

Convergence of the barycentre of measures from Fuchsian action groups

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Abstract. We prove the pointwise ergodic convergence of the sequence of barycentres of empirical measures which are defined from the action of Fuchsian groups and by maps valued in $CAT(0)$ -spaces. A result of this nature was established by Austin from actions of amenable groups and defining the empirical measures from Følner sequences. Here we define different sequences of barycentres, in particular we do not consider a topological structure on the group and Følner sequences.

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1 Introduction

The extension of the Birkhoff Ergodic Theorem to groups acting on probability spaces has been matter of relevant studies. Nevo and Stein [8] have proved the pointwise ergodic convergence for finite measure preserving actions of the free group F_r , $r \geq 2$, whereas Fujiwara and Nevo [5] have established the ergodic convergence for exponentially mixing actions of word-hyperbolic groups.

In reference [6] Lindenstrauss considered actions of amenable groups Γ on a Lebesgue space (X, μ) and maps $\varphi : X \rightarrow \mathbf{R}$, to analyze averages of the form

$$\frac{1}{m_\Gamma(E)} \int_E \varphi(\gamma x) dm_\Gamma(\gamma),$$

where $E \subset \Gamma$ and m_Γ is the Haar measure on Γ . In that article was proved that for adequate sequences (F_n) in Γ , called Følner sequences, there is a Γ -invariant map $\bar{\varphi}$ such that

$$(1.1) \quad \frac{1}{m_\Gamma(F_n)} \int_{F_n} \varphi(\gamma x) dm_\Gamma(\gamma) \rightarrow_n \bar{\varphi}(x), \quad \text{for } \mu - a.e.x. .$$

Bufetov and Series [3] applied a previous result by Bufetov [2] about convergence of Cesàro averages to prove the ergodic convergence of surface groups: let Γ be the fundamental group of a surface of genus $g \geq 2$ acting on probability space (X, μ) , the length of $\gamma \in \Gamma$, denoted $|\gamma|$, is the minimal number of generators needed to represent γ . Here $S(n)$ denotes the sphere of radius n , i.e.

$$S(n) = \{\gamma : |\gamma| = n\}.$$

For $\gamma \in \Gamma$, by T_γ is denoted the transformation on X given by $T = T_\gamma(x) = \gamma x$. Let the average

$$(1.2) \quad \mathcal{S}_{N,\varphi}(x) := \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{\text{card}S(n)} \sum_{\gamma \in S(n)} \varphi(T_\gamma(x)),$$

where $\varphi \in L^1(X, \mu)$. For a class of groups, by the Series coding [9], a partition \mathcal{I} of the boundary of the hyperbolic disc H^2 in intervals can be defined, which results a Markov partition. Then a transition matrix $(A_{I,J})_{I,J \in \mathcal{I}}$ can be obtained and a Markov symbolic space given by the sequences $I_{i_0} \dots I_{i_{n-1}}$, allowed by the matrix A . In [3] was established that 1.2 converges *a.e.* for groups in which Series coding can be done. When Γ acts ergodically on X holds:

$$\mathcal{S}_{N,\varphi}(x) \rightarrow_N \int \varphi d\mu,$$

for any $\varphi \in L^1(X, \mu)$ and *a.e.x.* In fact, the Series coding can be applied to a wide class of finitely generated Fuchsian groups; so the above ergodic convergence can be established with more generality, being the surfaces groups a particular case [3].

Further developments have considered maps valuated in more general spaces like $CAT(0)$ -spaces. Let us consider the empirical measures

$$(1.3) \quad \mathcal{E}_{N,\varphi}(x) := \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{\text{card}S(n)} \sum_{\gamma \in S(n)} \delta_{\varphi(T_\gamma(x))},$$

where δ is the point mass Dirac measure. Thus for $E \subset Y$

$$(1.4) \quad \mathcal{E}_{N,\varphi}(x)(E) := \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{\text{card}S(n)} \times \text{card}\{\gamma \in S(n) : \varphi(T_\gamma(x)) \in E\}.$$

For complementary literature about ergodic theorems for groups see, for example, [7] and references therein.

For $CAT(0)$ -spaces Y can be defined the *barycentre map*

$$\text{bar} : \mathcal{M}_2(Y) \rightarrow Y,$$

where $\mathcal{M}_2(Y)$ denotes the set of measures μ in Y with second finite moment, i.e.

$$\int_Y d(y, z)^2 d\mu(z) < \infty, \quad \text{for any } y \in Y.$$

This map replaces the ergodic averages in the real case.

In [1] Austin considered maps $\varphi : X \rightarrow Y$, with (X, ν) a probability space and Y a complete, separable, $CAT(0)$ -space, where these maps verify the condition $\int_X d(\varphi(x), y)^2 d\nu(x) < \infty$. The class of these maps is denoted by $L^2(X, Y, \nu)$. The barycentre map is defined on the set of measures on Y with second finite moment. Austin established the convergence of the barycentre of empirical measures from the Lindenstrauss ergodic averages of amenable action groups and for any $\varphi \in L^1(X, Y, \nu)$.

In this article we shall prove the barycentre convergence of the sequence empirical measures $\mathcal{E}_N, \varphi(x)$ using the ergodic convergence established by Bufetov and Series in the real case. The groups considered are Fuchsian groups acting on the hyperbolic disc H^2 with a symmetric set of generators S , so that are included surface group in our setting. Besides, in the context of this work is not considered, unlike in reference [1], a topological structure on the group and the existence of Følner sequences.

2 The result

Before stating our result we recall some basic definitions and notations

Definition 2.1. A geodesic space (Y, d) is a $CAT(0)$ -space if for every geodesic triangle Δ in X there is a comparison triangle $\overline{\Delta}$ in \mathbf{R}^2 , i.e. a triangle with sides of the same length as the sides of Δ , such that distances between points on Δ are less than or equal to the distances between corresponding points on $\overline{\Delta}$.

Let (X, ν) be a probability space and let $\varphi : X \rightarrow Y$ with Y a geodesic, complete, separable, $CAT(0)$ -space. Recall that by $L^2(X, Y, \nu)$ is denoted the class of maps $\varphi : X \rightarrow Y$ with $\int_X d(\varphi(x), y)^2 d\nu(x) < \infty$. Since Y is $CAT(0)$, complete, separable the space $L^2(X, Y, \nu)$ can be endowed with the metric

$$(2.1) \quad d_2(\varphi, \psi) := \sqrt{\int_X d(\varphi(x), \psi(x))^2 d\nu(x)}.$$

Definition 2.2. a) The *barycentre map* is that defined as follows: for any $\mu \in \mathcal{M}_2(Y)$, there is an unique $y \in Y$ which minimizes $\int_Y d(y, z)^2 d\mu(z)$ [4], thus is defined $bar(\mu) = y$.

b) A *coupling* of two measures $\mu_1, \mu_2 \in \mathcal{M}(Y)$ is a measure $m \in \mathcal{M}(Y \times Y)$ which on the first and second factor projects on μ_1, μ_2 respectively.

c) The *2-Wasserstein metric* is defined as

$$(2.2) \quad W_2(\mu_1, \mu_2) = \inf_{m \text{ coupling of } \mu_1, \mu_2 \in \mathcal{M}(Y)} \sqrt{\int_{Y \times Y} d(y, z)^2 dm(y, z)}.$$

For the proof of our result we shall use the convergence in the real case, therefore the groups considered must be such that Series coding can be done. They are non-elementary Fuchsian groups acting on the hyperbolic disc H^2 with a symmetric set of generators S . Besides, a fundamental domain \mathcal{R} for Γ must verify some conditions,

in particular when $|\partial\mathcal{R}| \geq 5$ a coding of Γ can be done. Here $|\partial\mathcal{R}|$ means the number of sides of the polygon \mathcal{R} . For details about Series coding see [9].

A matrix $A = (a_{i,j})$, $a_{i,j} \geq 0$, is irreducible if for any i, j there is a $n > 0$ such that $a_{i,j}^{(n)} > 0$, where $a_{i,j}^{(n)}$ is the i, j -entry of A^n . A matrix A is *strictly irreducible* if $A^t A$ is irreducible (A^t means the transpose of A). If $I, J \in \mathcal{I}$ then consider the equivalence relation $I \sim J$ if $f(I) \cap f(J) \neq \emptyset$, where f is the boundary map defined in [9]. Then A is strictly irreducible when there is just one equivalence class. When $|\partial\mathcal{R}| \leq 4$ to obtain a symbolic representation of elements of Γ it must be imposed that A be strictly irreducible [3].

Theorem 2.1. *Let (X, ν) be a probability space and Y a complete, separable, $CAT(0)$ -space. Let Γ be a non-elementary Fuchsian group acting on the hyperbolic disc H^2 with a symmetric set of generators S . Besides Γ acts on X by measure preserving actions T , which has a fundamental region \mathcal{R} with $|\partial\mathcal{R}| \geq 5$ or if $|\partial\mathcal{R}| < 5$ then the transition matrix from the Series coding is strictly irreducible. If $\varphi : X \rightarrow Y$ is in the class $L^2(X, Y, \nu)$ then the sequence $\{\text{bar}(\mathcal{E}_N, \varphi(x))\}$ converges to $\bar{\varphi}(x)$, for some T -invariant map $\bar{\varphi}$ and for any x , $\nu - a.e.$ The map $\bar{\varphi}$ is constant $\nu - a.e.$ when the group acts ergodically.*

In the proof will be followed an idea of Austin [1], which consists in firstly to establish the result for maps taking only a finite number of values. Then, using the separability of Y , the result is extended for more general functions. Now, since Y is separable, a map φ can be approximated, in the d_2 metric, by as sequence of finite-valued maps $\{\psi_n\}$. Let us fix a number n_0 , and let $\psi := \psi_{n_0}$ such that $d_2(\varphi, \psi) < \alpha^2$, it can be proved that the measure of set of the points x such that $\text{bar}(\mathcal{E}_N, \varphi(x))$ and $\text{bar}(\mathcal{E}_N, \psi(x))$ apart, can be controlled. This will be possible by a maximal ergodic theorem for real valued maps. For amenable locally compact groups, a maximal ergodic theorem was proved by Lindenstrauss [6], result that was used by Austin. Here we formulate a maximal ergodic theorem in our scheme, which can established from classical results in Ergodic Theory and a more natural way than in the setting of amenable groups.

Let U_Γ be the operator $U_\Gamma(\varphi)(x) := \frac{1}{\text{card}S(1)} \sum_{\gamma \in S(1)} \varphi(T_\gamma(x))$, $\varphi \in L^0(X, \nu)$, it

holds that $\int U_\Gamma(\varphi) d\nu = \int \varphi d\nu$. Let $\{I_n\}$ be a sequence of simple function approximating φ , then $\int U_\Gamma(\varphi) d\nu = \lim_{n \rightarrow \infty} \int U_\Gamma(I_n) d\nu = \lim_{n \rightarrow \infty} \int I_n d\nu = \int \varphi d\nu$. Thus if we set $\psi(x) = |\varphi(x)|^{1/p}$ then U_Γ acts by isometries on $L^p(X, \nu)$, $p \geq 1$. Besides U_Γ is a positive operator and its norm is bounded.

Proposition 2.2. *Let $\alpha > 0$, $\varphi \in L^1(X, \nu)$ and $Z_\alpha := \{x : \sup_{N \geq 1} \mathcal{S}_N \varphi(x) > \alpha\}$, then*

$$\nu(Z_\alpha) \leq \frac{1}{\alpha} \int_{Z_\alpha} \varphi d\nu.$$

Proof. We may assume that $\varphi \geq 0$, otherwise the claim can proved by considering positive and negative parts. Let $\psi \in L^1(X, \nu)$ and let $\psi_n = \psi(x) + U_\Gamma(\psi) + \dots + U_\Gamma^{n-1}(\psi)$, for $n \geq 1$ and $\psi_0 = \psi$. For $N \geq 1$ set

$$\Phi_N(x) = \Phi_{N,\varphi}(x) := \max_{0 \leq n \leq N-1} \{\psi_n(x)\}.$$

Firstly we shall show that when $\psi_n(x) \geq 0$ holds

$$\int_{\{x:\Phi_N(x)>0\}} \psi d\nu \geq 0.$$

We have $\Phi_N \geq \psi_n$, and since U_Γ is positive holds $U_\Gamma(\Phi_N) \geq U_\Gamma(\psi_n) = \psi_{n+1} - \psi$. Thus

$$\psi(x) + U_\Gamma(\Phi_N)(x) \geq \psi_{n+1}(x) \text{ when } \Phi_N(x) > 0,$$

so

$$\psi(x) + U_\Gamma(\Phi_N)(x) \geq \max_{0 \leq n \leq N} \{\psi_{n+1}(x)\} = \max_{1 \leq n \leq N} \{\psi_n(x)\}.$$

Therefore when $\Phi_N(x) > 0$

$$\psi(x) + U_\Gamma(\Phi_N)(x) = \Phi_N(x).$$

Let $A_N := \{x : \Phi_N(x) > 0\}$, then

$$\int_{A_N} \psi d\nu \geq \int_{A_N} \Phi_N d\nu - \int_{A_N} U_\Gamma(\Phi_N) d\nu.$$

Since $\Phi_N = 0$ outside A_N , $U_\Gamma(\Phi_N) \geq 0$ because U_Γ is positive and the norm of U_Γ is bounded, we have

$$\int_{A_N} \psi d\nu \geq \int_X \Phi_N d\nu - \int_{A_N} U_\Gamma(\Phi_N) d\nu \geq \int_X \Phi_N d\nu - \int_X U_\Gamma(\Phi_N) d\nu \geq 0.$$

We apply the above result to $\psi = \varphi - \alpha$, with $\alpha > 0$, and $\Phi_N = \max\left\{\frac{1}{n}(\varphi_n - \alpha)\right\} \geq 0$. Therefore Z_α can be written as

$$Z_\alpha = \bigcup_{N=0}^{\infty} \left\{x : \sup_{N \geq 1} \{\mathcal{S}_{N,\varphi}(x) - \alpha\} > 0\right\},$$

so that $\int_{Z_\alpha} \psi d\nu = \int_{Z_\alpha} (\varphi - \alpha) d\nu \geq 0$, and then $\int_{Z_\alpha} \varphi d\nu \geq \alpha \nu(Z_\alpha)$. \square

The following result, due to Austin [1], will be used in the proof of the theorem:

Lemma 2.3. *The barycentre map is controlled by the Wasserstein metric in the sense that*

$$(2.3) \quad d(\text{bar}(\mu_1), \text{bar}(\mu_2)) \leq W_2(\mu_1, \mu_2).$$

Proof of the Theorem. Let $\varphi, \psi \in L^2(X, Y, \nu)$, $\alpha > 0$ and $\Phi(x) = d(\varphi(x), \psi(x))^2$. By the above proposition applied to the real valued function Φ , we have

$$\begin{aligned} \nu\left(\left\{x : \sup_{N \geq 1} \{\mathcal{S}_{N,\Phi}(x)\} > \alpha^2\right\}\right) &\leq \frac{1}{\alpha^2} \int \Phi d\nu \leq \\ &\leq \frac{1}{\alpha^2} \int d(\varphi(x), \psi(x))^2 = \frac{1}{\alpha^2} d_2(\varphi, \psi)^2, \end{aligned}$$

and since $d(\text{bar}(\mathcal{E}_N, \varphi(x)), \text{bar}(\mathcal{E}_N, \psi(x))) \leq W_2(\mathcal{E}_N, \varphi(x), \mathcal{E}_N, \psi(x))$, one gets

$$\left\{ x : \sup_{N \geq 1} d(\text{bar}(\mathcal{E}_N, \varphi(x)), \text{bar}(\mathcal{E}_N, \psi(x))) > \alpha \right\} \subset \left\{ x : \sup_{N \geq 1} W_2(\mathcal{E}_N, \varphi(x), \mathcal{E}_N, \psi(x)) > \alpha \right\}.$$

Let us consider the measure $\mathcal{F}_{N, \varphi, \psi}(x) := \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{\text{card}S(n)} \sum_{\gamma \in S(n)} \delta_{(\varphi(T_\gamma(x)), \psi(T_\gamma(x)))}$, with $x, y \in X$, which is a joining between $\mathcal{E}_N, \varphi(x), \mathcal{E}_N, \psi(y)$. Thus by the definition of the W_2 metric

$$\begin{aligned} W_2(\mathcal{E}_N, \varphi(x), \mathcal{E}_N, \psi(x))^2 &\leq \sqrt{\int_{Y \times Y} d(\varphi(T_\gamma(x)), \psi(T_\gamma(x)))^2 d\mathcal{F}_{N, \varphi, \psi}(x)} \\ &\leq \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{\text{card}S(n)} \sum_{\gamma \in S(n)} \Phi(\gamma x), \end{aligned}$$

and so

$$(2.4) \quad \left\{ x : \sup_{N \geq 1} d(\text{bar}(\mathcal{E}_N, \varphi(x)), \text{bar}(\mathcal{E}_N, \psi(x))) > \alpha^2 \right\} \subset \left\{ x : \sup_{N \geq 1} \{\mathcal{S}_{N, \Phi}(x)\} > \alpha \right\}.$$

Therefore

$$(2.5) \quad \begin{aligned} \nu \left(\left\{ x : \sup_{N \geq 1} d(\text{bar}(\mathcal{E}_N, \varphi(x)), \text{bar}(\mathcal{E}_N, \psi(x))) > \alpha^2 \right\} \right) &\leq \nu \left(\left\{ x : \sup_{N \geq 1} \{\mathcal{S}_{N, \Phi}(x)\} > \alpha^2 \right\} \right) \\ &\leq \frac{1}{\alpha^2} d_2(\varphi, \psi)^2. \end{aligned}$$

Let $\{y_1, y_2, \dots, y_m\}$ be the image of φ and $A_i := \varphi^{-1}(y_i)$, $i = 1, 2, \dots, m$. Let $\{\nu_x\}$ be the ergodic decomposition of ν with respect to T , by the convergence theorem of Bufetov and Series, for any $\varepsilon > 0$, $0 \leq i \leq m$, there is a $N_0 = N_0(\varepsilon, x, i)$ such that for $N \geq N_0$

$$(2.6) \quad |\mathcal{S}_{N, I_{A_i}}(x) - \nu_x(A_i)| < \varepsilon,$$

for any ν -a.e. x . Let $m(x) := \sum_{i=1}^m \nu_x(A_i) \delta_{y_i}$, we have

$$W_2(\mathcal{E}_N, \varphi(x), m(x))^2 \leq \sum_{i=1}^m |\mathcal{S}_{N, I_{A_i}}(x) - \nu_x(A_i)| \left[\max_{i,j} d(y_i, y_j) \right]^2 < \varepsilon^2 \left[\max_{i,j} d(y_i, y_j) \right]^2,$$

for $N \geq N_0$. By the property that the barycentre is dominated by the W_2 -metric we have $d(\text{bar}(\mathcal{E}_N, \varphi(x)), \text{bar}(m(x))) < \varepsilon \times$ (a fixed quantity), therefore

$$\text{bar}(\mathcal{E}_N, \varphi(x)) \rightarrow \text{bar}(m(x)) := \bar{\varphi}(x) \text{ as } N \rightarrow \infty.$$

For the general case, let $\varphi \in L^2(X, Y, \nu)$, since Y is separable φ can be approximated, in the d_2 metric by mean a sequence of finite-valuated maps, i.e. there is a family

$\{\psi_n\}$ of finite-valuated maps such that for any $\alpha > 0$ there is a $n_0 = n_0(\varepsilon)$ and a $\psi := \psi_{n_0}$ with $d_2(\varphi, \psi) < \alpha^2$. By (13) we have

$$(2.7) \quad \nu \left(\left\{ x : \sup_{N \geq 1} d(\text{bar}(\mathcal{E}_N, \varphi(x)), \text{bar}(\mathcal{E}_N, \psi(x))) > \alpha^2 \right\} \right) \leq \frac{1}{\alpha^2} d_2(\varphi, \psi)^2 < \alpha^2.$$

Therefore the sequence $\{\text{bar}(\mathcal{E}_N, \varphi(x))\}$ oscillates on a set of measure at most α^2 , with α arbitrary, and since the sequence $\text{bar}(\mathcal{E}_N, \psi(x))$ is almost surely convergent to a map $\bar{\psi}(x)$, the sequence $\{\text{bar}(\mathcal{E}_N, \varphi(x))\}$ also converges, $\nu - a.e.$, for any x . \square

3 Conclusion

In this article we have proved the barycentre convergence of the sequence empirical measures $\mathcal{E}_N, \varphi(x)$ considering Fuchsian groups acting on the hyperbolic disc H^2 . In our context we define the sequence of empirical measures without using Følner sequences.

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