf{x})\,.$$

□

6.6. When the Bertino quasi-copula is a copula

Finally, we identify conditions that guarantee that the Bertino n-quasi-copula is an n-copula. To this end, we recall a result that can be found in [166]. Here, Preiss et al. constructed a particular Lipschitz-continuous function that serves to illustrate the concept of Clarke derivative of a function. As a by-product of their analysis, they have shown that for any Lipschitz-continuous function f, both differences $D^+f(t) - D_+f(t)$ and $D^-f(t) - D_-f(t)$ are bounded by

$$\frac{1}{2}\left(\sup\{D^+f(t) \mid t \in \mathbb{R}\} - \inf\{D^+f(t) \mid t \in \mathbb{R}\}\right)\,.$$

As a consequence of this bound, we have the following result.

Lemma 6.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing M-Lipschitz continuous function. Then for all $t \in \mathbb{R}$ it holds that*

$$D^+f(t) - D_+f(t) \leqslant M/2 \quad \textit{and} \quad D^-f(t) - D_-f(t) \leqslant M/2\,.$$

This result can easily be adapted for functions f defined on a subdomain of the real line.

To identify conditions that guarantee that the Bertino n-quasi-copula is an n-copula, we first analyse the case when the n-diagonal function is piecewise linear.

Theorem 6.4. *Let d be a piecewise linear n-diagonal function. Then the Bertino*

n-quasi-copula $B_{d,n}$ associated with d is an n-copula if and only if d is $n/(n-1)$-Lipschitz continuous.

We will split the proof of this theorem in two parts.

Proposition 6.8. *Let $B_{d,n}$ be a Bertino n-quasi-copula with piecewise linear diagonal section d and such that there exists an interval on which the slope of d is strictly greater than $n/(n-1)$. Then $B_{d,n}$ is a proper n-quasi-copula.*

Proof. We start by considering an interval $[a, c] \subseteq [0, 1]$ on which the diagonal section takes the form:

$$d(t) = b + \alpha(t - a)\,,$$

with $0 \leqslant \alpha \leqslant n$. Let k denote the floor of α, i.e., $k = \lfloor \alpha \rfloor$. Note that $k = n$ only when $\alpha = n$. It is easy to see that for $\mathbf{x} \in [a, c]^n$, it holds that

$$B_{d,n}(\mathbf{x}) = \begin{cases} b + \alpha(x_{(k+1)} - a) - kx_{(k+1)} + \sum\limits_{j=1}^{k} x_{(j)} & \text{, if } 0 \leqslant k \leqslant n-1\,, \\ b - na + \sum\limits_{j=1}^{n} x_{(j)} & \text{, if } k = n\,. \end{cases}$$

Consider an n-box of the form $\mathbf{P} = [a, a + \epsilon]^n$, with $\epsilon < c - a$. We observe that if a vertex $\mathbf{x}$ of $\mathbf{P}$ is such that $x_{(k+1)} = a$, then $B_{d,n}(\mathbf{x}) = b$, whereas if the vertex $\mathbf{x}$ is such that $x_{(k+1)} = a + \epsilon$, then $B_{d,n}(\mathbf{x}) = b + \alpha\epsilon - m\epsilon$, where m counts the number of coordinates of $\mathbf{x}$ that are equal to a. It follows that the $B_{d,n}$-volume of the given n-box is

$$\begin{aligned} V_{B_{d,n}}(\mathbf{P}) &= \sum_{j=0}^{k} (-1)^j \binom{n}{j} (b + \alpha\epsilon - j\epsilon) + \sum_{j=k+1}^{n} (-1)^j \binom{n}{j} b \\ &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} b + \sum_{j=0}^{k} (-1)^j \binom{n}{j} (\alpha\epsilon - j\epsilon)\,. \end{aligned} \tag{6.8}$$

Using Lemma 2.1, it follows that

$$V_{B_{d,n}}(\mathbf{P}) = \sum_{j=0}^{k} \left[(-1)^j \binom{n}{j} (\alpha\epsilon - j\epsilon) \right] = (-1)^k \binom{n-1}{k} \alpha\epsilon - (-1)^k \binom{n-2}{k-1} n\epsilon\,,$$

whence

$$V_{B_{d,n}}(\mathbf{P}) = \left(\frac{(-1)^k}{k!} \left[\alpha(n-1) - nk \right] \prod_{j=2}^{k} (n - j) \right) \epsilon\,.$$

This volume is positive if $(-1)^k(\alpha(n-1) - nk) \geqslant 0$, implying that the value of the slope α must be restricted to the following union of intervals

$$\alpha \in \left(\left[0, \frac{n}{n-1}\right] \cup \left[\frac{2n}{n-1}, \frac{3n}{n-1}\right] \cup \cdots \right) \cap [0, n]\,.$$

Next, we assume that on some interval $[c, e]$ the n-diagonal function d takes the following form:

$$d(t) = \begin{cases} b + \alpha(t-a) & , \text{if } t \in [c, a]\,, \\ b + \beta(t-a) & , \text{if } t \in [a, e]. \end{cases}$$

We will show that if the slope $\alpha \in [0, n/(n-1)]$ on $[c, a]$ and the slope β on $[a, d]$ is such that $\beta \geqslant 2$, there exists an n-box with negative volume. Let us denote $k = \lfloor \beta \rfloor$. Consider an n-box of the form $\mathbf{P} = [a-\epsilon, a+\delta]^n$ with $(2-\alpha)\epsilon = (\beta-2)\delta$ and such that $[a-\epsilon, a+\delta] \subset [c, e]$.

Similarly as in the first case, it can be shown that if a vertex $\mathbf{x}$ of $\mathbf{P}$ is such that $S(\mathbf{x}) = 1$ (just one coordinate is of the type $a - \epsilon$), the maximum value of $d(t) - (t - a + \epsilon)$ on $[a - \epsilon, a]$ is $b - \alpha\epsilon$ if $\alpha < 1$, or $b - \epsilon$ if $\alpha \geqslant 1$; while the maximum of $d(t) - (t - a + \epsilon)$ on $[a, a + \delta]$ is $b + \beta\delta - \delta - \epsilon$. From this and the equality $\beta\delta + \alpha\epsilon = 2(\delta + \epsilon)$, it follows that for such vertex, it holds that $B_{d,n}(\mathbf{x}) = b + \beta\delta - \delta - \epsilon$. For the vertices $\mathbf{x}$ of $\mathbf{P}$ such that $S(\mathbf{x}) = 2$ (exactly two coordinates are of the type a), the maximum value of $d(t) - (t - a + \epsilon)$ on $[a - \epsilon, a]$ is $b - \alpha\epsilon$; whereas on the interval $[a, a + \delta]$, the maximum value is $b + \beta\delta - 2(\delta + \epsilon)$. For a vertex $\mathbf{x}$ of this type, it then holds that $B_{d,n}(\mathbf{x}) = b - \alpha\epsilon = b + \beta\delta - 2(\delta + \epsilon)$. By continuing with this procedure, we obtain that for a vertex $\mathbf{x}$ such that $S(\mathbf{x}) = s$ with $s \geqslant 3$, it holds that $B_{d,n}(\mathbf{x}) = b - \alpha\epsilon$. Hence, the $B_{d,n}$-volume of this n-box is given by

$$\begin{aligned} V_{B_{d,n}}(\mathbf{P}) &= \beta\delta - n(\beta\delta - \delta - \epsilon) - \alpha - \epsilon \sum_{j=2}^{n} (-1)^j \binom{n}{j} \\ &= (1-n)(\beta\delta + \alpha\epsilon) + n(\delta + \epsilon) \\ &= (1-n)(2\delta + 2\epsilon) + n(\delta + \epsilon) \\ &= (2-n)(\delta + \epsilon)\,. \end{aligned}$$

Note that the last expression is strictly negative for $n \geqslant 3$.

To conclude, first we have shown that a slope smaller than or equal to 2 must be necessarily situated in the interval $[0, n/(n-1)]$, since otherwise there exists an n-box with negative volume. Next, we have shown that is not possible to change slope from a value in $[0, n/(n-1)]$ to a value greater than two. Since a piecewise linear n-diagonal function always starts off with a slope less than or equal to 1, it follows that if α takes values out of the interval $[0, n/(n-1)]$, i.e., there exists an interval on which the slope of d is greater than $n(n-1)$, then there exist an n-box with a negative $B_{d,n}$-volume, so that $B_{d,n}$ is a proper quasi-copula. □

Proposition 6.9. *If d is a piecewise linear n-diagonal function that is $n/(n-1)$-Lipschitz continuous, then the Bertino n-quasi-copula $B_{d,n}$ is an n-copula.*

Proof. Since the slope of d is always in $[0, n/(n-1)]$, it follows from Remark 6.3 that the value of the Bertino n-quasi-copula only depends on $x_{(1)}$ and $x_{(2)}$, i.e.,

$$B_{d,n}(\mathbf{x}) = \sup\{d(t) - t + x_{(1)} \mid t \in [x_{(1)}, x_{(2)}]\} = B_{d,2}(x_{(1)}, x_{(2)}).$$

We now proceed to compute the volume of an n-box $\mathbf{P}$ of the type given in Eq. (2.3). We have to distinguish four cases.

Case 1: If $1 < m_1 < n$, then clearly for any vertex $\mathbf{x}$ of the n-the box $\mathbf{P}$, it holds that the smallest and the second smallest ordered values can only be a_1 or b_1. Hence, for any vertex $\mathbf{x}$ the value $B_{d,n}(\mathbf{x})$ is one of the three possible values $B_{d,2}(b_1, b_1)$, $B_{d,2}(a_1, b_1)$ or $B_{d,2}(a_1, a_1)$. Note that for the vertices $\mathbf{x}$ such that $S(\mathbf{x}) = 0$ (all the coordinates are of the type b_j) the value is $B_{d,2}(b_1, b_1)$. When the vertices are such that $S(\mathbf{x}) = 1$ (just one coordinate is of the type a_j) then $m_1\binom{n-m_1}{0}$ vertices have one coordinate equal to a_1 and $\binom{n-m_1}{1}$ have one coordinate equal to some a_j with $j \neq 1$. Hence, for m_1 such vertices the value is $B_{d,2}(a_1, b_1)$, whereas for the remaining ones the value is $B_{d,2}(b_1, b_1)$. For the vertices such that $S(\mathbf{x}) = 2$ (exactly two coordinates are of the type a_j), the value $B_{d,2}(b_1, b_1)$ is assigned to $\binom{n-m_1}{2}$ vertices, the value $B_{d,2}(a_1, b_1)$ to $m_1\binom{n-m_1}{1}$ vertices, whereas in the remaining $\binom{m_1}{2}\binom{n-m_1}{0}$ vertices the value is $B_{d,2}(a_1, a_1)$. When the vertices are such that $S(\mathbf{x}) = 3$ the value $B_{d,2}(b_1, b_1)$ is assigned to $\binom{n-m_1}{3}$ vertices, the value $B_{d,2}(a_1, b_1)$ to $m_1\binom{n-m_1}{2}$ vertices, whereas the value $B_{d,2}(a_1, a_1)$ is assigned to the remaining $\binom{m_1}{3}\binom{n-m_1}{0} + \binom{m_1}{2}\binom{n-m_1}{1}$ vertices. Continuing this procedure, we obtain that for vertices $\mathbf{x}$ such that $S(\mathbf{x}) = s$ for any $s \in \{0, 1, ..., n\}$, the value $B_{d,2}(b_1, b_1)$ is assigned to $\binom{n-m_1}{s}$ vertices, the value $B_{d,2}(a_1, b_1)$ to $m_1\binom{n-m_1}{s-1}$ vertices, whereas the value $B_{d,2}(a_1, a_1)$ is assigned to the remaining $\sum\limits_{j=2}^{s}\binom{m_1}{j}\binom{n-m_1}{s-j}$ vertices. Hence, the $B_{d,n}$-volume of the given n-box $\mathbf{P}$ is given by

$$\begin{aligned} V_{B_{d,n}}(\mathbf{P}) = & \sum_{s=0}^{n}(-1)^s\binom{n-m_1}{s}B_{d,2}(b_1, b_1) \\ & + m_1\sum_{s=1}^{n}(-1)^s\binom{n-m_1}{s-1}B_{d,2}(a_1, b_1) \\ & + \sum_{s=2}^{n}(-1)^s\sum_{j=2}^{s}\binom{m_1}{j}\binom{n-m_1}{s-j}B_{d,2}(a_1, a_1). \end{aligned}$$

From Lemma 2.1, we immediately see that the first two sums add to zero. Changing the order of summation in the double sum leads to

$$
\begin{aligned}
V_{B_{d,n}}(\mathbf{P}) &= \sum_{S=2}^{n} (-1)^S \sum_{j=2}^{S} \binom{m_1}{j} \binom{n-m_1}{S-j} B_{d,2}(a_1, a_1) \\
&= \sum_{j=2}^{m_1} \binom{m_1}{j} \sum_{S=j}^{n-m_1+j} (-1)^S \binom{n-m_1}{S-j} B_{d,2}(a_1, a_1) \\
&= 0\,,
\end{aligned}
$$

where in the last step we once again invoked Lemma 2.1.

Case 2: If $m_1 = 1$, but $m_2 < n-1$, then for any vertex $\mathbf{x}$ of the n-box $\mathbf{P}$ it holds that the smallest ordered value is either a_1 or b_1 and the second smallest ordered value is either a_2 or b_2. Hence, there are exactly four values that the Bertino n-quasi-copula can take at the vertices of this n-box: $B_{d,2}(b_1, b_2)$, $B_{d,2}(b_1, a_2)$, $B_{d,2}(a_1, b_2)$ or $B_{d,2}(a_1, a_2)$. By doing a combinatorial analysis analogous to the previous case, we can compute the $B_{d,n}$-volume of the given n-box $\mathbf{P}$ as

$$
\begin{aligned}
V_{B_{d,n}}(\mathbf{P}) &= \sum_{S=0}^{n-m_2-1} (-1)^S \binom{n-m_2-1}{S} B_{d,2}(b_1, b_2) \\
&\quad + \sum_{S=1}^{n-m_2} (-1)^S \binom{n-m_2-1}{S-1} B_{d,2}(a_1, b_2) \\
&\quad + \sum_{S=1}^{n} (-1)^S \sum_{j=1}^{S} \binom{m_2}{j} \binom{n-m_2-1}{S-j} B_{d,2}(b_1, a_2) \\
&\quad + \sum_{S=2}^{n} (-1)^S \sum_{j=1}^{S-1} \binom{m_2}{j} \binom{n-m_2-1}{S-j} B_{d,2}(a_1, a_2) \\
&= \sum_{j=1}^{m_2} \binom{m_2}{j} \sum_{S=j}^{n-m_2-1+j} (-1)^S \binom{n-m_2-1}{S-j} B_{d,2}(b_1, a_2) \\
&\quad + \sum_{j=1}^{m_2} \binom{m_2}{j} \sum_{S=j+1}^{n-m_2+j} (-1)^S \binom{n-m_2-1}{S-j-1} B_{d,2}(a_1, a_2) \\
&= 0\,.
\end{aligned}
$$

Again, the $B_{d,n}$-volume of this n-box is zero.

Case 3: The case when $m_1 = 1$ and $m_2 = n-1$ bears some similarities with the previous cases, the main difference being that the values $B_{d,2}(b_1, b_2)$ and $B_{d,2}(a_1, b_2)$

appear only once when evaluating the $B_{d,n}$-volume, while the coefficients of the terms in $B_{d,2}(b_1, a_2)$ and $B_{d,2}(a_1, a_2)$ sum up to -1 and 1 respectively. Hence the $B_{d,n}$-volume is $B_{d,2}(b_1, b_2) - B_{d,2}(b_1, a_2) - B_{d,2}(a_1, b_2) + B_{d,2}(a_1, a_2)$ which is positive, since this is the $B_{d,2}$-volume of the box $[a_1, b_1] \times [a_2, b_2]$ and $B_{d,2}$ is a 2-copula.

Case 4: The last case remaining is when $m_1 = n$. This is the case of an n-dimensional hypercube $[a_1, b_1]^n$ centered around the main diagonal of the unit hypercube $[0, 1]^n$. First, we assume that the slope α of d is constant on $[a_1, b_1]$, i.e.,

$$d(t) = b + \alpha(t - a)\,.$$

In this case, there are only three values the Bertino n-quasi-copula $B_{d,n}$ can take at the vertices $\mathbf{x}$ of the n-box $\mathbf{P}$, namely $B_{d,2}(b_1, b_1)$, $B_{d,2}(a_1, b_1)$ or $B_{d,2}(a_1, a_1)$. It turns out of that the $B_{d,n}$-volume of the n-box $\mathbf{P}$ is given by

$$V_{B_{d,n}}(\mathbf{P}) = B_{d,2}(b_1, b_1) - nB_{d,2}(a_1, b_1) + (n-1)B_{d,2}(a_1, a_1)\,.$$

Clearly

$$B_{d,n}(a_1, a_1) = d(a_1)\,, \quad B_{d,2}(b_1, b_1) = d(b_1)\,,$$

whereas

$$B_{d,n}(a_1, b_1) = \begin{cases} b + \alpha(a_1 - a) & \text{, if } 0 \leqslant \alpha < 1\,, \\ b + \alpha(b_1 - a) - b_1 + a_1 & \text{, if } 1 \leqslant \alpha \leqslant n/(n-1)\,. \end{cases}$$

Hence, the $B_{d,n}$-volume of the n-box is

$$V_{B_{d,n}}(\mathbf{P}) = \begin{cases} \alpha(b_1 - a_1) & \text{, if } 0 \leqslant \alpha < 1\,, \\ (n - (n-1)\alpha)(b_1 - a_1) & \text{, if } 1 \leqslant \alpha \leqslant n/(n-1)\,. \end{cases}$$

Now, in case the slope of d is not constant on $[a_1, b_1]$, thanks to the piecewise linearity, we can decompose the hypercube $[a_1, b_1]^n$ as the union of n-boxes that are either centered on a part of the diagonal with constant slope or are not centered around the diagonal. The n-boxes of the first type are covered above while the n-boxes of the other types are covered by cases 1, 2 and 3, respectively. In all cases, the $B_{d,n}$-volume is positive. □

The next proposition is an immediate consequence of Proposition 6.7 and Theorem 6.4.

Proposition 6.10. *If an n-diagonal function d is $n/(n-1)$-Lipschitz continuous, then the Bertino n-quasi-copula $B_{d,n}$ is an n-copula.*

Example 6.11. Consider the n-diagonal function d given by $d(x) = \lambda x/[1 - (1 - \lambda)x]$ with $\lambda \in [1/n, 1]$. Simple computations show that this function is $n/(n-1)$-

Lipschitz continuous if and only if $\lambda \in [(n-1)/n, 1]$. In that case, it is easy to see that $B_{d,n}$ takes the following form

$$B_{d,n}(\mathbf{x}) = \begin{cases} d(x_{(1)}) & \text{, if } \left(\frac{1-x_{(2)}}{x_{(2)}}\right)\left(\frac{1-x_{(1)}}{x_{(1)}}\right) > \lambda\,, \\ d(x_{(2)}) - x_{(2)} + x_{(1)} & \text{, if } \left(\frac{1-x_{(2)}}{x_{(2)}}\right)\left(\frac{1-x_{(1)}}{x_{(1)}}\right) \leqslant \lambda\,. \end{cases}$$

We will show next that the Lipschitz condition in Proposition 6.10 becomes also a necessary condition when restricting to a broad class of n-diagonal functions, called regular n-diagonal functions in this work.

Definition 6.1. *An n-diagonal function d is called regular if*

(i) the derivative of d is continuous except at countably many points;

(ii) the derivative of d exists on some interval $[0, c]$, with $c > 0$.

Note that an n-diagonal function is not necessarily regular, although the conditions imposed may seem rather weak. For example, in [74] Fernández-Sánchez et al. construct an n-diagonal function d_0 such that for any open interval $]a, b[$, the sets $\{x \in]a, b[\mid d_0'(x) = 0\}$ and $\{x \in]a, b[\mid d_0'(x) = 2\}$ are not empty.

Since the value of the Bertino n-quasi-copula at a given point is the result of a maximization procedure, it comes in handy to impose the above regularity conditions in order to avoid undesirable situations, such as encountering an n-diagonal function such as d_0, which would require to compute the maximum of a nowhere monotone function (for more details on nowhere monotone functions, see [20]).

Proposition 6.11. *If the Bertino n-quasi-copula $B_{d,n}$ with diagonal section d is an n-copula and d is regular, then d is $n/(n-1)$-Lipschitz continuous.*

Proof. As in the case of a piecewise linear diagonal section, we will prove this theorem by contradiction. Suppose that the diagonal section of the Bertino n-copula $B_{d,n}$ is Lipschitz continuous with Lipschitz constant greater than $n/(n-1)$. Then, due to the regularity conditions, there exists a point a such that $d'(a) > n/(n-1)$ and a positive constant η such that for all $x \in]a, a+\eta[$, $d'(x)$ exists and $\lfloor d'(x) \rfloor = k$. Clearly, $d'(x) - k \geqslant 0$ and $d'(x) - (k+1) < 0$ for all $x \in]a, a+\eta[$. Hence, if $k < n$, then for any $\mathbf{x} \in]a, a+\eta[^n$, it holds that

$$B_{d,n}(\mathbf{x}) = d(x_{(k+1)}) - kx_{(k+1)} + \sum_{j=1}^{k} x_{(j)}\,.$$

In case $k = n$, d is a linear function on $]a, a+\eta[$, and hence, the proof is analogous to the piecewisely linear case. For any closed interval $[b, b+\epsilon] \subset]a, a+\eta[$, the

$B_{d,n}$-volume of the n-box $\mathbf{P} = [b, b+\epsilon]^n$ is

$$
\begin{aligned}
V_{B_{d,n}}(\mathbf{P}) &= \sum_{j=0}^{k} \left[(-1)^j \binom{n}{j} \left(d(b+\epsilon) - j\epsilon \right) \right] + \sum_{j=k+1}^{n} \left[(-1)^j \binom{n}{j} d(b) \right] \\
&= \frac{(-1)^k}{k!} [(n-1)(d(b+\epsilon) - d(b)) - nk\epsilon] \prod_{j=1}^{k} (n-j) \\
&= \frac{(-1)^k}{k!} [(n-1)d'(b)\epsilon - nk\epsilon + \vartheta(\epsilon)] \prod_{j=1}^{k} (n-j) \,, \qquad (6.9)
\end{aligned}
$$

where $\vartheta(\epsilon)$ is a function that satisfies $\lim_{\epsilon \to 0^+} \vartheta(\epsilon)/\epsilon = 0$. Hence, by making ϵ small enough, the sign of the latter expression only depends on the sign of $(-1)^k[(n-1)d'(b) - nk]$. Note the resemblance between Eq. (6.8) and Eq. (6.9). Therefore, applying the same reasoning as in the proof of Proposition 6.8, we can conclude that $d'(b)$ belongs to the following union of intervals

$$
d'(b) \in \left(\left[0, \frac{n}{n-1} \right] \cup \left[\frac{2n}{n-1}, \frac{3n}{n-1} \right] \cup \cdots \right) \cap [0, n] \,,
$$

since otherwise the considered volume $V_{B_{d,n}}(\mathbf{P})$ is strictly negative. Since for a regular n-diagonal function d there exists a constant $0 < c \leqslant 1$ such that $d'(t) \leqslant 1$ for any $t \in]0, c[$, where the derivative exists, it holds that the constant a defined as

$$
a = \inf\{t \in [0,1] \mid D^- d(t) \leqslant n/(n-1) \,, D_+ d(a) \geqslant 2n/(n-1)\} \,,
$$

is strictly positive. This constant a is well defined from the assumption we have made regarding the values the Lipschitz constant can take. Also, $a < 1$ by assumption,since otherwise d would be $n/(n-1)$-Lipschitz continuous, contradicting our hypothesis.

First suppose that $D^- d(a) = D_- d(a)$. We know that there exists a sequence of positive numbers $(\delta_m)_{m=1}^{\infty}$ that is strictly decreasing and converges to zero, such that

$$
\frac{d(a+\delta_m) - d(a)}{\delta_m} \longrightarrow D^+ d(a) \,.
$$

For such sequence, define $(\omega - D_- d(a))\epsilon_m = (D^+ d(a) - \omega)\delta_m$ with ω a constant in $]n/(n-1), 2[$, which implies that the coefficients of ϵ_m and δ_m are strictly positive. Hence the sequence $(\epsilon_m)_{m=1}^{\infty}$ is positive and converges to zero, while the sequence $([d(a) - d(a-\epsilon_m)]/\epsilon_m)_{m=1}^{\infty}$ converges to $D^- d(a) = D_- d(a)$.

We now prove that there exists $M_1 < \infty$ such that for all $m > M_1$, it holds that

$d(a+\delta_m) - [\delta_m + \epsilon_m] \geqslant d(a-\epsilon_m)$. Indeed, if no such M_1 would exist, then

$$1 > \frac{\delta_m}{\delta_m + \epsilon_m} \frac{d(a+\delta_m) - d(a)}{\delta_m} + \frac{\epsilon_m}{\delta_m + \epsilon_m} \frac{d(a) - d(a-\epsilon_m)}{\epsilon_m},$$

for all positive integers m. By taking limits, it follows that

$$1 \geqslant \frac{D^+d(a)[\omega - D_-d(a)]}{D^+d(a) - D_-d(a)} + \frac{D_-d(a)[D^+d(a) - \omega]}{D^+d(a) - D_-d(a)} = \omega\,,$$

which is a contradiction since $\omega > 1$. Now, we prove in a similar manner that there exists $M_2 < \infty$ such that for all $m > M_2$, it holds that $d(a+\delta_m) - 2[\delta_m + \epsilon_m] \leqslant d(a-\epsilon_m)$. Indeed, if no such M_2 would exist, then

$$2 < \frac{\delta_m}{\delta_m + \epsilon_m} \frac{d(a+\delta_m) - d(a)}{\delta_m} + \frac{\epsilon_m}{\delta_m + \epsilon_m} \frac{d(a) - d(a-\epsilon_m)}{\epsilon_m},$$

for all positive integers m. By taking limits, it follows that

$$2 \leqslant \frac{D^+d(a)[\omega - D_-d(a)]}{D^+d(a) - D_-d(a)} + \frac{D_-d(a)[D^+d(a) - \omega]}{D^+d(a) - D_-d(a)} = \omega\,,$$

which is a contradiction since $\omega < 2$. Now, consider an n-box of the form $\mathbf{P} = [a-\epsilon_m, a+\delta_m]^n$. For $m > \max\{M_1, M_2\}$, the $B_{d,n}$-volume of $\mathbf{P}$ is given by

$$\begin{aligned} V_{B_{d,n}}(\mathbf{P}) &= d(a+\delta_m) - n(d(a+\delta_m) - [\delta_m + \epsilon_m]) + (n-1)d(a-\epsilon_m) \\ &= n(\delta_m + \epsilon_m) - (n-1)[d(a+\delta_m) - d(a-\epsilon_m)] \\ &= n(\delta_m + \epsilon_m) - (n-1)[\delta_m D^+d(a) + \epsilon_m D_-d(a) + \vartheta(\delta_m)] \\ &= n(\delta_m + \epsilon_m) - (n-1)[\omega\delta_m + \omega\epsilon_m + \vartheta(\delta_m)] \\ &= (\delta_m + \epsilon_m)[n - (n-1)\omega] - (n-1)\vartheta(\delta_m) \\ &< 0\,, \end{aligned}$$

where the last inequality holds for all $m > M$ with M some finite constant $\geqslant \max(M_1, M_2)$. Hence, for the present case we have obtained the desired contradiction. The case when $D^+d(a) = D_+d(a)$ can be treated in a similar way.

Therefore, the only case remaining is when the four Dini derivatives are different, which is only possible when $n \geqslant 4$. We define $(\delta_m)_{m=1}^\infty$ and $(\epsilon_m)_{m=1}^\infty$ just like before, but with the constant ω such that $\omega \in]n/(n-1), 3/2[$.

While now we cannot guarantee the convergence of the sequence $([d(a) - d(a-\epsilon_m)]/\epsilon_m)_{m=1}^\infty$, we nonetheless know that there exists a subsequence $\left(\epsilon_{m_q}\right)_{q=1}^\infty$ such

that
$$\lim_{q\to\infty} \frac{d(a)-d(a-\epsilon_{m_q})}{\epsilon_{m_q}} = \liminf_{m\to\infty} \frac{d(a)-d(a-\epsilon_m)}{\epsilon_m} = D^*\,.$$
Clearly, $D_-d(a) \leqslant D^* \leqslant D^-d(a)$. Once again, we will prove that there exists $Q_1 < \infty$ such that for $q > Q_1$, it holds that $d(a+\delta_{m_q}) - [\delta_{m_q} + \epsilon_{m_q}] \geqslant d(a-\epsilon_{m_q})$. Indeed, if no such Q_1 would exist, then for all q it would hold that
$$1 > \frac{\delta_{m_q}}{\delta_{m_q}+\epsilon_{m_q}} \frac{d(a+\delta_{m_q})-d(a)}{\delta_{m_q}} + \frac{\epsilon_{m_q}}{\delta_{m_q}+\epsilon_{m_q}} \frac{d(a)-d(a-\epsilon_{m_q})}{\epsilon_{m_q}}\,.$$
By taking limits, it follows that
$$\begin{aligned}
1 &\geqslant \frac{D^+d(a)[\omega - D_-d(a)]}{D^+d(a)-D_-d(a)} + \frac{D^*[D^+d(a)-\omega]}{D^+d(a)-D_-d(a)}\\
&\geqslant \frac{D^+d(a)[\omega - D_-d(a)]}{D^+d(a)-D_-d(a)} + \frac{D_-d(a)[D^+d(a)-\omega]}{D^+d(a)-D_-d(a)}\\
&= \omega\,,
\end{aligned}$$
which is a contradiction since $\omega > 1$. Similarly, there exists $Q_2 < \infty$ such that for all $q > Q_2$, it holds that $d(a+\delta_{m_q}) - 2[\delta_{m_q}+\epsilon_{m_q}] \leqslant d(a-\epsilon_{m_q})$. Otherwise, for all integers q it would hold that
$$\frac{\delta_{m_q}}{\delta_{m_q}+\epsilon_{m_q}} \frac{d(a+\delta_{m_q})-d(a)}{\delta_{m_q}} + \frac{\epsilon_{m_q}}{\delta_{m_q}+\epsilon_{m_q}} \frac{d(a)-d(a-\epsilon_{m_q})}{\epsilon_{m_q}} > 2\,,$$
which by taking limits leads to
$$\begin{aligned}
2 &\leqslant \frac{D^+d(a)[\omega - D_-d(a)]}{D^+d(a)-D_-d(a)} + \frac{D^*[D^+d(a)-\omega]}{D^+d(a)-D_-d(a)}\\
&= \frac{D^+d(a)[\omega - D_-d(a)]}{D^+d(a)-D_-d(a)} + \frac{[D_-d(a)+D^*-D_-d(a)][D^+d(a)-\omega]}{D^+d(a)-D_-d(a)}\\
&\leqslant \omega + \frac{[D^-d(a)-D_-d(a)][D^+d(a)-\omega]}{D^+d(a)-D_-d(a)}\,.
\end{aligned}$$
Now using the fact that the rational function $g(x) = (x-e)/(x-c)$ with $c < e$ is increasing on $[e,\infty[$, we obtain the following bound
$$\omega + \frac{[D^-d(a)-D_-d(a)][D^+d(a)-\omega]}{D^+d(a)-D_-d(a)} \leqslant \omega + \frac{[D^-d(a)-D_-d(a)](n-\omega)}{n-D_-d(a)}\,. \tag{6.10}$$
Here, we need to consider 2 cases. First, if $D_-d(a) \leqslant n/(2n-2)$, then due to Lemma 6.1 we have that $D^-d(a) - D_-d(a) \leqslant n/(2n-2)$. From this, we get the

following bound for the right-hand side of Eq. (6.10)

$$\begin{aligned}
\omega + \frac{[D^- d(a) - D_- d(a)](n-\omega)}{n - D_- d(a)} &\leqslant \omega + \frac{n(n-\omega)}{[n - D_- d(a)](2n-2)} \\
&\leqslant \omega + \frac{n(n-\omega)(2n-2)}{(2n-2)(2n^2-3n)} \\
&= \omega + \frac{n-\omega}{2n-3}\,,
\end{aligned}$$

hence,

$$2 \leqslant \omega + \frac{n-\omega}{2n-3}\,,$$

which implies $\omega \geqslant 3/2$, again a contradiction. Second, if $n/(2n-2) < D_- d(a) < n/(n-1)$, then $0 \leqslant D^- d(a) - D_- d(a) \leqslant n/(n-1) - D_- d(a)$, and the right-hand side of Eq. (6.10) is bounded by

$$\begin{aligned}
\omega + \frac{[D^- d(a) - D_- d(a)](n-\omega)}{n - D_- d(a)} &\leqslant \omega + \frac{(n-\omega)[n-(n-1)D_- d(a)]}{[n - D_- d(a)](n-1)} \\
&= \frac{\omega(n^2-n) + [n^2 - n^2 D_- d(a) + nD_- d(a)]}{(n-1)[n - D_- d(a)]}\,.
\end{aligned}$$

From this inequality, we deduce that

$$2 - \frac{[n^2 - n^2 D_- d(a) + nD_- d(a)]}{(n-1)[n - D_- d(a)]} \leqslant \omega \frac{n}{[n - D_- d(a)]}\,,$$

which after some computations leads to

$$\omega \geqslant 1 + \frac{(n-1)D_- d(a)}{n} > \frac{3}{2}\,,$$

which yields again a contradiction.

Finally, we consider an n-box of the form $\mathbf{P} = [a - \epsilon_{m_q}, a + \delta_{m_q}]^n$. For $q > \max(Q_1, Q_2)$, the $B_{d,n}$-volume of $\mathbf{P}$ is given by

$$\begin{aligned}
V_{B_{d,n}}(\mathbf{P}) &= d(a+\delta_{m_q}) - n(d(a+\delta_{m_q}) - [\delta_{m_q} + \epsilon_{m_q}]) + (n-1)d(a - \epsilon_{m_q}) \\
&= n(\delta_{m_q} + \epsilon_{m_q}) - (n-1)[d(a+\delta_{m_q}) - d(a-\epsilon_{m_q})] \\
&= n(\delta_{m_q} + \epsilon_{m_q}) - (n-1)[\delta_{m_q} D^+ d(a) + \epsilon_{m_q} D^* + \vartheta(\delta_{m_q})] \\
&\leqslant n(\delta_{m_q} + \epsilon_{m_q}) - (n-1)[\delta_{m_q} D^+ d(a) + \epsilon_{m_q} D_- d(a) + \vartheta(\delta_{m_q})] \\
&= n(\delta_{m_q} + \epsilon_{m_q}) - (n-1)[\omega\delta_{m_q} + \omega\epsilon_{m_q} + \vartheta(\delta_{m_q})]
\end{aligned}$$

$$= (\delta_{m_q} + \epsilon_{m_q})[n - (n-1)\omega] - (n-1)\vartheta(\delta_{m_q})$$
$$< 0\,.$$

Hence, we can conclude that if $B_{d,n}$ is an n-copula, then d must be $n/(n-1)$-Lipschitz continuous. □

Example 6.12. Let $d(x) = x^{3/2}$. This function is an n-diagonal function for any positive integer n. The corresponding Bertino 3-quasi-copula is a 3-copula, while the corresponding Bertino 4-quasi-copula is a proper quasi-copula. Indeed, computing the volume of the box $[4/5, 9/10]^4$, we get a volume smaller than $-.0145$.

More generally, consider an integer $n \geqslant 2$, and consider the n-diagonal function $d(x) = x^{n/(n-1)}$, then $B_{d,n}$ is an n-copula. However, for $m > n$, $B_{d,m}$ is a proper m-quasi-copula. Indeed, the $B_{d,m}$-volume of the m-box $[\sqrt[n-1]{\frac{mn-n}{mn-m}}, 1]^m$ is negative.

Example 6.13. Let $d(x) = (nx-1)^+/(n-1)$. Clearly, d is the smallest n-diagonal function that is $n/(n-1)$-Lipschitz continuous. The corresponding Bertino n-copula is given by

$$B_{d,n}(\mathbf{x}) = \left(x_{(1)} + \frac{x_{(2)} - n + 1}{n-1}\right)^+ ,$$

which is in turn, the smallest Bertino n-copula.

As a final note, it can be proven that the greatest n-quasi-copula that has d as diagonal section is given by

$$A_{n,d}(\mathbf{x}) = \min\left(M_n(\mathbf{x}), \inf\{d(t) + \sum_{j=1}^{n}(x_j - t)^+ \mid t \in \left[x_{(1)}, x_{(n)}\right]\}\right) .$$

$A_{n,d}$ is not always an n-copula, not even in the bivariate case. The submodularity of the greatest function $U_{\mathbf{T}}$ 'justifies' why the greatest 2-quasi-copula with a given diagonal section, given by $\min(x, y, U_{\mathbf{T}}(x,y))$, is a 2-copula only for very restrictive diagonal sections (see [74, 158, 193]), since it is a submodular function truncated from above by the minimum operator. We expect even more restrictive conditions when trying to generalize this result to higher dimensions.

7 On the structure of the set of supermodular quasi-copulas

7.1. Introduction

This chapter consists of two separate parts. First, we see that the set of supermodular n-quasi-copulas when endowed with the uniform metric has similar properties as the set of n-copulas endowed with the uniform metric. For example, the set of supermodular n-copulas is a compact subset of the set of all continuous real-valued functions that have the n-box $[0,1]^n$ as its domain, and endowed with the uniform metric.

Next, we study the set of supermodular n-quasi-copulas equipped with the pointwise ordering of functions and show that the poset of n-quasi-copulas is more closely related to the poset of supermodular n-quasi-copulas than to the poset of n-copulas. More specifically, we show that the set of supermodular n-quasi-copulas is join-dense in the set of n-quasi-copulas, although it fails to be meet-dense. The results of the second part of this chapter can also be found in [7].

7.2. The metric space of supermodular quasi-copulas

In this section we will show that the set of supermodular n-quasi-copulas endowed with the metric induced by the L^∞ norm has interesting properties.

We will denote by $(\Xi([0,1]^n), d_\infty)$ the set of all continuous real-valued functions whose domain is $[0,1]^n$, endowed with the metric induced by the L^∞ norm. Note that any converging sequence of functions in $(\Xi([0,1]^n), d_\infty)$ converges uniformly. It is well known that $(\Xi([0,1]^n, d_\infty)$ is a complete metric space.

We now show that, just as the metric spaces $(\mathcal{C}_n, d_\infty)$ and $(\mathcal{Q}_n, d_\infty)$, several properties of the metric space $(\Xi([0,1]^n), d_\infty)$ are inherited by the metric space $(\mathcal{SQ}_n, d_\infty)$. The proof of the following theorem follow the same lines as the one for n-copulas and n-quasi-copulas (see for example, Section 1.7.2 in [69]).

Theorem 7.1. *Let $(Q_{n,i})_{i=1}^\infty$ be a sequence of supermodular n-quasi-copulas that converges pointwisely to a function S_n. Then $S_n \in \mathcal{SQ}_n$ and the sequence $(Q_{n,i})_{i=1}^\infty$ converges to S_n in $(\mathcal{SQ}_n, d_\infty)$.*

Proof. First, we will see that $S_n \in \mathcal{SQ}_n$. Let $\mathbf{x} \in [0,1]^n$ be such that $x_i = 0$ for some $i \in \{1, 2, ..., n\}$, then $Q_{n,i}(\mathbf{x}) = 0$ for any i. Hence,

$$\lim_{i \to \infty} Q_{n,i}(\mathbf{x}) = 0 = S_n(\mathbf{x}) .$$

Therefore, S_n satisfies (q1). Analogously, one can prove that S_n satisfies (q2). Now, we show that S_n is an increasing function. Let $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ be such that $\mathbf{x} \leqslant \mathbf{y}$. Then it holds that $Q_{n,i}(\mathbf{x}) \leqslant Q_{n,i}(\mathbf{y})$ for any i. Hence,

$$S_n(\mathbf{x}) = \lim_{i \to \infty} Q_{n,i}(\mathbf{x}) \leqslant \lim_{i \to \infty} Q_{n,i}(\mathbf{y}) = S_n(\mathbf{y}) .$$

Therefore, S_n is an increasing function. To prove that S_n is 1-Lipschitz continuous, note that for any $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ and any i it holds that

$$|Q_{n,i}(\mathbf{y}) - Q_{n,i}(\mathbf{x})| \leqslant \sum_{j=1}^{n} |y_j - x_j| .$$

Hence,

$$|S_n(\mathbf{y}) - S_n(\mathbf{x})| = \lim_{i \to \infty} |Q_{n,i}(\mathbf{y}) - Q_{n,i}(\mathbf{x})| \leqslant \sum_{j=1}^{n} |y_j - x_j| .$$

Consequently, S_n is 1-Lipschitz continuous and therefore an n-quasi-copula. Now we show that S_n is a supermodular function. Let $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, then for any i it holds that

$$Q_{n,i}(\mathbf{y} \vee \mathbf{x}) + Q_{n,i}(\mathbf{y} \wedge \mathbf{x}) \geqslant Q_{n,i}(\mathbf{y}) + Q_{n,i}(\mathbf{x}) .$$

Hence,

$$\begin{aligned} S_n(\mathbf{y} \vee \mathbf{x}) + S_n(\mathbf{y} \wedge \mathbf{x}) &= \lim_{i \to \infty} Q_{n,i}(\mathbf{y} \vee \mathbf{x}) + \lim_{i \to \infty} Q_{n,i}(\mathbf{y} \wedge \mathbf{x}) \\ &\geqslant \lim_{i \to \infty} Q_{n,i}(\mathbf{y}) + \lim_{i \to \infty} Q_{n,i}(\mathbf{x}) \\ &= S_n(\mathbf{y}) + S_n(\mathbf{x}) . \end{aligned}$$

Therefore, S_n is a supermodular n-quasi-copula. Finally, we prove that the convergence is uniform. Let $\epsilon > 0$. First, note that S_n is uniformly continuous on $[0, 1]^n$ since S_n is continuous and $[0, 1]^n$ is a compact set. Hence, for any $\mathbf{x} \in [0, 1]^n$ there exists an open ball centred around $\mathbf{x}$, say $]\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x}[$, such that for any $\mathbf{y} \in]\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x}[$ we have

$$|S_n(\mathbf{x}) \vee S_n(\mathbf{y})| < \frac{\epsilon}{2} . \tag{7.1}$$

Clearly, it holds that

$$\bigcup_{\mathbf{x} \in [0,1]^n}]\mathbf{a}_\mathbf{x}, \mathbf{b}_\mathbf{x}[= [0, 1]^n .$$

Since $[0, 1]^n$ is compact, there exists a finite collection of points $\mathbf{x}_1, \ldots, \mathbf{x}_m$ such

that
$$\bigcup_{j=1}^{m}]\mathbf{a}_{\mathbf{x}_j}, \mathbf{b}_{\mathbf{x}_j}[= [0,1]^n .$$

Since $(Q_{n,i})_{i=1}^{\infty}$ converges to S_n pointwisely, there exists $k(\epsilon)$ such that for any $i \geqslant k(\epsilon)$ and $j \in \{1, \ldots, m\}$, it holds that

$$\max\left(|S_n(\mathbf{a}_{\mathbf{x}_{j_0}}) - Q_{n,i}(\mathbf{a}_{\mathbf{x}_{j_0}})|, |S_n(\mathbf{b}_{\mathbf{x}_{j_0}}) - Q_{n,i}(\mathbf{b}_{\mathbf{x}_{j_0}})|\right) < \frac{\epsilon}{2} . \tag{7.2}$$

Now, for any $\mathbf{x} \in [0,1]^n$ let $\mathbf{x}_{j_0}$ be any point such that $\mathbf{x}_{j_0} \in]\mathbf{a}_{\mathbf{x}_{j_0}}, \mathbf{b}_{\mathbf{x}_{j_0}}[$. For any $i \geqslant k(\epsilon)$ we have the following inequalities

$$\begin{aligned} Q_{n,i}(\mathbf{x}) - S_n(\mathbf{x}) &= Q_{n,i}(\mathbf{x}) - S_n(\mathbf{b}_{\mathbf{x}_{j_0}}) + S_n(\mathbf{b}_{\mathbf{x}_{j_0}}) - S_n(\mathbf{x}) \\ &\leqslant Q_{n,i}(\mathbf{b}_{\mathbf{x}_{j_0}}) - S_n(\mathbf{b}_{\mathbf{x}_{j_0}}) + S_n(\mathbf{b}_{\mathbf{x}_{j_0}}) - S_n(\mathbf{x}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} , \end{aligned}$$

where the first inequality follows from the increasingness of $Q_{n,i}$ and the last inequality follows from Eqs. (7.1) and (7.2). Analogously,

$$\begin{aligned} Q_{n,i}(\mathbf{x}) - S_n(\mathbf{x}) &= Q_{n,i}(\mathbf{x}) - S_n(\mathbf{a}_{\mathbf{x}_{j_0}}) + S_n(\mathbf{a}_{\mathbf{x}_{j_0}}) - S_n(\mathbf{x}) \\ &\geqslant Q_{n,i}(\mathbf{a}_{\mathbf{x}_{j_0}}) - S_n(\mathbf{a}_{\mathbf{x}_{j_0}}) + S_n(\mathbf{a}_{\mathbf{x}_{j_0}}) - S_n(\mathbf{x}) \\ &> -\frac{\epsilon}{2} - \frac{\epsilon}{2} . \end{aligned}$$

Hence, for any $\mathbf{x}$ and $i > k(\epsilon)$ it holds that

$$|Q_{n,i}(\mathbf{x}) - S_n(\mathbf{x})| < \epsilon ,$$

thus the convergence is uniform. □

With the preceding theorem, we can prove the following result the proof of which also follows the same lines as the one for n-copulas and n-quasi-copulas.

Theorem 7.2. *The set $\mathcal{SQ}_n$ is a compact subset of* $(\Xi([0,1]^n), d_\infty)$.

Proof. First, recall that $(\Xi([0,1]^n), d_\infty)$ is a complete metric space. From Theorem 7.1 it follows that $\mathcal{SQ}_n$ is a complete metric space. Hence, from Proposition 2.4.1 in [48] it follows that $(\mathcal{SQ}_n, d_\infty)$ is also a complete metric space.

Since $\sup\{S_n(\mathbf{x}) \mid \mathbf{x} \in [0,1]^n , S_n \in \mathbf{SQ}_n\} \leqslant 1$ we conclude that $\mathcal{SQ}_n$ is uniformly bounded. Even more, it is equi-continuous since any supermodular n-quasi-copula is 1-Lipschitz continuous. From the Ascoli–Arzelà Theorem [48], it follows that $\mathcal{SQ}_n$ is totally bounded with respect to the metric induced by the L^∞ norm.

Since any complete and totally bounded metric space is compact [48], it follows that $\mathcal{SQ}_n$ is compact. □

7.3. The lattice structure of the set of supermodular quasi-copulas

In this section, we show that from a lattice-theoretical point of view, n-quasi-copulas are closer to supermodular n-quasi-copulas than to n-copulas. From Theorem 4.20 we know that $\mathcal{Q}_n$ is not order-isomorphic to the Dedekind-MacNeille completion of $\mathcal{C}_n$, i.e., not every n-quasi-copula can be written as the supremum (resp. infimum) of a set of n-copulas. A natural question is to ask whether this result is also true if we replace $\mathcal{C}_n$ by $\mathcal{SC}_n$ in the previous statement. In order to answer this question, we need some additional results.

First, we study a generalization to higher dimensions of one of the main results proven in [33]: for a given modular function $G : [0,1]^n \to [0,1]$, we will find conditions guaranteeing that the function $H_G : [0,1]^n \to [0,1]$ given by $H_G(\mathbf{x}) = \max(W_n(\mathbf{x}), G(\mathbf{x}))$ is a supermodular n-quasi-copula.

Proposition 7.1. *Let $G : [0,1]^n \to \mathbb{R}$ be an increasing 1-Lipschitz continuous function. Then $H_G = \max(W_n, G)$ is an n-quasi-copula if and only if $G(\mathbf{x}) \leqslant 0$ for any $\mathbf{x} \in [0,1]^n$ such that there exists $i \in \{1,2,\ldots,n\}$ with the property that $x_i = 0$ and $x_j = 1$ for all $i \neq j$.*

Proof. First suppose that H_G is an n-quasi-copula. Let $\mathbf{x} \in [0,1]^n$ be such that there exists $i \in \{1,2,\ldots,n\}$ with the property that $x_i = 0$ and $x_j = 1$ for all $i \neq j$. Since H_G satisfies (q1) and (q2), we have

$$0 = H_G(\mathbf{x}) = \max(W_n(\mathbf{x}), G(\mathbf{x})) \geqslant G(\mathbf{x})\,.$$

Hence, $G(\mathbf{x}) \leqslant 0$.

Now suppose that G has the property that $G(\mathbf{x}) \leqslant 0$ for any $\mathbf{x} \in [0,1]^n$ such that there exists $i \in \{1,2,\ldots,n\}$ with the property that $x_i = 0$ and $x_j = 1$ for all $i \neq j$. Clearly, H_G is increasing and 1-Lipschitz continuous since both W_n and G are.

We now prove that H_G satisfies (q1). Consider a point $\mathbf{x} \in [0,1]^n$ such that $x_i = 0$ for some $i \in \{1,2,...,n\}$. Then,

$$G(\mathbf{x}) \leqslant 0 = W_n(\mathbf{x})\,,$$

since W_n is an n-quasi-copula and G an increasing function. Hence, $H_G(\mathbf{x}) = \max(W_n(\mathbf{x}), G(\mathbf{x})) = W_n(\mathbf{x}) = 0$.

To prove that H_G satisfies (q2) we consider, without loss of generality, the point $\mathbf{x} = (x, 1, 1, \ldots, 1)$. We now show that $G(\mathbf{x}) \leqslant x$. Suppose the contrary, i.e., that $G(\mathbf{x}) > x$. Then

$$G(\mathbf{x}) = G(x, 1, 1, \ldots, 1) \geqslant G(x, 1, 1, \ldots, 1) - G(0, 1, \ldots, 1) > x - 0\,,$$

which contradicts the 1-Lipschitz continuity of G. Hence $G(\mathbf{x}) \leqslant x = W_n(\mathbf{x})$, and, as a consequence, $H_G(\mathbf{x}) = \max(W_n(\mathbf{x}), G(\mathbf{x})) = W_n(\mathbf{x}) = x$. Hence, H_G is an n-quasi-copula. □

We now show that if G is a modular function and satisfies the conditions of Proposition 7.1, then H_G is a supermodular n-quasi-copula.

Proposition 7.2. *Let $G : [0,1]^n \to \mathbb{R}$ be an increasing and modular 1-Lipschitz continuous function such that $G(\mathbf{x}) \leqslant 0$ for any $\mathbf{x} \in [0,1]^n$ such that there exists $i \in \{1, 2, \ldots, n\}$ with the property that $x_i = 0$ and $x_j = 1$ for all $i \neq j$. Then $H_G = \max(W_n, G)$ is a supermodular n-quasi-copula.*

Proof. From Proposition 7.1 it follows that H_G is an n-quasi-copula. Now we prove that it is also a supermodular function. Using the characterization given by Proposition 5.1, it suffices to prove that the 2-dimensional sections of H_G are supermodular. Without loss of generality, we only need to prove that for any $\mathbf{z} \in [0,1]^n$, the function $H^* : [0,1]^2 \to [0,1]$ defined by

$$H^*(x, y) = \max(W^*(x, y), G^*(x, y))$$

is 2-increasing, where

$$W^*(x, y) = W_n(x, y, z_3, \ldots, z_n)$$

and

$$G^*(x, y) = G(x, y, z_3, \ldots, z_n)\,.$$

From Corollary 5.1 and Proposition 5.1, we know that W^* is a supermodular function. Also note that G^* has the form described in Lemma 3.1. Hence, G^* is a modular function. Since for bivariate functions, supermodularity and 2-increasingness are equivalent, we will prove that H^* is a 2-increasing function. Let $\mathbf{P} = [x_1, x_2] \times [y_1, y_2]$. To prove that $V_{H^*}(\mathbf{P}) = H^*(x_2, y_2) - H^*(x_2, y_1) - H^*(x_1, y_2) + H^*(x_1, y_1) \geqslant 0$, we have to analyse 16 different cases.

Case 1: $W^*(x_2, y_2) \geqslant G^*(x_2, y_2)$, $W^*(x_2, y_1) \geqslant G^*(x_2, y_1)$, $W^*(x_1, y_2) \geqslant G^*(x_1, y_2)$ and $W^*(x_1, y_1) \geqslant G^*(x_1, y_1)$. The proof of this case is immediate from the supermodularity of W^*.

Case 2: $W^*(x_2, y_2) \geqslant G^*(x_2, y_2)$, $W^*(x_2, y_1) \geqslant G^*(x_2, y_1)$, $W^*(x_1, y_2) \geqslant G^*(x_1, y_2)$

and $W^*(x_1,y_1) < G^*(x_1,y_1)$. We have the following trivial inequalities

$$V_{H*}(\mathbf{P}) > W^*(x_2,y_2) - W^*(x_2,y_1) - W^*(x_1,y_2) + W^*(x_1,y_1) \geqslant 0\,.$$

Case 3: $W^*(x_2,y_2) \geqslant G^*(x_2,y_2)$, $W^*(x_2,y_1) \geqslant G^*(x_2,y_1)$, $W^*(x_1,y_2) < G^*(x_1,y_2)$ and $W^*(x_1,y_1) \geqslant G^*(x_1,y_1)$. First note that $W^*(x_1,y_1) = 0$, since otherwise

$$y_2 - y_1 = W^*(x_1,y_2) - W^*(x_1,y_1) < G^*(x_1,y_2) - G^*(x_1,y_1) \leqslant y_2 - y_1\,,$$

a contradiction to the 1-Lipschitz continuity of G^*. Hence, $W^*(x_1,y_1) = 0$. Now we analyse two subcases. First, if $W^*(x_2,y_1) = 0$, then

$$V_{H*}(\mathbf{P}) = W^*(x_2,y_2) - G^*(x_1,y_2) \geqslant 0\,,$$

since $G^*(x_1,y_2) \leqslant G^*(x_2,y_2) \leqslant W^*(x_2,y_2)$. Second, if $W^*(x_2,y_1) > 0$, then

$$V_{H*}(\mathbf{P}) = y_2 - y_1 - G^*(x_1,y_2) + 0 \geqslant y_2 - y_1 - G^*(x_1,y_2) + G^*(x_1,y_1) \geqslant 0\,,$$

where the last inequality is justified by the 1-Lipschitz continuity of G^*.

Case 4: $W^*(x_2,y_2) \geqslant G^*(x_2,y_2)$, $W^*(x_2,y_1) \geqslant G^*(x_2,y_1)$, $W^*(x_1,y_2) < G^*(x_1,y_2)$ and $W^*(x_1,y_1) < G^*(x_1,y_1)$. Note that $W^*(x_2,y_2) > 0$ and $W^*(x_2,y_1) > 0$, since

$$W^*(x_1,y_1) < G^*(x_1,y_1) \leqslant G^*(x_2,y_1) \leqslant W^*(x_2,y_1)$$

and

$$W^*(x_1,y_2) < G^*(x_1,y_2) \leqslant G^*(x_2,y_2) \leqslant W^*(x_2,y_2)\,.$$

Hence,

$$V_{H*}(\mathbf{P}) = y_2 - y_1 - G^*(x_1,y_2) + G^*(x_1,y_1) \geqslant 0\,.$$

Case 5: $W^*(x_2,y_2) \geqslant G^*(x_2,y_2)$, $W^*(x_2,y_1) < G^*(x_2,y_1)$, $W^*(x_1,y_2) \geqslant G^*(x_1,y_2)$ and $W^*(x_1,y_1) \geqslant G^*(x_1,y_1)$. This case is similar to case 3.

Case 6: $W^*(x_2,y_2) \geqslant G^*(x_2,y_2)$, $W^*(x_2,y_1) < G^*(x_2,y_1)$, $W^*(x_1,y_2) \geqslant G^*(x_1,y_2)$ and $W^*(x_1,y_1) < G^*(x_1,y_1)$. This case is similar to case 4.

Case 7: $W^*(x_2,y_2) \geqslant G^*(x_2,y_2)$, $W^*(x_2,y_1) < G^*(x_2,y_1)$, $W^*(x_1,y_2) < G^*(x_1,y_2)$ and $W^*(x_1,y_1) \geqslant G^*(x_1,y_1)$. We have

$$V_{H*}(\mathbf{P}) \geqslant G^*(x_2,y_2) - G^*(x_2,y_1) - G^*(x_1,y_2) + G^*(x_1,y_1) = 0\,,$$

where the last equality holds due to the modularity of G^*.

Case 8: $W^*(x_2,y_2) \geqslant G^*(x_2,y_2)$, $W^*(x_2,y_1) < G^*(x_2,y_1)$, $W^*(x_1,y_2) < G^*(x_1,y_2)$

and $W^*(x_1, y_1) < G^*(x_1, y_1)$. We have

$$V_{H*}(\mathbf{P}) \geqslant G^*(x_2, y_2) - G^*(x_2, y_1) - G^*(x_1, y_2) + G^*(x_1, y_1) = 0\,.$$

Case 9: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) \geqslant G^*(x_2, y_1)$, $W^*(x_1, y_2) \geqslant G^*(x_1, y_2)$ and $W^*(x_1, y_1) \geqslant G^*(x_1, y_1)$. We have

$$V_{H*}(\mathbf{P}) > W^*(x_2, y_2) - W^*(x_2, y_1) - W^*(x_1, y_2) + W^*(x_1, y_1) \geqslant 0\,.$$

Case 10: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) \geqslant G^*(x_2, y_1)$, $W^*(x_1, y_2) \geqslant G^*(x_1, y_2)$ and $W^*(x_1, y_1) < G^*(x_1, y_1)$. We have

$$V_{H*}(\mathbf{P}) > W^*(x_2, y_2) - W^*(x_2, y_1) - W^*(x_1, y_2) + W^*(x_1, y_1) \geqslant 0\,.$$

Case 11: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) \geqslant G^*(x_2, y_1)$, $W^*(x_1, y_2) < G^*(x_1, y_2)$ and $W^*(x_1, y_1) \geqslant G^*(x_1, y_1)$. First, note that $W^*(x_2, y_1) = 0$, since otherwise,

$$y_2 - y_1 = W^*(x_2, y_2) - W^*(x_2, y_1) < G^*(x_2, y_2) - G^*(x_2, y_1) \leqslant y_2 - y_1\,,$$

a contradiction. Hence, $W^*(x_2, y_1) = 0$ and thus also $W^*(x_1, y_1) = 0$. Now,

$$V_{H*}(\mathbf{P}) = G^*(x_2, y_2) - G^*(x_1, y_2) \geqslant 0\,,$$

since G^* is an increasing function.

Case 12: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) \geqslant G^*(x_2, y_1)$, $W^*(x_1, y_2) < G^*(x_1, y_2)$ and $W^*(x_1, y_1) < G^*(x_1, y_1)$. We show that this case cannot occur. Indeed, if the previous inequalities were true, then, as in case 11, it holds that $W^*(x_2, y_1) = W^*(x_1, y_1) = 0$. Then,

$$0 = W^*(x_1, y_1) < G^*(x_1, y_1) \leqslant G^*(x_2, y_1) \leqslant W^*(x_2, y_1) = 0\,,$$

a contradiction.

Case 13: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) < G^*(x_2, y_1)$, $W^*(x_1, y_2) \geqslant G^*(x_1, y_2)$ and $W^*(x_1, y_1) \geqslant G^*(x_1, y_1)$. This case is similar to case 11.

Case 14: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) < G^*(x_2, y_1)$, $W^*(x_1, y_2) \geqslant G^*(x_1, y_2)$ and $W^*(x_1, y_1) < G^*(x_1, y_1)$. This case cannot occur and the proof is analogous to the proof of case 12.

Case 15: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) < G^*(x_2, y_1)$, $W^*(x_1, y_2) < G^*(x_1, y_2)$ and $W^*(x_1, y_1) \geqslant G^*(x_1, y_1)$. We have

$$V_{H*}(\mathbf{P}) \geqslant G^*(x_2, y_2) - G^*(x_2, y_1) - G^*(x_1, y_2) + G^*(x_1, y_1) = 0\,.$$

Case 16: $W^*(x_2, y_2) < G^*(x_2, y_2)$, $W^*(x_2, y_1) < G^*(x_2, y_1)$, $W^*(x_1, y_2) < G^*(x_1, y_2)$ and $W^*(x_1, y_1) < G^*(x_1, y_1)$. The proof of this case is immediate from the modularity of G^*.

Hence, H^* is a supermodular function and therefore $H_G = \max(W_n, G)$ is a supermodular n-quasi-copula. ☐

With the previous proposition, we can easily prove the following result.

Proposition 7.3. *The n-quasi-copula $Q_{n,l,\mathbf{z},a}$ defined in Theorem 4.13 is a supermodular function.*

Proof. Let $\mathbf{z} \in [0,1]^n$ and $a \in [W_n(\mathbf{z}), M_n(\mathbf{z})]$. Define the function $G : [0,1]^n \to \mathbb{R}$ as $G(\mathbf{x}) = a - \sum_{j=1}^{n}(z_j - x_j)^+$. We will show that G satisfies the conditions of Proposition 7.2. Clearly, G is an increasing 1-Lipschitz continuous function.

We will now see that G satisfies that $G(\mathbf{x}) \leqslant 0$ for any $\mathbf{x} \in [0,1]^n$ such that there exists $i \in \{1, 2, \ldots, n\}$ with the property that $x_i = 0$ and $x_j = 1$ for all $i \neq j$. To prove the latter, consider, without loss of generality, the point $\mathbf{x} = (0, 1, 1, \ldots, 1)$. Then

$$G(\mathbf{x}) = a - \sum_{j=1}^{n}(z_j - x_j)^+ = a - (z_1 - 0) - \sum_{j=2}^{n}(z_j - 1)^+ = a - z_1 \leqslant 0\,,$$

where the last inequality follows from the condition $a \leqslant M_n(\mathbf{z})$. Finally, from Lemma 3.1 we can conclude that G is a modular function.

Hence, G satisfies all the conditions of Proposition 7.2 and therefore we conclude that $Q_{n,l,\mathbf{z},a} = \max(W_n, G)$ is a supermodular n-quasi-copula. ☐

Now, we focus our attention to the lattice structure of the set of supermodular n-quasi-copulas. We start by a simple generalization of Theorem 4.12.

Proposition 7.4. *The poset of supermodular n-quasi-copulas $\mathcal{SQ}_n$ is not a complete lattice.*

Proof. We follow the same steps as the proof of Theorem 4.12 in [72, 161]. Consider the 2-copulas $C_{2,1}(x_1, x_2) = \min(x_1, x_2, \max(0, x_1 - \frac{2}{3}, x_2 - \frac{1}{3}, x_1 + x_2 - 1))$ and $C_{2,2}(x_1, x_2) = C_{2,1}(x_2, x_1)$. It holds that $Q_{2,L} = C_{2,1} \vee C_{2,2}$ is a proper 2-quasi-copula, i.e., a 2-quasi-copula that is not supermodular. Now, consider the n-copulas $C_{n,1}$ and $C_{n,2}$ given by

$$C_{n,1}(\mathbf{x}) = C_{2,1}(x_1, x_2)\prod_{i=3}^{n} x_i$$

and
$$C_{n,2}(\mathbf{x}) = C_{2,2}(x_1, x_2) \prod_{i=3}^{n} x_i \,.$$

Since $C_{n,1}$ and $C_{n,2}$ are n-copulas, they are also supermodular functions [11]. But $Q_{n,L} = C_{n,1} \vee C_{n,2}$ is a proper n-quasi-copula that is not supermodular, since the section $Q_{n,L,\mathbf{b},B}(\mathbf{x}) = Q_{2,L}(x_1, x_2)$ is not a supermodular function, where $\mathbf{b} = (1, 1, \ldots, 1)$ and $B = \{1, 2\}$. Hence, $\mathcal{SQ}_n$ is not a complete lattice. □

We now show that $\mathcal{Q}_n$ is not meet-dense in $\mathcal{SQ}_n$.

Proposition 7.5. *For $n \geqslant 3$, there exists an n-quasi-copula $Q_{n,L}$ such that for any $A \subseteq \mathcal{SQ}_n$ it holds that $Q_{n,L} \neq \bigwedge_{\mathcal{Q}_n} A$.*

Proof. We will first analyse the case $n = 3$. We will consider the proper 3-quasi-copula used in [72] and in Corollary 4.2. Let $C_{3,1}$ be the 3-copula whose mass is distributed uniformly along the main diagonals of the 3-boxes $[0, 1/4]^3$, $[1/4, 1/2] \times [1/2, 3/4]^2$, $[1/2, 3/4] \times [1/4, 1/2]^2$ and $[3/4, 1]^3$; and let $C_{3,2}$ be the 3-copula whose mass is distributed uniformly along the main diagonals of the 3-boxes $[0, 1/4]^3$, $[1/4, 1/2] \times [1/2, 3/4] \times [1/4, 1/2]$, $[1/2, 3/4] \times [1/4, 1/2] \times [1/4, 1/2]$ and $[3/4, 1]^3$. Define $Q_{3,L}$ as $Q_{3,L} = C_{3,1} \vee C_{3,2}$. Note that $Q_{3,L}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$ and $Q_{3,L}(1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2} = Q_{3,L}(\frac{1}{2}, 1, \frac{1}{2})$.

In [72] it was proven that $Q_{3,L}$ is a proper 3-quasi-copula such that for any $A \subseteq \mathcal{C}_3$ it holds that $Q_{3,L} \neq \bigwedge_{\mathcal{Q}_3} A$. Using similar arguments, we now show that the same holds true when we consider the set of supermodular n-quasi-copulas instead of the set of n-copulas.

Suppose that the latter is not true, i.e., that there exists $A \subseteq \mathcal{SQ}_3$ such that $Q_{3,L} = \bigwedge_{\mathcal{Q}_3} A$. Then, for any $\epsilon > 0$, there exists a sequence of supermodular 3-quasi-copulas $(S_{3,i})_{i=1}^{\infty}$ such that for any i it holds that
$$\frac{1}{4} = Q_{3,L}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \leqslant S_{3,i}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) < Q_{3,L}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \epsilon = \frac{1}{4} + \epsilon \,.$$

Also note that for any i it holds that $S_{3,i}(1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2} = S_{3,i}(\frac{1}{2}, 1, \frac{1}{2})$, due to properties (q2) and (q3) of n-quasi-copulas and the inequalities $Q_{3,L}(1, \frac{1}{2}, \frac{1}{2}) \leqslant S_{3,i}(1, \frac{1}{2}, \frac{1}{2})$, $Q_{3,L}(\frac{1}{2}, 1, \frac{1}{2}) \leqslant S_{3,i}(\frac{1}{2}, 1, \frac{1}{2})$. Hence, we have the following inequalities
$$S_{3,i}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \geqslant \frac{1}{4} \,,$$
$$S_{3,i}\left(1, \frac{1}{2}, \frac{1}{2}\right) - S_{3,i}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) > \frac{1}{4} - \epsilon \,,$$
$$S_{3,i}\left(\frac{1}{2}, 1, \frac{1}{2}\right) - S_{3,i}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) > \frac{1}{4} - \epsilon \,,$$

$$S_{3,i}\left(1,1,\frac{1}{2}\right) - S_{3,i}\left(1,\frac{1}{2},\frac{1}{2}\right) - S_{3,i}\left(\frac{1}{2},1,\frac{1}{2}\right) + S_{3,i}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \geqslant 0\,.$$

Then, by adding each side of the previous four inequalities, we obtain

$$S_{3,i}\left(1,1,\frac{1}{2}\right) > \frac{3}{4} - 2\epsilon\,,$$

which contradicts property (q2) of n-quasi-copulas if one takes $\epsilon < \frac{1}{8}$. Hence, such set $A \subseteq \mathcal{SQ}_3$ does not exist.

For $n \geqslant 4$, consider the n-quasi-copula given by

$$Q_{n,L}(x_1, x_2, \ldots, x_n) = Q_{3,L}(x_1, x_2, x_3) \prod_{k=4}^{n} x_k\,.$$

Then, using the same arguments as those given for the case $n = 3$, we can conclude that for any $A \subseteq \mathcal{SQ}_n$ it holds that $Q_{n,L} \neq \bigwedge_{\mathcal{Q}_n} A$. $\square$

Remark 7.1. The copulas $C_{3,1}$ and $C_{3,2}$ first appeared in [23], to show that there does not necessarily exist a greatest extension of a finite n-subcopula to an n-copula for $n \geqslant 3$.

The following corollary is an immediate consequence of the previous proposition.

Corollary 7.1. *For $n \geqslant 3$, $\mathcal{Q}_n$ is not order-isomorphic to the Dedekind-MacNeille completion of $\mathcal{SQ}_n$.*

Even though $\mathcal{Q}_n$ is not order-isomorphic to the Dedekind-MacNeille completion of $\mathcal{SQ}_n$, it holds true that $\mathcal{SQ}_n$ is join-dense in $\mathcal{Q}_n$, as the following theorem shows.

Theorem 7.3. *A function $Q_n : [0,1]^n \to [0,1]$ is an n-quasi-copula if and only if there exists $A_{Q_n} \subseteq \mathcal{SQ}_n$ such that $Q_n = \bigvee_{\mathcal{Q}_n} A_{Q_n}$.*

Proof. Suppose that Q_n is an n-quasi-copula. For any $\mathbf{z} \in [0,1]^n$ define $a_{\mathbf{z},Q_n}$ as $a_{\mathbf{z},Q_n} = Q_n(\mathbf{z})$. Define the set A as

$$A = \{Q_{n,l,\mathbf{z},a_{\mathbf{z},Q_n}} \mid \mathbf{z} \in [0,1]^n\}\,.$$

Note that due Theorem 4.13 it holds for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$ that

$$Q_{n,l,\mathbf{y},a_{\mathbf{x},Q_n}}(\mathbf{x}) \leqslant Q_n(\mathbf{x}) = a_{\mathbf{x},Q_n} = Q_{n,l,\mathbf{x},a_{\mathbf{x},Q_n}}(\mathbf{x})\,.$$

By taking the supremum over all possible values of $\mathbf{y}$, it follows that

$$\sup\{Q_{n,l,\mathbf{y},a_{\mathbf{x},Q_n}}(\mathbf{x}) \mid \mathbf{y} \in [0,1]^n\} \leqslant Q_n(\mathbf{x}) = a_{\mathbf{x},Q_n} = Q_{n,l,\mathbf{x},a_{\mathbf{x},Q_n}}(\mathbf{x})\,.$$

Hence, it holds that $Q_n = \bigvee_{\mathcal{Q}_n} A$. Additionally, from Proposition 7.3 it follows that $A \subseteq \mathcal{SQ}_n$, proving the desired result.

The converse trivially follows from Theorem 4.12. $\square$

PART III

CONCLUSIONS

8 General conclusions

In this chapter, we summarize the main conclusions that can be drawn from each of the chapters of this dissertation.

First, in Chapter 1 we recalled several concepts that were useful for the development of this dissertation: the concept of an n-copula and several of its properties, the well-known Sklar theorem, some families of n-copulas, and some examples of measures of dependence.

In Chapter 2, we proposed a generalization of bivariate semilinear copulas in higher dimensions and constructed a class of symmetric n-copulas by linearly interpolating on segments connecting the main diagonal of the unit hypercube $[0,1]^n$ to one of its upper faces. For given diagonal functions $d_2, d_3, ..., d_n$, we found conditions that guarantee the existence of a semilinear n-copula $U_{\mathcal{D}_n}$ such that d_n is the diagonal section of $U_{\mathcal{D}_n}$, d_{n-1} is the diagonal section of the $(n-1)$-dimensional marginals of $U_{\mathcal{D}_{n-1}}$, etc. We note that the conditions become really restrictive as the dimension increases. However, up to the our knowledge, this is one of the first attempts to build a copula given both the diagonal section of the copula and the diagonal sections of all of its marginals, another one being the work of Mai et al. [133]. A couple of questions arise from the results obtained in Chapter 2.

For the first question, first note that while we were deriving Eq. (2.2), we used the fact that the $(n-1)$-dimensional marginal also belongs to the class of upper semilinear copulas. Hence, the question is: is it possible to obtain conditions for an arbitrary $(n-1)$-dimensional marginal, i.e., to characterize the n-diagonal functions and $(n-1)$-copulas such that it is possible to construct a semilinear n-copula from them?

The second question is related to the compatibility problem: given a set of diagonal functions $d_2, ..., d_n$, what are the necessary and sufficient conditions that guarantee the existence of a symmetric n-copula, such that the diagonal section of the n-copula is d_n, d_{n-1} is the diagonal section of the $(n-1)$-dimensional marginals of the n-copula, etc. This problem can be thought of as a 'symmetric diagonal compatibility problem', which is obviously a particular case of the 'copula compatibility problem'. Obviously, the conditions of Theorem 2.1 are sufficient to guarantee that the diagonal functions are compatible, but not necessary. This can be easily seen by considering the diagonal functions $d_2 = x^2, \ldots, d_n = x^n$, as they do not satisfy condition (i) of Theorem 2.1, but they are compatible as d_n is the diagonal section of the product copula, while the others are the diagonal sections of its marginals.

Next, in Chapter 3 we looked at radially symmetric copulas and proved a representation theorem for symmetric and radially symmetric copulas in terms of an auxiliary function H that does not necessarily need to be an n-copula; H has to satisfy the weaker condition that for any n-box $\mathbf{P}$ it holds that $V_H(\mathbf{P}) + V_H(\mathbf{1} - \mathbf{P}) \geqslant 0$. Additionally, we have used this representation in order to propose a method to construct radially symmetry copulas in higher dimensions. However, in general, it is difficult to verify whether the volume condition on H that guarantees that $S_{C_{n-1},H}$ is an n-copula is satisfied. Moreover, it could happen that there does not exist an n-copula such that all of its $(n-1)$-dimensional marginals are equal to C_{n-1}, i.e., they are not compatible.

In the case when it is possible to construct a symmetric n-copula such that its $(n-1)$-dimensional marginal is C_{n-1}, there are several ways to construct the function H. We have presented three possible options for H in the three-dimensional case: one inspired by the nesting of 2-copulas, another inspired by the $\star_{\mathbf{D}}$ product of copulas and the third one based on the product of copulas. While the results in Section 3.2 hold for any dimension n, the examples that we present are not easily generalized to higher dimensions, as there may not be a unique way to choose H, as highlighted in the discussion at the end of Section 3.3.

In Chapter 4, we recalled the definition of a quasi-copula as it was originally introduced. We also analysed numerous characterizations of quasi-copulas, while highlighting the differences between the bivariate case and the n-dimensional case.

Additionally, we have studied how quasi-copulas have been used in the literature to develop bounds on sets of copulas, while emphasizing again the differences between the bivariate case and the n-dimensional case. We also recalled some results concerning the mass distribution induced by quasi-copulas, starting by recalling that there are quasi-copulas that do not induce signed measures, and how the quasi-copulas that induce signed measures are 'small' from a Baire category point of view.

Finally, we recalled several applications of quasi-copulas in other fields different from statistics, for example, how quasi-copulas have been used as conjunctors in fuzzy probability calculus, their importance in fuzzy preference modelling and their applications as aggregation functions, including how they are used to extend fuzzy measures on specific subsets of the natural numbers.

Next, in Chapter 5 we studied the class of supermodular n-quasi-copulas for $n \geqslant 3$. The main result of this chapter is that some properties of 2-copulas that do not extend to higher dimensions, hold true for supermodular n-quasi-copulas. Examples of such properties are the supermodularity of the lower Fréchet-Hoeffding bound, as well as the characterization of supermodular Archimedean n-quasi-copulas in terms of a convex generator. It is worth to remark that most of the proofs presented in this chapter regarding supermodular n-quasi-copulas are similar to the proofs of the corresponding results in the framework of bivariate copulas. This suggests that regular n-quasi-copulas are closer to supermodular n-quasi-copulas than to n-copulas.

Additionally, we have noted that there are more classes of n-quasi-copulas in between n-quasi-copulas and n-copulas. With these new classes, we were able to characterize a certain class of n-quasi-copulas that are used in a generalization of the Lovász extension and the Owen extension of monotone games. We note that one of the classes, namely $(n-1)$-dim-increasing n-quasi-copulas, could be more useful to study n-copulas than regular n-quasi-copulas. However, the construction of $(n-1)$-dim-increasing n-quasi-copulas is not easy, since all of their $(n-1)$-marginals are $(n-1)$-copulas, and as a consequence, the construction of such class of n-quasi-copulas is closely related to the compatibility problem mentioned in Chapter 1.

In Chapter 6 we studied the smallest and the greatest Lipschitz continuous n-ary aggregation functions with a given diagonal section and we showed that several results from the bivariate case extend naturally to the multivariate case, such as the way these functions can be computed.

Additionally, we proved that the smallest Lipschitz continuous n-ary aggregation function with a given diagonal section is supermodular and the greatest Lipschitz continuous n-ary aggregation function with a given diagonal section is submodular. As a by-product we showed that this is another result of bivariate copulas that extends to the class of supermodular quasi-copulas in higher dimensions.

Finally, we showed that the Bertino n-quasi-copula is an n-copula if the diagonal section d is $n/(n-1)$-Lipschitz continuous. This condition is also necessary in the case of regular n-diagonal functions. It is still an open research topic to analyse whether or not the regularity condition can be weakened. Anyhow, the result is very restrictive, in the sense that as the dimension increases, the set of n-diagonal functions for which there exists an n-dimensional Bertino copula, gets smaller.

Thereafter, in Chapter 7 we showed that both the metric structure and the lattice structure of the set of supermodular n-quasi-copulas has similar properties as the lattice structure of the set of n-copulas. The proofs of several of the results for supermodular n-quasi-copulas follow the same lines as their n-copula counterparts. However, there is one big difference in the case of the lattice structure of the set of supermodular n-quasicopulas. This set is join-dense in the set of n-quasi-copulas, even though the set of n-quasi-copulas is not isomorphic to the Dedekind-MacNeille completion of $\mathcal{SQ}_n$. The latter result shows that n-quasi-copulas are more closely related to supermodular functions than to n-copulas.

As a general conclusion from this dissertation, we can see that both n-copulas and n-quasi-copulas can be studied from different point of views. In the case of n-copulas we studied two construction methods based on the diagonal section, and a construction method based on radial symmetry. In the case of n-quasi-copulas we studied the class of supermodular n-quasi-copulas, the smallest n-quasi-copula with a given diagonal section and the metric structure and the lattice structure of the set of supermodular n-quasi-copulas. These are just a few results that can be obtained when one looks through the kaleidoscope of n-copulas and n-quasi-copulas.

Summary

Copulas have become a valuable tool in multivariate statistics. Due to Sklar's theorem, it is possible to express a continuous multivariate distribution function in terms of its n univariate marginals by means of a unique n-copula. Consequently, n-copulas have become one of the most important tools for the study of certain types of non-parametric properties of random vectors, such as stochastic dependence.

The theory of copulas has been growing in the last years. Nowadays, there are many results on copulas that have been developed from different points of view. In this dissertation we also study n-copulas while looking at them from several points of view, hence the word 'kaleidoscopic' in the title. In the first three chapters of this dissertation, we summarize several results about n-copulas, and then propose two construction methods for n-copulas: one based on the diagonal section of an n-copula, and the other based on the property of radial symmetry.

In Chapter 1, we first recall the concept of an n-copula. We also review several important properties and results about n-copulas that are relevant for the development of this dissertation.

In Chapter 2, we generalize the well-known class of bivariate upper semilinear copulas to higher dimensions. These new upper semilinear n-copulas are constructed by linear interpolation on segments connecting the main diagonal of the unit hypercube $[0,1]^n$ to one of its upper faces. Later, we focus on the particular case where all the lower dimensional marginals are also upper semilinear copulas themselves, in which case the n-copula is actually constructed given its diagonal section and the diagonal sections of its lower dimensional marginals. For this construction method, we study which necessary and sufficient conditions on these diagonal sections guarantee that the upper semilinear construction method yields an n-copula.

In Chapter 3, we propose a construction method for n-copulas that are simultaneously symmetric and radially symmetric. To this end, we first prove a representation theorem for n-copulas that are simultaneously symmetric and radially symmetric. With the help of this representation theorem we propose a method to construct an n-ary symmetric function that is radially symmetric, starting from an $(n-1)$-copula and an n-ary auxiliary function. Next, we find the necessary and sufficient conditions on this auxiliary function that guarantee our construction method to result in a symmetric and radially symmetric n-dimensional copula. Finally, we restrict mainly to the trivariate case to examine several options for defining the auxiliary function.

Next, we turn our attention to the concept of an n-quasi-copula, a concept that

is closely connected to that of an n-copula. n-quasi-copulas have been mainly used to find best-possible bounds on arbitrary sets of n-copulas, remarkably in the bivariate case. In the following chapters of this dissertation we review several results about n-quasi-copulas; then we introduce two new classes of n-quasi-copulas to show that n-quasi-copulas are more closely related to supermodular functions than to n-copulas.

In Chapter 4, we discuss the role played by n-quasi-copulas in the study of n-copulas. We recall the concept of an n-quasi-copula, starting from the characterization of functions that can be derived from operations on random variables. Then, we review the several characterizations and properties that have been proven in the literature. We also highlight the applications of n-quasi-copulas in the study of n-copulas, such as their role in the study of bounds on sets of n-copulas. Special emphasis is placed on the differences between the bivariate case and the higher-dimensional setting ($n \geqslant 3$).

In Chapter 5, we introduce the classes of supermodular n-quasi-copulas and k-dimensionally-increasing n-quasi-copulas. We observe that some properties of 2-copulas that cannot be generalized to higher-dimensional copulas, hold true for supermodular n-quasi-copulas. Additionally, we show that k-dimensionally-increasing n-quasi-copulas play a role in a generalization of a volume-based characterization of bivariate copulas to higher dimensions.

Chapter 6 consists of two important parts. In the first part we work in the more general framework of n-ary aggregation functions. In particular, we study the smallest and the greatest M-Lipschitz continuous n-ary aggregation functions with a given diagonal section and generalize several results from the bivariate case to the higher-dimensional case while considering different Lipschitz constants. Then, we used the results obtained in the framework of n-ary aggregation functions to prove that the smallest n-quasi-copula with a given diagonal section, called the Bertino n-quasi-copula, is supermodular for any $n \geqslant 2$.

Subsequently, in the second part of Chapter 6, we study the Bertino n-quasi-copula in depth. We start by studying the marginal copulas of an n-dimensional Bertino n-quasi-copula and we show that all marginal n-quasi-copulas of an n-dimensional Bertino n-quasi-copula are Bertino n-quasi-copulas themselves. Later, we introduce the notion of a regular n-diagonal function and we characterise the sets of regular n-diagonal functions for which there exists an n-dimensional Bertino copula whose diagonal section coincides with the given n-diagonal function.

Chapter 7 also consists of two parts. First, we study the set of n-quasi-copulas from a metric-space point of view. We see that the set of supermodular n-quasi-copulas when endowed with the uniform metric has similar properties as the metric space of n-copulas endowed with the uniform metric. Second, we study the relationship between the poset of supermodular n-quasi-copulas and the posets of n-quasi-copulas and n-copulas. We show that the poset of supermodular n-quasi-copulas is not order-isomorphic to the Dedekind-MacNeille completion of the poset of n-copulas, although the structure of the poset of n-quasi-copulas is more closely related to that of the poset of supermodular n-quasi-copulas than that of the poset of n-copulas.

Finally, in Chapter 8 we summarize the results that we obtained in this thesis. We also discuss some interesting questions that arise from the research done during the development of this dissertation.

Nederlandstalige samenvatting

Copula's zijn een nuttig instrument geworden voor de multivariate statistiek. Vanwege het theorema van Sklar, is het mogelijk om een continue multivariate verdelingsfunctie uit te drukken in functie van zijn univariate marginalen aan de hand van een unieke n-copula. Bijgevolg zijn n-copula's één van de belangrijkste instrumenten geworden voor het bestuderen van enkele niet-parametrische eigenschappen van stochastische variabelen, zoals stochastische afhankelijkheid.

De theorie van copula's is in de laaste jaren blijven groeien. Tegenwoordig zijn er veel resulaten over copula's die vanuit verschillende gezichtspunten zijn ontwikkeld. In deze dissertatie bestuderen wij ook n-copula's terwijl we ze bekijken vanuit verschillende gezichtspunten, vandaar het woord 'kaleidoscopic' in de titel. In de eerste drie hoofdstukken van deze dissertatie vatten we verschillende resultaten over n-copula's samen, en daarna stellen we twee constructiemethoden voor n-copula's voor: één is gebaseerd op de diagonale sectie van een n-copula, en de andere is gebaseerd op de eigenschap van radiale symmetrie.

In Hoofdstuk 1 herhalen we het concept van een n-copula. We geven ook een overzicht van verschillende belangrijke eigenschappen en resultaten over n-copula's die relevant zijn voor de verdere uitwerking van deze dissertatie.

In Hoofdstuk 2 veralgemenen we de bekende klasse van bivariate bovensemilineaire copula's naar hogere dimensies. Deze bovensemilineaire n-copula's worden geconstrueerd door lineaire interpolatie op segmenten die de hoofddiagonaal van de n-dimensionale kubus $[0,1]^n$ verbinden met één van de bovenvlakken van deze kubus. Daarna concentreren we ons op het bijzondere geval wanneer alle lagerdimensionale marginalen ook behoren tot de klasse van bovensemilineaire copula's. In dit geval wordt de n-copula eigenlijk geconstrueerd met een gegeven diagonale sectie en de diagonale secties van al zijn lagerdimensionale marginalen. Voor deze constructiemethode bestuderen we de nodige en voldoende voorwaarden waaraan de diagonale secties moeten voldoen zodat de bovensemilineaire constructiemethode in een n-copula resulteert.

In Hoofdstuk 3 introduceren we een constructiemethode voor n-copula's die simultaan symmetrisch en radiaalsymmetrisch zijn. Hiertoe bewijzen we ten eerste een representatiestelling voor n-copula's die simultaan symmetrisch en radiaalsymmetrisch zijn. Met behulp van die representatiestelling introduceren we een constructiemethode voor een n-dimensionale symmetrische functie die radiaalsymmetrisch is, uitgaande van een radiaalsymmetrische $(n-1)$-copula en een n-dimensionale hulpfunctie. Daarna identificieren we in de nodige en voldoende voorwaarden waaraan de n-dimensionale hulpfunctie moet voldoen zodat onze constructiemethode een symmetrische en radiaalsymmetrische n-copula voortbrengt. Uiteindelijk beperken we ons tot het trivariate geval en onderzoeken we verschillende keuzes voor de definitie van de hulpfunctie.

Vervolgens richten we onze aandacht op het concept van een n-quasi-copula, een concept dat sterk verwant is met het concept van een n-copula. n-quasi-copula's worden meestal gebruikt om de bovengrenzen en ondergrenzen van willekeurige verzamelingen van n-copula's te bestuderen, in het bijzonder in het bivariate geval. In de volgende hoofdstukken van deze dissertatie geven we een overzicht van verschillende resultaten over n-quasi-copula's, daarna introduceren we twee nieuwe klassen van n-quasi-copula's om aan te tonen dat n-quasi-copula's sterker verwant zijn met supermodulaire functies dan met n-copula's.

In Hoofdstuk 4 bespreken wij de rol die wordt gespeeld door n-quasi-copula's in het bestuderen van n-copula's. Wij herhalen het concept van een n-quasi-copula, beginnend vanaf de representatie van functies die kunnen worden afgeleid van operaties die op stochastische variabelen worden toegepast. Daarna geven we een overzicht van de verschillende representaties en eigenschappen van n-quasi-copula's die in de literatuur zijn bewezen. We benadrukken ook de toepassingen van n-quasi-copula's in het bestuderen van n-copula's, zoals hun rol in het bestuderen van bovengrenzen en ondergrenzen van verzamelingen van n-copula's. Bijzondere aandacht wordt besteed aan de verschillen tussen het bivariate geval en het hoogdimensionale geval ($n \geqslant 3$).

In Hoofdstuk 5 introduceren we de klassen van supermodulaire n-quasi-copula's en k-dimensionaal stijgende n-quasi-copula's. We merken op dat enkele eigenschappen van 2-copula's die niet naar hoogdimensionale copula's kunnen worden veralgemeend, kunnen worden veralgemeend voor supermodulaire n-quasi-copula's. Bovendien tonen wij aan dat k-dimensionaal stijgende n-quasi-copula's een rol spelen in een veralgemening van een volume-gebaseerde representatie van bivariate copula's naar hogere dimensies.

Hoofdstuk 6 bestaat uit twee delen. In het eerste deel werken we in het algemenere kader van n-dimensionale aggregatiefuncties. In het bijzonder bestuderen we de kleinste en de grootste M-Lipschitz-continue n-dimensionale aggregatiefuncties met een gegeven diagonale sectie, en nadien veralgemenen we verschillende resultaten van het bivariate geval naar het hoogdimensionale geval waarbij we verschillende Lipschitz-constanten beschouwen. Daarna gebruiken we de resultaten die we in het kader van n-dimensionale aggregatiefuncties hebben verkregen om te bewijzen dat de kleinste n-quasi-copula met een gegeven diagonale sectie, die de Bertino n-quasi-copula genoemd wordt, een supermodulaire functie is voor elke $n \geqslant 2$.

In het tweede deel van Hoofdstuk 6 diepen we de Bertino n-quasi-copula uit. We beginnen met het bestuderen van de marginalen van een Bertino n-quasi-copula en dan tonen we aan dat alle marginalen van een Bertino n-quasi-copula ook behoren tot de klasse van Bertino quasi-copula's. Daarna introduceren we het concept van een reguliere n-dimensionale diagonale functie en karakteriseren we de reguliere n-dimensionale diagonale functies waarvoor er een n-dimensionale Bertino copula bestaat waarvan de diagonale sectie samenvalt met de gegeven n-diagonale functie.

Hoofdstuk 7 bestaat ook uit twee delen. Ten eerste bestuderen we de verzameling van n-quasi-copula's vanuit het gezichtspunt van metrische ruimtes. We tonen aan dat de verzameling van supermodulaire n-quasi-copula's met de uniforme metriek gelijkaardige eigenschappen heeft als de verzameling van n-copula's met de uniforme metriek. Ten tweede bestuderen we het verband tussen de partieel geordende verzameling van supermodulaire n-quasi-copula's en de partieel geordende verzamelingen van n-quasi-copula's en n-copula's. We tonen aan dat hoewel de partieel geordende verzameling van supermodulaire n-quasi-copula's niet orde-isomorf is met de Dedekind-MacNeille vervollediging van de partieel geordende verzameling van n-copula's, de structuur van de partieel geordende verzameling van n-quasi-copula's sterker verwant is met de partieel geordende verzameling van supermodulaire n-quasi-copula's dan met de partieel geordende verzameling van n-copula's.

Finaal vatten we in Hoofdstuk 8 de resultaten samen die we in deze dissertatie hebben bekomen. We bespreken ook interessante vragen die voortkomen uit het gevoerde onderzoek tijdens de uitwerking van deze dissertatie.

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Curriculum Vitae

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Education

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2010-2011: M.Sc. Mathematical sciences, Universidad Nacional Autónoma de México, Distrito Federal, México.

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Primary and secondary education

High school

2002–2005: Colegio México Bachillerato, Distrito Federal, México.

Middle school

1999–2002: Colegio St. John's, Distrito Federal, México.

Elementary school

1993–1999: Colegio St. John's, Distrito Federal, México.

Employment

Current employment

Full-time researcher at the Research Unit Knowledge-Based Systems, Department of Data Analysis and Mathematical Modelling, Faculty of Bioscience Engineering, Ghent University.

Teaching experience

- Lecturer of the bachelor course "Probability I", Faculty of Sciences, Universidad Nacional Autónoma de México (spring 2012, fall 2012, fall 2013, spring 2014).
- Lecturer of the bachelor course "Probability II", Faculty of Sciences, Universidad Nacional Autónoma de México (fall 2012, spring 2013).
- Lecturer of the bachelor course "Statistics I", Faculty of Sciences, Universidad Nacional Autónoma de México (fall 2013).
- Lecturer of the bachelor course "Stochastic Processes I", Faculty of Sciences, Universidad Nacional Autónoma de México (spring 2013, spring 2014).
- Diploma instructor of the modules "Introduction to Probability and Statistics" and "Advanced Probability", Diploma Solvency II, Actuary Hunters, México (spring and summer 2014).
- Diploma instructor of the module "Stochastic Simulation", Comisión Nacional de Seguros y Fianzas, México (winter 2012–spring 2013).

Other working expirience

- External consultant, Comisión Nacional de Seguros y Fianzas, México (2013–2014).
- Intership: Comisión Nacional de Seguros y Fianzas, México (2009– 2010).

Scientific output

Publications in international journals (ISI-papers)

- J.J. Arias-García and B. De Baets, *On the lattice structure of the set of supermodular quasi-copulas*, Fuzzy Sets and Systems, accepted March 2018, https://doi.org/10.1016/j.fss.2018.03.013.

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- J.J. Arias-García, H. De Meyer and B. De Baets, *On the construction of radially symmetric trivariate copulas*, In: Soft Methods for Data Science (M. Ángeles-Gil, M.B. Ferraro, M. Gagolewski, P. Giordani, P. Grzegorzewski, O. Hryniewicz and B. Vantaggi, eds.), Springer Berlin Heidelberg, 2010, 3–31.

Conference abstracts

- Intermediate classes between quasi-copulas and copulas in higher dimensions, Copulas and Their Applications, To commemorate the 75th birthday of Professor Roger B. Nelsen, Almería, Spain, July 3–5, 2017.
- A method to construct radially symmetric trivariate copulas: Salzburg Workshop on Dependence Models & Copulas, Salzburg, September 19-21, 2016..
- Construction of flipping-invariant functions in higher dimensions: 36th Linz Seminar on Fuzzy Set Theory (LINZ 2016), Linz, February 2–6, 2016.

Active participation at international scientific events

- Copulas and Their Applications, To commemorate the 75th birthday of Professor Roger B. Nelsen, organized by the University of Almeria, Almería, Spain, July 3–5, 2017.
- Salzburg Workshop on Dependence Models & Copulas, organized by the University of Salzburg, Salzburg, Austria, September 19-21, 2016.
- 8th International Conference on Soft Methods in Probability and Statistics (SMPS 2016), organized by La Sapienza University, Rome, Italy, September 12–14, 2016.
- 36th Linz Seminar on Fuzzy Set Theory (LINZ 2016), organized by the Johannes Kepler University, Linz, Austria, February 2–6, 2016.

Attendance at other international scientific events

- Recent Developments in Dependence Modelling with Applications in Finance and Insurance - Fourth Edition, organized by the Vrije Universiteit Brussel, Aegina, Greece, May 22-23, 2017.
- Dependence Modeling in Finance, Insurance and Environmental Science, organized by the Technical University of Munich, Munich, Germany, May 17–19, 2016.