EXTRAPOLATION FOR CLASSES OF WEIGHTS RELATED TO A FAMILY OF OPERATORS AND APPLICATIONS

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Abstract. In this work we give extrapolation results on weighted Lebesgue spaces for weights associated to a family of operators. The starting point for the extrapolation can be the knowledge of boundedness on a particular Lebesgue space as well as the boundedness of the a extremal space similar to BMO. This analysis can be applied to a variety of operators appearing in the context of a Schrödinger operator $(-\Delta + V)$ where V satisfies a reverse Hölder inequality. In that case the weights involved are a localized version of Muckenhoupt weights.

1. Introduction

Rubio de Francia's extrapolation result asserts that given a sublinear operator T, the knowledge of boundedness on a particular $L^{p_0}(w)$, with $1 < p_0 < \infty$, for every weight w in the Muckenoupt class A_{p_0}, it is enough to infer the boundedness of T on every $L^p(w)$, with $1 < p < \infty$ and w in A_p. This results appears by the first time in his celebrated work in [17]. From there many authors has extended and generalized that result (see [18], [16], [10] among others).

Recently, in [6] a simplified proof allow extensions to some weighted Banach function spaces, where vector valued inequalities appear naturally. The key tool is again Rubio de Francia's algorithm, based on the connection between the Hardy-Littlewood maximal function and Muckenhoupt weights.

In this work we deal with the extrapolation property for classes of weights that arise from the L^p boundedness of a one-parameter family of maximal operators, rather than a single operator like in the case of Muckenhoupt classes A_p. Our approach models the situation of weights appearing in [4], in the context of the analysis related to the Schrödinger operator $(-\Delta+V)$. In that case, as we shall see in Section 3 such classes of weights correspond to those w for which some member of a certain family of maximal operators $\{M^\theta\}_{\theta>0}$ is bounded on $L^p(w)$.

The structure of the paper is as follows. We start in Section 2 giving extrapolation results in a general framework of weights governed by a family of operators. The proofs are based on the techniques developed in [6], and include extrapolation from a fixed L^p with a finite p as well as extrapolation from the extreme L^∞ in the spirit of [12].

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In Section 3 we deal with the classes of weights appearing in the aforementioned work [4] and we check that those weights are in the hypothesis of the general theorems developed in Section 2.

In order to apply the extrapolation results of Section 2 from the extreme L^∞, we need to define a special sharp maximal function, since most of the interesting operators related to Schrödinger analysis are not bounded on L^∞. To this end, in Section 4, we prove a Fefferman-Stein type inequality that takes into account the structure of the appropriate version of BMO space introduced in [7] and [3].

Finally, in Section 5 we obtain weighted inequalities in the extreme L^∞ for operators that satisfy certain size and smooth conditions, as to include Riesz transform type operators of first and second order as well as imaginary powers of the Schrödinger operator, among others. In this way, we are able to apply the extrapolation results to obtain scalar and vector valued inequalities for such operators. We end with a similar analysis for the fractional integrals associated to the Schrödinger operator.

Let us remark that some of the weighted inequalities that we obtain from extrapolation were already known. Nevertheless, we believe that this method provides a simpler proof and besides it makes no use of the openness property of the classes of weights.

Let us now introduce some notation that will be used throughout this article.

Given a weight w we denote, as usual, $w(B) = \int_B w$.

For $0 < p < \infty$, by $L^p(w)$ we mean the set of measurable functions f of $\mathbb{R}^d$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^d} f(x)^p w(x) dx \right)^{1/p} < \infty.$$

When $w = 1$ we use the notation $L^p(w) = L^p$.

Also, $L^\infty(w)$ is defined as the set of measurable functions f of $\mathbb{R}^d$ such that

$$\|fw\|_{L^\infty} < \infty,$$

where $\|fw\|_{L^\infty}$ denotes the usual essential supremum of fw over $\mathbb{R}^d$.

2. A general setting of extrapolation

In this section we state a general theorem on extrapolation for boundedness on weighted Lebesgue spaces with weights associated to a family of sublinear operators. For a weight we mean a non-negative locally integrable function defined on $\mathbb{R}^d$.

Suppose we have a family of positive and sublinear operators $\{T_\theta\}_{\theta \in I}$, where I is a certain set of indexes, such that every T_θ is a bounded operator on $L^p(\mathbb{R}^d)$ for every $1 < p \le \infty$. Associated to a fixed $\theta \in I$ and $1 < p < \infty$, we define the family U_p^θ of those weights $w \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that T_θ maps $L^p(w)$ into itself, and we denote $[w]_{p,\theta} = \|T_\theta\|_{L^p(w)}$ the usual operator norm. In the case $p = 1$, U_1^θ is the family of weights $w \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that for some constant C, $T_\theta w \le Cw$, and $[w]_{1,\theta}$ is defined as the infimum of those C satisfying the inequality. We also call $U_p = \bigcup_{\theta \in I} U_p^\theta$ and $U_\infty = \bigcup_{p \ge 1} U_p$.

We shall further assume that those families satisfy the following basics properties resembling Muckenhoupt weights:

u1) $U_p \subset U_q$ when $1 \le p \le q$.

u2) If $w \in U_p^\theta$, for some $p > 1$ and $\theta \in I$, then there exists $\theta' = \theta'(p,\theta)$, such that $w^{1-p'} \in U_{p'}^{\theta'}$ and the constant $[w^{1-p'}]_{p',\theta'}$ depends on w only through $[w]_{p,\theta}$.

u3) If $w_1 \in U_1^{\theta_1}$ and $w_2 \in U_1^{\theta_2}$ for some $\theta_1, \theta_2 \in I$, then for every $p \geq 1$ there exists $\theta = \theta(p, \theta_1, \theta_2)$ such that $w_1 w_2^{1-p} \in U_p^\theta$ and the constant $[w_1 w_2^{1-p}]_{p,\theta}$ depends on w_1 and w_2 only through $[w_1]_{1,\theta_1}$ and $[w_2]_{1,\theta_2}$.

Remark 1. From u2) it follows that $w \in U_p$ if and only if $w^{1-p'} \in U_{p'}$. Also, property u3) says that if $w_1, w_2 \in U_1$, then $w_1 w_2^{1-p} \in U_p$.

Following [6] the results on extrapolation of this section will be stated in terms of pair of fuctions (f,g) that belong to $\mathscr{F}$, a family of pairs of measurable and non-negative functions. Through this work we shall use the symbol C to denote a constant that may differ from line to line and always will be independent of the pair $(f,g) \in \mathscr{F}$.

Given $p > 0$ and a weight $w \in U_q^\theta$, $q \geq 1$, $\theta \in I$, the expression

$$\int_{\mathbb{R}^d} f(x)^p w(x) dx \leq C \int_{\mathbb{R}^d} g(x)^p w(x) dx, \qquad (f,g) \in \mathscr{F}, \tag{1}$$

means that the inequality holds for every $(f,g) \in \mathscr{F}$ whenever the left hand side is finite, with a constant C depending on w only through $[w]_{q,\theta}$.

Under the previous setting we present one of the following result.

Theorem 1. *Let $1 \leq p_0 < \infty$ and suppose* (1) *holds with $p = p_0$ for every $w \in U_{p_0}$. Then* (1) *also holds for every p, $1 < p < \infty$, and every $w \in U_p$.*

Proof. The proof of this result follows the lines of [6]. Let $1 < p < \infty$, $\theta \in I$ and $w \in U_p^\theta$. By u2) it follows $w^{1-p'} \in U_{p'}^{\theta'}$ for some $\theta' \in I$ with $[w^{1-p'}]_{p',\theta'}$ depending on w only through $[w]_{p,\theta}$. Given $h_1 \in L^p(w)$ and $h_2 \in L^{p'}(w)$, both non-negative, following Rubio de Francia's algorithm (see [1] and [18]) we define the operators

$$\mathscr{R}h_1(x) = \sum_{k=0}^{\infty} \frac{T_\theta^k h_1(x)}{2^k \|T_\theta\|_{L^p(w)}^k}, \qquad \mathscr{R}'h_2(x) = \sum_{k=0}^{\infty} \frac{(T'_{\theta'})^k h_2(x)}{2^k \|T'_{\theta'}\|_{L^{p'}(w)}^k},$$

where $T'_{\theta'} f = T_{\theta'}(fw)/w$ and T_θ^k, for $k \geq 1$, is the k times composition of the operator T_θ and T_θ^0 is the identity (analogously for $(T'_{\theta'})^k$). The function $\mathscr{R}h_1$ satisfies

$$h_1 \leq \mathscr{R}h_1, \tag{2}$$

$$\|\mathscr{R}h_1\|_{L^p(w)} \leq C \|h_1\|_{L^p(w)}, \tag{3}$$

$$T_\theta(\mathscr{R}h_1) \leq 2\|T_\theta\|_{L^p(w)} \mathscr{R}h_1, \tag{4}$$

and for $\mathscr{R}'h_2$, we have

$$h_2 \leq \mathscr{R}'h_2, \tag{5}$$

$$\|\mathscr{R}'h_2\|_{L^{p'}(w)} \leq C \|h_2\|_{L^{p'}(w)}, \tag{6}$$

$$T_{\theta'}(w\mathscr{R}'h_2) \leq 2\|T'_{\theta'}\|_{L^{p'}(w)} w\mathscr{R}'h_2. \tag{7}$$

Now we fix $(f,g)\in\mathscr{F}$. We may assume, that f and g are non-zero and both are in $L^p(w)$ to consider

$$h_1=\frac{f}{\|f\|_{L^p(w)}}+\frac{g}{\|g\|_{L^p(w)}}.$$

Clearly $h_1\in L^p(w)$ and $\|h_1\|_{L^p(w)}\le 2$. Since $f\in L^p(w)$, by duality, there exists $h_2\in L^{p'}(w)$, $\|h_2\|_{L^{p'}(w)}=1$, such that

$$\|f\|_{L^p(w)}=\int_{\mathbb{R}^d} f(x)h_2(x)w(x)dx.$$

If we call $w_1=\mathscr{R}h_1$ and $w_2=w\mathscr{R}'h_2$, then from (5) and Hölder's inequality with respect to the measure w_2 it follows

$$\begin{aligned}\|f\|_{L^p(w)}&\le\int_{\mathbb{R}^d} f(x)w_1(x)^{-1/p_0'}w_1(x)^{1/p_0'}w_2(x)dx\\&\le\left(\int_{\mathbb{R}^d} f(x)^{p_0}w_1(x)^{1-p_0}w_2(x)dx\right)^{1/p_0}\left(\int_{\mathbb{R}^d} w_1(x)w_2(x)dx\right)^{1/p_0'}\\&=I\times II.\end{aligned}$$

We first estimate II. By Hölder's inequality with respect to the measure w and properties (3) and (6), we have

$$II\le\|\mathscr{R}h_1\|_{L^p(w)}^{1/p_0'}\|\mathscr{R}'h_2\|_{L^{p'}(w)}^{1/p_0'}\le 4^{1/p_0'}\|h_1\|_{L^p(w)}^{1/p_0'}\|h_2\|_{L^{p'}(w)}^{1/p_0'}\le 8^{1/p_0'}.$$

To estimate I we will apply the hypothesis with the weight $w_3=w_1^{1-p_0}w_2$. In fact, since $\|T'_{\theta'}\|_{L^{p'}(w)}\le[w]_{p,\theta}$, inequalities (4) and (7) say that the weights w_1 and w_2 belong to U_1^{θ} and $U_1^{\theta'}$ respectively with constants depending on w only through $[w]_{p,\theta}$. Therefore, from property u3), there exists some $\sigma=\sigma(p_0,\theta)$, such that $w_3\in U_{p_0}^{\sigma}$ with $[w_3]_{p_0,\sigma}$ depending on w only through $[w]_{p,\theta}$. In order to use the hypothesis we have to check $I<\infty$. From (2), we have $f\le\|f\|_{L^p(w)}w_1$, then

$$I\le\|f\|_{L^p(w)}\,II^{p_0'/p_0}\le 8^{1/p_0}\|f\|_{L^p(w)}<\infty.$$

Therefore applying (1) with $p=p_0$ and considering $g\le\|g\|_{L^p(w)}w_1$, which follows from (2), we obtain

$$\begin{aligned}I&\le C\left(\int_{\mathbb{R}^d} g(x)^{p_0}w_3(x)dx\right)^{1/p_0}\\&\le C\|g\|_{L^p(w)}\left(\int_{\mathbb{R}^d} w_1(x)w_2(x)dx\right)^{1/p_0}\le C8^{1/p_0}\|g\|_{L^p(w)},\end{aligned}$$

and the proof is finished for the case $p>1$.

The case $p=1$ follows easily considering $1/p_0'=0$.

□

A consequence of Theorem 1 is the following vector valued inequalities.

Corollary 1. *Let $1\le p_0<\infty$ and assume that (1) holds with $p=p_0$, for every $w\in U_{p_0}$. Then*

$$\left\|\left(\sum_i f_i^q\right)^{1/q}\right\|_{L^p(w)}\le C\left\|\left(\sum_i g_i^q\right)^{1/q}\right\|_{L^p(w)},\quad\{(f_i,g_i)\}_i\subset\mathscr{F},\tag{8}$$

holds for every p and q, $1<p,q<\infty$, and every $w\in U_p$.

Proof. Let $1 < p, q < \infty$ and consider the family $\mathscr{F}_q$ of pairs (F, G), where

$$F(x) = \left(\sum_i f_i(x)^q\right)^{1/q}, \qquad G(x) = \left(\sum_i g_i(x)^q\right)^{1/q},$$

with $\{(f_i, g_i)\}_i \subset \mathscr{F}$. Using Theorem 1 we have (1) with $p = q$, then for every $w \in U_q^\theta$, $\theta \in I$, there exists a constant C (depending on w only through $[w]_{q,\theta}$) such that

$$\|F\|_{L^q(w)}^q = \sum_i \int_{\mathbb{R}^d} f_i(x)^q w(x)dx \le C \sum_i \int_{\mathbb{R}^d} g_i(x)^q w(x)dx = C\|G\|_{L^q(w)}^q,$$

for all $(F, G) \in \mathscr{F}_q$. Now, we are again in the hypothesis of Theorem 1 with $p_0 = q$ for the family $\mathscr{F}_q$. Then we obtain

$$\left\|\left(\sum_i f_i^q\right)^{1/q}\right\|_{L^p(w)} = \|F\|_{L^p(w)} \le C\|G\|_{L^p(w)} = C\left\|\left(\sum_i g_i^q\right)^{1/q}\right\|_{L^p(w)}$$

for every $w \in U_p$ with the desired dependence of C.

□

The following corollary provides a norm inequality in $L^p(w)$, for $w \in U_\infty$ and $p > 0$.

Corollary 2. *Let $0 < p_0 < \infty$ and assume that (1) holds with $p = p_0$, for every $w \in U_\infty$. Then (1) holds for $0 < p < \infty$, and every $w \in U_\infty$.*

Proof. Fix r, $1 < r < \infty$, and consider the family $\mathscr{F}_0$, of those pairs $(f^{p_0/r}, g^{p_0/r})$ such that $(f, g) \in \mathscr{F}$. By the hypothesis, if $w \in U_r^\theta$ for some $\theta \in I$, then

$$\begin{aligned}\int_{\mathbb{R}^d} (f(x)^{p_0/r})^r w(x)dx &= \int_{\mathbb{R}^d} f(x)^{p_0} w(x)dx \le C \int_{\mathbb{R}^d} g(x)^{p_0} w(x)dx \\ &= C \int_{\mathbb{R}^d} (g(x)^{p_0/r})^r w(x)dx,\end{aligned}$$

for every $(f, g) \in \mathscr{F}$, with C depending on w only through $[w]_{r,\theta}$.

Therefore, we have proved inequality (1) with $p = r$ for the family $\mathscr{F}_0$ and weights in U_r. In this way, applying Theorem 1, it follows

$$\int_{\mathbb{R}^d} (f(x)^{p_0/r})^q w(x)dx \le C \int_{\mathbb{R}^d} (g(x)^{p_0/r})^q w(x)dx, \qquad (f^{p_0/r}, g^{p_0/r}) \in \mathscr{F}_0,$$

for every $1 < q < \infty$ and every $w \in U_q$.

Now, let $p > 0$ and $w \in U_\infty$. From from property u1), there exists $q > p/p_0$ such that $w \in U_q$. Taking $r = p_0 q/p > 1$, then $p = p_0 q/r$, and we obtain the desiered inequality.

□

An other a consequence of Theorem 1 we obtain as in [6] the following rescaled extrapolation type result.

Corollary 3. *Let $0 < r < p_0 < \infty$ and assume that (1) holds for $p = p_0$, and every $w \in U_{p_0/r}$. Then (1) and (8) holds for $r < p, q < \infty$, and every $w \in U_{p/r}$.*

Proof. We start denoting the family $\mathscr{F}_r$ as those pairs (f^r, g^r), with $(f,g) \in \mathscr{F}$. By the hypothesis, it is easy to see that (1) holds with exponent $p = p_0/r$ for the family $\mathscr{F}_r$. Now from Theorem 1 it follows

$$\int_{\mathbb{R}^d} f(x)^{rq} w(x)dx \le C \int_{\mathbb{R}^d} g(x)^{rq} w(x)dx; \qquad (f^r, g^r) \in \mathscr{F}_r,$$

for every $q > 1$ and every $w \in U_q$, with the constant C depending on w only through $[w]_{q,\theta}$ for some $\theta \in I$. Hence, given any $p > r$ and taking $q = p/r$, the resulst follows from the previous inequality. The vector valued case follows similarly using the result of Corollary 1.

□

In the next results the approach is to obtain inequalities as (1) without asking the finiteness of the left hand side. In this situation we will say that (1) holds *unrestricted*. This type of results are more in the spirit of Rubio de Francia's original results, that are useful to obtain boundedness of operators on weighted Lebesgue spaces.

Theorem 2. *Let $1 \le p_0 < \infty$ and assume that* (1) *holds unrestrited with $p = p_0$, for every $w \in U_{p_0}$. Then* (1) *holds unrestrited for every p, $1 < p < \infty$ and every $w \in U_p$.*

Proof. We shall apply Theorem 1 for the families $\mathscr{F}_n = \{(f_n, g) : \ (f,g) \in \mathscr{F}\}$, where $f_n(x) = \chi_{B(0,n)} \min\{f(x), n\}$, for every $n \in \mathbb{N}$. Since $f_n \le f$, it follows

$$\int_{\mathbb{R}^d} f_n(x)^{p_0} w(x)dx \le C \int_{\mathbb{R}^d} g(x)^{p_0} w(x)dx, \qquad (f,g) \in \mathscr{F},$$

for every $w \in U_{p_0}$ with the constant C independent of n and depending on w only through $[w]_{p_0,\theta}$ whenever $w \in U_{p_0}^\theta$, for some $\theta \in I$. Since the left hand side is finite, we may apply Theorem 1 to each family $\mathscr{F}_n$. Hence, for each $n \in \mathbb{N}$, if $0 < p < \infty$ and $w \in U_p$ it holds

$$\int_{\mathbb{R}^d} f_n(x)^p w(x)dx \le C \int_{\mathbb{R}^d} g(x)^p w(x)dx, \qquad (f,g) \in \mathscr{F}.$$

Now, if $(f,g) \in \mathscr{F}$, by the Monotone Convergence Theorem since $f_n \nearrow f$, we obtain

$$\int_{\mathbb{R}^d} f(x)^p w(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)^p w(x)dx \le C \int_{\mathbb{R}^d} g(x)^p w(x)dx.$$

□

Remark 2. Clearly as consequence of this result it followsshow (with the same proofs as before) the corresponding Corollary 1, Corollary 3 and Corollary 2 whitout the hypothesis of finiteness of the left hand side.

Now we will prove an extrapolation theorem similar to Theorem 1 starting from an inequality at the end point $p_0 = \infty$. As before, given a weight $w \in U_p^\theta$, $p \ge 1$, $\theta \in I$, the expression

$$\|fw\|_{L^\infty} \le C\|gw\|_{L^\infty}, \qquad (f,g) \in \mathscr{F}, \tag{9}$$

should be understood in the sense that the inequality holds for every $(f,g) \in \mathscr{F}$ whenever the left hand side is finite, with a constant C depending on w only through $[w]_{p,\theta}$.

Before presenting the next theorem we state the following technical lemma which proof can be found in [12] (see the corollary after Lemma 4).

Lema 1. *Let $1 < p < \infty$ and f in $L^p(w)$. There exists a positive function F in $L^p(w^{-1/(p-1)})$ such that*

$$\left(\int_{\mathbb{R}^d} F^p w^{-1/(p-1)}\right)^{1/p} \le 2$$

and

$$\left(\int_{\mathbb{R}^d} |f|^p w\right)^{1/p} = \|f w^{1/(p-1)} F^{-1}\|_{L^\infty}.$$

Theorem 3. *If (9) holds for every w satisfying $w^{-1} \in U_1$, then (1) holds for every p, $1 < p < \infty$, and every $w \in U_p$.*

Proof. Let $w \in U_p$ and $(f, g) \in \mathscr{F}$. We may suppose, without loss of generality that f and g belongs to $L^p(w)$. From Lemma 1 there exists non-negative functions F and G in $L^p(w^{-1/(p-1)})$ such that

$$\|F\|_{L^p(w^{-1/(p-1)})} \le 2, \tag{10}$$

$$\|f\|_{L^p(w)} = \|f w^{1/(p-1)} F^{-1}\|_{L^\infty}, \tag{11}$$

$$\|G\|_{L^p(w^{-1/(p-1)})} \le 2, \tag{12}$$

and

$$\|g\|_{L^p(w)} = \|g w^{1/(p-1)} G^{-1}\|_{L^\infty}. \tag{13}$$

Since $w \in U_p$ there exists $\theta \ge 0$ such that $\widetilde{T}_\theta$ maps $L^p(w^{-1/(p-1)})$ into itself, where $\widetilde{T}_\theta f = T(f w^{-1/(p-1)})/w^{-1/(p-1)}$. Now we follow Rubio de Francia's algorithm applied to $h = F + G$ defining

$$\mathscr{R}h(x) = \sum_{k=0}^{\infty} \frac{\widetilde{T}_\theta^k h(x)}{2^k \|\widetilde{T}_\theta\|^k_{L^p(w^{-1/(p-1)})}}.$$

From the definition of $\mathscr{R}$ it follows,

$$h \le \mathscr{R}h, \tag{14}$$

$$\|\mathscr{R}h\|_{L^p(w^{-1/(p-1)})} \le C\|h\|_{L^p(w^{-1/(p-1)})}, \tag{15}$$

and

$$\widetilde{T}_\theta(\mathscr{R}h) \le 2\|\widetilde{T}_\theta\|_{L^p(w^{-1/(p-1)})}\mathscr{R}h. \tag{16}$$

The last inequality implies $w^{-1/(p-1)}\,\mathscr{R}h \in U_1$ with

$$[w^{-1/(p-1)}]_{1,\theta} \le 2\|\widetilde{T}_\theta\|_{L^p(w^{-1/(p-1)})}$$

(similarly, as in Theorem 1, the quantity $\|\widetilde{T}_\theta\|_{L^p(w^{-1/(p-1)})}$ depends on w only through $[w]_{p,\theta}$).

Therefore, from (13), (14) and (9) we obtain

$$\begin{aligned} \|g\|_{L^p(w)} &= \|g w^{1/(p-1)} G^{-1}\|_{L^\infty} \ge \|g w^{1/(p-1)} (\mathscr{R}h)^{-1}\|_{L^\infty} \\ &\ge C\|f w^{1/(p-1)} (\mathscr{R}h)^{-1}\|_{L^\infty}, \end{aligned} \tag{17}$$

whenever $\|fw^{1/(p-1)}(\mathscr{R}h)^{-1}\|_{L^\infty} < \infty$, in fact by (14) and (11), we have

$$\|fw^{1/(p-1)}(\mathscr{R}h)^{-1}\|_{L^\infty} \leq \|fw^{1/(p-1)}F^{-1}\|_{L^\infty} = \|f\|_{L^p(w)} < \infty.$$

Finally,

$$\|f\|^p_{L^p(w)} \;\leq\; \|fw^{1/(p-1)}(\mathscr{R}h)^{-1}\|^p_{L^\infty}\|\mathscr{R}h\|^p_{L^p(w^{-1/(p-1)})} \leq C\|g\|^p_{L^p(w)},$$

where in the last inequality we have used (17), (15), (10) and (12).

□

Corollary 4. *Let $r > 0$ and suppose* (9) *holds for every w such that $w^{-r} \in U_1$. Then* (1) *holds for every $p > r$ and every $w \in U_{p/r}$.*

Proof. We start considering the family $\mathscr{F}_r$ of pairs (f^r, g^r) with $(f, g) \in \mathscr{F}$. Let w be such that $w^{-1} \in U_1$ and $(f, g) \in \mathscr{F}$. Using the hypothesis with $w^{1/r}$,

$$\|f^r w\|^{1/r}_{L^\infty} = \|fw^{1/r}\|_{L^\infty} \leq C\|gw^{1/r}\|_{L^\infty} = C\|g^r w\|^{1/r}_{L^\infty}.$$

Therefore, we have proved (9) for the family $\mathscr{F}_r$. In this way, applying Theorem 3, we have

$$\int_{\mathbb{R}^d} f(x)^{rq}w(x)dx \leq C\int_{\mathbb{R}^d} g(x)^{rq}w(x)dx, \qquad (f^r, g^r) \in \mathscr{F}_r,$$

for every $q > 1$ and every $w \in U_q$. Finally, given $p > r$ the result follows considering $q = p/r$.

□

Remark 3. Teorema 3 (and its corollary) can be proved without asking the finiteness of the left hand side in (9). To this end we may slightly modify the proofs taking $h = G$, since it is not necessary to verify $\|fw^{1/(p-1)}(\mathscr{R}h)^{-1}\|_{L^\infty} < \infty$.

In order to state our next results we introduce the following classes of weights. Given $1 \leq p, q < \infty$ we define,

$$U_{p,q} = \{w \in L^1_{\text{loc}}(\mathbb{R}^d) : w^{-p'} \in U_{1+p'/q}\}$$

and

$$U_{p,\infty} = \{w \in L^1_{\text{loc}}(\mathbb{R}^d) : w^{-p'} \in U_1\}.$$

Theorem 4. *Let $1 < s < \infty$. Suppose the expression*

$$\|fw\|_{L^\infty} \leq C\|g\|_{L^s(w^s)}, \qquad (f, g) \in \mathscr{F},$$

holds for every $w \in U_{s,\infty}$, whenever the left hand side is finite and where the constant C depends on w only throught $[w^{-s'}]_{1,\theta}$ for every θ such that $w^{-s'}$ belongs to U_1^θ. Then

$$\|f\|_{L^q(w^q)} \leq C\|g\|_{L^p(w^p)}, \qquad (f, g) \in \mathscr{F}, \tag{18}$$

holds for every p and q such that $1 < p < s$, $1/p - 1/q = 1/s$, and every $w \in U_{p,q}$, whenever the left hand side is finite. Moreover, the constant C in (18) *depends on w only throught $[w^{-p'}]^\theta_{1+p'/q}$ whenever $w^{-p'}$ belongs to $U^\theta_{1+p'/q}$, with $\theta \in I$.*

Proof. Let $1 < p < s$, $1/p - 1/q = 1/s$ and $w \in U_{p,q}$. Consider $f \in L^q(w^q)$ and $g \in L^p(w^p)$, and write

$$\|g\|_{L^p(w^p)} = \|(gw^{p'})^s\|^{1/s}_{L^{p/s}(w^{-p'})}.$$

It follows by duality that there exists a non-negative function G such that

$$\|G\|_{L^{\frac{p}{s-p}}(w^{-p'})} = 1 \tag{19}$$

and

$$\|g\|_{L^p(w^p)} = \left(\int_{\mathbb{R}^d} |g(x)w(x)^{p'}|^s G(x)^{-1} w(x)^{-p'} dx\right)^{1/s}. \tag{20}$$

On the other hand, from Lemma 1, there exists a non-negative function $F \in L^q(w^{-q'})$ such that

$$\|F\|_{L^q(w^{-q'})} \leq 2, \tag{21}$$

$$\|f\|_{L^q(w^q)} = \|f w^{q'} F^{-1}\|_{L^\infty}. \tag{22}$$

By the definition of the class $U_{p,q}$ we have $v = w^{-p'} \in U_{1+\frac{p'}{q}}$. If we denote $r = 1 + \frac{p'}{q}$, so $r' = q/s'$, from (19) and (21) it follows $\|G^{s'/s}\|_{L^{r'}(v)} = 1$ and $\|F^{s'} w^{p'-q's'}\|_{L^{r'}(v)} < 2^{s'}$, respectively.

Since $v \in U_r^\theta$, for some $\theta \in I$, there exists $\sigma = \sigma(r,\theta) \in I$ such that the operator $T'_\sigma f = T_\sigma(fv)/v$ maps $L^{r'}(v)$ into itself with

$$\|T'_\sigma\|_{L^{p'}(v)} \leq [v]_r^\theta. \tag{23}$$

We now proceed following Rubio de Francia's algorithm with $h = G^{s'/s} + F^{s'} w^{p'-q's'}$ and

$$\mathscr{R}' h(x) = \sum_{k=0}^{\infty} \frac{(T'_\sigma)^k h(x)}{2^k \|T'_\sigma\|^k_{L^{r'}(v)}}.$$

Thus, we have

$$h \leq \mathscr{R}' h, \tag{24}$$

$$\|\mathscr{R}' h\|_{L^{r'}(v)} \leq C\|h\|_{L^{r'}(v)}, \tag{25}$$

and

$$T_\sigma(\mathscr{R}' h v) \leq 2\|T'_\sigma\|_{L^{p'}(v)} \mathscr{R}' h v.$$

Last inequality and (23) asserts that the weight $(\mathscr{R}' h)w^{-p'}$ belongs to U_1 with

$$[(\mathscr{R}' h)w^{-p'}]_1^\sigma \leq 2[v]_r^\theta.$$

Thus, by the definition of $U_{s,\infty}$, the weight $u = (\mathscr{R}' h)^{-1/s'} w^{p'/s'}$ belongs to $U_{s,\infty}$ with constant depending on $w^{-p'}$ only through $[w^{-p'}]_r^\theta$.

Now, coming back to (20) and using (24), we obtain

$$\begin{aligned}
\|g\|_{L^p(w^p)} &= \left(\int_{\mathbb{R}^d} |g(x)|^s G(x)^{-1} w(x)^{p'(s-1)} dx\right)^{1/s} \\
&\geq \left(\int_{\mathbb{R}^d} |g(x)|^s u(x)^s dx\right)^{1/s} \\
&\geq C\|fu\|_{L^\infty},
\end{aligned}$$

where in the last inequality we have used the hypothesis with u under the assumption

$$\|fu\|_{L^\infty} < \infty. \tag{26}$$

In fact, from $F^{s'}w^{p'-q's'} \leq \mathscr{R}'h$, it follows

$$\begin{aligned}\|f(\mathscr{R}'h)^{-1/s'}w^{p'/s'}\|_{L^\infty} &= \|f[(\mathscr{R}'h)^{1/s'}w^{-p'/s'+q'}]^{-1}w^{q'}\|_{L^\infty}\\ &\leq \|fF^{-1}w^{q'}\|_{L^\infty} = \|f\|_{L^q(w^q)} < \infty.\end{aligned}$$

From (25) and $q = -p' + qp'/s'$, we have

$$\begin{aligned}\|f\|_{L^q(w^q)} &\leq \left(\int_{\mathbb{R}^d} \mathscr{R}'h(x)^{r'}v(x)dx\right)^{1/q} \|f(\mathscr{R}'h)^{-1/s'}w^{p'/s'}\|_{L^\infty}\\ &\leq C\|g\|_{L^p(w^p)}.\end{aligned}$$

□

3. Weighs associated to a critical radius function

In this section we deal with classes of weights that recently arised in conection to Schrödinger operators (see [4]). That classes will fit into the previous general context to obtain some applications. We call a *critical radius function* to any positive continuous function ρ with the property that there exist constants c_ρ and $N_0 \geq 1$ such that

$$c_\rho^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq c_\rho\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}, \tag{27}$$

for every $x, y \in \mathbb{R}^d$. Also, a ball $B(x,r) \subset \mathbb{R}^d$ will be called *critical* if $r = \rho(x)$.

Inequality (27) implies that if $\sigma > 0$ and $x, y \in \sigma B$, for some critical ball B, then

$$\rho(x) \leq C_\sigma \rho(y), \tag{28}$$

where $C_\sigma = c_\rho^2(1+\sigma)^{\frac{2N_0+1}{N_0+1}}$, and c_ρ is the constant appearing in (27).

In [8] the authors obtain that (27) gives the following decomposition of $\mathbb{R}^d$.

Proposition 1 (See [8])**.** *There exists a sequence of points x_j, $j \geq 1$, in $\mathbb{R}^d$, so that the family $Q_j = B(x_j, \rho(x_j))$, $j \geq 1$, satisfies*

i) $\cup_j Q_j = \mathbb{R}^d$.

ii) For every $\sigma \geq 1$ there exist constants C and N_1 such that, $\sum_j \chi_{\sigma Q_j} \leq C\sigma^{N_1}$.

Following [4] we present some classes of weights associated to a critical radius function ρ. Given $p > 1$ and $\theta \geq 0$ the class $A_p^{\rho,\theta}$ is defined as the set of weights w such that

$$\left(\int_B w\right)^{1/p}\left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \leq C|B|\left(1+\frac{r}{\rho(x)}\right)^{\theta}, \tag{29}$$

for every ball $B = B(x,r)$. The infimum of the constants in (29) will be denoted by $(w)_{p,\theta}$.

For the case $p = 1$, and $\theta \geq 0$ the class $A_1^{\rho,\theta}$ is defined as those weights w satisfying

$$\frac{1}{|B|}\int_B w \leq C\left(1+\frac{r}{\rho(x)}\right)^{\theta} \inf_B w, \tag{30}$$

for every ball $B = B(x,r)$. Also for this case the infimum of the constants in (30) will be denoted by $(w)_{1,\theta}$. We shall use the notation $A_p^\rho = \cup_{\theta\geq 0} A_p^{\rho,\theta}$, $p \geq 1$.

We also consider the calsses $A_p^{\rho,\mathrm{loc}}$, $1 < p < \infty$, defined as those w satisfying

$$\left(\int_B w\right)^{1/p}\left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \leq C|B|,$$

for every ball $B = B(x,r)$, with $r \leq \rho(x)$. We denote $A_\infty^{\rho,\mathrm{loc}} = \cup_{p>1} A_p^{\rho,\mathrm{loc}}$ (see [4] for details).

We remind an important property of these classes whose proof can be found in [el de classes of weights] (see Corollary 1 therein). Namely that for any $p > 1$ and $\gamma > 1$,

$$A_p^{\rho,\mathrm{loc}} = A_p^{\gamma\rho,\mathrm{loc}}. \tag{31}$$

It is not difficult to verify that if $p > 1$, then $A_p^\rho \subset A_p^{\rho,\mathrm{loc}}$. Next we present some properties of A_p^ρ classes.

Proposition 2. *Let $1 \leq p \leq q < \infty$.*

(i) If $w \in A_p^\rho$, then $w \in A_q^\rho$.
(ii) If $w \in A_p^\rho$, then $w^{1-p'} \in A_{p'}^\rho$.
(iii) If $w_1, w_2 \in A_1^\rho$, then $w_1 w_2^{1-p} \in A_p^\rho$.

Proof. Properties (i) and (ii) follows in the same way as one proceeds for Muckenhoupt's classes. In order to prove (iii), let $x_0 \in \mathbb{R}^d$, $r > 0$ and denote $B = B(x_0, r)$. As $w_1, w_2 \in A_1^\rho$, there exists a number $\theta \geq 0$ such that

$$w_i(B) \leq C|B|\left(1 + \frac{r}{\rho(x_0)}\right)^\theta \inf_B w_i,$$

for $i = 1, 2$. Thus

$$w_i(x)^{-1} \leq \sup_B w_i(x)^{-1} = \left(\inf_B w_i(x)\right)^{-1} \leq C\left(\frac{w_i(B)}{|B|}\right)^{-1}\left(1 + \frac{r}{\rho(x_0)}\right)^\theta.$$

Then, we have

$$\begin{aligned}
&\left(\int_B w_1 w_2^{1-p}\right)^{1/p}\left(\int_B (w_1 w_2^{1-p})^{-\frac{1}{p-1}}\right)^{1/p'} \\
&\quad\leq C w_1(B)^{1/p} w_2(B)^{(1-p)/p} |B|^{(p-1)/p}\left(1 + \frac{r}{\rho(x_0)}\right)^{(p-1)\theta/p} \\
&\qquad\times w_2(B)^{1/p'} w_1(B)^{1/(1-p)p'} |B|^{1/(p-1)p'}\left(1 + \frac{r}{\rho(x_0)}\right)^{\theta/(p-1)p'} \\
&\quad= C\left(1 + \frac{r}{\rho(x_0)}\right)^\theta.
\end{aligned} \tag{32}$$

□

Remark 4. It is worth mentioning that there is a precise control of the constants in properties (ii) and (iii) in Proposition 2. In fact, it follows easily from the definition of the class that $(w^{1-p'})_{p',\theta} = (w)_{p,\theta}$. With respect to (iii), it follows from (32) that $(w_1 w_2^{1-p'})_{p,\theta} \leq (w_1)_{1,\theta}(w_2)_{1,\theta}$.

For each $\theta \geq 0$, we define the maximal operator M^θ as

$$M^\theta f(x) = \sup_{B(x_0,r)\ni x} \left(1+\frac{r}{\rho(x_0)}\right)^{-\theta} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f|, \qquad f\in L^1_{\text{loc}}.$$

If for each $\theta \geq 0$ we denote $M^\theta = T_\theta$, we will see in the following proposition that A_p^ρ coincides with the U_p class of Section 2, for every $1<p<\infty$. Observe that $A_1^\rho = U_1$.

Notice that it is not true that for a fixed θ the class U_p^θ associated to M_θ coincides with A_p^θ.

Proposition 3. *Let $1<p<\infty$. A weight w belongs to A_p^ρ if and only if there exists $\theta\geq 0$ such that M^θ is bounded on $L^p(w)$.*

Proof. Let us start assuming that for a weight w there exist constants $\theta \geq 0$ and C such that

$$\int_{\mathbb{R}^d} |M^\theta f|^p w \leq C\int_{\mathbb{R}^d} |f|^p w, \tag{33}$$

for every f in $L^p(w)$.

Let $B=B(x,r)$ for some $x\in\mathbb{R}^d$ and $r>0$, and take $f=w^{-1/(p-1)}\chi_B$. By the definition of M^θ, and (33), we have

$$\left(1+\frac{r}{\rho(x)}\right)^{-\theta p}\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}}\right)^p\left(\int_B w\right) \leq C\int_{\mathbb{R}^d}|f|^p w = C\int_B w^{-\frac{1}{p-1}},$$

thus $w\in A_p^{\rho,\theta}$.

On the other hand, suppose now that $w\in A_p^\rho$. Then there exist $\theta\geq 0$ and C such that

$$\left(\int_B w\right)\left(\int_B w^{-\frac{1}{p-1}}\right)^{p-1} \leq C|B|^p\left(1+\frac{r}{\rho(x)}\right)^{\theta p}, \tag{34}$$

for every ball $B=B(x,r)$.

We will obtain the bound not for M^σ but for its equivalent centered version

$$\tilde{M}^\sigma f(x) = \sup_{r>0}\left(1+\frac{r}{\rho(x)}\right)^{-\sigma}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f|, \qquad f\in L^1_{\text{loc}},$$

since $M^\sigma f(x) \leq 2^d c_\rho \tilde{M}^{\sigma/(N_0+1)} f(x)$.

Let $\sigma\geq 0$ that we shall determine later. Observe that if $f\in L^p(w)$, we have

$$\tilde{M}^\sigma f(x) \leq M_1^\sigma f(x) + M_2^\sigma f(x),$$

where

$$M_1^\sigma f(x) = \sup_{r\leq\rho(x)}\left(1+\frac{r}{\rho(x)}\right)^{-\sigma}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f|,$$

and

$$M_2^\sigma f(x) = \sup_{r>\rho(x)}\left(1+\frac{r}{\rho(x)}\right)^{-\sigma}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f|.$$

Therefore, we have to check that M_1^σ and M_2^σ are bounded on $L^p(w)$.

Since for every $\sigma\geq 0$ and $x\in\mathbb{R}^d$,

$$M_1^\sigma f(x) \leq M_{\text{loc}} f(x) = \sup_{r\leq\rho(x_0),\, x\in B(x_0,r)} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|f|,$$

and $A_p^\rho \subset A_p^{\rho,\mathrm{loc}}$ the boundedness of M_1^σ follows from that of M_{loc} (see [4]).

In order to deal with M_2^σ, Let $\{Q_k\}_{k\geq 1}$ be a covering provided by Proposition 1. Now for $x \in Q_k$ we call $R_j = \{r : 2^{j-1}\rho(x) < r \leq 2^j\rho(x)\}$, and then we use (27) to obtain

$$\begin{aligned} M_2^\sigma f(x) &= \sup_{j\geq 1}\sup_{r\in R_j}\left(1+\frac{r}{\rho(x)}\right)^{-\sigma}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f| \\ &\leq 2^{d+\sigma}\sup_{j\geq 1}\frac{2^{-j(\sigma+d)}}{\rho(x)^d}\int_{B(x,2^j\rho(x))}|f| \\ &\leq C\sup_{j\geq 1}\frac{2^{-j(\sigma+d)}}{\rho(x_k)^d}\int_{c_jQ_k}|f|, \end{aligned}$$

with $c_j = 2^j(c_\rho 2^{N_0}+1)$.

Finally, from (34), we get

$$\begin{aligned} \int_{\mathbb{R}^d}|M_2^\sigma f|^p w &\leq \sum_{k\geq 1}\int_{Q_k}|M_2^\sigma f|^p w \\ &\leq C\sum_{k\geq 1}\sup_{j\geq 1}\frac{2^{-j(\sigma+d)p}}{\rho(x_k)^{dp}}\left(\int_{c_jQ_k}|f|\right)^p\left(\int_{Q_k}w\right) \\ &\leq C\sum_{k\geq 1}\sup_{j\geq 1}\frac{2^{-j(\sigma+d)p}}{\rho(x_k)^{dp}}\left(\int_{c_jQ_k}|f|^p w\right) \\ &\qquad\qquad \times\left(\int_{c_jQ_k}w^{-1/(p-1)}\right)^{p-1}\left(\int_{c_jQ_k}w\right) \\ &\leq C\sum_{k\geq 1}\sup_{j\geq 1}2^{-j(\sigma-\theta)p}\left(\int_{c_jQ_k}|f|^p w\right) \\ &\leq C\sum_{j\geq 1}2^{-j(\sigma-\theta)p}\left(\sum_{k\geq 1}\int_{c_jQ_k}|f|^p w\right) \\ &\leq C\left(\sum_{j\geq 1}2^{-j(\sigma p-\theta p-N_1)}\right)\int_{\mathbb{R}^d}|f|^p w, \end{aligned}$$

where the last series converges converges taking $\sigma > \theta + N_1/p$.

□

Remark 5. Observe that if we follow the constants in Proposition 3 we have

$$(w)_{p,\theta} \leq [w]_{p,\theta} \leq C(w)_{p,\sigma},$$

for $\sigma > (N_0+1)(\theta + N_1/p)$, and $C > 0$ independent of w.

Proposition 4. *The classes U_p, $1 \leq p < \infty$, satisfy u1), u2) and u3).*

Proof. Since $A_p^\rho = U_p$ (see Proposition 3), from the definition of A_p^ρ, property u1) follows in same way as for Muckenhoupt classes.

In order to verify u2), let $1 < p < \infty$, $\theta \geq 0$ and $w \in U_p^\theta$. From Proposition 3 we have $w \in A_p^{\rho,\theta}$, and then we use (ii) to obtain $w^{1-p'} \in A_{p'}^\rho$. Using again Proposition 3 we have $w^{1-p'} \in U_{p'}^{\theta'}$ with $\theta' > (N_0+1)(\theta + N_1/p)$ (see Proposition 3 for the meaning of the constants of this expression). Observe that due to Remark 4 and Remark 5 the contant $[w^{1-p'}]_{p',\theta'}$ depens on w only through $[w]_{p,\theta}$. Property u3) can be verified analogously.

□

4. Bounded mean oscillation results

As in the classical case most of th interesting operators do not preserve $L^\infty(w)$ even in the case $w = 1$.

Rather they map $L^\infty(w)$ into slightly larger sapces which are appropriated version of BMO, the John-Nirenberg space.

In this section we introduce these spaces characterizing them in terms of a suitable sharp maximal operator. Besides we present a local version of Fefferman-Stein inequality. With those tools we will be able to apply the general results on extrapolation of Section 2 to a variety of operators of our interest.

Given a weight w we define the space $BMO_\rho(w)$ as the set of functions f in $L^1_{\text{loc}}(\mathbb{R}^d)$ satisfying for some constant C,

$$\frac{\|\chi_B w\|_{L^\infty}}{|B|}\int_B |f - f_B| \le C, \qquad r < \rho(x), \tag{35}$$

$$\frac{\|\chi_B w\|_{L^\infty}}{|B|}\int_B |f| \le C, \qquad r = \rho(x), \tag{36}$$

for every ball $B \subset \mathbb{R}^d$, where $f_B = \frac{1}{|B|}\int_B f$.

A norm $\|f\|_{BMO_\rho(w)}$ in $BMO_\rho(w)$ is defined as the least constant satisfying (35) and (36).

If we take the limit case $\rho \equiv \infty$, the above definition gives one of the weighted versions of bounded mean oscillation spaces introduced by Muckenhoupt and Wheeden in [14]. Also let us notice that another version of these spaces was considered in [3] although both definitions coincide whenever the weight w is such that w^{-1} belongs to the Muckenhoupt class A_1.

The next definitions shall require to consider for some $\alpha > 0$ the family of balls

$$\mathcal{B}_{\rho,\alpha} = \{B(z,r):\ z \in \mathbb{R}^d,\ r \le \alpha\rho(z)\}. \tag{37}$$

Given $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define a localized version of the sharp function as

$$M^\sharp_{\text{loc}} f(x) = \sup_{x\in B\in\mathcal{B}_{\rho,1}} \frac{1}{|B|}\int_B |f(y) - f_B|dy + \sup_{x\in B=B(y,\rho(y))} \frac{1}{|B|}\int_B |f(y)|dy.$$

As it is expected, the space $BMO_\rho(w)$ can be described by means of $M^\sharp_{\text{loc}}$ defined above.

Lema 2. *Let w be a weight. If f belongs to L^1_{loc}, then*

$$\frac{1}{2}\|M^\sharp_{loc}(f)w\|_{L^\infty} \le \|f\|_{BMO_\rho(w)} \le \|M^\sharp_{loc}(f)w\|_{L^\infty}.$$

Proof. We start proving

$$\|f\|_{BMO_\rho(w)} \le \|M^\sharp_{\text{loc}}(f)w\|_{L^\infty}. \tag{38}$$

Now let $B = B(x_0, r)$ with $r < \rho(x_0)$. Then for almost every x in B we have

$$w(x)\left(\frac{1}{|B|}\int_B |f - f_B|\right) \le w(x)M^\sharp_{\text{loc}}(f)(x) \le \|M^\sharp_{\text{loc}}(f)w\|_{L^\infty}.$$

In the case $r = \rho(x_0)$, for almost every x in B,

$$w(x)\left(\frac{1}{|B|}\int_{\mathbb{R}^d}|f|\right) \leq w(x)M^{\sharp}_{\mathrm{loc}}(f)(x) \leq \|M^{\sharp}_{\mathrm{loc}}(f)w\|_{L^\infty},$$

and thus, taking supermum in x over B, we get (38).

For other inequality, if $B = B(x_0, r)$ with $r < \rho(x_0)$, for almost every x in B we have

$$w(x)\left(\frac{1}{|B|}\int_B |f - f_B|\right) \leq \|w\chi_B\|_{L^\infty}\left(\frac{1}{|B|}\int_B |f - f_B|\right) \leq \|f\|_{BMO_\rho(w)}. \tag{39}$$

Proceding in the same way for averages over balls of the form $B = B(y, \rho(y))$ we get the desired result.

□

Given $\alpha > 0$ we define the following maximal operators appearing in [5] for $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$M_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g|,$$

$$M^{\sharp}_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g - g_B|,$$

where $\mathcal{B}_{\rho,\alpha}$ is defined in (37).

A weighted version of Lemma 2 in [5] is presented as follows.

Lema 3. *If $1 < p < \infty$ and $w \in A^{\rho,loc}_\infty$, then there exist constants $\beta_0 > 0$ and $C > 0$ independent of p such that*

$$\|M_{\rho,\beta_0}g\|_{L^p(w)} \leq C\|M^{\sharp}_{loc}g\|_{L^p(w)},$$

for every $g \in L^1_{loc}(\mathbb{R}^d)$.

Proof. The proof is based on weighted Fefferman-Stein inequalities given in [15] for homogeneous type spaces of finite measure. We shall apply their result to balls. Let us remind the definitions of the maximal and sharp maximal functions for a fixed ball $Q \subset \mathbb{R}^d$, a function $g \in L^1(Q)$, as

$$M_Q g(x) = \sup_{x\in B\in\mathcal{F}(Q)} \frac{1}{|B\cap Q|}\int_{B\cap Q}|g|, \quad x \in Q, \tag{40}$$

and

$$M^{\sharp}_Q g(x) = \sup_{x\in B\in\mathcal{F}(Q)} \frac{1}{|B\cap Q|}\int_{B\cap Q}|g - g_{B\cap Q}|, \quad x \in Q, \tag{41}$$

where $\mathcal{F}(Q) = \{B(y, r) : y \in Q, r > 0\}$. The corresponding Muckenhoupt classes will be denoted by $A_p(Q)$, for $p \geq 1$.

Following the proof of Lemma 2 in [5], if Q is a critical ball with respect to ρ_0 and $x \in Q$, it is easy to see

$$M_{\rho,\beta_0}g(x) = M_{\rho_0,\beta}g(x) \leq M_{2Q}(g\chi_{2Q})(x). \tag{42}$$

Also if $x \in 2Q$, then for some constant C,

$$M^{\sharp}_{2Q}(g\chi_{2Q})(x) \leq CM^{\sharp}_{\rho_0,2}g(x). \tag{43}$$

We use a decomposition Q_k, $k \geq 0$, of the space $\mathbb{R}^d$ given by Proposition 1 associated to the critical radius function $\rho_0 = \rho/c_0$, where $c_0 = 4c_\rho 3^{N_0}$.

We denote $\beta = \frac{1}{2c_{\rho_0}^2}$, where c_{ρ_0} is the constant in (27) associated to ρ_0, and take $\beta_0 = \beta/c_0$.

Then, if we call $w_k = w\chi_{2Q_k}$, we have

$$\int_{\mathbb{R}^d} |M_{\rho,\beta_0}(g)|^p w \leq \sum_k \int_{Q_k} |M_{\rho,\beta_0}(g)|^p w_k.$$

Using (31) it is easy to check that if $w \in A_\infty^{\rho,\mathrm{loc}}$, then $w_k \in A_\infty(2Q_k)$, for every $k \geq 1$ with a constant independent of k. Therefore, we are able to apply Proposition 3.4 in [15] and inequalities (42) and (43), to obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |M_{\rho,\beta_0}(g)|^p w &\leq \sum_k \int_{2Q_k} |M_{2Q_k}(g\chi_{2Q_k})|^p w_k \\ &\leq C \sum_k \left\{ w_k(2Q_k)(g_{2Q_k})^p + \|M^\sharp_{2Q_k}(g\chi_{2Q_k})\|^p_{L^p(w_k,2Q_k)} \right\} \\ &\leq C \sum_k \left\{ w(2Q_k)\left(\frac{1}{|2Q_k|}\int_{2Q_k} |g|\right)^p + \int_{2Q_k} |M^\sharp_{\rho_0,2}(g)|^p w \right\}.\end{aligned} \tag{44}$$

Since $M^\sharp_{\rho_0,2} g = M^\sharp_{\rho,\frac{2}{c_0}} g \leq M^\sharp_{\rho,1} g$, and using the bounded overlapping due to Proposition 1, it follows

$$\begin{aligned}\sum_k \int_{2Q_k} |M^\sharp_{\rho_0,2} g|^p w &\leq \int_{\mathbb{R}^d} \Big(\sum_k \chi_{2Q_k}\Big) |M^\sharp_{\rho,1} g|^p w \\ &\leq C \int_{\mathbb{R}^d} |M^\sharp_{\rho,1} g|^p w \\ &\leq C \int_{\mathbb{R}^d} |M^\sharp_{\mathrm{loc}} g|^p w.\end{aligned}$$

In oder to deal with the other term of the left hand side of (44) we first see that due to (27) there exists a constant C independent of k such that

$$2Q_k \subset B(x, \rho(x)) \subset 2CQ_k,$$

for all $x \in 2Q_k$. From the previous inclusion and Proposition 1 it follows

$$\begin{aligned}&\sum_k w(2Q_k)\left(\frac{1}{|2Q_k|}\int_{2Q_k} |g(y)|dy\right)^p \\ &\qquad \leq C \sum_k \int_{2Q_k} w(x)\left(\frac{1}{\rho(x)^d}\int_{B(x,\rho(x))} |g(y)|dy\right)^p dx \\ &\qquad \leq C \int_{\mathbb{R}^d} w(x)\left(\frac{1}{\rho(x)^d}\int_{B(x,\rho(x))} |g(y)|dy\right)^p dx \\ &\qquad \leq C \int_{\mathbb{R}^d} |M^\sharp_{\mathrm{loc}} g(x)|^p w(x) dx.\end{aligned}$$

□

As a consequence of Lemma 3 and the fact that $g \leq M_{\rho,\beta} g$ for every $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\beta > 0$, we have the following corollary.

Corollary 5. *Let $1 < p < \infty$ and $w \in A_\infty^{\rho,loc}$. If g belongs to L^1_{loc}, then there exits a constant C such that*

$$\|g\|_{L^p(w)} \leq C\|M^\sharp_{loc} g\|_{L^p(w)}.$$

5. Some applications to Schrödinger settings

We begin this section dealing with an operator $\mathcal{T}$ with a kernel K (in the principal value sense) satisfying some Calderón-Zygmund or Hörmander type conditions.

Theorem 5. *Let $r > 0$ and $\mathcal{T}$ a bounded operator from $L^\infty(w)$ into $BMO^\rho(w)$ for any w such that $w^{-r} \in A_1^\rho$ with constants depending on w only through $(w^{-r})_{1,\theta}$ whenever $w^{-r} \in A_1^\theta$. Then $\mathcal{T}$ is bounded on $L^p(w)$ for $r < p < \infty$ and every $w \in A^\rho_{p/r}$. Furthermore,*

$$\left\|\left(\sum_i |\mathcal{T} f_i|^q\right)^{1/q}\right\|_{L^p(w)} \leq C\left\|\left(\sum_i |f_i|^q\right)^{1/q}\right\|_{L^p(w)}, \quad \{f_i\}_{i\in\mathbb{N}} \in L^p_{l^q}(w), \tag{45}$$

holds for $r < p < \infty$, $1 < q < \infty$, and every $w \in A^\rho_{p/r}$.

Proof. By the hypothesis on $\mathcal{T}$ and Lemma 2 we have

$$\|M^\sharp_{loc}(\mathcal{T} f)w\|_{L^\infty} \leq C\|fw\|_{L^\infty},$$

for every weight w such that $w^{-r} \in A_1^\rho$ in the sense of (9). Due to Proposition 3, Proposition 4 and Remark 5 we can apply Corollary 4 to conclude that $M^\sharp_{loc}\mathcal{T}$ is bounded on $L^p(w)$ whenever $r < p < \infty$ and $\omega \in A^\rho_{p/r}$. Hence, the boundedness of $\mathcal{T}$ follows from Corollary 5. Finally, vector valued inequalities are consequence of Corollay 3.

□

Proposition 5. *Let $\mathcal{T}$ be a bounded operator on L^p for every $p > 1$, with a kernel K satisfying:*

(i) For each $N > 0$ there exists C_N such that

$$|K(x,y)| \leq C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{1}{|x-y|^d}, \tag{46}$$

for every $x, y \in \mathbb{R}$.

(ii) There exists a constants C and $\delta > 0$ such that

$$|K(x,y) - K(x_0,y)| \leq C\frac{|x-x_0|^\delta}{|x-y|^{d+\delta}}, \tag{47}$$

for every $x, y \in \mathbb{R}^d$, whenever $|x - x_0| < \frac{|x-y|}{2}$.

Then, $\mathcal{T}$ is bounded from $L^\infty(w)$ into $BMO_\rho(w)$ for every weight w such that $w^{-1} \in A_1^\rho$.

Proof. Let $x_0 \in \mathbb{R}^d$, $r \leq \rho(x_0)$ and $B = B(x_0, r)$. We write $f = f_1 + f_2 + f_3$, with $f_1 = f\chi_{2B}$ and $f_3 = f\chi_{B(x_0,2\rho(x_0))^c}$.

We start estimating $\frac{1}{|B|}\int_B |\mathcal{T} f_1| dx$. As $w^{-1} \in A_1^{\rho,\theta}$, for some $\theta > 0$, from Lemma 5 in [4] there exists $\gamma > 1$ such that $w^{-\gamma} \in A_1^\rho$. Also, from Hölder's inequality with exponent γ and considering that $\mathcal{T}$ is bounded on L^γ, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |\mathcal{T}f_1(x)|dx &\leq \left(\frac{1}{|B|}\int_B |\mathcal{T}f_1(x)|^\gamma dx\right)^{1/\gamma}\\ &\leq C\left(\frac{1}{|B|}\int_{2B} |f(x)|^\gamma dx\right)^{1/\gamma}\\ &\leq C\|fw\|_{L^\infty}\left(\frac{1}{|2B|}\int_{2B} |w(x)|^{-\gamma} dx\right)^{1/\gamma}\\ &\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}.\end{aligned}$$

We now estimate $|\mathcal{T}f_3(x)|$ for $x\in B$. We denote $\widetilde{B}=B(x_0,\rho(x_0))$, $\widetilde{B}_k=2^k\widetilde{B}$ and let $y\in \widetilde{B}_{k+1}\setminus\widetilde{B}_k$. Since $\rho(x)\leq C\rho(x_0)$ and $|x_0-y|\leq 2|x-y|$, we have from (46) for any $N>0$,

$$|K(x,y)|\leq \frac{C}{|y-x_0|^d}\left(1+\frac{|y-x_0|}{\rho(x_0)}\right)^{-N}\leq C\frac{2^{-kN}}{|\widetilde{B}_{k+1}|}. \tag{48}$$

In addition, from the fact that $\omega^{-1}\in A_1^{\rho,\theta}$ for some $\theta\geq 0$, we obtain

$$\begin{aligned}\int_{\widetilde{B}_{k+1}}|f| &\leq \|f\omega\|_{L^\infty}\int_{\widetilde{B}_{k+1}}\omega^{-1}\leq C(1+2^{k+1})^\theta|\widetilde{B}_{k+1}|\frac{\|f\omega\|_{L^\infty}}{\|\omega\chi_{\widetilde{B}_{k+1}}\|_{L^\infty}}\\ &\leq C2^{k\theta}|\widetilde{B}_{k+1}|\frac{\|f\omega\|_{L^\infty}}{\|\omega\chi_B\|_{L^\infty}}.\end{aligned} \tag{49}$$

Now we use (48) and then (49) to get,

$$\begin{aligned}|\mathcal{T}f_3(y)| &\leq \sum_{k\geq 1}\int_{\widetilde{B}_{k+1}\setminus\widetilde{B}_k}|K(y,z)||f(z)|dz\\ &\leq C\sum_{k\geq 1}\frac{2^{-kN}}{|\widetilde{B}_{k+1}|}\int_{\widetilde{B}_{k+1}}|f|\\ &\leq C\frac{||f\omega||_{L^\infty}}{||\omega\chi_B||_{L^\infty}}\sum_{k\geq 1}2^{-k(N-\theta)},\end{aligned}$$

where the last series converges taking $N>\theta$.

Finally we will estimate $|\mathcal{T}f_2(x)-c_B|$, where $c_B=\mathcal{T}f_2(x_0)$. Denoting $B_k=2^kB$ and $k_0=\max\{k: 2^kr<2\rho(x_0)\}$, and since $|x-x_0|<|y-x_0|/2$ for every $y\in(2B)^c$, we use (47) to obtain

$$\begin{aligned}
|\mathcal{T} f_2(x)-c_B| &\leq \int_{2\widetilde{B}\setminus 2B}|f(y)||K(x_0,y)-K(x,y)|dy\\
&\leq Cr^{\delta}\int_{2\widetilde{B}\setminus 2B}\frac{|f(y)|}{|x_0-y|^{d+\delta}}dy\\
&\leq Cr^{\delta}\sum_{k=1}^{k_0}\frac{1}{(2^kr)^{d+\delta}}\int_{B_{k+1}}|f(y)|dy\\
&\leq C\|fw\|_{L^\infty}\sum_{k=1}^{k_0}\frac{2^{-k\delta}}{|B_{k+1}|}\int_{B_{k+1}}w(y)^{-1}dy\\
&\leq C\|fw\|_{L^\infty}\sum_{k=1}^{k_0}\frac{2^{-k\delta}}{\|w\chi_{B_{k+1}}\|_{L^\infty}}\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{\theta}\\
&\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}\left(\sum_{k\geq 1}2^{-k\delta}\right)
\end{aligned}$$

where we have use the hypothesis $w^{-1}\in A_1^{\rho,\theta}$ and the fact that $\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{\theta}\leq 5^{\theta}$ for every $k\leq k_0$.

□

Proposition 6. *Let $1<s<\infty$, and $\mathcal{T}$ be a bounded operator from $L^{s'}$ into $L^{s',\infty}$ with a kernel K satisfying:*

(i) For each $N>0$ there exists C_N such that

$$\left(\int_{R<|y-x_0|\leq 2R}|K(x,y)|^s dy\right)^{1/s}\leq C_N R^{-d/s'}\left(\frac{\rho(x_0)}{R}\right)^{N}, \tag{50}$$

for every ball $B=B(x_0,\rho(x_0))$, $x\in B$ and $R>2\rho(x_0)$.

(ii) There exists a constant C such that

$$\sum_{k\geq 1}(2^kr)^{d/s'}\left(\int_{B_{k+1}\setminus B_k}|K(x,y)-K(x_0,y)|^s dy\right)^{1/s}\leq C, \tag{51}$$

for every ball $B=B(x_0,r)$ and every $x\in B$, with $r\leq\rho(x_0)$ and $B_k=2^kB$, $k\in\mathbb{N}$.

Then, $\mathcal{T}$ is bounded from $L^\infty(w)$ into $BMO_\rho(w)$ for every weight w such that $w^{-s'}\in A_1^{\rho}$.

Proof. Let w be such that $w^{-s'}\in A_1^{\rho,\theta}$ for some $\theta\geq 0$. Let $x_0\in\mathbb{R}^d$ and $B=B(x_0,r)$ with $r\leq\rho(x_0)$. We write $f=f_1+f_2+f_3$, with $f_1=f\chi_{2B}$ and $f_3=f\chi_{B(x_0,2\rho(x_0))^c}$. We start estimating the average $\frac{1}{|B|}\int_B|\mathcal{T} f_1|dx$ using Kolmogorov's inequality and the hypothesis on w to get

$$\begin{aligned}\frac{1}{|B|}\int_B |\mathcal{T}f_1(x)|dx &\leq C\frac{|B|^{1-1/s'}}{|B|}\left(\int_{2B}|f(x)|^{s'}dx\right)^{1/s'}\\ &\leq C\|fw\|_{L^\infty}\left(\frac{1}{|2B|}\int_{2B}w(x)^{-s'}dx\right)^{1/s'}\\ &\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_{2B}\|_{L^\infty}}\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta/s'}\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}.\end{aligned}$$

For $x\in B$ we shall estimate $|\mathcal{T}f_3(x)|$. Denoting $\widetilde{B}_k=2^k\widetilde{B}$, where $\widetilde{B}=B(x_0,\rho(x_0))$, and considering that $w^{-s'}\in A_1^{\rho,\theta}$,

$$\begin{aligned}\|f\chi_{\widetilde{B}_{k+1}}\|_{L^{s'}}\leq \|fw\|_{L^\infty}\|w^{-1}\chi_{\widetilde{B}_{k+1}}\|_{L^{s'}}&\leq C(1+2^{k+1})^{\theta/s'}|\widetilde{B}_{k+1}|^{1/s'}\frac{\|fw\|_{L^\infty}}{\|w\chi_{\widetilde{B}_{k+1}}\|_{L^\infty}}\\ &\leq C2^{k(\theta+d)/s'}\rho(x_0)^{d/s'}\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}.\end{aligned}$$

Now we use (50) to obtain,

$$\begin{aligned}|\mathcal{T}f_3(x)|&\leq\sum_{k\geq1}\int_{\widetilde{B}_{k+1}\setminus\widetilde{B}_k}|K(x,y)||f(y)|dy\\ &\leq\sum_{k\geq1}\left(\int_{\widetilde{B}_{k+1}\setminus\widetilde{B}_k}|K(x,y)|^s dy\right)^{1/s}\left(\int_{\widetilde{B}_{k+1}}|f|^{s'}\right)^{1/s'}\\ &\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}\sum_{k\geq1}2^{k(\theta+d)/s'}\rho(x_0)^{d/s'}\left(\int_{\widetilde{B}_{k+1}\setminus\widetilde{B}_k}|K(x,y)|^s dy\right)^{1/s}\\ &\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}\sum_{k\geq1}2^{k(\theta-N)},\end{aligned}$$

where the last series converges choosing N large enough.

Finally, we estimate $|\mathcal{T}f_2(x)-c_B|$, with $c_B=\mathcal{T}f_2(x_0)$. Let $B_k=2^kB$ and $k_0=\max\{k:2^kr<2\rho(x_0)\}$. We use again that $w^{-s'}\in A_1^{\rho,\theta}$, to obtain for every $k\leq k_0$,

$$\begin{aligned}\|f\chi_{B_{k+1}}\|_{L^{s'}}&\leq\|fw\|_{L^\infty}\|w^{-1}\chi_{B_{k+1}}\|_{L^{s'}}\\ &\leq C\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{\theta/s'}|B_{k+1}|^{1/s'}\frac{\|fw\|_{L^\infty}}{\|w\chi_{B_{k+1}}\|_{L^\infty}}\\ &\leq C(2^kr)^{d/s'}\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}.\end{aligned}$$

Last inequality together with (51) implies

$$\begin{aligned}|\mathcal{T}f_2(x)-c_B| &\leq \int_{2\widetilde{B}\setminus 2B}|f(y)||K(x,y)-K(x_0,y)|dy\\ &\leq \sum_{k=1}^{k_0}\left(\int_{B_{k+1}\setminus B_k}|K(x,y)-K(x_0,y)|^s dy\right)^{1/s}\left(\int_{B_{k+1}}|f|^{s'}\right)^{1/s'}\\ &\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}\sum_{k=1}^{k_0}(2^k r)^{d/s'}\left(\int_{B_{k+1}\setminus B_k}|K(x,y)-K(x_0,y)|^s dy\right)^{1/s}\\ &\leq C\frac{\|fw\|_{L^\infty}}{\|w\chi_B\|_{L^\infty}}.\end{aligned}$$

□

Remark 6. Condition (50) is equivalent to assume that for every $N>0$ there exists $C_N>0$ such that

$$\left(\int_{2^k<\frac{|y-x_0|}{\rho(x_0)}\leq 2^{k+1}}|K(x,y)|^s dy\right)^{1/s}\leq C_N 2^{-k(N+d/s')}\rho(x_0)^{-d/s'}, \tag{52}$$

for all $x_0\in\mathbb{R}^d$.

We end this section considering a Schrödinger operator in $\mathbb{R}^d$ with $d\geq 3$,

$$\mathcal{L}=-\Delta+V$$

where $V\geq 0$ is a function that satisfies for some $q>d/2$, the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q}\leq\frac{C}{|B|}\int_B V(y)^q dy,$$

for every ball $B\subset\mathbb{R}^d$.

We shall apply Proposition 5 and Proposition 6 to some operators associated to $\mathcal{L}$ that were considered in [19] for the unweighted case.

Theorem 6. *If $V\in RH_q$ with $q>d$, then the operators $(-\Delta+V)^{-1/2}\nabla$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ are bounded from $L^\infty(w)$ into $BMO_\rho(w)$ for every weight w such that $w^{-1}\in A_1^\rho$.*

Proof. It was proven in [19] (see Theorem 0.8) that $(-\Delta+V)^{-1/2}\nabla$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ are Calderón-Zygmund operators. Moreover, their kernels satisfy (46) (see estimate (6.5) given in [19]). Therefore, the result follows from Proposition 5.

□

Theorem 7. *If $V\in RH_q$ with $q>d/2$, then the operator $(-\Delta+V)^{i\gamma}$ is bounded from $L^\infty(w)$ into $BMO_\rho(w)$ for every weight w such that $w^{-1}\in A_1^\rho$.*

Proof. As in the proof of Theorem 6 the result follows from Proposition 5, where the hypothesis are satisfied due to Theorem 0.4 and estimate (4.3) in [19].

□

Theorem 8. *If $V \in RH_q$ with $q > d/2$, then the operators $(-\Delta+V)^{-1/2}V^{1/2}$, $(-\Delta+V)^{-1}V$ and $(-\Delta+V)^{-1/2}\nabla$ are bounded from $L^\infty(w)$ into $BMO_\rho(w)$ for every weight w such that $w^{-s'} \in A_1^\rho$, with $s = 2q$, $s = q$ and $1/s = 1/q - 1/d$ when $q < d$, respectively.*

Proof. Denoting $T_1 = (-\Delta+V)^{-1/2}V^{1/2}$, $T_2 = (-\Delta+V)^{-1}V$ and $T_3 = (-\Delta+V)^{-1/2}\nabla$ we shall apply Proposition 6 to each T_j, $j = 1, 2, 3$. From [19] (see Theorem 5.10, Theorem 3.1 and Theorem 0.5), the operators T_1, T_2 y T_3 are bounded on L^p for $p \geq (2q)'$, $p \geq q'$ and $p \geq s'$, respectively.

The proof that the kernels of T_j, $j = 1, 2, 3$ satisfy condition (51) it is contained in Theorem 1 in [11].

Condition (52) for the kernel of T_3 follows from Lema 7 in [2]. Hence, it only left to verify (52) for the kernels of T_1 y T_2.

Let $x_0 \in \mathbb{R}^d$, $x \in B = B(x_0, \rho(x_0))$ and $B_k = 2^k B$. As $x \in B$, we have $\rho(x) \geq c\rho(x_0)$ for some constant c. Also, $|x-y| \geq 2^{k-1}\rho(x_0)$ for every $y \in B_{k+1}\setminus B_k$. Thus, from Lemma 2 and Lemma 3 in [11], for each $N > 0$, there exists $C > 0$ such that

$$|K_i(x,y)| \leq C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N} \frac{1}{|x-y|^{d-j}} V(y)^{j/2} \leq C \frac{2^{-kN}}{(2^k\rho(x_0))^{d-j}} V(y)^{j/2}$$

Therefore, since $V \in RH_q$, the measure $V(x)dx$ is doubling and from formula (19) in [2] we obtain

$$\begin{aligned}
&\left(\int_{B_{k+1}\setminus B_k} |K(x,y)|^{2q/j} dy\right)^{j/(2q)} \\
&\qquad \leq C \frac{2^{-kN}}{(2^k\rho(x_0))^{d-j}} \left(\int_{B_{k+1}} V(y)^q dy\right)^{j/(2q)} \\
&\qquad \leq C \frac{2^{-kN}}{(2^k\rho(x_0))^{d/(2q/j)'-j+dj/2}} \left(\int_{B_{k+1}} V(y) dy\right)^{j/2} \\
&\qquad \leq C \frac{2^{-kN}}{(2^k\rho(x_0))^{d/(2q/j)'-j+dj/2}} (2^k\rho(x_0))^{dj/2-j}(2^{kd\mu j/2}) \\
&\qquad \leq C 2^{-k(N-d\mu j/2)}(2^k\rho(x_0))^{-d/(2q/j)'},
\end{aligned}$$

thus we get (52) choosing N large enough. □

Theorem 9. *If $V \in RH_q$ with $q > d$, then the operators $(-\Delta+V)^{-1/2}\nabla$, $\nabla(-\Delta+V)^{-1/2}$ and $\nabla(-\Delta+V)^{-1}\nabla$ are bounded on $L^p(w)$, for $1 < p < \infty$ and for every weight $w \in A_p^\rho$. Moreover, they satisfy the vector valued inequality (45) with $r = 1$.*

Proof. Las consequencias se siguen del teorema 5 y los teoremas 6, 7, 8. □

Theorem 10. *If $V \in RH_q$ with $q > d/2$, then the operator $(-\Delta+V)^{i\gamma}$ is bounded on $L^p(w)$, for $1 < p < \infty$ and for every weight $w \in A_p^\rho$. Moreover, it satisfy the vector valued inequality (45) with $r = 1$.*

Theorem 11. *If $V \in RH_q$ with $q > d/2$, then the operators $(-\Delta+V)^{-1/2}V^{1/2}$, $(-\Delta+V)^{-1}V$ and $(-\Delta+V)^{-1/2}\nabla$ are bounded on $L^p(w)$, for $s' < p < \infty$ and for every weight $w \in A_{p/s'}^\rho$, with $s = 2q$, $s = q$ and $1/s = 1/q - 1/d$ when $q < d$, respectively. Moreover, they satisfy the vector valued inequality (45) with $r = s'$.*

We end this section with a result that proves boundedness of the fractional integral associated to $\mathcal{L}$ defined for a given α, with $0<\alpha<d$, as

$$\mathcal{I}_\alpha f(x)=\mathcal{L}^{-\alpha/2}f(x)=\int_0^\infty e^{-t\mathcal{L}}f(x)\,t^{\alpha/2}\,\frac{dt}{t},\quad x\in\mathbb{R}^d, \tag{53}$$

where $e^{-t\mathcal{L}}$, $t>0$ denotes the heat semigroup associated to $\mathcal{L}$ with kernel k_t. It is known (see [13] and [9]) that for a given $N>0$ and $0<\delta<\min(1,2-\frac{d}{q})$, there exists a constant C such that

$$k_t(x,y)\le C\,t^{-d/2}\,e^{-\frac{|x-y|^2}{C\,t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \tag{54}$$

for all x and y in $\mathbb{R}^d$, and also

$$|k_t(x,y)-k_t(x_0,y)|\le C\left(\frac{|x-x_0|}{\sqrt{t}}\right)^{\delta}t^{-d/2}\,e^{-\frac{|x-y|^2}{C\,t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \tag{55}$$

whenever $|x-x_0|<\sqrt{t}$.

Theorem 12. *Let $V\in RH_q$ with $q>d/2$. The operator $\mathcal{I}_\alpha$ is bounded from $L^{d/\alpha}(w^{d/\alpha})$ into $BMO_\rho(w)$, for every w such that $w^{-1}\in A_1^\rho$.*

Proof. Let $x_0\in\mathbb{R}^d$, and consider $B=B(x_0,r)$ with $r\le\rho(x_0)$. As in Theorem 6, we write $f=f_1+f_2+f_3$, with $f_1=f\chi_{2B}$ and $f_3=f\chi_{B(x_0,2\rho(x_0))^c}$.

Now, we call $B_k=B(x_0,2^kr)$. From Hölder's inequality with exponent d/α and considering θ such that $w^{-1}\in A_1^{\rho,\theta}$ we obtain,

$$\begin{aligned}
\int_{(2B)^c}\frac{f(y)}{|x_0-y|^M}dy &\le \frac{1}{r^M}\sum_{k\ge1}\frac{1}{2^{kM}}\int_{B_{k+1}}f(y)dy\\
&\le \frac{\|fw\|_{L^{d/\alpha}}}{r^M}\sum_{k\ge1}\frac{1}{2^{kM}}\left(\int_{B_{k+1}}w(y)^{-d/(d-\alpha)}dy\right)^{(d-\alpha)/d}\\
&\le C\frac{\|fw\|_{L^{d/\alpha}}}{r^M}\sum_{k\ge1}\frac{(2^{k+1}r)^{d-\alpha}}{2^{kM}\|w\chi_{B_{k+1}}\|_{L^\infty}}\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{\theta}\\
&\le C\frac{\|fw\|_{L^{d/\alpha}}}{\|w\chi_B\|_{L^\infty}}r^{-M+d-\alpha}\left(\sum_{k\ge1}2^{-k(-M+d-\alpha+\theta)}\right).
\end{aligned} \tag{56}$$

Let us start estimating $\frac{1}{|B|}\int_B|\mathcal{I}_\alpha f_1|dx$. Using again that $w^{-1}\in A_1^{\rho,\theta}$, it follows

$$\begin{aligned}
&\frac{1}{|B|}\int_B|\mathcal{I}_\alpha f_1(x)|dx\\
&\qquad\le \frac{C}{|B|}\int_{2B}|f(y)|\left(\int_B\frac{1}{|x_0-y|^{d-\alpha}}dx\right)dy=C|B|^{\alpha/d-1}\int_{2B}|f(y)|dy\\
&\qquad\le C|B|^{\alpha/d-1}\|fw\|_{L^{d/\alpha}}\left(\int_{2B}w(y)^{-d/(d-\alpha)}dy\right)^{(d-\alpha)/d}\\
&\qquad\le C|B|^{\alpha/d-1}|B|^{1-\alpha/d}\frac{\|fw\|_{L^{d/\alpha}}}{\|w\chi_{2B}\|_{L^\infty}}\left(1+\frac{2r}{\rho(x_0)}\right)^{\theta}\\
&\qquad\le C\frac{\|fw\|_{L^{d/\alpha}}}{\|w\chi_B\|_{L^\infty}}.
\end{aligned}$$

Observe that if $N \geq M > d-\alpha$, we have

$$\begin{aligned} \int_0^\infty \left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-N} t^{(M-d+\alpha)/2}\frac{dt}{t} \\ \leq \int_0^{\rho(x)^2} t^{(M-d+\alpha)/2}\frac{dt}{t} \\ + \rho(x)^N \int_{\rho(x)^2}^\infty t^{(-N+M-d+\alpha)/2}\frac{dt}{t} \\ \leq C\rho(x)^{M-d+\alpha}. \end{aligned} \tag{57}$$

Now we will estimate uniformly $|\mathcal{I}_\alpha f_3(x)|$, for every $x \in B$. For $x \in B$ and $y \in 2B^c$, we have $\rho(x) \leq C\rho(x_0)$ and $|x_0 - y| \leq 2|x-y|$. By using (54) as well as inequalities (57) and (56), we have for each $N > \theta + d - \alpha$,

$$\begin{aligned} |\mathcal{I}_\alpha f_3(x)| &\leq \int_{B(x_0,2\rho(x_0))^c} |f(y)| \int_0^\infty k_t(x,y)t^{\alpha/2}\frac{dt}{t}dy \\ &\leq C\int_{B(x_0,2\rho(x_0))^c} |f(y)|\bigg(\int_0^\infty e^{-\frac{|x_0-y|^2}{ct}}\left(1+\frac{\sqrt{t}}{\rho(x_0)}\right)^{-N} t^{(\alpha-d)/2}\frac{dt}{t}\bigg)dy \\ &\leq C\bigg(\int_{B(x_0,2\rho(x_0))^c} \frac{|f(y)|}{|x_0-y|^N}dy\bigg)\bigg(\int_0^\infty \left(1+\frac{\sqrt{t}}{\rho(x_0)}\right)^{-N} t^{(N+\alpha-d)/2}\frac{dt}{t}\bigg) \\ &\leq C\frac{\|fw\|_{L^{d/\alpha}}}{\|w\chi_B\|_{L^\infty}}\rho(x)^{-N+d-\alpha}\rho(x)^{N-d+\alpha} = C\frac{\|fw\|_{L^{d/\alpha}}}{\|w\chi_B\|_{L^\infty}}. \end{aligned}$$

Finally we will estimate $|\mathcal{I}_\alpha f_2(x) - c_B|$, uniformly in $x \in B$, with

$$c_B = \int_{\mathbb{R}^d} |f_2(y)| \int_{r^2}^\infty k_t(x_0,y)t^{\alpha/2}\frac{dt}{t}dy.$$

$$\begin{aligned} |\mathcal{I}_\alpha f_2(x) - c_B| \leq \int_0^{r^2} t^{\alpha/2}\int_{\mathbb{R}^d} |f_2(y)|k_t(x,y)dy\frac{dt}{t} \\ + \int_{r^2}^\infty t^{\alpha/2}\int_{\mathbb{R}^d} |f_2(z)||k_t(x,y)-k_t(x_0,y)|dy\frac{dt}{t} = I + II \end{aligned}$$

By using estimates (54) and (56), and the fact that $|x_0 - y| \leq 2|x-y|$, we have

$$\begin{aligned} I &\leq C\int_{(2B)^c} |f(y)|\bigg(\int_0^{r^2} e^{-\frac{|x_0-y|^2}{ct}} t^{(\alpha-d)/2}\frac{dt}{t}\bigg)dy \\ &\leq C\bigg(\int_0^{r^2} t^{(N+\alpha-d)/2}\frac{dt}{t}\bigg)\bigg(\int_{B(x_0,2r)^c} \frac{|f(z)|}{|x_0-z|^N}dz\bigg) \leq C\frac{\|fw\|_{L^{d/\alpha}}}{\|w\chi_B\|_{L^\infty}}, \end{aligned}$$

for every $N > \theta + d - \alpha$.

In order to estimate II we call $B_k = B(x_0, 2^k r)$ and let $k_0 = \max\{k : 2^k r < 2\rho(x_0)\}$. From estimates (55) and (56) and considering that $\left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^\gamma \leq 5^\gamma$ for every $k \leq k_0$, we obtain

$$
\begin{aligned}
II &\leq C|x-x_0|^{\delta} \int_{B(x_0,2\rho(x_0))\setminus B(x_0,2r)} |f(y)| \left(\int_{r^2}^{\infty} e^{-\frac{|x_0-y|^2}{ct}} t^{(\alpha-d-\delta)/2} \frac{dt}{t} \right) dy \\
&\leq C r^{\delta} \left(\int_0^{\infty} e^{-t} t^{(d+\delta-\alpha)/2} \frac{dt}{t} \right) \left(\int_{B(x_0,2\rho(x_0))\setminus B(x_0,2r)} \frac{|f(y)|}{|x_0-y|^{d+\delta-\alpha}} dy \right) \\
&\leq C r^{\delta} \sum_{k=1}^{k_0} \int_{B_{k+1}\setminus B_k} \frac{|f(y)|}{|x_0-y|^{d+\delta-\alpha}} dy \leq C r^{\delta} \sum_{k=1}^{k_0} \frac{1}{|B_k|^{d+\delta-\alpha}} \int_{B_{k+1}} |f(y)| dy \\
&\leq C \frac{\|fw\|_{L^{d/\alpha}}}{r^{d-\alpha}} \sum_{k=1}^{k_0} \frac{|B_{k+1}|^{(d-\alpha)/d}}{2^{k(d+\delta-\alpha)}} \left(1 + \frac{2^{k+1} r}{\rho(x_0)} \right)^{\theta} \|w\chi_{B_{k+1}}\|_{L^{\infty}}^{-1} \\
&\leq C \frac{\|fw\|_{L^{d/\alpha}}}{\|w\chi_B\|_{L^{\infty}}} \left(\sum_{k\geq 1} 2^{-k\delta} \right),
\end{aligned}
$$

and this finishes the proof of the theorem.

□

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