ON THE REMOVAL OF INFINITIES FROM DIVERGENT SERIES

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Abstract

The consequences of adopting other definitions of the concepts of sum and convergence of a series are discussed in the light of historical and epistemological contexts. We show that some divergent series appearing in the context of renormalization methods cannot be assigned finite values while preserving a minimum of consistency with standard summation, without at the same time obtaining contradictions, thus destroying the mathematical building (the conditions are known as Hardy's axioms). We finally discuss the epistemological costs of accepting these practices in the name of instrumentalism.

1 Introduction

The possibility of assigning a finite value to divergent series has made it to the news[1] in a way that is unusual for science or mathematics news. Indeed, this news and even some related Youtube videos seem to lie halfway between joke and serious matter but in the end it turns out that these contributions are intended (by their authors) to be serious. In textbooks (see e.g., Nesterenko & Pirozhenko (1997) below) and in informal conversations, the book by Hardy Hardy (1949) is mentioned as support for these procedures.

It is then compelling to assess the scientific relevance of these methods and in particular the issue of (logical) consistency of these methods in relation to the body of mathematical knowledge, as well as their epistemological implications. These are, hence, the main goals of this work.

We assume the reader to be familiar with the basic elements of the theory of series, as developed in analysis textbooks Apostol (1967). For the sake of completion, in Appendix A we review the relevant ingredients of the theory of series concerning this work.

In Section 2 we discuss the role of mathematics along the process of understanding our environement through the interplay between *abstraction* and *signification*. Following Hardy Hardy (1949) we discuss the necessary requirements that any definition for the sum of a series must have in order to have some chances of retaining signification. Section 3 contains the statement of the main results of this work i.e., the impossibility of retaining consistency (and thereafter signification) when assigning a finite value to divergent series (proofs are presented in Appendix B) under basic reasonability constraints. Further, we discuss the historical and epistemological issues related to these practices in Section 4, while Section 5 is devoted to conclusions. We advance that Hardy's book provides the basis for rejecting the assignment of finite values to divergent series that appears in the context of renormalization methods, rather than giving support to these practices.

2 Signification

Mathematics can be regarded as a process of successive abstractions originating in real-life situations. Thus, natural numbers abstract the process of counting objects and involve the abstraction of the concept of addition as well. Instead of saying: *one goat and another goat makes two goats and another goat makes three goats* … and the same for all indivisible objects, we say 1 + 1 + 1… abstracting away the

1See the New York edition of New York Times, page D6 of February 4, 2014, on-line at http://www.nytimes.com/2014/02/04/science/in-the-end-it-all-adds-up-to.html?smid=fb-share&_r=1

objects and *dressing* the final answer with them again (I counted *n goats*). We increase our ability of counting by moving to objects with more elaborated properties (e.g., divisible objects or portions of objects, debts, missing objects), producing the realms of integers, rationals, etc. The properties of addition are *extended* (i.e., defined in a broader context while preserving its original properties when used in the original context) to these higher levels of abstraction. The operations of abstraction and its inverse, dressing, relate mathematics to the material world. We call them more precisely abstraction and *signification*.

The process of abstraction is also known as *idealization* and in physics is historically linked with Galileo Galilei and his discussion of free fall. (Galilei 1638, pp. 205) (for an English translation see (Galilei 1914, pp. 170)) Insight on the process of signification can also be traced back to Galileo's words announcing that mathematics is the language of the universe Galilei (1623), thus recognising mathematics as belonging to the realm of the material world.

Hence, when addressing issues of the material world and its sciences, mathematical objects retain a specific signification. Any new mathematical object introduced along the investigation is related by abstraction and signification on one hand to the material world, on the other hand to mathematics where the object becomes context independent by the very process of abstraction. Thus, when counting apples we use the same addition as when we count goats, nodes in a vibrating string or smiling faces. The reciprocally inverse processes of *abstraction* and *signification* lie in the foundations of any attempt to understand the material world with mathematical tools, and cannot be disrupted nor neglected in any of its parts. In plain words, the extension of a mathematical concept beyond its current limits of applicability cannot destroy the previously developed processes of abstraction and signification.

2.1 Minimal Requirements

Let us suppose that we want to give up the usual concept of sum of infinite series, i.e., we give up the standard definition that extends the concept of finite sums via a limit process in order to encompass infinite sums. The goal of this process is to produce new (supposedly broader) definitions related to computing a finite number that somehow resembles summation, out of a sequence $\{a_0, a_1, a_2, \dots\}$ with divergent sum in the ordinary sense. We require however to keep as much as possible of the original properties of series summation, since finite sums and also the sum of *convergent* series cannot be given up as a fundamental part of our understanding.

Starting from the standard definition of sum of a series one arrives to several properties, among which we count

- For any real k, $\sum a_n = s \Rightarrow \sum k\, a_n = ks$.
- $\sum a_n = s$ and $\sum b_n = r \Rightarrow \sum (a_n + b_n) = s + r$.
- $\sum_{n=0} a_n = s \Leftrightarrow \sum_{n=1} a_n = s - a_0$.

Hardy Hardy (1949) proposes to transform these properties into axioms, dropping the standard definition.

The first two axioms state that multiplying the elements of a summable series by a constant k, the result is also a summable series with sum equal to the original sum s multiplyied by the constant k. The second, states that the sum of two summable series is also a summable series with corresponding sum. The third axiom states that the sum of a summable series is insensitive to breaking out of the series the first term (in fact any finite number of terms), summing it separately and adding the result to the remainder of the series.

Giving up the standard definition carries along that all properties of series that do not follow from the above axioms are lost as well, namely association, permutation and dilution (corresponding respectively to grouping adjacent terms, altering the order of the terms and interposing zeroes between terms, see Appendix A).

Also, the series symbol should be replaced by something new, since the original series symbol had received its meaning in Definition 2 (Appendix A). Let us adopt the notation[2] $Y(\{a_n\})$ to denote the new summation recipe. Each new method of assigning values to infinite sequences should provide its own definition (as well as signification) for this object.

We will distinguish those methods that sum convergent series and series diverging to infinity in the usual way, namely,

Definition 1 ((see p. 10 in Hardy (1949))). *(a) A method Y assigning a finite value to a series is called* regular *if this value coincides with the standard sum in the case of standard convergent series.*
(b) A regular method Y is called totally regular *if series diverging to* $\pm\infty$ *with the standard definition also diverge to* $\pm\infty$ *in* Y.

Axioms (A-C) may be regarded as a minimal compromise. The first two axioms extend the linearity of standard sum to infinite sums, while the third, called *stability*, along with Corollary 1 states that a finite portion of an infinite sum behaves as a standard finite sum. Any method which is **not** linear and stable cannot be seriously considered as an alternative to the sum of a series (it would be either not linear or not finitely related to ordinary sum). Also, in Hardy's (and our) view, regularity is a basic requirement: whichever method not complying with standard results for standard convergent problems cannot be seriously considered as an extension of the concept of sum.

3 Statement of Results

Many textbooks attempting to use the sum of divergent series discuss a few classes of series Nesterenko & Pirozhenko (1997). Two of these classes are discussed in the Theorems below (these being part of the central results of this work), since assigning a finite number to them leads to inconsistencies irrespective of the summation method adopted. For the other class, Euler's summation method is invoked. Euler's method was already addressed by Sierpiński and others about a century ago, see Appendix C.1 for details. For completion, we elaborate also on Cesaro's summation method Hardy (1949) in Appendix C.2.

Theorem 1. *Any method Y assigning a finite number to the expression* 1+1+1+1+1+… *is (i)* not totally regular, *(ii)* not regular *and (iii)* contradictory.

Theorem 2. *Any method Y assigning a finite number to the expression* 1+2+3+4+5+… *is (i)* not totally regular, *(ii)* not regular *and (iii)* contradictory.

By *contradictory* we mean that incompatible statements corresponding to $r = Y(\{a_n\}) = s$ for (real) numbers $r \neq s$ can be proved in this context. Proofs are given in Appendix B.

It goes without saying that we are speaking about well-posed methods Y where the assigned values for $Y(\{a_n\})$ are unique, whenever the sequence belongs to the domain of the method. Also, the definition of regularity naturally assumes that the sequences associated to all convergent series belong to the domain of whichever method Y is under consideration (even non-regular ones).

2The symbol Y is inspired in the Cyrillic word указ meaning decree or edict (formally: imposition).

4 Discussion

4.1 Historical Digression

Extending on Hardy's account, it is to be noted that the modern concept of limit was established by Cauchy around 1821. However, he could not solve the question of uniform convergence. In fact, it is said Lakatos (1976) that this issue worried Cauchy to the point of never publishing the second volume of his course of analysis, nor consenting to a reedition of the first. He eventually allowed the publication of the lecture notes of his classes by his friend and student Moigno in 1840 Moigno (1840). Again according to Lakatos, the distinction between point convergence and uniform convergence was unraveled by Seidel in 1847 Lakatos (1976), thus completing the approach of Cauchy. The modern way of regarding limits and convergence could be said to originate around 1847.

4.2 Epistemological Issues

The idea of substituting a definition with another one is not free from consequences. Definitions in mathematics may look arbitrary at a first glance but they are always motivated. Fundamentally, (a) they satisfy the need of filling a vacancy of content in critical places where precision is needed (however, since many textbooks present definitions without discussing the process for producing them, the epistemological requirements remain usually obscure) and (b) they are explicitly forbidden to be contradictory or logically inconsistent with the previously existing body of mathematics on which they rest. In addition, when dealing with the mathematisation of natural sciences, definitions carry a signification, which is the support for using that particular piece of mathematics in that particular science.

While we appreciate the exploration work around concepts that has been done over the years, we do not substitute a meaningful and established concept with something that is inequivalent to it in the common domain of application. Again, when understanding natural sciences, such substitution would disrupt the signification chain. In simple words, we do not replace a meaningful content with a meaningless one. This would be to depart from rationality, something that is positively rejected by mathematics as a whole as well as by science in general and by a large part of society.

Along the presentation, we cared to put limits to the possible relation between $Y(\{\cdot\})$ and ordinary sums. In the light of the proven Theorems, it is verified that such relation is feeble or absent. Hence, the very inspirational root of these techniques becomes divorced from its results and effects. As stated above, the alternative of giving up one of axioms (A-C) also destroys any possible relation to ordinary sums.

Replacements that assign $Y(\{n\}) = a$ or $Y(\{1\}) = b$ (with a,b real numbers) destroy the basis of mathematics, making it the same to have one goat that having a million goats.

We must emphasize that regularity is a necessary condition to preserve signification but it is not sufficient. Whatever replacement we attempt must provide a rationale for the method, preserving signification within mathematics (in the chain of abstractions it belongs) and in relation to natural sciences. For the case of series, signification is further destroyed along with properties such as association, permutation and dilution (corresponding respectively to grouping adjacent terms, altering the order of the terms and interposing zeroes between terms). The alternative of using one or another definition depending of the matter under study simply destroys the role mathematics as a whole. Instead of having *eternal and pure relations accessible by reason alone* Platon (360 AC), it will turn mathematics to be dependent of the context of use.

4.2.1 The Epistemology of Success

The issue of assigning a finite value to divergent series with methods that are not regular and are contradictory under Hardy's axioms is not only material of newspaper notes, discussion blogs or Youtube videos. It has actually reached the surface of society from articles and books published as scientific material. We support this statement by commenting on a couple of references. This issue is not just a feature of these two citations, but the standard procedure of a community: just read the references in

Birrell & Davies (1982) to find a large amount of practitioners of this community.

In Nesterenko & Pirozhenko (1997) we encounter an attempt to justify the use of the Riemann's Zeta function. The authors refer to Hardy's book for the actual method. They use axioms A and B and the zeta function to write equality between $\sum_{n=1}^{\infty} n$ and -1⁄12 (see their eq. (2.20)). The conclusion is evident: the method does not comply with Hardy's axioms. Furthermore, the result is false since to reach their conclusion the authors disregard a divergent contribution. Hence, the equal sign does not relate identical quantities as it should. The correct expression would be

$$Y(n) \equiv Y(1,2,3,4,\ldots) = \frac{-1}{12} \quad (1)$$

where Y must be understood as the method based on Riemann's Zeta function. Here, Theorem 2 applies.

Our second example is the book Birrell & Davies (1982) where on page 167 we read "The analytic continuation method converts a manifestly infinite series into a finite result" exemplifying this statement with the expression $Y(1) \equiv Y(\{1,1,1,\ldots\}) = 1/2$ (our notation, the authors use standard series notation) using the same procedure as Nesterenko & Pirozhenko (1997). On p. 165 this expression was given the value -1⁄2, probably a typo. The authors refer explicitly to Euler's method. We note though that by Theorem 1, any method assigning a finite number to such series is defective in the same way.

Needless to say, the authors do not use the symbol Y but they refer to the method as a "formal procedure". This way of expression places the issue within the ambiguities of language. If by formal we read *belonging to or constituting the form or essence of a thing*, we strongly disagree, since *essence* is the result of an abstracting (usually analytic) procedure Hegel (1971). However, if "formal" is intended as in its second accepted meaning: *following or according with established form, custom, or rule*, we agree, while observing that such social agreements are not a part of science.

In defense of such procedures it is usually said that the theories using them are among the most precise and successful in Physics. This argumentation claims, then, that questions of unicity of results, backward compatibility of a method with standard convergent series, or its relation to sums (let alone signification) are uninteresting. The value is assigned because in such a way one obtains a "correct" result. Hence they adhere to a (false) epistemology that Dirac called *instrumentalism* (Kragh 1990, page 185) and we plainly call *the epistemology of success*.

Hitting (what is claimed to be) the right answer is not equivalent to using the right method. One may hit a correct answer with a wrong method just by chance, by misunderstanding, or even by adaptation to the known answer, etc. Paraphrasing Feyerabend's *everything goes*: an idea may be welcome as a starting point without deeper considerations (within ethical limits, of course). However, for that idea to be called **scientific** it has to comply with the scientific method. Moreover, it has to comply with rationality. The attitude described by *something is right because it gives the correct answer* is dangerous in many levels. The mathematical attitude is actually the opposite (and logically inequivalent to it): *If it gives the wrong answer, either the assumptions or the method in use are incorrect* Popper (1959). This holds also for natural sciences, where in addition we have, through signification, a safe and independent method to distinguish wrong answers from right answers. Note that independency is crucial. Natural science makes predictions that can be tested independently of the theory involved. If verified, they give continued support to the theory, while if refuted they indicate where and why to correct it. As a contrast, a theory making predictions that can only be tested within itself obtains at best internal support for being consistent, but it never speaks about Nature since predictions are not independently testable. In any case, having the right answer is not a certificate of correctness (there may be an error somewhere else) whereas having **any** wrong answer is a certificate of incorrectness (the error is "there").

It is worth to keep in mind the attitude taken by the founding fathers of Quantum Electrodynamics:

> The shell game that we play [...] is technically called ’renormalization’. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It’s surprising that the theory still hasn’t been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate. **Richard Feynman**, 1985 Feynman (1983)

> I must say that I am very dissatisfied with the situation, because this so called good theory does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small - not neglecting it just because it is infinitely great and you do not want it! **Paul Dirac**. (Kragh 1990, page 195)

5 Concluding Remarks

Definitions are not arbitrary, any extension of an established operation (such as addition) needs to preserve the properties of the operation when applied to objects in the original domain of definition (regularity). We have shown in this sense that some methods proposed as extensions for the sum of convergent series (standard definition) fail on this regard. An important example are the proposed summations improperly justified by using a series for Riemann’s Zeta function outside its domain of validity. In some cases the use of Euler’s summation method on it is improper, in others whichever method one intends to use would yield an improper result. Moreover, regularity of a proposed replacement for series summation is not enough, the extension needs to preserve all the mathematical building-blocks it rests on.

It is important to indicate that attempts to justify renormalization such as that found in Nesterenko & Pirozhenko (1997) ought to be considered scientific attempts on the ground of Popper’s demarcationism Popper (1959) since by connecting to mathematical issues such as the problem of infinite series, they offer the rationale behind their procedures of scientific enquire. The outcome of the examination indicates that such replacements must be rejected.

The decision to expose the issues concerning the mathematical foundations of these matters to the general public should also be commented. The understanding of scientific matters is not reserved to an elite of practitioners that guard the “truth” of the subject as priests of a cult. Opening science to the scrutiny of the general public, including scientists outside the paradigm is simply correct, as discussed by Lakatos Lakatos (1976).

The following thesis should be considered: the more than 50 years that this matter has stayed without resolution is a demonstration that in closed elitist communities the interest of (return for) the community may very well have priority over the public (humane) interest.

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Appendix A: Background on Series

Sum of a Series

We follow here standard textbooks (Apostol 1967, p. 383-4) reviewing the definition of series and some of its basic properties, focusing on what will be relevant for the coming Sections.

Consider a sequence $\{a_0,a_1,a_2,\ldots\}$ and also its sequence of partial sums $S_n = \sum a_k$. This sequence of partial sums is called an *infinite series*, or simply *series*.

Definition 2. *If there exists a number s such that* $s = \lim_{N\to\infty} S_N$ *we say that the series* $\sum a_k$ *is convergent with sum s and write* $\sum a_k = s$. *Otherwise, we say that the series is* divergent.

Among divergent series we may distinguish those where the $\lim_{N\to\infty} S_N$ is $+\infty$ or $-\infty$ on the one hand (we may call them *divergent to infinity*) and those where this limit does not exist.

Definition 3. *A convergent series* $\sum a_k$ *is called* absolutely convergent *if the series* $\sum |a_k|$ *is convergent. Otherwise, it is called* conditionally convergent.

Basic Properties

We follow an approach inspired on the posthumous book by Hardy Hardy (1949) on divergent series. For convergent series, the following properties can be demonstrated as theorems:

- For any real k, $\sum a_n = s \Rightarrow \sum k\, a_n = ks$.
- $\sum a_n = s$ and $\sum b_n = r \Rightarrow \sum (a_n + b_n) = s + r$.
- $\sum_{n=0} a_n = s \Leftrightarrow \sum_{n=1} a_n = s - a_0$.

The proofs are a direct application of the properties of the limit of a sequence (in this case the sequence of partial sums).

Remark 1. *Properties (A) and (B) are recognised as* linearity *and (C) is called* stability. *They extend the natural properties of sums for the sequence of partial sums* S_N *all the way through the limit.*

An immediate generalisation of (C) to finitely many operations is the following

Corollary 1. *For any positive integer N,* $\sum a_n = s \Leftrightarrow \sum a_n = s - \sum a_k$.

We list here other natural properties of convergent series extrapolated from finite sums via the limit properties for the sequence of partial sums.

Corollary 2. *(a)* Associativity *If the series* $\sum_n a_n$ *has a (finite or infinite) sum, then the series* $\sum_k b_k$ *obtained via* $b_k = a_{2k} + a_{2k+1}$ *for some or all nonnegative integers k, has the same sum.*
(b1) Commutativity *If the series* $\sum_n a_n$ *has a (finite or infinite) sum then the series* $\sum_n b_n$ *obtained via* $(b_{2k},b_{2k+1}) = P(a_{2k},a_{2k+1}) = (a_{2k+1},a_{2k})$ *for some or all nonnegative integers k, has the same sum (P is the nontrivial permutation of* 2 *elements).*
(b2) For series having the commutativity property, finite compositions of permutations of order up to N (where N a positive integer) do not alter the sum of the series.
(c) Dilution *If the series with elements* $a_0,a_1,a_2,\ldots$ *has a (finite or infinite) sum then the series with elements* $a_0,0,a_1,0,a_2,0,\ldots$ *i.e., inserting a zero between some or all pairs of elements in the original sequence, has the same sum.*

Proof. For associativity, collecting up to N terms corresponds to picking a subsequence from the original sequence of partial sums, having thus the same limit. For (b1) every other partial sum coincides with the original ones. For (b2), the partial sums of the new series coincides with the original one every N steps. In between, they differ at most in a finite sum of terms that goes to zero for $k\to\infty$. Hence, both sequences of partial sums have the same limit. As for dilution, if one takes a convergent sequence $\{S_N\}$ and duplicates its terms: $S_1,S_1,S_2,S_2,\ldots$, the new sequence has the same limit as the original one. Hence, dilution does not alter the sum of the series. □

The above properties can be arbitrarily (but finitely) combined, without altering the sum of a series. However, more drastic rearrangements need not preserve the sum unless the series is absolutely convergent. In fact, any *rearrangement* of an absolutely convergent series produces a new series with the same sum as the original one. However, invariance in front of arbitrary rearrangements of terms does not hold for conditionally convergent series. This is the content of *Riemann rearrangement theorem* (Apostol 1967, p. 413).

Theorem (Riemann)**.** *Let $\sigma(n)$ be an injective function of the positive integers and K some real number. Suppose that $\{a_1,a_2,a_3,\ldots\}$ is a sequence of real numbers, and that $\sum a_n$ is conditionally convergent. Then there exists a* rearrangement $\sigma(n)$ *of the sequence such that $\sum a_{\sigma_n}=K$. The sum can also be rearranged to diverge to ±∞ or to fail to approach any limit.*

Remark 2. *A classical example of rearrangement is the series $a_n=\frac{(-1)^{n+1}}{n}, n\geq 1$. This alternating series sums to $s=\ln(2)$. Create a new series by dilution adding one zero before each element of the series and dividing by 2, i.e., $\{0,a_1/2,0,a_2/2,\ldots\}$. Combine then this series and the original one as in property (B). The new series has sum $s=\frac{3}{2}\ln(2)$. However, after disregarding the intermediate zeroes, the combined series is a rearrangement of the original one, where the negative terms (for n = 2k) appear every third element instead of every other element. The new series adds two positive numbers for every negative contribution, thus subtracting the negative contributions in a different way from that in the original series.*

Remark 3. *All conditionally convergent series can be decomposed into two monotonic series: one with the positive terms, diverging to +∞ and another with the negative terms, diverging to -∞. What Riemann's Rearrangement Theorem teaches us is that if one wishes to interpret the sum of such a series pictorially as the outcome from "cancellation of both infinities", then there is actually not one way to do it, but infinitely many, depending of the order in which the elements of the two participant series are picked up. However, each possible result is the unique limit of a specific sequence of partial sums.*

B Proofs of the main Theorems

Proof of Theorem 1. That the method *Y* is not totally regular is immediate since otherwise it should assign the value +∞ to the proposed expression. By (C), *Y* ({1,1,1,…}) has the same value as 0 + 1 + 1 + 1 + … (namely *Y* ({0,1,1,…})) and hence the termwise difference of both objects by (B) must satisfy: 1 + 0 + 0 + 0 + …≡ *Y* ({1,0,0,0,…}) = 0. Since 1 + 0 + 0 + 0 + … is a convergent series with sum 1 with the standard definition, we conclude that the method *Y* is not regular. As for contradiction, using (A) we obtain in the same way: -1 + 0 + 0 + 0 + …≡ *Y* ({-1,0,0,0,…}) = 0 thus establishing by (C) that 1 = *Y* ({0,0,0,0,…}) = -1 since both those numbers can be assigned as value for 0 + 0 + 0 + 0 + …, which by (C) belongs to the domain of the method *Y* . This result is contradictory with the whole body of mathematics since obviously 1≠ - 1. □

Proof of Theorem 2. That the method *Y* is not totally regular is immediate since otherwise it should assign the value +∞ to the proposed expression. As for regularity and consistency, we will prove that *Y* ({1,1,1,1, …}) belongs to the domain of *Y* and compute its value. Let *s* be the value of 1 + 2 + 3 + 4 + 5 + …, i.e., *Y* ({1,2,3,4,…}) = *s*. Then by (C), *Y* ({0,1,2,3,4,…}) = *s*+0 = *s* and also by (A), *Y* ({-0,-1,-2,-3,-4,…}) = -*s*. Hence, we obtain by (B): 1 + 1 + 1 + 1 + …≡ *Y* ({1,1,1,…}) = 0. Hence, this expression belongs to the domain of the method *Y* having the value zero and the previous theorem applies. Moreover, repeated application of (C) on 1 + 1 + 1 + 1 + …≡ Y ({1,1,1,…}) = 0, results in 1 + 1 + 1 + 1 + …≡ Y ({1,1,1,…}) = -N for any nonnegative integer N. This result provides an alternative proof of contradiction in Theorem 1. □

C Other Related Results

C.1 Euler's Continuation Method E Hardy (1949)

Abel's method and Euler's continuation method E are highlighted (Hardy 1949, p. 7) in Hardy's account of possible alternatives to standard summation. While the method E is invoked in textbooks Nesterenko & Pirozhenko (1997), already in Section 1.4 of Hardy's book it is noted that this method of assigning values for divergent series is not regular, and hence inappropriate. We may say that Euler glimpsed a possible approach in a moment in history where the concept of convergence was not fully developed (see Subection 4.1) while Abel formulated the idea with exhaustive precision. We comment on Hardy's observations for the sake of completeness, in order to give a more detailed historical framework.

Abel's second theorem (Borel 1928, p. 3) can be stated as:

Theorem (Abel)**.** *If* $\sum_n a_n = s$ *and* $\lim_{x\to 1^-} \sum_n a_n x^n = S$, *then* $s = S$.

In other words, if the series is convergent, with finite sum s, and if the power series has a (finite) limit S, then both numbers coincide.

Hardy's account of E is that if $\sum_n a_n z^n$ defines an analytic function $f(z)$ in some region of the complex plane such that the function is properly defined along a path from that region up to $z = 1$, then $\sum_n a_n = f(1)$. E is indeed less precise than Abel's theorem. Let us consider in which ways the assumptions of Abel's theorem may fail in the context of E. Either $\{a_n\}$ is a divergent series or $\lim_{z\to 1} \sum_n a_n z^n$ does not exist (now with $z \in \mathbf{C}$).

For the first case, consider the geometric series for $f(z) = (1 - z)^{-1}$ contrasting the series inspired in Riemann's zeta function: $g(z) = \sum_{n\geq 1} n^{1-z}$. Both series have disjoint domains of convergence in the complex plane. The point $z = 1$ lies outside the domain of convergence of any of the series. Hence, neither $g(1)$ nor $f(1)$ can be expressed by the corresponding sums in the respective rhs. However, in the context of Y () any of them could be taken to represent 1 + 1 + 1 + 1 + …, but while $g(1)$ is a finite number, $f(1)$ is not defined (diverges).

For the second case, namely that $\lim_{z\to 1} \sum_n a_n z^n$ does not exist (for $z \in \mathbf{C}$), Sierpiński Sierpiński (1916) gives an example of a convergent series $\sum_n a_n$ and a power series derived $H(z)$ from it (defining a function $f(z)$) such that while the power series has a limit for $x \to 1^-$ along the real axis (fully compatible with Abel's theorem), the limit does not exist along arbitrary paths $z \to 1$ in the complex plane.

Clearly, E cannot be used as a tool in this context. On the contrary, following Borel (1928) we may say that Abel's approach is exhaustive and there is no room for improvement. Different attempts to prove the converse of Abel's theorem after adding adequate additional hypotheses, have originated the branch of mathematics called Tauberian theorems.

Moreover, there are examples of power series having exactly the same shape in different regions of the complex plane, while defining different functions. Then there is not always an uniquely defined function $f(z)$ to be used as a means for summing series when one chooses to ignore the domains of convergence. For an example with the functions $f(z) = \pm \frac{1+z^2}{1-z^2}$, see (Hardy 1949, p.16).

This has more than anecdotic value, since Sierpiński series combined with E could be used to "destroy convergence". Let $\sum_n a_n z^n$ be Sierpiński's power series and $r = \sum_n a_n$ be the sum of the associated Sierpiński's series (see above). Let $\sum_n b_n = S$ be an absolutely convergent series, defining a power series $\sum_n b_n z^n$ with radius of convergence $R > 1$. The series $\sum_n (a_n + b_n) - r$ is absolutely convergent with sum S. Consider the power series $\sum_n (a_n + b_n) z^n - r$. We may use to sum the series correctly to S via the method of Abel's Theorem. However, E cannot compute the sum of this convergent series, since the power series is discontinuous for $z = 1$ (for the same reasons as in Sierpiński (1916)).

C.2 Associativity, Commutativity and Dilution

Let us now consider $Y(\{(-1)^n\}) \equiv Y(\{1,-1,1,-1,\ldots\})$ (where $n \geq 0$). Any method assigning a finite value s to it should satisfy $s = 1 - s$ by (A) and (C) and hence $s = 1/2$. There exist many totally regular methods for the purpose, the most famous of which is probably the *Cesaro sum*, defined as the limit of the the sequence of successive averages of partial sums, i.e., letting $Z_n = \left(\frac{1}{n}\right) \sum S_k$ for $n \geq 1$, (being S_k the partial sum of the first consecutive elements of the original series up to k) we have $Zn = \left(\frac{1}{2}\right) + \left(\frac{c}{n}\right)$, where $c = 0$ or 1 and the Cesaro sum of $Y(\{(-1)^n\})$ is $\lim_{n\to\infty} Z_n = 1/2$.

Lemma 1. *Cesaro sums fulfill* **none** *of the properties associativity, commutativity and dilution.*

Proof. For associativity just note that summing the elements of $Y(\{(-1)^n\})$ pairwise we obtain either 0 + 0 + 0 + … or 1 + 0 + 0 + … (starting the association in the first or in the second element of the original series), both having different Cesaro sums and both different from 1⁄2. For commutativity, permute the elements (a_n, a_{n+1}) for all odd n, obtaining 1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 + … whose Cesaro sum is unity. For dilution, insert one zero only after each positive element, obtaining the Cesaro sum 2⁄3. □

Lemma 2. *Any commutative method Y assigning a (finite) value to* $Y(\{(-1)^n\})$ *is contradictory.*

Proof. (A) and (C) force $Y(\{(-1)^n\}) = r = 1 - r$ and hence $r = 1/2$, regardless of the chosen commutative method Y. However, by permuting the elements pairwise, from $Y(\{(-1)^n\}) = r$, we obtain $Y(\{(-1)^{n+1}\}) = r$. Also, $Y(\{(-1)^{n+1}\}) = -r$ is obtained from the first expression by (A) and multiplication with -1. We obtain the contradiction $r = Y(\{(-1)^{n+1}\}) = -r$ with $r = 1/2$. □

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