Lifting Coalgebra Modalities and IMELL Model Structure to Eilenberg-Moore Categories

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Abstract

A categorical model of the multiplicative and exponential fragments of intuitionistic linear logic (IMELL), known as a *linear category*, is a symmetric monoidal closed category with a monoidal coalgebra modality (also known as a linear exponential comonad). Inspired by R. Blute and P. Scott's work on categories of modules of Hopf algebras as models of linear logic, we study Eilenberg-Moore categories of monads as models of IMELL. We define an IMELL lifting monad on a linear category as a Hopf monad – in the Bruguieres, Lack, and Virelizier sense – with a mixed distributive law over the monoidal coalgebra modality. As our main result, we show that the linear category structure lifts to Eilenberg-Moore categories of IMELL lifting monads. We explain how monoids in the Eilenberg-Moore category of the monoidal coalgebra modality can induce IMELL lifting monads and provide sources for such monoids. Along the way, we also define mixed distributive laws of bimonads over coalgebra modalities and lifting differential category structure to Eilenberg-Moore categories of exponential lifting monads.

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1 Introduction

Linear logic, as introduced by Girard [11], is a resource sensitive logic which due to its flexibility admits multiple different fragments and a wide range of applications. A categorical model of the multiplicative fragment of intuitionistic linear logic (IMILL) [2, 12] is a symmetric monoidal closed category, while a categorical model of IMILL with negation is a $*$-autonomous category [23]. Categories of modules of (cocommutative) Hopf algebras (over a commutative ring) are important and of interest, especially in representation theory, due in part as they are (symmetric) monoidal closed categories [7, 15]. Blute [5] and Scott [4] studied the idea of interpreting categories of modules of Hopf algebras as models of IMILL with negation and its non-commutative variant. If one were instead to look in a more general setting, categories of modules of cocommutative Hopf monoids in arbitrary symmetric monoidal closed categories are again symmetric monoidal closed categories and therefore models of IMILL. But for what

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kind of monoids in categorical models of the multiplicative and exponential fragments of intuitionistic linear logic (IMELL), is their categories of modules again a categorical model of IMELL?

The exponential fragment of IMELL adds in the exponential modality which is a unary connective ! — read as either "of course" or "bang" — admitting four structural rules [2, 19]: promotion, dereliction, contraction, and weakening. In terms of the categorical semantics: the exponential modality ! is interpreted as a monoidal coalgebra modality [3] (see Definition 14 below) – also known as a linear exponential comonad [22] – which in particular is a symmetric monoidal comonad (capturing the promotion and dereliction rules) such that for each object A, the exponential $!A$ comes equipped with a natural cocommutative comonoid structure (capturing the contraction and weakening rules). Categorical models of IMELL are known as linear categories [2, 17, 19], which are symmetric monoidal closed categories with monoidal coalgebra modalities.

We can now restate the question we aim to answer in this paper:

- **Question 1:** "For what kind of monoid A in a linear category, is the category of modules of A also a linear category?"

We have already discussed part of the answer regarding the symmetric monoidal closed structure of a linear category: the monoid A needs to be a cocommutative Hopf monoid. What remains to be answered is how to 'extend', or better yet 'lift', the monoidal coalgebra modality to the category of modules of A. Observing that if A is a monoid, then the endofunctor $A \otimes -$ is a monad (see Section 9) and so the category of modules of A corresponds precisely to the category of Eilenberg-Moore algebras of the monad $A \otimes -$. Therefore, we can further generalize the question we want answered:

- **Question 2:** "For what kind of monad on a linear category, is the Eilenberg-Moore category of algebras of that monad also a linear category?"

This now becomes a question of how to lift a comonad to the Eilenberg-Moore category of a monad. And the answer to this question brings us into the realm of distributive laws [1, 26].

Main Definitions and Results

The two main definitions of this paper are exponential lifting monads (Definition 17) and IMELL lifting monads (Definition 20). Briefly, an exponential lifting monad is a symmetric bimonad with a mixed distributive law over a monoidal coalgebra modality, while an IMELL lifting monad is an exponential lifting monad on a linear category which is also a Hopf monad. Proposition 16 provides a partial answer to the Question 2, while Theorem 21 provides the full answer. We summarize these two main results as follows:

- The Eilenberg-Moore category of an exponential lifting monad admits a monoidal coalgebra modality (Proposition 16).
- The Eilenberg-Moore category of an IMELL lifting monad is a linear category (Theorem 21).

Section 9 and Theorem 24 are dedicated to answering Question 1. Summarizing, where recall that for a monoid A, the category of modules over A can be seen as the Eilenberg-Moore category of the monad $A \otimes -$, we have the following two results:

- Monoids in the Eilenberg-Moore category of monoidal coalgebra modalities induce exponential lifting monads (Theorem 23).
- Monoids with antipodes in the Eilenberg-Moore category of monoidal coalgebra modalities of a linear category induce IMELL lifting monads (Theorem 24).

In the process of constructing and defining mixed distributive laws involving monoidal coalgebra modalities, we also discuss mixed distributive laws over the strictly weaker notion of coalgebra modalities and define coalgebra modality lifting monads (see Definition 12 below) in order to also discuss lifting differential category structure (see Section 11).

Conventions: In these notes, we will use diagrammatic order for composition: this means that the composite map $f; g$ is the map which first does f then g. Also, to simplify working in a symmetric monoidal category, we will instead work in a strict symmetric monoidal category, that is, we will suppress the unit and associativity isomorphisms. For a symmetric monoidal category we use $\otimes$ for the tensor product, K for the monoidal unit, and $\sigma : A \otimes B \to B \otimes A$ for the symmetry isomorphism.

2 Mixed Distributive Laws Between Monads and Comonads

Distributive laws between monads, which are natural transformations satisfying certain coherences with the monad structures, were introduced by Beck [1] in order to both compose monads and lift one monad to the other's Eilenberg-Moore category. By lifting we mean that the forgetful functor from the Eilenberg-Moore category to base category preserves the monad strictly. In fact, there is a bijective correspondence between distributive laws between monads and lifting of monads. From a higher category theory perspective, a distributive law is a monad on the 2-category of monads of a 2-category [24]. There are also several other notions of distributive laws involving monads and bijective correspondence with certain liftings [26]. Of particular interest for this paper are mixed distributive laws of monads over comonads [1] (see Definition 3 below). For a more detailed introduction on distributive laws and liftings see [26].

If only to introduce notation, we first quickly review the notions of monads and their algebras, and the dual notions of comonads and their coalgebras [14, Chapter VI].

▶ **Definition 1.** A **monad** on a category $\mathbb{X}$ is a triple (T, μ, η) consisting of a functor $\mathsf{T} : \mathbb{X} \to \mathbb{X}$ and two natural transformations $\mu : \mathsf{T}\mathsf{T}A \to \mathsf{T}A$ and $\eta : A \to \mathsf{T}A$ such that:

$$\mu; \mu = \mathsf{T}(\mu); \mu \qquad \eta; \mu = 1 = \mathsf{T}(\eta); \mu \tag{1}$$

A **T-algebra** for a monad (T, μ, η) is a pair (A, ν) consisting of an object A and a map $\nu : \mathsf{T}A \to A$ such that:

$$\mu; \nu = \mathsf{T}(\mu); \nu \qquad \eta; \nu = 1 \tag{2}$$

A **T-algebra morphism** $f : (A, \nu) \to (B, \omega)$ is a map $f : A \to B$ such that $\nu; f = \mathsf{T}(f); \omega$.

The category of T-algebras and T-algebra morphisms is called the **Eilenberg-Moore category of the monad** (T, μ, η) and is denoted $\mathbb{X}^{\mathsf{T}}$. There is a forgetful functor $\mathsf{U}^{\mathsf{T}} : \mathbb{X}^{\mathsf{T}} \to \mathbb{X}$, which is defined on objects as $\mathsf{U}^{\mathsf{T}}(A, \nu) = A$ and on maps as $\mathsf{U}^{\mathsf{T}}(f) = f$.

▶ **Definition 2.** Dually, a **comonad** on a category $\mathbb{X}$ is a triple $(!, \delta, \varepsilon)$ consisting of a functor $! : \mathbb{X} \to \mathbb{X}$ and two natural transformations $\delta : !A \to !!A$ and $\varepsilon : !A \to A$ such that the dual equations of a monad (1) hold. A **!-coalgebra** for a comonad $(!, \delta, \varepsilon)$ is a pair (A, ω), consisting of an object A and a map $\omega : A \to !A$ such that the dual equalities of (2) hold, while !-coalgebra morphisms are the dual analogue of T-algebra morphisms.

The category of !-coalgebras and !-coalgebra morphisms is called the **Eilenberg-Moore category of the comonad** $(!, \delta, \varepsilon)$ and is denoted $\mathbb{X}^!$. There is also a forgetful functor $\mathsf{U}^! : \mathbb{X}^! \to \mathbb{X}$.

▶ **Definition 3.** Let (T, μ, η) be a monad and $(!, \delta, \varepsilon)$ a comonad on the same category. A **mixed distributive law of** (T, μ, η) **over** $(!, \delta, \varepsilon)$ [26] is a natural transformation $\lambda : \mathsf{T}!A \to !\mathsf{T}A$ such that the following diagrams commute:

$$\begin{array}{ccccc} \mathsf{T}\mathsf{T}!A & \xrightarrow{\mathsf{T}(\lambda)} & \mathsf{T}!\mathsf{T}A & \xrightarrow{\lambda} & !\mathsf{T}\mathsf{T}A \\ {\scriptstyle \mu}\downarrow & & & & \downarrow{\scriptstyle !(\mu)} \\ \mathsf{T}!A & & \xrightarrow[\lambda]{\qquad\qquad} & & !\mathsf{T}A \end{array} \qquad \begin{array}{ccc} !A & \xrightarrow{\eta} & \mathsf{T}!A \\ & \searrow^{!(\eta)} & \downarrow{\scriptstyle \lambda} \\ & & !\mathsf{T}A \end{array} \tag{3}$$

$$\begin{array}{ccccc} \mathsf{T}!A & \xrightarrow{\mathsf{T}(\delta)} & \mathsf{T}!!A & \xrightarrow{\lambda} & !\mathsf{T}!A \\ {\scriptstyle \lambda}\downarrow & & & & \downarrow{\scriptstyle !(\lambda)} \\ !\mathsf{T}A & & \xrightarrow[\delta]{\qquad\qquad} & & !!\mathsf{T}A \end{array} \qquad \begin{array}{ccc} \mathsf{T}!A & \xrightarrow{\lambda} & !\mathsf{T}A \\ & \searrow^{\mathsf{T}(\varepsilon)} & \downarrow{\scriptstyle \varepsilon} \\ & & \mathsf{T}A \end{array} \tag{4}$$

As with distributive laws between monads, mixed distributive laws allow one to lift comonads to Eilenberg-Moore categories of monads and also to lift monads to Eilenberg-Moore categories of comonads, such that the respective forgetful functors preserve the monads or comonads strictly. In fact, mixed distributive laws are in bijective correspondence with these liftings:

▶ **Theorem 4.** *[25, Theorem IV.1] Let* (T, μ, η) *be a monad and* $(!, \delta, \varepsilon)$ *be a comonad on the same category* $\mathbb{X}$*. Then the following are in bijective correspondence:*

1. *Mixed distributive laws of* (T, μ, η) *over* $(!, \delta, \varepsilon)$*;*
2. *Liftings of the comonad* $(!, \delta, \varepsilon)$ *to* $\mathbb{X}^{\mathsf{T}}$*, that is, a comonad* $(\tilde{!}, \tilde{\delta}, \tilde{\varepsilon})$ *on* $\mathbb{X}^{\mathsf{T}}$ *such that the forgetful functor* U^{T} *preserves the comonad strictly, that is, the following equalities hold:*

   $$!; \mathsf{U}^{\mathsf{T}} = \mathsf{U}^{\mathsf{T}}; \tilde{!} \qquad \mathsf{U}^{\mathsf{T}}(\tilde{\delta}) = \delta \qquad \mathsf{U}^{\mathsf{T}}(\tilde{\varepsilon}) = \varepsilon$$

3. *Liftings of the monad* (T, μ, η) *to* $\mathbb{X}^{\mathsf{T}}$*, that is, a monad* $(\tilde{\mathsf{T}}, \tilde{\mu}, \tilde{\eta})$ *on* $\mathbb{X}^{!}$ *such that the forgetful functor* $\mathsf{U}^{!}$ *preserves the comonad strictly, that is, the following equalities hold:*

   $$\mathsf{T}; \mathsf{U}^{!} = \mathsf{U}^{!}; \mathsf{T} \qquad \mathsf{U}^{!}(\tilde{\mu}) = \mu \qquad \mathsf{U}^{!}(\tilde{\eta}) = \eta$$

We quickly recall part of how to construct liftings from mixed distributive laws (for more details see [26]). Let λ be a mixed distributive law of (T, μ, η) over $(!, \delta, \varepsilon)$. For a T-algebra (A, ν), the pair $(!A, \nu^\sharp)$ is a T-algebra where the map $\nu^\sharp : \mathsf{T}!A \to !A$ is defined as follows:

$$\nu^\sharp := \mathsf{T}!A \xrightarrow{\lambda} !\mathsf{T}A \xrightarrow{!(\nu)} !A \tag{5}$$

Dually, if (A, ω) is a !-coalgebra, then the pair $(\mathsf{T}A, \omega^\flat)$ is a !-coalgebra where the map $\omega^\flat : \mathsf{T}A \to !\mathsf{T}A$ is defined as follows:

$$\omega^\flat := \mathsf{T}A \xrightarrow{\mathsf{T}(\omega)} \mathsf{T}!A \xrightarrow{\lambda} !\mathsf{T}A \tag{6}$$

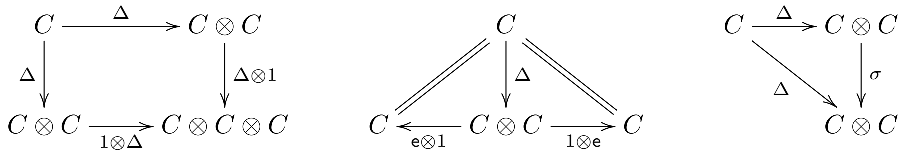
To see how to construct mixed distributive laws from liftings, see Appendix A.

3 Coalgebra Modalities

Coalgebra modalities were defined by Blute, Cockett, and Seely when they introduced differential categories [6] and are a strictly weaker notion of monoidal coalgebra modalities.

While monoidal coalgebra modalities are much more popular as they give categorical models of IMELL, coalgebra modalities have sufficient structure to axiomatize differentiation. Therefore, we believe it of interest to discuss liftings and mixed distributive laws over coalgebra modalities in order to also discuss lifting differential category structure (see Section 11). Interesting examples of coalgebra modalities which are not monoidal can be found in [9].

▶ **Definition 5.** In a symmetric monoidal category, a **cocommutative comonoid** is a triple (C, Δ, e) consisting of an object C, a map $\Delta : C \to C \otimes C$ called the **comultiplication**, and a map $\mathsf{e} : C \to K$ called the **counit** such that the following diagrams commute:



(7)

Coalgebra modalities are comonads with the added property that for each object A, the object $!A$ is naturally a cocommutative comonoid.

▶ **Definition 6.** A **coalgebra modality** [6] on a symmetric monoidal category is a quintuple $(!, \delta, \varepsilon, \Delta, e)$ consisting of a comonad $(!, \delta, \varepsilon)$, a natural transformation $\Delta : !A \to !A \otimes !A$, and a natural transformation $\mathsf{e} : !A \to K$ such that for each object A, the triple $(!A, \Delta, \mathsf{e})$ is a cocommutative comonoid and δ is a comonoid morphism, that is, $\delta; \Delta = \Delta; (\delta \otimes \delta)$ and $\delta; \mathsf{e} = \mathsf{e}$.

Requiring that Δ and e be natural transformations is equivalent to asking that for each map $f : A \to B$, the map $!(f) : !A \to !B$ is a comonoid morphism. Every !-coalgebra (A, ω) of a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ comes equipped with a cocommutative comonoid structure [19, 22] with comultiplication $\Delta^\omega : A \to A \otimes A$ and counit $\mathsf{e}^\omega : A \to K$ defined as follows:

$$\Delta^\omega := \; A \xrightarrow{\omega} !A \xrightarrow{\Delta} !A \otimes !A \xrightarrow{\varepsilon \otimes \varepsilon} A \otimes A \qquad \mathsf{e}^\omega := \; A \xrightarrow{\omega} !A \xrightarrow{\mathsf{e}} K \tag{8}$$

Notice that since δ is a comonoid morphism, when applying this construction to a cofree !-coalgebra $(!A, \delta)$ we re-obtain Δ and e, that is, $\Delta^\delta = \Delta$ and $\mathsf{e}^\delta = \mathsf{e}$. Furthermore, by naturality of Δ and e, every !-coalgebra morphisms becomes a comonoid morphism on the induced comonoid structures.

4 Symmetric Bimonads and Lifting Symmetric Monoidal Structure

In order to lift coalgebra modalities to an Eilenberg-Moore category of algebras over a monad, we must at least have that said Eilenberg-Moore category be a symmetric monoidal category such that the forgetful functor be a strict monoidal functor. To achieve this, the monad must also be a symmetric comonoidal monad, which we will here call a *symmetric bimonad* following Bruguieres, Lack, and Virelizier's terminology [7] (orginially introduced under the name Hopf monad by Moerdijk [21]). In short, a symmetric bimonad monad (see Defintion 8 below) is a monad whose underlying endofunctor is symmetric comonoidal such that certain extra compatibilities with the monad structure hold. For a higher category theory approach to the subject, we invite the curious reader to see [27].

▶ **Definition 7.** A **symmetric comonoidal endofunctor** – also known as a symmetric opmonoidal endofunctor [16] – on a symmetric monoidal category $\mathbb{X}$ is a triple $(\mathsf{T}, \mathsf{n}_2, \mathsf{n}_1)$

consisting of an endofunctor $\mathsf{T} : \mathbb{X} \to \mathbb{X}$, a natural transformation $\mathsf{n}_2 : \mathsf{T}(A \otimes B) \to \mathsf{T}A \otimes \mathsf{T}B$, and a map $\mathsf{n}_1 : \mathsf{T}K \to K$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathsf{T}(A\otimes B\otimes C) & \xrightarrow{\mathsf{n}_2} & \mathsf{T}(A\otimes B)\otimes \mathsf{T}C \\ {\scriptstyle \mathsf{n}_2}\downarrow & & \downarrow{\scriptstyle \mathsf{n}_2\otimes 1} \\ \mathsf{T}A\otimes \mathsf{T}(B\otimes C) & \xrightarrow[1\otimes \mathsf{n}_2]{} & \mathsf{T}A\otimes \mathsf{T}B\otimes \mathsf{T}C \end{array} \qquad \begin{array}{ccc} \mathsf{T}A & \xrightarrow{\mathsf{n}_2} & \mathsf{T}A\otimes \mathsf{T}K \\ {\scriptstyle \mathsf{n}_2}\downarrow & \searrow\!\!\!\!= & \downarrow{\scriptstyle 1\otimes \mathsf{n}_1} \\ \mathsf{T}K\otimes \mathsf{T}A & \xrightarrow[\mathsf{n}_1\otimes 1]{} & !A \end{array} \qquad \begin{array}{ccc} \mathsf{T}(A\otimes B) & \xrightarrow{\mathsf{T}(\sigma)} & \mathsf{T}(B\otimes A) \\ {\scriptstyle \mathsf{n}_2}\downarrow & & \downarrow{\scriptstyle \mathsf{n}_2} \\ \mathsf{T}A\otimes \mathsf{T}B & \xrightarrow[\sigma]{} & \mathsf{T}B\otimes \mathsf{T}A \end{array} \tag{9}$$

Of particular importance to us is that symmetric comonoidal endofunctors preserves cocommutative comonoids. Indeed, if (C, Δ, e) is a cocommutative comonoid, then the triple $(\mathsf{T}C, \mathsf{T}(\Delta); \mathsf{n}_2, \mathsf{T}(\mathsf{e}); \mathsf{n}_1)$ is a cocommutative comonoid.

▶ **Definition 8.** A **symmetric bimonad** [7] on a symmetric monoidal category is a symmetric comonoidal monad, that is, a quintuple $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ consisting of a monad (T, μ, η) and a symmetric comonoidal endofunctor $(\mathsf{T}, \mathsf{n}_2, \mathsf{n}_1)$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathsf{TT}(A\otimes B) & \xrightarrow{\mu} & \mathsf{T}(A\otimes B) \\ {\scriptstyle \mathsf{T}(\mathsf{n}_2)}\downarrow & & \\ \mathsf{T}(\mathsf{T}A\otimes \mathsf{T}B) & & \downarrow{\scriptstyle \mathsf{n}_2} \\ {\scriptstyle \mathsf{n}_2}\downarrow & & \\ \mathsf{TT}A\otimes \mathsf{TT}B & \xrightarrow[\mu\otimes\mu]{} & \mathsf{T}A\otimes \mathsf{T}B \end{array} \qquad \begin{array}{ccc} A\otimes B & \xrightarrow{\eta} & \mathsf{T}(A\otimes B) \\ & {\scriptstyle \eta\otimes\eta}\searrow & \downarrow{\scriptstyle \mathsf{n}_2} \\ & & \mathsf{T}A\otimes \mathsf{T}B \end{array} \qquad \begin{array}{ccc} \mathsf{TT}K & \xrightarrow{\mu} & \mathsf{T}K \\ {\scriptstyle \mathsf{T}(\mathsf{n}_1)}\downarrow & & \downarrow{\scriptstyle \mathsf{n}_1} \\ \mathsf{T}K & \xrightarrow[\mathsf{n}_1]{} & K \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{\eta} & \mathsf{T}K \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \mathsf{n}_1} \\ & & K \end{array} \tag{10}$$

One reason for the name bimonad is that bimonoids (the generalization of bialgebras for arbitrary symmetric monoidal categories) give rise to bimonads [7] as we will explain in Section 9.

As previously advertised, the Eilenberg-Moore category of a symmetric bimonad is a symmetric monoidal category. Define a symmetric monoidal structure on $\mathbb{X}^{\mathsf{T}}$ as follows: the monoidal unit is the pair (K, n_1), while for a pair of T-algebras (A, ν) and (B, ν'), their tensor product is defined as the pair $(A \otimes B, \nu \otimes^{\mathsf{T}} \nu')$ where $\nu \otimes^{\mathsf{T}} \nu'$ is defined as follows:

$$\nu \otimes^{\mathsf{T}} \nu' := \ \mathsf{T}(A \otimes B) \xrightarrow{\quad \mathsf{n}_2 \quad} \mathsf{T}A \otimes \mathsf{T}B \xrightarrow{\quad \nu\otimes\nu' \quad} A \otimes B \tag{11}$$

Therefore, the two left most diagrams of (10) are the statement that n_2 is a T-algebra morphism, while the right most diagrams state that (K, n_1) is a T-algebra. In fact, for a monad on a symmetric monoidal category, symmetric bimonad structures on the monad are in bijective correspondence with symmetric monoidal structures on the Eilenberg-Moore category which are strictly preserved by the forgetful functor [26].

5 Lifting Coalgebra Modalities

We now define the notion of mixed distributive laws between symmetric comonoidal endofunctors and coalgebra modalities, in order to lift coalgebra modalities to the Eilenberg-Moore category of symmetric bimonads.

▶ **Definition 9.** Let $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ be a symmetric bimonad and $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ be a coalgebra modality on the same symmetric monoidal category. A **mixed distributive law of** $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ **over** $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ is a mixed distributive law λ of (T, μ, η) over $(!, \delta, \varepsilon)$ such that λ is a comonoid morphism, that is, the following diagrams commute:

$$\begin{array}{ccc} \mathsf{T}!A & \xrightarrow{\lambda} & !\mathsf{T}A \\ \downarrow{\scriptstyle \mathsf{T}(\Delta)} & & \\ \mathsf{T}(!A \otimes !A) & & \downarrow{\scriptstyle \Delta} \\ \downarrow{\scriptstyle \mathsf{n}_2} & & \\ \mathsf{T}!A \otimes \mathsf{T}!A & \xrightarrow[\lambda \otimes \lambda]{} & !\mathsf{T}A \otimes !\mathsf{T}A \end{array} \qquad \begin{array}{ccc} \mathsf{T}!A & \xrightarrow{\lambda} & !\mathsf{T}A \\ \downarrow{\scriptstyle \mathsf{T}(\mathsf{e})} & & \downarrow{\scriptstyle \mathsf{e}} \\ \mathsf{T}K & \xrightarrow[\mathsf{n}_1]{} & K \end{array} \tag{12}$$

We first observe that these mixed distributive laws preserve the induced comonoid structure on !-coalgebras in the following sense:

▶ **Lemma 10.** *Let* $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ *be a symmetric bimonad and* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ *be a coalgebra modality on the same symmetric monoidal category, and let* λ *be a mixed distributive law of* $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ *over* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$*. If* (A, ω) *is a !-coalgebra, then the following diagrams commute:*

$$\begin{array}{ccc} \mathsf{T}A & \xrightarrow{\mathsf{T}(\Delta^{\omega})} & \mathsf{T}(A \otimes A) \\ & {\scriptstyle \Delta^{\omega^\flat}} \searrow & \downarrow{\scriptstyle \mathsf{n}_2} \\ & & \mathsf{T}A \otimes \mathsf{T}A \end{array} \qquad \begin{array}{ccc} \mathsf{T}A & \xrightarrow{\mathsf{T}(\mathsf{e}^{\omega})} & \mathsf{T}K \\ & {\scriptstyle \mathsf{e}^{\omega^\flat}} \searrow & \downarrow{\scriptstyle \mathsf{n}_1} \\ & & K \end{array}$$

where $\omega^\flat$ *is defined as in (6), and both* $\Delta^{\omega^\flat}$ *and* $\mathsf{e}^{\omega^\flat}$ *are defined as in (8).*

Proof. See Appendix B. ◀

Now we give the equivalence between liftings and mixed distributive laws of symmetric bimonads over coalgebra modalities.

▶ **Proposition 11.** *Let* $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ *be a symmetric bimonad and* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ *be a coalgebra modality on the same symmetric monoidal category* $\mathbb{X}$*. Then the following are in bijective correspondence:*

1. *Mixed distributive laws* $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ *over* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$*;*
2. *Liftings of* $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ *to* $\mathbb{X}^{\mathsf{T}}$*, that is, a coalgebra modality* $(\tilde{!}, \tilde{\delta}, \tilde{\varepsilon}, \tilde{\Delta}, \tilde{\mathsf{e}})$ *on* $\mathbb{X}^{\mathsf{T}}$ *which is a lifting of the underlying comonad* $(!, \delta, \varepsilon)$ *to* $\mathbb{X}^{\mathsf{T}}$ *(in the sense of Theorem 4) such that* $\mathsf{U}^{\mathsf{T}}(\tilde{\Delta}) = \Delta$ *and* $\mathsf{U}^{\mathsf{T}}(\tilde{\mathsf{e}}) = \mathsf{e}$*.*

Proof. See Appendix B. ◀

We give a name to symmetric bimonads with these mixed distributive laws.

▶ **Definition 12.** Let $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ be a coalgebra modality. A **coalgebra modality lifting monad** is a sextuple $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1, \lambda)$ consisting of a symmetric bimonad $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ and a mixed distributive law λ of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$.

Proposition 11 implies that the Eilenberg-Moore category a coalgebra modality lifting monad admits a coalgebra modality which is strictly preserved by the forgetful functor.

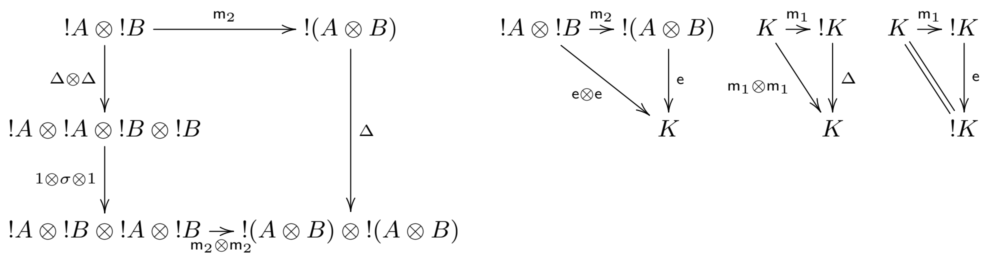
6 Monoidal Coalgebra Modalities

Monoidal coalgebra modalities can be described as coalgebra modalities whose underlying comonad is a symmetric monoidal comonad such that Δ and e are compatible with the symmetric monoidal comonad structure. Symmetric monoidal comonads are simply the dual notion of symmetric bimonads (Definition 8) and could therefore be called symmetric bicomonads. However the name symmetric monoidal comonad is used within the linear logic community and therefore we have elected to keep it here. Though it should be noted that the term bicomonad was used by Bruguieres, Lack, and Virelizier [7].

▶ **Definition 13.** A **symmetric monoidal comonad** is a quintuple $(!, \delta, \varepsilon, \mathsf{m}_2, \mathsf{m}_1)$ consisting of a comonad $(!, \delta, \varepsilon)$, a natural transformation $\mathsf{m}_2 : !A \otimes !B \to !(A \otimes B)$, and a map $\mathsf{m}_1 : K \to !K$ such that $(!, \mathsf{m}_2, \mathsf{m}_1)$ is a symmetric monoidal functor, that is, the dual diagrams of (9) commute, and such that δ and ε are monoidal transformations, that is, the dual diagrams of (10) commute.

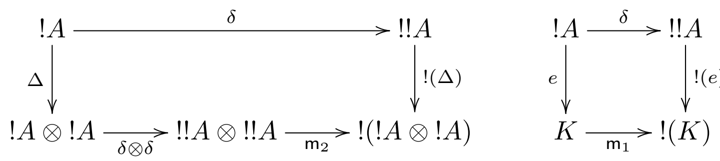
As this is the dual notion of symmetric bimonads, the Eilenberg-Moore category of a symmetric monoidal comonad is a symmetric monoidal category.

▶ **Definition 14.** A **monoidal coalgebra modality** [3] (also called a **linear exponential modality** [22]) on a symmetric monoidal category is a septuple $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ such that $(!, \delta, \varepsilon, \mathsf{m}_2, \mathsf{m}_1)$ is a symmetric monoidal comonad and $(!, \delta, \varepsilon, \Delta, e)$ is a coalgebra modality, and such that Δ and e are monoidal transformations, that is, the following diagrams commute:



(13)

and also that Δ and e are !-coalgebra morphisms, that is, the following diagrams commute:



(14)

Notice that the monoidal coalgebra modality requirement that Δ and e both be monoidal transformations is equivalent to asking that m_2 and m_1 are both comonoid morphisms. Furthermore, if (A, ω) is a !-coalgebra then Δ^ω and e^ω (as defined in (8)) are both !-coalgebra morphisms. This implies that the tensor product of the Eilenberg-Moore category of a monoidal coalgebra modality is in fact a product [22].

The most well known and common examples of monoidal coalgebra modalities are known as **free exponential modalities** [20]. Free exponential modalities can be described as monoidal coalgebra modalities with the added property that $!A$ is the cofree cocommtuative comonoid over A. In this case, !-coalgebras correspond precisely to the cocommutative comonoids of the symmetric monoidal category. Therefore, the Eilenberg-Moore category of a free exponential modality is equivalent to the category of cocommutative comonoids of the base symmetric monoidal category.

7 Lifting Monoidal Coalgebra Modalities

▶ **Definition 15.** Let $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ be a symmetric bimonad and $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ be a monoidal coalgebra modality on the same symmetric monoidal category. A **mixed distributive law** $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ **over** $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ is a mixed distributive law λ of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ such that the following diagrams commute:

$$\begin{array}{ccccc}
\mathsf{T}(!A \otimes !B) & \xrightarrow{\mathsf{n}_2} & \mathsf{T}!A \otimes \mathsf{T}!B & \xrightarrow{\lambda \otimes \lambda} & !\mathsf{T}A \otimes !\mathsf{T}B \\
\downarrow{\scriptstyle \mathsf{T}(\mathsf{m}_2)} & & & & \downarrow{\scriptstyle \mathsf{m}_2} \\
\mathsf{T}!(A \otimes B) & \xrightarrow[\lambda]{} & !\mathsf{T}(A \otimes B) & \xrightarrow[!(\mathsf{n}_2)]{} & !(\mathsf{T}A \otimes \mathsf{T}B)
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{T}K & & \xrightarrow{\mathsf{n}_1} & & K \\
\downarrow{\scriptstyle \mathsf{T}(\mathsf{m}_1)} & & & & \downarrow{\scriptstyle \mathsf{m}_1} \\
\mathsf{T}!K & \xrightarrow[\lambda]{} & !\mathsf{T}K & \xrightarrow[!(\mathsf{n}_1)]{} & !K
\end{array}
\qquad (15)$$

▶ **Proposition 16.** *Let $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ be a symmetric bimonad and $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ be a monoidal coalgebra modality on the same symmetric monoidal category $\mathbb{X}$. Then the following are in bijective correspondence:*

1. *Mixed distributive laws of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$;*
2. *Liftings of the monoidal coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ to $\mathbb{X}^{\mathsf{T}}$, that is, a monoidal coalgebra modality $(\tilde{!}, \tilde{\delta}, \tilde{\varepsilon}, \tilde{\Delta}, \tilde{\mathsf{e}}, \tilde{\mathsf{m}}_2, \tilde{\mathsf{m}}_1)$ on $\mathbb{X}^{\mathsf{T}}$ which is a lifting of the underlying coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ to $\mathbb{X}^{\mathsf{T}}$ (in the sense of Proposition 11) such that $\mathsf{U}^{\mathsf{T}}(\tilde{\mathsf{m}}_2) = \mathsf{m}_2$ and $\mathsf{U}^{\mathsf{T}}(\tilde{\mathsf{m}}_1) = \mathsf{m}_1$.*

Proof. See Appendix C. ◀

As before, we give a name to symmetric bimonads with these mixed distributive laws.

▶ **Definition 17.** Let $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ be a monoidal coalgebra modality. An **exponential lifting monad** is a sextuple $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1, \lambda)$ consisting of a symmetric bimonad $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ and a mixed distributive law λ of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$.

Proposition 16 implies that the Eilenberg-Moore category of an exponential lifting monad admits a monoidal coalgebra modality which is strictly preserved by the forgetful functor.

8 Lifting Linear Category Structure

Categorical models of IMELL are known as linear categories:

▶ **Definition 18.** A **linear category** [2] is a symmetric monoidal closed category with a monoidal coalgebra modality.

As linear categories are categorical models of IMELL [2, 17, 19], there is no shortage of examples of linear categories throughout the literature. Hyland and Schalk provide a very nice list of various kinds examples in [13, Section 2.4]. Linear categories whose monoidal coalgebra modality is in fact a free exponential modality are known as **Lafont categories** [19] – we discuss a particular example of a Lafont category at the end of Section 9.

The last piece of the puzzle is being able to lift the monoidal closed structure of a linear category to the Eilenberg-Moore category of our symmetric bimonad in such a way that the forgetful functor preserves the monoidal closed structure strictly. For this we turn to Bruguieres, Lack, and Virelizier's notion of a Hopf monad. Hopf monads were originally introduced by Bruguieres and Virelizier for monoidal categories with duals [8], but the definition of Hopf monads was later extended to arbitrary monoidal categories by the two previous authors and Lack [7]. We choose the later of the two as the definition is somewhat simpler. The left and right fusion operators of a symmetric bimonad $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ are the natural transformations h_l and h_r defined respectively as follows:

$$\mathsf{h}_l := \mathsf{T}(\mathsf{T}A \otimes B) \xrightarrow{\mathsf{n}_2} \mathsf{TT}A \otimes \mathsf{T}B \xrightarrow{\mu \otimes 1} \mathsf{T}A \otimes \mathsf{T}B$$

$$\mathsf{h}_r := \mathsf{T}(A \otimes \mathsf{T}B) \xrightarrow{\mathsf{n}_2} \mathsf{T}A \otimes \mathsf{TT}B \xrightarrow{1 \otimes \mu} \mathsf{T}A \otimes \mathsf{T}B$$

Notice that the fusion operators are T-algebra morphisms.

▶ **Definition 19.** A **symmetric Hopf monad** [7] on a symmetric monoidal category is a symmetric bimonad whose fusion operators are natural isomorphisms.

Extending on [7, Theorem 3.6], the Eilenberg-Moore category of a Hopf monad of a symmetric monoidal closed category, is again a symmetric monoidal closed category such that the forgetful functor preserve the symmetric monoidal closed structure strictly.

▶ **Definition 20.** A IMELL **lifting monad** on a linear category with monoidal coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ is an exponential lifting monad $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1, \lambda)$ whose underlying symmetric bimonad is also a symmetric Hopf monad.

▶ Remark. It is worth mentioning that in the definitions of a monoidal coalgebra modality monad and of an IMELL lifting monad, we do not require that the underlying endofunctors of these monads be linearly distributive functors between linear categories in the sense of Hyland and Schalk [13, Definition 4] or that of Melliés [18, Definition 9].

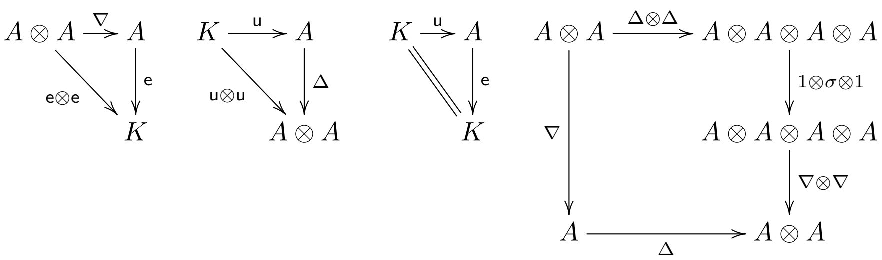
IMELL lifting monads provide us with following main result of this paper:

▶ **Theorem 21.** *The Eilenberg-Moore category of an* IMELL *lifting monad is a linear category such that the forgetful functor preserves the linear category structure strictly.*

9 What Monoids Give IMELL Lifting Monads?

As explained in the introduction, a particular example of Eilenberg-Moore categories we are interested are those arising as categories of modules over monoids. Indeed, endofunctors of the form $A \otimes -$ for some object A, admit a monad structure precisely when the object A is a monoid. Recall that a monoid of a monoidal category is a triple (A, ∇, u) consisting of an object A, a map $\nabla : A \otimes A \to A$ called the multiplication, and a map $\mathsf{u} : K \to A$ called the unit such that the dual of the left and center diagrams of (7) commute (in particular we do not require the multiplication to be commutative). For a monoid (A, ∇, u), the algebras of the monad $(A \otimes -, \nabla \otimes 1, \mathsf{u} \otimes 1)$ are more commonly known as (left) A-modules, and in this case, we denote the Eilenberg-Moore category instead by $\mathsf{MOD}(A)$.

▶ **Definition 22.** In a symmetric monoidal category, a **bimonoid** is a quintuple $(A, \nabla, \mathsf{u}, \Delta, \mathsf{e})$ such that (A, ∇, u) is a monoid, (A, Δ, e) is a comonoid, and the following diagrams commute:



(16)

A **Hopf monoid** in a symmetric monoidal category is a sextuple $(H, \nabla, \mathsf{u}, \Delta, \mathsf{e}, \mathsf{S})$ consisting of a bimonoid $(H, \nabla, \mathsf{u}, \Delta, \mathsf{e})$ and a map $\mathsf{S} : H \to H$ called the antipode such that the following diagram commutes:

$$\begin{array}{ccccc} & H \otimes H & \xrightarrow{1 \otimes \mathsf{S}} & H \otimes H & \\ {}^{\Delta}\nearrow & & & & \searrow^{\nabla} \\ H & \xrightarrow{\quad \mathsf{e} \quad} & K & \xrightarrow{\quad \mathsf{u} \quad} & H \\ {}_{\Delta}\searrow & & & & \nearrow_{\nabla} \\ & H \otimes H & \xrightarrow[\mathsf{S} \otimes 1]{} & H \otimes H & \end{array} \tag{17}$$

As previously hinted at, endofunctors of the form $A \otimes -$ admit a symmetric bimonad (resp. symmetric Hopf monad) structure precisely when the object A admits a bimonoid (resp. Hopf monoid) structure whose comultiplication is cocommutative. For details on these constructions see [7, Example 2.10].

Our goal is now to find bimonoids and Hopf monoids which induce exponential lifting monads and IMELL lifting monads. For this we turn to monoids in the Eilenberg-Moore categories of monoidal coalgebra modalities. A monoid in the Eilenberg-Moore category of a monoidal coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ can be seen as a quadruple $(A, \omega, \nabla^\omega, \mathsf{u}^\omega)$ consisting of a !-coalgebra (A, ω) and a monoid $(A, \nabla^\omega, \mathsf{u}^\omega)$ such that ∇^ω and u^ω are !-coalgebra morphisms, that is, the following diagrams commute:

$$\begin{array}{ccccc} A \otimes A & & \xrightarrow{\quad \nabla^\omega \quad} & & A \\ \downarrow{\scriptstyle \omega \otimes \omega} & & & & \downarrow{\scriptstyle \omega} \\ !A \otimes !A & \xrightarrow[\mathsf{m}_2]{} & !(A \otimes A) & \xrightarrow[!(\nabla^\omega)]{} & !A \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{\mathsf{u}^\omega} & A \\ \downarrow{\scriptstyle \mathsf{m}_1} & & \downarrow{\scriptstyle \omega} \\ !K & \xrightarrow[!(\mathsf{u}^\omega)]{} & !(A) \end{array} \tag{18}$$

However we just mentioned that $A \otimes -$ admit a symmetric bimonad structure if and only if A admits a bimonoid structure with cocommutative comultiplication. Therefore we could instead ask for bimonoids. But it turns out that we only need to ask for monoids instead! To see this, consider monoids in a cartesian monoidal category – which is a category with finite products regarded as a symmetric monoidal category. Every object in a cartesian monoidal category is a cocommutative comonoid and every map is a comonoid morphism. Therefore, since the bimonoid identities are equivalent to requiring that the multiplication and unit be comonoid morphisms, every monoid in a cartesian monoidal category is automatically a cocommutative bimonoid. Since the Eilenberg-Moore category of a monoidal coalgebra modality is a cartesian monoidal category [19, 22], every monoid will be a bimonoid with cocommutative comultiplication.

Following this observation, we obtain the main result of this section:

▶ **Theorem 23.** *Let $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ be a monoidal coalgebra modality on a symmetric monoidal category $\mathbb{X}$. Then the following are in bijective correspondence:*

1. *Monoids in $\mathbb{X}^!$;*
2. *Objects A such that the endofunctor $A \otimes -$ admits an exponential monad lifting structure*

whose mixed distributive law λ satisfies the following:

$$\begin{array}{ccc} A \otimes !X \otimes !Y & \xrightarrow{\lambda \otimes 1} & !(A \otimes X) \otimes !Y \\ {\scriptstyle 1 \otimes \mathsf{m}_2} \downarrow & & \downarrow {\scriptstyle \mathsf{m}_2} \\ A \otimes !(X \otimes Y) & \xrightarrow[\lambda]{} & !(A \otimes X \otimes Y) \end{array} \tag{19}$$

Therefore, each monoid in the Eilenberg-Moore category of a monoidal coalgebra modality induces an exponential lifting monad.

Proof. We only show how to construct one from the other as the proof is somewhat lengthy. Let $(A, \omega, \nabla^{\omega}, \mathsf{u}^{\omega})$ be a monoid in $\mathbb{X}^!$. Define the natural transformation (natural by construction) $\omega^{\natural} : A \otimes !X \to !(A \otimes X)$ as follows:

$$\omega^{\natural} := \ A \otimes !X \xrightarrow{\omega \otimes 1} !A \otimes !X \xrightarrow{\mathsf{m}_2} !(A \otimes X) \tag{20}$$

Then $\omega^{\natural}$ is a mixed distributive law of the symmetric bimonad structure of $A \otimes -$ over the monoidal coalgebra modality. Conversely, let λ be a mixed distributive law of the exponential lifting monad structure of $A \otimes -$. By applying (6) to the !-coalgebra (K, m_1), we obtain the !-coalgebra $(A, \mathsf{m}_1^{\flat})$ where recall:

$$\mathsf{m}_1^{\flat} := \ A \xrightarrow{1 \otimes \mathsf{m}_1} A \otimes !K \xrightarrow{\lambda} !A \tag{21}$$

Furthermore, the comonoid structure on A induced by the symmetric bimonad structure on $A \otimes -$ is precisely the same as the comonoid structure on A induced by the coalgebra modality from (8). It then follows that the multiplication and unit of A induced by the symmetric bimonad structure on $A \otimes -$ are !-coalgebra morphisms and therefore A admits a monoid structure in $\mathbb{X}^!$. ◀

▶ Remark. It is worth pointing out that commutivity of diagram (19) is only necessary for the bijective correspondence.

As a source of such monoids, since monoidal endofunctors preserve monoids (dual to what was discussed in Section 4), every monoid (A, ∇, u) of the base symmetric monoidal category $\mathbb{X}$ induces a monoid $(!A, \delta, \nabla^{\delta}, \mathsf{u}^{\delta})$ in $\mathbb{X}^!$ where $\nabla^{\delta} := \mathsf{m}_2; !(\nabla)$ and $\mathsf{u}^{\delta} := \mathsf{m}_1; !(\mathsf{u})$. In particular, since the monoidal unit K admits a canonical monoid structure, the quadruple $(!K, \delta, \mathsf{m}_2, \mathsf{m}_1)$ is a monoid in $\mathbb{X}^!$. Another source of such monoids is discussed in the next section.

Recall that in the special case of a free exponential modality, its Eilenberg-Moore category is equivalent to the category of cocommutative comonoids of the base symmetric monoidal category. Therefore, to give a monoid in this Eilenberg-Moore category is precisely to give a bimonoid with cocommutative comultiplication of the base symmetric monoidal category. In fact the category of monoids in the Eilenberg-Moore category of a free exponential modality is equivalent to the category of bimonoids with cocommutative comultiplication of the base category.

For a bimonoid, there is a unique antipode (if it exists) [15] which makes it into a Hopf monoid. Therefore we can easily extended Theorem 23 for IMELL lifting monads.

▶ **Theorem 24.** *Let $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ be a monoidal coalgebra modality of a linear category $\mathbb{X}$. Then the following are in bijective correspondence:*

1. *Monoids* $(A, \omega, \nabla^\omega, \mathsf{u}^\omega)$ *of* $\mathbb{X}^!$ *such that there exists an antipode* S *for the bimonoid* $(A, \nabla^\omega, \mathsf{u}^\omega, \Delta^\omega, \mathsf{e}^\omega)$*;*
2. *Objects* A *such that the endofunctor* $A \otimes -$ *admits an* IMELL *lifting monad structure whose mixed distributive law satisfies (19).*

Therefore, if A *is an object of a linear category which admits a monoid structure with antipode in the Eilenberg-Moore category of the monoidal coalgebra modality, then* $\mathsf{MOD}(A)$ *is a linear category.*

A source of such monoids with antipodes can be found in the next section (Theorem 28).

For Lafont categories – which recall are linear categories whose monoidal coalgebra modality is a free exponential modalitiy – to give a monoid as in Theorem 24 is precisely to give a Hopf monoid with cocommutative comultiplication. As an example, consider the category of vector spaces over a field K, which is a Lafont category where the construction of $!V$, which in this case is known as the cofree cocommutative K-coalgebra over V, can be found here [13, Section 2.4]. Particular examples of Hopf K-algebras [15] with cocommutative comultiplication include the polynomial rings $K[x_1, \ldots, x_n]$, the tensor algebra $\mathsf{T}(V)$ over a K-vector spaces V, the group K-algebra $K[G]$ over an arbitrary group G, and also the field K itself.

10 IMELL Lifting Monads from Additive Structure

We've already seen that monoids in the base category provide a source of monoids in the Eilenberg-Moore category of a monoidal coalgebra modality. In this section we turn to monoidal coalgebra modalities over additive symmetric monoidal categories to provide us with another source of monoids in the Eilenberg-Moore category of said monoidal coalgebra modalities. Here we mean "additive" in the Blute, Cockett, and Seely sense of the term [6], that is, to mean enriched over commutative monoids. In particular, we do not assume negatives (at least not yet...see Theorem 27) nor do we assume biproducts — which differs from other definitions of an additive category found in the literature [14].

▶ **Definition 25.** An **additive category** is a commutative monoid enriched category, that is, a category in which each hom-set is a commutative monoid with an addition operation $+$ and a zero 0, and such that composition preserves the additive structure, that is $k; (f + g); h = k; f; h + k; g; h$ and $0; f = 0 = f; 0$. An **additive symmetric monoidal category** is a commutative monoid enriched symmetric monoidal category, that is, symmetric monoidal category which is also an additive category in which the tensor product is compatible with the additive structure in the sense that $(f + g) \otimes h = f \otimes h + g \otimes h$ and $0 \otimes f = 0$.

It is worth mentioning that every additive category can be completed to a category with biproducts (which is itself an additive category), and similarly every additive symmetric monoidal category can be completed to a additive symmetric monoidal category with biproducts. For this reason, it is possible to argue [10] that one should always assume a setting with biproducts. The problem is that arbitrary coalgebra modalities do not necessarily extend to the biproduct completion. However, monoidal coalgebra modalities induce monoidal coalgebra modalities on the biproduct completion (see [9] for more details).

If $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ is a monoidal coalgebra modality on an additive symmetric monoidal category, then $!A$ comes equipped with a monoid structure [9, Theorem 19] where the multiplication ∇ and unit u are both !-coalgebra morphisms [9, Lemma 20], [10, Theorem 3.1]. Therefore we obtain the following:

▶ **Lemma 26.** *Every cofree coalgebra of a monoidal coalgebra modality on an additive symmetric monoidal category induces an exponential lifting monad. In particular, for each object A,* $\mathsf{MOD}(!A)$ *is an additive symmetric monoidal category with a monoidal coalgebra modality.*

As promised, we will now add negatives to the story of additive symmetric monoidal categories. In particular we will now show that cofree coalgebras of monoidal coalgebra modalities on additive symmetric monoidal categories are Hopf monoids precisely when the additive symmetric monoidal category also admits additive inverses, i.e. negatives. This statement should not be too surprising for two reasons. The first reason is that for a Hopf algebra, the antipode is the bialgebra convolution [15] inverse to the identity. The second reason is that monoidal coalgebra modalities on additive symmetric monoidal categories are strongly connected to the additive structure [9]. A category enriched over abelian groups can be seen as an additive category such that each map f admits an additive inverse, that is, a map $-f$ such that $f + (-f) = 0$. Actually, for an additive category to be enriched over abelian groups, one only requires that the identity maps 1 have additive inverses -1.

▶ **Proposition 27.** *Let* $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ *be a monoidal coalgebra modality on an additive symmetric monoidal category. Then there exists a natural transformation* $\mathsf{S} : !A \to !A$ *such that for each object A, the septuple* $(!A, \nabla, \mathsf{u}, \Delta, \mathsf{e}, \mathsf{S})$ *is a cocommutative Hopf monoid (where* ∇ *and* u *are defined as in [9]) if and only if the additive symmetric monoidal category is enriched over abelian groups.*

Proof. We only give how to construct antipodes from negatives and conversly negatives from antipodes. Suppose our additive symmetric monoidal category is enriched over abelian groups. Define the antipode $\mathsf{S} : !A \to !A$ as $\mathsf{S} := !(-1)$. Conversly, suppose there exists a natural transformation $\mathsf{S} : !A \to !A$ such that for each object A, the septuple $(!A, \nabla, \mathsf{u}, \Delta, \mathsf{e}, \mathsf{S})$ is a cocommutative Hopf monoid. As previously mentioned, it suffices to give an additive inverse for the identity morphisms. Then for each object A, define the map $-1_A : A \to A$ as follows:

$$-1_A := \quad A \xrightarrow{\mathsf{m}_1 \otimes 1} !K \otimes A \xrightarrow{\mathsf{S} \otimes 1} !K \otimes A \xrightarrow{\varepsilon \otimes 1} A \tag{22}$$

◀

Therefore we obtain the following:

▶ **Theorem 28.** *Every cofree coalgebra of a monoidal coalgebra modality of a linear category which is also an additive symmetric monoidal category enriched over abelian groups, induces an* IMELL *lifting monad. In particular, for each object A,* $\mathsf{MOD}(!A)$ *is a linear category which is also an additive symmetric monoidal category enriched over abelian groups.*

11 Lifting Differential Category Structure

In this final section, we briefly recall the notion of differential categories and discuss lifting differential category structure. For more details on differential categories see [3, 6, 9].

▶ **Definition 29.** A differential category [6] is an additive symmetric monoidal category with a coalgebra $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ equipped with a deriving transformation, that is, a natural transformation $\mathsf{d} : !A \otimes A \to !A$ satisfying the identities found in [6, Definition 2.5].

Similar to our discussion on lifting coalgebra modalities, in order to be able to lift differential category structure we will need that the Eilenberg-Moore category of our symmetric bimonad be an additive symmetric monoidal category. In order to achieve this, we will need the underlying endofunctor of our symmetric bimonad to be additive.

▶ **Definition 30.** An **additive functor** between additive categories is a functor which preserves the additive structure strictly, that is, a functor T such that $\mathsf{T}(f+g) = \mathsf{T}(f) + \mathsf{T}(g)$ and $\mathsf{T}(0) = 0$.

One can easily check that for a monad on an additive category whose underlying endofunctor is additive, that its Eilenberg-Moore category is also an additive category such that the forgetful functor preserves the additive structure strictly. Similarly, for a symmetric bimonad on an additive symmetric monoidal category whose underlying endofunctor is additive, its Eilenberg-Moore category is also an additive category such that the forgetful functor preserves the additive symmetric monoidal structure strictly. Luckily for us for any additive symmetric monoidal category, our favourite endofunctor $A \otimes -$ is additive for any object A.

▶ **Definition 31.** Let $\mathbb{X}$ be a differential category with coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ equipped with deriving transformation d, and let $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ be a symmetric bimonad on $\mathbb{X}$ whose underlying endofunctor is additive. A **mixed distributive law of** $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ **over** $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ **with deriving transformation** d is a mixed distributive law λ of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathsf{T}(!A \otimes A) & \xrightarrow{\mathsf{n}_2} & \mathsf{T}!A \otimes \mathsf{T}A & \xrightarrow{\lambda \otimes 1} & !\mathsf{T}A \otimes \mathsf{T}A \\ {\scriptstyle \mathsf{T}(\mathsf{d})}\downarrow & & & & \downarrow{\scriptstyle \mathsf{d}} \\ \mathsf{T}!A & & \xrightarrow[\lambda]{\qquad\qquad} & & !\mathsf{T}A \end{array} \tag{23}$$

▶ **Proposition 32.** *Let $\mathbb{X}$ be a differential category with coalgebra modality $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ equipped with deriving transformation d, and let $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ be a symmetric bimonad on $\mathbb{X}$ whose underlying endofunctor is additive. Then the following are in bijective correspondence:*
1. *Mixed distributive laws $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ with deriving transformation d ;*
2. *Liftings of d to $\mathbb{X}^{\mathsf{T}}$, that is, a deriving transformation $\tilde{\mathsf{d}}$ for the lifted coalgebra modality $(\tilde{!}, \tilde{\delta}, \tilde{\varepsilon}, \tilde{\Delta}, \tilde{\mathsf{e}})$ on $\mathbb{X}^{\mathsf{T}}$ from Proposition 11 such that $\mathsf{U}^{\mathsf{T}}(\tilde{\mathsf{d}}) = \mathsf{d}$.*

Proof. See Appendix D. ◀

In a differential category whose coalgebra modality is also a monoidal coalgebra modality, the deriving transformation and the monoidal coalgebra modality are compatible in the sense of [9, Theorem 25]. And therefore it follows that:

▶ **Theorem 33.** *In a differential category with a monoidal coalgebra modality, the category of modules over a monoid in the Eilenberg-Moore category of the monoidal coalgebra modality is a differential category.*

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A From Liftings to Mixed Distributive Laws

As we will need to know how to construct mixed distributive laws from liftings and vice-versa for the proofs of Propositions 11, 16, and 23, we quickly recall part of these constructions here (for more details see [26]). Constructing liftings from mixed distrubitve laws was discussed at the end of Section 2.

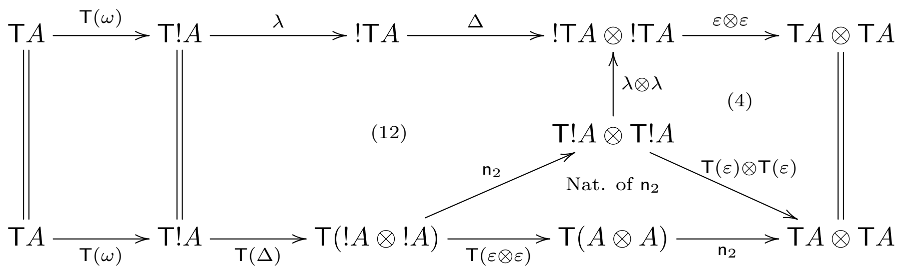
Let $(\tilde{!}, \tilde{\delta}, \tilde{\varepsilon})$ be a lifting of $(!, \delta, \varepsilon)$ to the Eilenberg-Moore category $\mathbb{X}^{\mathsf{T}}$ of a monad (T, μ, η). This implies that for each free T-algebra $(\mathsf{T}A, \mu)$ we have that $\tilde{!}(\mathsf{T}A, \mu) = (!\mathsf{T}A, \mu^\sharp)$ for some map $\mu^\sharp : \mathsf{T}!\mathsf{T}A \to !\mathsf{T}A$. Define the natural transformation $\lambda : \mathsf{T}!A \to !\mathsf{T}A$ as follows:

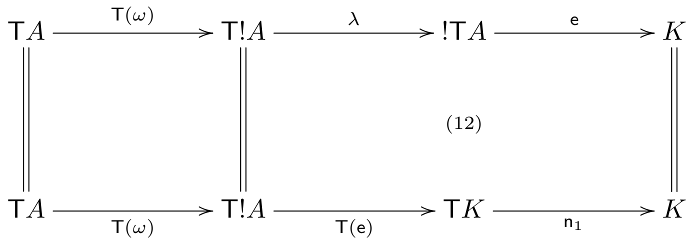
$$\lambda := \mathsf{T}!A \xrightarrow{\mathsf{T}!(\eta)} \mathsf{T}!\mathsf{T}A \xrightarrow{\mu^\sharp} !\mathsf{T}A \tag{24}$$

Then λ is a mixed distributive law of (T, μ, η) over $(!, \delta, \varepsilon)$.

B Proofs of Lemma 10 and Proposition 11

Proof of Lemma 10. The lemma follows from commutativity of the following diagrams:





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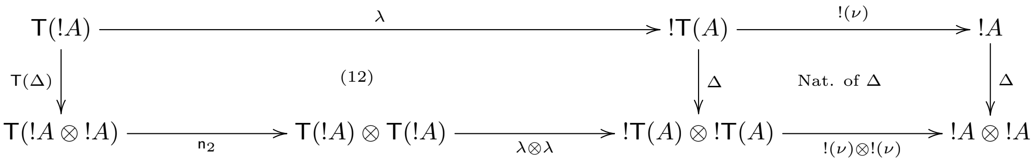
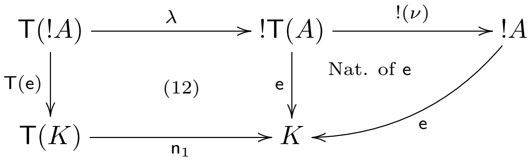
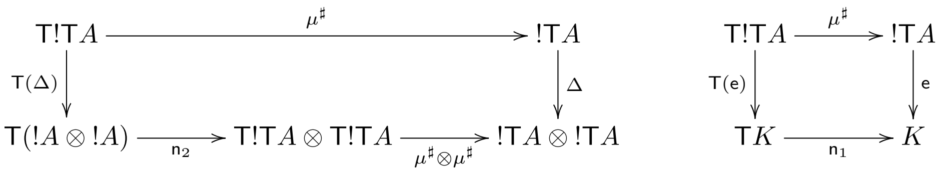
Proof of Proposition 11. The bijective correspondence will follow from Theorem 4. It remains to show that the induced lifting of the comonad from the mixed distributive law is also a lifting of the colagebra modality, and similarly for the mixed distributive law from the lifting of the coalgebra modality.
(1) $\Rightarrow$ (2): Let λ be a mixed distributive of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$. Consider the induced lifting of $(!, \delta, \varepsilon)$ from Theorem 4. To prove that we have a lifting of the coalgebra modality, it suffices to show that Δ and e are T-algebra morphisms. Then if (A, ν) is a T-algebra, commutativitiy of the following diagrams show that Δ and e are T-algebra

morphisms:



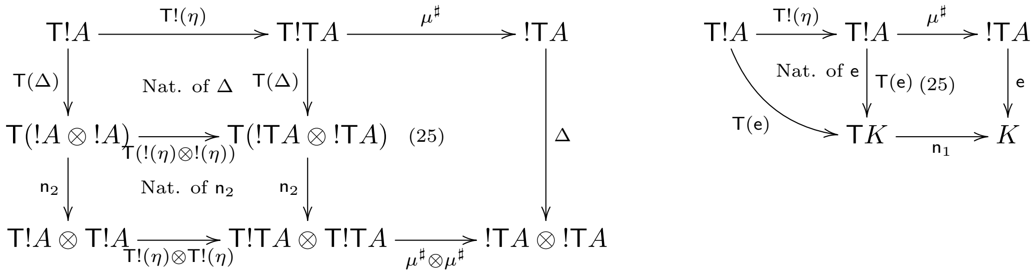


(2) ⇒ (1): Let $(\tilde{!}, \tilde{\delta}, \tilde{\varepsilon}, \tilde{\Delta}, \tilde{\mathsf{e}})$ be a lifting of $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ to $\mathbb{X}^{\mathsf{T}}$. This implies that Δ and e are T-algebra morphisms, which in particular for free T-algebras $(\mathsf{T}A, \delta)$, the following diagrams commute:



(25)

where recall $\mu^\sharp$ is the T-algebra structure of $\tilde{!}(\mathsf{T}A, \mu) = (!\mathsf{T}A, \mu^\sharp)$. Consider now the induced mixed distributive law λ of (T, μ, η) over $(!, \delta, \varepsilon)$ as defined in (24). Then that λ is also a mixed distributive law of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ follows from commutativity of the following diagrams:



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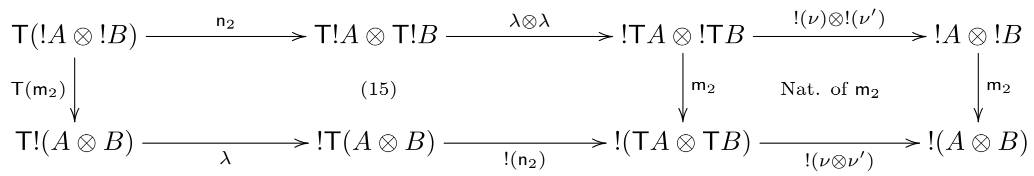
C Proof of Proposition 16

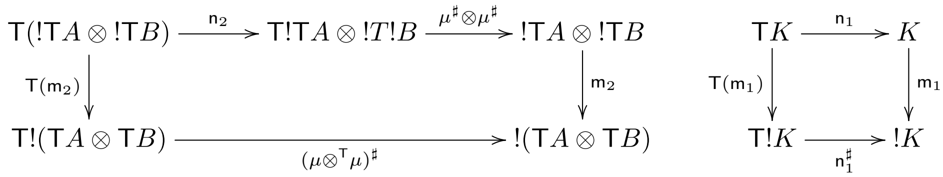
Proof of Proposition 16. We take the same approach as in the proof of Proposition 11. Again, the bijective correspondence will follow from Theorem 4.

(1) ⇒ (2): Let λ be a mixed distributive law of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$. Consider the induced lifting of $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ from Proposition 11. To prove that we have a lifting of the monoidal coalgebra modality, it suffices to show that m_2 and m_1 are T-algebra morphisms. The right diagram of (15) is precisely the statement that m_1 is a T-algebra morphism. Then if (A, ν) and (B, ν') are T-algebras, commutativitiy of the following diagrams

show that m_2 is a T-algebra morphism:

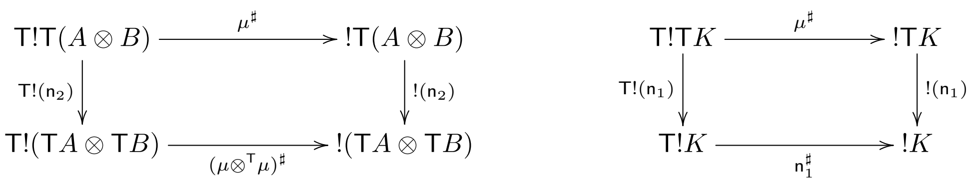


(2) $\Rightarrow$ (1): Let $(\tilde{!}, \tilde{\delta}, \tilde{\varepsilon}, \tilde{\Delta}, \tilde{\mathsf{e}}, \tilde{\mathsf{m}_2}, \tilde{\mathsf{m}_1})$ be a lifting of $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ to $\mathbb{X}^\mathsf{T}$. In particular, this implies that m_2 and m_1 are T-algebra morphisms. In particular for free T-algebras $(\mathsf{T}A, \mu)$ and the T-algebra (K, n_1), we have that the following diagrams commute:



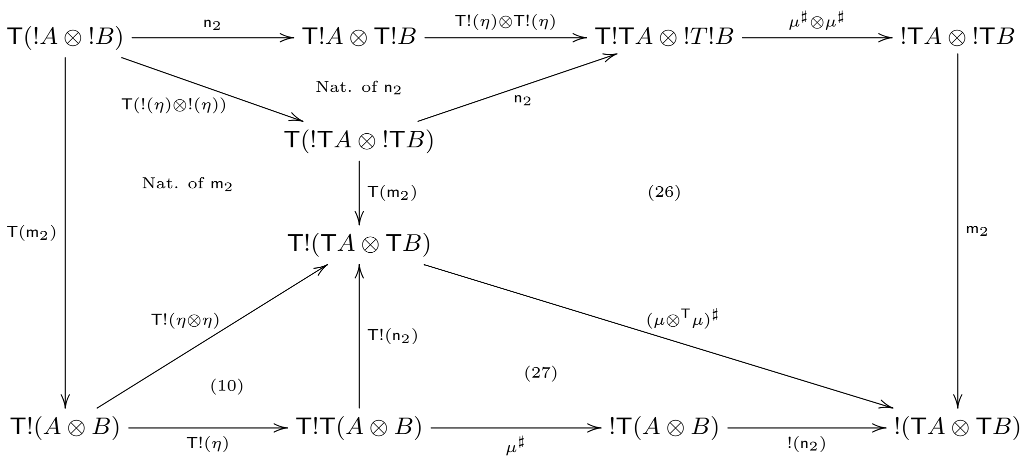
(26)

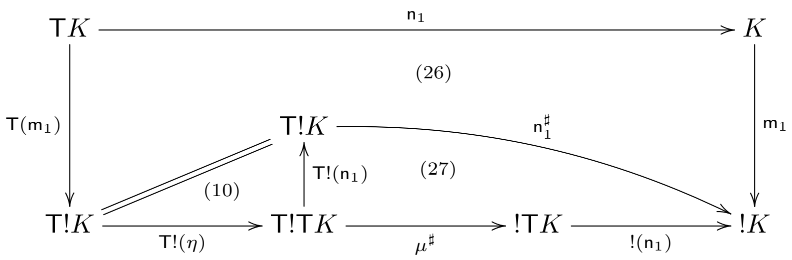
where recall for a T-algebra (A, ν), the map $\nu^\sharp$ is the induced T-algebra on $\tilde{!}(A, \nu) = (!A, \nu^\sharp)$, and $\mu \otimes^\mathsf{T} \mu$ is defined as in (11). Notice that since both n_2 and n_1 are T-algebra morphisms, the lifting implies that $!(\mathsf{n}_2)$ and $!(\mathsf{n}_1)$ are also, that is, the following diagrams commute:



(27)

Consider the induced mixed distributive law λ of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ from Proposition 11. Then that λ is a mixed distributive law of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e}, \mathsf{m}_2, \mathsf{m}_1)$ follows from commutativity of the following diagrams:





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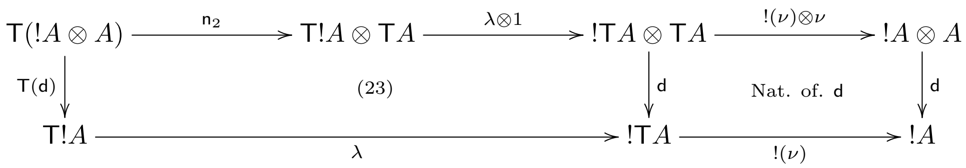
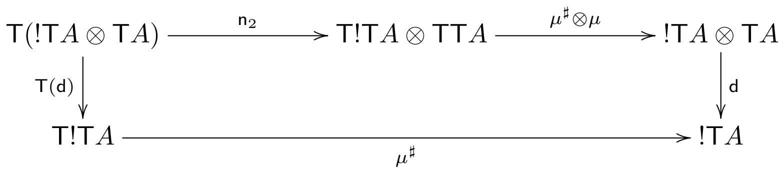
D Proof of Proposition 32

Proof Proposition 32. The bijective correspondence will follow immediately from Proposition 11. Therefore, it remains to show that we can obtain one from the other.

(1) ⇒ (2): Let λ be a mixed distributive law of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ with deriving transformation d. Consider the induced lifting of $(!, \varepsilon, \varepsilon, \Delta, \mathsf{e})$ from Proposition 11. To prove that we have a lifting of the deriving transformation, it suffices to show that d is a T-algebra morphism. Then if (A, ν) is a T-algebra, commutativitiy of the following diagram shows that d is a T-algebra morphism:

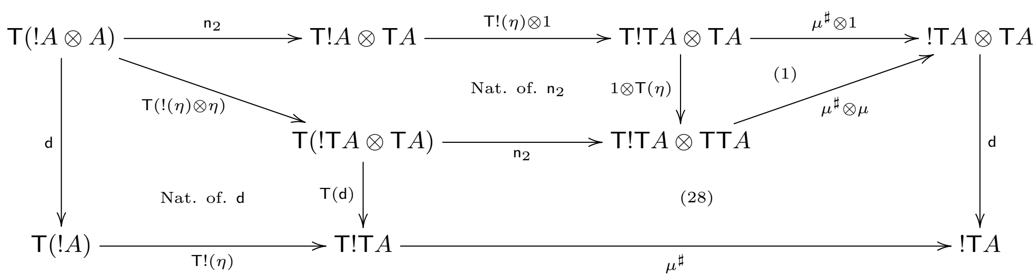


(2) ⇒ (1): Let $\tilde{\mathsf{d}}$ be a lifting of d to $\mathbb{X}^{\mathsf{T}}$. This implies that d is a T-algebra morphism, which in particular for free T-algebras $(\mathsf{T}A, \mu)$, the following diagram commutes:



(28)

Consider now the induced mixed distributive law λ of $(\mathsf{T}, \mu, \eta, \mathsf{n}_2, \mathsf{n}_1)$ over $(!, \delta, \varepsilon, \Delta, \mathsf{e})$ from Proposition 11. Then that λ satisfies the extra necessary condition follows from commutativity of the following diagram:



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