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FREE MONADIC TARSKI AND MMI_3-ALGEBRAS

Abstract. MMI_3-algebras are a generalization of the monadic Tarski algebras as defined by A. Monteiro and L. Iturrioz, and a particular case of the MMI_{n+1}-algebras defined by A. Figallo. They can also be seen as monadic three-valued Łukasiewicz algebras without a first element. By using this point of view, and the free monadic extensions, we construct the free MMI_3-algebras on a finite number of generators, and indicate the coordinates of the generators. As a byproduct, we also obtain a construction of the free monadic Tarski algebras.

1. Introduction

Monadic Tarski algebras were defined by A. Monteiro and L. Iturrioz in [9, 10]. MMI_3-algebras are a generalization of the monadic Tarski algebras, and a particular case of the MMI_{n+1}-algebras defined by A. Figallo in [4]. MMI_3-algebras may also be regarded as monadic three-valued Łukasiewicz algebras [11] without a first element.

The goal of this article is to present a new method for determining the structure of free monadic Tarski and MMI_3-algebras, and also their number of elements. The result for monadic Tarski algebras is a byproduct of our work on the MMI_3-algebras. We proceed by first taking a look at three-valued Łukasiewicz algebras and monadic three-valued Łukasiewicz algebras, so we begin by displaying these varieties of algebras and the relationships among them. In Section 3, we outline our plan to obtain the free MMI_3-algebras using the free monadic extensions developed in [13], which we carry out in following section. Finally, we show how other known results can be obtained from ours. The method presented in this article is different from the one used by A. Figallo, A. Suardíaz and A. Ziliani in [6].

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2. Preliminaries

Tarski algebras are a generalization of boolean algebras: if $(A, \wedge, \vee, -, 0, 1)$ is a Boolean algebra and we define $x \to y = -x \vee y$ for all $x, y \in A$, then $(A, \to, 1)$ is a Tarski algebra as defined below.

Definition 2.1. An algebra $(A, \to, 1)$ of type $(2, 0)$ is a Tarski algebra if:

M1) $1 \to x \approx x$.
M2) $x \to x \approx 1$.
M3) $x \to (y \to z) \approx (x \to y) \to (x \to z)$.
M4) $(x \to y) \to y \approx (y \to x) \to x$.

It is well known that if a Tarski algebra A has least element 0, then A is a Boolean algebra, where the Boolean complement of $a \in A$ is $-a = a \to 0$ and the infimum of the elements a and $b \in A$ is $a \wedge b = -(b \to -a)$.

One can consider an additional operation on Tarski algebras:

Definition 2.2. [9] An algebra $(A, \to, \forall, 1)$ of type $(2, 1, 0)$ is said to be a monadic Tarski algebra if $(A, \to, 1)$ is a Tarski algebra and:

Q1) $\forall 1 \approx 1$.
Q2) $\forall x \to x \approx 1$.
Q3) $\forall((x \to \forall y) \to \forall y) \approx (\forall x \to \forall y) \to \forall y$.
Q4) $\forall(x \to y) \to (\forall x \to \forall y) \approx 1$.

Three-valued Łukasiewicz algebras, on the other hand, are one of the many generalizations of boolean algebras. In this case, an intermediate truth value between 0 and 1 is considered.

Definition 2.3. [8] A three-valued Łukasiewicz algebra (from now on L_3-algebra) is an algebra $(A, \wedge, \vee, \sim, \nabla, 1)$ of type $(2, 2, 1, 1, 0)$ where $(A, \wedge, \vee, 1)$ is a distributive lattice with greatest element 1 and satisfying:

L1) $\sim\sim x \approx x$.
L2) $\sim (x \wedge y) \approx \sim x \vee \sim y$.
L3) $\sim x \vee \nabla x \approx 1$.
L4) $\sim x \wedge \nabla x \approx x \wedge \sim x$.
L5) $\nabla(x \wedge y) \approx \nabla x \wedge \nabla y$.

As usual we set $0 = \sim 1$, and this is the least element of the lattice A. Defining $\Delta x = \sim \nabla \sim x$, we get a dual operator satisfying: $\sim x \wedge \Delta x \approx 0$, $\sim x \vee \Delta x \approx x \vee \sim x$ and $\Delta(x \vee y) \approx \Delta x \vee \Delta y$. We denote by $\mathbf{T}$ the three-valued Łukasiewicz algebra $\{0, c, 1\}$, where $0 < c < 1$, $\sim 0 = 1$, $\sim c = c$, $\nabla 0 = 0$, $\nabla c = \nabla 1 = 1$ and $\mathbf{B}$ is the subalgebra formed by the elements 0 and 1.

Definition 2.4. [11] A monadic three-valued Łukasiewicz algebra (from now on ML_3-algebra) is an L_3 algebra with an additional unary operator $\exists$

satisfying the equations: $\exists 0 \approx 0$, $x \approx x \wedge \exists x$, $\exists(x \wedge \exists y) \approx \exists x \wedge \exists y$, $\nabla \exists x \approx \exists \nabla x$ and $\Delta \exists x \approx \exists \Delta x$.

It follows immediately that $\exists 1 \approx 1$ and $\exists (x \vee y) \approx \exists x \vee \exists y$. Furthermore, defining $\forall x = \sim \exists \sim x$, the following equations are satisfied as well: $\forall 1 \approx 1$, $x \approx x \vee \forall x$, $\forall (x \vee \forall y) \approx \forall x \vee \forall y$, $\Delta \forall x \approx \forall \Delta x$ and $\nabla \forall x \approx \forall \nabla x$.

Definition 2.5. [4] An MI_3-algebra is an algebra $(A, \to, \sigma_1, \sigma_2, 1)$ of type $(2, 1, 1, 0)$ satisfying:

M1) $1 \to x \approx x$.
M2) $(x \to y) \to ((y \to z) \to (x \to z)) \approx 1$.
M3) $(x \to y) \to y \approx (y \to x) \to x$.
M4) $((x \to y) \to (y \to x)) \to (y \to x) \approx 1$.
M5) $(x \to (x \to y)) \to x \approx x$.
M6) $\sigma_1 x \to y \approx x \to (x \to y)$.
M7) $(\sigma_j x \to (\sigma_j x \to y)) \to (\sigma_j x \to y) \approx 1$, $j = 1, 2$.
M8) $\sigma_j(\sigma_k x \to \sigma_k y) \approx \sigma_k x \to \sigma_k y$, $1 \leq j, k \leq 2$.
M9) $(\sigma_1 x \to \sigma_1 y) \to ((\sigma_2 x \to \sigma_2 y) \to (x \to y)) \approx 1$.
M10) $\sigma_2 x \to ((\sigma_1 x \to \sigma_1(x \to y)) \to \sigma_1(x \to y)) \approx 1$.
M11) $\sigma_1(x \to y) \to (\sigma_j x \to \sigma_j y) \approx 1$, $j = 1, 2$.

MI_3 algebras are to L_3 algebras as Tarski algebras are to boolean algebras. In this context, the following definition of MMI_3-algebras is the analogue to Monadic Tarski algebras.

Definition 2.6. [6] An MMI_3-algebra is an MI_3-algebra with an additional operator $\forall$ satisfying:

M12) $\forall x \to x \approx 1$.
M13) $\forall((x \to \forall y) \to \forall y) \approx (\forall x \to \forall y) \to \forall y$.
M14) $\forall(x \to y) \to (\forall x \to \forall y) \approx 1$.
M15) $\forall(\forall x \to \forall y) \approx \forall x \to \forall y$.
M16) $\forall \sigma_j x \approx \sigma_j \forall x$, $j = 1, 2$.

Given elements a, b in an MMI_3-algebra A, we will denote as usual $a \leq b$ if and only if $a \to b = 1$. It is well known that A is an upper semilattice with 1 as its greatest element and the join of two elements a and b given by $a \vee b = (a \to b) \to b$.

The relationships between these classes of algebras was established in [6] and can be summarized as follows:

If A is an MI_3-algebra with least element 0, then defining $\sim x = x \to 0$, $x \wedge y = \sim(\sim x \vee \sim y)$ and $\nabla x = \sigma_2 x$, it turns out that $(A, \wedge, \vee, \sim, \nabla, 1)$ is an L_3-algebra.

On the other hand, if A is an L_3-algebra, letting $\sigma_1 x = \sim \nabla \sim x$, $\sigma_2 x = \nabla x$ and $x \to y = (\nabla \sim x \vee y) \wedge (\nabla y \vee \sim x)$, we get an MI_3-algebra $(A, \to, \sigma_1, \sigma_2, 1)$.

Given an ML_3-algebra A and defining $\sigma_1 x = \sim \nabla \sim x$,$\sigma_2 x = \nabla x$, $\forall x = \sim \exists \sim x$ and $x \to y = (\nabla \sim x \vee y) \wedge (\nabla y \vee \sim x)$, we get an MMI_3-algebra $(A, \to, \sigma_1, \sigma_2, \forall, 1)$.

An MMI_3-algebra A that has a least element 0, can be turned into an ML_3-algebra $(A, \wedge, \vee, \sim, \nabla, \exists, 1)$ by definig $\sim x = x \to 0$, $\nabla x = \sigma_2 x$, $\exists\, x = \sim \forall \sim x$ and $x \wedge y = \sim (\sim x \vee \sim y)$.

$$\begin{tikzcd}
& L_3 \arrow[dl, "-0"'] \arrow[dr, "+\forall"] & \\
MI_3 \arrow[dr, "+\forall"'] & & ML_3 \arrow[dl, "-0"] \\
& MMI_3 &
\end{tikzcd}$$

We will denote by $F_V(n)$ the free algebra in the variety V over a finite number n of generators.

Our main goal is to determine the structure of $F_{MMI_3}(n)$. In [6] it was proven that $F_{MMI_3}(n)$ can be obtained from $F_{ML_3}(n)$, so we will first construct $F_{ML_3}(n)$ with an eye on how to calculate the coordinates of the generators. We proceed by first constructing $F_{L_3}(n)$ and then its free monadic extension [13], $FME(F_{L_3}(n))$, which turns out to be $F_{ML_3}(n)$. In this way, we obtain a formula for the number of elements in $F_{ML_3}(n)$. As a byproduct, we obtain, by a different method, the same formula found in [12] for the number of elements in the free monadic Tarski algebra with n generators.

3. $F_{L_3}(n)$ and its free monadic extension

3.1. Free L_3-algebras

To determine $F_{L_3}(n)$, we use the following theorem:

Theorem 3.1. [3] *Let A be an algebra; then the free algebra in the variety generated by A, with a set of generators of cardinality α, is obtained as follows: Let I be a set of cardinal α and for each $i \in I$define a mapping $g_i : A^I \to A$ by*

$$g_i((a_j)_{j \in I}) = a_i.$$

Then the subalgebra of A^{A^I} generated by the elements g_i, $i \in I$, is the free algebra on the variety generated by A, on those elements.

Roberto Cignoli, proved in [2] that every L_3-algebra is a subdirect product of a family of subalgebras of the algebra $\mathbf{T}$, that is, either $\mathbf{T}$ or $\mathbf{B}$. Therefore, we can apply Cohn's theorem, obtaining the free L_3-algebra as the subalgebra of $\mathbf{T}^{\mathbf{T}^n}$ generated by the elements $g_1, \ldots, g_n \in \mathbf{T}^n$.

We see the algebra $\mathbf{T}^{\mathbf{T}^n}$ as a product $\prod_{\mathbf{T}^n} \mathbf{T}$, so that each element has 3^n coordinates.

Example 3.1. For $n = 2$, we write in a column the $3^2 = 9$ elements of $\mathbf{T}^2$ and in the following columns the values of g_1 and g_2:

x	$g_1(x)$	$g_2(x)$
$(0,0)$	0	0
$(0,c)$	0	c
$(0,1)$	0	1
$(c,0)$	c	0
(c,c)	c	c
$(c,1)$	c	1
$(1,0)$	1	0
$(1,c)$	1	c
$(1,1)$	1	1

It is easy to see that the subalgebra of $\mathbf{T}^{\mathbf{T}^2}$ generated by these two elements is isomorphic to $\mathbf{B} \times \mathbf{T} \times \mathbf{B} \times \mathbf{T}^3 \times \mathbf{B} \times \mathbf{T} \times \mathbf{B}$, since the element c cannot be obtained in $\mathbf{T}$ by doing operations on 0 and 1, so in the axes of the product where the functions g_i only take the value 0 or 1, the generated subalgebra is isomorphic to $\mathbf{B}$.

For the free L_3-algebra with n generators, it is easy to calculate how many axes are isomorphic to $\mathbf{B}$ (that is, 2^n of them) and the rest of them are isomorphic to $\mathbf{T}$ so $F_{L_3}(n) = \mathbf{B}^{2^n} \times \mathbf{T}^{3^n - 2^n}$.

3.2. Free monadic extensions

Definition 3.1. An ML_3-algebra L is a *free monadic extension* of a L_3-algebra A (noted $L = FME_{L_3}(A)$) if:

L_1) A is a subalgebra of L,
L_2) L is the monadic subalgebra generated by A,
L_3) every homomorphism of L_3-algebras g from A to an ML_3-algebra C can be extended to a (necessarily unique) monadic homomorphism f from L to C.

Now we briefly review the construction of free monadic extensions of finite L_3-algebras given in [13] and that was based on P. Halmos' method for boolean algebras presented in [7].

Definition 3.2. An hemimorphism is a map from an L_3-algebra A to an L_3-algebra A' such that for all $x, y \in A$:

$h_1)$ $h(0) = 0$, $h_2)$ $h(x \vee y) = h(x) \vee h(y)$,
$h_3)$ $h(\nabla x) = \nabla h(x)$, $h_4)$ $h(\Delta x) = \Delta h(x)$,
$h_5)$ $h(1) = 1$.

Definition 3.3. Given algebras A, A', we say that an homomorphism y *precedes* the hemimorphism v if for all $x \in A$, $y(x) \leq v(x)$. We denote this with $y \leq v$.

Lemma 3.1. [13] *In a finite, non trivial L_3-algebra A, every hemimorphism from A to $\mathbf{T}$ is the join of all the homomorphisms preceding it.*

Starting with a finite L_3-algebra $A = \mathbf{B}^j \times \mathbf{T}^k$, we build $FME_{L_3}(A)$ as follows.

Note that the boolean elements form a boolean algebra, which we denote by $B(A)$ and in this case is isomorphic to $\mathbf{B}^{j+k}$. Each of the atoms of this boolean algebra determine an homomorphism from A to $\mathbf{T}$. Let $Y = Hom_{L_3}(A, \mathbf{T})$, which by the previous consideration has $j + k$ elements. Let V be the set of all hemimorphisms from A to $\mathbf{T}$. By Lemma 3.1, each hemimorphism is the join of all the homomorphisms preceding it, so we can calculate that V has $2^{j+k} - 1$ elements and if $X = \{(y, v) : y \in Y, v \in V, y \leq v\}$ then $|X| = (j + k)2^{j+k-1}$.

Let L be the L_3-algebra:

$$L = \prod_{(y,v) \in X} v(A).$$

We let $X_{\mathbf{B}} = \{(y, v) \in X : v(A) = \mathbf{B}\}$ and $X_{\mathbf{T}} = X \backslash X_{\mathbf{B}}$. A is immersed in L through the monomorphism h that for each $a \in A$ and each coordinate $(y, v) \in X$ yields the element

$$ha(y, v) = y(a).$$

The quantifier $\exists$ is defined for all $p \in L$ through the formula

$$\exists p(y, v) = \bigvee \{p(u, v) : u \in Y, u \leq v\}.$$

In particular, for $a \in A$, using Lemma 3.1, we get

$$\exists ha(y, v) = \bigvee_{u \leq v} (ha)(u, v) = \bigvee_{u \leq v} u(a) = v(a). \tag{1}$$

Then $L = FME_{L_3}(h(A))$ is isomorphic to

$$\mathbf{B}^{j \cdot 2^{j-1}} \times \mathbf{T}^{(j+k) \cdot 2^{j+k-1} - j \cdot 2^{j-1}}.$$

In particular, if we take A to be $F_{L_3}(n)$, which is isomorphic to $\mathbf{B}^{2^n} \times \mathbf{T}^{3^n-2^n}$, we get that L constructed as above is isomorphic to

$$\mathbf{B}^{2^{(2^n+n-1)}} \times \mathbf{T}^{3^n \cdot 2^{(3^n-1)} - 2^{(2^n+n-1)}}.$$

This is then $FME_{L_3}(F_{L_3}(n))$, which turns out to be $F_{ML_3}(n)$, and is generated by the elements $h(g_1), \ldots, h(g_n)$, where $g_1, \ldots, g_n$ are the free generators of $F_{L_3}(n)$.

Example 3.2. When $n = 1$, we have a single generator $g = (0, c, 1)$ for $F_{L_3}(1)$, which is isomorphic to $\mathbf{B} \times \mathbf{T} \times \mathbf{B}$. We can describe the set X as follows: Let $Y = \{y_1, y_2, y_3\}$, where each y_i is the 0 homomorphism having as kernel the principal filter generated by one of the atoms of $B(A)$. We can build the set V of hemimorphisms as joins of homomorphisms:

$$V = \{y_1, y_2, y_3, v_1 = y_1 \vee y_2, v_2 = y_1 \vee y_3, v_3 = y_2 \vee y_3, v_4 = y_1 \vee y_2 \vee y_3\}.$$

The elements of X and the values of $hg(x)$, for each $x \in X$, are in the following table:

x	$h(g)(x)$
(y_1, y_1)	0
(y_1, v_1)	0
(y_1, v_2)	0
(y_1, v_4)	0
(y_2, y_2)	c
(y_2, v_1)	c
(y_2, v_3)	c
(y_2, v_4)	c
(y_3, y_3)	1
(y_3, v_2)	1
(y_3, v_3)	1
(y_3, v_4)	1

So we have the coordinates of $h(g)$ in $L = \mathbf{B} \times \mathbf{T} \times \mathbf{B} \times \mathbf{T^5} \times \mathbf{B^2} \times \mathbf{T^2}$.

4. Free MMI$_3$-algebras

In [6], it is proved that the free MMI_3-algebras with n generators $F_{MMI_3}(n)$ can be obtained from $F_{ML_3}(n)$. If $a_1, a_2, \ldots, a_n$, are a set of free generators of $F_{ML_3}(n)$ then

$$F_{MMI_3}(n) = \bigcup_{i=1}^{n} [\Delta \forall a_i). \tag{2}$$

Here as usual, $[y)$ denotes the set of all the elements x of the algebra such that $y \leq x$.

We are using as generators for $F_{ML_3}(n)$ the images under h of the generators of $F_{L_3}(n)$ that we described in Section 3.1. Using equation (1), we compute $\Delta\forall hg_i(y,v) = \Delta \sim \exists \sim hg_i(y,v) = \Delta \sim v(\sim g_i)$.

$$v(\sim g_i) = \begin{cases} 0, & \text{iff } y^*(\sim g_i) = 0, \text{ for all } y^* \leq v \\ 1, & \text{iff } y^*(\sim g_i) = 1, \text{ for some } y^* \leq v \\ c, & \text{otherwise.} \end{cases}$$

$$v(\sim g_i) = \begin{cases} 0, & \text{iff } y^*(g_i) = 1, \text{ for all } y^* \leq v \\ 1, & \text{iff } y^*(g_i) = 0, \text{ for some } y^* \leq v \\ c, & \text{otherwise.} \end{cases}$$

$$\sim v(\sim g_i) = \begin{cases} 1, & \text{iff } y^*(g_i) = 1, \text{ for all } y^* \leq v \\ 0, & \text{iff } y^*(g_i) = 0, \text{ for some } y^* \leq v \\ c, & \text{otherwise.} \end{cases}$$

$$\Delta \sim v(\sim g_i) = \begin{cases} 1, & \text{iff } y^*(g_i) = 1, \text{ for all } y^* \leq v \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.1. Coming back to $F_{ML_3}(1)$ from Example 3.2, the following table shows the value of $\Delta\forall hg$ and intermediate computation steps:

x	$h(g)$	$v(\sim g)$	$\sim v(\sim g)$	$\Delta \sim v(\sim g)$
(y_1, y_1)	0	1	0	0
(y_1, v_1)	0	1	0	0
(y_1, v_2)	0	1	0	0
(y_1, v_4)	0	1	0	0
(y_2, y_2)	c	c	c	0
(y_2, v_1)	c	1	0	0
(y_2, v_3)	c	c	c	0
(y_2, v_4)	c	1	0	0
(y_3, y_3)	1	0	1	1
(y_3, v_2)	1	1	0	0
(y_3, v_3)	1	c	c	0
(y_3, v_4)	1	1	0	0

Next, we want to compute the number of elements in $[\Delta\forall hg_i)$. Since this is an increasing subset of an MMI_3-algebra, we know it is of the form $2^b \cdot 3^t$. Since the coordinates of $\Delta\forall hg_i$ are all 0 or 1, b is the number of coordinates equal to 0 on boolean axes and t on the three-valued ones. To count those null coordinates, we consider the complementary sets

$$R^{(i)} = \{(y, v) \in X : y^*(g_i) = 1 \text{ for all } y^* \leq v\}.$$

To calculate the cardinal of $R^{(i)}$, we count the number of homomorphisms $y \in Y$ such that $y(g_i) = 1$. Since there are n generators, there are 3^n homomorphisms, from which a third, that is 3^{n-1}, are 1 when valued at g_i. By Lemma 3.1, to count the hemimorphisms in these conditions, we take all the ways we can choose j homomorphisms from among those 3^{n-1}, so

$$|R^{(i)}| = \sum_{j=1}^{3^{n-1}} \binom{3^{n-1}}{j} j = 3^{n-1} \cdot 2^{3^{n-1}-1}.$$

Notice that the value we obtained here does not depend on the index i, so $|R^{(i)}| = |R^{(j)}|$ for all $1 \leq i, j \leq n$.

To discriminate between the boolean axes and the three-valued ones, we define $R^{(i)}_{\mathbf{B}} = R^{(i)} \cap X_{\mathbf{B}}$ and $R^{(i)}_{\mathbf{T}} = R^{(i)} \backslash R^{(i)}_{\mathbf{B}}$. From the $3^{n-1} \cdot 2^{3^{n-1}-1}$ pairs in $R^{(i)}$, how many are in $X_{\mathbf{B}}$? $F_{ML_3}(n)$ has 3^n axes, from which 2^n are boolean axes and in 2^{n-1} of them the coordinates of g_i are 1, so reasoning as above we obtain

$$|R^{(i)}_{\mathbf{B}}| = \sum_{j=1}^{2^{n-1}} \binom{2^{n-1}}{j} \cdot j = 2^{2^{n-1}+n-2}$$

and therefore

$$|R^{(i)}_{\mathbf{T}}| = 3^{n-1} \cdot 2^{3^{n-1}-1} - 2^{2^{n-1}+n-2}.$$

From the above calculations, we get:

$$|[\Delta\forall hg_i)| = 2^{|X_{\mathbf{B}}| - |R^{(i)}_{\mathbf{B}}|} \cdot 3^{|X_{\mathbf{T}}| - |R^{(i)}_{\mathbf{T}}|}. \tag{3}$$

Again, the values obtained for $|R^{(i)}_{\mathbf{B}}|$ and $|R^{(i)}_{\mathbf{T}}|$, do not depend on the index i.

By the inclusion-exclusion principle, we now calculate

$$\Big|\bigcup_{i=1}^{n} [\Delta\forall hg_i)\Big| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \leq j_1 < \ldots < j_i \leq n} \Big|\bigcap_{s=1}^{i} [\Delta\forall hg_{j_s})\Big|. \tag{4}$$

Using the remark following Example 4.1, we calculate

$$\Big|\bigcap_{s=1}^{i} [\Delta\forall hg_{j_s})\Big| = \Big|\Big[\bigvee_{s=1}^{i} \Delta\forall hg_{j_s}\Big]\Big| = 2^{b(n,i,j_1,\ldots,j_i)} \cdot 3^{t(n,i,j_1,\ldots,j_i)},$$

where for each $1 \le i \le n$,

$$b(n,i,j_1,\ldots,j_i) = |X_{\mathbf{B}}| - \Big|\bigcup_{s=1}^{i} R_{\mathbf{B}}^{(j_s)}\Big|,$$

$$t(n,i,j_1,\ldots,j_i) = |X_{\mathbf{T}}| - \Big|\bigcup_{s=1}^{i} R_{\mathbf{T}}^{(j_s)}\Big|.$$

Using the inclusion-exclusion principle,

$$\Big|\bigcup_{s=1}^{i} R_{\mathbf{B}}^{(j_s)}\Big| = \sum_{h=1}^{i}(-1)^{h+1} \sum_{1\le j_1<\ldots<j_h\le n} \Big|\bigcap_{s=1}^{h} R_{\mathbf{B}}^{(j_s)}\Big|.$$

So we need to compute $|\bigcap_{s=1}^{h} R_{\mathbf{B}}^{(j_s)}|$. It is easy to see that

$$\bigcap_{s=1}^{h} R_{\mathbf{B}}^{(j_s)} = \{(y,v) \in X_{\mathbf{B}} : \text{ for all } y^* \le v,\ y^*(g_{j_1}) = \cdots = y^*(g_{j_h}) = 1\}.$$

For each h and $j_1,\ldots,j_h$, the number of homomorphisms y^* satisfying the condition is 2^{n-h}, and following the previous reasoning, the number of elements in $\bigcap_{s=1}^{h} R_{\mathbf{B}}^{(j_s)}$ is

$$v(n,h) = 2^{n-h}\cdot 2^{2^{n-h}-1} = 2^{2^{n-h}+n-h-1},$$

and again this number is independent from the chosen $j_1, j_2, \ldots, j_h$, so we may as well take the first h indices and calculate:

$$\Big|\bigcup_{s=1}^{i} R_{\mathbf{B}}^{(s)}\Big| = \sum_{h=1}^{i}(-1)^{h+1}\binom{i}{h}\Big|\bigcap_{j=1}^{h} R_{\mathbf{B}}^{(j)}\Big|,$$

$$b(n,i) = 2^{2^n+n-1} - \sum_{h=1}^{i}(-1)^{h+1}\binom{i}{h}v(n,h) = \sum_{h=0}^{i}(-1)^{h}\binom{i}{h}v(n,h).$$

In a similar way, we compute $|\bigcap_{j=1}^{h} R_{\mathbf{T}}^{(j)}| = |\bigcap_{j=1}^{h} R^{(j)}| - |\bigcap_{j=1}^{h} R_{\mathbf{B}}^{(j)}|$. For this we first note that

$$\Big|\bigcap_{j=1}^{h} R^{(j)}\Big| = \sum_{s=1}^{3^{n-h}}\binom{3^{n-h}}{s}\cdot s = 3^{n-h}\cdot 2^{3^{n-h}-1} = u(n,h).$$

Now using again the inclusion-exclusion principle,

$$\Big|\bigcup_{s=1}^{i} R_{\mathbf{T}}^{(s)}\Big| = \sum_{s=1}^{i}(-1)^{s+1}\binom{i}{s}\Big|\bigcap_{j=1}^{s} R_{\mathbf{T}}^{(j)}\Big|,$$

and since

$$\left|\bigcap_{j=1}^{s} R_{\mathbf{T}}^{(j)}\right| = u(n,s) - v(n,s) = 3^{n-s} \cdot 2^{3^{n-s}-1} - 2^{n-s} \cdot 2^{2^{n-s}-1},$$

we get that

$$\left|\bigcup_{s=1}^{i} R_{\mathbf{T}}^{(s)}\right| = \sum_{s=1}^{i} (-1)^{s+1} \binom{i}{s} (u(n,s) - v(n,s)).$$

Since we also have $|X_{\mathbf{T}}| = |X| - |X_{\mathbf{B}}| = 3^n \cdot 2^{3^n-1} - 2^{2^n+n-1} = u(n,0) - v(n,0)$, we can write:

$$t(n,i) = \sum_{s=0}^{i} (-1)^{s} \binom{i}{s} (u(n,s) - v(n,s)).$$

Finally, we can put all together:

$$\begin{aligned} |F_{MMI_3}(n)| &= \left|\bigcup_{i=1}^{n} [\Delta\forall h g_i)\right| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \le j_1 < \ldots < j_i \le n} \left|\bigcap_{s=1}^{i} [\Delta\forall h g_{j_s})\right| \\ &= \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} 2^{b(n,i)} \cdot 3^{t(n,i)}, \end{aligned}$$

where

$$b(n,i) = \sum_{h=0}^{i} (-1)^{h} \binom{i}{h} 2^{2^{n-h}+n-h-1}$$

and

$$t(n,i) = \sum_{h=0}^{i} (-1)^{h} \binom{i}{h} 3^{n-h} \cdot 2^{3^{n-h}-1} - 2^{2^{n-h}+n-h-1}.$$

Example 4.2. For $n = 1$, we just use the coordinates of $\Delta\forall hg$ that we calculated in Example 4.1. $F_{ML_3}(1)$ is isomorphic to $\mathbf{B}^4 \times \mathbf{T}^8$, and $\Delta\forall hg$ is 1 only at one of its boolean axes. Therefore $|F_{MMI_3}(1)| = 2^3 \cdot 3^8$.

For $n = 2$ we calculate $|[\Delta\forall h g_1) \cap [\Delta\forall h g_2)| = |[\Delta\forall h g_1 \vee \Delta\forall h g_2)|$.

$$(\Delta\forall h g_1 \vee \Delta\forall h g_2)(y,v) = 1 \text{ iff } y^*(g_1) = 1 \text{ or } y^*(g_2) = 1 \text{ for all } y^* \le v.$$

This is, iff $(y,v) \in R^{(1)} \cup R^{(2)}$.

$$|R^{(1)} \cup R^{(2)}| = |R^{(1)}| + |R^{(2)}| - |R^{(1)} \cap R^{(2)}|.$$

Since there is only one homomorphism y^* such that $y^*(g_1) = y^*(g_2) = 1$, we obtain $|R^{(1)} \cap R^{(2)}| = 1$.

Therefore,

$$|R^{(1)} \cup R^{(2)}| = 2 \cdot 3 \cdot 2^{3-1} - 1 = 23.$$

Using a similar reasoning, $|R_{\mathbf{B}}^{(1)} \cap R_{\mathbf{B}}^{(2)}| = 1$.

Then

$|F_{MMI_3}(2)| = 2 \cdot 2^{|X_{\mathbf{B}}|-|R_{\mathbf{B}}|} \cdot 3^{|X_{\mathbf{T}}|-|R_{\mathbf{T}}|} - 2^{|X_{\mathbf{B}}|-|R^{(1)}_{\mathbf{B}} \cup R^{(2)}_{\mathbf{B}}|} \cdot 3^{|X_{\mathbf{T}}|-|R^{(1)}_{\mathbf{T}} \cup R^{(2)}_{\mathbf{T}}|}$,

where $|X_{\mathbf{B}}| = 2^2 \cdot 2^{2^2-1} = 32$; $|X_{\mathbf{T}}| = |X| - |X_{\mathbf{B}}| = 3^2 \cdot 2^{3^2-1} - 2^2 \cdot 2^{2^2-1} = 2^5 \cdot (9 \cdot 2^3 - 1) = 2272$; $|R_{\mathbf{B}}| = 2^{2^{2-1}+2-2} = 2^2 = 4$; $|R_{\mathbf{T}}| = 3^{2-1} \cdot 2^{3^{2-1}-1} - 2^{2^{2-1}+2-2} = 3 \cdot 4 - 4 = 8$.

The unions are computed as follows: $|R^{(1)}_{\mathbf{B}} \cup R^{(2)}_{\mathbf{B}}| = |R^{(1)}_{\mathbf{B}}| + |R^{(2)}_{\mathbf{B}}| - |R^{(1)}_{\mathbf{B}} \cap R^{(2)}_{\mathbf{B}}| = 4+4-1 = 7$; $|R^{(1)}_{\mathbf{T}} \cup R^{(2)}_{\mathbf{T}}| = |R^{(1)} \cup R^{(2)}| - |R^{(1)}_{\mathbf{B}} \cup R^{(2)}_{\mathbf{B}}| = 23-7 = 16$, so replacing with the corresponding figures, we finally get:

$|F_{MMI_3}(2)| = 2 \cdot 2^{32-4} \cdot 3^{2272-8} - 2^{32-7} \cdot 3^{2272-16} = 2 \cdot 2^{28} \cdot 3^{2264} - 2^{25} \cdot 3^{2256}$.

5. Conclusions

We have presented a new method to determine the structure and number of elements of the free MMI_3-algebras with a finite number of generators. This method is interesting because it provides with easier formulas than those presented in [6] (those formulas can also be seen in the abstract published in [1]), as well as an explicit description of the generators and minimal elements of those algebras.

As a byproduct of the construction presented here, one may also obtain the structure and generators of the free monadic Tarski algebras $F_{MT}(n)$, using a different method from those used in [5] and [12]. In order to do this, one starts with the free boolean algebra, which can be obtained using Cohn's theorem and then calculate its free monadic extension just as in [7]. The set $X_{\mathbf{T}}$ is then empty and the calculations of the sets $X_{\mathbf{B}}$ and $R^{(i)}_{\mathbf{B}}$ stand, so we get as in [12]:

$$|F_{MT}(n)| = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} 2^{\sum_{h=0}^{i}(-1)^h \binom{i}{h} 2^{2^{n-h}+n-h-1}}.$$

We hope to extend some of these results to other varieties of interest.

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