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Note on generalization of Jackiw-Pi vortices

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Abstract – We analyze two Abelian Higgs systems with nonstandard kinetic terms. First, we consider a model involving the Maxwell term. For a particular choice of the nonstandard kinetics, we are able to obtain generalized Jackiw-Pi vortices. We, also, analyze a second model which is a generalization of the Jackiw-Pi model. In this case we show that the system support the Nielsen-Olesen vortices as solutions.

Introduction. – The two-dimensional Abelian Higgs model coupled to gauge fields, whose dynamics is dictated by a Maxwell term, admits topological vortex solutions [1–4]. Furthermore, with a specific choice of coupling constants, the model presents particular properties such as supersymmetry extension [5–7] and the reduction of the field equations to first-order differential equations, often called Bogomolnyi equations [8]. In addition, Chern-Simons-Higgs theory in $(2 + 1)$-dimensions without Maxwell term presents Bogomolnyi equations for a sixth-order potential [9,10]. In this model there exist both topological and non-topological solitons. These solitons carry not only magnetic flux, but also electric charge.

In the recent years, theories with nonstandard kinetic term, named k-field models, have received much attention. The k-field models are mainly in connection with effective cosmological models [11,12] as well as with the strong-interaction physics, strong gravitational waves and dark matter. One interesting aspect to analyze in these models concerns their topological structure. In the context of soliton solution, several works have be done showing that the k-theories can support topological soliton solutions both in models of matter and in gauged models [13–24]. These solitons have certain features such as their characteristic size, which are not necessarily those of the standard models [25]. Other interesting aspects are that they do not interact at large distances and they are, in general, not self-dual.

In this letter we are interested in studying a generalization of the Abelian Maxwell-Higgs model, by introducing nonstandard kinetic terms in the Lagrangian. We introduce the nonstandard dynamics by a function ω, which depends on the Higgs field. We study the Bogomolnyi limit for such system. We are able to obtain, for a particular choice of the ω and the potential term, a generalization of the Bogomolnyi equations corresponding to a nonrelativistic Chern-Simons-Higgs theory usually known as Jackiw-Pi model [26,27]. We construct the solutions of these equations and we show that the Jackiw-Pi vortices may be obtained as a particular case. As a particular feature, we show that our vortex solutions have no electric charge. This is a difference with the solutions of the Jackiw-Pi model.

We further propose a generalization of the Jackiw-Pi model. In particular, will show that choosing a suitable ω, the Bogomolnyi equations of the Maxwell-Higgs model are obtained. The soliton solutions of these equations are identical in form to the Nielsen-Olesen vortices. Nevertheless, our vortex solution have electric charge, a feature which is not present in the usual Nielsen-Olesen vortices.

Generalized Jackiw-Pi vortices from the Abelian Maxwell-Higgs model. – Following the works cited in refs. [17–24], we start by considering a generalized $(2 + 1)$-dimensional Maxwell model coupled to the bosonic field ϕ. The dynamics of this model is described by the action

$$S = \int d^3x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right.$$
$$\left. + \omega(\rho) \left[\frac{1}{2} |D_0 \phi|^2 - \frac{1}{2} |D_i \phi|^2 \right] - V(\rho) \right). \quad (1)$$

The covariant derivative, here, is defined as

$$D_\mu = \partial_\mu + ie A_\mu \qquad (\mu = 0, 1, 2). \quad (2)$$

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The metric tensor is $g^{\mu\nu} = (1, -1, -1)$ and $\epsilon^{\mu\nu\lambda}$ is the totally antisymmetric tensor such that $\epsilon^{012} = 1$.

Notice, that we have replaced the usual kinetic term $|D_i\phi|^2$ by a more generalized term $\omega(\rho)|D_i\phi|^2$, where $\omega(\rho)$ is a dimensionless function of the complex scalar field ϕ. Here, $V(\rho)$ is the scalar field potential to be determined below, being $\rho = \phi^\dagger\phi$.

The variation of this action yields the field equations,

$$\frac{\partial\omega(\rho)}{\partial\phi^*}|D_0\phi|^2 - \frac{\partial\omega(\rho)}{\partial\phi^*}|D_i\phi|^2 + \omega(\rho)D_0D^0\phi$$
$$-\omega(\rho)D_iD^i\phi - 2\frac{\partial V}{\partial\phi^*} = 0, \tag{3}$$
$$\partial^\nu[F_{i\nu}] = -ej_i\omega(\rho),$$
$$\partial^\nu[F_{0\nu}] = -ej_0\omega(\rho),$$

where $j_0 = \frac{1}{2i}(\phi^*D_0\phi - (D_0\phi)^*\phi)$ and $j_i = \frac{1}{2i}(\phi^*D_i\phi - (D_i\phi)^*\phi)$. From the last equation in (3), we see that the $A_0 = 0$ gauge may be chosen in a consistent way for static solutions.

Here, we are interested in time-independent soliton solutions that ensure the finiteness of the action (1). These are the stationary points of the energy which, for the static field configuration, reads

$$E = \int d^2x \left(\frac{1}{2}B^2 + \frac{1}{2}\omega(\rho)|D_i\phi|^2 + V(\rho)\right). \tag{4}$$

In order to find self-dual soliton solutions, we consider the following choice for the function $\omega(\rho)$:

$$\omega(\rho) = (n+1)\rho^n, \tag{5}$$

where n is a real number that satisfies $n \geq 0$.

To proceed, we need the fundamental identity

$$|D_i\phi|^2 = |(D_1 \pm iD_2)\phi|^2 \mp eB|\phi|^2 \pm \epsilon^{ik}\partial_ij_k, \tag{6}$$

where i and k may take the values 1 and 2. Then, we may rewrite the energy (4) as

$$E = \int d^2x \left(\frac{1}{2}B^2 + \frac{(n+1)}{2}\rho^n|(D_1 \pm iD_2)\phi|^2\right.$$
$$\left.\mp \frac{(n+1)}{2}\rho^{n+1}eB + V(\rho) \pm \frac{(n+1)}{2}\rho^n\epsilon^{ik}\partial_ij_k\right). \tag{7}$$

The last term of this integral may be integrated by parts. Indeed, after a bit of algebra, it is not difficult to show that

$$\int d^2x\epsilon^{ik}\rho^n\partial_ij_k = \int d^2xe\left(\epsilon^{ik}\rho^nA_k\partial_i\rho + \epsilon^{ik}\rho^n\partial_iA_k\rho\right), \tag{8}$$

which may be rewritten in a more suitable form,

$$\int d^2x\epsilon^{ik}\rho^n\partial_ij_k = \int d^2xe\left(\frac{\epsilon^{ik}}{n+1}A_k\partial_i\rho^{n+1} + \epsilon^{ik}\rho^{n+1}\partial_iA_k\right). \tag{9}$$

Again, integrating by parts, we have

$$\int d^2x\epsilon^{ik}\rho^n\partial_ij_k = e\int d^2x\epsilon^{ik}\partial_iA_k\rho^{n+1}\left(-\frac{1}{n+1} + 1\right) =$$
$$\frac{ne}{n+1}\int d^2xB\rho^{n+1}. \tag{10}$$

Thus, using (10), the energy (7) is rewritten as

$$E = \int d^2x\left(\frac{(n+1)}{2}\rho^n|(D_1 \pm iD_2)\phi|^2\right.$$
$$\left.\mp \frac{e}{2}\rho^{n+1}B + V(\rho) + \frac{1}{2}B^2\right). \tag{11}$$

In the case that we choose

$$V(\rho) = \frac{e^2}{8}\rho^{2(n+1)}, \tag{12}$$

we can write the last two terms of (11) as

$$V(\rho) + \frac{1}{2}B^2 = \frac{1}{2}\left[B \mp \frac{e}{2}\rho^{(n+1)}\right]^2 \pm \frac{e}{2}B\rho^{(n+1)}. \tag{13}$$

Then,

$$E = \int d^2x\left(\frac{1}{2}\left[B \mp \frac{e}{2}\rho^{(n+1)}\right]^2\right.$$
$$\left.+ \frac{(n+1)}{2}\rho^n|(D_1 \pm iD_2)\phi|^2\right). \tag{14}$$

The energy is bounded below by zero and the bound is saturated by the fields satisfying the first-order equations,

$$\phi^n(D_1 \pm iD_2)\phi = 0,$$
$$B = \pm\frac{e}{2}\rho^{(n+1)}. \tag{15}$$

Here, it is interesting to analyze these equations when $n = 0$. In that case the eqs. (15) becomes

$$(D_1 \pm iD_2)\phi = 0,$$
$$B = \pm\frac{e}{2}\rho. \tag{16}$$

which may be compared with the self-duality equations of the Jackiw-Pi model [26]. Indeed, if we choose the plus sign in the second of equations (16) we arrive to the Bogomolnyi equations of the Jackiw-Pi model,

$$(D_1 \pm iD_2)\phi = 0,$$
$$B = \frac{e}{2}\rho. \tag{17}$$

Also, if $n = 0$, the potential term (12) becomes a ϕ^4 potential as in the Jackiw-Pi model. To solve the Bogomolnyi equations of the Jackiw-Pi model one usually decomposes the scalar field ϕ into its phase and magnitude:

$$\phi = \rho^{\frac{1}{2}}e^{i\alpha}. \tag{18}$$

Then the first of the self-duality equations (17) determines the gauge field

$$A_i = \frac{1}{2ie\rho}\left(\mp i\epsilon^{ij}\partial_j\rho - \phi^\dagger\partial_i\phi + \phi\partial_i\phi^\dagger\right) \qquad (19)$$

everywhere away from the zeros of the scalar field. Thus, using (19) the second self-duality equation in (17) reduces to a nonlinear elliptic equation for the scalar field density ρ,

$$\pm e^2\rho = \nabla^2 \log\rho. \qquad (20)$$

This elliptic equation, known as the Liouville equation, is exactly solvable,

$$\rho = \frac{2}{e^2}\nabla^2 \log\left(1 + |f|^2\right), \qquad (21)$$

where $f = f(z)$ is a holomorphic function of $z = x_1 + ix_2$. General radially symmetric solutions may be obtained by taking $f(z) = (\frac{z_0}{z})^N$. Then, we have

$$\rho = \frac{8N^2}{e^2 r_0^2}\frac{\left(\frac{r}{r_0}\right)^{2(N-1)}}{\left(1 + \left(\frac{r}{r_0}\right)^{2N}\right)^2}. \qquad (22)$$

This vanishes as $r \to \infty$ and is nonsingular at the origin for $|N| \geq 0$ but for $|N| > 0$, the vector potential behaves as $A_i(r) \sim -\partial_i\alpha \mp 2(N-1)\epsilon_{ij}\frac{x^j}{r^2}$. Therefore, we can avoid singularities in the vector potential at the origin if we choose the phase of ϕ to be $\alpha = \pm\theta(N-1)$. Then, the self-dual ϕ field is

$$\phi = \frac{\sqrt{2}2N}{er_0}\left(\frac{(\frac{r}{r_0})^{N-1}}{1 + (\frac{r}{r_0})^{2N}}\right)e^{\pm i(N-1)\theta}. \qquad (23)$$

Requiring that ϕ be single-valued we find that N must be an integer, and for ρ to decay at infinity we require that N be positive.

This procedure may be used to solve eqs. (15). Indeed, we can define a new field $\psi = \phi^{n+1}$. Thus, it is not difficult to show that eqs. (15) may be rewritten, in terms of ψ, as

$$\begin{aligned}(D_1' \pm iD_2')\psi &= 0, \\ B &= \pm\frac{e}{2}(\psi^\dagger\psi),\end{aligned} \qquad (24)$$

where $D_i' = \frac{1}{n+1}\partial_i + ieA_i$. So, the set of equations (24) is basically the same as (16), and, therefore, it has the same solution, *i.e.*,

$$\psi = \frac{\sqrt{2}2N}{\sqrt{e^2(n+1)}r_0}\left(\frac{(\frac{r}{r_0})^{N-1}}{1 + (\frac{r}{r_0})^{2N}}\right)e^{\pm i(N-1)\theta}. \qquad (25)$$

Notice that in the denominator of (25) there appears the factor $(n+1)$. We can rewrite (25) in a more compact form,

$$\psi = g(r)e^{\pm i(N-1)\theta}. \qquad (26)$$

Then, the field ϕ is

$$\phi = (g(r))^{\frac{1}{n+1}}e^{\pm im\theta} \qquad (27)$$

being $m = \frac{(N-1)}{n+1}$. Since, ϕ must be single-valued we require that $\frac{(N-1)}{n+1}$ be an integer. It is interesting to note that our solutions are electrically neutral, *i.e.* $j_0 = 0$ for our solutions. This constitute a difference with the Jackiw-Pi vortices, since it is well known that these vortices are electrically charged. Thus, in the particular case $n = 0$, we have that our solutions are mathematically identical to the Jackiw-Pi vortices, although from the physical point of view our vortices do not have electric charge.

Another interesting point refers to the possible self-dual equations obtained from the model described by the action (1). Indeed, we can obtain another set of self-dual equations if the self-dual potential is chosen to be

$$V(\rho) = \frac{e^2}{8}(1 - \rho^{(n+1)})^2. \qquad (28)$$

In this case it is not difficult to see that the energy (11) may be rewritten as

$$\begin{aligned}E = \int \mathrm{d}^2x\Bigg(&\frac{1}{2}[B \pm \frac{e}{2}(1 - \rho^{(n+1)})]^2 \mp \frac{e}{2}B \\ &+ \frac{(n+1)}{2}\rho^n|(D_1 \pm iD_2)\phi|^2\Bigg).\end{aligned} \qquad (29)$$

Then, we see that the energy is bounded below by a multiple of the magnitude of the magnetic flux (for positive flux we choose the lower signs, and for negative flux we choose the upper signs):

$$E \geq \frac{e}{2}|\Phi|. \qquad (30)$$

In order that the energy be finite the covariant derivative must vanish asymptotically. This fixes the behavior of the gauge field A_i and implies a nonvanishing magnetic flux:

$$\Phi = \int \mathrm{d}^2xB = \oint_{|x|=\infty} A_i\mathrm{d}x^i = 2\pi N, \qquad (31)$$

where N is a topological invariant which takes only integer values. The bound is saturated by fields satisfying the first-order self-duality equations:

$$\begin{aligned}\phi^n(D_1 \pm iD_2)\phi &= 0, \\ B &= \mp\frac{e}{2}(1 - \rho^{(n+1)}).\end{aligned} \qquad (32)$$

This constituted a new set of self-duality equations for the action (1) of the paper. It is interesting to note that in the case $n = 0$, eqs. (32) become the well-known equations of the Abelian Higgs model [4]. This is an expected result since in that case eq. (5) is

$$\omega(\rho) = 1 \qquad (33)$$

and the potential term (28) becomes the usual symmetry breaking potential of the Higgs model. Therefore, model (1) results the Higgs model.

Finally, it is worth mentioning that due to the potential term (12), the soliton solutions of eqs. (15) are nontopological. On the other hand, solutions of (32) are topological due to the form of the potential (28).

Nielsen-Olesen vortices from a generalized Jackiw-Pi model. – Suppose that instead of action (1) we had a generalization of the Jackiw-Pi model,

$$S = \int d^3x \left(\frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right.$$
$$\left. + i\omega(\rho)\phi^* D_0\phi - \frac{1}{2m}|D_i\phi|^2 - V(\rho) \right). \quad (34)$$

As in model (1), $\omega(\rho)$ is a positive-definite dimensionless function of the complex scalar field ϕ.

The equations of motion for this system are given by

$$i\left(\frac{\partial\omega(\rho)}{\partial\phi^*}\phi^* D_0\phi + \omega(\rho)D_0\phi \right) = -\frac{1}{2m}D_i^2\phi + \frac{\partial V(\rho)}{\partial\phi^*},$$
$$B = \frac{e}{\kappa}\omega(\rho)\rho, \quad (35)$$
$$E^i = -\frac{1}{\kappa}\epsilon^{ij}j_i,$$

where the second equation is the Gauss law.

The theory may be described in terms of the Hamiltonian formulation as

$$H = \int d^2x \left(\frac{1}{2m}|D_i\phi|^2 + V(\rho) \right) \quad (36)$$

which may be rewritten using the Gauss law and the identity (6) in the form

$$E = \int d^2x \left(\frac{1}{2m}|D_\pm\phi|^2 \mp \frac{e^2}{2m\kappa}\omega(\rho)\rho^2 + V(\rho) \right). \quad (37)$$

Here, it will be interesting to consider the following $\omega(\rho)$ function:

$$\omega(\rho) = \pm\kappa(\rho-1)\rho^{-1}. \quad (38)$$

Thus, we have two possible cases:

$$\omega(\rho) = \kappa(\rho-1)\rho^{-1}, \qquad \omega(\rho) = -\kappa(\rho-1)\rho^{-1}. \quad (39)$$

The first case corresponds to the upper signs in the energy (37), the second to the lower signs.

Then, we can introduce (38) into (37) to obtain

$$E = \int d^2x \left(\frac{1}{2m}|D_\pm\phi|^2 \mp \frac{e^2}{2m}(\rho-1)\rho + V(\rho) \right). \quad (40)$$

For $\omega(\rho)$ taking the form of the formula (38), the Gauss law of eq. (35) becomes

$$B = \frac{e}{\kappa}\omega(\rho)\rho = \pm e(\rho-1). \quad (41)$$

Now, suppose that we choose a symmetry breaking ϕ^4 potential as is usual in Maxwell-Higgs model,

$$V(\rho) = \frac{e^2}{2m}(\rho-1)^2. \quad (42)$$

Then, the energy (40) becomes

$$E = \int d^2x \left(\frac{1}{2m}|D_\pm\phi|^2 \mp \frac{e}{2m}B \right), \quad (43)$$

Here, the situation is very similar to the case of Nielsen-Olesen and Chern-Simons-Higgs vortices, in the sense that these models are bounded below by a multiple of magnitude of the magnetic flux (for positive flux we choose the lower signs, and for negative flux we choose the upper signs):

$$E \geq \frac{e}{2m}|\Phi|. \quad (44)$$

Then, the topological bound is saturated by fields satisfying the first-order self-duality equations,

$$D_\pm\phi = (D_1 \pm iD_2)\phi = 0, \quad (45)$$
$$B = \pm e(\rho-1). \quad (46)$$

Here, we have a couple of equations which are identical to the self-duality equations of the Abelian Maxwell-Higgs model [8]. Nevertheless, our vortices are physically different from the Nielsen-Olesen solutions [4]. Indeed, our vortices not only carry magnetic flux, but also $U(1)$ charge. From the Gauss law (41), we know that $B = \frac{e}{\kappa}\omega(\rho)\rho$ and from the Noether theorem we have that the conserved charge associated to $U(1)$ transformation

$$\delta\phi = -ie\phi \quad (47)$$

is

$$Q_{U(1)} = \int d^2x j^0 = \int d^2x \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\delta\phi = e\int d^2x\omega(\rho)\rho. \quad (48)$$

Thus, the magnetic flux is proportional to the $U(1)$ charge,

$$\Phi = \frac{1}{\kappa}Q_{U(1)}, \quad (49)$$

which is a particularity of the Chern-Simons models [9].

So, starting from a generalization of the Jackiw-Pi model we were able to obtain the self-duality equations of the Maxwell-Higgs model. However, our vortices are different from the Nielsen-Olesen vortices. The difference lies in the fact that, here, our vortices not only carry magnetic flux, as in the Higgs model, but also $U(1)$ charge. To conclude, we note that, as is well known, the solution of the self-duality equations of the Maxwell-Higgs model, are topological vortices.

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